# Homework 5

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### 7.7

Show that the solution to 7.18 is given by 7.17. As a further generalization, what is the solution to the following problem: minimizing with respect the symmetric matrix $\boldsymbol{S}$:

$$||\hat{\boldsymbol{\Sigma}}_\lambda - \boldsymbol{S}||_F^2 \quad \text{such that} \quad \lambda_{\min}(\boldsymbol{S}) \geq \delta$$

We can show the solution to 7.18 is given by 7.17 since:

$$\begin{aligned} ||\hat{\boldsymbol{\Sigma}}_\lambda - \boldsymbol{S}||_F^2 &= tr[(\hat{\boldsymbol{\Sigma}}_\lambda - \boldsymbol{S})(\hat{\boldsymbol{\Sigma}}_\lambda - \boldsymbol{S})^T] \\ &= tr(\hat{\boldsymbol{\Sigma}}\hat{\boldsymbol{\Sigma}}^T) - tr(\hat{\boldsymbol{\Sigma}}\boldsymbol{S}^T) - tr(\boldsymbol{S}\hat{\boldsymbol{\Sigma}}^T) + tr(\boldsymbol{S}\boldsymbol{S}^T) \\ &= tr(\hat{\boldsymbol{\Sigma}}\hat{\boldsymbol{\Sigma}}^T) - 2tr(\hat{\boldsymbol{\Sigma}}\boldsymbol{S}^T) + tr(\boldsymbol{S}\boldsymbol{S}^T) \end{aligned}$$

Since we eventually want to constrain $\boldsymbol{S}$ to positive definite matrices, let's represent $\boldsymbol{S}$ as $\boldsymbol{V}\boldsymbol{V}^T$, leading to:

$$tr(\hat{\boldsymbol{\Sigma}}\hat{\boldsymbol{\Sigma}}^T) - 2tr(\hat{\boldsymbol{\Sigma}}\boldsymbol{S}\boldsymbol{S}^T) + tr(\boldsymbol{V}\boldsymbol{V}^T\boldsymbol{V}\boldsymbol{V}^T)$$

Taking the derivative of this relative to $\boldsymbol{V}$ results in $-4\boldsymbol{V}^T\hat{\boldsymbol{\Sigma}} + 4\boldsymbol{V}^T\boldsymbol{V}\boldsymbol{V}^T = 0$. We can simplify and solve this by letting $\boldsymbol{V} = \Gamma^T(\Lambda^+)^{1/2}$. Since our covariance matrix is symmetric and can be broken down into $\Gamma^T\Lambda\Gamma$, then:

$$\begin{gathered} -4[(\Lambda^+)^{1/2}\Gamma\Gamma^T(\Lambda^+ + \Lambda^-)\Gamma] + 4(\Lambda^+)^{1/2}\Gamma\Gamma^T(\Lambda^+)^{1/2}(\Lambda^+)^{1/2}\Gamma \\ -4[(\Lambda^+)^{1/2}(\Lambda^+ + \Lambda^-)\Gamma] + 4(\Lambda^+)^{1/2}\Lambda^+\Gamma \\ -4[(\Lambda^+)^{1/2}\Lambda^+\Gamma] + 4(\Lambda^+)^{1/2}\Lambda^+\Gamma \end{gathered}$$

Our positive definite matrice $\boldsymbol{S} = \boldsymbol{V}\boldsymbol{V}^T$ thus minimizes the function, which can also be represented by 7.17 where:

$$\hat{\boldsymbol{\Sigma}}^+ = \boldsymbol{\Lambda}^T diag(\lambda_1^+, \ldots, \lambda_p^+)\boldsymbol{\Lambda}$$

### 7.10

Let $\hat{\boldsymbol{\Sigma}}$ be an estimated volatility matrix of a true volatility. Show that for any portfolio allocation $\boldsymbol{w}$, the relative estimation error is bounded by:

$$|\frac{\boldsymbol{w}^T\hat{\boldsymbol{\Sigma}}\boldsymbol{w}}{\boldsymbol{w}^T\boldsymbol{\Sigma}\boldsymbol{w}} - 1| \leq ||\boldsymbol{\Sigma}^{-1/2}\hat{\boldsymbol{\Sigma}}\boldsymbol{\Sigma}^{-1/2} - \boldsymbol{I}_p||$$

