Implicit Functions

Jay Waddell

February 20, 2020

1 A1

Suppose that the continuously differentiable function $f : \mathbb{R} \to \mathbb{R}$ has the property that there is some positive number c such that $f'(x) \geq c$ for every $x \in \mathbb{R}$. Show that f is bijective.

Because c is positive, $f'(x) \geq c \implies f'(x) \neq 0$ for all $x \in \mathbb{R}$. Therefore we can apply the Inverse Function Theorem[1] at every point on f, and it follows that f is a bijection. $\square$

2 A2

For the mapping $\mathbf{F}$: $\mathbb{R}^2 \to \mathbb{R}^2$, $\mathbf{F}(x, y) = (x + x^2 + e^{x^2y^2}, -x + y + \sin(xy))$, apply the Inverse Function Theorem at the point $(x_0, y_0) = (0, 0)$ and calculate the partial derivatives of the components of the inverse mapping at the point $(u_0, v_0) = \mathbf{F}$(0,0).

We calculate the derivative matrix of $\mathbf{F}$ at (0,0).

$$\mathbf{DF}(x, y) = \begin{pmatrix} 1 + 2x + 2xy^2e^{x^2y^2} & 2x^2ye^{x^2y^2} \\ -1 + y\cos(xy) & 1 + x\cos(xy) \end{pmatrix} \implies \mathbf{DF}(0, 0) = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$$

Now we use Cramer's Rule[2] to find $\mathbf{DF}^{-1}$(0,0). Note that det($\mathbf{DF}$(0,0)) = $(1 \cdot 1) - (0 \cdot -1) = 1$.

$$\mathbf{DF}^{-1}(0, 0) = \frac{1}{1}\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

The partials at (u_0, v_0) are simply just the entries of $\mathbf{DF}^{-1}(0, 0)$.

[1] Inverse Function Theorem: Let $\mathcal{O}$ be an open subset of $\mathbb{R}^n$ and suppose that the mapping $\mathbf{F}$: $\mathcal{O} \to \mathcal{R}^n$ is continuously differentiable. Let x_* be a point in $\mathcal{O}$ at which the derivative matrix $\mathbf{DF}(x_*)$ is invertible. Then there is a neighborhood U of the point x_* and a neighborhood V of its image $\mathbf{F}(x_*)$ such that the mapping $\mathbf{F}$: $U \to V$ is one-to-one and onto. Moreover, the inverse mapping $\mathbf{F}^{-1}$: $V \to U$ is also continuously differentiable, and for a point $\mathbf{y}$ in V, if $\mathbf{x}$ is the point in U at which $\mathbf{F}(\mathbf{x}) = \mathbf{y}$, then $\mathbf{DF}^{-1}(\mathbf{y})=[\mathbf{DF}(\mathbf{x})]^{-1}$.

[2] Cramer's Rule: An $n \times n$ matrix $\mathbf{A}$ is invertible if and only if $\det(\mathbf{A}) \neq 0$. Moreover, if $\det(\mathbf{A}) \neq 0$, then $(\mathbf{A}^{-1})_{ij}=\frac{1}{\det(\mathbf{A})}((-1)^{i+j}\det(\mathbf{A})^{ji})$.

3 A3

Consider the equation $f(x, y) = (x^2 + y^2 - 2)(x^2 - y^2) = 0$, (x, y) in $\mathbb{R}^2$.

3.1

Compute partial derivatives to show that $\nabla f(x, y) = \mathbf{0}$ and hence that the assumptions of Dini's Theorem[3] do not hold at each of the following solutions: (0,0), (1,1), (1,-1), (-1,-1), (-1,1).

