Differentiability

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1 A1

Define $f(x, y, z) = x^2 + y^2 + z^2$ for $(x, y, z) \in \mathbb{R}^3$. Find the affine function that is a first-order approximation to the function $f : \mathbb{R}^3 \to \mathbb{R}$ at the point (0,0,0).

It is easy to see that $f_x = 2x, f_y = 2y$, and $f_z = 2z$. Now it is clear that the first-order approximation to f is trivial.

$$f_{approx}(0, 0, 0) = f(0, 0, 0) + f_x(0, 0, 0)x + f_y(0, 0, 0)y + f_z(0, 0, 0)z = 0 + 0 \cdot x + 0 \cdot y + 0 \cdot z = 0$$

2 A2

Suppose $f : \mathbb{R}^2 \to \mathbb{R}$ and $g : \mathbb{R}^2 \to \mathbb{R}$ are continuously differentiable. Find the conditions for these functions to be first order approximations of each other at the point (0,0).

If f and g are first-order approximations of each other at the point (0,0) then we have the following restrictions on f and g:

$$\lim_{(x,y)\to(0,0)} \frac{f(x, y) - g(x, y)}{\sqrt{x^2 + y^2}} = 0$$

$$\lim_{(x,y)\to(0,0)} \frac{f(x, y) - (f(0, 0) + f_x(0, 0)x + f_y(0, 0)y)}{\sqrt{x^2 + y^2}} = 0$$

$$\lim_{(x,y)\to(0,0)} \frac{g(x, y) - (g(0, 0) + g_x(0, 0)x + g_y(0, 0)y)}{\sqrt{x^2 + y^2}} = 0$$

Subtracting the second and third equations yields:

$$\lim_{(x,y)\to(0,0)} \left(\frac{f(x, y) - g(x, y)}{\sqrt{x^2 + y^2}} + \frac{(g(0, 0) - f(0, 0)) + (g_x(0, 0) - f_x(0, 0))x + (g_y(0, 0) - f_y(0, 0))y}{\sqrt{x^2 + y^2}} \right) = 0$$

The limit of the first term in the sum is 0, so the limit of the second term in the sum must also equal 0. Any limit in the form of the second term only equals 0 if the numerator is 0, so $f(0, 0) = g(0, 0)$, $f_x(0, 0) = g_x(0, 0)$, and $f_y(0, 0) = g_y(0, 0)$.

3 A3

Define $f : \mathbb{R}^2 \to \mathbb{R}$:

$$f(x,y) = \begin{cases} \sin\left(\frac{y^2}{x}\right) \cdot \sqrt{x^2+y^2} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

3.1

Show that $f : \mathbb{R}^2 \to \mathbb{R}$ is continuous at the point (0,0) and has directional derivatives in every direction at the point (0,0).

To show that f is continuous at (0,0), we will show that $\lim_{(x,y)\to(0,0)} f(x,y) = 0$. We will use the fact that $-1 \leq \sin\left(\frac{y^2}{x}\right) \leq 1$.

$$\lim_{(x,y)\to(0,0)} -\sqrt{x^2+y^2} \leq \lim_{(x,y)\to(0,0)} f(x,y) \leq \lim_{(x,y)\to(0,0)} \sqrt{x^2+y^2} \implies 0 \leq \lim_{(x,y)\to(0,0)} f(x,y) \leq 0$$

This means the limit is 0 and f is continuous at the point (0,0). f has directional derivatives in all directions at (0,0) if $\lim_{t\to 0} \frac{1}{t}(f(ct,dt) - f(0,0))$ exists for all vectors $\langle c,d \rangle$. The directional derivative in any direction $\langle 0,d \rangle = 0$, so we will only consider vectors $\langle c,d \rangle$ with $c \neq 0$:

$$\lim_{t\to 0} \frac{\sin\left(\frac{d^2t^2}{ct}\right) \cdot \sqrt{c^2t^2+d^2t^2}}{t} = \lim_{t\to 0} \sin\left(\frac{d^2t}{c}\right) \cdot \sqrt{c^2+d^2} = 0$$

It follows that the directional derivative of f in any arbitrary direction $\langle c,d \rangle$ at (0,0) is 0. $\square$

3.2

Show that there is no plane that is tangent to the graph of $f : \mathbb{R}^2 \to \mathbb{R}$ at the point (0,0,$f(0,0)$).

