Rebalancing Risk

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Abstract

While a routinely rebalanced portfolio such as a 60-40 equity-bond mix is commonly employed by many investors, most do not understand that the rebalancing strategy adds risk. Rebalancing is similar to starting with a buy and hold portfolio and adding a short straddle (selling both a call and a put option) on the relative value of the portfolio assets. The option-like payoff to rebalancing induces negative convexity by magnifying drawdowns when there are pronounced divergences in asset returns. The expected return from rebalancing is compensation for this extra risk. We show how a higher-frequency momentum overlay can reduce the risks induced by rebalancing by improving the timing of the rebalance. This smart rebalancing, which incorporates a momentum overlay, shows relatively stable portfolio weights and reduced drawdowns.

Keywords: Fixed weights, 60-40, drift weights, constant weights, rebalanced portfolio, rebalancing, negative skewness, negative skew, short straddle, negative gamma, momentum overlay.

JEL: G11, G13

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Introduction

Constant-mix strategies, such as allocating 60% of the value of a portfolio to equities and 40% to bonds, are used extensively by pension funds and other long-term investors. In such strategies, the investor periodically rebalances the portfolio so that each asset class is a constant fraction of portfolio value rather than allowing the allocation to drift as asset prices change.

Constant-mix strategies have intuitive motivations, but we believe key properties of these strategies are poorly understood by many investors. In particular, *rebalancing can magnify drawdowns* when there are pronounced divergences in asset performance. Such divergences are usually driven by equities, and in late 2008 and early 2009, some rebalanced strategies underperformed passive strategies by hundreds of basis points.

In traditional 60/40 portfolios, the vast majority of the risk (ca. 85%) comes from the allocation to equities.[1] Equity indices, in addition to generally having much higher volatilities than bond indices, have fat downside tails, i.e. large negative returns and sharp drawdowns occur more frequently than large positive returns. Constant-mix strategies not only inherit these properties, but actually exacerbate the drawdowns.

We demonstrate how the opposing return profile of a momentum overlay can help to mitigate the added risk introduced by the rebalancing process. Although momentum trading itself has produced long-term positive returns, even without any assumptions of positive performance, momentum acts to improve the risk and drawdown properties of the constant-mix portfolio according to both theoretical and historical analyses.

We provide a technical appendix which gives analytic explanations for the results described in the main body of the paper. This appendix provides the theoretical arguments to elucidate how the rebalancing process can change the risk properties of the portfolio and how a momentum overlay can help. The appendix is provided principally to demonstrate that the results in this paper are of an analytic nature and are not dependent on the particular realisation of history seen in the empirical data. If one can take this on trust, reading the appendix is purely optional.

[1] Over the last ten years the average annualised volatility of the S&P500 from daily closes has been 20.6%. The average volatility of the 10 year US Treasury bond has been 6.3%. Source: Bloomberg.

Fixed weights vs. drift weights

A passive *buy-and-hold* strategy is an alternative to a constant-mix strategy. Under buy-and-hold, one simply buys a portfolio (e.g. 60% equities/40% bonds) and holds it without trading for a period of time.[2] The capital allocation within the portfolio will obviously drift as market prices change, but the investor remains passive. In contrast, the active rebalancing in the constant-mix strategy will restore the target fixed weights. We shall also call the buy-and-hold approach a *drift-weight* portfolio, to be contrasted with *fixed weights* under rebalancing.

The motivations behind a constant-mix strategy are straightforward. First, drift weights may become extreme as assets diverge. In 2013, the S&P500 delivered a total return of 31.9%, while the S&P 7-10 Year U.S. Treasury bond index declined by 6.1%[3]. A 60/40 portfolio would have drifted over 2013 to a 68/32 portfolio, and, with a repetition of these returns in 2014, a 74/26 portfolio. In practice, few if any investors remain passive indefinitely as drift weights become extreme. At some point, the investor is likely to adjust the weights. The constant-mix strategy sensibly puts this adjustment process on a regular schedule.

The so-called "rebalancing premium" is a second motivation for the constant-mix strategy. The premium refers to the extra returns that the rebalancing process generates under certain circumstances. Let's consider how such profits might arise. Divergent asset performance will cause the weights to differ from the target allocation. To restore the desired allocation, the investor must buy some of the underperforming assets and sell some of the outperformers. "Buying low and selling high" has an intuitive appeal, and *provided there is mean reversion in relative asset performance,* rebalancing to fixed weights may generate positive returns.

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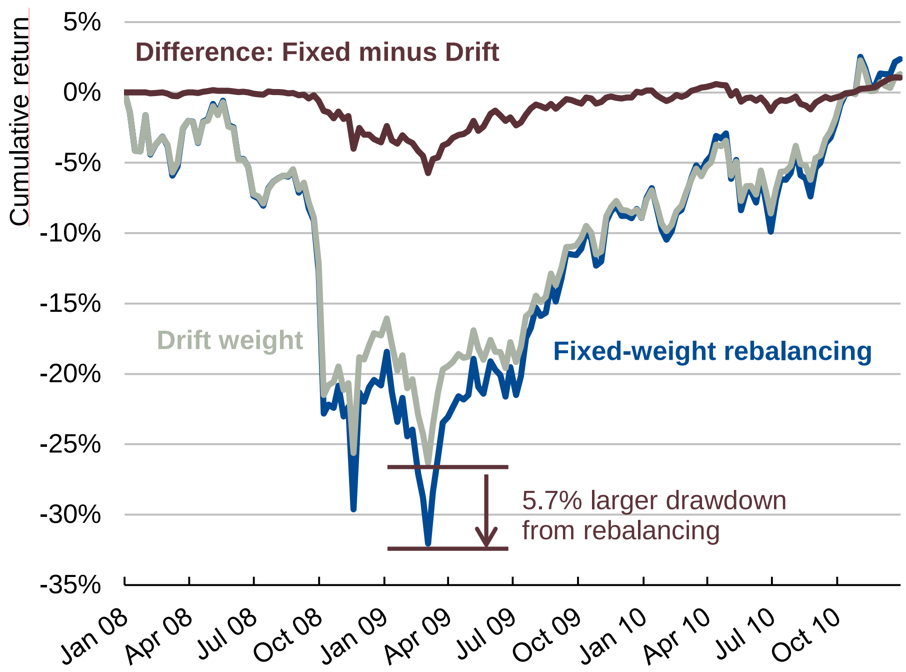
[2] Dividends and interest are usually reinvested.

[3] Source: Bloomberg.

On the other hand, *when assets diverge strongly over time*, fixed-weight rebalancing may cause losses relative to buy-and-hold as the rebalancing continues to increase the allocation to the underperforming asset. Figures 1 and 2 show the performance of constant-mix strategies vs. drift-weights during two bear markets in equities: 2008/9 and 2000/2. The portfolios start with 60% allocations to the S&P500 and 40% allocations to the 10-year US Treasury bond. Rebalancing is done monthly. In both periods, rebalancing exacerbated losses and increased the drawdowns by about 600 bps. To be sure, the markets recovered in the subsequent years, and the constant mix portfolio eventually caught up with the passive portfolio.

Figure 1: Rebalancing magnified drawdown in 2008-2010

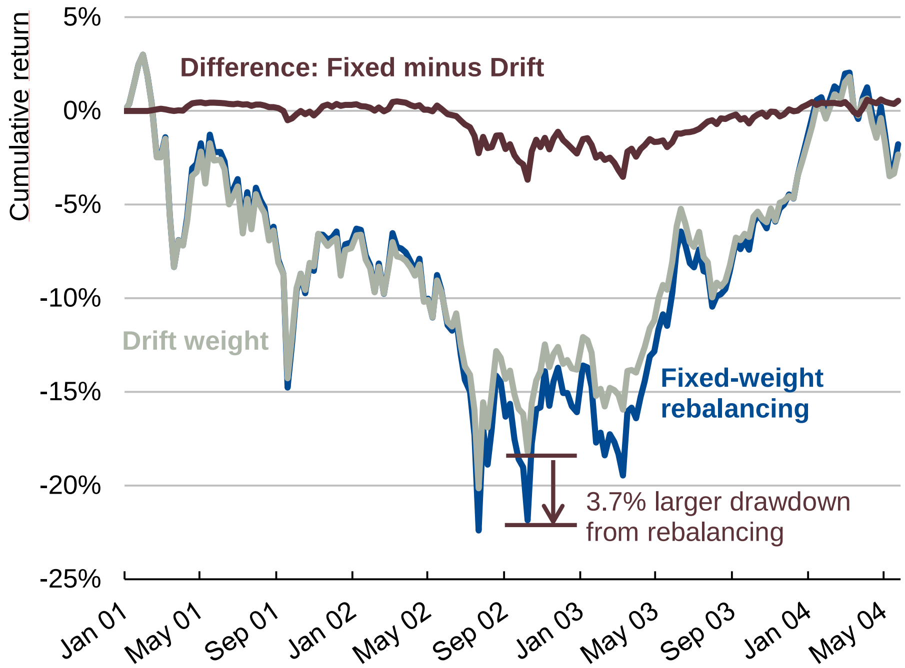
Cumulative returns to 60/40 portfolios of the S&P500 and 10-year US Treasury bond



Source: Reuters. Date range: January 2008 to December 2010. Monthly rebalancing.

Figure 2: Rebalancing magnified drawdown in 2001-2004

Cumulative return to 60/40 portfolios of the S&P500 and 10-year US Treasury bond



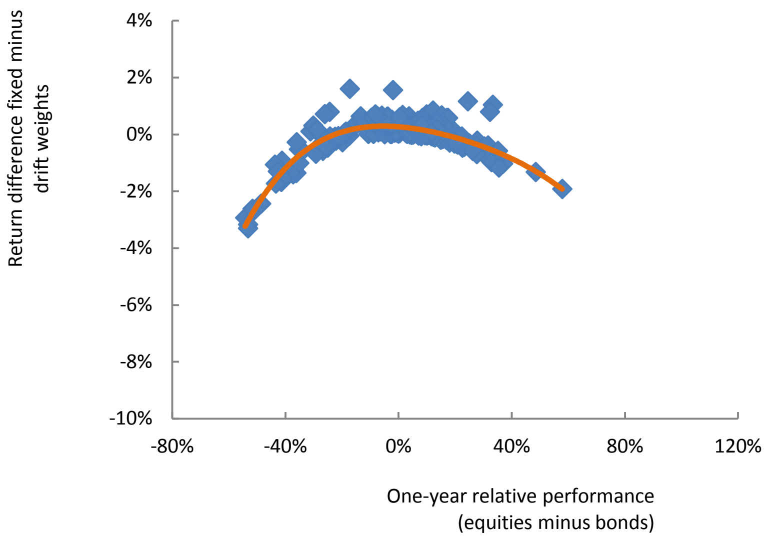
Source: Reuters. Date range: January 2001 to May 2004. Monthly rebalancing.

