Quiz 13

1. Evaluate the integral $\iint\limits_D |\cos(x+y)|\,\mathrm{d}\sigma$, here D is the region bounded by $y=x, y=0, x=\dfrac{\pi}{2}$.

Hint: $\iint\limits_D |\cos(x+y)|\,\mathrm{d}\sigma$

$$=\iint\limits_{D_1}\cos(x+y)\mathrm{d}\sigma+\iint\limits_{D_2}\cos(x+y)\mathrm{d}\sigma=\int_0^{\pi/4}dy\int_y^{\pi/2-y}\cos(x+y)dx-\int_{\pi/4}^{\pi/2}dx\int_{\pi/2-x}^{x}\cos(x+y)dy$$

$$=\int_0^{\pi/4}(1-\sin 2y)\,dy-\int_{\pi/4}^{\pi/2}(\sin 2x-1)\,dx=\left(y+\frac{\cos 2y}{2}\right)\Bigg|_0^{\pi/4}+\left(x+\frac{\cos 2x}{2}\right)\Bigg|_{\pi/4}^{\pi/2}$$

$$=\left(y+\frac{\cos 2y}{2}\right)\Bigg|_0^{\pi/2}=\frac{\pi}{2}-1.$$

2. Evaluate the integral $\iint\limits_D (y-x)^2\mathrm{d}\sigma$， $D=\{(x,y)|y\le R+x, x^2+y^2\le R^2, y\ge 0(R>0)\}$.

Hint: $D: 0\le y\le R, y-R\le x\le\sqrt{R^2-y^2}$,

$$\iint\limits_D (y-x)^2\mathrm{d}\sigma=\int_0^R dy\int_{y-R}^{\sqrt{R^2-y^2}}\left(x^2+y^2-2xy\right)dx$$

$$=\int_0^R dy\int_{y-R}^{0}(x-y)^2\,dx+\int_0^{\pi/2}d\theta\int_0^R\left(r^2-2r^2\sin\theta\cos\theta\right)rdr$$

$$=\int_0^R\frac{R^3-y^3}{3}dy+\frac{R^4}{4}\int_0^{\pi/2}(1-2\sin\theta\cos\theta)d\theta=\frac{\pi}{8}R^4.$$

3. If $f(x)$ is continuous on $[a,b]$, n is a positive integer, try to prove that

$$\int_a^b \mathrm{d}y\int_a^y (y-x)^{n-1}f(x)\mathrm{d}x=\frac{1}{n}\int_a^b (b-x)^n f(x)\mathrm{d}x\ .$$

Hint: LHS $=\iint\limits_D (y-x)^{n-1}f(x)\mathrm{d}x\mathrm{d}y=\int_a^b\mathrm{d}x\int_x^b (y-x)^{n-1}f(x)\mathrm{d}y$

$$=\int_a^b\frac{1}{n}(y-x)^n f(x)\Big|_x^b\,\mathrm{d}x=\frac{1}{n}\int_a^b (b-x)^n f(x)\mathrm{d}x$$ =RHS.

4. Let D be a closed bounded plane region, $f(x,y)$ and $g(x,y)$ are continuous on D, $g(x,y)$ is positive on D, try to prove that: $\exists(\xi,\eta)\in D$, s. t.

$$\iint_D f(x,y)g(x,y)d\sigma = f(\xi,\eta)\iint_D g(x,y)d\sigma .$$

Hint: Since $m \le f(x,y) \le M$, and $g(x,y)$ is positive on D, we have

$m \cdot g(x,y) \le f(x,y)g(x,y) \le M \bullet g(x,y)$.

Then

$$\iint_D m\cdot g(x,y)dx \le \iint_D f(x,y)g(x,y)dx \le \iint_D M \bullet g(x,y)dx$$

That is $m \le \dfrac{\iint_D f(x,y)g(x,y)d\sigma}{\iint_D g(x,y)d\sigma} \le M$

$\exists(\xi,\eta)\in D$, s. t. $f(\xi,\eta) = \dfrac{\iint_D f(x,y)g(x,y)d\sigma}{\iint_D g(x,y)d\sigma}$, i.e.

