Vector Calculus — IA Lent

1 Differential Geometry of Curves

- Parametrisation of a curve, under some coordinate system (e.g. Cartesian): a function $\mathbf{x} : [a, b] \to \mathbb{R}^3$ with

$$\mathbf{x}(t) = \begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix}$$

- In Cartesians: Curve differentiable = components differentiable
- Curve is regular: $|\mathbf{x}'(t)| \neq 0$
- Arc length of curve:
  1. Partition the interval $[a, b]$
  2. Calculate sum of straight line lengths at partition points
  3. Take limit as maximum Δt approaches zero
- Piecewise smooth curve: break down into integral of each piece
- Line element:

$$\begin{aligned} \mathrm{d}s &= |\mathbf{x}'(t)| \, \mathrm{d}t \\ &= \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}z}{\mathrm{d}t}\right)^2} \, \mathrm{d}t \end{aligned}$$

- Tangent, normal, binormal, curvature, torsion

2 Integration

- Line integral: parametrise curve C by $[a, b] \ni t \to \mathbf{x}(t)$
  - Scalar field $f(\mathbf{x})$
    Scalar arc-length element: $\mathrm{d}s = |\mathbf{x}'(t)| \, \mathrm{d}t$:

$$\int_C f(\mathbf{x}) \, \mathrm{d}s = \int_a^b f(\mathbf{x}(t)) \, |\mathbf{x}'(t)| \, \mathrm{d}t$$

- Vector field $\mathbf{F}(\mathbf{x})$
  Vector line element: $\mathrm{d}\mathbf{x} = \mathbf{x}'(t)\,\mathrm{d}t$:

$$\int_C \mathbf{F}(\mathbf{x}) \cdot \mathrm{d}\mathbf{x} = \int_a^b \mathbf{F}(\mathbf{x}(t)) \cdot \mathbf{x}'(t)\,\mathrm{d}t$$

- If the curve C is a closed loop: closed integral/ circulation about C

$$\oint_C \mathbf{F} \cdot \mathrm{d}\mathbf{x}$$

- Area integral: given a region D
  Scalar field $f(\mathbf{x})$
  Scalar area element (if D is divided into rectangular elements)
  $\mathrm{d}A = \mathrm{d}x\,\mathrm{d}y$

$$\begin{aligned}\iint_D f(\mathbf{x})\,\mathrm{d}A &= \int_y \int_{X_y} f(x,y)\,\mathrm{d}x\,\mathrm{d}y \\ &= \int_x \int_{Y_x} f(x,y)\,\mathrm{d}y\,\mathrm{d}x\end{aligned}$$

  where $X_y = \{x : (x,y) \in D\}$ (visualise horizontal strips),
  and $Y_x = \{y : (x,y) \in D\}$ (vertical strips)

- Jacobian: If $x = x(u,v)$ and $y = y(u,v)$ are smooth bijection from region D' in (u,v) plane to region D in (x,y) plane, then

$$\iint_D f(x,y)\,\mathrm{d}x\,\mathrm{d}y = \iint_{D'} f(x(u,v),y(u,v)) \left|\frac{\partial(x,y)}{\partial(u,v)}\right| \mathrm{d}u\,\mathrm{d}v$$

$$\text{i.e. } \mathrm{d}x\,\mathrm{d}y = |J|\ \mathrm{d}u\,\mathrm{d}v$$

$$\text{where } J = \frac{\partial(x,y)}{\partial(u,v)} = \det\begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v}\end{pmatrix} = \det\left(\frac{\partial \mathbf{x}}{\partial u}\middle|\frac{\partial \mathbf{x}}{\partial v}\right)$$

  ($|J|$ is the "scale factor" for area)

- Volume integral

$$\iiint_V f(\mathbf{x}) dV$$

  Analogue for Jacobian:

$$J = \frac{\partial(x,y,z)}{\partial(u,v,w)} = \det\left(\frac{\partial \mathbf{x}}{\partial u}\middle|\frac{\partial \mathbf{x}}{\partial v}\middle|\frac{\partial \mathbf{x}}{\partial w}\right)$$