As proven in 7.8:

$$tr[\hat{\boldsymbol{\Sigma}}\boldsymbol{\Sigma}^{-1} - \boldsymbol{I}_p]^2 = ||\boldsymbol{\Sigma}^{-1/2}\hat{\boldsymbol{\Sigma}}\boldsymbol{\Sigma}^{-1/2} - \boldsymbol{I}_p||_F^2$$

Thus:

$$tr[\hat{\boldsymbol{\Sigma}}\boldsymbol{\Sigma}^{-1} - \boldsymbol{I}_p] = ||\boldsymbol{\Sigma}^{-1/2}\hat{\boldsymbol{\Sigma}}\boldsymbol{\Sigma}^{-1/2} - \boldsymbol{I}_p||_F$$

We can then simplify:

$$
\begin{aligned}
\left|\frac{\boldsymbol{w}^T\hat{\boldsymbol{\Sigma}}\boldsymbol{w}}{\boldsymbol{w}^T\boldsymbol{\Sigma}\boldsymbol{w}} - 1\right| &= \left|\frac{\boldsymbol{w}^T(\hat{\boldsymbol{\Sigma}} - \boldsymbol{\Sigma})\boldsymbol{w}}{\boldsymbol{w}^T\boldsymbol{\Sigma}\boldsymbol{w}}\right| \\
&= \left|\left|\frac{\hat{\boldsymbol{\Sigma}} - \boldsymbol{\Sigma}}{\boldsymbol{\Sigma}}\right|\right| \\
&= ||\hat{\boldsymbol{\Sigma}}\boldsymbol{\Sigma}^{-1} - \boldsymbol{I}_p|| \\
&\leq ||\hat{\boldsymbol{\Sigma}}\boldsymbol{\Sigma}^{-1} - \boldsymbol{I}_p||_F \\
&\leq tr[\hat{\boldsymbol{\Sigma}}\boldsymbol{\Sigma}^{-1} - \boldsymbol{I}_p] \\
&= ||\boldsymbol{\Sigma}^{-1/2}\hat{\boldsymbol{\Sigma}}\boldsymbol{\Sigma}^{-1/2} - \boldsymbol{I}_p||_F
\end{aligned}
$$

**7.11**

Suppose that we have 100 investable stocks, labeled as 1 through 100 and classified as "Consumer Non-durables", "Consumer durables", "Manufacturing", "Energy", "Business equipment", "Telecommunications", "Shops", "Health", "Utilities", and "Others". Let $w_1, \ldots, w_{100}$ be the portfolio weights. If the first 10 stocks are labeled as "Consumer Non-durables", the second 10 stock are in "Consumer durables", etc, write down the constraints of the portfolio.

1. the "health stocks" are no more than 15% and "energy stocks" are no more than 30%.
2. no exposure to Telecommunications.
3. exposure to "Consumer durables" but gross exposure is zero.

**7.12**

Let the study period be Jan 2001 to Jan 2015. Apply the sample covariance matrix, the FF 3 factor model, and the RiskMetrics with $\lambda = .94$ to obtain the time varying covariance matrix for Dell, Ford, GE, IBM, Intel, J&J, Merck, 3-mo Tres, and S&P500 at the beginning of each month. Optimize the portfolio and holds for the next 21 days. Compute the risk of such a portfolio and compare it with the equally weighted portfolio.

