Dynamics and Relativity IA Lent

1 Newtonian Dynamics

- Particle: negligible size, specified by position, mass, charge etc
- Frame of reference: an origin and a set of axes to measure position etc
- Velocity, acceleration, momentum, etc
- Newton's Laws of Motion
    1. There exist inertial frames, where $\mathbf{F} = 0 \implies \ddot{\mathbf{r}} = 0$
    2. $\mathbf{F} = \frac{\mathrm{d}t}{\mathrm{d}(}m\mathbf{v})$
    3. Action-reaction
- Inertial frames are not unique, can be obtained by boost, translation (in space and time), rotation and reflection: Galilean transformations
- Laws of physics are Galilean invariant (in Newtonian)
- There is no absolute velocity
- Given $\mathbf{F}(t), \mathbf{r}(0), \dot{\mathbf{r}}(0)$, behaviour of particle is deterministic

2 Dimensional Analysis

- L, M, T (and temperature perhaps), dimensions must be consistent in equation
- Solve relation by solving linear equations of powers of quantities

3 Forces

- Potential: in 1D force depends on position only $\implies F = \frac{\mathrm{d}V}{\mathrm{d}x}$
- In 3D $\nabla \times \mathbf{F} = \mathbf{0} \implies$ F is conservative, i.e. $\mathbf{F} = -\nabla V$
- Conservation of energy under conservative force
- Fixing energy gives first order DE

- Equilibrium $\iff V' = 0$, stability determined by V'' or gradient and hessian matrix
- Angular momentum $\mathbf{L} = m\mathbf{r} \times \dot{\mathbf{r}}$, conserved if and only if torque $\tau = \mathbf{r} \times \mathbf{F} = 0$
- Central force depend only on distance $V = V(r)$, force points at radial direction, angular momentum is conserved
- Gravity: inverse square law; gravitational potential $\Phi_g(\mathbf{x}) = -\frac{GM}{r}$, gravitational field $\mathbf{g} = -\frac{GM}{r^2}\hat{\mathbf{r}}$
- Electromagnetic force: Lorentz Force Law, attraction/repulsion depends on sign of charges
- Friction: dry friction (static, kinetic), fluid drag (linear, quadratic), terminal velocity, projectile motion example

4 Orbital Motion

- Use polar coordinates to describe motion in an orbit, central mass/charge at origin:

$$\begin{aligned}
\mathbf{r} &= r\mathbf{e_r} \\
\dot{\mathbf{r}} &= \dot{r}\mathbf{e_r} + r\dot{\theta}\mathbf{e}_{\boldsymbol{\theta}} \\
\ddot{\mathbf{r}} &= (\ddot{r} - r\dot{\theta}^2)\mathbf{e_r} + (2\dot{r}\dot{\theta} + r\ddot{\theta})\mathbf{e}_{\boldsymbol{\theta}}
\end{aligned}$$

- Gravity/EM are central forces $\implies$ force points in radial direction, i.e. angular momentum conserved, motion is in a plane
- Let $h = \frac{L}{m}$ = angular momentum per unit mass (determined by initial conditions), acts as a constant for the DE
- Effective potential V_{eff}

$$\begin{aligned}
E &= \frac{1}{2}m\left|\dot{\mathbf{r}}\right|^2 + V(r) \\
&= \frac{1}{2}m\dot{r}^2 + \underbrace{\frac{1}{2}\frac{mh^2}{r^2} + V(r)}_{V_{\text{eff}}}
\end{aligned}$$

- Treat the motion as a 1D problem in terms of r, the distance from origin and V_{eff}
- Stability of circular orbits related to nature of stationary point of V_{eff}
- The orbit equation: use the substitution $u = \frac{1}{r}$ and $\frac{\mathrm{d}}{\mathrm{d}t} = \frac{h}{r^2}\frac{\mathrm{d}}{\mathrm{d}\theta}$, Newton's second law (for the radial motion) becomes

$$m(\ddot{r} - r\dot{\theta}^2) = F(r)$$
$$\implies \frac{\mathrm{d}[}{\mathrm{d}2}]u\theta + u = -\frac{1}{mh^2u^2}\,F\left(\frac{1}{u}\right)$$

- Kepler problem: force in orbit equation given by inverse square law; this shows that orbits are circular, elliptical, parabolic or hyperbolic, characterised by eccentricity
- Kepler's Laws for Planetary Motion:
  1. Elliptical orbits with the sun at one focus
  2. Equal areas: $\frac{1}{2}r^2\dot{\theta}$ is constant (conservation of angular momentum)
  3. $P^2 \propto a^3$, where P = period of orbital motion, a = length of semi-major axis
- Rutherford Scattering: similar framework but repulsive: can find angle of deflection forces

