Measure and Infinite Sums

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1 L^2 space

Let f be defined by the Fourier series $f(x) = \sum_{n=1}^{\infty} \frac{1}{n^2} \cos(nx)$.

1.1

Prove that this series f converges uniformly to f on $[0, 2\pi]$.

We prove this by showing the limit does not depend on x.

$$\lim_{N\to\infty} |f(x) - f_N(x)| = \lim_{N\to\infty} \left| \sum_{n=1}^{\infty} \frac{1}{n^2} \cos(nx) - \sum_{n=1}^{N} \frac{1}{n^2} \cos(nx) \right| = \lim_{N\to\infty} \left| \sum_{n=N+1}^{\infty} \frac{1}{n^2} \cos(nx) \right|$$

We know that $|\cos(nx)| \leq 1$, so

$$\lim_{N\to\infty} \left| \sum_{n=N+1}^{\infty} \frac{1}{n^2} \cos(nx) \right| \leq \lim_{N\to\infty} \sum_{n=N+1}^{\infty} \frac{1}{n^2}$$

Now by the integral test, the result directly follows.

$$\lim_{N\to\infty} \sum_{n=N+1}^{\infty} \frac{1}{n^2} = \lim_{N\to\infty} \int_{N+1}^{\infty} \frac{dx}{x^2} = \lim_{N\to\infty} \left[\left(-\frac{1}{x} \right) \Bigg|_{N+1}^{\infty} \right] = \lim_{N\to\infty} \frac{1}{N+1} = 0 \quad \square$$

1.2

Show $f \in L^2([0, 2\pi], \mathbb{C})$ and find an expression for $\|f\|_{L^2}$.

Because this series converges, the partial sums are all finite and therefore $f(x) \in L^2([0, 2\pi], \mathbb{C})$. The Fourier series defined has $a_0 = 0, a_n = \frac{1}{n^2}, b_n = 0$. Because two of these coefficients are 0, calculating the norm $\|f\|$ using Parseval's formula is very simple. We have

$$\|f\|_{L^2} = \pi \left(\frac{0}{2} + \sum_{n=1}^{\infty} \left(\left(\frac{1}{n^2} \right)^2 + 0^2 \right) \right) = \sum_{n=1}^{\infty} \frac{\pi}{n^4}$$

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Addendum: from §4.3, we actually know this value is equal to $\frac{\pi^5}{90}$.

2 Gram-Schmidt process

Let $g_0, g_1, g_2, ...$ be linearly independent vectors in an inner product space. Inductively define

$$h_0 = g_0, \phi_0 = \frac{h_0}{\|h_0\|}, h_n = g_n - \sum_{k=0}^{n-1} \langle g_n, \phi_n \rangle \phi_k, \phi_n = \frac{h_n}{\|h_n\|}, ...$$

Show that $\phi_0, \phi_1, \phi_2, ...$ are orthonormal. Why must we assume that the g's are linearly independent?

We can see that $h_1 = g_1 - \langle g_1, \phi_0 \rangle$, and it follows that $\langle h_1, \phi_0 \rangle = 0$. h_1 may not have unit length, so $\langle \phi_1, \phi_0 \rangle = 0$, and therefore orthonormal. Inductively we can apply this identical process to show that $\langle \phi_n, \phi_{n+1} \rangle = 0$. Because each phi_n is orthonormal to the next sequential ϕ_{n+1}, all the $\phi_n s$ are orthonormal. If the g's were not linearly independent, then we would be able to write some ϕs in terms of other ones, and this construction would fall apart.

3 Orthonormal families on $[0, \ell]$

3.1

Suppose $\phi_0(x), \phi_1(x), ...$ are orthonormal functions on $[0, 2\pi]$. Show that the functions

$$\psi_n(x) = \sqrt{\frac{2\pi}{l}} \phi_n \left(\frac{2\pi x}{l} \right)$$

are orthonormal on $[0, l]$.

