作业四

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成绩: ________

第 1 题得分: ________. 对于自由 Dirac 场，证明下列非等时反对易关系

$$\{\psi_\alpha(x), \bar{\psi}_\beta(0)\} = i\left(\gamma_\mu \frac{\partial}{\partial x_\mu} - m\right)_{\alpha\beta} D(x)$$

其中

$$D(x) = \int \frac{d^3\vec{k}}{(2\pi)^3} e^{i\vec{k}\cdot\vec{x}} \frac{\sin\omega t}{\omega}$$

$$\omega = \sqrt{\vec{k}^2 + m^2}$$

解: Dirac 场算符为:

$$\psi_\alpha(\vec{r}, t) = \frac{1}{\sqrt{\Omega}} \sum_{\vec{p},s} \left[a_{\vec{p},s}(t)\left(u_{\vec{p},s}\right)_\alpha e^{i\vec{p}\cdot\vec{r}} + b^\dagger_{\vec{p},s}(t)\left(v_{\vec{p},s}\right)_\alpha e^{-i\vec{p}\cdot\vec{r}}\right].$$

对于自由 Dirac 场，由 Heisenberg 运动方程可知:

$$a_{\vec{p},s}(t) = a_{\vec{p},s} e^{-iE_p t} = a_{\vec{p},s} e^{-i\omega t}, \quad a^\dagger_{\vec{p},s}(t) = a^\dagger_{\vec{p},s} e^{iE_p t} = a^\dagger_{\vec{p},s} e^{i\omega t};$$
$$b_{\vec{p},s}(t) = b_{\vec{p},s} e^{-iE_p t} = b_{\vec{p},s} e^{-i\omega t}, \quad a^\dagger_{\vec{p},s}(t) = b^\dagger_{\vec{p},s} e^{iE_p t} = b^\dagger_{\vec{p},s} e^{i\omega t}.$$

从而场算符为:

$$\psi_\alpha(\vec{r}, t) = \frac{1}{\sqrt{\Omega}} \sum_{\vec{p},s} \left[a_{\vec{p},s}\left(u_{\vec{p},s}\right)_\alpha e^{i(\vec{p}\cdot\vec{r}-\omega t)} + b^\dagger_{\vec{p},s}\left(v_{\vec{p},s}\right)_\alpha e^{-i(\vec{p}\cdot\vec{r}-\omega t)}\right];$$
$$\bar{\psi}_\beta(0,0) = \frac{1}{\sqrt{\Omega}} \sum_{\vec{p},s} \left[a^\dagger_{\vec{p},s}\left(u^\dagger_{\vec{p},s}\gamma_4\right)_\beta + b_{\vec{p},s}\left(v^\dagger_{\vec{p},s}\gamma_4\right)_\beta\right]$$

可以计算非等时反对易关系如下:

$$\begin{aligned}
\{\psi_\alpha(x), \bar{\psi}_\beta(0)\} &= \frac{1}{\Omega} \sum_{\vec{p},s;\vec{p}',s'} \left\{a_{\vec{p},s}\left(u_{\vec{p},s}\right)_\alpha e^{i(\vec{p}\cdot\vec{r}-\omega t)} + b^\dagger_{\vec{p},s}\left(v_{\vec{p},s}\right)_\alpha e^{-i(\vec{p}\cdot\vec{r}-\omega t)},\ a^\dagger_{\vec{p}',s'}\left(u^\dagger_{\vec{p}',s'}\gamma_4\right)_\beta + b_{\vec{p},s}\left(v^\dagger_{\vec{p}',s'}\gamma_4\right)_\beta\right\} \\
&= \frac{1}{\Omega} \sum_{\vec{p},s} \left[\left(u_{\vec{p},s}\right)_\alpha \left(u^\dagger_{\vec{p},s}\gamma_4\right)_\beta e^{-i\omega t} + \left(v_{-\vec{p},s}\right)_\alpha \left(v^\dagger_{-\vec{p},s}\gamma_4\right)_\beta e^{i\omega t}\right] e^{i\vec{p}\cdot\vec{r}} \\
&= \frac{1}{\Omega} \sum_{\vec{p}} \left[\left(\sum_s u_{\vec{p},s} u^\dagger_{\vec{p},s}\beta\right)_{\alpha\beta} e^{-i\omega t} + \left(\sum_s v_{-\vec{p},s} v^\dagger_{-\vec{p},s}\beta\right)_{\alpha\beta} e^{i\omega t}\right] e^{i\vec{k}\cdot\vec{r}} \\
&= \frac{1}{\Omega} \sum_{\vec{p}} \left[\left(\frac{\not{p}+m}{2p_0}\right)_{\alpha\beta} e^{-i\omega t} + \left(\frac{-\not{p}-m}{2p_0}\right)_{\alpha\beta} e^{i\omega t}\right] e^{i\vec{k}\cdot\vec{r}} \\
&= \frac{1}{\Omega} \sum_{\vec{p}} \left[\left(\frac{-\not{p}-m}{\omega}\right)_{\alpha\beta} i\sin\omega t\right] e^{i\vec{k}\cdot\vec{r}} \\
&= \frac{1}{\Omega} \sum_{\vec{p}} i\left(\gamma_\mu \frac{\partial}{\partial x_\mu} - m\right)_{\alpha\beta} \frac{\sin\omega t}{\omega} e^{i\vec{k}\cdot\vec{r}} \\
&= i\left(\gamma_\mu \frac{\partial}{\partial x_\mu} - m\right)_{\alpha\beta} \int \frac{d^3\vec{k}}{(2\pi)^3} e^{i\vec{k}\cdot\vec{x}} \frac{\sin\omega t}{\omega}.
\end{aligned}$$