We first rewrite f as $f(x, y) = x^4 + x^2y^2 - 2x^2 - x^2y^2 - y^4 + 2y^2 = x^4 - y^4 - 2x^2 + 2y^2$. Now we calculate the partials:

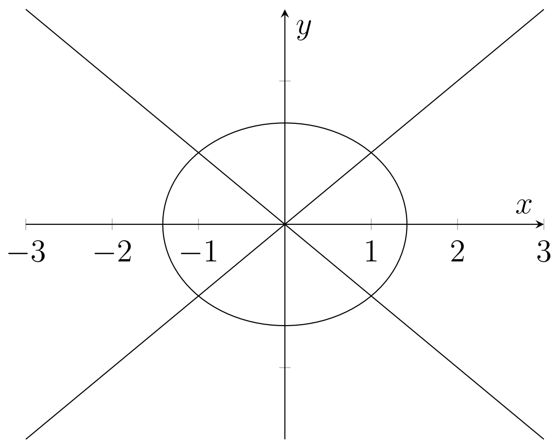
$$\frac{\partial f}{\partial x} = 4x^3 - 4x = 4x(x^2 - 1), \quad \frac{\partial f}{\partial y} = -4y^3 + 4y = -4y(y^2 - 1)$$

Hence $f_x = 0$ and $f_y = 0$ only when $x, y \in$ (-1,0,1). However, the only (x, y) pairs composed of elements in the aforementioned set that also satisfy $f(x, y) = 0$ are the 5 listed above.

3.2

By graphing the set of solutions of this equation, show that the conclusions of Dini's Theorem do not hold at each of the solutions listed above.

The graph of f is shown below:



For sake of simplicity, consider the failure point (0,0). $f(0, 0 + h) = 0$ has solutions for two distinct positive values of h, and it follows that any function g that satisfies $f(x, g(x)) = 0$ in the neighborhood of any failure point of f must not be well defined.

[3]Dini's Theorem: Let $\mathcal{O}$ be an open subset of the plane $\mathbb{R}^2$ and suppose that the function $f : \mathcal{O} \to \mathbb{R}$ is continuously differentiable. Let (x_0, y_0) be a point in $\mathcal{O}$ at which $f(x_0, y_0) = 0$ and $f_y(x_0, y_0) \neq 0$. Then there is a positive number r and a continuously differentiable function $g : I \to \mathbb{R}$, where I is the open interval $(x_0 - r, x_0 + r)$, such that $f(x, g(x)) = 0$ for all $x \in I$ and whenever $|x - x_0| < r, |y - y_0| < r$, and $f(x, y) = 0$, then $y = g(x)$. Moreover, $f_x(x, g(x)) + f_y(x, g(x))g'(x) = 0$ for all $x \in I$.

4 A4

Let $f : \mathbb{R}^2 \to \mathbb{R}$ be continuously differentiable ($f \in C^1(\mathbb{R}^2)$). Let $g : \mathbb{R} \to \mathbb{R}$ be continuously differentiable ($g \in C^1(\mathbb{R})$). Assume $\forall x \in \mathbb{R}, f(x, g(x)) = 0$.

4.1

Calculate $\frac{d}{dx} f(x, g(x))$ using the Chain Rule.

$$\frac{d}{dx} f(x, g(x)) = D_1 \cdot \frac{d}{dx} x + D_2 \cdot \frac{d}{dx} g(x) = \frac{\partial f}{\partial x}(x, g(x)) + \frac{\partial f}{\partial y}(x, g(x)) g'(x)$$

4.2

Assume $\frac{\partial f}{\partial y}(x_0, g(x_0)) \neq 0$ for some $x_0 \in \mathbb{R}$. Calculate $g'(x_0)$.

$$\frac{\partial f}{\partial x}(x_0, g(x_0)) + \frac{\partial f}{\partial y}(x_0, g(x_0)) g'(x_0) = 0 \implies g'(x_0) = -\frac{\frac{\partial f}{\partial x}(x_0, g(x_0))}{\frac{\partial f}{\partial y}(x_0, g(x_0))}$$

5 A5

Let A be a 2×4 matrix. Consider the linear equations $A\mathbf{x} = \mathbf{0}$. Assume A has rank 2.