To show orthogonality for two arbitrary functions ψ_m, ψ_n, we compute an integral.

$$\int_0^l \frac{2\pi}{l} \phi_m \left(\frac{2\pi}{l} x \right) \phi_n \left(\frac{2\pi}{l} x \right) dx$$

But the ϕs are orthogonal, so this integral is 0. Therefore the ψs are also orthogonal.

3.2

Write the family obtained by modifying $\frac{1}{\sqrt{2\pi}}, \frac{\sin(nx)}{\sqrt{\pi}}, \frac{\cos(mx)}{\sqrt{\pi}}$, or, alternatively, $\frac{e^{inx}}{\sqrt{2\pi}}$ to $[0, l]$ as in §3.1. The only change needed is swapping the denominators to $\sqrt{\frac{2\pi}{l}}$ or $\sqrt{\frac{\pi}{l}}$ as opposed to $\sqrt{2\pi}$ or $\sqrt{pi}$.

4 Parseval's theorem

Assume for the moment that the functions $\frac{1}{\sqrt{2\pi}}, \frac{\sin(nx)}{\sqrt{\pi}}, \frac{\cos(mx)}{\sqrt{\pi}}$ are complete on the interval $[0, 2\pi]$.

4.1

Apply this to the function x to show that $x = \pi - 2\sum_{n=1}^{\infty} \frac{\sin(nx)}{n}$.

First we find the coefficient a_0:

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} x dx = \frac{1}{\pi}\left[\frac{x^2}{2}\right]\Bigg|_0^{2\pi} = \frac{1}{\pi}\cdot\left(\frac{4\pi^2}{2} - 0\right) = 2\pi$$

Next we find a_n :

$$a_n = \frac{1}{\pi}\int_0^{2\pi} x\cos(nx)dx = \frac{1}{\pi}\left[\frac{\cos(nx)}{n^2} + \frac{x\sin(nx)}{n}\right]\Bigg|_0^{2\pi} = \frac{1}{\pi}\cdot\left(\frac{1}{n^2} + 0 - \frac{1}{n^2} - 0\right) = \frac{1}{\pi}\cdot 0 = 0$$

Lastly we find b_n :

$$b_n = \frac{1}{\pi}\int_0^{2\pi} x\sin(nx)dx = \frac{1}{\pi}\left[\frac{\sin(nx)}{n^2} - \frac{x\cos(nx)}{n}\right]\Bigg|_0^{2\pi} = \frac{1}{\pi}\left(0 - \frac{2\pi}{n} - 0 + 0\right) = -\frac{2\pi}{n}\cdot\frac{1}{\pi} = -\frac{2}{n}$$

Now we simply plug these coefficients in.

$$f(x) = x = \frac{2\pi}{2} + \sum_{n=1}^{\infty}\left(0\cdot\cos(nx) - \frac{2}{n}\sin(nx)\right) = \pi - 2\sum_{n=1}^{\infty}\frac{\sin(nx)}{n}$$

4.2

Using the Fourier coefficients found in §4.1, apply Parseval's relation to show that $\frac{\pi^2}{6} = \sum_{n=1}^{\infty}\frac{1}{n^2}$.

By Parseval's formula, we have:

$$\frac{(2\pi)^2}{2} + \sum_{n=1}^{\infty}\left(0^2 + \left(-\frac{2}{n}\right)^2\right) = 2\pi^2 + 4\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{1}{\pi}\int_0^{2\pi} x^2 dx$$

We integrate and rearrange terms to get the desired sum.

$$\frac{1}{\pi}\int_0^{2\pi} x^2 dx = \frac{1}{\pi}\left[\frac{x^3}{3}\right]\Bigg|_0^{2\pi} = \frac{1}{\pi}\cdot\left(\frac{8\pi^3}{3} - 0\right) = \frac{8\pi^2}{3}$$

$$\implies \frac{8\pi^2}{3} = 2\pi^2 + 4\sum_{n=1}^{\infty}\frac{1}{n^2}$$

$$\implies \frac{2\pi^2}{3} = 4\sum_{n=1}^{\infty}\frac{1}{n^2}$$

$$\implies \frac{\pi^2}{6} = \sum_{n=1}^{\infty}\frac{1}{n^2}$$

4.3

Use the same procedure on x^2 to get $\frac{\pi^4}{90} = \sum_{n=1}^{\infty}\frac{1}{n^4}$.

