作业三

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成绩: __________

第 1 题得分: ________. (i) 证明下列旋量波函数的表达式:

$$u_{\vec{p},s}=\eta\begin{pmatrix}\sqrt{\frac{E_p+m}{2E_p}}\varphi_s\\ 2s\sqrt{\frac{E_p-m}{2E_p}}\varphi_s\end{pmatrix}\quad v_{-\vec{p},s}=\eta'\begin{pmatrix}-2s\sqrt{\frac{E_p-m}{2E_p}}\varphi_s\\ \sqrt{\frac{E_p+m}{2E_p}}\varphi_s\end{pmatrix}$$

其中 $E_p=\sqrt{\vec{p}^2+m^2}$，η,η' 为任意相因子，$|\eta|=|\eta'|=1$，2×1 列矩阵 φ_s 为 $\vec{\tau}\cdot\hat{p}$ 的本征矢量，即

$$\vec{\tau}\cdot\hat{p}\varphi_s=2s\varphi_s$$

(ii) 讨论上述旋量波函数的非相对论极限。

解: (i) 旋量波函数 $u_{\vec{p},s}$ 满足

$$\begin{cases}(\vec{\alpha}\cdot\vec{p}+\beta m)u_{\vec{p},s}=E_pu_{\vec{p},s},\\ \vec{\sigma}\cdot\hat{p}u_{\vec{p},s}=2su_{\vec{p},s}.\end{cases}$$

其中 $\vec{\alpha}=\rho_1\vec{\sigma}=\begin{pmatrix}0&\vec{\tau}\\ \vec{\tau}&0\end{pmatrix},\beta=\begin{pmatrix}I&0\\0&-I\end{pmatrix}$。令 $u_{\vec{p},s}=\begin{pmatrix}u_1\\u_2\end{pmatrix}$，其中 u_1,u_2 分别表示两个 2×1 列矩阵。则上述方程可以写为:

$$\begin{pmatrix}m&\vec{\tau}\cdot\vec{p}\\ \vec{\tau}\cdot\vec{p}&-m\end{pmatrix}\begin{pmatrix}u_1\\u_2\end{pmatrix}=E_p\begin{pmatrix}u_1\\u_2\end{pmatrix};\quad\begin{pmatrix}\vec{\tau}\cdot\hat{p}&0\\0&\vec{\tau}\cdot\hat{p}\end{pmatrix}\begin{pmatrix}u_1\\u_2\end{pmatrix}=2s\begin{pmatrix}u_1\\u_2\end{pmatrix}.$$

可以验证:

$$\eta\left(m\sqrt{\frac{E_p+m}{2E_p}}+(2s)^2|\vec{p}|\sqrt{\frac{E_p-m}{2E_p}}\right)\varphi_s=\eta\sqrt{\frac{E_p+m}{2E_p}}\left(m+|\vec{p}|\frac{E_p-m}{\sqrt{E_p^2-m^2}}\right)\varphi_s=\eta\sqrt{\frac{E_p+m}{2E_p}}\varphi_s;$$

$$\eta\left(2s|\vec{p}|\sqrt{\frac{E_p+m}{2E_p}}-m2s\sqrt{\frac{E_p-m}{2E_p}}\right)\varphi_s=\eta(2s)\sqrt{\frac{E_p-m}{2E_p}}\left(|\vec{p}|\frac{E_p+m}{\sqrt{E_p^2-m^2}}-m\right)\varphi_s=\eta2s\sqrt{\frac{E_p-m}{2E_p}}\varphi_s.$$

其中，利用了 $\sqrt{E_p^2-m^2}=|\vec{p}|$ 和 $2s=\pm1,4s^2=1$。
同理，对于 $v_{-\vec{p},s}$，Dirac Hamiltonian 为:

$$\begin{cases}(\vec{\alpha}\cdot\vec{p}+\beta m)v_{-\vec{p},s}=-E_pu_{-\vec{p},s},\\ \vec{\sigma}\cdot\hat{p}v_{-\vec{p},s}=2sv_{-\vec{p},s}.\end{cases}$$

