1 Complex Variable

- Integral $F(z) = \int_C f(z,t)\,\mathrm{d}t$ defines an analytic function when
  - $f(z,t)$ continuous jointly in z and t
  - Integral converges uniformly in each compact set of the range of z
  - $f(z,t)$ analytic in z for each t
- Analytic continuation, uniqueness, radius of convergence up to closest singularity; might have natural boundary
- Cauchy Principal Value ($\mathcal{P}$): limit of definite integral to an undefined integral
- Hilbert Transform:

  $$\mathcal{H}(f)(y) = \frac{1}{\pi}\mathcal{P}\left(\int_{-\infty}^{\infty} \frac{f(x)}{x-y}\,\mathrm{d}x\right)$$

  Property:

  $$\mathcal{H}(e^{i\omega x})(y) = \begin{cases} ie^{i\omega y} & ,\omega > 0 \\ -ie^{i\omega y} & ,\omega < 0 \end{cases}$$

  So $\mathcal{H}^2 = -\mathrm{Id}$.
- Kramers Kronig Relations: for analytic $f(z) = u(x,y)+iv(x,y)$, (decaying quick enough as $z \to \infty$)

  $$\mathcal{H}(u(x,0)) = -v(x,0)$$
  $$\mathcal{H}(v(x,0)) = u(x,0)$$

- Multivalued functions: branch point/cuts, defining arcsin, ln.
- Elliptic function: doubly periodic meromorphic function, e.g. Weierstrass $\wp$ function

  Finitely many zeroes and poles in a cell; if no poles, then is constant

2 Special Functions

- Gamma:

$$\begin{aligned}\Gamma(z) &= \int_0^\infty t^{z-1}e^{-t}\,\mathrm{d}t && \text{, for } \mathrm{Re}(z)>0 \text{ (Integral)}\\ &= \lim_{n\to\infty}\frac{n!n^z}{z(z+1)\dots(z+n)} && \text{, for } z\in\mathbb{C}\setminus\{0,-1,-2,\dots\} \text{ (Euler product)}\\ &= \frac{1}{z}\prod_{n=1}^\infty\frac{(1+\frac{1}{n})^z}{1+\frac{z}{n}} && \text{, for } z\in\mathbb{C}\setminus\{0,-1,-2,\dots\} \text{ (Gauss product)}\\ &= \frac{1}{2i\sin\pi z}\int_{-\infty}^{(0^+)} t^{z-1}e^t\,\mathrm{d}t && \text{, for } z\in\mathbb{C}\setminus\{0,-1,-2,\dots\} \text{ (Hankel representation)}\end{aligned}$$

Weierstrass product (with $\gamma = \lim_{n\to\infty}\left(1+\frac{1}{2}\cdots+\frac{1}{n}-\log n\right)$)

$$\frac{1}{\Gamma(z)} = ze^{\gamma z}\prod_{k=1}^\infty\left(1+\frac{z}{k}\right)e^{-z/k}$$

Reflection formula:

$$\Gamma(z)\Gamma(1-z) = \pi\mathrm{cosec}(\pi z) \text{ for } z\notin\mathbb{Z}$$

Generalised Factorial $\Gamma(n+1)=n!$: unique extension (with some extra properties)

- Beta

$$\begin{aligned}B(p,q) &= \int_0^1 t^{p-1}(1-t)^{q-1}\,\mathrm{d}t \text{ for } \mathrm{Re}(p),\mathrm{Re}(q)>0\\ &= \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}\\ &= \frac{-1}{4}e^{\pi i(p+q)}\mathrm{cosec}(\pi p)\mathrm{cosec}(\pi q)\int_{\text{Pochhammer}} t^{p-1}(1-t)^{q-1}\,\mathrm{d}t\end{aligned}$$

- Zeta

$$\begin{aligned}\zeta(z) &= \sum_{n=1}^\infty n^{-z}, \text{ for } \mathrm{Re}(z)>1\\ &= \frac{\Gamma(1-z)}{2\pi i}\int_{-\infty}^{(0^+)}\frac{t^{z-1}}{e^{-t}-1}\,\mathrm{d}t\\ &= 2^z\pi^{z-1}\sin\left(\frac{\pi z}{2}\Gamma(1-z)\zeta(1-z)\right) \quad = \prod_p\frac{1}{1-p^{-z}}, \text{ for } \mathrm{Re}(z)>1\end{aligned}$$

Trivial zeros $\zeta(n)=0$ at $n=-2,-4,-6,\dots$

3 Solution of DEs by transform methods

- Integral transform $\int_\gamma K(z,t)f(t)\,\mathrm{d}t$ with some kernel K, e.g. Laplace, Euler Mellin
- Choose function, then choose contour such that boundary terms vanish
- Laplace transform, Fourier Transform
- Note causality, stability (tend to zero for large t)
- Nyquist stability criterion (?)

4 2nd order linear ODE in complex plane

- Classify ordinary points, regular singular points, irregular singular points
- Power series solution, indicial equation
- Fuchsian equation: at most 3 RSPs, transform
- Papperitz equation, transform rule (p.54), hypergeometric equation