We use similar processes as those in §4.1 and §4.2. We start with finding a_0, which is actually an integral we already computed in §4.2.

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} x^2 dx = \frac{8\pi^2}{3}$$

Next we find a_n :

$$a_n = \frac{1}{\pi}\int_0^{2\pi} x^2\cos(nx)dx = \frac{1}{\pi}\left[\frac{2x\cos(nx)}{n^2} + \left(\frac{x^2}{n} - \frac{2}{n^3}\right)\sin(nx)\right]\Bigg|_0^{2\pi} = \frac{1}{\pi}\cdot\left(\frac{4\pi}{n^2} + 0 - 0 - 0\right) = \frac{4}{n^2}$$

The clear next step is to find b_n :

$$b_n = \frac{1}{\pi}\int_0^{2\pi} x^2\sin(nx)dx = \frac{1}{\pi}\left[\frac{2x\sin(nx)}{n^2} + \left(\frac{2}{n^3} - \frac{x^2}{n}\right)\cos(nx)\right]\Bigg|_0^{2\pi}$$

$$= \frac{1}{\pi}\cdot\left(0 + \left(\frac{2}{n^3} - \frac{4\pi^2}{n}\right) - 0 - \left(\frac{2}{n^3} - 0\right)\right) = -\frac{4\pi^2}{n}\cdot\frac{1}{\pi} = -\frac{4\pi}{n}$$

From these coefficients, Parseval's formula gives us the following:

$$\frac{\left(\frac{8\pi^2}{3}\right)^2}{2} + \sum_{n=1}^{\infty}\left(\left(\frac{4}{n^2}\right)^2 + \left(-\frac{4\pi}{n}\right)^2\right) = \frac{32\pi^4}{9} + 16\sum_{n=1}^{\infty}\left(\frac{1}{n^4} + \frac{\pi^2}{n^2}\right) = \frac{1}{\pi}\int_0^{2\pi} x^4 dx$$

We integrate and rearrange just like in §4.2 to get the result.

$$\frac{1}{\pi}\int_0^{2\pi} x^4 dx = \frac{1}{\pi}\left[\frac{x^5}{5}\right]\Bigg|_0^{2\pi} = \frac{1}{\pi}\cdot\frac{32\pi^5}{5} = \frac{32\pi^4}{5}$$

$$\implies \frac{32\pi^4}{5} = \frac{32\pi^4}{9} + 16\sum_{n=1}^{\infty}\left(\frac{1}{n^4} + \frac{\pi^2}{n^2}\right)$$

$$\implies \frac{2\pi^4}{5} = \frac{2\pi^4}{9} + \sum_{n=1}^{\infty}\left(\frac{1}{n^4} + \frac{\pi^2}{n^2}\right)$$

$$\implies \frac{2\pi^4}{5} = \frac{2\pi^4}{9} + \sum_{n=1}^{\infty}\frac{1}{n^4} + \pi^2\sum_{n=1}^{\infty}\frac{1}{n^2}$$

Here we use the result from §4.2.

$$\implies \frac{2\pi^4}{5} = \frac{2\pi^4}{9} + \sum_{n=1}^{\infty}\frac{1}{n^4} + \frac{\pi^4}{6}$$

Note that the lowest common denominator of all three fractions is 90.

$$\implies \sum_{n=1}^{\infty}\frac{1}{n^4} = \frac{36\pi^4}{90} - \frac{20\pi^4}{90} - \frac{15\pi^4}{90} = \frac{\pi^4}{90}$$

5 Trigonometric Fourier series

5.1

Show that the nth partial sum of the trigonometric Fourier series of a (real or complex) function equals the nth partial sum of the exponential series.