The performance difference between fixed weights and drift weights shows the characteristic return profile (sometimes called "negative convexity") of an investor who has sold both put and call options (on the relative performance of stocks and bonds). This profile shows positive, but limited, returns under small moves and large negative returns under large moves. The driver is the divergence in stock and bond returns because differences in return change the allocation. Put another way, when assets perform equally, there is no need for rebalancing trades, even if the returns are large.

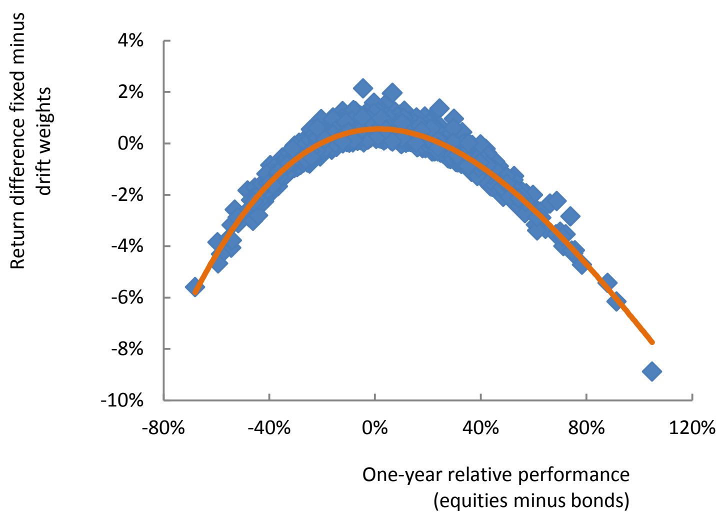
To illustrate, Figure 3 shows the difference in one-year returns between fixed-weight rebalancing and drift weights using both empirical (Panel A) and simulated (Panel B) data. The empirical analysis uses overlapping one-year returns for 60/40 strategies with monthly rebalancing on the S&P500 and 10y US Treasuries, using data from 1990 to present. The simulation takes two asset returns using lognormal distributions with volatilities of 20% and 6% and a constant -20% correlation. The constant-mix strategy often earns a small rebalancing premium, but at the risk of occasionally sharply underperforming buy-and-hold.

Figure 3: Fixed weights vs. drift weights: Performance difference over one year

A. One-year returns on empirical data
Fixed-weight rebalancing ***minus*** drift weights



B. One-year returns on simulated data
Fixed-weight rebalancing ***minus*** drift weights



Source: Reuters, Man calculations. Date range: January 1990 to February 2014. Monthly rebalancing.

Why does fixed-weight rebalancing have higher drawdowns?

Buying losers and selling winners is the essence of rebalancing to fixed weights. Market dynamics ultimately determine whether such a trading strategy is profitable. A trend-less environment, especially one characterised by mean reversion, would clearly be friendly to this strategy – losers would bounce back, high-flyers would come down to earth, and rebalancing investors would gain from their trades.

It is also clear that trending markets, e.g. where equities *keep* losing relative to bonds, pose a problem for rebalancing. A sustained trend means one buys an asset on the way down, "catching a falling knife" in the popular parlance. Trending (or "time-series momentum") is in fact a well-established property of markets over the last century and remains an important trading signal.[4] One ignores trends at one's own peril.

Figures 1-3 show *rebalancing is no free lunch but represents an implicit bet against diverging asset performance.* The cost occurs when markets trend and rebalanced portfolios experience deeper drawdowns than buy-and-hold portfolios. For small values of stock-bond return divergence, fixed-weight rebalancing modestly outperforms buy-and-hold, as one can see in Figure 3. But for large divergences, the underperformance of the constant mix is marked.

In the Appendix, we use analytic methods to derive the aforementioned risk properties of rebalancing and buy-and-hold portfolios. Since the derivation is a mathematical proof, no past returns are used. The risk properties in fact arise from the dynamic style of trading rather than any special properties of the underlying markets.

[4] "Momentum in Futures Market", Norges Bank Investment Management, Discussion Note #1-2014.

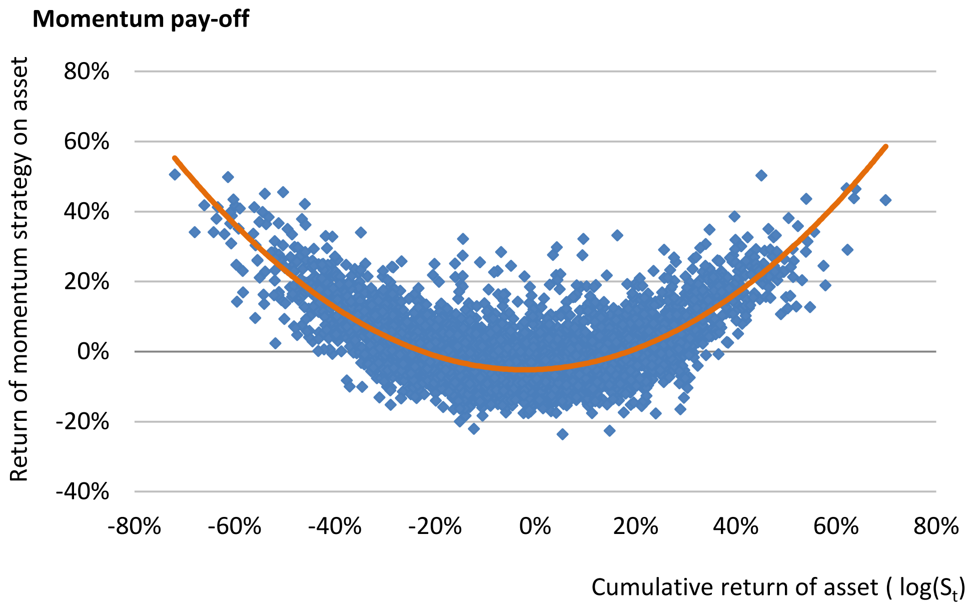
Momentum has the opposite characteristics to fixed-weight rebalancing

Momentum trading, in contrast to rebalancing, involves the purchase of winners and sale of losers, while maintaining a given volatility exposure. We have previously shown in an article in *Risk* Magazine[5] that momentum trading produces a positively convex and positively skewed distribution of returns. This result was also derived *analytically*. Remarkably the positive convexity and skewness do not depend on strong assumptions about the underlying markets. The dynamic style of trading gives rise to the positive skewness, which translates into reduced drawdowns relative to passive strategies.

Figure 4 shows the pay-off profile of momentum trading on a single simulated asset. One-year asset returns are simulated using daily lognormal distributions for returns (with 20% annualised volatility and zero serial correlation). Then we use a moving average cross-over system, to be described in the next section. Intriguingly, it looks like the exact opposite of the profile for the performance difference between fixed- and drift-weights. One important distinction is that x-axis of Figure 4 represents the absolute performance of the single asset whereas Figure 3 uses the relative performance of stocks and bonds. No fees or trading costs were imposed in the simulation. As shown in the above cited article, the shape of the payoff profile is a general property of momentum systems.

[5] "Momentum strategies offer a positive point of skew," *Risk*, August 2012.

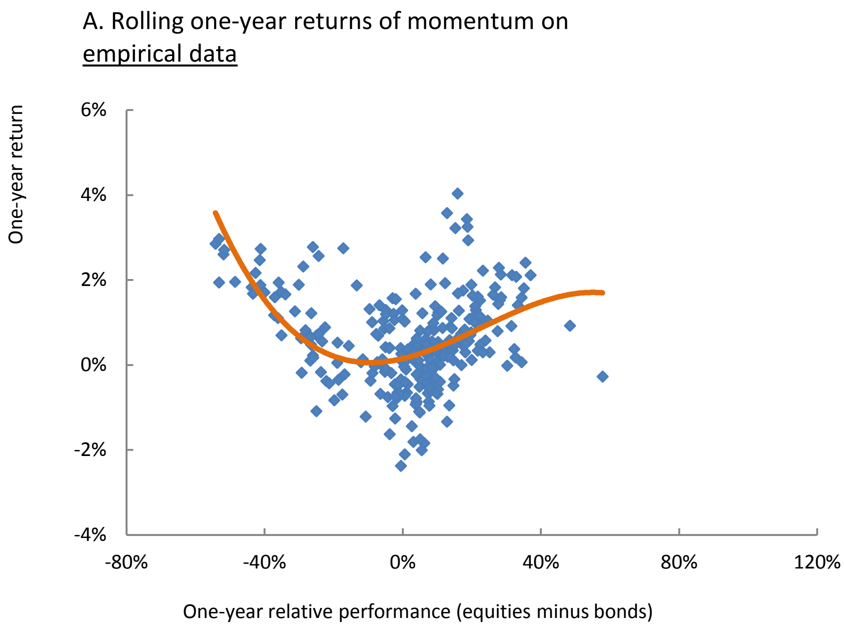
Figure 4: Momentum trading on a single asset over one-year periods, simulation

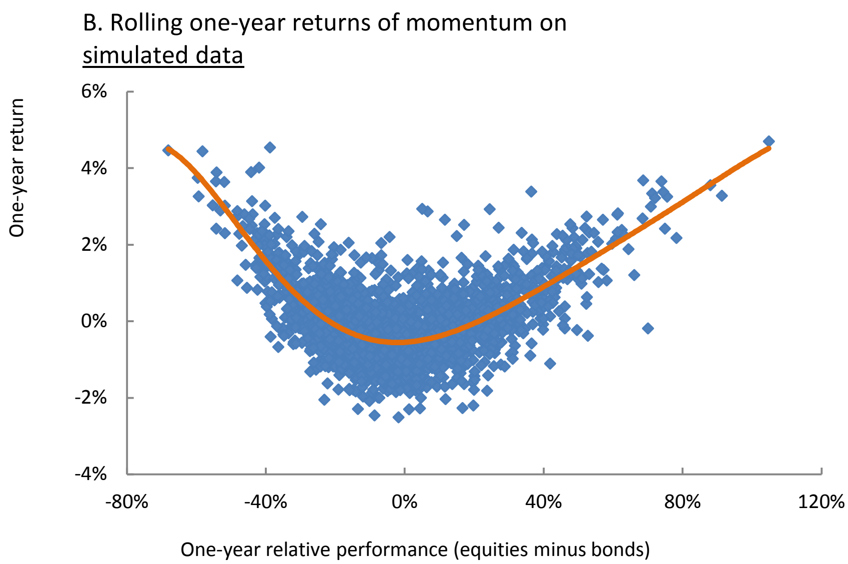


Source: Man calculations.

