SOLID PHYSICS II

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1 I would like you to do a bit of the extended reading on the keyword of $g - factor$. There is no need to derive the origin it, nevertheless, you should answer the following questions: what is g-factor? Where does it come from? How was it measured?

g 因子为磁矩与角动量量子数乘玻尔磁矩的比值。

$$\boldsymbol{\mu}_J = g_J \frac{e}{2m} \boldsymbol{J}.$$

其中 $\boldsymbol{J}$ 为任意角动量，对于轨道角动量，$g_L = 1$；对于自旋角动量，$g_S = 2$。

可以通过测量在稳恒磁场 B_0 下的能级劈裂 $\Delta E = g\mu_B B_0$，从而反推出 g 因子。

考虑电磁场中的 Dirac 方程：

$$\gamma_\mu \left(\frac{\partial}{\partial x_\mu} - ieA_\mu \right) \psi + m\psi = 0$$

在非相对论近似下，$E \approx m + \frac{p^2}{2m}$, 令 $\psi = e^{-imt}\chi$, 方程化为：

$$i\frac{\partial \chi}{\partial t} = [-i\vec{\alpha} \cdot (\vec{\nabla} - ie\vec{A}) + (\beta - 1)m + eA_0]\chi.$$

令 $\chi = \begin{pmatrix} \varphi \\ \varphi' \end{pmatrix}$, 方程可化为：

$$i\frac{\partial \varphi}{\partial t} = -\frac{1}{2m}(\vec{\nabla} - ie\vec{A})^2\varphi - \frac{e}{2m}\vec{\tau} \cdot \vec{B}\varphi + eA_0\varphi.$$

取自旋算符 $\vec{S} = \frac{1}{2}\vec{\tau}$,Bohr 磁矩 $\mu_B = \frac{e}{2m}$，方程为：

$$i\frac{\partial \varphi}{\partial t} = -\frac{1}{2m}(\vec{\nabla} - ie\vec{A})^2\varphi - g\mu_B\vec{S} \cdot \vec{B}\varphi + eA_0\varphi.$$

即自旋回磁比 (gyromagnetic radio)$g = 2$.

2 You should have a basic feeling of the units that are mentioned in the field of magnetism. Unfortunately, the magnetic units are quite a mess and confusing, due to historical reasons. In general, two systems are used (you usually see a mixture of the two in literatures), i.e., cgs and SI units. Try to compile a list for the conversion for the following quantities: magnetic field, energy, magnetic moment and magnetisation.

	基本单位制	常用单位及单位换算
磁感应强度 B	$1T = 1kg \cdot A^{-1} \cdot s^{-2}$	$1T = 10^4 Gs$
磁场强度 H	$1A \cdot m^{-1}$	$1A \cdot m^{-1} = 4\pi \times 10^{-3} Oe$
能量 E	$1J = 1kg \cdot m^2 \cdot s^{-2}$	$1eV \approx 1.6 \times 10^{-19} J$
磁矩 μ	$1A \cdot m^2$	$1\mu_B = 9.27 \times 10^{-24} A \cdot m^2$
磁化强度 M	$1A \cdot m^{-1}$	

其中 $\mu_B = \frac{e\hbar}{2m_e}$；磁化强度为单位体积内的磁矩。

3 Try to prove that g_J (called Landé g-factor) is in fact the add-up effect of $\hat{J} = \hat{L} + \hat{S}$, such that

$$g_J = \frac{3}{2} + \frac{S(S+1) - L(L+1)}{2J(J+1)} \tag{1}$$

By the definition, the g-factor is

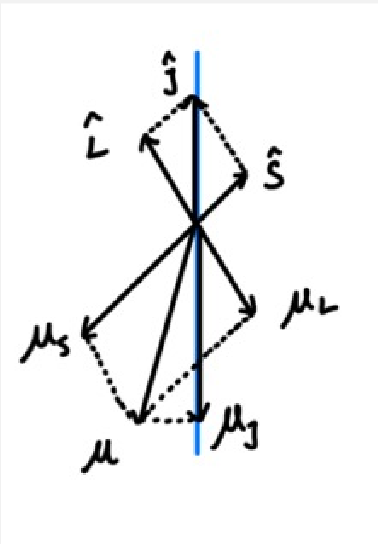
$$g_J = \frac{\mu_J/\mu_B}{J},$$

where we let $\hbar = 1$. According to Law of Cosines, we could determine μ_J as

$$\mu_J = \mu_L \cos(\vec{L}, \vec{J}) + \mu_S \cos(\vec{S}, \vec{J}).$$

Like the figure, we could determine that

$$\cos(\vec{L}, \vec{J}) = \frac{\hat{L}^2 + \hat{J}^2 - \hat{S}^2}{2\hat{L}\hat{J}},$$
$$\cos(\vec{S}, \vec{J}) = \frac{\hat{S}^2 + \hat{J}^2 - \hat{L}^2}{2\hat{S}\hat{J}},$$