We recall the following two formulae:

$$\sin(nx) = \frac{e^{inx} - e^{-inx}}{2i}, \quad \cos(nx) = \frac{e^{inx} + e^{-inx}}{2}$$

Next we recall the general formula for the partial sum of a trigonometric Fourier series:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{N} (a_n \cos(nx) + b_n \sin(nx))$$

Using the equations for sine and cosine, we get:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{N} \left(a_n \frac{e^{inx} + e^{-inx}}{2} + b_n \frac{e^{inx} - e^{-inx}}{2i} \right) = \frac{a_0}{2} + \sum_{n=1}^{N} \left(\left(\frac{a_n}{2} + \frac{b_n}{2i} \right) e^{inx} + \left(\frac{a_n}{2} - \frac{b_n}{2i} \right) e^{-inx} \right)$$

We can see that when splitting the imaginary and real parts of the equation, we arrive at the exponential series.

5.2

Write the corresponding series on $[-\pi, \pi]$. Shown above.

5.3

Show that if, on $[-\pi, \pi]$, f is even (that is, $f(x) = f(-x)$), then in the trigonometric Fourier series, all $b_n = 0$. The series is then called the *cosine series.*

Let g be an even function and let h be an odd function. Let $q(x) = g(x)h(x)$, $q(-x) = g(-x)h(-x)$. Using the properties of even and odd functions, $g(-x)h(-x) = (g(x))(-h(x)) = -g(x)h(x) = -q(x)$. Therefore q is an odd function.

We know that f is even, and $\sin(nx)$ is an odd function for all n. Therefore $f(x)\sin(nx)$ is an arbitrary odd function from above. Let us call it $O(x)$. We now calculate b_n :

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} O(x)dx$$

$$\implies \pi b_n = \int_{-\pi}^{0} O(x)dx + \int_{0}^{\pi} O(x)dx$$

Now let us manipulate the first integral. We use the fact that O is odd, and substitute $u = -x$ in the third integral.

$$\int_{-\pi}^{0} O(x)dx = \int_{-\pi}^{0} -O(-x)dx = \int_{\pi}^{0} O(u)du = -\int_{0}^{\pi} O(x)dx$$

It is clear now that $b_n = 0$.

5.4

Repeat §5.3 for f odd; that is, if $f(-x) = -f(x)$, show that all $a_n = 0$. The series is then called the *sine series.*

The proof for §5.3 suffices for this problem as well, as f is odd and $\cos(nx)$ is even. It follows their product is odd, and $a_n = 0$.

6 Measure 0

Decide whether each of the following sets has measure 0. Give a short justification.

6.1

$A = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$

This set has measure 0 as we can cover the circle with annuli of arbitrarily small width.

6.2

The xy-plane in $\mathbb{R}^3$

This set has measure 0 as we can contain the entire xy-plane in a generalized rectangle of arbitrarily small volume.

6.3

The interval $[0, 1]$ in $\mathbb{R}$

This set does not have measure 0 as there are uncountably many points between 0 and 1, and thus it would take a union of uncountably many intervals of arbitrarily small width to cover the set.

6.4

The set of irrational numbers in $[0, 1]$

The justification for this is almost identical to the justification of the set described in §6.3. There are uncountably many irrationals between 0 and 1, and thus we cannot cover them all with countably many arbitrarily small intervals.

6.5

The boundary ∂A of a set A of measure 0 in $\mathbb{R}^n$

The boundary of a set of measure 0 will have measure 0. If A is closed, then $\partial A \subseteq A$ and obviously ∂A will have measure 0. If A is open (and nonempty) in $\mathbb{R}^n$, then A contains an open interval and therefore has positive measure.

7 Fourier coefficients

Let $\{\phi_0, \phi_1, ...\}$ be an infinite orthonormal system in the inner product space V. Let $f \in V$. Prove that $\lim_{n\to\infty}\langle f, \phi_n\rangle = 0$.

The expression $\langle f, \phi_n\rangle$ represents a Fourier coefficient. If $\lim_{n\to\infty}\langle f, \phi_n\rangle \neq 0$, then the corresponding Fourier series would not be able to converge, and therefore not be a Fourier series.

8 Integral of a periodic function under translation

Let $f : \mathbb{R} \to \mathbb{R}$ be a 2π periodic function that is continuous. Show that $\int_0^{2\pi} f = \int_{-\pi}^{\pi} f$.

