Notes on Integration

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1 The absolute value of an integrable function

1.1

Let $k : \mathbb{R}^n \to \mathbb{R}$ and $h : \mathbb{R} \to \mathbb{R}$. Let $D(k) = \{\mathbf{x} \in \mathbb{R}^n \mid k \text{ is discontinuous at } \mathbf{x}\}$ be the set of discontinuities of k. Prove that if h is continuous, then $D(h \circ k) \subset D(k)$.

It is equivalent to prove that if k is continuous at x, then $h \circ k$ is continuous at x. The composition of two continuous functions is continuous, so this statement is true at any point x where $k(x)$ is locally continuous. $\square$

1.2

Let A be a bounded subset of $\mathbb{R}^n$ and $f : A \to \mathbb{R}$ integrable. Assume $D(\hat{f})$ has Jordan content 0. Prove that $|f|$ is integrable and that $\left|\int_A f\right| \leq \int_A |f|$.

$D(\hat{f})$ has Jordan content 0 $\implies$ $\left|\hat{f}\right|$ is integrable on I $\implies$ $|f|$ is integrable on A. If we look at integrals as sums, then $\left|\int_A f\right| = |\sum x_k|$ and $\int_A |f| = \sum |x_k|$. by the triangle inequality, $|\sum x_k| \leq \sum |x_k| \implies \left|\int_A f\right| \leq \int_A |f|$. $\square$

2 The boundary of a set of Jordan content 0

2.1

Let S and F be subsets of $\mathbb{R}^n$ such that $S \subseteq F$. If F is closed, show that $\partial S \subseteq F$.

F is closed $\implies$ F contains all of its accumulation points. If $\mathbf{x} \in \partial S$, then $\mathbf{x}$ is an accumulation point of S or $\mathbf{x} \in S$. $F \subseteq S \implies$ an accumulation point of $S \in F$, as F contains all of its accumulation points. Therefore $\partial S \subseteq F$. $\square$

2.2

Use §2.1 and the fact that the union of a finite number of generalized rectangles is closed to show that if S has Jordan content 0, then ∂S also has Jordan content 0.

Let F be a finite union of generalized rectangles, which is by definition closed. Let $S \subseteq F$. By §2.1, $\partial S \in F$, and therefore ∂S has Jordan Content 0. $\square$

3 The integral with respect to one variable

Assume $g : [a, b] \times [c, d] \to \mathbb{R}$ is continuous. Prove that the function $[c, d] \ni y \mapsto \int_a^b g(x, y)dx$ is continuous on $[c, d]$.

Let $\epsilon > 0$. We have the following expression for $|f(x) - f(x_0)|$.

$$\left|\int_a^b g(x,y)dx - \int_a^b (x_0,y_0)dx\right| = \left|\int_a^b (g(x,y) - \int_a^b (x_0,y_0))dx\right| \leq \int_a^b |g(x,y) - g(x_0,y_0)|dx.$$

We need this integral to evaluate to a value less than ϵ, as to satisfy the $\epsilon - \delta$ definition of continuity. We have that g is continuous, therefore the $\epsilon - \delta$ definition of continuity is satisfied for g. We can pick such a y_0 such that $|g(x, y) - g(x_0, y_0)|$ is bounded by $\frac{\epsilon}{2(b-a)}$, and therefore the integral would evaluate to a value less than ϵ. It follows from the $\epsilon - \delta$ definition of continuity that this mapping is continuous. $\square$

4 A function that is discontinuous at exactly the nonzero rationals in [0,1]

The Dirichlet function, which is 1 on the rationals in $[0, 1]$, is discontinuous at every point of $[0, 1]$, rational or irrational. In this example, you will find a function whose points of discontinuity are exactly the nonzero rationals in $[0, 1]$. Define $f : [0, 1] \to \mathbb{R}$ by:

$$f(x) = \begin{cases} 0 & \text{if } x \text{ is irrational or } 0 \\ \frac{1}{q} & \text{if } x = \frac{p}{q} \text{ is rational in lowest terms.} \end{cases}$$

4.1

Show that f is discontinuous at all rational numbers $\neq 0$ in $[0, 1]$.

Let $x \in [0, 1]$ be rational. For any x_0 in the interval $(x - \delta, x + \delta)$, we need $|f(x) - f(x_0)| < \epsilon$. But for irrational x_0, we would need $\frac{1}{q} < \epsilon$, which is not necessarily always true. Therefore f is discontinuous at all rational numbers $\neq 0$ in $[0, 1]$. $\square$

4.2

Let a be an irrational number and let $x_k = \frac{p_k}{q_k}$ be a sequence of rational numbers in lowest terms converging to a. Prove that the set of denominators $\{q_k\}_{k=1}^{\infty}$ is unbounded.