5 Rotating frames of Reference

- Equation of motion between rotating frame and inertial frame For frame of reference rotating about a fixed axis:
- Coriolis force: (only exists for moving object, velocity perceived in the rotating frame)

$$-2m\boldsymbol{\omega} \times \dot{\mathbf{x}}$$

- Centrifugal force: (of order $O(\omega^2)$)

$$-m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{x})$$

- Euler force: (only exists for changing angular velocity)

$$-m\dot{\boldsymbol{\omega}} \times \mathbf{x}$$

6 System of Particles

- A system of N particles, each one ($1 \leq i \leq N$) associated with mass (m_i) and position ($\mathbf{x}_i(t)$).
- 2nd law on each particle:

$$\dot{\mathbf{p}}_i = \mathbf{F}_i^{\text{ext}} + \sum_{j \neq i} \mathbf{F}_{ij}$$

($\mathbf{F}_{ij}$ being force acting on i due to j)

- 3rd law on each pair of particles:

$$\mathbf{F}_{ij} = -\mathbf{F}_{ji}$$

- Define centre of mass

$$\mathbf{R} = \frac{\sum m_i \mathbf{x}_i}{\sum m_i} = \frac{1}{M} \sum m_i \mathbf{x}_i$$

(mass-weighted average position)

- Total momentum:

$$\mathbf{P} = \sum m_i \dot{\mathbf{x}}_i = M\dot{\mathbf{R}}$$

- 2nd law applies to a system since internal forces cancel:

$$\dot{\mathbf{P}} = \sum_i \mathbf{F}_i^{\text{ext}} + \cancelto{0}{\sum_i \sum_{j \neq i} \mathbf{F}_{ij}}$$

- Conservation of total momentum

$$\sum_i \mathbf{F}_i^{\text{ext}} = 0 \implies \dot{\mathbf{P}} = 0$$

- Total angular momentum (about the origin):

$$\mathbf{L} = \sum_i \mathbf{x}_i \times \mathbf{p}_i$$

- Condition for conservation of angular momentum:

$$\begin{aligned}
\dot{\mathbf{L}} &= \sum_i \left(\mathbf{x}_i \times \dot{\mathbf{p}}_i + \cancelto{0}{\dot{\mathbf{x}}_i \times \mathbf{p}_i} \right) \\
&= \sum_i \left(\mathbf{x}_i \times \left(\mathbf{F}_i^{\text{ext}} + \sum_{j \neq i} \mathbf{F}_{ij} \right) \right) \\
&= \underbrace{\sum_i \mathbf{x}_i \times \mathbf{F}_i^{\text{ext}}}_{\text{torque } \boldsymbol{\tau}} + \sum_{i<j} (\mathbf{x}_i - \mathbf{x}_j) \times \mathbf{F}_{ij}
\end{aligned}$$

  Angular momentum conserved if $\boldsymbol{\tau} = 0$ and $\mathbf{F}_{ij} \parallel \mathbf{x}_i - \mathbf{x}_j$

- Separate motion of particle:

$$\mathbf{x}_i = \underbrace{\mathbf{R}}_{\text{motion of CoM}} + \underbrace{\mathbf{y}_i}_{\text{relative to CoM}}$$

$$\sum_i m_i \mathbf{y}_i = \mathbf{0}$$

- Total kinetic energy:

$$\begin{aligned}
T &= \frac{1}{2} \sum_i m_i \dot{\mathbf{x}}_i \cdot \dot{\mathbf{x}}_i \\
&= \frac{1}{2} \sum_i m_i \dot{\mathbf{R}}^2 + \frac{1}{2} \sum_i m_i \dot{\mathbf{y}}_i^2 + \cancelto{0}{\sum_i m_i \dot{\mathbf{R}} \cdot \dot{\mathbf{y}}_i} \\
&= \underbrace{\frac{1}{2} M \dot{\mathbf{R}}^2}_{\text{KE as if mass is concentrated at CoM}} + \underbrace{\frac{1}{2} \sum_i m_i \dot{\mathbf{y}}_i^2}_{\text{KE of particles about CoM}}
\end{aligned}$$

- Change in KE along path:

$$\begin{aligned}
\Delta T &= \sum_i \int_{C_i} \left(\mathbf{F}_i^{\text{ext}} + \sum_{j \neq i} \mathbf{F}_{ij} \right) \cdot \mathrm{d}\mathbf{x}_i \\
&= \sum_i \int_{C_i} \mathbf{F}_i^{\text{ext}} \cdot \mathrm{d}\mathbf{x}_i + \sum_i \sum_{j \neq i} \int_{C_i} \mathbf{F}_{ij} \cdot \mathrm{d}\mathbf{x}_i
\end{aligned}$$

  Energy conserved if all external and internal forces are conservative:

$$\begin{aligned}
\mathbf{F}_i^{\text{ext}} &= -\nabla_i V_i^{\text{ext}} \\
\mathbf{F}_{ij} &= -\nabla_i V_{ij}(|\mathbf{x}_i - \mathbf{x}_j|)
\end{aligned}$$

