Quiz 15

1. Find the general solution of $xy' = y\ln y$. 直接分离变量

Hint: $x\dfrac{dy}{dx} = y\ln y$

Then $\dfrac{dy}{y\ln y} = \dfrac{dx}{x}$

$$\int \frac{dy}{y\ln y} = \int \frac{d\ln y}{\ln y} = \int \frac{dx}{x} \Rightarrow \ln\ln y = \ln x + \ln c \Rightarrow \ln y = cx, y = e^{cx}.$$

2. Solve the equation $(y+\sqrt{x^2+y^2})\mathrm{d}x - x\mathrm{d}y = 0 (x>0), y(1)=0$.

Hint: Let $u = \dfrac{y}{x}$, then $y = ux, y' = u + xu'$

The equation is $\dfrac{dy}{dx} = \dfrac{y+\sqrt{x^2+y^2}}{x}$, i.e. $y' = \dfrac{y}{x} + \sqrt{1+\left(\dfrac{y}{x}\right)^2} = u + \sqrt{1+u^2} = u + xu'$

$\sqrt{1+u^2} = x\dfrac{du}{dx}$, $\dfrac{du}{\sqrt{1+u^2}} = \dfrac{dx}{x}$, then $\displaystyle\int \frac{du}{\sqrt{1+u^2}} = \int \frac{dx}{x}$

$\ln\left(u+\sqrt{1+u^2}\right) = \ln x + \ln c, y + \sqrt{x^2+y^2} = cx^2$

Since $y(1)=0$, $0+\sqrt{1+0} = c, \Rightarrow c = 1, y + \sqrt{x^2+y^2} = x^2$.

3. Find the general solution to $y' = \dfrac{y-x+1}{y+x+5}$.

Hint: Let $x = X+a, y = Y+b$, then $y' = \dfrac{dY}{dX} = \dfrac{Y-X+b-a+1}{Y+X+b+a+5}$

And let $b-a+1=0, b+a+5=0 \Rightarrow b=-3, a=-2$, $x = X-2, y = Y-3$

Let $Y = uX, \Rightarrow Xu' + u = \dfrac{u-1}{u+1}, Xu' = \dfrac{u-1}{u+1} - u = \dfrac{-1-u^2}{u+1}$

so $\displaystyle\int \frac{(u+1)du}{1+u^2} = \int \left(\frac{u}{1+u^2} + \frac{1}{1+u^2}\right)du = -\int \frac{dX}{X},$

$\dfrac{1}{2}\ln\left(1+u^2\right) + \arctan u = -\ln X - \dfrac{1}{2}\ln c, e^{-2\arctan u} = cX^2\left(1+u^2\right)$

$e^{-2\arctan\frac{y+3}{x+2}} = c\left[(x+2)^2 + (y+3)^2\right].$

4. Find the general solution of the following equations

(1) $(x+2y)^2 y' = 1$;

Hint：Let $u=x+2y$, then $u'=1+2y'=1+\frac{2}{u^2}=\frac{u^2+2}{u^2}$, i.e. $\frac{u^2}{u^2+2}du=dx$,

$$\Rightarrow \int \frac{u^2+2-2}{u^2+2}du=\int dx \Rightarrow u-\sqrt{2}\arctan\frac{u}{\sqrt{2}}=x+c$$

$$x+2y-\sqrt{2}\arctan\frac{x+2y}{\sqrt{2}}=x+c, 2y-c=\sqrt{2}\arctan\frac{x+2y}{\sqrt{2}}.$$

(2) $x\frac{\mathrm{d}y}{\mathrm{d}x}+y=x^2+3x+2$;

Hint： Since the equation can be rewritten as $\frac{\mathrm{d}(xy)}{\mathrm{d}x}=x^2+3x+2,$

$$xy=\int(x^2+3x+2)dx=\frac{1}{3}x^3+\frac{3}{2}x^2+2x+c, y=\frac{1}{3}x^2+\frac{3}{2}x+2+\frac{c}{x}$$