where $\hat{L}^2 = L(L+1), \hat{S}^2 = S(S+1)$ and $\hat{J}^2 = J(J+1)$. Let $\mu_J = g_J \mu_B \hat{J}$, applying $\mu_L = g_L \mu_B \hat{L} = \mu_B \hat{L}$ and $\mu_S = g_S \mu_B \hat{S} = 2\mu_B \hat{S}$, we could get

$$g_J = g_L \frac{\hat{L}^2 + \hat{J}^2 - \hat{S}^2}{2\hat{J}^2} + g_S \frac{\hat{S}^2 + \hat{J}^2 - \hat{L}^2}{2\hat{J}^2} = \frac{3}{2} + \frac{S(S+1) - L(L+1)}{2J(J+1)}.$$

4 The magnetic moment that is predicted from Hund's Rules works quite well for rare-earth elements, but not for 3d-transition metals. Try to obtain these moments. (Hint: the magnetic moment is usually expressed in the unit of μ_B): $Cu^{2+}(3d^9), Eu^{3+}(4f^6), Gd^{3+}(4f^7), Dy^{3+}(4f^9)$ and $Tm^{3+}(4f^{12})$.

$Cu^{2+}(3d^9)$

l: -2 -1 0 1 2

↑↓ ↑↓ ↑↓ ↑↓ ↑

$L = 2$

$S = \frac{1}{2}$

more than half $\Rightarrow J = L + S = \frac{5}{2}$

$g = \frac{3}{2} + \frac{\frac{3}{4} - 6}{\frac{35}{2}} = \frac{6}{5}$

$\mu = \frac{6}{5} \cdot \sqrt{\frac{5}{2} \cdot \frac{7}{2}}\, \mu_B = \frac{3}{5}\sqrt{35}\, \mu_B$

$Eu^{3+}(4f^6)$

l: -3 -2 -1 0 1 2 3

↑ ↑ ↑ ↑ ↑ ↑ —

$L = 3$

$S = 3$

Less than half $\Rightarrow J = |L - S| = 0 \Rightarrow \mu = 0$

$Gd^{3+}(4f^7)$

l: -3 -2 -1 0 1 2 3

↑ ↑ ↑ ↑ ↑ ↑ ↑

$L = 0$

$S = \frac{7}{2}$

$\Rightarrow J = L + S = \frac{7}{2}$

$g = \frac{3}{2} + \frac{1}{2} = 2$

$\mu = \sqrt{63}\, \mu_B$

$Dy^{3+}(4f^9)$

l: -3 -2 -1 0 1 2 3

↑↓ ↑↓ ↑ ↑ ↑ ↑ ↑

$L = 5$

$S = \frac{5}{2}$

more than half $\Rightarrow J = L + S = \frac{15}{2}$

$g = \frac{3}{2} + \frac{\frac{35}{4} - 30}{2 \cdot \frac{15}{2} \cdot \frac{17}{2}} = \frac{4}{3}$

$\mu = \frac{2}{3}\sqrt{255}\, \mu_B$

$Tm^{3+}(4f^{12})$

l: -3 -2 -1 0 1 2 3

↑↓ ↑↓ ↑↓ ↑↓ ↑↓ ↑ ↑

$L = 5$

$S = 1$

more than half $\Rightarrow J = L + S = 6$

$g = \frac{3}{2} + \frac{2 - 30}{2 \cdot 6 \times 7} = \frac{7}{6}$

$\mu = \frac{7}{6}\sqrt{42}\, \mu_B$

5 E.M. wave has an extended and colorful spectrum, spanning from static point charge (frequency $= 0$ Hz), as field, rf radiation, microwave. terahertz, infrared, optical light, ultraviolet, soft x-rays, tender x-rays, hard x-rays and further. Compile a table that contains all regions, and label them using the following units at the same time: (1) eV; (2) Hz; (3) nm. You should have a basic feeling of their amplitudes as an experimentalist.

