Quiz 14

1. Evaluate $\oint_L \sqrt{x^2+y^2}\,\mathrm{d}s$, and L is $x^2+y^2=ax, a>0$.

Hint: $x=r\cos t, y=r\sin t \Rightarrow r^2=ar\cos t, r=a\cos t, x=a\cos^2 t,$

$$y=a\cos t\sin t, t\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right], dx=-a\sin 2tdt, dy=a\cos 2tdt, ds=adt$$

$$\oint_L \sqrt{x^2+y^2}\,\mathrm{d}s=\int_{-\pi/2}^{\pi/2}\sqrt{a^2\cos^2 t}\,adt=2\int_0^{\pi/2} a^2\cos tdt=2a^2\sin t\Big|_0^{\pi/2}=2a^2$$

2. Evaluate $\int_\Gamma x\mathrm{d}x+y\mathrm{d}y+(x+y-1)\mathrm{d}z$, and Γ is a straight segment from $(1,1,1)$ to $(2,3,4)$.

HINT $\Gamma:\frac{x-1}{2-1}=\frac{y-1}{3-1}=\frac{z-1}{4-1}, x=1+t, y=1+2t, z=1+3t, t:0\to 1$

$$\text{The integral}=\int_0^1\left[(1+t)+2(1+2t)+3(1+t+1+2t-1)\right]dt$$

$$=\int_0^1(6+14t)dt=\left(6t+7t^2\right)\Big|_0^1=13$$

3.Find $I=\int_L(\mathrm{e}^x\sin y-y^3)\mathrm{d}x+(\mathrm{e}^x\cos y+x^3)\mathrm{d}y$, and L is the arc $x=-\sqrt{a^2-y^2}$ from $A(0,-a)$ to $B(0,a)$.

Hint:

L+ $BA: x=0, x:a\to -a$,

$$I=\int_L(\mathrm{e}^x\sin y-y^3)\mathrm{d}x+(\mathrm{e}^x\cos y+x^3)\mathrm{d}y=-\iint_D 3\left(x^2+y^2\right)d\sigma-\int_a^{-a}\cos ydy$$

$$=-3\int_{\pi/2}^{3\pi/2}d\theta\int_0^a r^3dr+\int_{-a}^{a}\cos ydy=-\frac{3\pi a^4}{4}+2\sin a$$

4. Show that $\oint_L\frac{x\mathrm{d}y-y\mathrm{d}x}{x^2+y^2}$ equals 2π or 0 accordingly as the origin is inside or outside L, here L is a smooth simple closed curve in its positive direction.

Hint:

5. Evaluate $\iint\limits_{\Sigma}\left(x^2+y^2+z^2\right)\mathrm{d}S$, and Σ is the sphere $x^2+y^2+z^2=2ax$.

Hint: Σ : $x=a\pm\sqrt{a^2-y^2-z^2}, x_y=\dfrac{\pm y}{\sqrt{a^2-y^2-z^2}}, x_y=\dfrac{\pm z}{\sqrt{a^2-y^2-z^2}}$

$$dS=\sqrt{1+z_x^2+z_y^2}dxdy=\frac{a}{\sqrt{a^2-y^2-z^2}}dxdy\ ,\quad D:0\le r\le a, 0\le\theta\le 2\pi$$

$$\text{The integral}=\iint\limits_{\Sigma_1}2ax\mathrm{d}S+\iint\limits_{\Sigma_2}2ax\mathrm{d}S=\iint\limits_{D}\frac{2a^2\left(a+\sqrt{a^2-y^2-z^2}\right)}{\sqrt{a^2-y^2-z^2}}dxdy$$

$$+\iint\limits_{D}\frac{2a^2\left(a-\sqrt{a^2-y^2-z^2}\right)}{\sqrt{a^2-y^2-z^2}}dxdy=4a^3\iint\limits_{D}\frac{dxdy}{\sqrt{a^2-y^2-z^2}}=4a^3\int_0^{2\pi}d\theta\int_0^a\frac{2rdr}{2\sqrt{a^2-r^2}}$$

$$=-8\pi a^3\int_0^a\frac{d\left(a^2-r^2\right)}{2\sqrt{a^2-r^2}}=-8\pi a^3\cdot\sqrt{a^2-r^2}\Big|_0^a=8\pi a^4$$

6. Let $\vec{\mathrm{F}}=\left(2x+3y\right)\vec{\mathrm{i}}-\left(xz+y\right)\vec{\mathrm{j}}+\left(y^2+2z\right)\vec{\mathrm{k}}$, Σ is a sphere which-+ center is (3,-1,2) and radius is 3, the normal is outward.