In Figure 5, we apply momentum to stocks and bonds individually, and graph the performance as a function of the divergence between stocks and bonds. Panel A uses overlapping one-year returns to the S&P500 and the 10-year US Treasury bond, while Panel B uses simulated data as in Figure 3.

Figure 5: Rolling one-year returns of momentum strategy on US stocks and bonds





Source: Reuters, Man calculations. Date range: January 1990 to February 2014.

Momentum to the rescue?

In our analysis adding a momentum overlay to a constant-mix portfolio improves the risk properties of the portfolio.

The concept of an overlay is familiar to many pension plans, where overlays are commonly used, particularly on the liability side. But on the asset side, momentum overlays can be used to hedge the drawdown risks exacerbated by rebalancing. *The overlay tends to gain from the very trends that hurt the constant-mix strategy, but imposes little cost in other environments.*

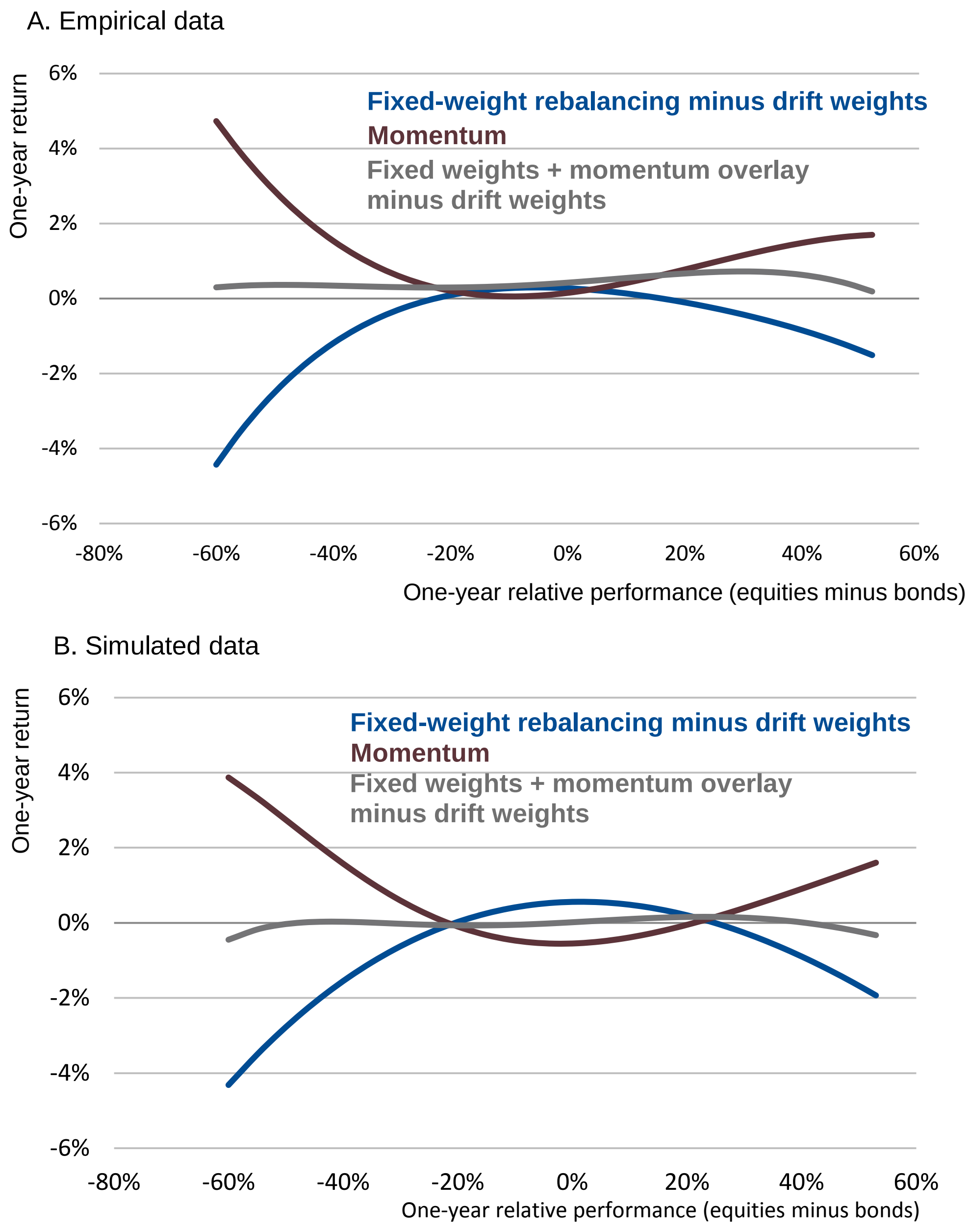
This hedging property is seen in both empirical data and in simulations. Figure 6 illustrates the cumulative return of assets with fixed-weight rebalancing combined with a 10% momentum overlay[6], as a function of asset divergence. The momentum overlay is applied as individual trend-following strategies on the individual assets, with the risk allocation set to 10% of the constant-mix portfolio.

The momentum overlay uses a moving average cross-over model employed at AHL. It has a target volatility of 10% and turns over positions, on average, 4 times a year. The model has a number of features – including applying a response function to the raw signal and scaling by volatility. While these features may improve performance, the essential story remains the same even for simpler momentum models.

In addition, Table 1 shows the impact of varying amounts of momentum allocation on a constant mix portfolio using empirical data. We see, for example, that increasing the momentum overlay modestly increases the Sharpe ratio and improves the skewness, at both rebalancing frequencies. Moreover the maximum drawdown is reduced materially, depending on the amount of the overlay.

[6] We chose 10% as a reasonable initial case to consider, we discuss other levels later.

Figure 6: Impact of momentum overlay on fixed-weight rebalancing



Source: Reuters, Man calculations. Date range: January 1990 to February 2014.

The effect of adding the momentum overlay is to neutralise the negative tails from rebalancing seen in Figure 3.

Table 1: Impact of momentum overlays on constant-mix portfolios

Rebalance frequency	Momentum allocation	Annualised return	Annualised volatility	Sharpe Ratio	Skewness	Correlation to rebalanced portfolio without overlay	Max drawdown depth
Monthly	0%	3.5%	9%	0.40	-0.99	1.00	-34.4%
	10%	4.2%	9%	0.45	-0.8	1.00	-31.9%
	20%	4.8%	10%	0.50	-0.62	0.98	-29.6%
	40%	5.8%	10%	0.58	-0.3	0.93	-25.3%
Quarterly	0%	3.7%	9%	0.42	-0.9	1.00	-33.0%
	10%	4.3%	9%	0.47	-0.72	1.00	-29.5%
	20%	4.8%	9%	0.52	-0.55	0.98	-27.4%
	40%	5.9%	10%	0.60	-0.26	0.93	-23.6%

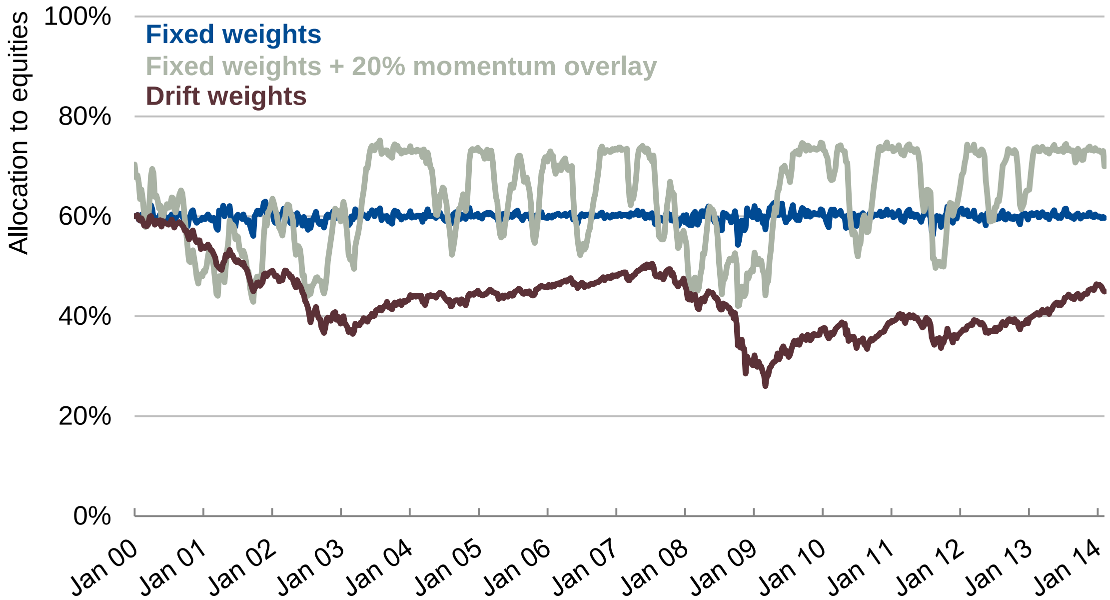
Source: Reuters, Man calculations. Date range: January 2000 to February 2014.

Why don't rebalancing and momentum just cancel?

Doesn't adding momentum to a constant-mix portfolio merely get you back to where you started, a passive buy-and-hold portfolio?

In our view the answer is an emphatic no. Consider the S&P500 and 10-year Treasuries. Figure 7 shows the portfolio weight on equities at each point in time, since 2000, under different portfolio construction strategies. The blue line comes from a 60/40 monthly rebalanced portfolio, the grey line shows the same portfolio with a 20% momentum overlay, and the brown line illustrates a simple drift-weight portfolio starting at the same 60/40 allocation. Over short time-frames, the portfolio with the momentum overlay can differ quite markedly from the standard rebalanced portfolio. Over the long-run, however, the portfolio with the overlay clearly tracks the long-term desired allocation. In contrast, the drift-weight portfolio drops to equity allocations as low as 25% by 2009. The rebalanced portfolio with momentum overlay, on the other hand, stays between 40% and 75%, while the rebalanced portfolio without overlay is locked around 60% equities.

