Analysis 1 IA Lent

Want a rigorous foundation from limits, convergence, continuity, differentiation and integration (and more)

1 Real Numbers

- Real numbers $\mathbb{R}$: totally ordered field with the least upper bound property (LUBP):
  A set S is non empty and bounded above $\implies$ S has least upper bound (supremum) in $\mathbb{R}$
- Definition of upper bound and least upper bound
- Axiom of Archimedes: two equivalent versions $(\forall M)(\exists N \in \mathbb{N}) N > M$
  $(\forall \varepsilon > 0)(\exists N \in \mathbb{N}) \frac{1}{N} < \varepsilon$
- $\mathbb{Q}$ is dense in $\mathbb{R}$: given reals $a < b$, $\exists$ rational $\frac{p}{q} \in (a, b)$

2 Sequence and Series

Real Sequences

- A sequence (a_n) in X: a function $a : \mathbb{N} \to X$ (assigns an element in X to each natural number)
- Convergence: $(z_n) \to z \iff (\forall \varepsilon > 0)(\exists N)(\forall n > N)\ |z_n - z| < \varepsilon$;
  given any $\varepsilon > 0$, can find N beyond which, z_n stay close to z
- Uniqueness of limits: $(z_n) \to a$ and $(z_n) \to b \implies a = b$
- Laws of limits: add, subtract, multiply, reciprocal
- If a **real** sequence (x_n) converges and $\forall n : x_n \geq a$, then $x_n \to x \geq a$
- Squeeze Theorem (for **real** sequences): If sequences $a_n \to c$, $b_n \to c$ and $\forall n : a_n \leq c_n \leq b_n$, then $c_n \to c$
- Bounded sequences, monotone sequences, subsequences

- Monotone Sequence Theorem (bounded above + monotone increasing $\implies$ convergence): supremum is limit
- Sequence converges $\implies$ any subsequence converges
- Monotone increasing sequence has a convergent subsequence $\implies$ whole sequence converges
- Bolzano-Weierstrass Theorem (lion hunting): every bounded sequence has a convergent subsequence
- Cauchy Sequences: Cauchy $\iff$ convergent in $\mathbb{R}$ (General principle of convergence)
- Subsequence of Cauchy $\to a \implies$ Cauchy $\to a$
- Cauchy completeness
- (Nested interval property: in Gowers' notes)
- ($\limsup$ and $\liminf$, converge $\iff \limsup = \liminf$)

Complex Sequences

- Definition of sequence, convergence are the same (absolute value to modulus)
- Bounded sequence: same definition
- Ordering does not exist: no Squeeze Theorem, Monotone Sequence Theorem
- Have Bolzano-Weierstrass Theorem for complex numbers
- Have Cauchy sequences $\iff$ convergent in $\mathbb{C}$

Series

- Series = infinite sums, convergence in terms of sequence of partial sums
- Series converge $\implies$ n-th term $\to$ zero
- Convergence tests:

- Comparison test (real only):
  sequence bounded above and monotone increasing
- Absolute convergence (real and complex):
  Real: separate into series with +ve/−ve terms, both converge by comparison, then difference also converges;
  (converges absolutely $\iff$ converges to the same limit for any rearrangement)
  Complex: real and imaginary parts converge $\implies$ complex series converge
- Strong comparison (complex):
  use real comparison + absolute
- Alternating series test (real):
  odd partial sums are bounded above and monotone decreasing, thus converge and even partial sums follow
- Ratio test (complex):
  sum of absolute value bounded above by a converging geometric series; bounded below by a GS diverging to ∞
- Cauchy condensation test (real):
  $\sum_{n=1}^{\infty} a_n$ converges $\iff$ $\sum_{k=1}^{\infty} 2^k a_{2^k}$ converges
- (Abel's Test, Integral test, n-th root test)

3 Limits and continuity

Limits (of functions)

- Limit of function in $\mathbb{C}$: $\lim\limits_{z\to a} f(z) = c$
  $\iff$ can make $f(z)$ as close to c as possible by making z sufficiently close to a
  $\iff$ can approach c by $f(z_n)$ with any sequence $z_n \to a$
  i.e. $(\forall\varepsilon > 0)(\exists\delta > 0)\colon 0 < |z - a| < \delta \implies 0 < |f(z) - c| < \varepsilon$
  i.e. $\forall\,(z_n) : z_n \in \mathbb{C}\backslash\{a\} \wedge z_n \to a \implies f(z_n) \to c$
- Laws of limits follow from that of sequences
- Limit point: can get as close to the point as needed (e.g. by sequences) in subsets $A \subset \mathbb{C}$