(∇_i = directional derivative along path C_i)
Then conservation of energy:

$$E = T + \sum_i V_i^{\text{ext}} + \frac{1}{2}\sum_{i,j} V_{ij}(|\mathbf{x}_i - \mathbf{x}_j|)$$

- Two-body problem
- Variable mass, rocket equation

7 Rigid bodies

- Rigid body: Distance between any two particles remain the same (no stretching, only translation of the whole object or rotation at the same angular velocity)
- For an object rotating about an axis through origin with angular speed ω (angular velocity $\boldsymbol{\omega} = \omega\hat{\mathbf{n}}$, direction given by right hand rule):

$$\dot{\mathbf{x}} = \boldsymbol{\omega} \times \mathbf{x}$$

- Kinetic energy of the whole object (wrt origin, lying on axis of rotation):

$$\begin{aligned} T &= \sum_i \frac{1}{2} m_i (\boldsymbol{\omega} \times \mathbf{x_i})^2 \\ &= \frac{1}{2} \underbrace{\left(\sum_i m_i (\hat{\mathbf{n}} \times \mathbf{x}_i)^2\right)}_{I = \sum_i m_i x_{i\perp}^2} \omega^2 \end{aligned}$$

- Angular momentum in terms of moment of inertia:

$$\begin{aligned} \mathbf{L} &= \sum_i \mathbf{x}_i \times (m_i \dot{\mathbf{x}}_i) \\ &= \sum_i m_i \mathbf{x}_i \times (\boldsymbol{\omega} \times \mathbf{x_i}) \\ &= \omega \sum_i m_i \mathbf{x}_i \times (\hat{\mathbf{n}} \times \mathbf{x_i}) \\ \therefore \mathbf{L} \cdot \hat{\mathbf{n}} &= \omega \sum_i m_i \times (\hat{\mathbf{n}} \times \mathbf{x_i})^2 \\ &= I\omega \end{aligned}$$

- Translating everything to integrals (continuum):

	Discrete	Continuum (3D version)
Total Mass	$M = \sum_i m_i$	$M = \int \rho(\mathbf{x})\mathrm{d}V$
Centre of Mass	$\mathbf{R} = \frac{1}{M}\sum_i m_i \mathbf{x}_i$	$\mathbf{R} = \frac{1}{M}\int \rho(\mathbf{x})\,\mathbf{x}\mathrm{d}V$
Moment of Inertia	$I = \sum_i m_i x_{i\perp}^2$	$I = \int \rho(\mathbf{x})\, x_\perp^2 \mathrm{d}V$

- Perpendicular Axis Theorem:
For *a lamina*, with z-axis pointing in normal direction, x, y, z-axes mutually orthogonal:
$$I_z = I_x + I_y$$

- Parallel Axis Theorem:
Given axis through centre of mass, and a parallel axis distance d apart,
$$I = I_{CoM} + Md^2$$

- Inertia tensor of object about origin:
$$\mathcal{I}_{ij} = \int \rho(\mathbf{x})(x_k x_k \delta_{ij} - x_i x_j)\mathrm{d}V$$
$$\mathrm{KE} = \boldsymbol{\omega}^T \mathcal{I} \boldsymbol{\omega},$$
Moment of inertia about any axis through origin: $I = \hat{\mathbf{n}}^T \mathcal{I} \hat{\mathbf{n}}$

- Motion about CoM: break down to motion of CoM and rotation about CoM: if body rotates with angular velocity $\boldsymbol{\omega}$ about CoM:
$$\dot{\mathbf{x}}_i = \dot{\mathbf{R}} + \boldsymbol{\omega} \times (\mathbf{x}_i - \mathbf{R})$$
$$\mathrm{KE} = \frac{1}{2}M\dot{\mathbf{R}} \cdot \dot{\mathbf{R}} + \frac{1}{2}I\omega^2$$

- Angular velocity about any particle/point in the moving body is the same (e.g. tip of a stick): take point $\mathbf{Q}$ moving with body:
$$\mathbf{Q} = \mathbf{R} + (\mathbf{Q} - \mathbf{R})$$
$$\dot{\mathbf{Q}} = \dot{\mathbf{R}} + \boldsymbol{\omega} \times (\mathbf{Q} - \mathbf{R})$$
Then any particle $\mathbf{r}_i$:
$$\dot{\mathbf{r}}_i = \dot{\mathbf{R}} + \boldsymbol{\omega} \times (\mathbf{r}_i - \mathbf{R})$$
$$\therefore \dot{\mathbf{r}}_i = \dot{\mathbf{Q}} + \boldsymbol{\omega} \times (\mathbf{r}_i - \mathbf{Q})$$
In particular if Q is fixed (pivot point), then total KE = rotational KE about pivot (no translational).

8 Special Relativity

- Postulates, Lorentz transformations
- Simultaneity, causality, time dilation, length contraction
- Velocity addition
- Invariant interval, proper time
- Monkowski space, 4-vectors