作业二

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姓名: 董 建 宇
学号: 2019511017
成绩: ________

第 1 题得分: ________. 证明自由 Hermitian 标量场的对易关系

$$[\varphi(\vec{r}_1,t_1),\varphi(\vec{r}_2,t_2)]=-iD(x_1-x_2)$$

可以写成下列形式:

$$D(x_1-x_2)=\int\frac{d^3\vec{k}}{(2\pi)^3}e^{i\vec{k}\cdot(\vec{r}_1-\vec{r}_2)}\frac{\sin\omega(t_1-t_2)}{\omega}$$

$$\omega=\sqrt{\vec{k}^2+m^2}$$

而且 $D(x)$ 满足下列偏微分方程:

$$\left(-\frac{\partial^2}{\partial t^2}+\nabla^2-m^2\right)D(x)=0$$

和当 $t=0$ 时的初始条件:

$$D(x)=0,\dot{D}(x)=\delta^3(\vec{r})$$

其中 $x_1=(\vec{r}_1,it_1),x_2=(\vec{r}_2,it_2),x=(\vec{r},it)$

解: 由 Fourier 展开可以得:

$$\varphi(\vec{r},t)=\sum_{\vec{k}}\frac{1}{\sqrt{2\omega\Omega}}\left(a_{\vec{k}}e^{i\vec{k}\cdot\vec{r}-i\omega t}+a_{\vec{k}}^\dagger e^{-i\vec{k}\cdot\vec{r}+i\omega t}\right).$$

则可以计算

$$\begin{aligned}
[\varphi(\vec{r}_1,t_1),\varphi(\vec{r}_2,t_2)]=&\frac{1}{2\Omega}\sum_{\vec{k},\vec{k}'}\frac{1}{\sqrt{\omega\omega'}}[a_{\vec{k}}e^{i\vec{k}\cdot\vec{r}_1-i\omega t_1}+a_{\vec{k}}^\dagger e^{-i\vec{k}\cdot\vec{r}_1+i\omega t_1},a_{\vec{k}'}e^{i\vec{k}'\cdot\vec{r}_2-i\omega t_2}+a_{\vec{k}'}^\dagger e^{-i\vec{k}'\cdot\vec{r}_2+i\omega t_2}]\\
=&\frac{1}{2\Omega}\sum_{\vec{k}}\frac{1}{\omega}\left(e^{i\vec{k}\cdot(\vec{r}_1-\vec{r}_2)-i\omega(t_1-t_2)}-e^{-i\vec{k}(\vec{r}_1-\vec{r}_2)+i\omega(t_1-t_2)}\right)\\
=&\frac{1}{2\Omega}\sum_{\vec{k}}\frac{1}{\omega}\left(e^{i(\vec{k}\cdot\vec{r}-\omega t)}-e^{-i(\vec{k}\cdot\vec{r}-\omega t)}\right)\\
=&\sum_{\vec{k}}\frac{i}{\Omega}\frac{\sin(\vec{k}\cdot\vec{r}-\omega t)}{\omega}\\
=&i\int\frac{d^3\vec{k}}{(2\pi)^3}\frac{\sin(\vec{k}\cdot\vec{r})\cos(\omega t)-\cos(\vec{k}\cdot\vec{r})\sin(\omega t)}{\omega}
\end{aligned}$$

由于 $\frac{\sin(\vec{k}\cdot\vec{r})\cos(\omega t)}{\omega}$ 关于 $\vec{k}$ 是奇函数，所以体积分为零，因此上述对易子可进一步计算得:

$$\begin{aligned}
[\varphi(\vec{r}_1,t_1),\varphi(\vec{r}_2,t_2)]=&-i\int\frac{d^3\vec{k}}{(2\pi)^3}\frac{\cos(\vec{k}\cdot\vec{r})\sin(\omega t)}{\omega}\\
=&-i\int\frac{d^3\vec{k}}{(2\pi)^3}\frac{(\cos(\vec{k}\cdot\vec{r})+i\sin(\vec{k}\cdot\vec{r}))\sin(\omega t)}{\omega}\\
=&-i\int\frac{d^3\vec{k}}{(2\pi)^3}\frac{e^{i\vec{k}\cdot\vec{r}}\sin(\omega t)}{\omega}\\
=&-iD(x_1-x_2)
\end{aligned}$$

则有

$$D(x_1-x_2)=\int\frac{d^3\vec{k}}{(2\pi)^3}e^{i\vec{k}\cdot(\vec{r}_1-\vec{r}_2)}\frac{\sin\omega(t_1-t_2)}{\omega}$$

可以验证:

$$\left(-\frac{\partial^2}{\partial t^2}+\nabla^2-m^2\right)D(x)=\int\frac{d^3\vec{k}}{(2\pi)^3}\left(e^{i\vec{k}\cdot\vec{r}}\frac{\omega^2\sin\omega t}{\omega}-\vec{k}^2e^{i\vec{k}\cdot\vec{r}}\frac{\sin\omega t}{\omega}-m^2e^{i\vec{k}\cdot\vec{r}}\frac{\sin\omega t}{\omega}\right)$$

$$= \int \frac{d^3\vec{k}}{(2\pi)^3}(\omega^2 - \vec{k}^2 - m^2)e^{i\vec{k}\cdot\vec{r}}\frac{\sin\omega t}{\omega}$$

由 $\omega = \sqrt{\vec{k}^2 + m^2}$ 可知：

$$\left(-\frac{\partial^2}{\partial t^2} + \nabla^2 - m^2\right)D(x) = 0$$

下面验证初始条件，当 $t = 0$ 时，容易发现：

$$D(x) = 0.$$

计算对时间的一阶导数可得：

$$\dot{D}(x) = \int \frac{d^3\vec{k}}{(2\pi)^3}e^{i\vec{k}\cdot\vec{r}} = \delta^3(\vec{r}).$$

□