CSYS5010 Introduction to Complex Systems

Week 6 Dynamical Systems I

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Review of Last Lecture

- Performing multiple runs in NetLogo
  - BehaviorSpace tool, Mathematica Link
- Analysis of the results
  - Finding percolation phase transition in Forest Fires
  - Fitting exponential to the wealth distribution of the Simple Economy model
  - Determining R0 in the epidemic model and finding its critical value
- Visualizing the results
  - Plotting in Excel and Mathematica
  - Making animations
- Concepts of Verification, Validation and Replication

This week session outcomes

- Equation based modelling – dynamical systems
- Discrete dynamical systems – iterative maps
  - Tutorial examples: exponential growth, logistic map, Lotka-Volterra equations
- Continuous dynamical systems – differential equations
  - Tutorial examples: simple harmonic motion, SIR epidemic models
- Analysis of dynamical systems
  - Time series plots
  - Phase plot diagram
  - Cobweb diagram
  - Finding equilibrium (fixed) points
  - Determining asymptotic behaviour
  - Determining stability of fixed points

Ch. 3-7, Hiroki Sayama, "Introduction to the Modeling and Analysis of Complex Systems", Geneseo, NY: Open SUNY Textbooks, 2015.

Equation based modelling

The time evolution of the system is expressed mathematically using equations.

Deterministic systems

- Continuous time – ordinary and partial differential equations
- Discrete time – difference equations (discrete maps)

Stochastic (with randomness) systems

- Continuous time – stochastic differential equations
- Discrete time – Markov chains

We will look at **ordinary differential equations** and discrete **maps.**

Dynamical systems

Dynamical system is a system whose state is uniquely specified by a set of variables and whose behaviour is described by predefined rules.

Discrete dynamical system

Time evolutions is given by a **map**

$$x_t = F(x_{t-1}, t)$$

This is called a difference equation, a recurrence relation or iterative map.

Continuous dynamical systems

Time evolution is given by a **flow**

$$\frac{dx}{dt} = F(x, t)$$

This is called a differential equation.

Dynamical systems

Question: Have you learned of any models in the natural or social sciences that are formulated as either discrete-time or continuous-time dynamical systems? If so, what are they? What are the assumptions behind those model

Question: What are some appropriate choices for state variables in the following systems?

- Population growth?
- Swinging pendulum?
- Motion of celestial bodies?

Discrete time models

A map

$$x_t = F(x_{t-1}, t)$$

Produces a series of values of state variable x over time, starting with initial condition x_0

$$\{x_0, x_1, x_2, x_3, \dots\}$$

This is called a **time series.**

Ch. 4, Hiroki Sayama, "Introduction to the Modeling and Analysis of Complex Systems", Geneseo, NY: Open SUNY Textbooks, 2015.

Discrete time models

Exercise: Consider a system that is made of two interacting components: A and B. Each component takes one or two possible states: Blue or Red. Their behaviours are determined by the following rules:

- A tries to stay the same color as B.
- B tries to be the opposite color of A.

1) Write down the state transition functions $F_A(s_A, s_B)$ and $F_B(s_A, s_B)$ for this system, where s_A and s_B are the states of A and B respectively
2) Produce a time series (s_A, s_B) starting with initial condition that both components are blue.

Classification of Discrete time models

Linear system: A dynamical equation whose rules involve just a linear combination of state variables (a constant times a variable, a constant, or their sum).

Nonlinear system: Anything else (e.g., equation involving squares, cubes, radicals, trigonometric functions, etc., of state variables).

Classification of Discrete time models

First-order system: A difference equation whose rules involve state variables of the immediate past (at time $t-1$) only.

Higher-order system: Anything else.

Classification of Discrete time models

Autonomous system: A dynamical equation whose rules don't explicitly include time t or any other external variables.

Non-autonomous system: A dynamical equation whose rules do include time t or other external variables explicitly.

Classification of Discrete time models

Non-autonomous, higher-order difference equations can always be converted into autonomous, first-order forms, by introducing additional state variables.

Simulating Discrete dynamical models

Three essential steps in a computer program

Initialize: set the initial state of the system

Update: define the rules that you will be using to update the state of the system at every time step

Observe: record the time series.

Tutorial 1a: simulating exponential growth model

Tutorial 1b: simulating discrete model with multiple variables

Designing discrete dynamical model

Tips

1. Use existing model as starting points.
2. Implement each model assumption one by one.
3. To implement a new assumption, first identify which part of the model equation represents the quantity you are about to change, replace it with an unknown function, and then design the function.
4. Adopt the simplest mathematical form.
5. Once your equation is complete, check if it behaves as you desired. Test its behaviour with extreme values assigned to variables and/or parameters.

