Name________ Student ID ________ School________ Major/Class________ Seat No.________

(DON NOT WRITE YOUR ANSWER IN THIS AREA)

Seal line

WARNING: MISBEHAVIOR AT EXAM TIME WILL LEAD TO SERIOUS CONSEQUENCE.

SCUT Final Exam

***Probability and Statistics* Exam Paper B (2018-2019-2)**

Notice: **1. Make sure that you have filled the form on the left side of the seal line.**
2. Write your answers on the exam paper.
3. This is a close-book exam.
4. The exam with full score of 100 points lasts 120 minutes.

Question No.	I	II	III	IV	V	VI	VII	VIII	Sum
Score									

Score

I. Multiple choice. Choose the one alternative that best completes the statement or answers the question. (5 questions, 3 points per question, 15 points in total)

1. Suppose A and B are random event with $P(A\cap B)>0$. Then $P(A|A\cap B)=$ (D)

   (A) $P(A)$ (B) $P(A\cap B)$ (C) $P(A\cup B)$ (D) 1

2. Let X be a random variables taking value on $(-1,1)$. Which of the following could be the density function of f? (A)

   (A) $f(x)=\begin{cases}\frac{1}{2}, & -1<x<1,\\ 0, & \text{otherwise}\end{cases}$ (B) $f(x)=\begin{cases}2, & -1<x<1,\\ 0, & \text{otherwise}\end{cases}$

   (C) $f(x)=\begin{cases}x, & -1<x<1,\\ 0, & \text{otherwise}\end{cases}$ (D) $f(x)=\begin{cases}x^2, & -1<x<1,\\ 0, & \text{otherwise}\end{cases}$

3. Let $X\sim N(-1,2)$ and $Y\sim N(1,3)$ be independent. Then $X+2Y\sim$ (B)

   (A) $N(1,8)$ (B) $N(1,14)$ (C) $N(1,22)$ (D) $N(1,40)$

4. Suppose X and Y are independent random variables and $E(X),E(Y)$ exist. If $U=\max\{X,Y\}$ and $V=\min\{X,Y\}$, then $E(UV)=$ (B)

   (A) $E(U)E(V)$ (B) $E(X)E(Y)$ (C) $E(U)E(Y)$ (D) $E(X)E(V)$

5. Let $X_1,\cdots,X_n$ be a sample coming from the population X with $E(X)=0$ and $\mathrm{Var}(X)=\sigma^2$. Write the sample mean and the sample variance by $\bar{X}$ and S^2 respectively. Then an unbiased estimator of σ^2 is (C)

   (A) $n\bar{X}^2+S^2$ (B) $\frac{1}{3}n\bar{X}^2+S^2$ (C) $\frac{1}{2}n\bar{X}^2+\frac{1}{2}S^2$ (D) $\frac{1}{4}n\bar{X}^2+\frac{1}{4}S^2$

Score

II. Fill-in-the-blanks. (5 questions, 3 points per question, 15 points in total)

1. Suppose A and B are mutually exclusive random events with $P(A) = 0.2$ and $P(A \cup B) = 0.5$. Then $P(B) =$ ___0.3___.

2. Let $X \sim \mathrm{Bin}(2, p)$ and $Y \sim \mathrm{Bin}(3, p)$ be two binomial random variables. If $P(X \geq 1) = \frac{5}{9}$, then $P(Y \geq 1)$= ___$\frac{19}{27}$___.

3. Let X be a Poisson random variables with parameter λ and $E[(X-1)(X-2)] = 1$. Then $\lambda =$ ___1___.

4. Let X be a random variable with density function

$$f(x) = \begin{cases} ce^{-\frac{x}{5}}, & x \geq 0, \\ 0, & x < 0, \end{cases}$$

then $c =$ ___$\frac{1}{5}$___.

5. Suppose the population is normal distributed with mean μ and standard deviation 1. A sample of size 16 is chosen from that population. If the sample mean $\bar{x} = 40$, then a 95% confidence interval for μ is ___$(39.51, 40.49)$___. ($z_{.025} = 1.96, z_{.05} = 1.64$)

Score

III. (12 points) Choose a number X from $1, 2, 3$ with equal probability; then we choose a number Y from $1, 2, \cdots, X$ with equal probability.

(a) Find the probability mass function of (X, Y).

(b) Find the marginal probability mass function of X and Y.

Solution.

The p.m.f. of (X, Y) and their marginal p.m.f. are described in the following table

X \ Y	1	2	3	f_X
1	$\frac{1}{3}$	0	0	$\frac{1}{3}$
2	$\frac{1}{6}$	$\frac{1}{6}$	0	$\frac{1}{3}$
3	$\frac{1}{9}$	$\frac{1}{9}$	$\frac{1}{9}$	$\frac{1}{3}$
f_Y	$\frac{11}{18}$	$\frac{5}{18}$	$\frac{1}{9}$	

Score

IV. (12 points) Let $X_1, \cdots, X_4$ be independent Bernoulli random variables with $P(X_i = 0) = 0.6$ and $P(X_i = 1) = 0.4$ $(i = 1,2,3,4)$. Find the probability mass function of

$$X = \begin{vmatrix} X_1 & X_2 \\ X_3 & X_4 \end{vmatrix}.$$

Solution.