Figure 7: Allocation to equities under fixed-weight rebalancing (with and without momentum overlay) and drift weights



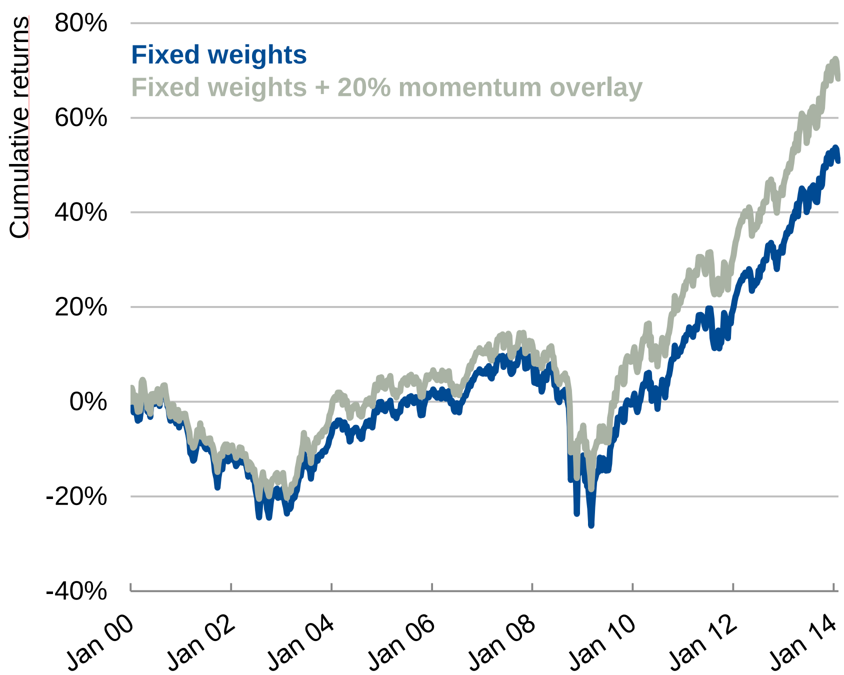
Source: Reuters, Man calculations. Date range: January 2000 to February 2014. Monthly rebalancing.

In any case, a passive buy-and-hold portfolio may not be a realistic option for most investors over the long-term, since drift weights may become extreme. Rebalancing, whether on a regular schedule or when drift weights become intolerably imbalanced, is nearly inevitable. The pertinent question, then, is what investors should do about the increased drawdown risk from this rebalancing. Our analysis indicates that a momentum overlay allows one to maintain a sensible rebalancing schedule without increased drawdown risk.

While rebalancing and momentum signals treat outperforming (and underperforming) assets in broadly opposite ways, with rebalancing selling winners and momentum buying them, there are important differences in the trading. We have observed that the momentum overlay, operating on a daily time-scale, effectively acts as a timing strategy around the monthly or quarterly rebalance. For example, the overlay will resist buying too aggressively into equities in times of market stress (easing the psychological pressure on the rebalancer). But, as divergences in asset performance stabilise, the momentum overlay will cut out and allow the full rebalance to occur. That is instead of undoing the rebalance, the momentum overlay, acting on a shorter-time scale, seeks to improve the timing of the rebalance. Asset returns have historically displayed autocorrelation, especially in times of market stress, and the momentum overlay allows the portfolio to take this into account as part of the rebalance process.

The positive impact of the overlay on portfolio performance can be seen in Figures 8, 9 and 10. Figure 8 shows the cumulative returns and drawdowns of a 60/40 constant mix portfolio with varying amounts of overlay. Performance and risk properties improve as the overlay is increased.

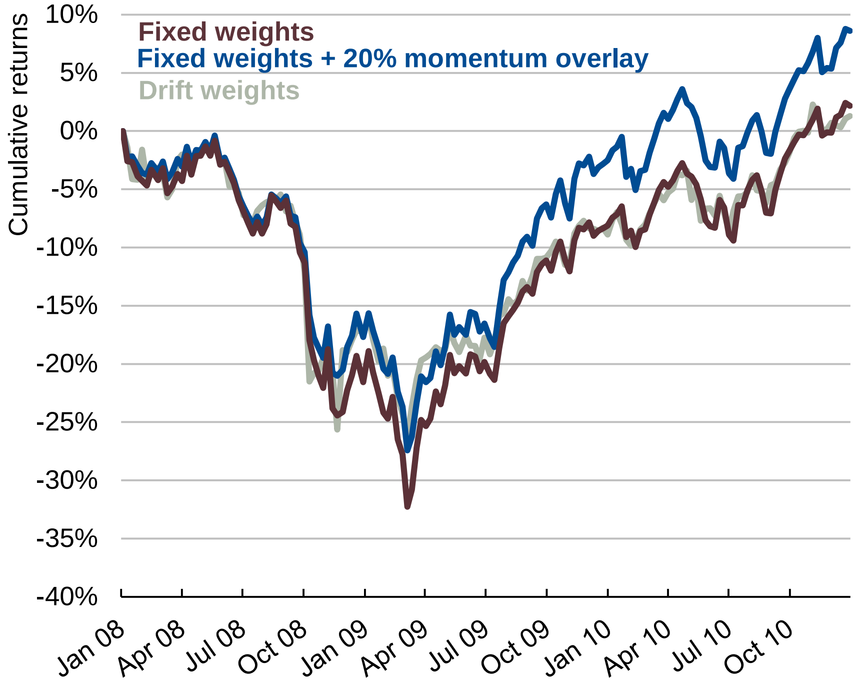
Figure 8: Constant-mix portfolio + momentum overlay on stocks and bonds



Source: Reuters, Man calculations. Date range: January 2000 to February 2014. Monthly rebalancing.

In Figures 9 and 10, we see how the overlay helps to mitigate the negative impact of rebalancing. Indeed, we see that the overlay reduces the drawdowns in both 2001-2004 and 2008-2010 and nearly replicates drift-weight performance. In the case of the latter episode, fixed weight rebalancing plus the overlay outperforms the other methods by the summer of 2009.

Figure 9: Impact of momentum overlay in 2008-2010



Source: Reuters, Man calculations. Date range: January 2008 to December 2010. Monthly rebalancing.

Figure 10: Impact of momentum overlay in 2001-2004

Source: Reuters, Man calculations. Date range: January 2001 to May 2004. Monthly rebalancing.

Conclusion

Rebalancing is sometimes touted as a costless way to enhance returns by means of the so-called rebalancing premium. While a premium can be earned when relative asset prices do not diverge strongly, diverging asset prices can lead to marked underperformance. Portfolio rebalancing can exacerbate drawdowns during major trends, as in 2008-2009 and 2000-2002. The performance loss from rebalancing reached, in those examples, 600 bps at the bottom (Figures 1 and 2).

The driver behind the worst drawdowns is the fact that rebalancing represents a kind of anti-momentum trading in which one buys underperforming assets and sells outperforming assets. The active trading associated with rebalancing introduces "negative convexity" – a return profile in which large divergences in asset performance can cost the investor a lot of money in return for a small rebalancing premium when asset prices do not diverge.

Rebalancing, however, is here to stay. Passive portfolio construction in which weights are allowed to drift is not a realistic alternative, because allocations can get extreme after only a few years. So one must rebalance in some way and must therefore deal with the effects of rebalancing on drawdowns and tail risk.

Adding a momentum overlay can help to mitigate the added risk introduced by the rebalancing process.

- First, a momentum overlay can introduce an opposing return profile, generating gains when trends are stronger and asset performance diverges.
- Second, momentum operates on a much shorter time-scale than the rebalancing process and effectively acts as a timing strategy around the monthly or quarterly rebalances.
- Third, both theoretical and empirical analyses indicate that introducing a momentum overlay to a rebalanced portfolio reduces drawdowns and improves risk-adjusted performance.
- Finally, momentum trading itself has historically produced positive returns over long periods of time.[7]

[7] "Momentum in Futures Market", Norges Bank Investment Management, Discussion Note #1-2014.

However, even without any assumptions of positive performance, our analysis shows that momentum acts to improve the risk and drawdown properties of the constant-mix portfolio.

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Appendix

First, we derive analytic results for fixed-weight rebalancing of a risky asset (e.g., a stock index) and cash. We show that the final payoff of the strategy is negatively convex in comparison with drift weights, and has inferior drawdown characteristics.

Second, we derive analytic results for momentum trading and show that momentum returns are positively convex in the underlying.

Thirdly, we look at combining a fixed-weight rebalanced portfolio with a momentum overlay and show how this overlay can reduce the negative convexity, worst returns and maximum drawdowns.

Finally, we introduce a second risky asset, say bonds. We derive similar but more complicated analytic results in the two-asset case.

1. Fixed-weight rebalance of one asset and cash

We begin with the simpler case of rebalancing between a single risky asset and cash. In practice this simplification is representative of the more complex two-asset case because equity risk dominates and cash is a reasonable proxy for bonds in this context. Later we examine the more complex case of two risky assets. Although the algebra is more involved the key ideas remain very similar.

We model the price of a single risky asset S_t using a drift-less geometric Brownian motion:[8]

$$\frac{dS_t}{S_t} = \sigma dW_t$$

$$S_t = S_0 e^{\sigma W_t - \frac{1}{2}\sigma^2 t}$$

Where σ is the asset volatility and W_t is a Wiener process.

[8] The stochastic calculus used below necessarily assumes continuous rebalancing. In practice, and as described in the main the section of this paper, rebalancing takes place at discrete frequencies – e.g. monthly. Although the discrete rebalancing can be seen as an approximation of the continuous case derived here, our empirical and Monte Carlo results demonstrate that this approximation is valid. For ease of exposition we also assume the risk-free rate is zero throughout.

Suppose our wealth at t is X_t, with a fixed $m \in [0, 1]$ invested in equity and $1 - m$ in cash. Following a similar derivation to Perold and Sharpe,[9] we obtain:

$$\frac{dX_t}{X_t} = m\frac{dS_t}{S_t}$$

Applying Ito's lemma to $\log X_t$ and taking the exponent, we get:

$$Eq.\ 1 \qquad X_t = X_0 e^{m\sigma W_t - \frac{1}{2}m^2\sigma^2 t} = X_0\left(\frac{S_t}{S_0}\right)^m e^{\frac{1}{2}(m-m^2)\sigma^2 t}$$

The final pay-off, is proportional to $\left(\frac{S_t}{S_0}\right)^m$, the terminal stock price divided by the initial price, raised to the *m^th^* power.

