

## 1 Differential Geometry of Curves

- Parametrisation of a curve, under some coordinate system (e.g. Cartesian): a function  $\mathbf{x} : [a, b] \rightarrow \mathbb{R}^3$  with

$$\mathbf{x}(t) = \begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix}$$

- In Cartesians: Curve differentiable = components differentiable
- Curve is regular:  $|\mathbf{x}'(t)| \neq 0$
- Arc length of curve:
  1. Partition the interval  $[a, b]$
  2. Calculate sum of straight line lengths at partition points
  3. Take limit as maximum  $\Delta t$  approaches zero
- Piecewise smooth curve: break down into integral of each piece
- Line element:

$$\begin{aligned} ds &= |\mathbf{x}'(t)| dt \\ &= \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} dt \end{aligned}$$

- Tangent, normal, binormal, curvature, torsion

## 2 Integration

- Line integral: parametrise curve  $C$  by  $[a, b] \ni t \rightarrow \mathbf{x}(t)$ 
  - Scalar field  $f(\mathbf{x})$
  - Scalar arc-length element:  $ds = |\mathbf{x}'(t)| dt$ :

$$\int_C f(\mathbf{x}) ds = \int_a^b f(\mathbf{x}(t)) |\mathbf{x}'(t)| dt$$

- Vector field  $\mathbf{F}(\mathbf{x})$   
Vector line element:  $d\mathbf{x} = \mathbf{x}'(t) dt$ :

$$\int_C \mathbf{F}(\mathbf{x}) \cdot d\mathbf{x} = \int_a^b \mathbf{F}(\mathbf{x}(t)) \cdot \mathbf{x}'(t) dt$$

- If the curve  $C$  is a closed loop: closed integral/ circulation about  $C$

$$\oint_C \mathbf{F} \cdot d\mathbf{x}$$

- Area integral: given a region  $D$   
Scalar field  $f(\mathbf{x})$   
Scalar area element (if  $D$  is divided into rectangular elements)  
 $dA = dx dy$

$$\begin{aligned} \iint_D f(\mathbf{x}) dA &= \int_y \int_{X_y} f(x, y) dx dy \\ &= \int_x \int_{Y_x} f(x, y) dy dx \end{aligned}$$

where  $X_y = \{x : (x, y) \in D\}$  (visualise horizontal strips),  
and  $Y_x = \{y : (x, y) \in D\}$  (vertical strips)

- Jacobian: If  $x = x(u, v)$  and  $y = y(u, v)$  are smooth bijection from region  $D'$  in  $(u, v)$  plane to region  $D$  in  $(x, y)$  plane, then

$$\iint_D f(x, y) dx dy = \iint_{D'} f(x(u, v), y(u, v)) \left| \frac{\partial^2(x, y)}{\partial(u \partial v)} \right| du dv$$

$$\text{i.e. } dx dy = |J| du dv$$

$$\text{where } J = \frac{\partial^2(x, y)}{\partial(u \partial v)} = \det \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix} = \det \left( \frac{\partial \mathbf{x}}{\partial u} \middle| \frac{\partial \mathbf{x}}{\partial v} \right)$$

( $|J|$  is the "scale factor" for area)

- Volume integral

$$\iiint_V f(\mathbf{x}) dV$$

Analogue for Jacobian:

$$J = \frac{\partial^3(x, y, z)}{\partial(u \partial v \partial w)} = \det \left( \frac{\partial \mathbf{x}}{\partial u} \middle| \frac{\partial \mathbf{x}}{\partial v} \middle| \frac{\partial \mathbf{x}}{\partial w} \right)$$

Examples:

Cylindrical polars:

$$dV = dx dy dz = \rho d\rho d\theta dz$$

Spherical polars:

$$dV = dx dy dz = r^2 \sin \theta dr d\theta d\phi$$

- Surface integral:

Given surface defined by  $S = \{\mathbf{x} : f(\mathbf{x}) = 0\}$ :  $\nabla f$  is normal to the surface

Given parametrised surface  $S = \{\mathbf{x} : \mathbf{x}(u, v) : (u, v) \in D\}$  for some region  $D$  in  $(u, v)$  plane

Normal vector (of unit length)

$$\mathbf{n} = \frac{\frac{\partial \mathbf{x}}{\partial u} \times \frac{\partial \mathbf{x}}{\partial v}}{\left| \frac{\partial \mathbf{x}}{\partial u} \times \frac{\partial \mathbf{x}}{\partial v} \right|}$$

Scalar and vector area element:

$$dS = \left| \frac{\partial \mathbf{x}}{\partial u} \times \frac{\partial \mathbf{x}}{\partial v} \right| du dv$$

$$d\mathbf{S} = \frac{\partial \mathbf{x}}{\partial u} \times \frac{\partial \mathbf{x}}{\partial v} du dv = \mathbf{n} dS$$

### 3 Div, Grad, Curl and Laplacian: In Cartesians

- Gradient (of scalar field):  $\nabla f$

$$[\nabla f]_i = \frac{\partial f}{\partial x_i}$$

- Gradient of vector field
- Directional derivative:  $D_{\mathbf{v}} f = \mathbf{v} \cdot \nabla f$

$$\mathbf{v} \cdot \nabla f = v_i \frac{\partial f}{\partial x_i}$$

- Divergence:  $\nabla \cdot \mathbf{F}$

$$\nabla \cdot \mathbf{F} = \frac{\partial F_i}{\partial x_i}$$

- Curl:  $\nabla \times \mathbf{F}$

$$[\nabla \times \mathbf{F}]_i = \epsilon_{ijk} \frac{\partial x_j}{\partial F_k}$$

- Laplacian (scalar):  $(\text{div grad}) \nabla^2 f = \nabla \cdot \nabla f$

$$\nabla^2 f = \frac{\partial f}{\partial x_i} x_i$$

- Laplacian (vector field):  $\nabla^2 \mathbf{F} = \nabla(\nabla \cdot \mathbf{F}) - \nabla \times (\nabla \times \mathbf{F})$

$$[\nabla^2 \mathbf{F}]_i = (\nabla^2 F_i) \mathbf{e}_i$$

- $\text{div} \circ \text{curl} = \nabla \cdot \nabla \times \mathbf{F} = 0$
- $\text{curl} \circ \text{grad} = \nabla \times \nabla f = \mathbf{0}$

## 4 Integration Theorems

- Green's Theorem:

If  $P$  and  $Q$  are continuously differentiable scalar fields on  $A \subset \mathbb{R}^2$  and  $\partial A$  is made of collection of smooth curves, then

$$\oint_{\partial A} P dx + Q dy = \iint_A \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx dy$$

- Stokes' Theorem:

If  $\mathbf{F}(\mathbf{x})$  is a continuously differentiable vector field,  $S$  an orientable surface with  $\partial S$  piecewise regular boundary, then

$$\int_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S} = \oint_{\partial S} \mathbf{F} \cdot d\mathbf{x}$$

- Divergence/Gauss' Theorem:

(3D) If  $\mathbf{F}(\mathbf{x})$  is a continuously differentiable vector field,  $V$  a volume with  $\partial V$  piecewise regular boundary, then

$$\int_V \nabla \cdot \mathbf{F} dV = \int_{\partial V} \mathbf{F} \cdot d\mathbf{S}$$

- (2D): (normal points out of the region)

$$\int_D \nabla \cdot \mathbf{F} dA = \oint_{\partial D} \mathbf{F} \cdot \mathbf{n} ds$$

## 5 Maxwell's Equations

## 6 Poisson's Equation

- Laplace's equation (Forcing = 0)
- Dirichlet and Neumann condition, with uniqueness

