Methods IB

1 Self-adjoint ODEs

Fourier Series

- In the vector space of 'nice enough' periodic functions: $\mathbb{R} \to \mathbb{R}$ with inner product $\langle f, g\rangle = \int_{-L}^{L} f(x)g(x)\mathrm{d}x$, the functions $\cos\left(\frac{n\pi x}{L}\right)$ and $\sin\left(\frac{n\pi x}{L}\right)$ form a countable complete orthonormal basis
- Dirichlet conditions:
  1. Finitely many bouned discontinuties
  2. Finitely many extrema in one period
  3. Absolutely integrable (Bounded?)
- Fourier series:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)\right),$$

$$\text{where } a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)\mathrm{d}x,$$
$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)\mathrm{d}x$$

- Half-range series
- Fourier series converge to average of left and right limits at discontinuities
- Order of coefficients related to differentiability of f and discontinuties: $f(p)$ discts $\implies a_n, b_n = O(n^{-(p+1)})$
- Can always integrate, but not always differentiate

Can also formulate using complex exponential as basis over periodic function $\mathbb{R} \to \mathbb{C}$ with inner product $\langle f, g\rangle = \int_{-L}^{L} f^*(x)g(x)\mathrm{d}x$:

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{\frac{in\pi x}{L}}, \quad \text{where } c_n = \frac{1}{2L}\int_{-L}^{L} f(x)e^{\frac{-in\pi x}{L}}\mathrm{d}x$$

- Parseval's Theorem:

$$\int_{-L}^{L} |f|^2 \, \mathrm{d}x = 2L \sum_{n=-\infty}^{\infty} |c_n|^2 = L \left(\frac{a_0^2}{2} + \sum_{n=1}^{\infty} (a_n^2 + b_n^2) \right)$$

Sturm-Liouville Theory

Background: (DE, LA)

- General solution to DE = Particular + Complementary (n linearly indep and spanning), uniqueness by boundary/initial conditions
- On inner product space V, a self-adjoint map (i.e. hermitian matrices for finite dim) $M : V \to V$ satisfies $\langle Mu, v \rangle = \langle u, Mv \rangle \, \forall u, v, \in V$
- Self-adjoint maps are orthogonally diagonalisable, can use eigenvector basis to solve $Mx = b$ (get coeffs)

S-L Theory: Consider linear differential operators as linear maps between inner product function spaces:

Aim: Solve 2nd order linear ODE:

$$\mathcal{D}y = \alpha y'' + \beta y' + \gamma y = f, \quad a \leq x \leq b$$

and two boundary conditions

- If $\alpha \neq 0$, any eigenvalue problem ($\mathcal{D}y = \lambda y$) becomes $\mathcal{L}y = -(py')' + qy = \lambda w y$, weight function w with countably many zeroes
- S-L form is self-adjoint iff $[p\,(\bar{y_1}' y_2 - \bar{y_1} y_2')]_a^b = 0$, for all y_1, y_2 that satisfy BC, includes many types (e.g. homogeneous Dirichlet, Neumann, mixed)

Good properties of self-adjoint operators:

- If $\mathcal{L}$ is self-adjoint, get countably infinite, complete basis of orthogonal (orthonormal) simple eigenfunctions with real eigenvalues: $\mathcal{L}y_n = \lambda_n w y_n$, orthogonal in the sense $\langle u, v \rangle_w = \int_a^b w(\bar{u}v)\mathrm{d}x$
- Can use basis to solve for particular integral, if forcing has same BC as solution

- Parseval's Theorem for eigenfunction expansion: if $f(x) = \sum_{n\in\mathbb{N}} a_n y_n$, then $\int_a^b w\,(f(x))^2\,\mathrm{d}x = \sum_{n\in\mathbb{N}} (a_n)^2 \left\|y_n\right\|_w^2$ (norm of y_n is integral with weight)