Let $\phi(x,y) = ax + by$ be a tangent plane of f at (0,0). By definition, we have:

$$\lim_{(x,y)\to(0,0)} \frac{f(x,y) - ax - by}{\sqrt{x^2+y^2}} = 0$$

But we know any limit of the form $\lim_{(x,y)\to(0,0)} \frac{c+ax+by}{\sqrt{x^2+y^2}}$ equals 0 only if $c = a = b = 0$. It follows that $\phi(x,y) = 0$ and ϕ does not describe a tangent plane. $\square$

4 A4

Suppose that the continuous function $f : \mathbb{R}^2 \to \mathbb{R}$ has a tangent plane at the point (0,0,$f(0,0)$). Prove that $f : \mathbb{R}^2 \to \mathbb{R}$ has directional derivatives in all directions at the point (0,0).

The graph of the tangent plane of f at (0,0) has the form $\phi(x, y) = ax + by + f(0, 0)$. We know ϕ then has the following property:

$$\lim_{(x,y)\to(0,0)} \frac{f(x, y) - \phi(x, y)}{\sqrt{x^2 + y^2}} = 0$$

We can approach (0,0) along any path and the limit will always be 0. We choose the path $y = \frac{d}{c}x$ and substitute.

$$\lim_{x\to 0} \frac{f(x, \frac{d}{c}x) - (a + \frac{bd}{c})x - f(0, 0)}{\sqrt{x^2 + \frac{d^2}{c^2}x^2}} = \lim_{x\to 0} \frac{f(x, \frac{d}{c}x) - f(0, 0)}{\frac{x}{c}\sqrt{c^2 + d^2}} - \frac{ac + bd}{\sqrt{c^2 + d^2}} = 0$$

Rearrange and set $t = \frac{x}{c}$:

$$ac + bd = \lim_{x\to 0} \frac{f(x, \frac{d}{c}x) - f(0, 0)}{\frac{x}{c}} = \lim_{t\to 0} \frac{f(ct, dt) - f(0, 0)}{t}$$

Thus the directional derivative in the arbitrary direction $\langle c, d\rangle$ at (0,0) is $ac + bd$. $\square$

5 A5

Consider the following assertions for a function $f : \mathbb{R}^2 \to \mathbb{R}$:
a. The function $f : \mathbb{R}^2 \to \mathbb{R}$ is continuously differentiable.
b. The function $f : \mathbb{R}^2 \to \mathbb{R}$ has directional derivatives in all directions in each point in $\mathbb{R}^2$.
c. The function $f : \mathbb{R}^2 \to \mathbb{R}$ has first-order partial derivatives at each point in $\mathbb{R}^2$.
Explain the implications among these assertions.

Implication 1: A implies B. This is asserted by the Directional Derivative Theorem[1].
Implication 2: B implies A. This is also asserted by the Directional Derivative Theorem.
Implication 3: A implies C. This is the most obvious implication. If a function is continuously differentiable, we can always find the first-order partials.
Implication 4: C does not imply A. Consider the following function:

$$f(x, y) = \begin{cases} \frac{x^2y}{x^2+y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$$

f is clearly has first order partials at each point in $\mathbb{R}^2$, but f is not continuous at (0,0) and it follows that f is not continuously differentiable.
Implication 5: B implies C. This too is asserted by the Directional Derivative Theorem.
Implication 6: C does not imply B. Suppose $f_x = 0$ and $f_y = 0$. We cannot find a directional derivative in any direction that is not an axis.

[1]Directional Derivative Theorem: Let $\mathcal{O}$ be an open subset of $\mathbb{R}^n$ and suppose that the function $f : \mathcal{O} \to \mathbb{R}$ is continuously differentiable. Then for each point $\mathbf{x}$ in $\mathcal{O}$ and each nonzero point $\mathbf{p}$ in $\mathbb{R}^n$, the function $f : \mathcal{O} \to \mathbb{R}$ has a directional derivative in the direction $\mathbf{p}$ at the point $\mathbf{x}$ that is given by the formula: $\frac{\partial f}{\partial \mathbf{p}}(\mathbf{x}) = \sum_{i=i}^{n} p_i \frac{\partial f}{\partial x_i}(\mathbf{x})$.