$$\iint_D f(x,y)g(x,y)d\sigma = f(\xi,\eta)\iint_D g(x,y)d\sigma .$$

5. Evaluate $\iint_D e^{\frac{y}{x+y}} d\sigma$ here D is the region bounded by $y+x=1, y=0, x=0.$

Hint:

$$\iint_D e^{\frac{y}{x+y}} d\sigma = \int_0^{\frac{\pi}{2}} e^{\frac{\sin\theta}{\cos\theta+\sin\theta}} d\theta \int_0^{\frac{1}{\cos\theta+\sin\theta}} r dr = \frac{1}{2}\int_0^{\frac{\pi}{2}} e^{\frac{\sin\theta}{\cos\theta+\sin\theta}} \frac{1}{(\cos\theta+\sin\theta)^2} d\theta$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{2}} e^{\frac{\sin\theta}{\cos\theta+\sin\theta}} d\left(\frac{\sin\theta}{\cos\theta+\sin\theta}\right) = \frac{1}{2}\left(e^{\frac{\sin\theta}{\cos\theta+\sin\theta}}\right)\Bigg|_0^{\frac{\pi}{2}} = \frac{1}{2}(e-1).$$

6. Evaluate $\iiint_S xz\mathrm{d}x\mathrm{d}y\mathrm{d}z$, here S is the solid bounded by $x=y, y=1, z=0$ and $z=x^2$.

Hint: S : $0\le y\le 1, 0\le x\le y$, $0\le z\le x^2$

$$\iiint_S xz\mathrm{d}x\mathrm{d}y\mathrm{d}z = \int_0^1 dy\int_0^y dx\int_0^{x^2} xzdz = \frac{1}{2}\int_0^1 dy\int_0^y x^5 dx = \frac{1}{12}\int_0^1 y^6 dy = \frac{1}{84} .$$

7. Evaluate $\iiint\limits_S z^3 \mathrm{d}v$, where S is the solid bounded by $z=\sqrt{2-x^2-y^2}$ and $z=x^2+y^2$.

Hint:

$$\iiint\limits_S z^3\mathrm{d}v=\int_0^{2\pi}\mathrm{d}\theta\int_0^1\mathrm{d}r\int_{r^2}^{\sqrt{2-r^2}} rz^3\mathrm{d}z=\int_0^{2\pi}\mathrm{d}\theta\int_0^1 r\mathrm{d}r\int_{r^2}^{\sqrt{2-r^2}} z^3\mathrm{d}z=\int_0^{2\pi}\mathrm{d}\theta\int_0^1 r\left(\frac{z^4}{4}\right)_{r^2}^{\sqrt{2-r^2}}\mathrm{d}r$$

$$=\frac{1}{4}\int_0^{2\pi}\mathrm{d}\theta\int_0^1\left(4r-4r^3+r^5-r^9\right)\mathrm{d}r=\frac{8}{15}\pi.$$

Or

$$\iiint\limits_S z^3\mathrm{d}v=\int_0^1 z^3\mathrm{d}z\iint\limits_{D_1(z)}\mathrm{d}x\mathrm{d}y+\int_0^{\sqrt{2}} z^3\mathrm{d}z\iint\limits_{D_2(z)}\mathrm{d}x\mathrm{d}y$$

$$=\int_0^1 z^3\cdot\pi z\mathrm{d}z+\int_0^{\sqrt{2}} z^3\cdot\pi(2-z^2)\mathrm{d}z=\frac{8}{15}\pi.$$

8. Evaluate $\iiint\limits_S (x^3+xy^2)\mathrm{d}v$, where S is the solid bounded by $x^2+(y-1)^2=1$, $z=0$, and $z=1$.

Hint:

$$u=x=r\cos\theta, v=y-1=r\sin\theta, \Rightarrow dv=dudvdz=rdrd\theta dz, 0\le r\le 1, 0\le\theta\le 2\pi, 0\le z\le 2$$

$$\iiint\limits_S (x^3+xy^2)\mathrm{d}v$$

$$=\iiint\limits_{\Omega_1}\left[\mathrm{u}^3+u(v+1)^2\right]\mathrm{dudvdz}=\int_0^{2\pi}d\theta\int_0^1\left[r^3\cos^3\theta+r\cos\theta\left(r\sin\theta+1\right)^2\right]rdr\int_0^2 dz$$

$$=2\int_0^{2\pi}d\theta\int_0^1\left(r^4\cos^3\theta+r^4\sin^2\theta\cos\theta+2r^3\cos\theta\sin\theta+r^2\cos\theta\right)dr$$

$$=2\int_0^{2\pi}\left(\frac{1}{5}r^5\cos^3\theta+\frac{1}{5}r^5\sin^2\theta\cos\theta+\frac{1}{2}r^4\cos\theta\sin\theta+\frac{1}{3}r^3\cos\theta\right)\Bigg|_0^1 d\theta$$

$$=\frac{2}{5}\int_0^{2\pi}\left(1-\sin^2\theta\right)d\sin\theta=\frac{4}{5}\pi.$$