Tutorial 2: designing population growth model

Continuous dynamical systems

Given by differential equations

$$\frac{dx}{dt} = F(x,t)$$

where $\frac{dx}{dt}$ is formally defined as

$$\frac{dx}{dt} = \lim_{\Delta t \to 0} \frac{x(t+\Delta t) - x(t)}{\Delta t}$$

Differential equations are solved by integration, producing a continuous time series solution $x(t)$.

Ch. 6, Hiroki Sayama, "Introduction to the Modeling and Analysis of Complex Systems", Geneseo, NY: Open SUNY Textbooks, 2015.

Classification of continuous dynamical systems

Similar to discrete dynamical systems.

We will focus on autonomous, first order differential equation since

Non-autonomous, higher-order differential equations can always be converted into autonomous, first-order forms by introducing additional state variables.

Solving differential equations

- Sometimes it is possible to solve differential equation analytically.
- Numerical methods are used when analytic solution is not available.
  - e.g. Euler forward method, Runge-Kutta methods etc.

Tutorial 3: Solving simple harmonic motion equation analytically and numerically.

Analysing dynamical models

Techniques:

- Finding equilibrium (fixed) points
- Phase space visualization
- Variable rescaling
- Determining asymptotic stability of a linear system
- Determining stability around equilibrium points of a non-linear system

Ch. 7, Hiroki Sayama, "Introduction to the Modeling and Analysis of Complex Systems", Geneseo, NY: Open SUNY Textbooks, 2015.

Finding equilibrium points

An **equilibrium point** in phase space is a state of the system, which remains invariant under the application of the dynamical map or flow.

Given a dynamical system described a differential equation

$$\frac{dx}{dt} = F(x)$$

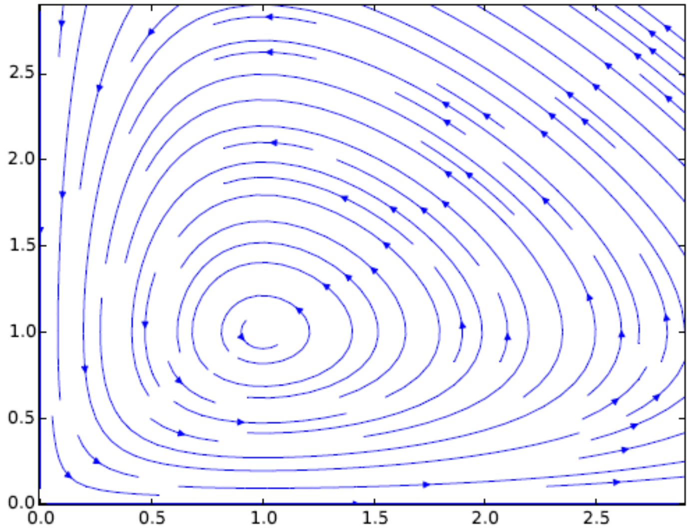
The equilibrium points are found by solving

$$0 = F\left(x_{eq}\right)$$

Phase space visualization

A **phase space diagram** visualizes the flows in phase space i.e. trajectories starting from various initial conditions. They can be effective in two dimensional and three dimensional system, but become progressively more difficult to draw in higher dimensions.

e.g.



Phase space plot for a Lotka-Volterra continuous model. Take from Sayama, p 114, fig 7.1

Variable rescaling

Variable rescaling should be applied to identify the parameter that determine the features of the phase space. e.g.

Consider the following logistic equation

$$\frac{dx}{dt} = rx(1 - \frac{x}{K})$$

Apply a rescale in x and t.

$$x \to \alpha x'$$
$$t \to \beta x'.$$

In terms of the rescaled variables the original equation becomes

$$\frac{dx'}{dt'} = r\beta x'(1 - \frac{\alpha x'}{K})$$

Choose α and β such that the equation becomes as simple as possible. In this case, choose $\beta = 1/r$ and $K = \alpha$. With this choice the equation is

$$\frac{dx'}{dt'} = x'(1 - x').$$

There are no parameter in the rescaled version of logistic equation! Thus changing (r,K) does not change the essential features of the phase space.

Asymptotic stability of linear systems

Consider a **linear** dynamical system given by equation

$$\frac{dx}{dt} = Ax$$

where x is a state vector and A is a matrix.

The asymptotic stability of the system can be inferred from the real part of the dominant eigenvalue of the matrix A i.e. from $\mathrm{Re}(\lambda_d)$.

Dominant eigenvalue is the eigenvalue with the largest real component.

Asymptotic stability of linear systems

Three cases

- $\text{Re}(\lambda_d) > 0$: The system is unstable diverging to infinity.
- $\text{Re}(\lambda_d) < 0$: The system is stable converging to the origin.
- $\text{Re}(\lambda_d) = 0$: The system is stable, but the dominant eigenvector component is conserved, and therefore the system may converge to a non-zero equilibrium point.