First, we get the data we need:

```
start <- as.Date("2001-01-01")
end <- as.Date("2015-01-01")
assets <- c(# "DVMT" # DELL is DROPPED since not provided by Quantmod
  "F",
  "GE",
  "IBM",
  "INTC",
  "JNJ",
  "MRK",
  "SPY")
getSymbols(assets, from = start, to = end)
# get the 3 mo tresury data
getSymbols("DGS3MO", src = "FRED")
# combine all info
```

```
closing.prices <- merge.xts(DGS3MO,
                            F[, 4],
                            GE[, 4],
                            IBM[, 4],
                            INTC[, 4],
                            JNJ[, 4],
                            MRK[, 4],
                            SPY[, 4])
# filter out to only dates of interest
data <- closing.prices["2001-01-01/2015-01-01"]
# save
saveRDS(data, "~/workspace/st790-financial-stats/hw5/covdata.rds")
```

Now we can reorganize the data within our desired time frame:

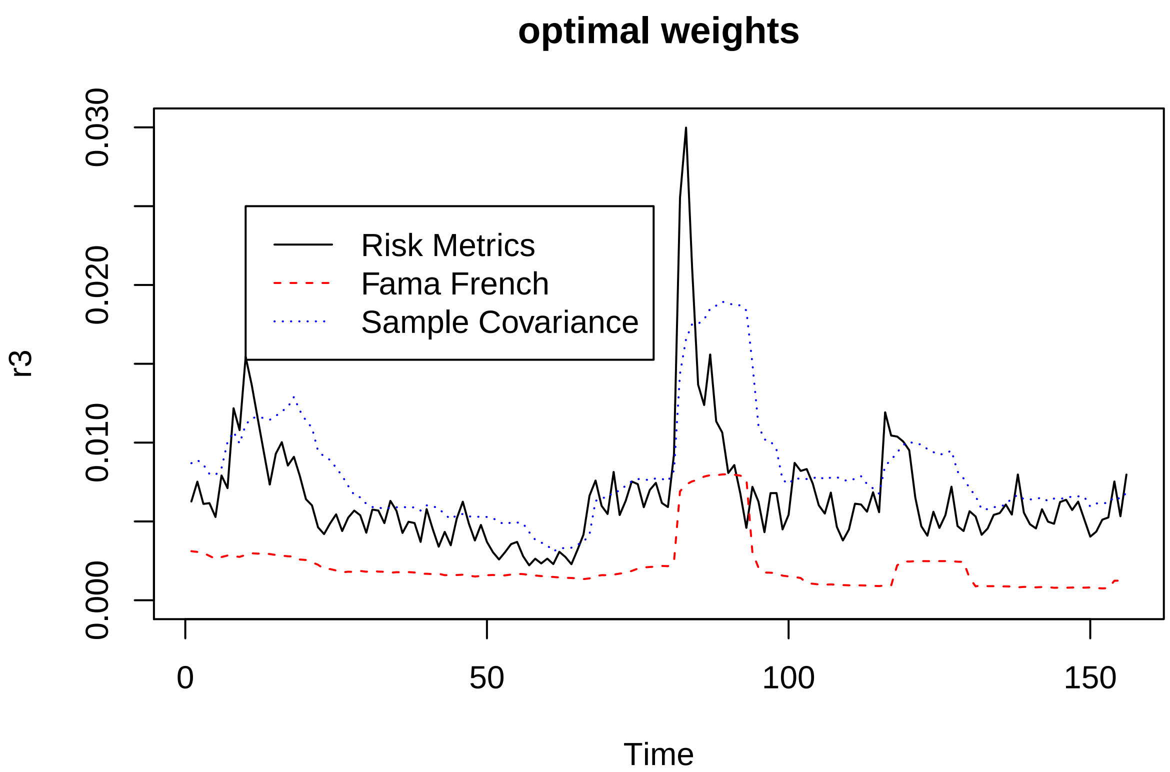
```
data <- readRDS("~/workspace/st790-financial-stats/hw5/covdata.rds")

# break it down monthly
by_month <- data.frame()
for(i in 1:ncol(data)){
  temp <- data[, i]
  monthly <- monthlyReturn(temp)
  by_month <- cbind(by_month, monthly)
  colnames(by_month)[i] <- strsplit(names(temp), "[.]")[[1]][1]
}
by_month <- na.omit(by_month)
# get the Fama French data
FF <- matrix(scan("~/workspace/st790-financial-stats/hw5/F-F_Research_Data_Factors_daily.txt", s
# match dates
D <- time(by_month)
D <- paste0(substr(D, 1, 4),
            substr(D, 6, 7),
            substr(D, 9, 10))
ind <- rep(0, length(D))
for(i in 1:length(ind)){
  ind[i] = (1:dim(FF)[1])[D[i] == FF[,1]]
}
ff <- FF[ind, ]
# break it down daily
dat <- na.omit(data)
by_day <- data.frame()
for(i in 1:ncol(dat)){
  temp <- dat[, i]
  daily <- dailyReturn(temp)
  by_day <- cbind(by_day, daily)
  colnames(by_day)[i] <- strsplit(names(temp), "[.]")[[1]][1]
}
```

```r
by_day <- na.omit(by_day)
# get the FF by day
days <- time(by_day)
days <- paste0(substr(days, 1, 4),
            substr(days, 6, 7),
            substr(days, 9, 10))
ind2 <- rep(0, length(days))
for(i in 1:length(ind2)){
  ind2[i] = (1:dim(FF)[1])[days[i] == FF[,1]]
}
ff_days <- FF[ind2, ]
saveRDS(by_day, "~/workspace/st790-financial-stats/hw5/byday.rds")
saveRDS(ff_days, "~/workspace/st790-financial-stats/hw5/ff_byday.rds")
saveRDS(by_month, "~/workspace/st790-financial-stats/hw5/bymonth.rds")
saveRDS(ff, "~/workspace/st790-financial-stats/hw5/ff_bymonth.rds")
```