For a function f of period 2π, we have that $f(x) = f(x + 2\pi)$ for all $x \in \mathbb{R}$. Define $g(x) = \int_x^{x+2\pi} f(x)dx$. $\frac{dg}{dx} = f(x + 2\pi) - f(x) = 0$, from previous identity. Because g' is identically 0, there exists a $c \in \mathbb{R}$ such that $g(x) = c$ for all $x \in \mathbb{R}$. Thus $g(0) = g(-\pi)$, and the proof is done. □

9 Parseval's theorem (2)

9.1

Compute the trigonometric Fourier series of $f(x) = |x|$ on $[-\pi, \pi]$.

$|x|$ is an even function, so $b_n = 0$. We compute a_0 :

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} |x|dx = \frac{2}{\pi}\int_0^{\pi} xdx = \frac{2}{\pi}\left[\frac{x^2}{2}\right]\Bigg|_0^{\pi} = \frac{2}{\pi}\cdot\left(\frac{\pi^2}{2} - 0\right) = \pi$$

Next we compute a_n :

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} |x|\cos(nx)dx = \frac{2}{\pi}\int_0^{\pi} x\cos(nx)dx = \frac{2}{\pi}\left[\frac{\cos(nx)}{n^2} + \frac{x\sin(nx)}{n}\right]\Bigg|_0^{\pi}$$

The sine term here is identically 0 for all $n\pi$, but the cosine term could be -1 or 1, depending on the parity of n. Therefore we have

$$a_n = \frac{2}{\pi}\cdot\left(\frac{(-1)^n}{n^2} - \frac{1}{n^2}\right) = \begin{cases} 0 & x \text{ is even} \\ -\frac{4}{\pi n^2} & x \text{ is odd} \end{cases}$$

Using $(2n-1)$ as an expression to distinguish parity, we finally have the Fourier series for $|x|$.

$$f(x) = |x| = \frac{\pi}{2} - \sum_{n=1}^{\infty} \frac{4\cos(nx)}{(2n-1)^2\pi}$$

9.2

Prove that

$$1 + \frac{1}{3^4} + \frac{1}{5^4} + \frac{1}{7^4} + ... = \frac{\pi^4}{96}.$$

We use Parseval's formula using the coefficients from §9.1.

$$\frac{\pi^2}{2} + \sum_{n=1}^{\infty}\left(\frac{4}{(2n-1)^2}\right)^2 = \frac{1}{\pi}\int_{-\pi}^{\pi} |x|^2 dx = \frac{1}{\pi}\left[\frac{x^3}{3}\right]\Bigg|_{-\pi}^{\pi} = \frac{1}{\pi}\cdot\left(\frac{\pi^3}{3} + \frac{\pi^3}{3}\right) = \frac{2\pi^2}{3}$$

We rearrange terms to get the desired sum.

$$\frac{2\pi^2}{3} = \frac{\pi^2}{2} + \sum_{n=1}^{\infty}\frac{16}{(2n-1)^4} \implies \frac{\pi^2}{6} = 16\sum_{n=1}^{\infty}\frac{1}{(2n-1)^4} \implies \frac{\pi^2}{96} = \sum_{n=1}^{\infty}\frac{1}{(2n-1)^4}$$

10 Principal value integral

Let $f : [-1, 1] \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} \frac{1}{x} & x \in [-1, 0) \cup (0, 1] \\ 0 & x = 0 \end{cases}$$

10.1

Explain why $f : [-1, 1] \to \mathbb{R}$ is not integrable. $\lim_{x \to 0^+} f = \infty$ and $\lim_{x \to 0^-} f = -\infty$, and we have two points in the interval [-1,1] where f blows up. Therefore f is not integrable.

10.2

Show that f has a principal value integral on $[-1, 1]$ and find it. The principal value integral is as follows:

$$\lim_{\epsilon \to 0} \int_{-1+\epsilon}^{1-\epsilon} \frac{1}{x} dx = \lim_{\epsilon \to 0} 0 = 0$$

We can say this because f is an odd function and the region $[-1 + \epsilon, 1 - \epsilon]$ is symmetric.