Hint: $\Phi=\iint\limits_{\Sigma}\left(2x+3y\right)\mathrm{d}y\mathrm{d}z-\left(xz+y\right)\mathrm{d}z\mathrm{d}x+\left(y^2+2z\right)\mathrm{d}x\mathrm{d}y=\iiint\limits_{\Omega}\left(2-1+2\right)dv$

$$=3\cdot\frac{4\pi}{3}\cdot3^3=108\pi$$

7. Find the circulation of $A=(x-z)\vec{\mathrm{i}}+(x^3+yz)\vec{\mathrm{j}}-3xy^2\vec{\mathrm{k}}$ along the closed curve Γ (Γ is counterclockwise from the direction of the vector $\vec{\mathrm{k}}$) here Γ is the circle

$z=2-\sqrt{x^2+y^2}, z=-2$.

Hint: $\oint\limits_{\Gamma}\vec{A}d\vec{l}=\oint\limits_{\Gamma}\left(x-z\right)\mathrm{d}x+\left(x^3+yz\right)\mathrm{d}y-3xy^2\mathrm{d}z$

$$=\iint\limits_{\Sigma}\left(-6xy-y\right)\mathrm{d}y\mathrm{d}z+\left(-1+3y^2\right)\mathrm{d}z\mathrm{d}x+\left(3x^2-0\right)\mathrm{d}x\mathrm{d}y\overset{z=-2}{=}\iint\limits_{\Sigma}3x^2\mathrm{d}x\mathrm{d}y$$

$$=\frac{3}{2}\iint\limits_{\Sigma}\left(x^2+y^2\right)\mathrm{d}x\mathrm{d}y=\frac{3}{2}\int_0^{2\pi}d\theta\int_0^4 r^3dr=3\pi\cdot\frac{1}{4}\cdot4^4=192\pi$$

8. Find the divergence and the curl of $\vec{A}=\left\{4xyz,-xy^2,x^2yz\right\}$ at M (1, -1, 2).

Hint: $div\vec{A} = 4yz - 2xy + x^2 y, div\vec{A}\big|_M = -8 + 2 - 1 = -7$

$rot\vec{A} = \{x^2 z, 4xy - 2xyz, -y^2 - 4xy\}, C = rot\vec{A}\big|_M = \{2, 0, -9\}$

9. $\iint_\Sigma (y^2 + z)\mathrm{d}y\mathrm{d}z + (x + z^2)\mathrm{d}z\mathrm{d}x + (y + x^2)\mathrm{d}x\mathrm{d}y$ =____________, here Σ is the sphere $x^2 + y^2 + z^2 = 1$, with outward normal.

Hint: 0.

10. Find $\oint_L \frac{|y|}{x^2 + y^2 + z^2}\mathrm{d}s$, here L is defined by $\begin{cases} x^2 + y^2 + z^2 = 4a^2 \\ x^2 + y^2 = 2ax \end{cases}$, $z \ge 0, a > 0$.

Hint: L: $\begin{cases} x^2 + y^2 + z^2 = 4a^2 \\ x^2 + y^2 = 2ax \end{cases} \Rightarrow \rho = 2a, \cos\theta = \sin\varphi \Rightarrow$

$x = 2a\cos^2\theta, y = 2a\sin\theta\cos\theta, z = 2a\sin\theta, \theta \in [0, \pi]$

The integral $= \oint_L \frac{|y|}{4a^2}\mathrm{d}s = 2\int_0^{\pi/2} \sin\theta\cos\theta\sqrt{1 + \cos^2\theta}\,d\theta = -\int_1^0 \sqrt{1 + t}\,dt = \frac{2}{3}\left(2\sqrt{2} - 1\right)$