- Bessel's inequality, for incomplete set of eigenfunctions

- Least squares approx:
  Using a finite set of eigenfunctions $S_N(x) = \sum_{n=1}^{N} b_n y_n$ to approximate $f(x) = \sum_{n\in\mathbb{N}} a_n y_n$: choosing $b_n = a_n$ minimises mean squared error $\varepsilon_N = \int_a^b w\,[f - S_N]^2\,\mathrm{d}x$

Can solve forced problems Taking product with Green's functions 'inverts' L

2 PDEs, Separation of variables

Wave equation

$$\text{1D: } y_{tt} = c^2 y_{xx}$$
$$\text{2D: } \frac{\partial^2 \phi}{\partial t^2} = c^2 \nabla^2 \phi, c = \sqrt{T/\mu}$$

- SOV: if $y = X(x)T(t)$ two sides depends on different variables $\implies$ both sides constant.

- If given BC: homogeneous Dirichlet at end points, SOV give eigenvalue problems; solutions must be quantised sine wave, called the normal modes

$$y(x,t) = \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{n\pi ct}{L}\right) + b_n \sin\left(\frac{n\pi ct}{L}\right)\right) \sin\left(\frac{n\pi x}{L}\right)$$

- IC: initial position and velocity; determine the coefficient of each separable solution using Fourier

- Energy = sum of squares of coeffs, indep of time (use orthogonality)

- Reflection at boundary of changing mass density, three waves: incident, transmitted and reflected; y, y' continuous at boundary at all time; phase shift

Drum problem:

- Wave equation in 2D plane polars, with fixed circular boundary conditions $u(r=1,\theta,t)=0$
  Suppose solutions separable: $u = R(r)\Theta(\theta)T(t)$

- Equation for R gives Bessel's equation

$$z^2\frac{\mathrm{d}^2R}{\mathrm{d}z^2} + z\frac{\mathrm{d}R}{\mathrm{d}z} + (z^2 - m^2)R = 0$$

- Power series solutions: Bessel/Neumann functions $(J_m(x), Y_m(x))$

- Regularity (boundedness) at r = 0 forces only Bessel function; so solution for space component:

$$X_{mn}(r,\theta) = J_m(j_{mn}r)(A_n\cos(m\theta) + B_n\sin(m\theta))$$

- General space solution = linear combination of X_{mn}:

- Solution for u:

$$\begin{aligned} u(r,\theta,t) = &\sum_{n=1}^{\infty} J_0(j_{0n}r)\left(A_{0n}\cos(j_{0n}ct) + C_{0n}\sin(j_{0n}ct)\right) \\ &\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} J_m(j_{mn}r)(A_{mn}\cos(m\theta) + B_{mn}\sin(m\theta))\left(\cos(j_{mn}ct)\right) \\ +&\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} J_m(j_{mn}r)(C_{mn}\cos(m\theta) + D_{mn}\sin(m\theta))\left(\sin(j_{mn}ct)\right) \end{aligned}$$

  IC determine coefficients by orthogonality

Diffusion equation

- Flux proportional to diffusion gradient, and opposite direction:

$$\mathbf{q} = -k\nabla\theta$$

- Changes in total heat = - (net flux outwards)

- Diffusion equation:

$$\frac{\partial\theta}{\partial t} = D\nabla^2\theta$$

- Alternative derivation from random walk
- Error function is a solution without BC; good initial approximation for solution with BCs
- SOV: reformulate problem into homogeneous, get transient solution

Laplace's equation

- Cartesian, Plane polar, Cylindrical polar, Spherical polar
- Spherical polar, assume axisymmetric, equation for Θ give Legendre's equation by substitution $x = \cos\theta \in [-1, 1]$:

$$-\frac{\mathrm{d}}{\mathrm{d}x}\left((1-x^2)\frac{\mathrm{d}\Theta}{\mathrm{d}x}\right) = \lambda\Theta$$

- Series solution, bounded at endpoints $\implies$ solutions are polynomials, with $\lambda_l = l(l+1)$
- Legendre polynomials:
  Scaling convention $P_l(1) = 1$, orthogonal,

$$\int_{-1}^{1} P_n(x)\mathrm{d}x = \frac{2}{2n+1}$$

- Generating function

3 Generalised function, Inhom ODE

- Dirac Delta: $\delta(x)$: defined by its property

$$\int_{-\infty}^{\infty} \delta(x)f(x)\mathrm{d}x = f(0)$$

  (sampling property)
- Dirac Delta as a limit of discrete functions

$$\delta_n(x) = \begin{cases} 0 & , |x| > 1/n \\ n/2 & , |x| \le 1/n \end{cases}$$