(3) $\frac{\mathrm{d}y}{\mathrm{d}x}-3xy=xy^2$. 伯努利方程

Hint： $\frac{1}{y^2}\frac{\mathrm{d}y}{\mathrm{d}x}-3\frac{x}{y}=x, -\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{y}\right)-3\frac{x}{y}=x, \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{y}\right)+3\frac{x}{y}=-x,$

Let

$$u=\frac{1}{y}, \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x}+3xu=-x,$$

Consider the standard equation, and

$$P(x)=3x, Q(x)=-x,$$

$$u=\frac{1}{y}=e^{-\frac{3}{2}x^2}\left[-\int\frac{1}{3}e^{\frac{3}{2}x^2}d\left(\frac{3}{2}x^2\right)x+C\right]=e^{-\frac{3}{2}x^2}\left[-\frac{1}{3}e^{\frac{3}{2}x^2}+C\right]=Ce^{-\frac{3}{2}x^2}-\frac{1}{3}.$$

(4) $\mathrm{e}^y\mathrm{d}x+(x\mathrm{e}^y-2y)\mathrm{d}y=0$. 全微分方程（又名恰当方程）

Hint： $\frac{\partial \mathrm{e}^y}{\partial y}=e^y=\frac{\partial(x\mathrm{e}^y-2y)}{\partial x}$,

then $u=\int_{(0,0)}^{(x,y)}\mathrm{e}^y\mathrm{d}x+(x\mathrm{e}^y-2y)\mathrm{d}y=\int_0^x e^0dx+\int_0^y(x\mathrm{e}^y-2y)\mathrm{d}y=x\mathrm{e}^y-y^2$

The general solution of the equation is $xe^y-y^2=c$.

(5) $y''(\mathrm{e}^x+1)+y'=0$. 可降解的二阶方程之一

Hint: $p = y', \Rightarrow y'' = \frac{dp}{dx}, (e^x+1)\frac{dp}{dx} + p = 0$

And $\frac{dp}{p} = -\frac{dx}{e^x+1}$,

then $\int \frac{dp}{p} = -\int \frac{e^x+1-e^x}{e^x+1}dx = -x + \ln\left(e^x+1\right) + \ln c_1$, $p = c_1\frac{e^x+1}{e^x} = \frac{dy}{dx}$

$dy = c_1\frac{e^x+1}{e^x}dx \Rightarrow y = c_1\int\frac{e^x+1}{e^x}dx = c_1\int\left(1+e^{-x}\right)dx$

So we get $y = c_2 + c_1\left(x - e^{-x}\right)$

(6) $y'' = (y')^3 + y'$. 可降阶的二阶方程之二

Hint: $p = y', \Rightarrow y'' = \frac{dp}{dx} = \frac{dp}{dy}\cdot\frac{dy}{dx} = p\frac{dp}{dy}, p\frac{dp}{dy} = p^3 + p$

Then $\frac{dp}{p^2+1} = dy$,

$y = \int\frac{1}{p^2+1}dp = \arctan p - c_1, p = \tan\left(y + c_1\right) = \frac{dy}{dx}$,

$\cot\left(y+c_1\right)dy = dx$,

$x + c_2 = \int\cot\left(y+c_1\right)dy = \ln\left|\sin\left(y+c_1\right)\right|$, $\sin\left(y+c_1\right) = \pm e^{x+c_2}$

(7) $4y'' - 12y' + 9y = 0$； 二阶常系数齐次线性微分方程

Hint: The auxiliary equation is

$$4r^2 - 12r + 9 = 0, r_{1,2} = \frac{-(-12)\pm\sqrt{(-12)^2 - 4\cdot4\cdot9}}{2\cdot4} = \frac{3\pm\sqrt{9-9}}{2} = \frac{3}{2}$$

The general solution is $y = \left(c_1 + c_2x\right)e^{\frac{3}{2}x}$

(8) $y'' + 6y' + 13y = 0$； 二阶常系数齐次线性微分方程

Hint: The auxiliary equation is

$$r^2 + 6r + 13 = 0, r_{1,2} = \frac{-6\pm\sqrt{6^2 - 4\cdot1\cdot13}}{2} = -3\pm2i, \alpha = -3, \beta = 2$$

The general solution is $y = \left(c_1\cos 2x + c_2\sin 2x\right)e^{-3x}$.

