Quiz 12

1. Let $f(x,y)=\begin{cases}(x+y)^2\sin\dfrac{1}{x^2+y^2}, & x^2+y^2\neq 0\\ 0, & x^2+y^2=0\end{cases}$

1) Find $\dfrac{\partial f}{\partial x}$, $\dfrac{\partial f}{\partial y}$.

2) Determine the continuity of $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$ at the origin.

3) Determine the differetiability of $f(x,y)$ at the origin.

Hint: (1) $x^2+y^2\neq 0, f(x,y)=(x+y)^2\sin\dfrac{1}{x^2+y^2}$

$$\frac{\partial f}{\partial x}=2(x+y)\sin\frac{1}{x^2+y^2}+(x+y)^2\cos\frac{1}{x^2+y^2}\cdot\frac{-2x}{\left(x^2+y^2\right)^2},$$

$$\frac{\partial f}{\partial y}=2(x+y)\sin\frac{1}{x^2+y^2}+\frac{-2y(x+y)^2}{\left(x^2+y^2\right)^2}\cos\frac{1}{x^2+y^2}$$

If $x^2+y^2=0$,

$$f_x(0,0)=\lim_{x\to 0}\frac{x^2\sin\frac{1}{x^2}-0}{x}=0, f_y(0,0)=\lim_{y\to 0}\frac{y^2\sin\frac{1}{y^2}-0}{y}=0$$

(2) $\dfrac{\partial f}{\partial x}$, $\dfrac{\partial f}{\partial y}$ are not continuous at the origin, but

(3) $$\lim_{\rho\to 0}\frac{f(x,y)-f(0,0)-f_x(0,0)x-f_y(0,0)y}{\sqrt{x^2+y^2}}=\lim_{\rho\to 0}\frac{(x+y)^2\sin\frac{1}{x^2+y^2}-0-0\cdot x-0\cdot y}{\sqrt{x^2+y^2}}$$

$$=\lim_{\rho\to 0}\left(\frac{\rho^2\sin\frac{1}{\rho}}{\rho}+\frac{2xy\sin\frac{1}{x^2+y^2}}{\sqrt{x^2+y^2}}\right)=0, \because |2xy|\le x^2+y^2$$

The function is differentiable at the origin.

2. Let $z=xf\left(\dfrac{y}{x}\right)+yg(x,\dfrac{x}{y})$, f'', g'' both exist, try to find $\dfrac{\partial^2 z}{\partial x\partial y}$.

Hint: $\frac{\partial z}{\partial x}=f+xf'\cdot\frac{-y}{x^2}+yg_1'\cdot 1+yg_2'\cdot\frac{1}{y}=f-\frac{y}{x}f'\cdot+yg_1'+g_2'$

Then $\frac{\partial^2 z}{\partial x\partial y}=\frac{\partial}{\partial y}\left(f-\frac{y}{x}f'\cdot+yg_1'+g_2'\right)$

$$=f'\cdot\frac{1}{x}-\frac{1}{x}f'-\frac{y}{x}f''\cdot\frac{1}{x}+g_1'+yg_{12}''\cdot\frac{-x}{y^2}+g_{22}''\cdot\frac{-x}{y^2}=-\frac{y}{x^2}f''+g_1'-\frac{x}{y}g_{12}''-\frac{x}{y^2}g_{22}''$$

3. If $z^x=y^z$, try to find $\mathrm{d}z$.

Hint: $dz=\frac{z^2dy-yz\ln zdx}{xy-yz\ln y}$

4. Assume $z=f(x,y),x=\varphi(y,z)$, and f,φ are differentiable.Please find $\frac{\mathrm{d}z}{\mathrm{d}x}$.

Hint: $dz=f_1dx+f_2dy,dx=\varphi_1dy+\varphi_2dz,$

So $dy=\frac{-\varphi_2dz+dx}{\varphi_1},dz=f_1dx+f_2\cdot\frac{-\varphi_2dz+dx}{\varphi_1},\varphi_1dz=\varphi_1f_1dx-f_2\varphi_2dz+f_2dx$

$$(\varphi_1+f_2\varphi_2)dz=(\varphi_1f_1+f_2)dx,\frac{dz}{dx}=\frac{\varphi_1f_1+f_2}{\varphi_1+f_2\varphi_2}$$

5. Find the directional derivative of $u=\mathrm{e}^x\cos(yz)$ at $(0,0,0)$ in the direction of $\vec{l}=\{2,1,-2\}$.

Hint: $\frac{2}{3}$

6. Please find the angle between the gradient of $u=x^2+y^2-z^2$ at $A(a,0,0)$ and the gradient of $u=x^2+y^2-z^2$ at $B(0,a,0)$.

