Seat No.______ Major/Class______ School______ Student ID Name______

(DONOT WRITE YOUR ANSWER IN THIS AREA)

Seal Line

Warning: MISBEHAVIOR AT EXAM TIME WILL LEAD TO SERIOUS CONSEQUENCE!

SCUT Final Exam

2019-2020-2 《Calculus II》Exam Paper A

Solution Manual

Notice:

1. **Make sure that you have filled the form on the left hand side of seal line.**
2. **Write your answers on the exam paper.**
3. **This is a close-book exam.**
4. **The exam has the full score of 100 points and lasts 120 minutes.**

Question No.	1-5	6-12	13-22	Sum
Score				

I. Answer the questions. $(3' \times 5 = 15')$

1. Interchange the integral orders, then $\int_{-1}^{0} dy \int_{-y}^{1} f(x,y)dx + \int_0^1 dy \int_{\sqrt{y}}^1 f(x,y)dx$.

   Answer $\int_0^1 dx \int_{-x}^{x^2} f(x,y)dy$;

2. Suppose $ye^{-x} + z\sin x = 0$, find $\partial z/\partial x$

   Answer $\frac{ye^{-x} - z\cos x}{\sin x} = -z\frac{\sin x + \cos x}{\sin x}$;

3. Find $\operatorname{div}(\vec{F})$ and $\operatorname{curl}(\vec{F})$ if $\vec{F} = x^2yz\vec{i} + 3xyz^3\vec{j} + (x^2 - z^2)\vec{k}$

   Answer $\operatorname{div}(\vec{F}) = 2xyz + 3xz^3 - 2z$

   $\operatorname{curl}(\vec{F}) = -9xyz^2\vec{i} + (x^2y - 2x)\vec{j} + (3yz^3 - x^2z)\vec{k}$;

4. Find f such that $\vec{F} = \nabla f$, while

$$\vec{F} = \left(45x^4y^2 - 6y^6 + 3\right)\mathbf{i} + \left(18x^5y - 36xy^5 + 7\right)\mathbf{j}$$

   Answer $f(x,y) = 9x^5y^2 - 6xy^6 + 3x + 7y + C$;

5. Does the limit $\lim\limits_{(x,y)\to(0,0)} \frac{xy+\cos x}{xy-\cos x}$ exist?

   Answer Yes. It equals -1 ;

II. Finish the following questions. (6-11: $6' \times 6 = 36'; 12 : 7' \times 1 = 7'$)

6. Find the equation of the plane through (6,2,-1) and perpendicular to the line of intersection of planes $4x - 3y + 2z + 5 = 0$ and $3x + 2y - z + 11 = 0$.

   Solution:

   Let (x, y, z) be an arbitrary point. Then (x, y, z) lies in the plane Σ that we are going to find if and only if the vector $(x-6)\vec{i} + (y-2)\vec{j} + (z+1)\vec{k}$ is perpendicular to the line of the intersection of planes $4x - 3y + 2z + 5 = 0$ and $3x + 2y - z + 11 = 0$. Equivalently, the vector $(x-6)\vec{i} + (y-2)\vec{j} + (z+1)\vec{k}$ and the normal vectors

$$\vec{n}_1 = 4\vec{i} - 3\vec{j} + 2\vec{k} \text{ and } \vec{n}_2 = 3\vec{i} + 2\vec{j} - 1\vec{k}$$

   lie in the same plane. Hence, the equation of the plane Σ is

$$\begin{vmatrix} x-6 & y-2 & z+1 \\ 4 & -3 & 2 \\ 3 & 2 & -1 \end{vmatrix} = 0.$$

   Equivalently,

$$-(x-6) + 10(y-2) + 17(z+1) = 0$$

   or

$$x - 10y - 17z = 3.$$

7. Find the minimize $z = x - \frac{x^3}{8} - \frac{y^2}{3}$ subject to $\frac{x^2}{16} + y^2 = 1$.

   Solution:

   We use the method of Lagrange's multiplier.

