Markov Chains   IB Mich

Definition, basic properties

- State space: I, all possible states
- Markov Chain: A sequence of r.v.s $(X_n)_n \geq 0$ with the Markov Property:

$$\mathbb{P}[X_{n+1} = i_{n+1} \mid X_0 = i_0, \dots X_n = i_n] = \mathbb{P}[X_{n+1} = i_{n+1} \mid X_n = i_n]$$

  $\forall n \geq 0,\ i_0, \dots i_{n+1} \in I$ (only depend on last step)

- Homogeneous: $\mathbb{P}[X_{n+1} = j \mid X_n = i] = \mathbb{P}[X_1 = j \mid X_0 = i]$ (same from n to $n+1$ as 0 to 1)
- Markov(λ, P) has:
  1. Initial distribution $\lambda = (\lambda_i)_{i \in I}$
  2. Transition matrix P, where $p_{ij} = P(X_1 = j \mid X_0 = i)$, stochastic matrix
- Equivalent characterisation of Markov chains:

$$\mathbb{P}[X_0 = i_0, \dots, X_n = i_n] = \lambda_{i_0} p_{i_0 i_1} \dots p_{i_{n-1} i_n}$$

- Conditional on $X_m = 1$, starting Markov Chain at a later fixed point $(X_{m+n})_{n \geq 0}$ is Markov(δ_i, P), independent on earlier positions
- Formula for probabilities at n-th step:

$$\mathbb{P}[X_n = j] = (\lambda P^n)_j$$

$$\mathbb{P}_i[X_n = j] = \mathbb{P}[X_n = j | X_0 = i] = (P^n)_{ij}$$

- Computing transition probabilities $p_{ij}^{(n)}$:
  Find eigenvalues (diagonalise/JNF), have

$$p_{ij}^{(n)} = a_1 \lambda_1^n + \dots a_N \lambda_N^n,$$

  if eigenvalues have multiplicities, coefficients become polynomials in n

Class structures

- Lead to ($P_i[X_n = j \text{ for some n}] > 0$), communicates with (equiv rel), communicating classes, irreducible chain (single communicating class)
- Closed subset (only lead to things within), absorbing state (i is absorbing if $\{i\}$ is closed)

Hitting and absorption probabilities

- Hitting time (random variable): $H^A : \Omega \to \{0, 1, 2, \dots\} \cup \{\infty\}$, with

$$H^A(\omega) = \inf\{n \geq 0 : X_n(\omega) \in A\}$$

- Hitting probability (scalar):

$$h_i^A = \mathbb{P}_i[X_n \in A \text{ for some n}] = \mathbb{P}_i[\text{hit A}]$$

(called absorption probability if A is closed)

- Mean hitting time: $k_i^A = \mathbb{E}_i[H_i^A] = \mathbb{E}_i[\text{time to hit } A]$
- Solving for h_i^A: minimal non-negative solution to

$$\begin{cases} h_i^A = 1 & i \in A \\ h_i^A = \sum_{j \in I} p_{ij} h_j^A & i \notin A \end{cases}$$

- Solving for k_i^A: minimal non-negative solution to

$$\begin{cases} k_i^A = 0 & i \in A \\ k_i^A = 1 + \sum_{j \notin A} p_{ij} k_j^A & i \notin A \end{cases}$$

Strong Markov Property

- A random variable T is a stopping time if the event $\{T = n\}$ only depends on $X_0, X_1, \dots X_n$ for all $n \in \mathbb{N}$
- First passage time $T_i = \inf\{n \geq 1 : X_n = i\}$
- Strong Markov property: if (X_n) is Markov(λ, P), and T is a stopping time, then conditional on $T < \infty$ and $X_T = i$, (X_{T+n}) is Markov(δ_i, P) and independent on $X_0, \dots X_T$

Recurrence and Transience

- A state i is
  - Recurrent: $\mathbb{P}_i[X_n = i \text{ for infinitely many } n] = 1$
    $\iff \mathbb{P}_i[T_i < \infty] = 1$
    $\iff \sum_n p_{ii}^{(n)} = \infty$
  - Transient: $\mathbb{P}_i[X_n = i \text{ for infinitely many } n] = 0$
    $\iff \mathbb{P}_i[T_i < \infty] < 1$
    $\iff \sum_n p_{ii}^{(n)} < \infty$
- Class properties
- Every recurrent class is closed
- Every finite closed class is recurrent
- Irreducible and recurrent: for all $j \in I$, $\mathbb{P}[T_j < \infty] = 1$.

Invariant Measures

- For finite state space I, if for some $i \in I$, $p_{ij}^{(n)} \to \pi_j$ as $n \to \infty$ for all $j \in I$, then π_j is an invariant distribution
- $\gamma_i^k = E_k\left[\sum_{n=0}^{T_k-1} 1_{X_n=1}\right]$, expected time spent in state i between two visits to k
- Invariant and recurrent: properties of γ^k (p.21)

Positive recurrent: $E_i[T_i] < \infty$ Convergence to equilibrium