Seat No.______ Major/Class______ School______ Student ID______ Name______

(DON NOT WRITE YOUR ANSWER IN THIS AREA)

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WARNING: MISBEHAVIOR AT EXAM TIME WILL LEAD TO SERIOUS CONSEQUENCE.

SCUT Midterm Exam

***Probability and Statistics* 2022-2023-(2) Exam Paper A**

Notice: **1. Make sure that you have filled the form on the left side of the seal line.**
2. Write your answers on the exam paper.
3. This is a close-book exam.
4. The exam with full score of 100 points lasts 120 minutes.

Question No.	I	II	III	IV	V	VI	VII	Sum
Score								

I. (5 questions, 8 points each, 40 points in total)

1. Consider n-digit numbers where each digit is one of the 10 integers $0, 1, \ldots, 9$. How many such numbers are there for which
   (a) no two consecutive digits are equal?
   (b) 0 appears as a digit a total of i times, $i = 0, \ldots, n$?

2. Suppose that n balanced dice are rolled. Determine the probability that the number j will appear exactly n_j times $(j = 1, \ldots, 6)$, where $n_1 + n_2 + \cdots + n_6 = n$.

3. An elevator in a building starts with five passengers and stops at seven floors. If every passenger is equally likely to get off at each floor and all the passengers leave independently of each other, what is the probability that no two passengers will get off at the same floor?

4. Among a group of 200 students, 137 students are enrolled in a mathematics class, 50 students are enrolled in a history class, and 124 students are enrolled in a music class. Furthermore, the number of students enrolled in both the mathematics and history classes is 33, the number enrolled in both the history and music classes is 29, and the number enrolled in both the mathematics and music classes is 92. Finally, the number of students enrolled in all three classes is 18. We shall determine the probability that a student selected at random from the group of 200 students will be enrolled in at least one of the three classes.

5. Suppose that a person types n letters, types the addresses on n envelopes, and then places each letter in an envelope in a random manner. Let X be the number of letters that are placed in the correct envelopes. We shall find the mean of X.

II. (10 points) An urn contains 12 balls, of which 4 are white. Three players-A, B, and C-successively draw from the urn, A first, then B, then C, then A, and so on. The winner is the first one to draw a white ball. Find the probability of winning for each player if

1. each ball is replaced after it is drawn;
2. the balls that are withdrawn are not replaced.

III. (10 points) Suppose that a box contains five coins and that for each coin there is a different probability that a head will be obtained when the coin is tossed. Let p_i denote the probability of a head when the ith coin is tossed $(i = 1, \ldots, 5)$, and suppose that $p_1 = 0$, $p_2 = 1/4$, $p_3 = 1/2$, $p_4 = 3/4$ and $p_5 = 1$.

1. Suppose that one coin is selected at random from the box and when it is tossed once, a head is obtained. What is the probability that the ith coin was selected $(i = 1, \ldots, 5)$?
2. Suppose that one coin is selected at random from the box and is tossed repeatedly until a head is obtained. If the first head is obtained on the fourth toss, what is the probability that the ith coin was selected $(i = 1, \ldots, 5)$?

VI. (10 points)

1. Suppose that Z is a standard normal random variable. Let $Y = aZ + b$ where $a > 0$ and b are constants. Show that $Y \sim N(b, a^2)$.
2. Let $Y \sim N(\mu, \sigma^2)$. Show that $\frac{Y-\mu}{\sigma}$ has a standard normal distribution.

V. (10 points) The random variable X is said to have *Yule-Simons distribution* if

$$P\{X = n\} = \frac{c}{n(n+1)(n+2)}, \quad n \geq 1.$$

1. Find c.
2. Find $E(X)$ and $E(X^2)$.

VI. (10 points) Suppose that the radius X of a circle is a random variable having the following p.d.f.:

$$f(x) = \begin{cases} \frac{1}{8}(3x+1), & \text{for } 0 < x < 2; \\ 0, & \text{otherwise.} \end{cases}$$

1. Determine the p.d.f. of the area of the circle.
2. Construct a random variable $Y = g(X)$ (in function of X) such that Y has the uniform distribution on the interval $(0, 1)$.

VII. (10 points) Let Z be the rate at which customers are served in a queue. Assume that Z has the p.d.f.

$$f(z) = \begin{cases} 2e^{-2z}, & \text{for } z \geq 0; \\ 0, & \text{otherwise.} \end{cases}$$

1. Find the p.d.f. of the average waiting time $T = 1/Z$.

2. Let F be the c.d.f. of T. Find the p.d.f. of $F(T)$.