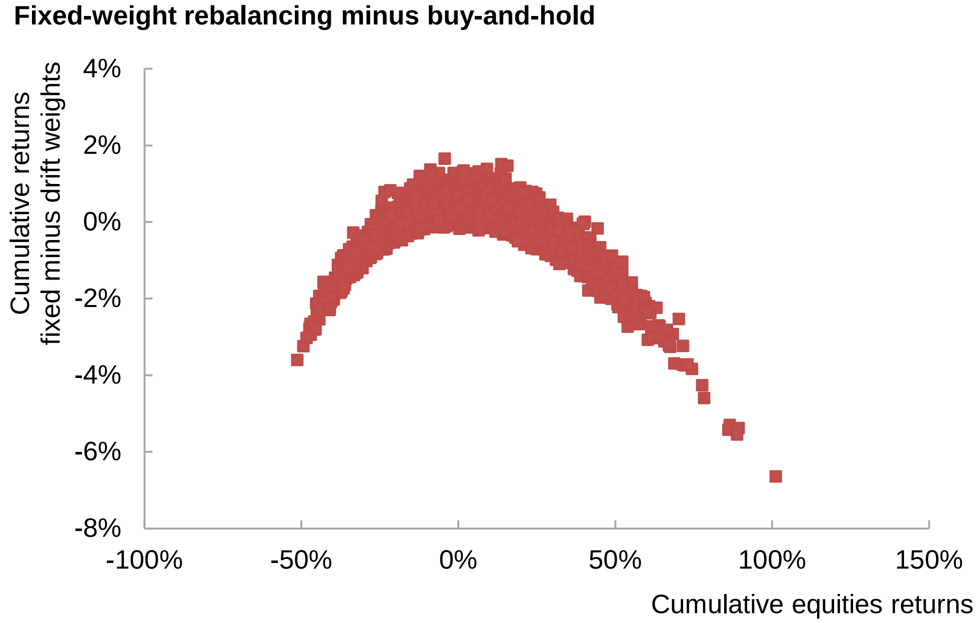
Equation 2 shows the Taylor expansion of X_t around $S_t = S_0$. When $m < 1$, the term $\left(\frac{S_t}{S_0} - 1\right)^2$ has a negative coefficient, which leads to negative convexity of the payoff as shown in Figure 11:

$$Eq. 2 \quad X_t = cX_0\left(\frac{S_t}{S_0}\right)^m = cX_0\left\{1 + m\left(\frac{S_t}{S_0} - 1\right) + \frac{1}{2}m(m-1)\left(\frac{S_t}{S_0} - 1\right)^2 + R_3\right\}$$
$$\approx cX_0\left[1 + m\left(\frac{S_t}{S_0} - 1\right) + \frac{1}{2}m(m-1)\left(\frac{S_t}{S_0} - 1\right)^2\right]$$

Where $c = e^{\frac{1}{2}(m-m^2)\sigma^2 t}$ and $R_3 = O\left[\left(\frac{S_t}{S_0} - 1\right)^3\right]$ is the remainder term of the expansion.

[9] "Dynamic Strategies for Asset Allocation" Financial Analysts Journal, 1988

Figure 11: Dynamic rebalancing strategy returns (monthly rebalancing frequency, 1-year horizon and 20% vol for equities)



Source: Man calculations, Monte Carlo simulation.

In contrast to Eq. 2, the final wealth of a drift-weight portfolio with *m* initially allocated to the risky asset is $X_0\left[m\frac{S_t}{S_0}+(1-m)\right]$. Thus the fixed-weight portfolio has lognormally distributed final wealth whereas the drift-weight portfolio has a shifted lognormal distribution.

The point is, in the drift-weight case a fixed proportion *m* of the *initial* wealth is exposed over time to the risky asset *S.* Thus the skewness of the wealth distribution in the drift-weight case comes only from this allocation of *m* to a lognormally distributed sub-portfolio with volatility $\sigma\sqrt{t}$. In contrast, Eq. 2 shows that in the fixed-weight case the entire wealth is exposed to the m^{th} power of the asset return, which has is lognormally distributed with volatility $m\sigma\sqrt{t}$.

The skewness of a general lognormal distribution with volatility σ is given by the following formula:

$$\gamma_1=\left(e^{\sigma^2}+2\right)\sqrt{e^{\sigma^2}-1}$$

Skewness is invariant with scaling and offset, thus the skewness of the drift-weight portfolio is independent of *m*:

$$Skew(DriftWeight) = \left(\mathrm{e}^{\sigma^2 t} + 2\right)\sqrt{\mathrm{e}^{\sigma^2 t} - 1}\,.$$

However *m* permeates the equation for skewness of the fixed weight rebalanced portfolio:

$$Skew(FixedWeight) = \left(e^{m^2\sigma^2 t} + 2\right)\sqrt{e^{m^2\sigma^2 t} - 1}.$$

The partial derivative with respect to *m* of this expression for the skewness of the fixed weight rebalance is given by:

$$\frac{\partial\left(e^{m^2\sigma^2 t} + 2\right)\sqrt{e^{m^2\sigma^2 t} - 1}}{\partial m} = 3\,\frac{m\sigma^2 t e^{2m^2\sigma^2 t}}{\sqrt{e^{m^2\sigma^2 t} - 1}}$$

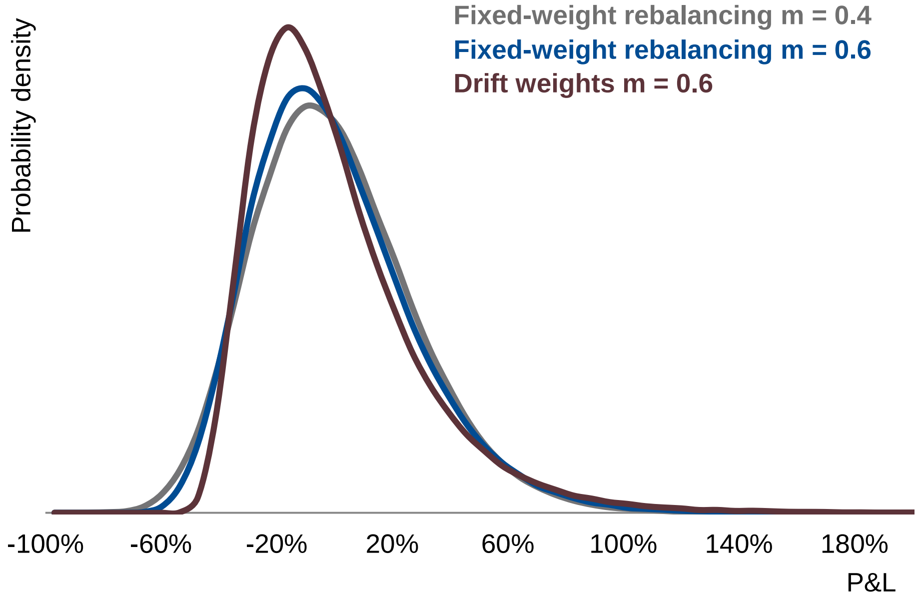
This term is always positive. Thus it is clear that skewness of the fixed-weight portfolio decreases as *m* decreases. Therefore the rebalancing process (reducing *m* from 1) reduces the skewness (i.e. fattens the left tail) in comparison to the original lognormal distribution of the drift-weight portfolio. This is not surprising given the negatively convex payoff we showed in Eq. 2 and Figure 11.

These changes in skewness and convexity have implications in terms of drawdowns and worst returns compared to a simple drift weight (buy and hold) process. We demonstrate this via Monte Carlo simulations below.

In Figure 12 we illustrate the distribution of final pay-offs under different levels of *m*. The final pay-off of the dynamically rebalanced strategy has a fatter left tail in comparison to the drift-weight strategy. This is as we would expect due to the less favourable skewness properties of the dynamic strategy.

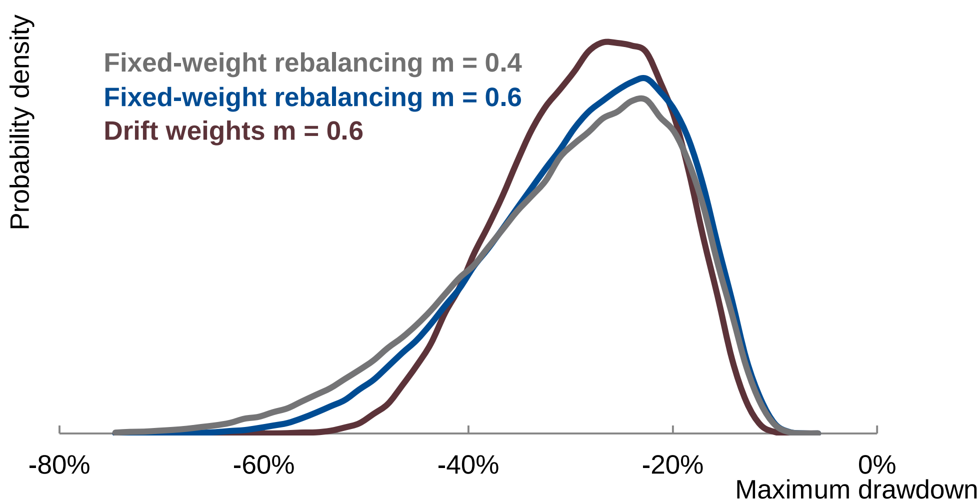
In Figure 13 we show the maximum drawdown experienced during the lifetime of the strategy. The drawdowns are larger for the dynamically rebalanced portfolio, as expected. These results are due to the negative convex exposure and reduced skewness from the dynamic rebalancing.

Figure 12: Distribution of final pay-offs for m = 0.4, 0.6, 5-year horizon



Source: Man calculations, Monte Carlo simulation.

Figure 13: Worst drawdowns for m = 0.4, 0.6 and 1.0, 5-year horizon



Source: Man calculations, Monte Carlo simulation.

2. Trend following

We have shown elsewhere[10] that trend-following has positive skewness by construction. Intuitively this happens because trend following has natural stops: it reduces positions when it loses money and only builds positions when it makes money. Furthermore a trend following strategy has a pay-off convex in the underlying due to the embedded optionality.

The expected P&L of trading momentum on an asset *S* from time *0* to *t* is a function of the terminal price S_t:

$$Eq.3 \qquad E\left(P\&L_{mom,t}\middle|S_t\right) = Y_0\psi(t)\left[\left(\ln\frac{S_t}{S_0}\right)^2 - \sigma^2 t\right]$$

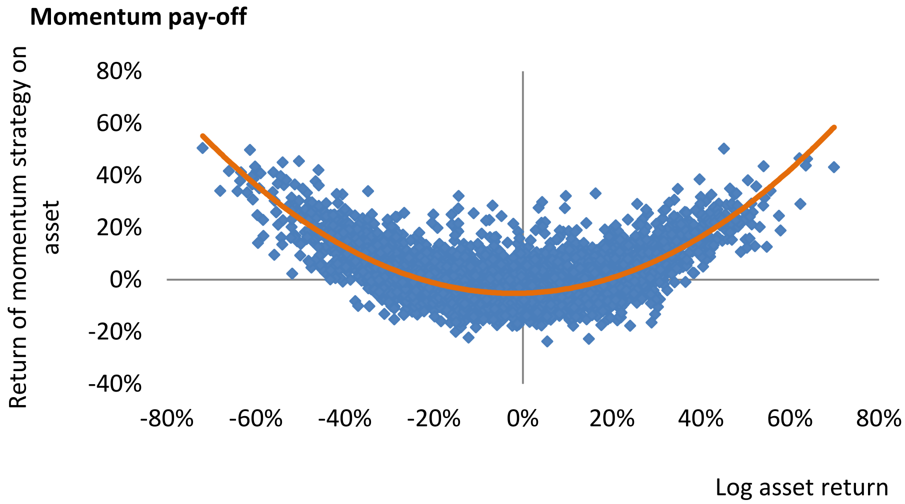
where Y_0 is the initial risk allocation to the momentum strategy and $\psi(t)$ is a (positively valued) function on t, dependent the design of the signal[11]. In this paper the momentum system is a simplified version of the moving average cross-over system in use at AHL. This formula makes explicit both the convexity of the momentum system and the optionality – including the time decay represented by the $\sigma^2 t$ term.