Examples:
Cylindrical polars:

$$\mathrm{d}V = \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \rho\,\mathrm{d}\rho\,\mathrm{d}\theta\,\mathrm{d}z$$

Spherical polars:

$$\mathrm{d}V = \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}x = r^2 \sin\theta\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\phi$$

- Surface integral:
  Given surface defined by $S = \{\mathbf{x} : f(\mathbf{x}) = 0\}$: ∇f is normal to the surface
  Given parametrised surface $S = \{\mathbf{x} : \mathbf{x}(u,v) : (u,v) \in D\}$ for some region D in (u,v) plane
  Normal vector (of unit length)

$$\mathbf{n} = \frac{\frac{\partial \mathbf{x}}{\partial u} \times \frac{\partial \mathbf{x}}{\partial v}}{\left|\frac{\partial \mathbf{x}}{\partial u} \times \frac{\partial \mathbf{x}}{\partial v}\right|}$$

  Scalar and vector area element:

$$\mathrm{d}S = \left|\frac{\partial \mathbf{x}}{\partial u} \times \frac{\partial \mathbf{x}}{\partial v}\right| \mathrm{d}u\,\mathrm{d}v$$

$$\mathrm{d}\mathbf{S} = \frac{\partial \mathbf{x}}{\partial u} \times \frac{\partial \mathbf{x}}{\partial v}\,\mathrm{d}u\,\mathrm{d}v = \mathbf{n}\,\mathrm{d}S$$

3 Div, Grad, Curl and Laplacian: In Cartesians

- Gradient (of scalar field): ∇f

$$[\nabla f]_i = \frac{\partial f}{\partial x_i}$$

- Gradient of vector field
- Directional derivative: $D_{\mathbf{v}} f = \mathbf{v} \cdot \nabla f$

$$\mathbf{v} \cdot \nabla f = v_i \frac{\partial f}{\partial x_i}$$

- Divergence: $\nabla \cdot \mathbf{F}$

$$\nabla \cdot \mathbf{F} = \frac{\partial F_i}{\partial x_i}$$

- Curl: $\nabla \times \mathbf{F}$

$$[\nabla \times \mathbf{F}]_i = \epsilon_{ijk} \frac{\partial x_j}{\partial F}_k$$

- Laplacian (scalar): (div grad) $\nabla^2 f = \nabla \cdot \nabla f$

$$\nabla^2 f = \frac{\partial f}{\partial x_i} x_i$$

- Laplacian (vector field): $\nabla^2 \mathbf{F} = \nabla(\nabla \cdot \mathbf{F}) - \nabla \times (\nabla \times \mathbf{F})$

$$[\nabla^2 \mathbf{F}]_i = (\nabla^2 F_i)\mathbf{e}_i$$

- div $\circ$ curl $= \nabla \cdot \nabla \times \mathbf{F} = 0$
- curl $\circ$ grad $= \nabla \times \nabla f = \mathbf{0}$

4 Integration Theorems

- Green's Theorem:
  If P and Q are continuously differentiable scalar fields on $A \subset R^2$ and ∂A is made of collection of smooth curves, then

$$\oint_{\partial A} P \,\mathrm{d}x + Q \,\mathrm{d}y = \iint_A \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \mathrm{d}x \,\mathrm{d}y$$

- Stokes' Theorem:
  If $\mathbf{F}(\mathbf{x})$ is a continuously differentiable vector field, S an orientable surface with ∂S piecewise regular boundary, then

$$\int_S (\boldsymbol{\nabla} \times \mathbf{F}) \cdot \mathrm{d}\mathbf{S} = \oint_{\partial S} \mathbf{F} \cdot \mathrm{d}\mathbf{x}$$

- Divergence/Gauss' Theorem:
  (3D) If $\mathbf{F}(\mathbf{x})$ is a continuously differentiable vector field, V a volume with ∂V piecewise regular boundary, then

$$\int_V \nabla \cdot \mathbf{F} \,\mathrm{d}V = \int_{\partial V} \mathbf{F} \cdot \mathrm{d}\mathbf{S}$$

- (2D): (normal points out of the region)

$$\int_D \nabla \cdot \mathbf{F} \,\mathrm{d}A = \oint_{\partial D} \mathbf{F} \cdot \mathbf{n} \,\mathrm{d}s$$