5.1

Use the Implicit Function Theorem[4] to prove that $A\mathbf{x} = \mathbf{0}$ has an infinite number of solutions.

Let B be an invertible 2×2 submatrix of A. $\mathbf{D_x}(\mathbf{Bx}) = B$, so $A\mathbf{x}$ has an invertible 2×2 partial derivative submatrix. Therefore we can apply the Implicit Function Theorem to assert that two components of the solution can be expressed in terms of the other two, and therefore $A\mathbf{x} = \mathbf{0}$ has an infinite number of solutions. $\square$

5.2

Use results from linear algebra to prove that $A\mathbf{x} = \mathbf{0}$ has an infinite number of solutions.

Let B be an invertible 2×2 submatrix of A. We can multiply B to both sides of our equation to yield a new equality.

$$A\mathbf{x} = \mathbf{0} \implies B \cdot A\mathbf{x} = B \cdot \mathbf{0} = \mathbf{0} \implies \begin{pmatrix} 1 & 0 & a & b \\ 0 & 1 & c & d \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

This gives $x_1 = -ax_3 - bx_4$ and $x_2 = -cx_3 - dx_4$. As x_1 and x_2 are functions of x_3 and x_4, the equation $A\mathbf{x} = \mathbf{0}$ has an infinite number of solutions. $\square$

---

[4] Implicit Function Theorem: Let n and k be positive integers, let $\mathcal{O}$ be an open subset of $\mathbb{R}^{n+k}$, and suppose that the mapping $\mathbf{F} : \mathcal{O} \to \mathbb{R}^k$ is continuously differentiable. At the point $(\mathbf{x}_0, \mathbf{y}_0)$ in $\mathcal{O}$, suppose that $\mathbf{F}(\mathbf{x}_0, \mathbf{y}_0) = \mathbf{0}$ and that the $k \times k$ partial derivative matrix $\mathbf{D_y F}(\mathbf{x}_0, \mathbf{y}_0)$ is invertible. Then there is a positive number r and a continuously differentiable mapping $\mathbf{G} : \mathcal{B} \to \mathbb{R}^k$, where $\mathcal{B} = \mathcal{B}_r(\mathbf{x}_0)$, such that $\mathbf{F}(\mathbf{x}, \mathbf{G}(\mathbf{x})) = \mathbf{0}$ for all points $\mathbf{x}$ in $\mathcal{B}$, and whenever $\|\mathbf{x} - \mathbf{x}_0\| < r$, $\|\mathbf{y} - \mathbf{y}_0\| < r$, and $\mathbf{F}(\mathbf{x}, \mathbf{y}) = \mathbf{0}$, then $\mathbf{y} = \mathbf{G}(\mathbf{x})$. Moreover, $\mathbf{D_x F}(\mathbf{x}, \mathbf{G}(\mathbf{x})) + \mathbf{D_y F}(\mathbf{x}, \mathbf{G}(\mathbf{x})) \cdot \mathbf{DG}(\mathbf{x}) = \mathbf{0}$ for all points $\mathbf{x}$ in $\mathcal{B}$.

6 B1

Consider the following system of equations:

$$(uv)^4 + (u+s)^3 + t = 0, \quad \sin(uv) + e^{v+t^2} - 1 = 0, \quad (u,v,s,t) \in \mathbb{R}^4$$

Use the Implicit Function Theorem to analyze the solutions near the solution $\mathbf{0}$.