Figure 14 depicts a Monte Carlo simulation of trading a moving-average cross-over momentum system on an asset S_t following a Brownian motion. The convexity of the payoff in Eq. 3 is clear to see.

[10] "Momentum strategies offer a positive point of skew," *Risk*, August 2012.

[11] The derivation of this is out of the scope of this paper and will be published in future paper.

Figure 14: Momentum strategy return



Source: Man calculations, Monte Carlo simulation.

However Figure 14 shows the momentum return as a function of the log asset return. We can see the relationship with simple asset returns by simplifying Eq. 3 with a Taylor expansion:

$$Eq.4 \qquad E\left(P\&L_{mom,t}\middle|S_t\right) = Y_0\psi(t)\left\{\left(\frac{S_t}{S_0}-1\right)^2 - \sigma^2 t + R_3'\right\}$$

Where $R_3' = O\left[\left(\frac{S_t}{S_0}-1\right)^3\right]$ is the remainder term.

3. Combining Momentum and Rebalancing

Using Eq. 4 we can compare the performance of the momentum strategy with the P&L from monthly rebalancing of the asset and cash (Eq. 2). We express allocation to momentum as $Y_0 = kX_0$, i.e. the risk allocation to momentum is a fixed proportion *k* of the initial wealth X_0: [12]

[12] In reality rather than having the momentum allocation as a fixed proportion of the initial wealth, the momentum allocation itself would be dynamically rebalanced so as always to account for a fixed proportion k of total current wealth. This is very much a second order effect and we work with a constant momentum allocation to keep the mathematics tractable.

$$E\left(P\&L_{fixed\ weight\ rebalance+mom,t}\middle|S_t\right)$$

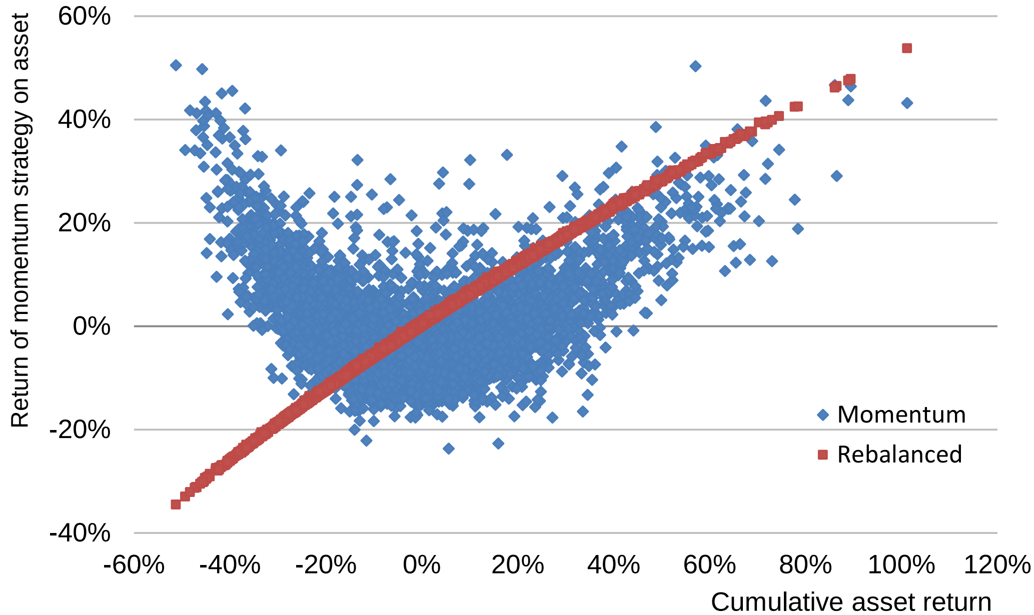
$$= \underbrace{cX_0\left\{1+m\left(\frac{S_t}{S_0}-1\right)+\frac{1}{2}m(m-1)\left(\frac{S_t}{S_0}-1\right)^2+R_3\right\}-X_0}_{rebalance}+\underbrace{Y_0\psi(t)\left[\left(\frac{S_t}{S_0}-1\right)^2-\sigma^2t+R_3^{'}\right]}_{momentum}$$

$$= X_0\left\{cm\left(\frac{S_t}{S_0}-1\right)+\left[k\psi(t)+\frac{1}{2}cm(m-1)\right]\left(\frac{S_t}{S_0}-1\right)^2+\xi(t)+R_3^{''}\right\}$$

where $R_3'' = O\left[\left(\frac{S_t}{S_0}-1\right)^3\right]$ collects all remainder terms and $\xi(t) = e^{\frac{1}{2}(m-m^2)\sigma^2 t}-1-k\psi(t)\sigma^2 t \approx \frac{1}{2}(m-m^2)\sigma^2 t - k\psi(t)$, is approximately the deterministic time decay term for quadratic term $\left(\frac{S_t}{S_0}-1\right)^2$ in the equation. This equation provides us with the theoretical basis for offsetting the negative convexity of the rebalancing we showed in Figure 6.

In Figure 15 we show together the pay-offs of rebalancing and momentum. Note the change in the x-axis from Figure 14 where we have moved from log asset returns to simple percentage returns (as given by Eq. 4). We see that the rebalanced stocks-and-cash strategy has an upward slope with respect to the terminal stock value and a negatively convex shape. In contrast, momentum on S_t has positive convexity.

Figure 15: The final pay-off of momentum, momentum with monthly regearing and dynamic rebalancing with m=0.6



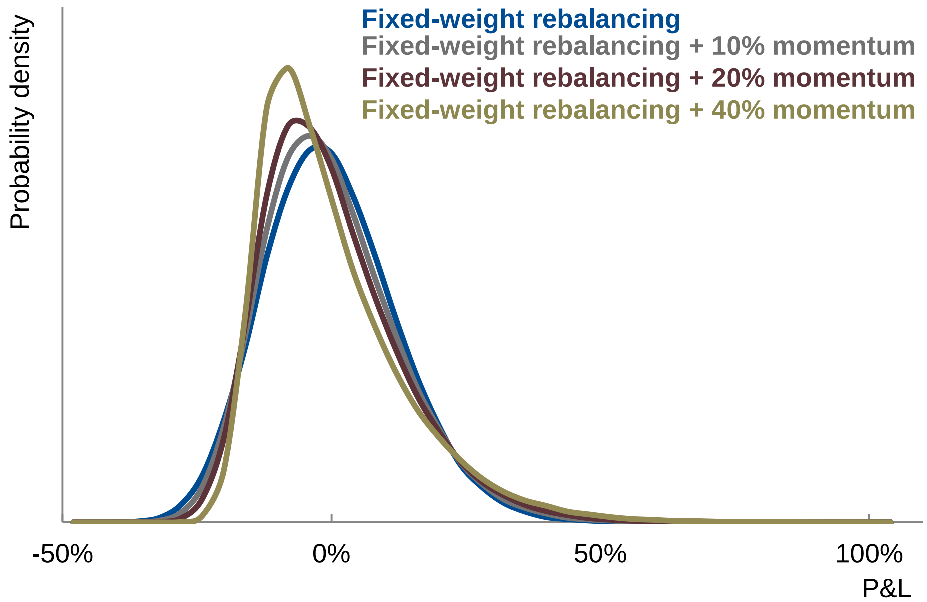
Source: Man calculations, Monte Carlo simulation.

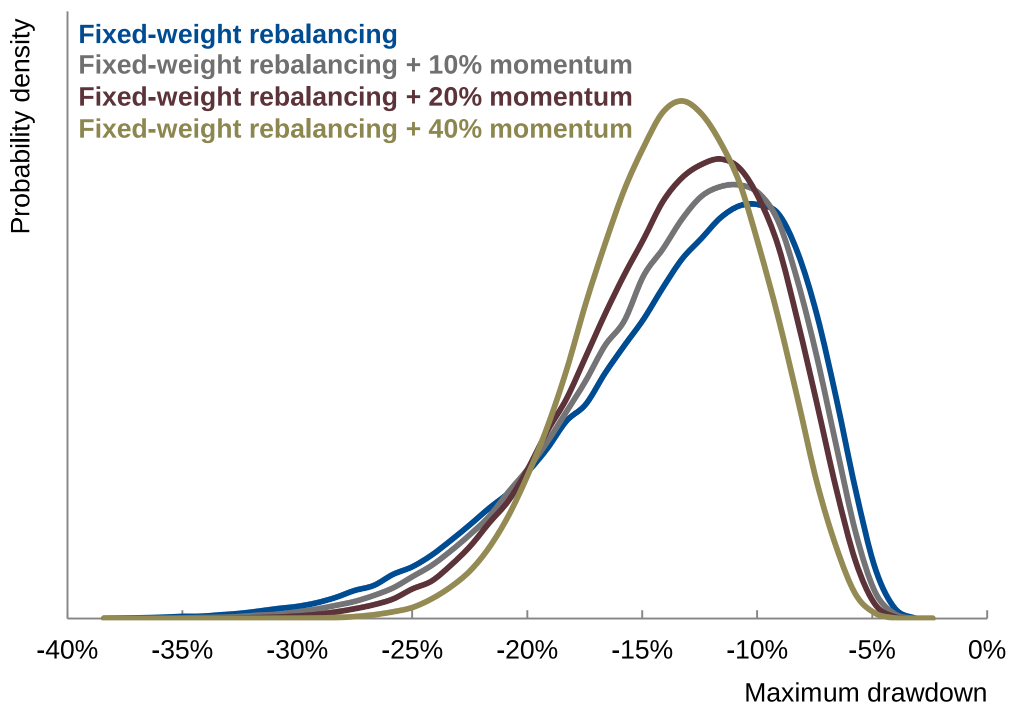
By offsetting the negative convexity of the rebalance process, the addition of a momentum system to a fixed-weight rebalanced portfolio (in the one asset case) produces distributions with a thinner left tail.

These facts translate directly into smaller drawdowns. Figures 16 and 17 illustrate the results of Monte Carlo simulations over 1 and 5 year horizons. In both figures addition of momentum reduces the left tail, thickens the right tail and increases the positive skewness of the P&L. The potential maximum drawdown is also noticeably reduced on the left tail with increasing allocation to momentum. In other words the fixed weight rebalance portfolio is less likely to experience extremes of drawdowns with allocation to momentum.[13]

[13] Contrary to first impressions the mean of the return distribution is unchanged, with the shift of the mode to the left compensated for with the reduced left tail and increased right tail. Indeed since all of these Monte-Carlo simulations are Martingales the mean must by definition be zero.