5 Maxwell's Equations

6 Poisson's Equation

- Laplace's equation (Forcing = 0)
- Dirichlet and Neumann condition, with uniqueness

7 Cartesian Tensors

- Definition: A tensor T of rank n has components $T_{\underbrace{ij\ldots k}_{n\text{ indices}}}$ transforms from the right handed orthonormal basis $\mathbf{e}_i$ to $\mathbf{e}'_i$ under the law

  $$T'_{ij\ldots k} = R_{ip}R_{jq}\ldots R_{kr}T_{pq\ldots r}$$

  (where R_ij are components of a rotation matrix)

- Scalar (rank 0), vector (rank 1), linear map
- Scaling and adding tensors (of same order) give tensors
- Tensor product: $(T \otimes S)_{\underbrace{ij\ldots k}_{n\text{ indices}}\underbrace{pq\ldots r}_{m\text{ indices}}} = T_{ij\ldots k}S_{pq\ldots r}$
- Contraction: contracting on indices i and j: $S_{\underbrace{p\ldots q}_{n-2}} = \delta_{ij}T_{ijp\ldots q}$
- Symmetric, antisymmetric (in components in a pair of components); totally symmetric/antisymmetric
- Tensor field of rank n: $T_{ij\ldots k}(\mathbf{x}_0)$ gives a tensor at each point in space (say $\mathbb{R}^3$), e.g. scalar field, vector field
- Differentiating tensor fields:

  $$\frac{\partial}{\partial x'_i} = R_{ij}\frac{\partial}{\partial x_j}$$

  (each derivative gives a factor R)

- m-th partial derivative of rank n tensor is a tensor of rank $m+n$:

  $$\left(\frac{\partial}{\partial x_p}\right)\left(\frac{\partial}{\partial x_q}\right)\ldots\left(\frac{\partial}{\partial x_r}\right)T_{ij\ldots k}(\mathbf{x})$$

- Divergence theorem:
$$\int_V \frac{\partial T_{ij\ldots k\ldots l}}{\partial x_k} \mathrm{d}V = \int$$
$\mathrm{T}_{ij\ldots k\ldots l} n_k \mathrm{d}S$

- Rank 2 tensor: decompose (uniquely) into symmetric and antisymmetric parts:
$$T_{ij} = \underbrace{\frac{1}{2}(T_{ij} + T_{ji})}_{S_{ij}} + \underbrace{\frac{1}{2}(T_{ij} - T_{ji})}_{A_{ij}}$$
And the antisymmetric part
$$A_{ij} = \varepsilon_{ijk}\omega_k$$
where $\omega_i = \frac{1}{2}\varepsilon_{ijk}T_{jk}$.
Symmetric part:
$$S_{ij} = \underbrace{P_ij}_{\text{traceless}} + \underbrace{\frac{1}{3}\delta_{ij}S_kk}_{\text{isotropic}}$$

- For any symmetric second rank tensor, there exist a choice of right handed Cartesian coords where the matrix is diagonal (real symmetric matrix can be orthogonally diagonalised)

- Isotropic tensor: invariant under any change of bases :
$$T'_{ij\ldots k} = R_{ip}R_{jq}\ldots R_{kr}T_{pq\ldots r} = T_{ij\ldots k}$$

- Correspondence between multi-linear maps and tensors:
If a multi-linear map (well-defined, independent of basis) is given by
$$\underbrace{\mathbb{R}^3 \times \mathbb{R}^3 \times \cdots \times \mathbb{R}^3}_{n} \to \mathbb{R}$$
$$(\mathbf{a}, \mathbf{b}, \ldots \mathbf{c}) \to T_{ij\ldots k}a_i b_j c_k$$
Then $T_{ij\ldots k}$ is a rank n tensor.

- Quotient theorem:
Given array $T_{i\ldots jp\ldots q}$, if
$$v_{i\ldots j} := T_{i\ldots jp\ldots q}u_{p\ldots q}$$
is a tensor for *any* tensor $u_{p\ldots q}$, then $T_{i\ldots jp\ldots q}$ are components of a tensor.