(9) $2y''+y'-y=2\mathrm{e}^x$；二阶常系数非齐次线性微分方程之一

Hint：The auxiliary equation is

$$2r^2+r-1=0, r_{1,2}=\frac{-1\pm\sqrt{1^2-4\cdot1\cdot(-1)}}{2\cdot2}=\frac{-1\pm3}{4}, r_1=\frac{1}{2}, r_2=-1$$

The free term is $f(x)=2e^x$,comparing with $f(x)=P_n(x)e^{\lambda x}$ we have $n=0,\lambda=1$

Which implies $k=0$， we assume $y^*=x^kQ_n(x)e^{\lambda x}=ae^x, y^{*\prime}=ae^x, y^{*\prime\prime}=ae^x$

And we get $2a+a-a=2\Rightarrow a=1$

The general solution is $y=c_1e^{\frac{x}{2}}+c_2e^{-x}+e^x$

(10) $y''+4y=x\cos x$； 二阶常系数非齐次线性微分方程之二（此类不考）

Hint：The auxiliary equation is

$r^2+4=0, r_{1,2}=\pm 2i$

Since $f(x)=x\cos x$, consider $f(x)=e^{\alpha x}\left[P_m(x)\cos\beta x+P_l(x)\sin\beta x\right]$,

we have $n=\max\{m,l\}=1, \lambda=i$, $k=0$， we assume

$$y^*=x^k\left[{}_1Q_n(x)\cos\beta x+{}_2Q_n(x)\sin\beta x\right]e^{\alpha x}=(ax+b)\cos x+(cx+d)\sin x,$$

$$y^{*\prime}=(-ax-b+c)\sin x+(cx+d+a)\cos x, y^{*\prime\prime}=(-ax-b+2c)\cos x-(cx+d+2a)\sin x$$

substituting into the equation, we have

$$3ax+3b+2c=x, 3cx+3d-2a=0\Rightarrow a=\frac{1}{3}, c=0, b=0, d=\frac{2}{9}$$

$$y=c_1\cos 2x++c_2\sin 2x+\frac{1}{3}\cos x+\frac{2}{9}\sin x$$.

5. Let $f(x)=\sin x-\int_0^x(x-t)f(t)dt$, $f(x)$ is continuous, please find $f(x)$.

Hint: $f(x)=\sin x-x\int_0^x f(t)dt+\int_0^x tf(t)dt\Rightarrow f(0)=0$

$$f'(x)=\cos x-\int_0^x f(t)dt\Rightarrow f'(0)=\cos 0=1, \quad f''(x)+f(x)=-\sin x$$

The auxiliary equation is $r^2+1=0, r_{1,2}=\pm i$

Since $f(x)=-\sin x$, consider $f(x)=e^{\alpha x}\left[P_m(x)\cos\beta x+P_l(x)\sin\beta x\right]$（此类不考）

Then $n=\max\{m,l\}=0,\lambda=i$, $k=1$， let

$y^*=x^k\left[{}_1Q_n(x)\cos\beta x+{}_2Q_n(x)\sin\beta x\right]e^{\alpha x}=ax\cos x+bx\sin x,$

$y^{*\prime}=(a+bx)\cos x+(b-ax)\sin x, y^{*\prime\prime}=(2b-ax)\cos x+(-2a-bx)\sin x$

Substituting into the original equation, we have $2b\cos x-2a\sin x=-\sin x\Rightarrow b=0,a=\dfrac{1}{2}$

$f(x)=c_1\sin x+c_2\cos x+\dfrac{x}{2}\cos x$ ， and $f(0)=0\Rightarrow c_2=0$

$f'(x)=c_1\cos x-c_2\sin x-\dfrac{x}{2}\sin x+\dfrac{1}{2}\cos x$ ， and $f'(0)=1\Rightarrow c_1=\dfrac{1}{2}$

Then $f(x)=\dfrac{1}{2}\sin x+\dfrac{x}{2}\cos x$

6. If $z=f(\mathrm{e}^x\sin y)$ is a solution to the equation $\dfrac{\partial^2 z}{\partial x^2}+\dfrac{\partial^2 z}{\partial y^2}=z\mathrm{e}^{2x}$, please find the solution $z=f(\mathrm{e}^x\sin y)$.