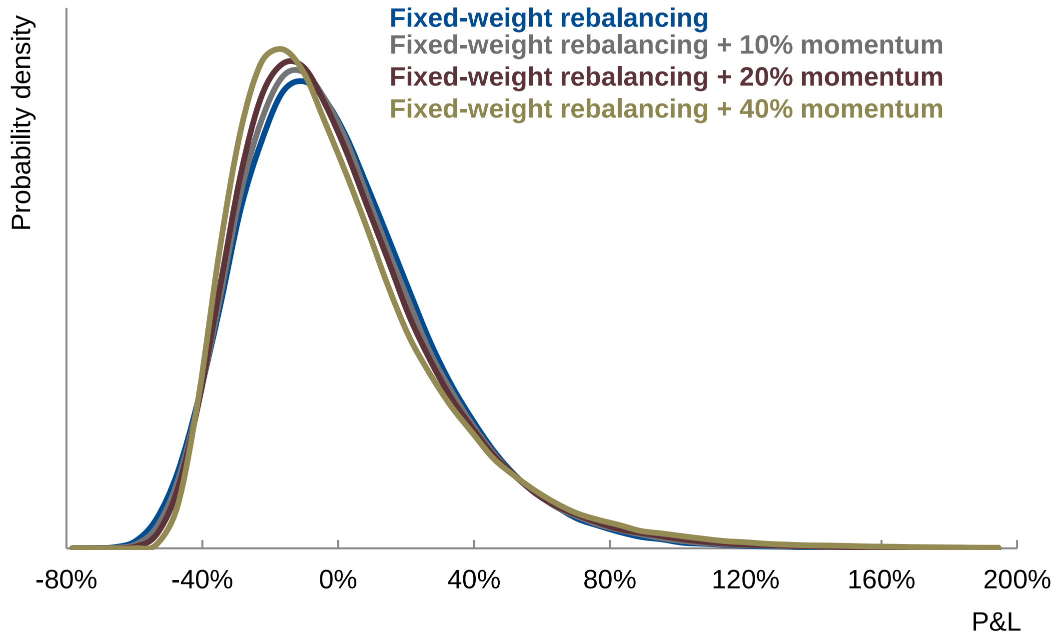
Figure 16: Terminal pay-off and worst drawdowns for fixed-weight rebalancing and momentum, 1-year horizon

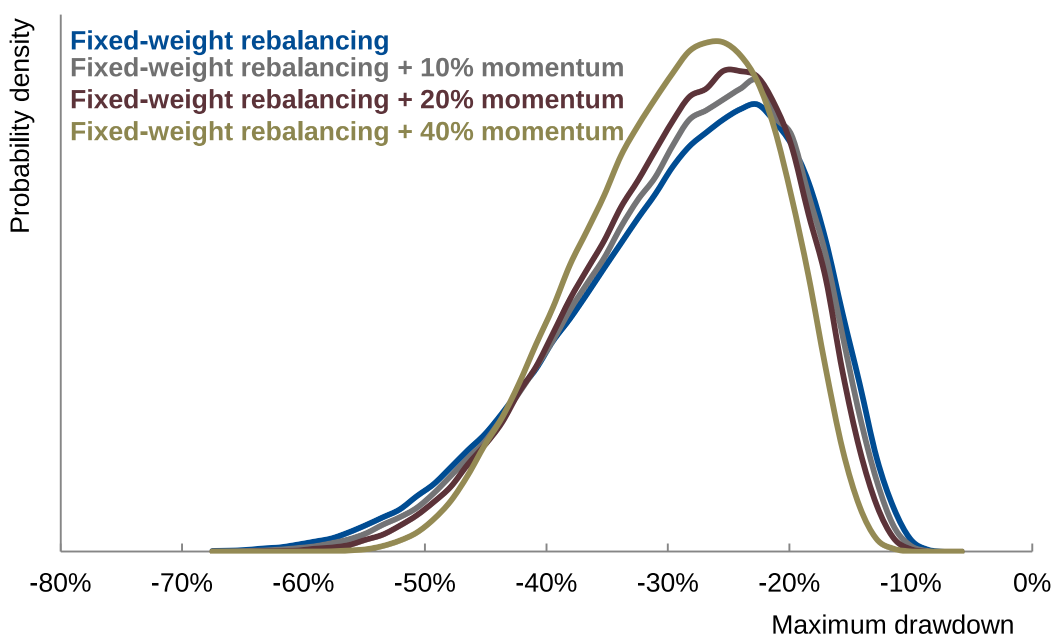




Source: Man calculations, Monte Carlo simulation.

Figure 17: Terminal pay-off and worst drawdowns for fixed-weight rebalancing and momentum, 5-year horizon





Source: Man calculations, Monte Carlo simulation.

4. Fixed-weight rebalancing of two assets

In this section we derive analytic results for the two asset case. The main difference with the one asset case is that the fixed-weight rebalance induces negative convexity in the divergence between the assets as well as in the individual asset returns.

We consider the case of two risky assets, S_t and B_t, both of which are drift-less with constant volatility. This simplifies the model and highlights the key points. By taking drift-less assets we avoid introducing a constant trend which would act to bias results in favour of momentum.

Although both S_t and B_t are assumed to be drift-less, one can nevertheless extend these results to a portfolio of stocks and bonds. With appropriate numeraire (e.g. money market account) the price processes of stocks and bonds are drift-less in the presence of non-zero interest rates.

In addition, within a stock-bond portfolio, the duration of the bond allocation is generally maintained around a certain target. Hence the constant volatility assumption for B_t is reasonable in this case.

In any case – since the purpose of this paper is compare various forms of fixed-weight and drift-weight portfolio – the differences *between* these portfolios introduced by non-zero drift and any stochastic rate element will be relatively insignificant to the main results.

Having justified those simplifications we have:

$$\frac{dS_t}{S_t} = \sigma_S dW_{S,t} \quad S_t = S_0 e^{\sigma_S W_{S,t} - \frac{1}{2}\sigma_S^2 t}$$
$$\frac{dB_t}{B_t} = \sigma_B dW_{B,t} \quad B_t = B_0 e^{\sigma_B W_{B,t} - \frac{1}{2}\sigma_B^2 t}$$

We assume the return correlation is ρ, and we have:

$$\frac{dX_t}{X_t} = m\frac{dS_t}{S_t} + n\frac{dB_t}{B_t}$$
$$\frac{dX_t}{X_t} = m\sigma_S dW_{S,t} + n\sigma_B dW_{B,t}$$

Again applying Ito's lemma to $\ln X_t$

$$d\ln X_t = \frac{1}{X_t}dX_t - \frac{1}{2}X_t^{-2}(dX_t)^2$$
$$= m\sigma_S dW_{S,t} + n\sigma_B dW_{B,t} - \frac{1}{2}(m^2\sigma_S^2 + n^2\sigma_B^2 + 2\rho mn\sigma_S\sigma_B)dt$$

and

$$X_t = X_0 e^{m\sigma_S W_{S,t} + n\sigma_B W_{B,t} - \frac{1}{2}(m^2\sigma_S^2 + n^2\sigma_B^2 + 2\rho mn\sigma_S\sigma_B)t}$$
$$= X_0\left(\frac{S_t}{S_0}\right)^m\left(\frac{B_t}{B_0}\right)^n e^{\frac{1}{2}[(m-m^2)\sigma_S^2 + (n-n^2)\sigma_B^2 + 2\rho mn\sigma_S\sigma_B]t}$$
$$= cX_0\left(\frac{S_t}{S_0}\right)^m\left(\frac{B_t}{B_0}\right)^n$$

where $c = e^{\frac{1}{2}[(m-m^2)\sigma_S^2 + (n-n^2)\sigma_B^2 + 2\rho mn\sigma_S\sigma_B]t}$

The portfolio, in addition to the obvious delta exposure to S_t and B_t, also has negative gamma to the movement of S_t and B_t. When $n + m=1$, and dropping higher order terms, we get:

$$X_t = cX_0\left(\frac{S_t}{S_0}\right)^m\left(\frac{B_t}{B_0}\right)^n$$
$$\frac{X_t}{X_0} \approx c\left[1 + m\left(\frac{S_t}{S_0} - 1\right) + n\left(\frac{B_t}{B_0} - 1\right) - \frac{1}{2}mn\left(\frac{S_t}{S_0} - \frac{B_t}{B_0}\right)^2\right]$$

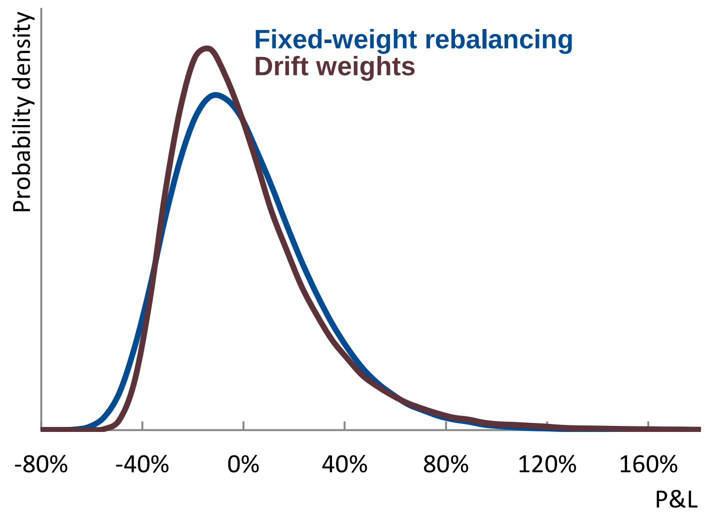
where $c = e^{\frac{1}{2}[(m-m^2)\sigma_S^2 + (n-n^2)\sigma_B^2 + 2\rho mn\sigma_S\sigma_B]t}$.

Thus the difference between this and the one asset case in Eq 2. is the addition of the bond return: $\mathrm{n}\left(\frac{\mathrm{B_t}}{\mathrm{B_0}} - 1\right)$ and the change in quadratic term from $\left(\frac{\mathrm{S_t}}{\mathrm{S_0}} - 1\right)^2$ to $\left(\frac{\mathrm{S_t}}{\mathrm{S_0}} - \frac{\mathrm{B_t}}{\mathrm{B_0}}\right)^2$. Since we expect the bond return to be small in comparison to the equity return, this justifies the earlier approximation.

For completeness we show results from Monte Carlo simulations of the two-asset case. In this case we have modeled the stock and bond components as a geometric brownian motions with a volatilities of 20% and 6% respectively.