Continuity

- Continuous as $z = a$:
  $f(x) \to f(y)$ as $x \to y$
  $\iff$ either not limit point or $\lim_{z \to a} f(z) = f(a)$
  $\iff$ any sequence $z_n \to a \implies f(z_n) \to f(a)$
- Continuous at all points in a set = continuous
- (Continuous induction)
- Intermediate value theorem:
  Given $f : [a, b] \to \mathbb{R}$ continuous, $f(a) < 0 < f(b)$, then $\exists c \in [a, b]$ with $f(c) = 0$ (Open or closed interval in result does not matter)
- Maximum Value Theorem:
  Given $f : [a, b] \to \mathbb{R}$ continuous, then f is bounded and $\exists c \in [a, b]$ with $f(c) > f(x) \forall x \in [a, b]$ (i.e. f attains its inf and sup in the interval) (Need closed interval, as f may attain extrema at the bounds)
- Continuous bijection (must be either strictly increasing or decreasing) have continuous inverse
- (Cover of a set)

4 (Real) Differentiation

- A function $f : [a, b] \to \mathbb{R} : f$ is differentiable at $x = a$:
  $\lim_{x \to a} \frac{f(x)-f(a)}{x-a} = c = f'(a)$ for some $c \in \mathbb{R}$
- Alternative characterisation: f differentiable at x with derivative $f'(x)$ if: $f(x + h) = f(x) + hf'(x) + h\alpha(h)$, where $\alpha(h) \to 0$ as $h \to 0$
- Differentiable $\implies$ continuous
- Sum rule, product rule, chain rule, quotient rule (prove that the remaining error term goes to 0)
- Global/local maxima/minima, interior point of an interval
- Differentiable at a local maximum or minimum interior point $c \implies f'(c) = 0$ (Reverse implication not true)

- Rolle's Theorem:
  Given $f : [a, b] \to \mathbb{R}$ continuous on $[a, b]$, and differentiable on (a, b), if $f(a) = f(b) = 0$, then $\exists c \in (a, b)$ with $f'(c) = 0$

- Mean value theorem:
  Given $f : [a, b] \to \mathbb{R}$ continuous on $[a, b]$, and differentiable on (a, b), then $\exists c \in (a, b)$ with $f'(c) = \frac{f(b)-f(a)}{b-a}$

  Use MVT when we know everything about f' but want to deduce sth about f

- Derivative of constant/increasing/strictly increasing function

- Inverse function theorem:
  Given $f : I \to \mathbb{R}$ (I an interval) and $f'(x) > 0 \forall x \in I$, then $f^{-1} : f(I) \to I$ is continuous, differentiable, with derivative

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(y))}$$

- (L'Hôpital's Rule(s))

5 Power series

Want best polynomial approximation to functions, one way is match first k derivatives using a (unique) k-th degree polynomial:

- The k-th Taylor polynomial of f centred at a:

$$p_k(x) = \sum_{i=0}^{k} \frac{f^{(i)}(a)}{i!}(x-a)^i$$

- Taylor's Theorem (Lagrange Remainder):
  For k-times differentiable function f:

$$f(a+h) = \underbrace{\sum_{i=0}^{k-1} \frac{f^{(i)}(a)}{i!}h^i}_{p_{k-1}(a)} + \underbrace{\frac{f^{(k)}(a)}{k!}h^k}_{\text{remainder/error}} \text{ , for some } c \in (a, x)$$

- Differentiability classes: C^n = n-times differentiable with continuous n-th derivative
- Taylor Series:
$$\sum_{i=0}^{\infty} \frac{f^{(i)}(a)}{i!}(x-a)^i$$

Complex Differentiation

- Complex differentiability: same definition, more restrictive (have nice properties, see future courses)
- (Partial differentiation: relation to complex differentiation by writing real part and imaginary parts as bivariate functions, Cauchy Riemann equations, complex differentiable functions satisfy Laplace's equation)
- Complex differentiable with $F'(z) = 0$ for $z \in B_r(c) = \{z \in \mathbb{C} : |z - c| < r\}$
$\implies F$ is constant in $B_r(c)$
- Radius of convergence:
$$R = \sup\left\{|z| : \sum a_n z^n \text{ converges}\right\},$$
$|z| < R \implies$ converges (strong comparison)
$|z| > R \implies$ diverges (by definition)
- Find radius of convergence by ratio test or n-th root test

Want to differentiate power series term by term:

- If $f(z) = \sum a_n z^n$, $g(z) = \sum n a_n z^{n-1}$, then f and g have the same radius of convergence
- Inside circle of convergence,
$$f(z+h) - f(z) - hg(z) = \sum a_n((z+h)^n - z^n - nhz^{n-1})$$
the whole thing is bounded for sufficiently small h, thus f is complex differentiable (in fact, infinitely differentiable)
- Derivatives of partial sums also approach the derivative of the Taylor Series