   Let

$$L(x, y, \lambda) = x - \frac{x^3}{8} - \frac{y^2}{3} + \lambda(\frac{x^2}{16} + y^2 - 1).$$

   The points (x, y) where z takes the minimum value subject to $\frac{x^2}{16} + y^2 = 1$ satisfy

$$\begin{cases} \frac{\partial L}{\partial x} = 1 - \frac{3}{8}x^2 + \frac{1}{8}\lambda x = 0, \\ \frac{\partial L}{\partial y} = -\frac{2}{3}y + 2\lambda y = 0, \\ \frac{\partial L}{\partial \lambda} = \frac{x^2}{16} + y^2 - 1 = 0. \end{cases}$$

Solve the equation system, we obtain

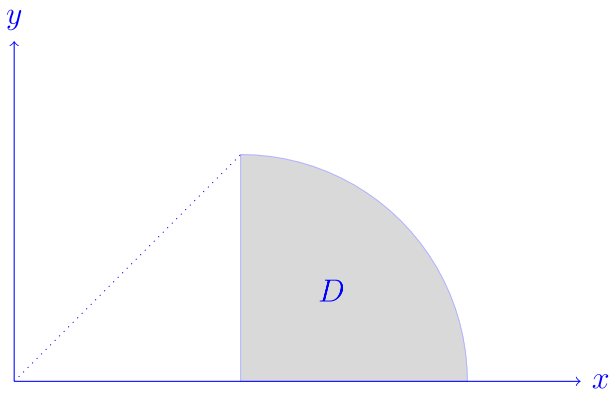
$$\begin{cases} x = 4, \\ y = 0, \\ \lambda = 10, \end{cases} \quad \begin{cases} x = -4, \\ y = 0, \\ \lambda = -10, \end{cases} \quad \begin{cases} x = \frac{1-\sqrt{865}}{18}, \\ y^2 = \frac{2159+\sqrt{865}}{2592}, \\ \lambda = \frac{1}{3}, \end{cases} \quad \begin{cases} x = \frac{1+\sqrt{865}}{18}, \\ y^2 = \frac{2159-\sqrt{865}}{2592}, \\ \lambda = \frac{1}{3}, \end{cases}$$

where z takes values -4, 4, $\frac{-6479-865\sqrt{865}}{23328}$, $\frac{-6479-865\sqrt{865}}{23328}$, respectively. Hence, the minimum value of z subject to $\frac{x^2}{16} + y^2 = 1$ is -4.

8. Evaluate $\int_1^2 \int_0^{\sqrt{2x-x^2}} (x^2+y^2)^{-1/2}\, dydx$.

Solution:

Let D be the region as shown in the picture



Using the polar coordinates, the region D can be represented as

$$D := \{(r,\theta) : 0 \le \theta \le \frac{\pi}{4}, \frac{1}{\cos\theta} \le r \le 2\cos\theta\}.$$

Hence,

$$\begin{aligned} \int_1^2 \int_0^{\sqrt{2x-x^2}} (x^2+y^2)^{-1/2}\, dydx &= \iint\limits_D (x^2+y^2)^{-1/2}\, dA \\ &= \int_0^{\frac{\pi}{4}} \mathrm{d}\theta \int_{\frac{1}{\cos\theta}}^{2\cos\theta} \frac{1}{r} \cdot r\mathrm{d}r \\ &= \int_0^{\frac{\pi}{4}} 2\cos\theta - \frac{1}{\cos\theta} \mathrm{d}\theta \\ &= 2\sin\theta\Big|_0^{\frac{\pi}{4}} - \frac{1}{2}\ln\frac{1+\sin\theta}{1-\sin\theta}\Bigg|_0^{\frac{\pi}{4}} \\ &= \sqrt{2} - \frac{1}{2}\ln\frac{1+\frac{1}{\sqrt{2}}}{1-\frac{1}{\sqrt{2}}} \\ &= \sqrt{2} - \ln(1+\sqrt{2}). \end{aligned}$$

9. Suppose that a differentiable function $f(x, y)$ satisfies $f(tx, ty) = tf(x, y)$ for all $t > 0$. Show that $f(x, y) = x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y}$.

Solution:

Since $f(tx, ty) = tf(x, y)$ for all $t > 0$, take partial derivatives with respect to t on both sides of the equation, we obtain that

$$xf_1(tx, ty) + yf_2(tx, ty) = f(x, y).$$

for all $t > 0$. Let $t = 1$, we conclude that

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = f(x, y).$$

10. $\oint\limits_C (x^2 + 4xy)\, dx + (2x^2 + 3y)\, dy$, where C is the ellipse $9x^2 + 16y^2 = 144$ with counter-clockwise direction.