其中，利用了 $p_0 = \sqrt{\vec{k}^2 + m^2} = \omega$ 和 $\not{p} = -i\gamma_m u p_\mu = -\gamma_\mu \frac{\partial}{\partial x_\mu}$。
即有:

$$\left\{\psi_\alpha(x), \bar{\psi}_\beta(0)\right\} = i\left(\gamma_\mu \frac{\partial}{\partial x_\mu} - m\right)_{\alpha\beta} D(x).$$

□

第 2 题得分: ________. 证明 $\bar{\psi}(x)\gamma_\mu\psi(x)$ 为一 Lorentz 矢量。

解: 进行坐标变换之后满足:

$$\psi'(x') = D(a)\psi(x);$$

可以计算得:

$$\bar{\psi}'(x') = \psi'^{\dagger}\gamma_4 = \psi^{\dagger}(x)D^{\dagger}(a)\gamma_4 = \psi^{\dagger}(x)\gamma_4 D^{-1}(a).$$

则有:

$$\bar{\psi}'(x')\gamma_\mu\psi'(x') = \psi^{\dagger}(x)\gamma_4 D^{-1}(a)\gamma_\mu D(a)\psi(x) = \psi^{\dagger}(x)\gamma_4 a_{\mu\nu}\gamma_\nu\psi(x) = a_{\mu\nu}\psi^{\dagger}(x)\gamma_4\gamma_\nu\psi(x).$$

即 $\bar{\psi}(x)\gamma_\mu\psi(x)$ 为一 Lorentz 矢量。 □

第 3 题得分: ________. 证明电荷共轭旋量 $\psi^c(x) = \gamma_2\psi^*(x)$ 与 $\psi(x)$ 的 Lorentz 变换相同。

解: 要证明 $\psi^c(x) = \gamma_2\psi^*(x)$ 与 $\psi(x)$ 的 Lorentz 变换相同，即需要证明存在 $D(a)$ 满足:

$$\psi'(x') = D(a)\psi(x); \quad \psi'^c(x') = D(a)\psi^c(x).$$

第一个方程两边取复共轭，有:

$$\psi'^*(x') = D^*(a)\psi^*(x).$$

带入第二个方程得:

$$\psi'^c(x') = \gamma_2\psi'^*(x') = \gamma_2 D^*(a)\psi^*(x) = D(a)\gamma_2\psi^*(x).$$