可以验证:

$$\eta'\left(-2sm\sqrt{\frac{E_p-m}{2E_p}}+2s|\vec{p}|\sqrt{\frac{E_p+m}{2E_p}}\right)\varphi_s=\eta'(-2s)\sqrt{\frac{E_p-m}{2E_p}}\left(m-|\vec{p}|\frac{E_p+m}{\sqrt{E_p^2-m^2}}\right)\varphi_s=-E_p\eta'(-2s)\sqrt{\frac{E_p-m}{2E_p}}\varphi_s;$$

$$\eta'\left(-(2s)^2|\vec{p}|\sqrt{\frac{E_p-m}{2E_p}}-m\sqrt{\frac{E_p+m}{2E_p}}\right)\varphi_s=-\eta'\sqrt{\frac{E_p+m}{2E_p}}\left(|\vec{p}|\frac{E_p-m}{\sqrt{E_p^2-m^2}}+m\right)\varphi_s=-\eta'\sqrt{\frac{E_p+m}{2E_p}}\varphi_s.$$

(ii) 对于非相对论极限，有 $m>>|\vec{p}|$，即有:

$$\frac{E_p}{m}=\sqrt{1+\frac{\vec{p}^2}{m^2}}=1+\frac{\vec{p}^2}{2m^2}+\mathcal{O}(\vec{p}^2/m^2).$$

则有:

$$\begin{cases}\dfrac{E_p+m}{2E_p}=\dfrac{1}{2}+\dfrac{1}{2(E_p/m)}=\dfrac{1}{2}+\dfrac{1}{2}\left(1-\dfrac{\vec{p}^2}{2m^2}\right)=1-\dfrac{\vec{p}^2}{4m^2};\\ \dfrac{E_p-m}{2E_p}=\dfrac{1}{2}-\dfrac{1}{2(E_p/m)}=\dfrac{1}{2}-\dfrac{1}{2}\left(1-\dfrac{\vec{p}^2}{2m^2}\right)=\dfrac{\vec{p}^2}{4m^2}.\end{cases}$$

进而有:

$$\begin{cases} \sqrt{\dfrac{E_p+m}{2E_p}} = \sqrt{1-\dfrac{\vec{p}^2}{4m^2}} = 1-\dfrac{\vec{p}^2}{8m^2}; \\ \sqrt{\dfrac{E_p-m}{2E_p}} = \dfrac{|\vec{p}|}{2m}. \end{cases}$$

则上述旋量波函数的非相对论极限为:

$$u_{\vec{p},s} = \eta \begin{pmatrix} \sqrt{1-\frac{\vec{p}^2}{4m^2}}\varphi_s \\ s\frac{|\vec{p}|}{m}\varphi_s \end{pmatrix} \quad v_{-\vec{p},s} = \eta' \begin{pmatrix} -s\frac{|\vec{p}|}{m}\varphi_s \\ \sqrt{1-\frac{\vec{p}^2}{4m^2}}\varphi_s \end{pmatrix}$$

□

第 2 题得分: ________. 证明

$$\sum_s u_{\vec{p},s} u_{\vec{p},s}^\dagger \beta = \frac{\not{p}+m}{2p_0}$$
$$\sum_s v_{\vec{p},s} v_{\vec{p},s}^\dagger \beta = \frac{\not{p}-m}{2p_0}$$

其中 $\not{p} = -i\gamma_\mu p_\mu$, $p_4 = ip_0 = i\sqrt{\vec{p}^2+m^2}$。

解: 可以计算得:

$$u_{\vec{p},s} u_{\vec{p},s}^\dagger \beta = \begin{pmatrix} \sqrt{\frac{E_p+m}{2E_p}}\varphi_s \\ 2s\sqrt{\frac{E_p-m}{2E_p}}\varphi_s \end{pmatrix} \begin{pmatrix} \sqrt{\frac{E_p+m}{2E_p}}\varphi_s^\dagger & 2s\sqrt{\frac{E_p-m}{2E_p}}\varphi_s^\dagger \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} = \begin{pmatrix} \frac{E_p+m}{2E_p}\varphi_s\varphi_s^\dagger & -s\frac{|\vec{p}|}{E_p}\varphi_s\varphi_s^\dagger \\ s\frac{|\vec{p}|}{E_p}\varphi_s\varphi_s^\dagger & -\frac{E_p-m}{2E_p}\varphi_s\varphi_s^\dagger \end{pmatrix}$$

