Generalized Rectangles

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1 B2

Find $\lim_{n\to\infty}(\sum_{k=1}^{n} \frac{k}{n^2+k^2})$.

Consider the Riemann Sum for $\int_0^1 \frac{x}{1+x^2}dx$ using the partition P_n of $[0,1]$, where P_n divides $[0,1]$ into n intervals of length $\frac{1}{n}$. Assume that $c_i = \frac{i}{n}$ for $1 \leq i \leq n$. By definition of Riemann Sum, we have the following:

$$\lim_{n\to\infty} R(f, P_n, C_n) = \lim_{n\to\infty} \left[\left(\frac{1}{n}\right) \cdot \left(\frac{\frac{1}{n}}{1+\frac{1}{n^2}} + \frac{\frac{2}{n}}{1+\frac{2^2}{n^2}} + ... + \frac{\frac{n}{n}}{1+\frac{n^2}{n^2}} \right) \right]$$

$$= \lim_{n\to\infty} \left(\sum_{k=1}^{n} \frac{\frac{1}{n}\cdot\frac{k}{n}}{1+\frac{k^2}{n^2}} \right) = \lim_{n\to\infty} \left(\sum_{k=1}^{n} \frac{k}{n^2+k^2} \right)$$

By the Riemann Sum Convergence Theorem, we know this limit is equal to the aforementioned integral.

$$\lim_{n\to\infty} \left(\sum_{k=1}^{n} \frac{k}{n^2+k^2} \right) = \int_0^1 \frac{x^2}{1+x^2}dx = \frac{1}{2}[\ln(1+x^2)]\Big|_0^1 = \frac{1}{2}\ln(2)$$

2 B3

Suppose that the function $f : [a, b] \to \mathbb{R}$ is continuous and let P be any partition of its domain $[a, b]$. Show that there is a Riemann sum $R(f, P, C)$ that equals $\int_a^b f$.

By the Mean Value Theorem For Integrals, we know that there exists a $c_i \in [x_{i-1}, x_i]$ such that $\frac{1}{x_i - x_{i-1}} \int_{x_{i-1}}^{x_i} f(x)dx = f(c_i) \implies \int_{x_{i-1}}^{x_i} f(x)dx = f(c_i)(x_i - x_{i-1})$. By definition, we have $R(f, P, C) = \sum_{i=1}^{n} f(c_i)(x_i - x_{i-1}) = \sum_{i=1}^{n} \int_{x_{i-1}}^{x_i} f = \int_a^b f$. $\square$

3 B4

For the rectangle $\mathbf{I} = [0,1] \times [0,1]$ in the plane $\mathbb{R}^2$, define the function $f : \mathbf{I} \to \mathbb{R}$ by $f(x,y) = xy$ for (x,y) in $\mathbf{I}$. Use the Archimedes-Riemann Theorem to evaluate $\int_{\mathbf{I}} f$.

Let P_n be the partition of $[0,1]$ into n intervals of length $\frac{1}{n}$. Define $\mathbf{P}_n = (P_n, P_n)$. Each rectangle $\mathbf{J} = [\frac{i-1}{n}, \frac{i}{n}] \times [\frac{j-1}{n}, \frac{j}{n}] \in \mathbf{P}_n$ has volume $\frac{1}{n^2}$. We also have $m(f, \mathbf{J}) = \frac{(i-1)(j-1)}{n^2}$ and $M(f, \mathbf{J}) = \frac{ij}{n^2}$. We calculate the upper and lower sums:

$$U(f, \mathbf{P}_n) = \sum_{1 \le i,j \le n} \left(\frac{ij}{n^2}\right)\left(\frac{1}{n^2}\right) = \frac{1}{n^4}\sum_{i=1}^{n} i \sum_{j=1}^{n} j = \frac{1}{n^4}\left(\frac{n(n+1)}{2}\right)^2$$

$$L(f, \mathbf{P}_n) = \sum_{1 \le i,j \le n} \left(\frac{(i-1)(j-1)}{n^2}\right)\left(\frac{1}{n^2}\right) = \frac{1}{n^4}\sum_{i=1}^{n}(i-1)\sum_{j=1}^{n}(j-1) = \frac{1}{n^4}\left(\frac{n(n-1)}{2}\right)^2$$

We notice that $\lim_{n\to\infty} U(f, \mathbf{P}_n) = \lim_{n\to\infty} L(f, \mathbf{P}_n) = \frac{1}{4}$. Because the upper and lower sums converge to the same value, $\int_{\mathbf{I}} f = \frac{1}{4}$.

4 B5

Let $\mathbf{I}$ be a generalized rectangle in $\mathbb{R}^n$ and let $\epsilon > 0$. Show that there is a generalized rectangle $\mathbf{J}$ that is contained in the interior of $\mathbf{I}$ and has the property that vol $\mathbf{I}$ $-$ vol $\mathbf{J} < \epsilon$.

Let $\mathbf{I} = [a_1, b_1] \times ... \times [a_n, b_n]$ and let $\mathbf{J} = [a_1 + \frac{1}{k}, b_1 - \frac{1}{k}] \times ... \times [a_n + \frac{1}{k}, b_n - \frac{1}{k}] \in \mathbf{I}$. We now calculate the difference in volumes:

$$\text{vol } \mathbf{I} - \text{vol } \mathbf{J} = \prod_{i=1}^{n}(b_i - a_i) - \prod_{i=1}^{n}\left(b_i - a_i - \frac{2}{k}\right)$$

We take the limit as k tends towards infinity:

$$\lim_{k\to\infty}(\text{vol } \mathbf{I} - \text{vol } \mathbf{J}) = \lim_{k\to\infty}\left(\prod_{i=1}^{n}(b_i - a_i) - \prod_{i=1}^{n}\left(b_i - a_i - \frac{2}{k}\right)\right) = \left(\prod_{i=1}^{n}(b_i - a_i) - \prod_{i=1}^{n}(b_i - a_i)\right) = 0 < \epsilon$$