6 B1

Suppose that the mapping $\mathbf{F}\colon \mathbb{R}^n \to \mathbb{R}^m$ is continuously differentiable and that the derivative matrix $\mathbf{DF}(\mathbf{x})$ at each point $\mathbf{x}$ in $\mathbb{R}^n$ has all its entries equal to 0. Prove that the mapping $\mathbf{F}\colon \mathbb{R}^n \to \mathbb{R}^m$ is constant.

For any $f_i \in \mathbf{F}$, we know $f_{i_{x_1}} = f_{i_{x_2}} = ... = f_{i_{x_n}} = 0$. Let $\mathbf{u}$ and $\mathbf{v}$ be points in $\mathbb{R}^n$. By the Mean Value Theorem, there exists a value θ such that $f_i(\mathbf{u}) - f_i(\mathbf{v}) = f_{i_{x_1}}(\theta, \theta, ..., \theta) + f_{i_{x_2}}(\theta, \theta, ..., \theta) + ... + f_{i_{x_n}}(\theta, \theta, ..., \theta)$. But every first-order partial of f_i is 0, so $f_i(\mathbf{u}) - f_i(\mathbf{v}) = \sum_{k=1}^{n} 0 = 0$, so $f_i(\mathbf{u}) = f_i(\mathbf{v}) \implies f_i$ is constant. As every component of $\mathbf{F}$ is constant, $\mathbf{F}$ is constant. $\square$

7 B2

Suppose that the mapping $\mathbf{F}\colon \mathbb{R}^n \to \mathbb{R}^m$ is continuously differentiable and that there is a fixed $m \times n$ matrix $\mathbf{A}$ so that $\mathbf{DF}(\mathbf{x}) = \mathbf{A}$ for every $\mathbf{x}$ in $\mathbb{R}^n$. Prove that there is some $\mathbf{c}$ in $\mathbb{R}^m$ so that $\mathbf{F}(\mathbf{x}) = \mathbf{Ax} + \mathbf{c}$ for every $\mathbf{x}$ in $\mathbb{R}^n$. Restate the result for the case when $n = m = 1$.

Define a mapping $\mathbf{G}\colon \mathbb{R}^n \to \mathbb{R}^m$ by $\mathbf{G}(\mathbf{x}) = \mathbf{F}(\mathbf{x})$ - $\mathbf{Ax}$. $\mathbf{DG}(\mathbf{x}) = \mathbf{A}$ - $\mathbf{A} = 0$ and $\mathbf{G}(\mathbf{x})$ is constant. We can then substitute $\mathbf{G}(\mathbf{x})$ for some constant vector $\mathbf{c}$, which yields the equality $\mathbf{c} = \mathbf{F}(\mathbf{x})$ - $\mathbf{Ax} \implies \mathbf{F}(\mathbf{x}) = \mathbf{Ax} + \mathbf{c}$. $\square$

If $n = m = 1$, $f : \mathbb{R} \to \mathbb{R}$ is a single-variable function. The above result translated in this simpler context is that if $f'(x) = a$, then $f(x) = ax + c$ for some $a, c \in \mathbb{R}$.

8 B3

Suppose that the function $h : \mathbb{R}^3 \to \mathbb{R}$ is continuously differentiable. Define the function $\eta : \mathbb{R}^3 \to \mathbb{R}$ by $\eta(u, v, w) = (3u + 2v)h(u^2, v^2, uvw)$ for (u, v, w) in $\mathbb{R}^3$. Find $D_2\eta(u, v, w)$.

