Quiz 9 Infinite Series

1.Determine convergence or divergence for each of the series.

(1) $\sum_{n=1}^{\infty} \frac{1}{\ln(n+1)}$;

Hint: $\lim_{n\to\infty} \frac{1}{\ln(n+1)} \cdot n = \lim_{x\to+\infty} \frac{x}{\ln(x+1)} = \lim_{x\to+\infty} \frac{1}{1/(x+1)} = +\infty$

(2) $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n+1}} \ln \frac{n+2}{n}$;

Hint: $\lim_{n\to\infty} \frac{1}{\sqrt{n+1}} \ln \frac{n+2}{n} \cdot n^{\frac{3}{2}} = \lim_{n\to\infty} \frac{1}{\sqrt{n+1}} \ln\left(1+\frac{2}{n}\right) \cdot n^{\frac{3}{2}} = \lim_{n\to\infty} \frac{2}{\sqrt{1+\frac{1}{n}}} = 2$

(3) $\sum_{n=1}^{\infty} \frac{n^4+1}{n!}$;

Hint: $\rho = \lim_{n\to\infty} \frac{(n+1)^4+1}{(n+1)!} \cdot \frac{n!}{n^4+1} = \lim_{n\to\infty} \frac{1}{n+1} \cdot \frac{\left(1+n^{-4}\right)^4 + n^{-4}}{1+n^{-4}} = 0 < 1$

(4) $\sum_{n=1}^{\infty} \frac{1}{na+b}, (a,b>0)$;

Hint: $\lim_{n\to\infty} \frac{1}{na+b} \cdot n = \lim_{n\to\infty} \frac{1}{a+bn^{-1}} = \frac{1}{a}$

(5) $\sum_{n=1}^{\infty} n\left(\sqrt[n]{3}-1\right)^n$;

Hint: Let $y = x\left(3^{\frac{1}{x}}-1\right)^x$, $\ln y = x\left[\frac{\ln x}{x} + \ln\left(3^{\frac{1}{x}}-1\right)\right]$, and

$\lim_{x\to+\infty} \ln y = \lim_{x\to+\infty} x\left[\frac{\ln x}{x} + \ln\left(3^{\frac{1}{x}}-1\right)\right] = -\infty, \lim_{x\to+\infty} y = \lim_{n\to\infty} n\left(\sqrt[n]{3}-1\right)^n = 0$.

Then let $y = x^3\left(3^{\frac{1}{x}}-1\right)^x$, $\ln y = x\left[\frac{3\ln x}{x} + \ln\left(3^{\frac{1}{x}}-1\right)\right]$,

$\lim_{x\to+\infty} x^3\left(3^{\frac{1}{x}}-1\right)^x = \lim_{n\to\infty} n\left(\sqrt[n]{3}-1\right)^n \cdot n^2 = 0$,

$\sum_{n=1}^{\infty} \frac{1}{n^2}, p = 2 > 1$ converges, the original series converges.

(6) $\sum_{n=1}^{\infty}\left(\frac{b}{a_n}\right)^n$, here a_n, a, b are positive real numbers, and $\lim_{n\to\infty} a_n = a$

Hint: $\rho = \lim_{n\to\infty}\sqrt[n]{u_n} = \lim_{n\to\infty}\sqrt[n]{\left(\frac{b}{a_n}\right)^n} = \lim_{n\to\infty}\frac{b}{a_n} = \frac{b}{a}$.

(7) $\sum_{n=1}^{\infty}\int_0^{\frac{1}{n}}\frac{\sqrt{x}}{x^2+1}\mathrm{d}x$;

Hint: By Mean Value Theorem for integration $\int_0^{\frac{1}{n}}\frac{\sqrt{x}}{x^2+1}\mathrm{d}x = \frac{\sqrt{\xi}}{\xi^2+1}\cdot\frac{1}{n}, 0\le\xi\le\frac{1}{n}$

Then we have $0\le\int_0^{\frac{1}{n}}\frac{\sqrt{x}}{x^2+1}\mathrm{d}x \le \frac{\sqrt{\xi}}{n} \le \frac{1}{n\sqrt{n}}$.

(8) $\sum_{n=1}^{\infty}\frac{(-1)^n}{n-\ln n}$;

Hint:Let $f(x) = \frac{1}{x-\ln x}$, $f'(x) = -\frac{1-\frac{1}{x}}{(x-\ln x)^2} < 0, x > 1$, i.e. f is decreasing.

And $\lim_{n\to\infty}\frac{1}{n-\ln n} = 0$. The series converges.

But $\lim_{n\to\infty}\frac{1}{n-\ln n}\cdot n = 1$, the series is conditionally convergent.

(9) $\sum_{n=1}^{\infty}\sin \pi\sqrt{R^2+n^2}$.

Hint: $u_n = \sin\pi\sqrt{R^2+n^2} = (-1)^{n-1}\sin\pi\left(n-\sqrt{R^2+n^2}\right) = (-1)^n\sin\frac{\pi R^2}{n+\sqrt{R^2+n^2}}$,

When n is large enough, the series is an alternating series. And $|u_n| = \sin\frac{\pi R^2}{n+\sqrt{R^2+n^2}}$ is decreasing, $\lim_{n\to\infty}|u_n| = 0$, $\sum_{n=1}^{\infty}\sin\pi\sqrt{R^2+n^2}$ converges.

