Partitions

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1 A1

Suppose that the bounded function $f : [a, b] \to \mathbb{R}$ has the property that $f(x) \geq 0$ for all x in $[a, b]$. Prove that $\underline{\int_a^b} f \geq 0$.

For any partition P of $[a, b]$, $L(f, P) = \sum_{i=1}^{n} m_i(x_i - x_{i-1})$. Because $f(x) \geq 0$ for all x in $[a, b]$, each m_i is nonnegative, and $L(f, P) \geq 0$. $\underline{\int_a^b} f = \sup\{L(f, P)\} \geq 0 \implies \underline{\int_a^b} f \geq 0$. $\square$

2 A2

2.1

For a partition $P = [x_0, ..., x_n]$ of the interval $[a, b]$, show that $\sum_{i=1}^{n}[x_i - x_{i-1}]^2 \leq [b - a] \cdot \text{gap } P$.

We use the definition of gap to develop several inequalities.

$$x_i - x_{i-1} \leq \text{gap } P \implies (x_i - x_{i-1})^2 \leq (x_i - x_{i-1}) \cdot \text{gap } P$$

$$\implies \sum_{i=1}^{n}(x_i - x_{i-1})^2 \leq \sum_{i=1}^{n}(x_i - x_{i-1}) \cdot \text{gap } P = \text{gap } P \cdot \sum_{i=1}^{n}(x_i - x_{i-1})$$

We now expand the rightmost sum.

$$\sum_{i=1}^{n}(x_i - x_{i-1}) = (x_1 - a) + (x_2 - x_1) + ... + (x_{n-1} - x_{n-2}) + (b - x_{n-1})$$

$$= b - a + (x_1 - x_1) + ... + (x_{n-1} - x_{n-1}) = b - a$$

We substitute this result into the above equality and we are done.

2.2

Suppose that the function $f : [a, b] \to \mathbb{R}$ is Lipschitz; that is, that there is a constant $c \geq 0$ such that $|f(u) - f(v)| \leq c|u - v|$ for all points u, v in $[a, b]$. For a partition P of $[a, b]$, prove that $0 \leq U(f, P) - L(f, P) \leq c[b - a] \cdot \text{gap } P$.

Let m_i and M_i be the infimum and supremum of each interval within P respectively. By definition of Darboux sums, we have

$$U(f, P) - L(f, P) = \sum_{i=1}^{n} (M_i - m_i)(x_{i+1} - x_i)$$

and

$$U(f, P) \geq L(f, P) \implies U(f, P) - L(f, P) \geq 0.$$

For any pair of values x_{i+1} and x_i, $x_{i+1} - x_i \leq \text{gap } P$. Lipschitz functions are continuous, so we can invoke the Extreme Value Theorem to assert that for each pair of points x_{i+1} and x_i, $f(x_{i+1}) = M_i$ and $f(x_i) = m_i$. We now have

$$\sum_{i=1}^{n} (M_i - m_i)(x_{i+1} - x_i) \leq \text{gap } P \cdot \sum_{i=1}^{n} (f(x_{i+1}) - f(x_i)).$$

By definition of Lipschitz, $f(x_{i+1}) - f(x_i) \leq c|x_{i+1} - x_i|$. Therefore,

$$\text{gap } P \cdot \sum_{i=1}^{n} (f(x_{i+1}) - f(x_i)) = c \cdot \text{gap } P \cdot \sum_{i=1}^{n} (x_{i+1} - x_i).$$

From §2.1, we know this sum is equal to $b - a$, and the desired inequality directly follows. □

2.3

Use the Darboux sum difference estimate in §2.2 and the Archimedes-Riemann Theorem[1] to show that a Lipschitz function is integrable.

Let P_n be the partition of $[a, b]$ where each partition interval has equal length, specifically $\frac{b-a}{n}$. We now use the inequality from §2.2.

$$0 \leq U(f, P) - L(f, P) \leq c[b - a] \cdot \text{gap } P \implies 0 \leq U(f, P_n) - L(f, P_n) \leq \frac{c(b-a)^2}{n}$$

Now we apply a limit to both sides.

$$0 \leq \lim_{n\to\infty} (U(f, P_n) - L(f, P_n)) \leq \lim_{n\to\infty} \frac{c(b-a)^2}{n} = 0$$

Because $\lim_{n\to\infty}(U(f, P_n) - L(f, P_n)) = 0$, f is integrable by the Archimedes-Riemann Theorem.

[1]Archimedes-Riemann Theorem: Let $f : [a, b] \to \mathbb{R}$ be a bounded function. Then f is integrable on $[a, b]$ if and only if there is a sequence $[P_n]$ of partitions of the interval $[a, b]$ such that $\lim_{n\to\infty} [U(f, P_n) - L(f, P_n)] = 0$. Moreover, for any such sequence of partitions, $\lim_{n\to\infty} L(f, P_N) = \int_a^b f$ and $\lim_{n\to\infty} U(f, P_N) = \int_a^b f$.