First we notice that we must use the Product Rule.

$$\frac{\partial \eta}{\partial v} = \frac{\partial}{\partial v}(3u + 2v) \cdot h(u^2, v^2, uvw) + \frac{\partial h}{\partial v} \cdot (3u + 2v) = 2v \cdot h(u^2, v^2, uvw) + \frac{\partial h}{\partial v} \cdot (3u + 2v)$$

Let $x = u^2, y = v^2, z = uvw$. We now use the Chain Rule.

$$\frac{\partial h}{\partial v} = \frac{\partial h}{\partial x} \cdot \frac{\partial x}{\partial v} + \frac{\partial h}{\partial y} \cdot \frac{\partial y}{\partial v} + \frac{\partial h}{\partial z} \cdot \frac{\partial z}{\partial v} = \frac{\partial h}{\partial x} \cdot 0 + \frac{\partial h}{\partial y} \cdot 2v + \frac{\partial h}{\partial z} \cdot uw$$

Lastly we substitute into our original expression.

$$\frac{\partial \eta}{\partial v} = 2v \cdot h(u^2, v^2, uvw) + (D_2h(u^2, v^2, uvw)2v + D_3h(u^2, v^2, uvw)uw)(3u + 2v)$$

9 B4

In this problem, we introduce an important definition:
Definition 1. Let $\mathcal{O} \subset \mathbb{R}^n$ be open and let $F : \mathcal{O} \to \mathbb{R}^n$. Let $\mathbf{x} \in \mathcal{O}$. Then F is **differentiable** at $\mathbf{x}$ if there is an $m \times n$ matrix A such that:

$$\lim_{\mathbf{h} \to \vec{0}} \frac{\|F(\mathbf{x} + \mathbf{h}) - [F(\mathbf{x}) + A\mathbf{h}]\|}{\|\mathbf{h}\|} = \lim_{\mathbf{u} \to \mathbf{x}} \frac{\|F(\mathbf{u}) - [F(\mathbf{x}) + A(\mathbf{u} - \mathbf{x})]\|}{\|\mathbf{u} - \mathbf{x}\|} = 0$$

Use this definition to show that if F is differentiable at $\mathbf{x}$ then F is continuous at $\mathbf{x}$.

F is continuous at $\mathbf{x}$ if $\lim_{\mathbf{h}\to\vec{0}} F(\mathbf{x}+\mathbf{h}) = F(\mathbf{x})$. We note that:

$$\lim_{\mathbf{h}\to\vec{0}} \frac{\|F(\mathbf{x}+\mathbf{h}) - [F(\mathbf{x}) + A\mathbf{h}]\|}{\|\mathbf{h}\|} = 0 \implies \lim_{\mathbf{h}\to\vec{0}} \frac{F(\mathbf{x}+\mathbf{h}) - [F(\mathbf{x}) + A\mathbf{h}]}{\|\mathbf{h}\|} = \vec{0}$$

Now we write $F(\mathbf{x}+\mathbf{h})$ as follows:

$$F(\mathbf{x}+\mathbf{h}) = \frac{F(\mathbf{x}+\mathbf{h}) - [F(\mathbf{x}) + A\mathbf{h}]}{\|\mathbf{h}\|} \cdot \|\mathbf{h}\| + [F(\mathbf{x}) + A\mathbf{h}]$$

We take the limit as $\mathbf{h} \to \vec{0}$:

$$\lim_{\mathbf{h}\to\vec{0}} \left(\frac{F(\mathbf{x}+\mathbf{h}) - [F(\mathbf{x}) + A\mathbf{h}]}{\|\mathbf{h}\|} \cdot \|\mathbf{h}\| + [F(\mathbf{x}) + A\mathbf{h}] \right) = \vec{0} \cdot 0 + F(\mathbf{x}) + A \cdot \vec{0} = F(\mathbf{x})$$

It directly follows that F is continuous at $\mathbf{x}$. □

10 B5

Let g and h be continuously differentiable functions and let $c > 0$.
Define $u(x,t) = g(x-ct) + h(x+ct)$. Prove that u satisfies the wave equation: $c^2 u_{xx} = u_{tt}$.

First we differentiate with respect to x:

$$u_x = g'(x-ct) + h'(x+ct)$$

$$u_{xx} = g''(x-ct) + h''(x+ct)$$

Now we differentiate with respect to t:

$$u_t = -c \cdot g'(x-ct) + c \cdot h'(x+ct)$$

$$u_{tt} = c^2 \cdot g''(x-ct) + c^2 \cdot h''(x+ct) = c^2(g''(x-ct) + h''(x+ct))$$

It is then clear that u satisfies the wave equation. □