Hint: Since $\dfrac{\partial z}{\partial x}=f'(u)e^x\sin y,\dfrac{\partial^2 z}{\partial x^2}=f''(u)\left(e^x\sin y\right)^2+f'(u)e^x\sin y$

$\dfrac{\partial z}{\partial y}=f'(u)e^x\cos y,\dfrac{\partial^2 z}{\partial y^2}=f''(u)\left(e^x\cos y\right)^2+f'(u)e^x(-\sin y)$

$\dfrac{\partial^2 z}{\partial x^2}+\dfrac{\partial^2 z}{\partial y^2}=f''(u)e^{2x}=f(u)\mathrm{e}^{2x},\Rightarrow f''(u)-f(u)=0$

The auxiliary equation is $r^2-1=0,r_1=1,r_2=-1,f(u)=c_1e^u+c_2e^{-u}$

Then the solution is $z=f(\mathrm{e}^x\sin y)=c_1e^{\mathrm{e}^x\sin y}+c_2e^{-\mathrm{e}^x\sin y}$

7. Let $f(x)\in C^1(-\infty,+\infty)$, and satisfies

$f(t)=2\iint\limits_{x^2+y^2\le t^2}(x^2+y^2)f(\sqrt{x^2+y^2})\mathrm{d}x\mathrm{d}y+t^4$. Please find $f(x)$.

Hint: $f(t)=2\int_0^{2\pi}d\theta\int_0^t r^2f(r)rdr+t^4=4\pi\int_0^t r^3f(r)dr+t^4$

Then $f'(t)=4\pi t^3 f\left(t\right)+4t^3, f\left(0\right)=0$, i.e. $f'(t)-4\pi t^3 f\left(t\right)=4t^3$,

$$f\left(t\right)=e^{-\int\left(-4\pi t^3\right)}\left(\int 4t^3 e^{\int\left(-4\pi t^3\right)}dt+c\right)=e^{\pi t^4}\left(-\frac{1}{\pi}e^{-\pi t^4}+c\right)=-\frac{1}{\pi}+ce^{\pi t^4}$$

And $f\left(0\right)=0$, we have $c=\frac{1}{\pi}, f\left(t\right)=\frac{1}{\pi}e^{\pi t^4}-\frac{1}{\pi}, f\left(x\right)=\frac{1}{\pi}e^{\pi x^4}-\frac{1}{\pi}$

8. If $\varphi(x)\in C(-\infty,+\infty)$, $\varphi'(0)$ exists, and $\varphi(x+y)=\varphi(x)\varphi(y)$, please try to find $\varphi(x)$.

Hint: $\varphi(x+0)=\varphi(x)\varphi(0),\Rightarrow \varphi\left(0\right)=1$

$$\varphi'(x)=\lim_{y\to 0}\frac{\varphi(x+y)-\varphi(x)}{y}=\lim_{y\to 0}\frac{\varphi(x)\varphi(y)-\varphi(x)}{y}=\varphi(x)\lim_{y\to 0}\frac{\varphi(y)-\varphi(0)}{y}$$

Then $\varphi'(x)=\varphi(x)\varphi'(0)$, $\int\frac{d\varphi(x)}{\varphi(x)}=\int\varphi'(0)dx=\varphi'(0)x+\ln c$

That is $\varphi(x)=ce^{\varphi'(0)x}$, and $\varphi\left(0\right)=1$, we have $c=1,\varphi(x)=e^{\varphi'(0)x}$.