Solution:

Let D be the region inside the curve C. By Green's Theorem,

$$\int\limits_C (x^2 + 4xy)dx + (2x^2 + 3y)dy = \iint\limits_D 4x - 4x\mathrm{d}A = 0.$$

11. Evaluate $\int\limits_C (1 - y^2)\, ds$, C is the quarter circle from $(0, -1)$ to $(1, 0)$ center at the origin.

Solution:

The curve has the parametric equations

$$x = \cos t, \qquad y = \sin t, \qquad -\frac{pi}{2} \leq t \leq 0.$$

Hence,

$$\begin{aligned}\int\limits_C (1 - y^2)\, ds &= \int_{-\frac{\pi}{2}}^{0} (1 - \sin^2 t)\sqrt{x'(t)^2 + y'(t)^2}\mathrm{d}t \\ &= \int_{-\frac{\pi}{2}}^{0} (1 - \sin^2 t)\mathrm{d}t \\ &= \int_{-\frac{\pi}{2}}^{0} \frac{1 + \cos 2t}{2}\mathrm{d}t \\ &= \frac{\pi}{4}.\end{aligned}$$

12. Evaluate $\int_{-3}^{3}\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}}\int_{-\sqrt{9-x^2-z^2}}^{\sqrt{9-x^2-z^2}}(x^2+y^2+z^2)^{3/2}\,dydzdx$.

Solution:

Using the spherical coordinates,

$$\begin{aligned}&\int_{-3}^{3}\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}}\int_{-\sqrt{9-x^2-z^2}}^{\sqrt{9-x^2-z^2}}\left(x^2+y^2+z^2\right)^{3/2}\mathrm{d}y\mathrm{d}z\mathrm{d}x\\ =&\iiint\limits_{x^2+y^2+z^2\le 9}\left(x^2+y^2+z^2\right)^{3/2}\mathrm{d}V\\ =&\int_0^{2\pi}\mathrm{d}\theta\int_0^{\pi}\mathrm{d}\phi\int_0^3\rho^3\cdot\rho^2\sin\phi\mathrm{d}\rho\\ =&2\pi\cdot 2\cdot\frac{1}{6}3^6\\ =&486\pi.\end{aligned}$$

III. Please select 6 questions from the following 10 questions $(7'\times 6=42')$

（请从下面的 10 道题中选择 6 道题目来回答，并把答案写在试卷上）

13. Test for the convergence or divergence $\sum\limits_{n=1}^{\infty}\frac{n}{n5^n+2}$.

Solution:

For $n\ge 1$,

$$\frac{n}{n5^n+2}\le\frac{n}{n5^n}=\frac{1}{5^n}.$$

Note that the geometric series $\sum\limits_{n=1}^{\infty}\frac{1}{5^n}$ converges and $\sum\limits_{n=1}^{\infty}\frac{n}{n5^n+2}$ is a positive series.

By the Comparison Test, the series $\sum\limits_{n=1}^{\infty}\frac{n}{n5^n+2}$ converges.

14. Find the convergence set for the power series $\sum\limits_{n=0}^{\infty}\frac{(x-1)^n}{(n+1)^2}$

Solution:

The power series has the radius of the convergence

$$R=\lim_{n\to\infty}\frac{\frac{1}{(n+1)^2}}{\frac{1}{(n+2)^2}}=\lim_{n\to\infty}\frac{(n+2)^2}{(n+1)^2}=1.$$

When $x-1=1$, the series $\sum\limits_{n=0}^{\infty}\frac{1}{(n+1)^2}$ converges.

When $x-1=-1$, the series $\sum\limits_{n=0}^{\infty}\frac{(-1)^n}{(n+1)^2}$ also converges.

Hence, its convergence set is

$$\{x : -1 \leq x - 1 \leq 1\} = [0, 2].$$

15. Solve differential equation $y'' + y = \sec x$.

Solution:

This is a non-homogeneous linear differential equation with constant coefficients. Its associated homogeneous equation

$$y'' + y = 0$$

has the solution

$$y = C_1 \sin x + C_2 \cos x.$$

Suppose that the equation $y'' + y = \sec x$ has a solution

$$y^* = C_1(x) \sin x + C_2(x) \cos x,$$

provided that the functions $C_1(x)$ and $C_2(x)$ satisfy the equations

$$\begin{cases} C_1'(x) \sin x + C_2'(x) \cos x = 0, \\ C_1'(x) \cos x - C_2'(x) \sin x = \sec x, \end{cases}$$