## 7 Cartesian Tensors

- Definition: A tensor  $T$  of rank  $n$  has components  $\underbrace{T_{ij\dots k}}_{n \text{ indices}}$  transforms from the right handed orthonormal basis  $\mathbf{e}_i$  to  $\mathbf{e}'_i$  under the law

$$T'_{ij\dots k} = R_{ip}R_{jq}\dots R_{kr}T_{pq\dots r}$$

(where  $R_{ij}$  are components of a rotation matrix)

- Scalar (rank 0), vector (rank 1), linear map
- Scaling and adding tensors (of same order) give tensors
- Tensor product:  $(T \otimes S)_{\underbrace{ij\dots k}_{n \text{ indices}} \underbrace{pq\dots r}_{m \text{ indices}}} = T_{ij\dots k}S_{pq\dots r}$
- Contraction: contracting on indices  $i$  and  $j$ :  $\underbrace{S_{p\dots q}}_{n-2} = \delta_{ij}T_{ijp\dots q}$
- Symmetric, antisymmetric (in components in a pair of components); totally symmetric/antisymmetric
- Tensor field of rank  $n$ :  $T_{ij\dots k}(\mathbf{x}_0)$  gives a tensor at each point in space (say  $\mathbb{R}^3$ ), e.g. scalar field, vector field
- Differentiating tensor fields:

$$\frac{\partial}{\partial x'_i} = R_{ij} \frac{\partial}{\partial x_j}$$

(each derivative gives a factor  $R$ )

- $m$ -th partial derivative of rank  $n$  tensor is a tensor of rank  $m + n$ :

$$\left(\frac{\partial}{\partial x_p}\right)\left(\frac{\partial}{\partial x_q}\right)\cdots\left(\frac{\partial}{\partial x_r}\right)T_{ij\dots k}(\mathbf{x})$$

- Divergence theorem:

$$\int_V \frac{\partial T_{ij\dots k\dots l}}{\partial x_k} dV = \int_{\frac{\partial V}{\partial V}} T_{ij\dots k\dots l} n_k dS$$

- Rank 2 tensor: decompose (uniquely) into symmetric and antisymmetric parts:

$$T_{ij} = \underbrace{\frac{1}{2}(T_{ij} + T_{ji})}_{S_{ij}} + \underbrace{\frac{1}{2}(T_{ij} - T_{ji})}_{A_{ij}}$$

And the antisymmetric part

$$A_{ij} = \varepsilon_{ijk}\omega_k$$

where  $\omega_i = \frac{1}{2}\varepsilon_{ijk}T_{jk}$ .

Symmetric part:

$$S_{ij} = \underbrace{P_{ij}}_{\text{traceless}} + \underbrace{\frac{1}{3}\delta_{ij}S_kk}_{\text{isotropic}}$$

- For any symmetric second rank tensor, there exist a choice of right handed Cartesian coords where the matrix is diagonal (real symmetric matrix can be orthogonally diagonalised)
- Isotropic tensor: invariant under any change of bases :

$$T'_{ij\dots k} = R_{ip}R_{jq}\dots R_{kr}T_{pq\dots r} = T_{ij\dots k}$$

- Correspondence between multi-linear maps and tensors:  
If a multi-linear map (well-defined, independent of basis) is given by

$$\underbrace{\mathbb{R}^3 \times \mathbb{R}^3 \times \cdots \times \mathbb{R}^3}_n \rightarrow \mathbb{R}$$

$$(\mathbf{a}, \mathbf{b}, \dots \mathbf{c}) \rightarrow T_{ij\dots k}a_ib_jc_k$$

Then  $T_{ij\dots k}$  is a rank  $n$  tensor.

- Quotient theorem:  
Given array  $T_{i\dots jp\dots q}$ , if

$$v_{i\dots j} := T_{i\dots jp\dots q}u_{p\dots q}$$

is a tensor for *any* tensor  $u_{p\dots q}$ , then  $T_{i\dots jp\dots q}$  are components of a tensor.