Define $\mathbf{F} : \mathbb{R}^4 \to \mathbb{R}^2$ by $\mathbf{F}(u,v,s,t) = ((uv)^4 + (u+s)^3 + t,\ \sin(uv) + e^{v+t^2} - 1)$. We now calculate the derivative matrix $\mathbf{DF}$ at the point (0,0,0,0).

$$\mathbf{DF}(u,v,s,t) = \begin{pmatrix} 4u^3v^4 + 3u^2 + 6us + 3s^2 & 4u^4v^3 & 3s^2 + 6us + 3u^2 & 1 \\ vcos(uv) & ucos(uv) + e^{v+t^2} & 0 & 2te^{v+t^2} \end{pmatrix}$$

$$\implies \mathbf{DF}(0,0,0,0) = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

It follows that the 2×2 submatrix of $\mathbf{DF}(\mathbf{0})$ given by the v and t columns is invertible. We apply the Implicit Function Theorem to pick a positive number r and continuously differentiable functions $g : \mathcal{B} \to \mathbb{R}$, and $h : \mathcal{B} \to \mathbb{R}$ where $\mathcal{B} = \mathcal{B}_r(0,0)$, such that if $u^2 + s^2 < r^2$, then $(u, g(u,s), s, h(u,s))$ is a solution.

7 B2

Consider the equation:

$$e^{xy} + x^2 + 2y - 1 = 0, \quad (x,y) \in \mathbb{R}$$

Use the Implicit Function Theorem to analyze the solutions near the solution $\mathbf{0}$.

Define $f : \mathbb{R}^2 \to \mathbb{R}$ by $f(x,y) = e^{xy} + x^2 + 2y - 1$. We take the derivative with respect to y and evaluate at (0,0).

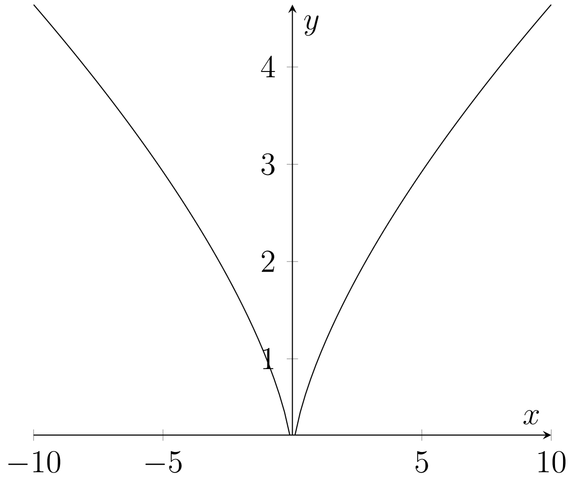
$$\frac{\partial f}{\partial y}(x,y) = xe^{xy} + 2 \implies \frac{\partial f}{\partial y}(0,0) = 0e^0 + 2 = 2 \neq 0$$

We apply the Implicit Function Theorem to find a continuously differentiable function $g : I \to \mathbb{R}$ where I is an open interval containing 0, such that if $x \in I, (x, g(x))$ is a solution.

8 B3

Graph the solutions of $y^3 - x^2 = 0$, $(x, y) \in \mathbb{R}^2$. Does the Implicit Function Theorem apply at the point (0,0)? Does this equation define one of the components of a solution (x,y) as a function of the other component?

Define $f : \mathbb{R}^2 \to \mathbb{R}$ by $f(x, y) = y^3 - x^2$. We notice that all first-order partial derivatives of f evaluated at the point (0,0) are 0, thus the Implicit Function Theorem does not apply at this solution point. A graph of $f(x, y) = 0$ is shown below.



As we can see, $y^3 - x^2 = 0 \iff y = x^{\frac{2}{3}}$. In other words, we can explicitly find a solution for one component of the equation in terms of the other.