3 A3

Define

$$f(x) = \begin{cases} x & \text{if the point } x \text{ in [0,1] is rational,} \\ 0 & \text{if the point } x \text{ in [0,1] is irrational.} \end{cases}$$

Prove that $\underline{\int_0^1} f = 0$ and $\overline{\int_0^1} f = \frac{1}{2}$.

Let P_n be the the partition of $[0,1]$ where each partition interval has an equal length of $\frac{1}{n}$. $\underline{\int_0^1} f = \lim_{n\to\infty} L(f, P_n)$ and $\overline{\int_0^1} f = \lim_{n\to\infty} U(f, P_n)$. Because the rationals and irrationals are dense in $\mathbb{R}$, if m_i and M_i are defined as the infimum and supremum values respectively, then $m_i = 0$ and $M_i = x_i$. First we calculate the lower bound.

$$\lim_{n\to\infty} L(f, P_n) = \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} (0) = 0$$

Now we calculate the upper bound, which is less trivial.

$$\lim_{n\to\infty} U(f, P_n) = \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} (x_i) = \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \frac{i}{n} = \lim_{n\to\infty} \frac{1}{n^2} \cdot \frac{n(n+1)}{2} = \frac{1}{2}$$

4 A4

A function $f : [a, b] \to \mathbb{R}$ is called a step function if there is a partition of $[a, b]$ so that on each open partition interval the function is constant. Prove that a step function is integrable.

Let $f : [a, b] \to \mathbb{R}$ be a step function. By additivity, we can say the following.

$$\int_a^b f = \int_a^{c_1} C_1 + \int_{c_1}^{c_2} C_2 + ... + \int_{c_{n-1}}^{c_n} C_n$$

On each interval (c_i, c_{i+1}), f is equal to some constant C_i. Constant functions are clearly integrable, and by additivity again it directly follows that f is integrable over $[a, b]$. $\square$

5 B1

Find

$$\lim_{n\to\infty} \left[\frac{1}{\sqrt{n \cdot n}} + \frac{1}{\sqrt{n(n+1)}} + ... + \frac{1}{\sqrt{n(n+n)}} \right].$$

Let $f(x) = \frac{1}{\sqrt{1+x}}$. This is integrable on $[0,1]$ and $\int_0^1 \frac{1}{\sqrt{1+x}} dx = 2\sqrt{1+x}\Big|_0^1 = 2\sqrt{2} - 2$. Let P_n be the partition that splits $[0,1]$ into n intervals of length $\frac{1}{n}$. Consider the Riemann Sum for the integral $\int_0^1 \frac{1}{\sqrt{1+x}} dx$ with $c_i = \frac{i}{n}$ for $1 \le i \le n$.

$$\lim_{n\to\infty} R(f, P_n, C_n) = \lim_{n\to\infty} \left[\left(\frac{1}{n}\right) \cdot \left(\frac{1}{\sqrt{1+\frac{1}{n}}} + \frac{1}{\sqrt{1+\frac{2}{n}}} + ... + \frac{1}{\sqrt{1+\frac{n}{n}}} \right) \right]$$

$$= \lim_{n\to\infty} \left[\frac{1}{\sqrt{n(n+1)}} + ... + \frac{1}{\sqrt{n(n+n)}} \right] = \lim_{n\to\infty} \left[\frac{1}{\sqrt{n \cdot n}} + \frac{1}{\sqrt{n(n+1)}} + ... + \frac{1}{\sqrt{n(n+n)}} \right]$$

We can append the first term to the limit because $\lim_{n\to\infty} \frac{1}{\sqrt{n\cdot n}} = 0$. By the Riemann Sum Convergence Theorem[2], the limit is equal to the integral we originally evaluated.

6 B2

Let $f : [a, b] \to \mathbb{R}$ be bounded.

6.1

Assume f is continuous on $[a, b]$, $f(x) \geq 0$ for all $x \in [a, b]$ and assume $f(x_0) > 0$ for some $x_0 \in [a, b]$. Prove that $\int_a^b f > 0$.

Because f is continuous on $[a, b]$ and $f(x_0) > 0$, there exists an $\epsilon > 0$ such that for any $x_I \in I$, where I is the open interval $(x_0 - \epsilon, x_0 + \epsilon)$, $f(x_I) > 0$. Let $g(x) = 0$ on the interval $[a, b]$. For all $x_I \in I$, $f(x_I) > g(x_I)$, so by monotonicity, $\int_I f > \int_I g = 0$. Using additivity, $\int_a^b f = \int_a^{x-\epsilon} f + \int_{x+\epsilon}^b f + \int_I f$. All three terms in this sum are nonnegative, and the third term is strictly positive, therefore $\int_a^b f > 0$. $\square$

6.2

Is the conclusion of §2.1 true if one assumes f is integrable on $[a, b]$, $f(x) \geq 0$ for all $x \in [a, b]$ and assumes $f(x_0) > 0$ for some $x_0 \in [a, b]$? If so, prove it, and if not, provide a counterexample.