- Dirac Delta as a limit of continuous Gaussian functions, mean 0, var ε^2:

$$\delta_\varepsilon(x) = \frac{1}{\varepsilon\sqrt{\pi}} e^{-x^2/\varepsilon^2}$$

- Properties: $\delta(ax)$, $\delta(f(x))$, integral of $\delta'(x)$
- Write Dirac Delta as a Fourier series (Dirac comb): if $f(x) = \delta(x)$ on $-L < x < L$, then

$$\delta(x) = \frac{1}{2L} \sum_{n=-\infty}^{\infty} c_n e^{in\pi x/L}$$

- Using Dirac Delta to define Green's function: solution to

$$\mathcal{L}G(x;\xi) = \delta(x-\xi)$$

with homogeneous BC

4 Fourier Transform

Fourier Transform:

$$\tilde{f}(k) = \mathcal{F}[f](k) = \int_{-\infty}^{\infty} f(x) e^{-ikx} \mathrm{d}x$$

Inverse Fourier Transform:

$$f(x) = \mathcal{F}^{-1}[\tilde{f}](x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{f}(k) e^{ikx} \mathrm{d}k$$

(Dual property)

Properties:

- FT of xf(x)
- FT of f': $\tilde{f}'(k) = ik\tilde{f}(k)$
- Convolution: $h(x) = (f*g)(x) = \int_{-\infty}^{\infty} f(x-u)g(u)\mathrm{d}u$ iff $\tilde{h} = \tilde{f}\tilde{g}$ Parseval's Theorem:

$$\int_{-\infty}^{\infty} |f(u)|^2 \, \mathrm{d}u = \frac{1}{2\pi} \int_{\infty}^{\infty} \left|\tilde{f}(k)\right|^2 \mathrm{d}k$$

- FT of trig, Heaviside, Delta
- Express Dirac Delta as a FT: *delta* $\longleftrightarrow 1$

5 PDEs

Cauchy problem: include BVP/IVP as auxiliary data (Cauchy data)

Well posed problem:

- Solution exists
- Solution is unique
- Solution depends continuously on auxiliary data

Method of characteristics

1st order, Change coordinates along characteristics and along boundary curve

Classification of 2nd order PDEs

- General form with coefficient functions, compute value of $b^2 - ac$ to classify:
  - $>$0: hyperbolic, two real characteristics, e.g. wave
  - =0: parabolic, one real characteristics, e.g. heat
  - $>$0: elliptic, complex characteristics, e.g. Laplace
- Equations for characteristics
- Hyperbolic equation: use two sets of characteristics as coordinates, canonical form for the equation
- d'Alembert's solution for wave equation: change variable to $x + ct$ and $x - ct$, get solution

$$u = f(x + ct) + g(x - ct)$$

Solving PDEs with FT Heat equation: FT wrt x, solve, convolution of IC and source function/fundamental solution/diffusion kernel

$$S_d(x,t) = \frac{1}{\sqrt{4\pi Dt}} e^{\frac{-x^2}{4Dt}}$$

Duhamel's principle:

- (Hyperbolic/parabolic, i.e. wave/heat equation)
- Greens function (forcing is product of deltas);

$$G = H(t - \tau)S_d(x - \xi, t - \tau)$$

- Built in causal relationship

Poisson's equation (forced Laplace's):

- Higher dim delta function (higher dim intergation)
- Solve with delta forcing:

$$\nabla^2 G(\mathbf{r}; \mathbf{r_0}) = \delta(\mathbf{r} - \mathbf{r_0})$$

- Divergence theorem; get formula for free space Green's function (different in 3D and 2D)

Method of Images

Do something outside of domain to eliminate boundary condition (e.g. another heat source)