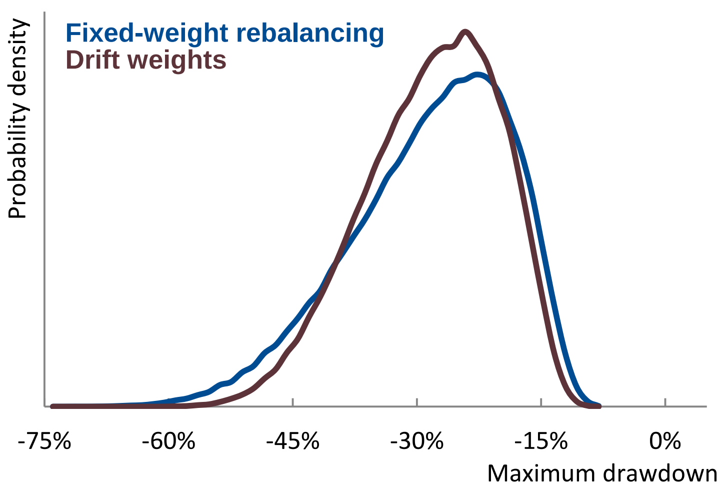
These simulations confirm the earlier results showing increased worst returns and maximum drawdowns of the fixed weight reblancing in comparison to the drift weights (Figure 18 and 19), and the ability of a momentum overlay to improve these risk characteristics (Figure 20 and 21).

Figure 18: Distribution of pay-off, one year, fixed weights (60/40) and drift weights (60/40) of two assets



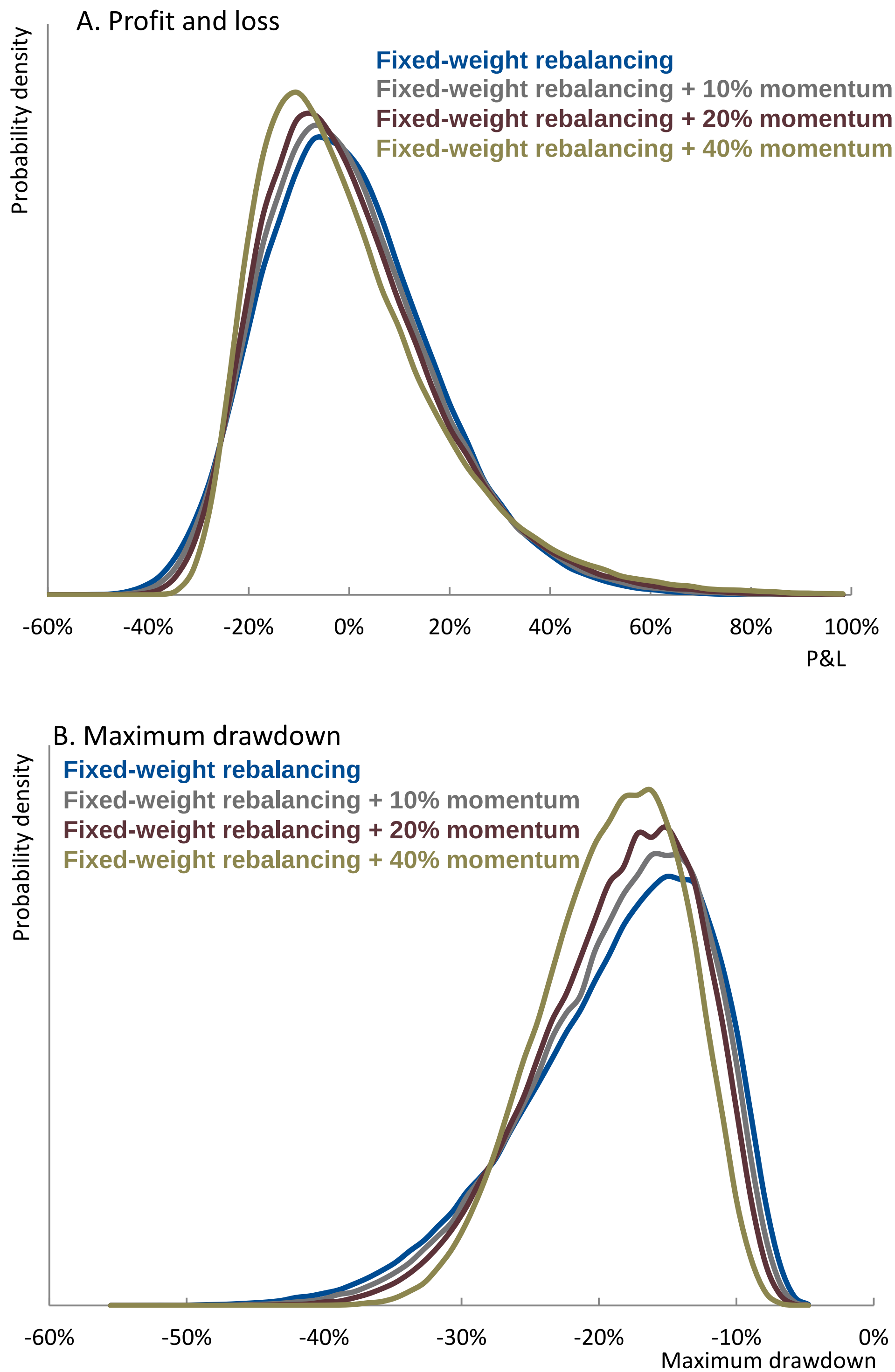
Source: Man calculations, Monte Carlo simulation.

Figure 19: Distribution of worst drawdowns, one year, fixed weights (60/40) and drift weights (60/40) of two assets



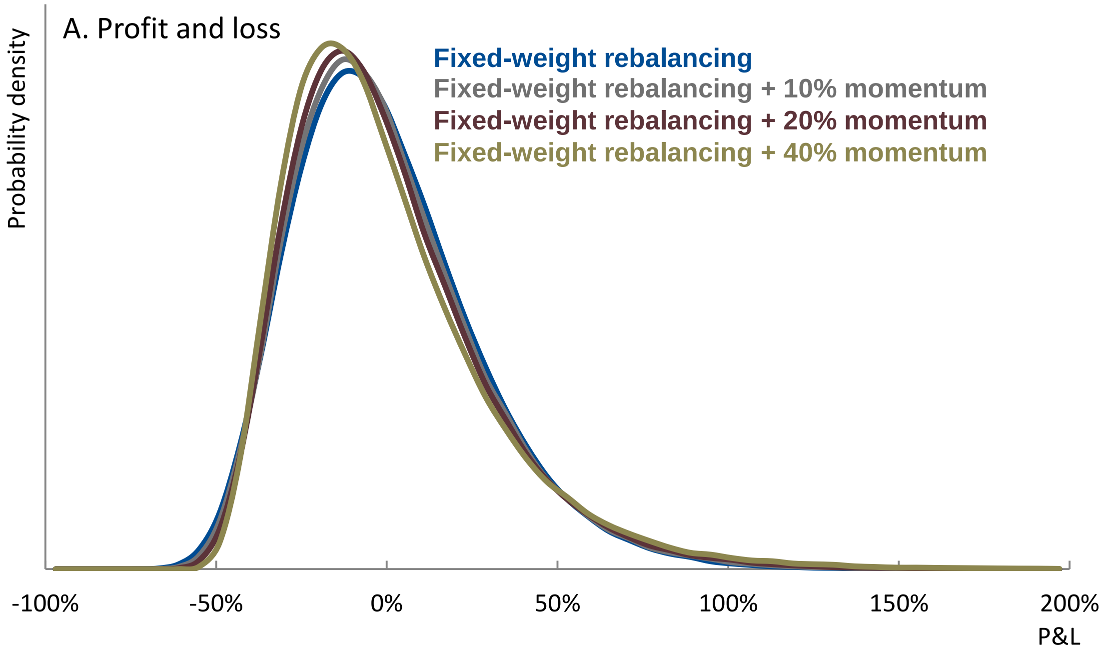
Source: Man calculations, Monte Carlo simulation.

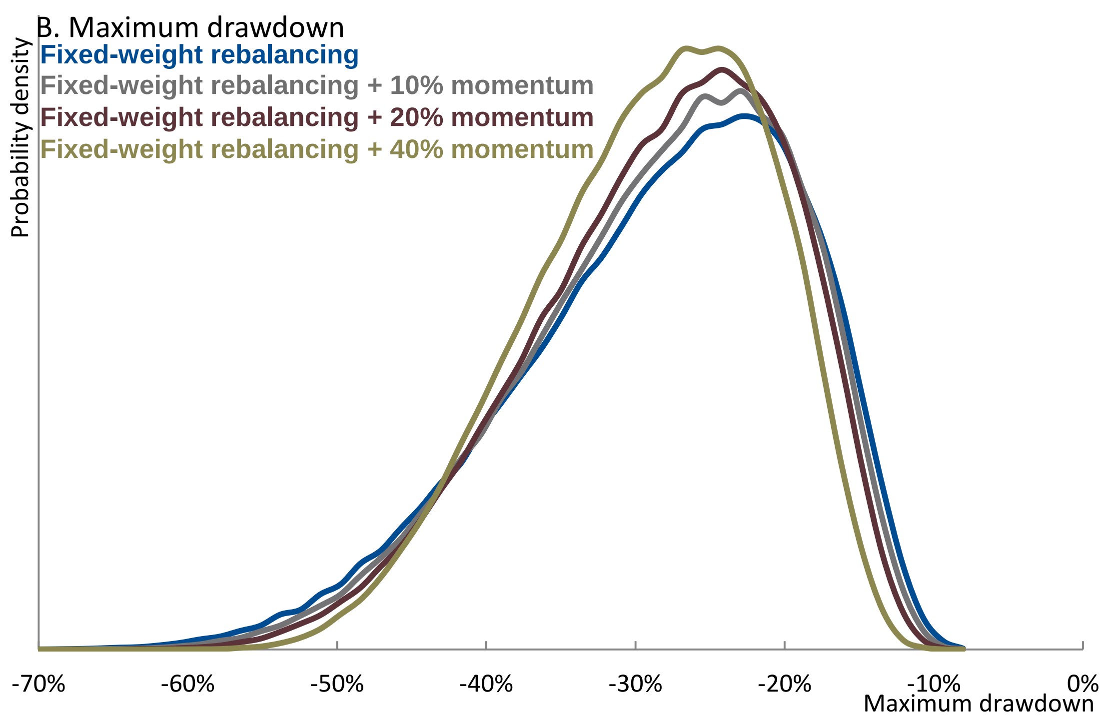
Figure 20: Terminal pay-off and worst drawdowns for fixed-weight rebalancing and momentum, 1-year horizon



Source: Man calculations, Monte Carlo simulation.

Figure 21: Terminal pay-off and worst drawdowns for fixed-weight rebalancing and momentum, 5-year horizon





Source: Man calculations, Monte Carlo simulation.