Solve it, we obtain

$$C_1'(x) = 1, \quad C_2'(x) = \tan x.$$

It yields

$$C_1(x) = x, \quad C_2(x) = -\ln|\cos x|.$$

Consequently, we find that

$$y^* = x \sin x - \cos x \ln|\cos x|$$

is a solution to $y'' + y = \sec x$. Hence, its general solution is

$$y = C_1 \sin x + C_2 \cos x + x \sin x - \cos x \ln|\cos x|.$$

16. Solve differential equation $y'''' - 2y''' + 5y'' = 0$. Solution:

This is a homogeneous linear ordinary differential equation with constant coefficients. Its characteristic equation is

$$0 = \lambda^4 - 2\lambda^3 + 5\lambda^2 = \lambda^2(\lambda - 1 + 2i)(\lambda - 1 - 2i)$$

where $i=\sqrt{-1}$. Hence, the solution to the differential equation is

$$C_1+C_2x+C_3e^x\sin 2x+C_4e^x\cos 2x,$$

where C_1,C_2,C_3,C_4 are constant numbers.

17. Let $z=xf\left(xy,\frac{y}{x}\right)$, and f has the second-order continuous partial derivatives, find $\frac{\partial z}{\partial y}$, $\frac{\partial^2 z}{\partial y\partial x}$.
Solution:

$$\begin{aligned}\frac{\partial z}{\partial y}=&xf_1(xy,\frac{y}{x})\cdot x+xf_2(xy,\frac{y}{x})\cdot\frac{1}{x}\\=&x^2f_1(xy,\frac{y}{x})+f_2(xy,\frac{y}{x}).\\\frac{\partial^2 z}{\partial y\partial x}=&\frac{\partial^2 z}{\partial x\partial y}\\=&2xf_1(xy,\frac{y}{x})+x^2\left(f_{11}(xy,\frac{y}{x})\cdot y+f_{12}(xy,\frac{y}{x})\cdot\left(-\frac{y}{x^2}\right)\right)\\&+f_{21}(xy,\frac{y}{x})\cdot y+f_{22}(xy,\frac{y}{x})\cdot\left(-\frac{y}{x^2}\right)\\=&2xf_1(xy,\frac{y}{x})+x^2yf_{11}(xy,\frac{y}{x})-\frac{y}{x^2}f_{22}(xy,\frac{y}{x}).\end{aligned}$$

18. Let $z=f(u,x,y)$, $u=xe^y$, and f has second-order continuous partial derivatives, find $\frac{\partial^2 z}{\partial x\partial y}$
Solution:

$$\begin{aligned}\frac{\partial z}{\partial y}=&\frac{\partial f}{\partial u}\frac{\partial u}{\partial y}+\frac{\partial f}{\partial y}\\=&xe^y\frac{\partial f}{\partial u}+\frac{\partial f}{\partial y}.\\\frac{\partial^2 z}{\partial x\partial y}=&e^y\frac{\partial f}{\partial u}+xe^y\left(\frac{\partial^2 f}{\partial u^2}\frac{\partial u}{\partial x}+\frac{\partial^2 f}{\partial x\partial u}\right)+\left(\frac{\partial^2 f}{\partial u\partial y}\frac{\partial u}{\partial x}+\frac{\partial^2 f}{\partial x\partial y}\right)\\=&e^y\frac{\partial f}{\partial u}+xe^y\left(e^y\frac{\partial^2 f}{\partial u^2}+\frac{\partial^2 f}{\partial x\partial u}\right)+\left(e^y\frac{\partial^2 f}{\partial u\partial y}+\frac{\partial^2 f}{\partial x\partial y}\right)\\=&e^y\frac{\partial f}{\partial u}+xe^{2y}\frac{\partial^2 f}{\partial u^2}+xe^y\frac{\partial^2 f}{\partial x\partial u}+e^y\frac{\partial^2 f}{\partial u\partial y}+\frac{\partial^2 f}{\partial x\partial y}.\end{aligned}$$

19. Find $\iiint\limits_{\Omega} z dv$, and Ω is bounded by $x^2+y^2=1$ and $z=0, z=1$.

$$\begin{aligned}\iiint\limits_{\Omega} z \mathrm{d}v = & \iint\limits_{x^2+y^2\leq 1} \mathrm{d}A \int_0^1 z\mathrm{d}z \\ = & \iint\limits_{x^2+y^2\leq 1} \frac{1}{2}\mathrm{d}A \\ = & \frac{\pi}{2}.\end{aligned}$$