即只需要证明:

$$\gamma_2 D^*(a) = D(a)\gamma_2.$$

考虑无穷小 Lorentz 变换的 $D(a) = 1 + \frac{i}{4}\varepsilon_{\mu\nu}\sigma_{\mu\nu}$。其中 $\varepsilon_{ij} = -\varepsilon_{ji}$ 为实数；$\varepsilon_{j4} = -\varepsilon_{4j}$ 为虚数；$\varepsilon_{\mu\mu} = 0$。则可以得到 $D(a)$ 为:

$$D(a) = 1 + \frac{i}{4}(\varepsilon_{ij}\sigma_{ij} + \varepsilon_{j4}\sigma_{j4} + \varepsilon_{4j}\sigma_{4j}) = 1 + \frac{i}{2}\left(\sum_{i<j}\varepsilon_{ij}\sigma_{ij} + \sum_{j=1}^{3}\varepsilon_{j4}\sigma_{j4}\right).$$

由 Pauli 矩阵的表达式可以计算:

$$\gamma_1^* = -\gamma_1; \quad \gamma_2^* = \gamma_2; \quad \gamma_3^* = -\gamma_3; \quad \gamma_4^* = \gamma_4.$$

进而可以计算:

$$\sigma_{12}^* = -\frac{1}{2i}(\gamma_1^*\gamma_2^* - \gamma_2^*\gamma_1^*) = \sigma_{12}; \quad \sigma_{13}^* = -\sigma_{13}; \quad \sigma_{23}^* = \sigma_{23}; \quad \sigma_{14}^* = \sigma_{14}; \quad \sigma_{24}^* = -\sigma_{24}; \quad \sigma_{34}^* = \sigma_{34}$$

则有:

$$D^*(a) = 1 - \frac{i}{2}\left(\varepsilon_{12}\sigma_{12} - \varepsilon_{13}\sigma_{13} + \varepsilon_{23}\sigma_{23} - \varepsilon_{14}\sigma_{14} + \varepsilon_{24}\sigma_{24} - \varepsilon_{34}\sigma_{34}\right).$$

可以计算得:

$$\begin{aligned}
\gamma_2\sigma_{12} &= \frac{1}{2i}(\gamma_2\gamma_1\gamma_2 - \gamma_2^2\gamma_1) = \frac{1}{2i}(\gamma_2\gamma_1 - \gamma_1\gamma_2)\gamma_2 = -\sigma_{12}\gamma_2;\\
\gamma_2\sigma_{13} &= \frac{1}{2i}(\gamma_2\gamma_1\gamma_3 - \gamma_2\gamma_3\gamma_1) = \frac{1}{2i}(\gamma_1\gamma_3 - \gamma_3\gamma_1)\gamma_2 = \sigma_{13}\gamma_2;\\
\gamma_2\sigma_{23} &= \frac{1}{2i}(\gamma_2^2\gamma_3 - \gamma_2\gamma_3\gamma_2) = \frac{1}{2i}(\gamma_3\gamma_2 - \gamma_2\gamma_3)\gamma_2 = -\sigma_{23}\gamma_2;\\
\gamma_2\sigma_{14} &= \frac{1}{2i}(\gamma_2\gamma_1\gamma_4 - \gamma_4\gamma_1\gamma_2) = \frac{1}{2i}(\gamma_1\gamma_4 - \gamma_4\gamma_1)\gamma_2 = \sigma_{14}\gamma_2;\\
\gamma_2\sigma_{24} &= \frac{1}{2i}(\gamma_2^2\gamma_4 - \gamma_2\gamma_4\gamma_2) = \frac{1}{2i}(\gamma_4\gamma_2 - \gamma_2\gamma_4)\gamma_2 = -\sigma_{24}\gamma_2;\\
\gamma_2\sigma_{34} &= \frac{1}{2i}(\gamma_2\gamma_3\gamma_4 - \gamma_2\gamma_4\gamma_3) = \frac{1}{2i}(\gamma_3\gamma_4 - \gamma_4\gamma_3)\gamma_2 = \sigma_{34}\gamma_2.
\end{aligned}$$

从而有:

$$\gamma_2 D^*(a) = \gamma_2 + \frac{i}{2}(\varepsilon_{12}\sigma_{12} + \varepsilon_{13}\sigma_{13} + \varepsilon_{23}\sigma_{23} + \varepsilon_{14}\sigma_{14} + \varepsilon_{24}\sigma_{24} + \varepsilon_{34}\sigma_{34})\gamma_2 = D(a)\gamma_2.$$

综上所述，电荷共轭旋量 $\psi^c(x) = \gamma_2\psi^*(x)$ 与 $\psi(x)$ 的 Lor □