We then get the useful functions: exp, log, trig using power series, and define π using periodicity of trig

6 Integration

- Dissection: a finite subset

$$D = \{a_0, a_1, ..., a_n | a = a_0 < a_1 < \cdots < a_n = b\} \subset [a, b]$$

- Mesh = $\max\{a_i - a_{i-1}\}$
- Upper sum, Lower sum:

$$U(f, D) = \sum_{i=1}^{n} (a_i - a_{i-1}) \sup\{f(x) | a_{i-1} \leq x \leq a_i\}$$
$$L(f, D) = \sum_{i=1}^{n} (a_i - a_{i-1}) \inf\{f(x) | a_{i-1} \leq x \leq a_i\}$$

($\therefore$ integrable $\implies$ sup and inf exists $\implies$ boundedness)

- For any dissection $D \subset D'$, (using a finer dissection)

$$U(f, D) \geq U(f, D')$$
$$L(f, D) \leq L(f, D')$$

- Any D_1, D_2: (compare with the common refinement $D_i \cup D_2$)

$$U(f, D_1) \geq L(f, D_2)$$

- So upper sums are bounded below, lower sums bounded above, i.e.

$$U(f) = \inf_D U(f, D) \text{ and } L(f) = \sup_D L(f, D) \text{ exist.}$$

- $U(f) = L(f) \implies$ Riemann integrable: $\int_a^b f(x) \mathrm{d}x = U(f) = L(f)$ (Boundedness is necessary)
- Riemann's Integrability Criterion:

Riemann integrable $\iff \forall \varepsilon > 0 \exists$ dissection D s.t. $U(f, D) - L(f, D) < \varepsilon$

- Any increasing f on $[a, b]$ is integrable

Properties of the integral

- Linearity: if $f, g : [a,b] \to \mathbb{R}$ integrable, then
$$\int_a^b (\lambda f(x) + \mu g(x))\mathrm{d}x = \lambda \int_a^b f(x)\mathrm{d}x + \mu \int_a^b g(x)\mathrm{d}x$$
- If $f(x) \leq g(x) \forall x \in [a,b]$, then
$$\int_a^b f(x)\mathrm{d}x \leq \int_a^b g(x)\mathrm{d}x$$
- f integrable $\implies$ $|f|$ integrable
- Additivity: if $f : [a,b] \to \mathbb{R}$ is integrable, then the restriction of f to $[a,c]$ and $[c,b]$ are integrable for any $c \in [a,b]$ and
$$\int_a^b f(x)\mathrm{d}x + \int_a^c f(x)\mathrm{d}x = \int_c^b f(x)\mathrm{d}x$$
- Continuous $\implies$ Riemann integrable (on $[a,b]$)
- Integrable on $[a,b]$ $\iff$ integrable on $[a,c],[c,b]$ for $c \in (a,b)$
-
$$\left| \int_a^b f(x)\mathrm{d}x \right| \leq \int_a^b |f(x)|\mathrm{d}x$$
- Product of integrable is integrable
- (Uniform continuity)
- (Bounded on $[a,b]$ and continuous on (a,b) $\implies$ integrable)
- (Can restrict to only using uniformly spaced dissections: will get same results)

Fundamental Theorem of Calculus

- (V1) Suppose $f : [a,b] \to \mathbb{R}$ continuous, and define $F = \int_a^x f(x)\mathrm{d}x$, then F is differentiable and $F'(x) = f(x)$
i.e. has anti-derivative
- (V2) Notes: F is C^1 $\implies$ $\int_a^b F'(t)\mathrm{d}t = F(b) - F(a)$
- (FTC also true for F differentiable with F' integrable)
- (Differentiable $\not\Longrightarrow$ derivative integrable)

Stuff that follow from FTC

- Integration by substitution: chain rule
- Integration by parts: product rule
- Taylor's theorem with integral remainder:
  Given $f : [a, x] \to \mathbb{R}$ is C^k, then

$$f(x) = \underbrace{\sum_{i=0}^{k-1} \frac{f^{(i)}(a)}{i!}(x-a)^i}_{p_{k-1}(x)} + \underbrace{\int_a^x \frac{(x-a)^{k-1}}{(k-1)!} f^{(k)}(t)\mathrm{d}t}_{\text{remainder}}$$

- Integral Test:
  Given $f : [1, \infty] \to \mathbb{R}$ decreasing and non negative, $\sum_{n=1}^{\infty} f(n)$ converges $\iff \int_1^{\infty} f(x)\mathrm{d}x$ converges
- Improper integrals