Differential Equations IA Mich

1 Basic Calculus

- Limits: ϵ-δ definition, algebra of limits, left/right handed limits
- Derivative: definition, chain/product/Leibniz's rule
- Order of magnitude: Big O, little o notation
- Taylor Series/polynomial
- L'Hopital's rule
- Integration, FTC, indefinite integral, u-sub, trig-sub, by parts
- Multivariate functions: Partial derivative, multivariate chain rule

2 First Order Linear ODE

- Eigenfunction, logarithm
- Homogeneous, heterogeneous, forcing term
- General solution $= y_c + y_p$ (complementary function + particular integral)
- <u>Integrating factor</u>: $y'' + p(x)y' + q(x)y = f(x)$, use IF $= e^{\int p(x)dx}$

3 First Order Non-Linear ODE

- Separable equations
- <u>Exact equations</u>: can write $Q(x,y)\frac{\mathrm{d}f}{\mathrm{d}x} + P(x,y) = 0$ as $\frac{\mathrm{d}f}{\mathrm{d}x}(x,y) = 0$
- Quick check for exact equations over a simply connected domain: $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$
- Isocline: lines on which derivative is the same
- Stability of fixed points: perturbation analysis
- Phase portrait: represent solution by showing directions/growth directions

4 Higher Order Linear ODE

- Linear ODE: general solution $= y_c + y_p$ by superposition
- Linearly independent solutions: n many for n-th order ODE
- 2nd Order ODE with constant coefficients: use characteristic equation to determine eigenfunction; detuning
- Reduction of order given complementary function $y_1(x)$, find $y_2(x)$ by letting $y_2(x) = u(x)y_1(x)$
- Phase space: each point describes the state of a system; linearly independent solutions
- Wronskian; Abel's Theorem; using the Wronskian gives a first order DE for y_2
- Equidimensional equations: scaling invariant; in the form $ax^2y''+bxy'+cy = 0$; eigenfunction $y = x^k$; or introduce substitution $z = \ln x$
- Variation of parameters: given solution vectors for homogeneous equation, let $y_p = uy_1 + vy_2$
- Damped oscillating systems: $\ddot{y}+2\kappa\dot{y}(t)+y(t) = f(t)$: types of damping; free/forced motion gives transient/long time response
- Resonance: $\ddot{y} + \omega_0^2 y = \sin(\omega_0 t)$: detuning by having forcing term approach ω_0: gives $y_p = \frac{-t}{2\omega_0}\cos\omega_0 t$
- Dirac Delta Function; Heaviside Step Function; Ramp function; DE with these as forcing: continuity/jump conditions
- Series solutions (Method of Frobenius): given

  $$p(x)y'' + q(x)y' + r(x)y = 0,$$

  the point $x = x_0$ is

  - Ordinary point: if $\frac{q(x)}{p(x)}$ and $\frac{r(x)}{p(x)}$ have Taylor series about $x = x_0$, i.e. analytic

  If $\frac{q}{p}$ and $\frac{r}{p}$ do not have Taylor series, then rewrite the DE as

  $$P(x)(x-x_0)^2y'' + Q(x)(x-x_0)y' + R(x)y = 0,$$

- Regular singular point: if $\frac{Q}{P}$ and $\frac{R}{P}$ have Taylor series, i.e. $\frac{q}{p}(x-x_0)$ and $\frac{r}{p}(x-x_0)^2$ have Taylor series
- Irregular singular point: otherwise

The solutions:

- Ordinary point: Use the series $y = \sum_{n=0}^{\infty} a_n(x-x_0)^n$; solve recurrence relation
- Regular Singular point: use $y = \sum_{n=0}^{\infty} a_n(x-x_0)^{n+\sigma}$, for $\sigma \in \mathbb{C}$, solve indicial equation, giving roots σ_1, σ_2 such that $\mathrm{Re}(\sigma_1) \geq \mathrm{Re}(\sigma_2)$:
- If $\sigma_1 - \sigma_2$ is not an integer $\implies$ two independent solutions
- If $\sigma_1 - \sigma_2$ is an integer $\implies$ there are two solutions in the form

$$y_1 = \sum_{n=0}^{\infty} a_n(x-x_0)^{n+\sigma_1}$$
$$y_2 = y_1 \ln x + \sum_{n=0}^{\infty} b_n(x-x_0)^{n+\sigma_2}$$

5 Multivariate Functions

- Directional derivative, gradient; maximum rate of change, direction of steepest ascent
- Stationary points: max/min, saddle point
- Multivariate Taylor series, Hessian matrix; eigenvalues of Hessian matrix determines the type of stationary points
  - all +ve: min
  - all −ve: max
  - both + and −: saddle
  - has eigenvalue zero: degenerate, need higher order terms
- An n-th order ODE can be expressed as a system of n first order ODEs
- Matrix methods: with $\dot{\mathbf{Y}} = M\mathbf{Y} + \mathbf{F}$, find eigenvalues and eigenvectors; then consider behaviour of $\mathbf{Y}$ when it is an eigenvector

- Behaviour around stationary points depend on eigenvalues: stable/unstable node, saddle node, stable/unstable spiral, center
- Autonomous system (e.g. predator prey): behaviour do not depend on time
- Linearise a system of DEs to analyse stability

PDEs

- First order wave equation: $y(x,t)$ satisfying $\frac{\partial y}{\partial t} - c\frac{\partial y}{\partial x} = 0$: method of characteristics, lines on which $\frac{\mathrm{d}y}{\mathrm{d}t}$ is constant
- Second order wave equation: $y(x,t)$ satisfying $\frac{\partial^2 y}{\partial t^2} - c^2\frac{\partial^2 y}{\partial x^2} = 0$: solution is $y = f(x+ct) + g(x-ct)$
- Diffusion Equation: $\frac{\partial y}{\partial t} = \kappa\frac{\partial^2 y}{\partial x^2}$: introduce similarity variable (by dimensional analysis), reduce PDE to ODE

6 Discrete Equation

- Numerical Integration: express derivative as fraction, find recurrence relation and then take limit
- Series solutions: let power series, gather terms to obtain recurrence relation
- Stability of fixed points: compare magnitude of error term in successive iterations