But $\lim_{n\to\infty}\sin\frac{\pi R^2}{n+\sqrt{R^2+n^2}}\cdot n = \lim_{n\to\infty}\frac{\pi R^2}{1+\sqrt{R^2n^{-2}+1}} = \frac{\pi R^2}{2}$, the series is conditionally convergent.

2. Determine convergence or divergence for the infinite series $\sum_{n=1}^{\infty}\frac{1}{(2n-1)2^n}$, if it converges

please find its sum.

Hint: $R=\sqrt{\lim\limits_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right|}=\sqrt{\lim\limits_{n\to\infty}\left|\frac{1}{2n-1}\cdot\frac{2n+1}{1}\right|}=1$, and $\pm\sum\limits_{n=1}^{\infty}\frac{1}{2n-1}$ diverge, the convergent set is $(-1,1)$.

Let $s(x)=\sum\limits_{n=1}^{\infty}\frac{x^{2n-1}}{2n-1}$, $s(0)=0, s'(x)=\sum\limits_{n=1}^{\infty}x^{2n-2}=\frac{1}{1-x^2}$,

$$s(x)=\int_0^x s'(t)dt=\int_0^x\frac{1}{1-t^2}dt=\frac{1}{2}\ln\frac{1+x}{1-x}, x\in(-1,1)$$

$$\frac{1}{\sqrt{2}}\in(-1,1), \sum_{n=1}^{\infty}\frac{1}{(2n-1)2^n}=\sqrt{2}\cdot s\left(\frac{1}{\sqrt{2}}\right)=\frac{\sqrt{2}}{2}\ln\frac{\sqrt{2}+1}{\sqrt{2}-1}=\frac{\sqrt{2}}{2}\ln\left(3+2\sqrt{2}\right).$$

3. Represent $\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{e}^x-1}{x}\right)$ in power series in x, and try to prove that $\sum\limits_{n=1}^{\infty}\frac{n}{(n+1)!}=1$.

Hint: $\frac{e^x-1}{x}=\sum\limits_{n=1}^{\infty}\frac{1}{n!}x^{n-1}, x\in(-\infty,+\infty)$

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{e}^x-1}{x}\right)=\left(\sum_{n=1}^{\infty}\frac{1}{n!}x^{n-1}\right)'=\sum_{n=2}^{\infty}\frac{n-1}{n!}x^{n-2}=\sum_{n=1}^{\infty}\frac{n}{(n+1)!}x^{n-1}, x\in(-\infty,+\infty)$$

$$\sum_{n=1}^{\infty}\frac{n}{(n+1)!}=\sum_{n=1}^{\infty}\frac{n}{(n+1)!}x^{n-1}\Bigg|_{x=1}=\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{e}^x-1}{x}\right)\Bigg|_{x=1}=\frac{x\mathrm{e}^x-\left(\mathrm{e}^x-1\right)}{x^2}\Bigg|_{x=1}=1.$$

4. Represent $f(x)=|\sin x|, \quad (-\pi<x\le\pi)$ in Fourier series.

Hint: $a_0=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)dx=\frac{2}{\pi}\int_0^{\pi}\sin xdx=-\frac{2\cos x}{\pi}\Big|_0^{\pi}=\frac{4}{\pi}$

$$a_n=\frac{2}{\pi}\int_0^{\pi}\sin x\cos nxdx=\frac{\left[\frac{-\cos(1+n)x}{1+n}+\frac{-\cos(1-n)x}{1-n}\right]}{\pi}\Bigg|_0^{\pi}=\frac{2-2(-1)^{n+1}}{\pi\left(1-n^2\right)}$$

$$b_n=\frac{1}{\pi}\int_{-\pi}^{\pi}|\sin x|\sin nxdx=0$$

$$f(x)=\frac{2}{\pi}+\sum_{n=1}^{\infty}\frac{2+(-1)^n}{1-n^2}\cos nx, x\in(-\infty,+\infty).$$

5. If the second derivative of $f(x)$ is well-defined on $[0,1]$, c is a point in $(0,1)$.

$\left|f(x)\right| \le a, \left|f''(x)\right| \le b$, here a, b are non-negative constant. Try to prove that $\left|f'(c)\right| \le 2a + \frac{b}{2}$.

Hint $f(x) = f(c) + f'(c)(x-c) + \frac{1}{2}f''(\xi)(x-c)^2$, and

$$f(0) = f(c) + f'(c)(-c) + \frac{1}{2}f''(\xi_1)(-c)^2, \xi_1 \in (0,c)$$

$$f(1) = f(c) + f'(c)(1-c) + \frac{1}{2}f''(\xi_2)(1-c)^2, \xi_2 \in (c,1)$$

So

$$f(1) - f(0) = f'(c) + \frac{1}{2}f''(\xi_2)(1-c)^2 - \frac{1}{2}f''(\xi_1)c^2$$

$$\left|f'(c)\right| \le 2a + \frac{b}{2}\left[1 - 2c(1-c)\right] \le 2a + \frac{b}{2}.$$