Assume the set of denominators $\{q_k\}_{k=1}^{\infty}$ is bounded. This means $\exists M$ such that $\forall k$, $q_k \leq M$. We also know that $\forall k$, $x_k \leq 1 \implies p_k \leq q_k \implies \{p_k\}_{k=1}^{\infty}$ is bounded. This means that both $\{p_k\}_{k=1}^{\infty}$ and $\{q_k\}_{k=1}^{\infty}$ are finite sets, and thus any infinite sequence x_k can take on at most a finite number of values. By the pigeonhole principle, at least one of these values must occur an infinitely many times. Therefore we have two cases. In the first case, $\{x_k\}$ oscillates and therefore does not converge, which goes against our assumption that $\{x_k\}$ converges. In the second case, $\{x_k\}$ trails with the same value occurring infinitely many times. This means $\{x_k\} \to x_k$ for some k, thus $\{x_k\}$ converges to a rational value. In either case, $\{x_k\}$ does not converge to an irrational value, so our initial assumption is incorrect. Therefore $\{q_k\}_{k=1}^{\infty}$ is unbounded. $\square$

4.3

Prove that if the sequence $x_k = \frac{p_k}{q_k}$ converges to an irrational number, then $\frac{1}{q_k}$ converges to 0.

By §4.2, if x_k converges to an irrational number, then the set of denominators $\{q_k\}_{k=1}^{\infty}$ is unbounded, and $\{q_k\}_{k=1}^{\infty} \to \infty$. Therefore $\{\frac{1}{q_k}\}_{k=1}^{\infty} \to 0$. $\square$

4.4

Prove that f is continuous at 0.

For any irrational value x_0 in the in the interval $(0, \delta)$, the $\epsilon - \delta$ definition of continuity trivially holds because $|0 - x_0| < \delta \implies |0 - 0| < \epsilon$. For $x_0 \in \mathbb{Q}$ in the interval, $|x_0| < \delta \implies \left|\frac{1}{q}\right| < \epsilon$. But from §4.3, we know that $\{\frac{1}{q_k}\}_{k=1}^{\infty} \to 0$, and thus we can find an x_0 in the interval such that $\left|\frac{1}{q}\right| < \epsilon$. Therefore f is continuous at 0.

5 Interchanging the order of integration

5.1

Evaluate the integral

$$\int_0^3 \int_{y^2}^9 y \cos{(x^2)} dx dy.$$

First we change the order of integration then evaluate the inner integral.

$$\int_0^3 \int_{y^2}^9 y \cos{(x^2)} dx dy = \int_0^9 \int_0^{\sqrt{x}} y \cos{(x^2)} dy dx = \int_0^9 \frac{x}{2} cos(x^2) dx$$

Now we can evaluate this single integral using standard methods.

$$\int_0^9 \frac{x}{2} cos(x^2) dx = \frac{\sin(81)}{4}$$

5.2

Suppose that the function $f : [a, b] \to \mathbb{R}$ is continuous. Prove that

$$2 \int_a^b \left[f(x) \int_x^b f(y) dy \right] dx = \left[\int_a^b f(x) dx \right]^2 .$$

.

We rewrite the expression on the right.

$$\left[\int_a^b f(x) dx \right]^2 = \left(\int_a^b f(x) dx \right) \left(\int_a^b f(y) dy \right) = \int_a^b \int_a^b f(x) f(y) dy dx$$

We expand the integral on the left.

$$2 \int_a^b \left[f(x) \int_x^b f(y) dy \right] dx = \int_a^b \int_x^b f(x) f(y) dy dx + \int_a^b \int_x^b f(x) f(y) dy dx$$

We change x and y in one of the two integrals in the sum.

$$\int_a^b \int_x^b f(x)f(y)dydx + \int_a^b \int_x^b f(x)f(y)dydx = \int_a^b \int_y^b f(x)f(y)dxdy + \int_a^b \int_x^b f(x)f(y)dydx$$

We change the order of integration and use additivity.

$$\int_a^b \int_y^b f(x)f(y)dxdy + \int_a^b \int_x^b f(x)f(y)dydx = \int_a^b \int_a^x f(x)f(y)dydx + \int_a^b \int_x^b f(x)f(y)dydx$$

$$= \int_a^b \int_a^b f(x)f(y)dydx$$

We can now see that the left and right sides of this equality are equivalent. $\square$