对于等号右侧, 可以计算:

$$\frac{\not{p}+m}{2p_0} = \frac{1}{2E_p}(-i\vec{\gamma}\cdot\vec{p} - i\gamma_4 p_4 + m) = \frac{1}{2E_p}\begin{pmatrix} E_p+m & -\vec{\tau}\cdot\vec{p} \\ \vec{\tau}\cdot\vec{p} & m-E_p \end{pmatrix}$$

注意到:

$$\sum_s s|\vec{p}|\varphi_s\varphi_s^\dagger = \sum_s \tau\cdot\vec{p}\varphi_s\varphi_s^\dagger = \tau\cdot\vec{p}\sum_s \varphi_s\varphi_s^\dagger = \tau\cdot\vec{p}.$$

其中, 利用了 $\sum_s \varphi_s\varphi_s^\dagger = I$。则有:

$$\sum_s u_{\vec{p},s} u_{\vec{p},s}^\dagger \beta = \frac{1}{2E_p}\begin{pmatrix} (E_p+m)\sum_s \varphi_s\varphi_s^\dagger & \sum_s -2s|\vec{p}|\varphi_s\varphi_s^\dagger \\ \sum_s 2s|\vec{p}|\varphi_s\varphi_s^\dagger & (m-E_p)\sum_s \varphi_s\varphi_s^\dagger \end{pmatrix} = \frac{1}{2E_p}\begin{pmatrix} E_p+m & -\tau\cdot\vec{p} \\ \tau\cdot\vec{p} & m-E_p \end{pmatrix} = \frac{\not{p}+m}{2p_0}$$

同理, 对于第二个方程, 可以计算:

$$v_{\vec{p},s} v_{\vec{p},s}^\dagger \beta = \begin{pmatrix} -2s\sqrt{\frac{E_p-m}{2E_p}}\varphi_s \\ \sqrt{\frac{E_p+m}{2E_p}}\varphi_s \end{pmatrix} \begin{pmatrix} -2s\sqrt{\frac{E_p-m}{2E_p}}\varphi_s^\dagger & \sqrt{\frac{E_p+m}{2E_p}}\varphi_s^\dagger \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} = \begin{pmatrix} \frac{E_p-m}{2E_p}\varphi_s\varphi_s^\dagger & s\frac{|\vec{p}|}{E_p}\varphi_s\varphi_s^\dagger \\ -s\frac{|\vec{p}|}{E_p}\varphi_s\varphi_s^\dagger & -\frac{E_p+m}{2E_p}\varphi_s\varphi_s^\dagger \end{pmatrix}$$

求和得到:

$$\sum_s v_{\vec{p},s} v_{\vec{p},s}^\dagger \beta = \frac{1}{2E_p}\begin{pmatrix} (E_p-m)\sum_s \varphi_s\varphi_s^\dagger & \sum_s 2s|\vec{p}|\varphi_s\varphi_s^\dagger \\ \sum_s -2s|\vec{p}|\varphi_s\varphi_s^\dagger & -(E_p+m)\sum_s \varphi_s\varphi_s^\dagger \end{pmatrix} = \frac{1}{2E_p}\begin{pmatrix} E_p-m & \tau\cdot\vec{p} \\ -\tau\cdot\vec{p} & -E_p-m \end{pmatrix}$$

对于等号右侧, 可以计算:

$$\frac{\not{p}-m}{2p_0} = \frac{1}{2E_p}(-i\vec{\gamma}\cdot\vec{p} - i\gamma_4 p_4 - m) = \frac{1}{2E_p}\begin{pmatrix} E_p-m & \vec{\tau}\cdot\vec{p} \\ -\vec{\tau}\cdot\vec{p} & -E_p-m \end{pmatrix} = \sum_s v_{\vec{p},s} v_{\vec{p},s}^\dagger \beta.$$

□