Hint: $gradu\big|_A=\{2x,2y,-2z\}\big|_A=\{2a,0,0\}$

$gradu\big|_B=\{2x,2y,-2z\}\big|_B=\{0,2a,0\}$

$$\cos\varphi=\frac{\left|gradu\big|_A\cdot gradu\big|_B\right|}{\left|gradu\big|_A\right|\cdot\left|gradu\big|_B\right|}=0\Rightarrow\varphi=\frac{\pi}{2}$$

7. Let $\begin{cases} z = x^2 + y^2 \\ x^2 + 2y^2 + 3z^2 = 20 \end{cases}$, please find $\frac{\mathrm{d}y}{\mathrm{d}x}$ and $\frac{\mathrm{d}z}{\mathrm{d}x}$.

Hint: $\begin{cases} dz = 2xdx + 2ydy \\ 2xdx + 4ydy + 6zdz = 0 \end{cases} \Rightarrow \begin{cases} dz - 2xdx = 2ydy \\ 2xdx + 2(dz - 2xdx) + 6zdz = 0 \end{cases}$

$\Rightarrow \begin{cases} -2xdx + (2+6z)dz = 0 \\ xdx + 2ydy + 3z(2xdx + 2ydy) = 0 \end{cases} \Rightarrow \frac{dz}{dx} = \frac{x}{1+3z}, \frac{dy}{dx} = -\frac{x+6xy}{2y+6yz}$

8. Please find the equation of the tangent line and the equation of the normal plane of the curve $x = \sin^2 t, y = \sin t \cos t, z = \cos^2 t$ at $t = \frac{\pi}{4}$.

Hint: The point is $\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right), \vec{T} = \left\{2\sin t \cos t, \cos^2 t - \sin^2 t, -2\cos t \sin t\right\}\Big|_{\frac{\pi}{4}} = \{1, 0, -1\}$,

The equation of the tangent line is $\frac{x-\frac{1}{2}}{1} = \frac{y-\frac{1}{2}}{0} = \frac{z-\frac{1}{2}}{-1}, \begin{cases} x+z-1=0 \\ y = \frac{1}{2} \end{cases}$,

And the equation of the normal plane is $x - \frac{1}{2} - \left(z - \frac{1}{2}\right) = 0, x - z = 0$

9. Find the equation of the tangent plane of the surface $ax^2 + by^2 + cz^2 = 1(abc \neq 0)$ at the point (x_0, y_0, z_0).

Hint:: $\vec{n} = \{2ax_0, 2by_0, 2cz_0\} // \{ax_0, by_0, cz_0\}$

The equation of the tangent plane is $ax_0x + by_0y + cz_0z = 1$,

And the equation of the normal is $\frac{x-x_0}{ax_0} = \frac{y-y_0}{by_0} = \frac{z-z_0}{cz_0}$.

10. Find the maximum and minimum values of $f(x,y) = x^2 - y^2$ when $(x,y) \in \{(x,y) \mid x^2 + y^2 \leq 4\}$.

Hint: Let $f_x = 2x = 0, f_y = 2y = 0 \Rightarrow x = y = 0$

On the boundary, let $L = x^2 - y^2 + \lambda(x^2 + y^2 - 4)$,

$$L_x = 2x + 2\lambda x = 0, L_y = -2y + 2\lambda y = 0,\ x^2 + y^2 - 4 = 0$$

If $\lambda = 0$, then we get $x = 0, y = 0,$ but $x^2 + y^2 - 4 = 0$， the solution is not right.

If $\lambda \neq 0$, then we have $x = 0, y = \pm 2,$ or $x = \pm 2, y = 0,$

Since $f(0,0) = 0, f(\pm 2, 0) = 4, f(0, \pm 2) = -4$

The maximum value is 4， and the minimum value is -4.