	S.P.C.	ac field	rf radiation	microwave
eV	0	2.07×10^{-13}	$4.13 \times 10^{-10} \sim 1.24 \times 10^{-4}$	$4.14 \times 10^{-6} \sim 1.24 \times 10^{-3}$
Hz	0	50	$3 \times 10^{5} \sim 3 \times 10^{10}$	$10^{9} \sim 3 \times 10^{11}$
nm	∞	6×10^{15}	$10^{12} \sim 10^{7}$	$3 \times 10^{8} \sim 1 \times 10^{6}$

	terahertz	infrared	optical light
eV	$4.14 \times 10^{-4} \sim 4.14 \times 10^{-2}$	$1.24 \times 10^{-3} \sim 1.63$	$1.59 \sim 3.10$
Hz	$10^{11} \sim 10^{13}$	$3 \times 10^{11} \sim 3.95 \times 10^{14}$	$3.85 \times 10^{14} \sim 7.5 \times 10^{14}$
nm	$3 \times 10^{6} \sim 3 \times 10^{4}$	$10^{6} \sim 760$	$780 \sim 400$

	ultraviolet	soft x-rays	hard x-rays
eV	$3.10 \sim 124$	$124 \sim 1.24 \times 10^{4}$	$1.24 \times 10^{4} \sim 1.24 \times 10^{5}$
Hz	$7.5 \times 10^{14} \sim 3 \times 10^{16}$	$3 \times 10^{16} \sim 3 \times 10^{18}$	$3 \times 10^{18} \sim 3 \times 10^{19}$
nm	$400 \sim 10$	$10 \sim 0.1$	$0.1 \sim 0.01$

6 In continuum approximation, anisotropic exchange interaction is made up by components of Lifshitz invariance:

$$L_{i,j}^{(k)} = m_i \frac{\partial m_j}{\partial x_k} - m_j \frac{\partial m_i}{\partial x_k}, \tag{2}$$

where i, j, k are index of the dimensionality. The energy density is therefore constructed as:

$$w = D_1 L^{(x)} + D_2 L^{(y)} + D_3 L^{(z)}. \tag{3}$$

In cubic systems without inversion centre, $D_1 = D_2 = D_3 = D$ is a scalar that parameterizes the strength of the anisotropic exchange. Now we are facing the *competing interaction* right away, as meanwhile, there exists exchange interaction in the form of $A(\nabla \mathbf{m})^2$. Try to prove that under such scheme, a one-dimensional helical order is the true ground state.

For the cubic systems, $D_1 = D_2 = D_3 = D$, the density of Hamiltonian could be written as

$$\omega = D\vec{m} \cdot (\nabla \times \vec{m}) + A(\nabla \vec{m})^2.$$

考虑螺旋态

$$\vec{m}(x) = (0, \cos(q_h x), \sin(q_h x)).$$

可以计算:

$$D\vec{m} \cdot (\nabla \times \vec{m}) = -Dq_h; \;\; A(\nabla \vec{m})^2 = Aq_h^2.$$

则能量密度可以写为:

$$\omega = Aq_h^2 - Dq_h$$

能量取极值有:

$$\frac{d\omega}{dq_h} = 2Aq_h - D = 0.$$

解得:

$$q_h = \frac{D}{2A}.$$

即螺旋态 $\vec{m}(x) = (0, \cos(q_h x), \sin(q_h x))$ 为基态，当 $q_h = \frac{D}{2A}$.

7 If two neighbouring spins $\hat{\mathbf{S}}_i$ and $\hat{\mathbf{S}}_j$ exchange in a way of $\hat{H} = -2J\hat{\mathbf{S}}_i \cdot \hat{\mathbf{S}}_j$, prove that

$$\frac{d\langle \hat{\mathbf{S}}_i \rangle}{dt} = \frac{2J}{\hbar}\langle \hat{\mathbf{S}}_i \times \hat{\mathbf{S}}_j \rangle. \tag{4}$$

