Coordinate Systems

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1 Hyperbolic coordinates

Define $U = \{(x, y) \mid x > 0, y > 0\}$ and define $\phi : U \to \mathbb{R}^2$ by $\phi(x, y) = (x^2 - y^2, xy)$ for (x, y) in U. Show that $\phi : U \to \mathbb{R}^2$ is a smooth change of variables.

The derivative matrix of ϕ is

$$\begin{bmatrix} 2x & -2y \\ y & x \end{bmatrix}$$

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We see that $\det(\mathbf{D}\phi) = (2x)(x) - (-2y)(y) = 2x^2 + 2y^2$. x and y are strictly positive, so the determinant of the derivative matrix of ϕ is also strictly positive and therefore $\phi : U \to \mathbb{R}^2$ is a continuous, invertible mapping. It follows from definition that $\phi : U \to \mathbb{R}^2$ is a smooth change of variables.

2 Integration using hyperbolic coordinates

Define $D = \{(x, y) \mid x > 0, y > 0, 1 < x^2 - y^2 < 9, 2 < xy < 4\}$. For a continuous function $f : D \to \mathbb{R}$, use the hyperbolic coordinates from §1 to show that $\int_D [x^2 + y^2] dxdy = 8$.

From §1 and The Change Of Variables Theorem,

$$\int_D [x^2 + y^2] dxdy = \int_1^9 \int_2^4 \frac{x^2 + y^2}{2x^2 + 2y^2} dvdu = \frac{1}{2} \int_1^9 \int_2^4 dvdu = \frac{1}{2}(9 - 1)(4 - 2) = 8$$

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3 Uniform limit of differentiable functions is not differentiable

For each natural number n and each number $x \in (-1, 1)$, define $f_n(x) = \sqrt{x^2 + \frac{1}{n}}$ and define $f(x) = |x|$. Prove that the sequence (f_n) converges uniformly on the open interval $(-1, 1)$ to the function f.

First we show that $f_n(x)$ converges to $|x|$ pointwise:

$$\lim_{n\to\infty} \sqrt{x^2 + \frac{1}{n}} = \sqrt{x^2 + \lim_{n\to\infty} \frac{1}{n}} = \sqrt{x^2} = |x|$$

Now we show the convergence is uniform, using $f(x) = |x| = \sqrt{x^2}$.

$$\lim_{n\to\infty} \sqrt{x^2 + \frac{1}{n}} - \sqrt{x^2} = \lim_{n\to\infty} \frac{x^2 + \frac{1}{n} - x^2}{\sqrt{x + \frac{1}{n}} + \sqrt{x^2}} \leq \lim_{n\to\infty} \frac{\frac{1}{n}}{\sqrt{\frac{1}{n}}} = \lim_{n\to\infty} \sqrt{\frac{1}{n}}$$

$$\lim_{n\to\infty} \sqrt{x^2 + \frac{1}{n}} - \sqrt{x^2} \leq \lim_{n\to\infty} \sqrt{\frac{1}{n}} = 0 \implies \lim_{n\to\infty} \sqrt{x^2 + \frac{1}{n}} - \sqrt{x^2} = 0$$

This function is continuously differentiable, as $f_n'(x) = \frac{x}{\sqrt{x^2+\frac{1}{n}}}$. This does not contradict Theorem 9.33.

4 $f_n \to f$ pointwise but $\int f_n \not\to \int f$

For each natural number n and each number $x \in [0, 1]$, define $f_n(x) = nxe^{-nx^2}$. Prove that the sequence (f_n) converges pointwise on the interval $[0, 1]$ to the constant function 0, but that the sequence of integrals $(\int_0^1 f_n)$ does not converge to 0.

First we show that $f_n(x)$ converges to the constant function 0 pointwise:

$$\lim_{n\to\infty} nxe^{-nx^2} = \frac{1}{x} \lim_{n\to\infty} \frac{nx^2}{e^{nx^2}} = \frac{1}{x} \cdot 0 = 0$$

Now we show that $\int_0^1 f_n$ does not converge to 0.

$$\int_0^1 nxe^{-nx^2} dx = -\frac{1}{2}e^{-nx^2}\Big|_0^1 = -\frac{1}{2}e^{-n} + \frac{1}{2}$$

$$\lim_{n\to\infty} \int_0^1 nxe^{-nx^2} = \lim_{n\to\infty} \left(-\frac{1}{2}e^{-n} + \frac{1}{2}\right) = \frac{1}{2} \neq 0$$

This does not contradict Theorem 9.32.

5 Term-by-term integration

Express the sum of the series $\sum_{n=1}^{\infty} \frac{x^n}{n^2}$ as an integral.

$$\frac{x^n}{n^2} = \int_0^x \frac{t^{n-1}}{n} dt \implies \sum_{n=1}^{\infty} \frac{x^n}{n^2} = \int_0^x \sum_{n=1}^{\infty} \frac{t^{n-1}}{n} dt = \int_0^x -\frac{\ln(1-x)}{x}$$