Homework 2

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5.2

Let s_A and s_B be the Sharpe ratio of portfolios A and B. Let r_A and r_B be the expected returns of these two portfolios, with standard deviation denoted by σ_A and σ_B. Assume that through self financing, portfolio A borrows $(\sigma_B/\sigma_A - 1)$ at risk-free rate r_f to leverage so that its risk is now the same as that of portfolio B.

Show that the excess return of leveraged investment in portfolio A is larger than the expected return of portfolio B if $s_A > s_B$. This shows that the Sharpe ratio measures the efficiency of a portfolio.

5.3

Suppose that three mutual funds (conservative, growth and aggressive) have annual log-returns of 15%, 20% and 30% with volatility of 20%, 30% and 50% respectively. The correlation between any of the 2 funds is 0 and the risk-free rate is 5%.

```r
rf <- .05
vol <- c(.20, .30, .50)
r <- c(.15, .20, .30)
expected_return <- .15
Y <- r - rf # excess returns
gamma <- diag(x = 1, nrow = 3)
vol <- diag(vol, nrow = 3)
Sigma <- vol %*% gamma %*% vol
partial_alpha <- as.vector(solve(Sigma) %*% Y )
A <- sum(partial_alpha * Y)/(expected_return - rf)
```

1. What is the min variance portfolio with these 3 mutual funds? > The min variance is given by $\sigma^{*2} = \alpha^* \Sigma \alpha^*$
2. Find the optimal portfolio allocation among the 3 mutual funds, if the expected return is set at 15%. Give the associated standard deviation of this portfolio.
3. Compute the Sharpe ratio for the portfolio in A. How does it compare with that in B?

5.10

Let $\mathbf{y}$ be the excess returns of risky assets. Let $X = \mathbf{ff}^T\mathbf{y}$ be a portfolio with allocation vector $\mathbf{ff}$. Denote by $\Sigma = \text{var}(\mathbf{y})$ and $\mu = \mathbb{E}(\mathbf{y})$. Consider the following decomposition:

$$\mathbf{y} = \alpha + \beta X + \epsilon \quad \mathbb{E}(\epsilon) = 0 \quad \text{cov}(\epsilon, X) = 0$$

1. Show that if $\mathbf{ff} = c\Sigma^{-1}\mu$ then $\alpha = 0$. 2. Conversely if $\alpha = 0$, there exists a constant c such that $\mathbf{ff} = c\Sigma^{-1}\mu_0$

5.13

Consider the following portfolio optimization problem with a risk-free asset having return r_0:

$$\min \mathbf{ff}^T\Sigma\mathbf{ff} \quad \text{such that} \quad \mathbf{ff}^T\bar{\ } + (1-\mathbf{ff}^T\mathbf{1})r_0 = \mu$$

1. The optimal solution is $\mathbf{ff} = P^{-1}(\mu - r_0)\Sigma^{-1}\mu_0$ where $P = \bar{\ }_0^T\Sigma^{-1}\bar{\ }_0^T$ is the squared Sharpe ratio, and $\bar{\ }_0 = \bar{\ } - r_0\mathbf{1}$ is the vector of excess returns. 2. The variance of this portfolio is $\sigma^2 = (\mu - r_0)^2/P$. 3. When $r_0 < \mu$, show that $r_0 + P^{1/2}\sigma = \mu$, namely the optimal allocation for the risky asset $\mathbf{ff}$ is the tangent portfolio.

5.14