由 Heisenberg 运动方程可知:

$$\begin{aligned}
\frac{d\langle \hat{S}_i^x \rangle}{dt} =& \frac{i}{\hbar}\langle [\hat{H}, \hat{S}_i^x] \rangle \\
=& \frac{-2iJ}{\hbar}\langle [\hat{S}_i^x \hat{S}_j^x + \hat{S}_i^y \hat{S}_j^y + \hat{S}_i^z \hat{S}_j^z, \hat{S}_i^x] \rangle \\
=& \frac{-2J}{\hbar}\langle \hat{S}_i^z \hat{S}_j^y - \hat{S}_i^y \hat{S}_j^z \rangle \\
=& \frac{-2J}{\hbar}\langle \hat{S}_i \times \hat{S}_j \rangle_x.
\end{aligned}$$

同样的，有:

$$\frac{d\langle \hat{S}_i^y \rangle}{dt} = \frac{-2J}{\hbar}\langle \hat{S}_i \times \hat{S}_j \rangle_y; \ \frac{d\langle \hat{S}_i^z \rangle}{dt} = \frac{-2J}{\hbar}\langle \hat{S}_i \times \hat{S}_j \rangle_z.$$

即:

$$\frac{d\langle \hat{\mathbf{S}}_i \rangle}{dt} = \frac{2J}{\hbar}\langle \hat{\mathbf{S}}_i \times \hat{\mathbf{S}}_j \rangle.$$

8 As a traditional exercise, you should work out the spin wave dispersion spectrum for a two-dimensional ferromagnetic square lattice (set the lattice constant of a).

Hamiltonian 为:

$$H = -2J\sum_{i,j} \hat{\vec{S}}_{ij} \cdot (\hat{\vec{S}}_{ij+1} + \hat{\vec{S}}_{i+1j}).$$

由 Heisenberg 运动方程，可得:

$$\begin{aligned}
&\frac{dS_{ij}^x}{dt} = \frac{i}{\hbar}\langle [\hat{H}, \hat{S_{ij}^x}] \rangle \\
&= -\frac{2iJ}{\hbar}\langle [S_{ij}^y(S_{ij+1}^y + S_{i+1j}^y + S_{ij-1}^y + S_{i-1j}^y) + S_{ij}^z(S_{ij+1}^z + S_{i+1j}^z + S_{ij-1}^z + S_{i-1j}^z), S_{ij}^x] \rangle \\
&= -\frac{2iJ}{\hbar}\langle -iS_{ij}^z(S_{ij+1}^y + S_{i+1j}^y + S_{ij-1}^y + S_{i-1j}^y) + iS_{ij}^y(S_{ij+1}^z + S_{i+1j}^z + S_{ij-1}^z + S_{i-1j}^z) \rangle
\end{aligned}$$

将所有自选 z 分量近似为 S，方程为：

$$\frac{dS_{ij}^x}{dt} = \frac{2J}{\hbar}(4SS_{ij}^y - S(S_{ij+1}^y + S_{i+1j}^y + S_{ij-1}^y + S_{i-1j}^y)).$$

同样的，对于 S_{ij}^y 可以列出如下方程：

$$\frac{dS_{ij}^y}{dt} = \frac{2J}{\hbar}(S(S_{ij+1}^x + S_{i+1j}^x + S_{ij-1}^x + S_{i-1j}^x) - 4SS_{ij}^x).$$

考虑行波解：

$$S_{ij}^x = Ae^{i(\vec{k}\cdot\vec{r}_{ij}-\omega t)},\ S_{ij}^y = Be^{i(\vec{k}\cdot\vec{r}_{ij}-\omega t)}$$

带入上述方程可得：

$$\begin{cases} i\omega A + \dfrac{2JS}{\hbar}(4 - 2\cos k_x a - 2\cos k_y a)B = 0; \\ \dfrac{2JS}{\hbar}(4 - 2\cos k_x a - 2\cos k_y a)A - i\omega B = 0. \end{cases}$$

方程对 A, B 有非零解，则需要系数行列式等于 0，即

$$\begin{vmatrix} i\omega & \frac{2JS}{\hbar}(4 - 2\cos k_x a - 2\cos k_y a) \\ \frac{2JS}{\hbar}(4 - 2\cos k_x a - 2\cos k_y a) & -i\omega \end{vmatrix}$$

$$= \omega^2 - \left(\frac{2JS}{\hbar}(4 - 2\cos k_x a - 2\cos k_y a)\right)^2 = 0.$$

则自旋波色散为：

$$\omega = \frac{4JS}{\hbar}(2 - \cos k_x a - \cos k_y a).$$