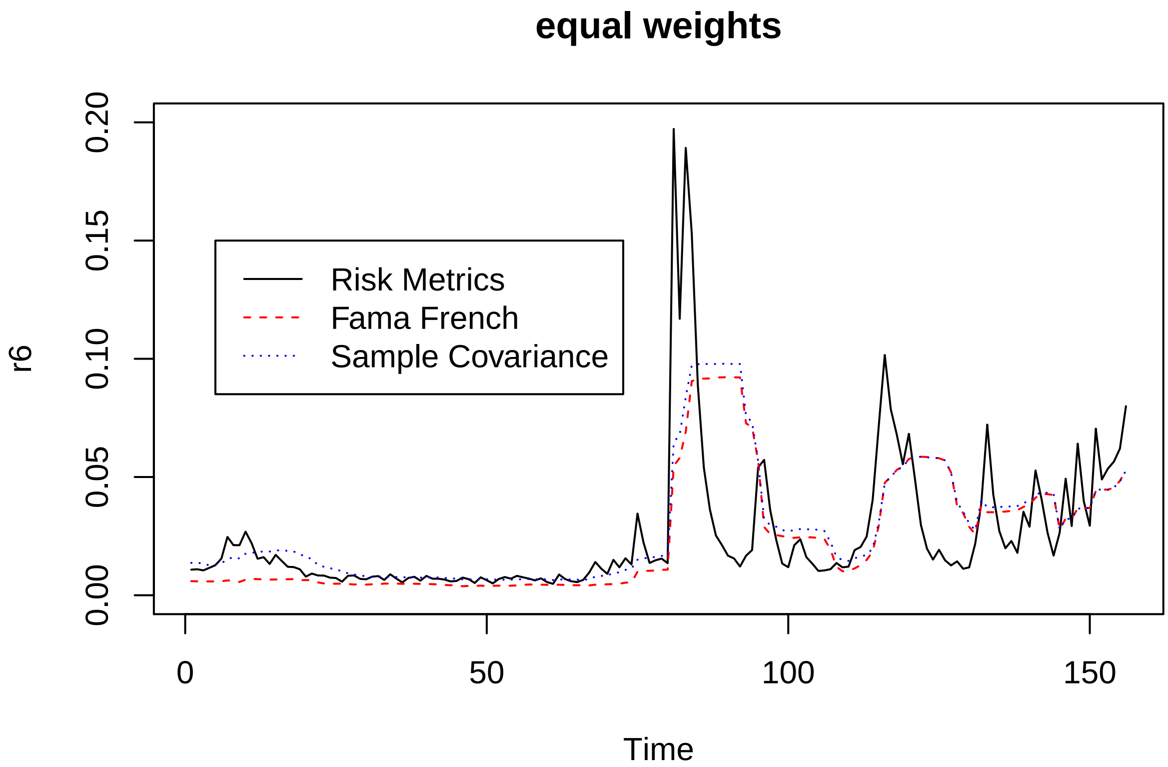
With our data in the desired format, we can now calculate our rolling covariances:

```r
by_day <- readRDS("~/workspace/st790-financial-stats/hw5/byday.rds")
ff <- readRDS("~/workspace/st790-financial-stats/hw5/ff_byday.rds")
by_day[is.infinite(by_day)] <- 0
by_month <- readRDS("~/workspace/st790-financial-stats/hw5/bymonth.rds")
# risk metrics
n <- nrow(by_day)
lambda <- .94 # for daily
Sigma <- array(0, c(8, 8, n+1))
for(i in 1:n){
  tmp <- as.numeric(c(as.numeric(by_day[i, ])))
  Sigma[,,i+1] <- lambda*Sigma[,,i] + (1-lambda)*tmp %*% t(tmp)
}
# only need optimize monthly
dates_of_interest <- time(by_month)
# start window in 2002
dates_of_interest <- dates_of_interest[12:length(dates_of_interest)]
r1 <- r2 <- r3 <- r4 <- r5 <- r6 <- numeric(length(dates_of_interest))
equal_weight <- rep(1, 8)/8
for(i in 1:length(dates_of_interest)){
  temp_date <- dates_of_interest[i]
  # find the correct index to get 252 rolling window
  temp_D <- paste0(substr(temp_date, 1, 4),
            substr(temp_date, 6, 7),
            substr(temp_date, 9, 10))
  temp_i <- (1:dim(ff)[1])[temp_D == ff[,1]]
  X <- ff[(temp_i - 252):temp_i, 2:4]
  Y <- by_day[(temp_i - 252):temp_i, ]
  # method 1: sample covariance
  SigmaS <- cov(Y)
```

```r
  # method 2: fama french
  resid <- resid(lsfit(X, Y))
  N <- dim(Y)[1] # length of window
  SigmaE <- t(resid) %*% resid / (N-4)
  # method 3: risk metrics
  SigmaR <- Sigma[,,temp_i]
