# Jordan Content

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## 1 The integrals of two functions that are equal except on a set of Jordan content 0

For $\mathbf{I}$ a generalized rectangle in $\mathbb{R}^n$, let $A$ be a subset of $\mathbf{I}$ of Jordan content 0 and suppose that the integrable functions $f : \mathbf{I} \to \mathbb{R}$ and $g : \mathbf{I} \to \mathbb{R}$ are such that $f(\mathbf{x}) = g(\mathbf{x})$ for $\mathbf{x}$ in $\mathbf{I} \backslash A$. Show that $\int_{\mathbf{I}} f = \int_{\mathbf{I}} g$.

Let $h(\mathbf{x}) = f(\mathbf{x}) - g(\mathbf{x}) = 0$ for all $\mathbf{x}$ in $\mathbf{I} \backslash A$. We then know $\int_{\mathbf{I}} h = 0$. $f$ and $g$ are both integrable, and by additivity, $\int_{\mathbf{I}}(f - g) = 0 \implies \int_{\mathbf{I}} f - \int_{\mathbf{I}} g = 0 \implies \int_{\mathbf{I}} f = \int_{\mathbf{I}} g$. $\square$

## 2 Integral on a closed rectangle versus on its interior

Let $I$ be a generalized rectangle in $\mathbb{R}^n$ and let the function $f : I \to \mathbb{R}$ be integrable. Denote the interior of $I$ by $D$. Show that the restriction $f : D \to \mathbb{R}$ is integrable and that $\int_I f = \int_D f$.

Let $\hat{f}$ be the zero extension of $f$. On $D$, $U(\hat{f}, P_k) - L(\hat{f}, P_k) = U(f, P_k) - L(f, P_k)$. Therefore, $\lim_{k\to\infty}[U(\hat{f}, P_k) - L(\hat{f}, P_k)] = \lim_{k\to\infty}[U(f, P_k) - L(f, P_k)] = 0$ and $\hat{f}$ is integrable. $D$ is a bounded subset of $\mathbb{R}^n$ and $D \subset I$, so $\int_I f = \int_D \hat{f}$. We now have $f : D \to \mathbb{R}$ is integrable. From §A3, the boundary of a generalized rectangle has Jordan content 0. Therefore, $\int_I f = \int_D f + \int_{\partial D} = \int_D f$.

## 3 A function that is constant on a subset

Let $g : \mathbb{R}^n \to \mathbb{R}$.

### 3.1

Assume $g$ is constant on an open set $\mathcal{O} \subset \mathbb{R}^n$. Prove that $g : \mathbb{R}^n \to \mathbb{R}$ is continuous at all $\mathbf{x} \in \mathcal{O}$.

Pick an $\epsilon > 0$. Because $g$ is constant on $\mathcal{O}$, $g(\mathbf{x}) - g(\mathbf{y}) = 0 < \epsilon$ for all $\mathbf{x}, \mathbf{y} \in \mathcal{O}$. Because $|g(\mathbf{x}) - g(\mathbf{y})| < \epsilon$ is always true, $\exists \delta > 0$ such that $|\mathbf{x} - \mathbf{y}| < \delta \implies |g(\mathbf{x}) - g(\mathbf{y})| < \epsilon$, and $g$ is continuous for all $\mathbf{x} \in \mathcal{O}$.

### 3.2

Now, assume $g : \mathbb{R}^n \to \mathbb{R}$ is constant on an *arbitrary set* $\mathcal{O}$ *in* $\mathbb{R}^n$. Is $g$ necessarily continuous for all $\mathbf{x} \in \mathcal{O}$? Either prove this or provide a counterexample.

Let $U = (0,1) \times [-\infty, \infty] \cup (2,3) \times [-\infty, \infty]$. A constant function on this set is not continuous, because there is a gap from $x = 1$ and $x = 2$.

## 4 Closure and interior of a set of Jordan content 0

Let $S \subset \mathbb{R}^n$ have Jordan content 0.

### 4.1

Prove that the closure of $S$, $\mathrm{cl}(S)$, has Jordan content 0.

The boundary of a set with Jordan Content 0 has Jordan Content 0, and $\mathrm{cl}(S) = S \cup \partial S$. The union of finitely many sets with Jordan content 0 has Jordan content 0, so $\mathrm{cl}(S)$ has Jordan content 0.

### 4.2

Prove that $\mathrm{int}(S) = \varnothing$.

$S$ has Jordan content 0, so by §5, $S$ has volume 0. $\mathrm{vol}(S) = 0 \implies \mathrm{int}(S) = \varnothing$. $\square$

## 5 Jordan content 0 versus volume 0

Let $A$ be a bounded subset of $\mathbb{R}^n$. Prove that $A$ has Jordan content zero if and only if $A$ has volume and $\mathrm{vol}(A) = 0$.

Assume $A$ has Jordan content 0. That means there exists a finite set of rectangles $R_i$ such that $A \subset \cup R_i$ and $\mathrm{vol}(\cup R_i) < \epsilon)$ for all $\epsilon > 0$. Hence $\mathrm{vol}(A)$ ¡ $\epsilon$ for all $\epsilon \implies \mathrm{vol}(A) = 0$. $\square$

Assume $A$ has volume 0. By definition, $\mathrm{vol}(A) = \int_A 1_A = \int_I \hat{1}_A$, where $1_A : A \to \mathbb{R}$ is constant with value 1 on $A$. Let $f = \hat{1}_A$. By hypothesis, $\int_I f$ is integrable and $\int_I f = 0$. Assume $\epsilon > 0$. Because $f$ is integrable on $I$, by Riemann, there must exist a partition $P$ of $I$ such that $\mathrm{Osc}(f, P) < \epsilon \implies \sum_{J \in P}(M_J(f) - m_J(f))\mathrm{vol}(J) < \epsilon$. This defines a summation of finitely many generalized rectangles that cover $A$ with volume less than $\epsilon$. Therefore $A$ has Jordan content 0. $\square$