Stability around equilibrium points

The following procedure can be used to determine stability of fixed points. We focus on a two component system for concreteness.

Given a non-linear dynamical system

$$\frac{dx_1}{dt} = F_1(x_1, x_2)$$
$$\frac{dx_2}{dt} = F_2(x_1, x_2)$$

Step 1: Find the equilibrium points (x_{1eq}, x_{2eq}).

Step 2: Write down the Jacobian matrix at the chose equilibrium point

$$J = \begin{bmatrix} \frac{dF_1}{dx_1} & \frac{dF_1}{dx_2} \\ \frac{dF_2}{dx_1} & \frac{dF_2}{dx_2} \end{bmatrix}$$

Step 3: Calculate the eigenvalues of the J.

Stability around equilibrium points

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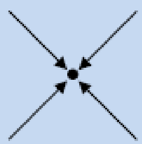
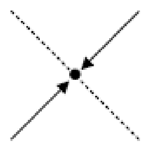
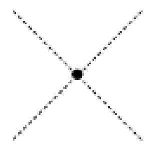
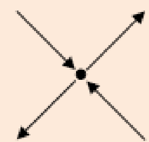
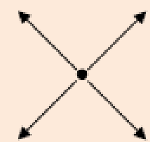
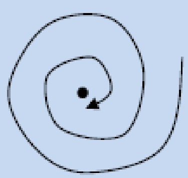
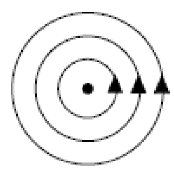
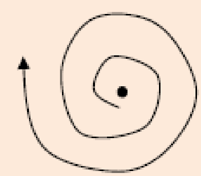
Step 3: Calculate the eigenvalues of the J.

Step 4: If the real part of the dominant eigenvalue is

- > 0 then the equilibrium point is unstable
- < 0 then the equilibrium point is stable
- $= 0$ then the equilibrium point may be neutral (Lyapunov stable)

In addition, if there are complex conjugate eigenvalues involved, oscillatory dynamics are going on around the equilibrium point. If those complex conjugate eigenvalues are the dominant ones, the equilibrium point is called a stable or unstable *spiral focus* (or a neutral center if the point is neutral).

Stability around equilibrium points

	Stable $\mathrm{Re}(\lambda_d) < 0$	Lyapunov stable* $\mathrm{Re}(\lambda_d) = 0$		Unstable $\mathrm{Re}(\lambda_d) > 0$	
Real eigenvalues	Stable point $\mathrm{Re}(\lambda_1) < 0$, $\mathrm{Re}(\lambda_2) < 0$ 	Neutral point $\mathrm{Re}(\lambda_1) = 0$, $\mathrm{Re}(\lambda_2) < 0$ 	$\mathrm{Re}(\lambda_1) = 0$, $\mathrm{Re}(\lambda_2) = 0$ 	Saddle point $\mathrm{Re}(\lambda_1) > 0$, $\mathrm{Re}(\lambda_2) < 0$ 	Unstable point $\mathrm{Re}(\lambda_1) > 0$, $\mathrm{Re}(\lambda_2) > 0$ 
Complex conjugate eigenvalues	 $\mathrm{Re}(\lambda_1) = \mathrm{Re}(\lambda_2) < 0$ Stable spiral focus	 $\mathrm{Re}(\lambda_1) = \mathrm{Re}(\lambda_2) = 0$ Neutral center		 $\mathrm{Re}(\lambda_1) = \mathrm{Re}(\lambda_2) > 0$ Unstable spiral focus	

Stability around equilibrium points

$\boxed{\textit{Exercise 7.16}}$ Consider the following differential equations that describe the interaction between two species called *commensalism* (species x benefits from the presence of species y but doesn't influence y):

$$\frac{dx}{dt} = -x + rxy - x^2 \tag{7.71}$$

$$\frac{dy}{dt} = y(1-y) \tag{7.72}$$

$$x \geq 0, \quad y \geq 0, \quad r > 1 \tag{7.73}$$

1. Find all the equilibrium points.
2. Calculate the Jacobian matrix at the equilibrium point where $x > 0$ and $y > 0$.
3. Calculate the eigenvalues of the matrix obtained above.
4. Based on the result, classify the equilibrium point into one of the following: Stable point, unstable point, saddle point, stable spiral focus, unstable spiral focus, or neutral center.

Summary

We've looked at

- Deterministic discrete dynamical systems (discrete maps)
- Continuous dynamical systems (differential equations)

Including ideas for how to construct the equations (using predator-prey example) and several tools for analysis:

- Solutions using computers
- Phase space diagram
- Find equilibrium points
- Stability analysis around fixed points
- Equation rescaling

Next week: bifurcations, chaos …

Questions