9. Evaluate $\iiint\limits_S \sqrt[4]{x^2+y^2+z^2}\mathrm{d}v$, where S is the solid bounded by $x^2+y^2+z^2=z$.

Hint:

$$\iiint_S \sqrt[4]{x^2+y^2+z^2}\mathrm{d}v = \iiint_S \sqrt[4]{\rho^2}\cdot\rho^2\sin\varphi\mathrm{d}\rho\mathrm{d}\varphi\mathrm{d}\theta = \int_0^{2\pi}\mathrm{d}\theta\int_0^{\frac{\pi}{2}}\sin\varphi\mathrm{d}\varphi\int_0^{\cos\varphi}\rho^{\frac{5}{2}}\mathrm{d}\rho$$

$$=\frac{2}{7}2\pi\int_0^{\frac{\pi}{2}}\sin\varphi\cos^{\frac{7}{2}}\varphi\mathrm{d}\varphi = \frac{8}{63}\pi.$$

10. Evaluate $\iiint_S \mathrm{e}^{\sqrt{\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}}}\mathrm{d}v$, here S is the solid bounded by $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$.

Hint: Let $x = a\rho\cos\theta\sin\varphi, y = b\rho\sin\theta\sin\varphi, z = c\rho\cos\varphi \Rightarrow$

$$dv = abc\rho^2\sin\varphi d\rho d\theta d\varphi, 0\le\rho\le 1, 0\le\theta\le 2\pi, 0\le\varphi\le\pi$$

$$\iiint_S \mathrm{e}^{\sqrt{\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}}}\mathrm{d}v$$

$$=\int_0^{2\pi}d\theta\int_0^{\pi}d\varphi\int_0^{1}e^{\rho}\cdot abc\rho^2\sin\varphi d\rho = 2\pi abc\left(-\cos\varphi\right)\Big|_0^{\pi}\int_0^1\rho^2 de^{\rho} = 4\left(e-2\right)\pi abc$$

11. The radius of the sphere C is R. And the center of the sphere C is on the sphere K: $x^2+y^2+z^2\le a^2$.

Please find the maximum value of the area of the sphere C inside the sphere K.

Hint:

$\begin{cases} x^2+y^2+z^2=a^2, \\ x^2+y^2+(z-a)^2=R^2, \end{cases}$ implies $x^2+y^2=\frac{R^2}{4a^2}\left(4a^2-R^2\right)$, the projection of the cross section on xy plane is

$$D_{xy} := \left\{(x,y,z)\,|\,x^2+y^2=\frac{R^2}{4a^2}\left(4a^2-R^2\right), z=0\right\}$$

The equation of the sphere inside K is $z = a-\sqrt{R^2-x^2-y^2}$,then the area of this part of sphere is

$$S(R) = \iint_{D_{xy}}\sqrt{1+z_x^2+z_y^2}\mathrm{d}x\mathrm{d}y = \iint_{D_{xy}}\frac{R}{\sqrt{R^2-x^2-y^2}}\mathrm{d}x\mathrm{d}y$$

$$=\int_0^{2\pi}\mathrm{d}\theta\int_0^{\frac{R}{2a}\sqrt{4a^2-R^2}}\frac{rR\mathrm{d}r}{\sqrt{R^2-r^2}} = 2\pi R^2-\frac{\pi R^3}{a}(0<R<2a)$$

And $S'(R) = 4\pi R-\frac{3\pi R^2}{a}, S''(R) = 4\pi-\frac{6\pi R}{a}$.

Let $S'(R)=0 \Rightarrow R=\frac{4}{3}a$, and $S''(\frac{4}{3}a) = -4\pi < 0$. When $R=\frac{4}{3}a$, we are done.