The possible values of X are $-1, 0, 1$. Then

$$\begin{aligned}
P(X=-1) &= P(X_1X_4 = 0, X_2X_3 = 1) \\
&= P(X_1X_4 = 0)P(X_2X_3 = 1) \qquad \text{by independence} \\
&= (1-0.4^2)(0.4^2) = 0.1344. \\
P(X=0) &= P(X_1X_4 = 0, X_2X_3 = 0) \\
&= P(X_1X_4 = 0)P(X_2X_3 = 0) \qquad \text{by independence} \\
&= (1-0.4^2)(1-0.4^2) = 0.7312. \\
P(X=1) &= P(X_1X_4 = 1, X_2X_3 = 0) \\
&= P(X_1X_4 = 1)P(X_2X_3 = 0) \qquad \text{by independence} \\
&= (0.4^2)(1-0.4^2) = 0.1344.
\end{aligned}$$

Score

V. (12 points) Let X be a random variable with probability density function

$$f(x) = \begin{cases} a\cos(x), & |x| \le \frac{\pi}{2}, \\ 0, & \text{otherwise}. \end{cases}$$

(a) Find the constant a; (b) Calculate $P(0 < X < \frac{\pi}{4})$; (c) Calculate $\mathrm{Var}(X)$.

Solution.

(a) $\int_{-\infty}^{+\infty} f(x)\,dx = 1 \implies a = 0.5$.

(b) $P(0 < X < \frac{\pi}{4}) = \int_0^{\frac{\pi}{4}} 0.5\cos(x)\,dx = \frac{\sqrt{2}}{4}$.

(c) $E(X) = \int_{-\infty}^{+\infty} xf(x)\,dx = 0$ since $xf(x)$ is an odd function. The second moment is

$$\begin{aligned}
E(X^2) &= \int_{-\infty}^{+\infty} x^2 f(x)\,dx = \frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} x^2\cos(x)\,dx = \frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} x^2\,d\sin(x) \\
&= \frac{1}{2}x^2\sin(x)\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} - \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} x\sin(x)\,dx \\
&= \frac{\pi^2}{4} - 2.
\end{aligned}$$

Then $\mathrm{Var}(X) = \frac{\pi^2}{4} - 2$.

Score

VI. (12 points) Suppose X and Y are independent random variables with density functions

$$f_X(x) = \begin{cases} 1, & 0 < x < 1 \\ 0, & \text{otherwise.} \end{cases}, \quad f_Y(x) = \begin{cases} e^{-y}, & y > 0 \\ 0, & \text{otherwise.} \end{cases}$$

Find the density function of $Z := X + Y$.

Solution.

By independence, the joint density is

$$f(x,y) = f_X(x) f_Y(y) = \begin{cases} e^{-y}, & 0 < x < 1, y > 0, \\ 0, & \text{otherwise.} \end{cases}$$

The distribution function $F_Z(z) = P(Z \le z) = 0$ when $z \le 0$. When $0 < z < 1$,

$$F_Z(z) = \iint_{x+y \le z} f(x,y)\, dx\, dy = \int_0^z \int_0^{z-x} e^{-y}\, dy\, dx = z - 1 + \frac{1}{e^z}.$$

When $z \ge 1$,

$$F_Z(z) = \iint_{x+y \le z} f(x,y)\, dx\, dy = \int_0^1 \int_0^{z-x} e^{-y}\, dy\, dx = 1 + (1-e)\frac{1}{e^z}.$$

The p.m.f. of Z is

$$f_Z(z) = \begin{cases} 0, & z \le 0, \\ 1 - e^{-z}, & 0 < z < 1, \\ (e-1)e^{-z}, & z \ge 1. \end{cases}$$

Score

VII. (12 points) Suppose that $X_1, \cdots, X_n$ form a random sample from the uniform distribution on the interval $[0, \theta]$, where the value of the parameter θ is unknown $(\theta > 0)$.

(a) Find maximum likelihood estimator of θ.

(b) Use the method of moments to find an estimator of θ.

Solution.

(a) The likelihood function is

$$L(\theta) = \prod_{i=1}^{n} f(x_i; \theta) = \frac{1}{\theta^n}, \qquad 0 \leq x_i \leq \theta.$$

Then $\log L(\lambda) = -n \log \theta$ and

$$\frac{d \log L}{d\lambda} = -\frac{n}{\theta} < 0.$$

We have the M.L.E. $\hat{\theta} = \max\{x_1, \cdots, x_n\}$.

(b) Since

$$E(X) = \frac{\theta}{2},$$

we have $\hat{\theta} = 2\bar{X}$.

Score

VIII. (10 points) The drying time of certain type of paint under specified test condition is known to be normally distributed with mean value 75 min and standard deviation 9 min. Consider testing $H_0 : \mu = 75$ versus $H_a : \mu < 75$ based on a sample of size 25. A level 0.05 test is used. ($z_{0.025} = 1.96$, $z_{0.05} = 1.65$, $z_{0.1} = 1.28$, $\Phi(1.09) = 0.8612$, $\Phi(-0.29) = 0.3859$.)

(a) If $\bar{x} = 71.6$, what is the decision?

(b) What is $\beta(74)$, the probability of a type II error when $\mu = 74$?

(c) What value of n is necessary to ensure that $\beta(74) = 0.10$?

Solution.

(a) $z = \frac{71.6-75}{9/\sqrt{25}} \approx -1.89 \leq -1.65$. Reject H_0.

(b) The probability of making a type II error when $\mu = 74$ is

$$\begin{aligned}\beta(74) &= 1 - \Phi\left(-1.65 + \frac{75-74}{9/\sqrt{25}}\right) \\ &= 1 - \Phi(1.09) \\ &\approx 1 - 0.8621 \approx 0.14.\end{aligned}$$

(c) Since $z_{.1} = 1.28$, the requirement that the level .05 test also have $\beta(74) = .1$ necessitates

$$n = \left[\frac{9(1.65+1.28)}{75-74}\right]^2 \approx 695.37.$$

So $n = 696$ should be used.