Cosmology II

1 Expanding Universe

- FRW Metric:

$$\begin{aligned} \mathrm{d}s^2 &= -c^2\,\mathrm{d}t^2 + a^2(t)\left[\frac{1}{1-kr^2/R^2}\,\mathrm{d}r^2 + r^2\,\mathrm{d}\Omega^2\right] \\ &= -c^2\,\mathrm{d}t^2 + a^2(t)R^2\left[\mathrm{d}\chi^2 + S_k(\chi)^2\,\mathrm{d}\Omega^2\right] \end{aligned}$$

  where

$$S_k(\chi) = \begin{cases} \sin\chi & k=+1 \\ \chi & k=0 \\ \sinh\chi & k=-1 \end{cases}$$

- Physical coordinates, co-moving coordinate, peculiar velocity, Hubble's Law

$$\mathbf{v}_{\text{phys}} = H_0\mathbf{x}_{\text{phys}}$$

- Redshift parameter $1+z = \frac{1}{a(t)}$
- Particle horizon (how far we can see/be influenced up to Big Bang), cosmological event horizon (how far we can influence up for $t \to \infty$ or t_{BC})
- Conformal time

$$\tau = \int^t \frac{\mathrm{d}t'}{a(t')}$$

- Luminosity distance: wavelength redshifted, energy density scaled down by redshift:

$$l = \frac{L}{4\pi R^2 S_k(\chi)^2(1+z)^2}$$

- Deceleration parameter

$$q(t) = -\frac{\ddot{a}a}{\dot{a}^2}$$

- Equation of states
  - Matter $P=0$
  - Radiation $P=\frac{1}{3}\rho$

- Cosmological constant $P = -\rho$

- Continuity equation

$$\dot{\rho} + 3H(\rho + P) = 0$$

- Friedmann equation

$$H^2 = \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3c^2}\rho - \frac{kc^2}{R^2 a^2}$$

- Raychaudhuri equation

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3c^2}(\rho + 3P)$$

- Critical density

Cosmological solutions

- Single component solutions
  - Einstein de-Sitter
  - Radiation dominated
  - Milne Universe
  - de Sitter space: $\Lambda > 0$
  - anti-de Sitter space: $\Lambda < 0$
  - (Cosmological constant + matter)
- Critical density: (when $\dot{H}^2 = \frac{8\pi G}{3c^2}\rho_{\text{crit}}$)
- Singularity theorem
- Virial theorem: system of particles under gravitational potential, time-average PE and KE satisfy

$$\overline{V} = -\frac{1}{2}\overline{T}$$

- Inflation: flatness problem, horizon problem, accelerating phase: , inflation field, slow roll

2 Hot Universe

- Statistical Physics (ensembles, Boson/Fermions)
- Last scattering of photon forms CMB, redshift from T_{last} and λ_{last}, temperature scale as $1/a$

Recombination

- Saha Equation

$$\frac{n_H}{n_e^2} = \left(\frac{2\pi\hbar}{m_e k_B T}\right)^{3/2} e^{\beta E_{\text{bind}}}$$

Ionisation ratio $X_3 = \frac{n_e}{n_B}$

$$\frac{1 - X_e}{X_e^2} = \eta \frac{2\zeta(3)}{\pi^2} \left(\frac{2\pi k_B T}{m_e c^2}\right)^{3/2} e^{\beta E_{\text{bind}}}$$

Interaction rate Γ Recombination occurs when $\Gamma \approx H$

- Effective number of relativistic series

$$g_\star = \sum_{\text{boson}} g_i + \frac{7}{8} \sum_{\text{Fermion}} g_i$$

- Energy density from relaivistic species

$$\rho = g_\star \frac{\pi^2}{30\hbar^3 c^3} (k_B T)^4$$

3 Structure Formation

- Fluid equations

$$\frac{\partial n}{\partial t} + \nabla \cdot (n\mathbf{u}) = 0$$
$$mn\left(\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u}\right) = -\nabla P - mn\nabla\Phi$$
$$\nabla^2\Phi = 4\pi G m n$$