9 B4

We will write points in $\mathbb{R}^{n+k}$ as $(\mathbf{x},\mathbf{y}) = (x_1, x_2, ..., x_n, y_1, y_2, ..., y_k)$. Let $\mathbf{F}$: $\mathbb{R}^{n+k} \to \mathbb{R}^n$ be continuously differentiable, $\mathbf{F} \in C^1(\mathbb{R}^{n+k}, \mathbb{R}^n)$, and assume $\mathbf{F}(\mathbf{x}_0,\ \mathbf{y}_0) = \mathbf{0}$. Assume the partial derivative matrix $\mathbf{D_y F}(\mathbf{x}_0,\ \mathbf{y}_0)$ is invertible. Prove that there are an infinite number of solutions $(\mathbf{x},\mathbf{y}) \in \mathbb{R}^{n+k}$ to $\mathbf{F}(\mathbf{x},\mathbf{y}) = \mathbf{0}$.

We can apply the Implicit Function Theorem to $\mathbf{F}$ because $\mathbf{DF}(\mathbf{x}_0,\ \mathbf{y}_0)$ has an invertible submatrix, namely $\mathbf{D_y F}(\mathbf{x}_0,\ \mathbf{y}_0)$. This means k components of a solution of $\mathbf{F}(\mathbf{x},\mathbf{y}) = \mathbf{0}$ can be written in terms of the other n components, and therefore there are infinitely many solutions to this equation. $\square$

10 B5

10.1

Let $\mathbf{F}$: $\mathbb{R}^2 \to \mathbb{R}^2$ be continuously differentiable, $\mathbf{F} \in C^1(\mathbb{R}^2, \mathbb{R}^2)$. Assume $\mathbf{DF}(0{,}0) = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$. Prove that $\mathbf{F}$ is invertible near (0,0).

First we check that the determinant of $\mathbf{DF}(0{,}0)$ is nonzero: $\det(\mathbf{DF}(0{,}0)) = 1{\cdot}4 - 2{\cdot}3 = -2 \neq 0$. The conditions for the Inverse Function Theorem are met and $\mathbf{F}$ is invertible near (0,0). $\square$

10.2

Find $\mathbf{DF}^{\text{-}1}(\mathbf{F}(0{,}0))$.

We use Cramer's Rule to calculate the inverse of $\mathbf{DF}(0{,}0)$.

$$\mathbf{DF}^{\text{-}1}(\mathbf{F}(0,0)) = \frac{1}{-2}\begin{pmatrix} 4 & -2 \\ -3 & 1 \end{pmatrix} = \begin{pmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{pmatrix}$$

.

10.3

Let $\mathbf{F}$: $\mathbb{R}^2 \to \mathbb{R}^3$ be continuously differentiable, $\mathbf{F} \in C^1(\mathbb{R}^2, \mathbb{R}^3)$. Assume $\mathbf{DF}(0{,}0) = \begin{pmatrix} 1 & 2 \\ 4 & 6 \\ 3 & 4 \end{pmatrix}$. Prove that $\mathbf{F}$ is injective near (0,0).

From part 5.1, we can see that the submatrix of $\mathbf{DF}(0{,}0)$ given by $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ is invertible. By the Inverse Function Theorem, $\begin{bmatrix} f_1 \\ f_3 \end{bmatrix}$: $\mathbb{R}^2 \to \mathbb{R}^2$ is bijective near (0,0). Note that the composition of any two injective functions g and h is injective.

$$(g \circ h)(x) = (g \circ h)(y) \iff h(x) = h(y) \iff x = y$$

Consider the map $\mathbf{G} : \mathbb{R}^2 \to \mathbb{R}^3$ by $\mathbf{G}(\mathbf{x}) = \mathbf{x}$. $\mathbf{G}$ is clearly injective. From what is shown above, it directly follows that $\begin{bmatrix} f_1 \\ f_3 \end{bmatrix} \circ \mathbf{G}$ is injective near (0,0), or equivalently, $f_1 : \mathbb{R}^2 \to \mathbb{R}^3$ and $f_3 : \mathbb{R}^2 \to \mathbb{R}^3$ are injective near (0,0). Because at least one of the components of $\mathbf{F}$ is injective near (0,0), $\mathbf{F}$ is injective near (0,0). $\square$