20. Find $I=\iiint\limits_{\Omega} \frac{dv}{x^2+y^2+z^2}$, Ω is bounded by $z=1+\sqrt{1-x^2-y^2}$ and $z=1$.

Solution:

Using the spherical coordinate system,

$$\begin{aligned}\iiint\limits_{\Omega} \frac{\mathrm{d}v}{x^2+y^2+z^2} = & \int_0^{2\pi} \mathrm{d}\theta \int_0^{\frac{\pi}{4}} \mathrm{d}\phi \int_{\frac{1}{\cos\phi}}^{2\cos\phi} \frac{1}{\rho^2}\rho^2 \sin\phi \mathrm{d}\rho \\ =& 2\pi \int_0^{\frac{\pi}{4}} \left(2\cos\phi - \frac{1}{\cos\phi}\right) \sin\phi \mathrm{d}\phi \\ =& 2\pi \left(\sin^2\phi + \ln\cos\phi\right) |_0^{\frac{\pi}{4}} \\ =& \pi(1-\ln 2).\end{aligned}$$

21. Find $\iint\limits_{\Sigma} xz^2dydz + (x^2y-z^2)\, dzdx + (2xy+y^2z)\, dxdy$, Σ is hemisphere (half a sphere) $z=\sqrt{a^2-x^2-y^2}$ with upside direction.

Solution:

Let Σ_0 be the portion of the xy-plane inside the circle $x^2+y^2 \leqslant a^2$ with downward direction. Then Σ and Σ_0 make a closed surface together which encloses the solid region Ω.

By Gauss' Divergence Theorem,

$$\begin{aligned}\iint\limits_{\Sigma\cup\Sigma_0} xz^2\mathrm{d}y\mathrm{d}z + (x^2y-z^2)\mathrm{d}z\mathrm{d}x + (2xy+y^2z)\mathrm{d}x\mathrm{d}y = & \iiint\limits_{\Omega} z^2+x^2+y^2\mathrm{d}V \\ = & \int_0^{2\pi} \mathrm{d}\theta \int_0^{\frac{\pi}{2}} \mathrm{d}\phi \int_0^a \rho^2\cdot\rho^2\sin\phi\mathrm{d}\rho \\ = & 2\pi\cdot 1\cdot\frac{1}{5}a^5 \\ = & \frac{2}{5}\pi a^5.\end{aligned}$$

The surface Σ_0 has the unit normal vector $(0, 0, -1)$. Hence,

$$\iint\limits_{\Sigma_0} xz^2\mathrm{d}y\mathrm{d}z + (x^2y - z^2)\mathrm{d}z\mathrm{d}x + (2xy + y^2z)\mathrm{d}x\mathrm{d}y = \iint\limits_{x^2+y^2\leq a^2} 2xy\mathrm{d}x\mathrm{d}y = 0.$$

Hence,

$$\iint\limits_{\Sigma} xz^2\mathrm{d}y\mathrm{d}z + (x^2y - z^2)\mathrm{d}z\mathrm{d}x + (2xy + y^2z)\mathrm{d}x\mathrm{d}y = \frac{2}{5}\pi a^2.$$

22. Line integral $\int\limits_L xy^2dx + y\varphi(x)dy$ is independent of path, and $\varphi(x)$ is derivative, $\varphi(0) = 0$ find $\int_{(0,0)}^{(1,1)} xy^2dx + y\varphi(x)dy$.

Solution:

Let $P(x, y) = xy^2$ and $Q(x, y) = y\varphi(x)$. Since the line integral $\int\limits_L P(x, y)\mathrm{d}x + Q(x, y)\mathrm{d}y$ is independent of path, the vector field $P(x, y)\vec{i} + Q(x, y)\vec{j}$ is conservative. Hence,

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}.$$

i.e.,

$$2xy = y\varphi'(x).$$

It follows that $\varphi'(x) = 2x$. Observing that $\varphi(0) = 0$, we obtain that $\varphi(x) = x^2$.

Now, the differential form

$$P(x, y)\mathrm{d}x + Q(x, y)\mathrm{d}y = xy^2\mathrm{d}x + yx^2\mathrm{d}y = \mathrm{d}\left(\frac{1}{2}x^2y^2\right).$$

Therefore,

$$\int\limits_L xy^2\mathrm{d}x + yx^2\mathrm{d}y = \frac{1}{2}x^2y^2\Big|_{(0,0)}^{(1,1)} = \frac{1}{2}.$$

Please write your answers on the exam paper.