Define

$$f(x) = \begin{cases} 0 & h \neq x_0 \\ 1 & h = x_0 \end{cases}$$

This is a counterexample to the above claim, as $\int_a^b f = \int_a^{x_0} f + \int_{x_0}^b = 0 + 0 = 0$.

7 B3

Let $a < b$ and $c \in (a, b)$. Assume $f : [a, b] \to \mathbb{R}$ is bounded and integrable on both $[a, c]$ and $[c, b]$. Prove that f is integrable on $[a, b]$ and $\int_a^b f = \int_a^c f + \int_c^b f$.

Let $P_1 = \{x_0, x_1, ..., x_m\}$ be a partition of $[a, c]$ and $P_2 = \{x_m, x_{m+1}, ..., x_n\}$ be a partition of $[c, b]$. Let $P = P_1 \cup P_2$ be a partition of $[a, b]$. Therefore,

$$\sum_{i=1}^{n} m_i \Delta x_i = \sum_{i=1}^{m} m_i \Delta x_i + \sum_{i=m+1}^{n} m_i \Delta x_i \implies L(f, P) = L(f, P_1) + L(f, P_2)$$

and

$$\sum_{i=1}^{n} M_i \Delta x_i = \sum_{i=1}^{m} M_i \Delta x_i + \sum_{i=m+1}^{n} M_i \Delta x_i \implies U(f, P) = U(f, P_1) + U(f, P_2).$$

[2]Riemann Sum Convergence Theorem: Suppose that the function $f : [a, b] \to \mathbb{R}$ is integrable. For each natural number n, let P_n be a partition of $[a, b]$ and let $R(f, P_n, C_n)$ be a Riemann sum. If $\lim_{n\to\infty}$ gap $P_n = 0$, then $\lim_{n\to\infty} R(f, P_n, C_n) = \int_a^b f$.

We subtract to get $U(f,P)-L(f,P) = U(f,P_1)-L(f,P_1)+U(f,P_2)-L(f,P_2)$. Because f is integrable over $[a,c]$, $\lim_{m\to\infty}(U(f,P_1)-L(f,P_1)) = 0$ and because f is integrable over $[c,b]$, $\lim_{n\to\infty}(U(f,P_2)-L(f,P_2)) = 0$. The following directly follows.

$$\lim_{m,n\to\infty}(U(f,P)-L(f,P)) = \lim_{m,n\to\infty}(U(f,P_1)-L(f,P_1)+U(f,P_2)-L(f,P_2)) = 0$$

Therefore f is integrable over $[a,b]$ and $\int_a^b f = \int_a^c f + \int_c^b f$. $\square$

8 B4

The Second Fundamental Theorem has a somewhat more general form than we have stated: For an integrable function $f:[a,b]\to\mathbb{R}$, we define $F(x)\equiv\int_a^x f$ for all x in $[a,b]$. Then at each point x_0 in (a,b) at which the function $f:[a,b]\to\mathbb{R}$ is continuous, the function $F:[a,b]\to\mathbb{R}$ is differentiable and $F'(x_0) = f(x_0)$. Prove this.

We first use the definition of the derivative to evaluate $F'(x)$.

$$F'(x) = \lim_{x\to x_0}\frac{F(x)-F(x_0)}{x-x_0} = \lim_{x\to x_0}\frac{\int_a^x f(t)dt - \int_a^{x_0} f(t)dt}{x-x_0} = \lim_{x\to x_0}\frac{\int_{x_0}^x f(t)dt}{x-x_0}$$

By the Mean Value Theorem for Integrals, we can select a point $c(x)$ between x_0 and x such that $\frac{F(x)-F(x_0)}{x-x_0} = f(c(x))$. f is continuous at x_0, so $\lim_{x\to x_0} f(c(x)) = f(x_0)$. Finally we have $F'(x_0) = \lim_{x\to x_0}\frac{F(x)-F(x_0)}{x-x_0} = \lim_{x\to x_0} f(c(x)) = f(x_0)$. $\square$

9 B5

Prove that a bounded function $f:[a,b]\to\mathbb{R}$ is integrable if and only if for each $\epsilon>0$, there is a partition P of $[a,b]$ such that $U(f,P)-L(f,P)<\epsilon$.

If f is integrable, then $\lim_{n\to\infty}(U(f,P)-L(f,P)) = 0$. Therefore for all $\epsilon>0$, there exists a value N such that for all $n>N$, $U(f,P)-L(f,P)<\epsilon$.

If we assume that $U(f,P)-L(f,P)<\epsilon$ for all $\epsilon>0$, then we can create a sequence of partitions such that $U(f,P)-L(f,P) = \frac{1}{n}$ for all $n>0$. We take the limit on both sides.

$$\lim_{n\to\infty}(U(f,P)-L(f,P)) = \lim_{n\to\infty}\frac{1}{n} = 0$$

It follows that f is integrable. $\square$