- Perturbations from homogeneous solution $n = \bar{n}, P = \bar{P}$ (without gravitational field) give wave equation with sound speed

$$c_s^2 = \frac{1}{m}\frac{\partial P}{\partial \rho} =$$

  Solutions are non-dispersive

- With gravity, get

$$\left(\frac{\partial^2}{\partial t^2} - c_s^2 \nabla^2 - 4\pi G m \bar{n}\right) \delta n = 0$$

  Wave are dispersive:

$$\begin{aligned} \omega^2 &= c_s^2 k^2 - 4\pi G m \bar{n} \\ &= c_s^2 (k^2 - k_J^2) \end{aligned}$$

  Jeans wavenumber

$$k_J^2 = 4\pi G m \bar{n} / c_s^2$$

  For $k \ll k_J$, ω is imaginary, so have exponential solution (Jeans instability); $k > k_J$, then solution are oscillating

  Timescale for pressure to develop: $t_{\text{pressure}} \sim \frac{R}{c_s}$; timescale for collapse $\tau \sim \sqrt{\frac{1}{4\pi G m \bar{n}}}$

- In expanding universe, derivatives change (with $\mathbf{r} = a\mathbf{x}$)

$$\begin{aligned} \nabla_{\mathbf{r}} &= \frac{1}{a} \nabla_{\mathbf{x}} \\ \left.\frac{\partial}{\partial t}\right|_{\mathbf{r}} &= \left.\frac{\partial}{\partial t}\right|_{\mathbf{x}} - H \mathbf{x} \cdot \nabla_{\mathbf{x}} \end{aligned}$$

  The fluid equations in co-moving coords become (v = peculiar velocity)

$$\begin{aligned} \frac{\partial n}{\partial t} + 3Hn + \frac{1}{a} \nabla \cdot (n\mathbf{v}) &= 0 \\ mna \left(\frac{\partial}{\partial t} + \frac{\mathbf{v}}{a} \cdot \nabla\right) \mathbf{u} &= -\nabla P - mn \nabla \Phi \\ \nabla^2 \Phi &= 4\pi G m n a^2 \end{aligned}$$

- Under perturbation, density contrast

$$\delta = \frac{\delta n}{\bar{n}} = \frac{\delta \rho}{\bar{\rho}}$$

Background flow $\mathbf{u} = Ha\mathbf{x}$, spatially uniform pressure gives Poisson equation

$$\nabla^2 \Phi = -3\ddot{a}a$$

This gives acceleration equation

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3} mn$$

Perturb from background flow: linearised equations

$$\begin{aligned}
\dot{\delta} &= -\frac{1}{a} \nabla \cdot \mathbf{v} \\
m\bar{n}a\left(\dot{\mathbf{v}} + H\mathbf{v}\right) &= -\nabla \delta P - m\bar{n}\nabla\delta\Phi \\
\nabla^2 \delta\Phi &= 4\pi G m a^2 \bar{n} \delta
\end{aligned}$$

Combining to give

$$\ddot{\delta}(\boldsymbol{x}, t) + 2\dot{\delta}(\boldsymbol{x}, t) - c_s^2 \left(\frac{1}{a^2}\nabla^2 + k_J^2\right) \delta(\boldsymbol{x}, t) = 0$$

Fourier transform:

$$\ddot{\delta}(\boldsymbol{k}, t) + 2\dot{\delta}(\boldsymbol{k}, t) - c_s^2 \left(k_J^2 - \frac{k^2}{a^2}\right) \delta(\boldsymbol{k}, t) = 0$$

Small wavelength behave as damped oscillator Long wavelength $\frac{k}{a} \ll k_J$, drop k term For matter dominated universe, get equidimensional equation with constant coeffs, power law solutions $t^{2/3}$ or t^-1