  # optimize
  result1 <- optimalPortfolio(SigmaS, control = list(type = "minvol"))
  result2 <- optimalPortfolio(SigmaE, control = list(type = "minvol"))
  result3 <- optimalPortfolio(SigmaR, control = list(type = "minvol"))
  # compute risk
  r1[i] <- sqrt(result1 %*% SigmaS %*% result1)
  r2[i] <- sqrt(result2 %*% SigmaE %*% result2)
  r3[i] <- sqrt(result3 %*% SigmaR %*% result3)
  # compare to equal weight
  r4[i] <- sqrt(equal_weight %*% SigmaS %*% equal_weight)
  r5[i] <- sqrt(equal_weight %*% SigmaE %*% equal_weight)
  r6[i] <- sqrt(equal_weight %*% SigmaR %*% equal_weight)
}
# viz
plot(r3, type = "l", ylim = c(0, .03), xlab = "Time", ylabe = "Risk",
     lty = 1)
lines(r2, col = "red", lty =2 )
lines(r1, col = "blue", lty=3)
legend(10, .025, legend = c("Risk Metrics",
                            "Fama French",
                            "Sample Covariance"),
       col=c("black", "red", "blue"),
       lty=1:3)
title("optimal weights")
```




```
# viz
plot(r6, type = "l", ylim = c(0, .2), xlab = "Time", ylabe = "Risk",
     lty = 1)
lines(r5, col = "red", lty =2 )
lines(r4, col = "blue", lty=3)
legend(5, .15, legend = c("Risk Metrics",
                          "Fama French",
                          "Sample Covariance"),
       col=c("black", "red", "blue"),
       lty=1:3)
title("equal weights")
```


equal weights
r6
0.00
0.05
0.10
0.15
0.20
0
50
100
150
Time
Risk Metrics
Fama French
Sample Covariance