Fluid II

1 IB Recap

- Continuum Hypothesis, Eulerian/Lagrangian derivative
- Mass conservation, Momentum conservation
- KBC, DBC

2 Newtonian Viscous Flow

- Viscosity definition
- Rate of strain and vorticity: symmetric and antisymmetric parts of $\frac{\partial u_i}{\partial x_j}$, write $\Omega_{ij} = -\varepsilon_{ijk}\Omega_k$, vorticity as twice angular velocity
- Volume forces and surface force (traction), tensor relation between traction and stress tensor: $\tau_i = \sigma_{ij} n_j$ by force balance
- Symmetry of σ by moment balance
- Constitutive equation for Newtonian fluid: linear, instantaneous and isotropic:
$$\sigma_{ij} = -p\delta_{ij} + 2\mu e_{ij}$$
- Cauchy momentum equation:
$$\rho\left(\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j}\right) = F_i + \frac{\partial \sigma_{ij}}{\partial x_j}$$
- For Newtonian fluids, this gives N-S equations
$$\rho\left(\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u}\right) = -\nabla p + \mu\nabla^2\mathbf{u} + \mathbf{F}$$
$$\rho\left(\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j}\right) = -\frac{\partial p}{\partial x_i} + \mu\frac{\partial^2 u_i}{\partial x_j\,\partial x_j} + F_i$$
- Boundary conditions: no slip ($\mathbf{u}$ continuous), stress continuous ($\sigma_{ij} n_j$)

- Energy equation: for fixed volume V

$$\frac{d}{dt}\int_V \frac{1}{2}\rho\,|u|^2\,dV = -\oint_S \frac{1}{2}\rho u^2(u_j n_j)dS + \oint_S u_i\sigma_{ij}n_j dS + \int_V F_i u_i dV - \int_V \frac{\partial u_i}{\partial x_j}\sigma_{ij}dV$$

Change in KE = energy flux + work done due to surface and volume force + stress dissipation

- Non-dimensionalising: Reynolds number ($\mathrm{Re} = \frac{\rho U L}{\mu}$) and Strouhal number ($\mathrm{St} = \frac{L}{UT}$)

3 Unidirectional Flows

Velocity only has one component, $\frac{\mathrm{d}p}{\mathrm{d}x}$ constant

- Plane Couette Flow (shear flow with $\frac{\mathrm{d}p}{\mathrm{d}x} = 0$)
- Plane Poiseuille Flow ($\frac{\mathrm{d}p}{\mathrm{d}x}$) consider flow rate, traction
- Stokes problems: Flow due to impulsively starting plane and oscillating plane
- Velocity diffuses (diffusive scaling $y \sim \sqrt{\nu t}$), similarity solution
- Viscous penetration length for oscillating case
- Flow between cylinders: solve using polars

4 Stokes Flows

- $\mathrm{Re} = 0$: Stokes equations

$$\mu\nabla^2\mathbf{u} = \nabla p - \mathbf{F}, \quad \nabla\cdot\mathbf{u} = 0$$

- Solve for flow and pressure, then compute stress, traction, torque
- Properties: instantaneous, linear, reversible in time and space
- Stokes drag: $\mathbf{F} = 6\pi\mu a\mathbf{U}$

Theoretical Results

- 'Useful Lemma': for Stokes flow $\mathbf{u}^S$ and admissable flow $\mathbf{u}$ (incompressible) in the same volume

$$\int_V 2\mu e^S_{ij} e_{ij} dV = \int_S \sigma^S_{ij} u_i n_j dS$$

- Minimum dissipation theorem: all incompressible flows with the same Dirichlet BCs, Stokes flow has minimum dissipation

- Stokes flow is unique

- Geometric bounding: use minimum dissipation to bound $D = U_i \int_S \sigma_{ij} n_j dS = -\mathbf{F} \cdot \mathbf{U}$

- Reciprocal theorem for two Stokes flow in the same volume V with different BCs on S

$$\int_S u^2_i \sigma^1_{ij} n_j dS = \int_S u^1_i \sigma^2_{ij} n_j dS = \int_V 2\mu e^1_{ij} e^2_{ij} dV$$

- By linearity,

$$\begin{pmatrix} \mathbf{F} \\ \mathbf{G} \end{pmatrix} = \begin{pmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{pmatrix} \begin{pmatrix} \mathbf{U} \\ \mathbf{\Omega} \end{pmatrix}$$

  and the general resistance matrix must be symmetric by reciprocal theorem

- Use symmetry/isometry of resistance tensors

Flow in a corner (2D)

- Use stream function $u_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta}$, $u_\theta = \frac{\partial \psi}{\partial r}$

$$\nabla \cdot \mathbf{u} = \frac{1}{r}\frac{\partial}{\partial r}(r u_r) + \frac{1}{r}\frac{\partial}{\partial \theta}(u_\theta) = 0$$

  Vorticity $\omega = \nabla \times \mathbf{u} = (0, 0, -\nabla^2 \psi)$

  Taking curl of Stokes equation (gradient term drops out) gives $-\nabla^2 \omega = \nabla^2 \nabla^2 \psi = 0$

- Source flow in a corner: some fixed volume flux $Q = \int r u_r d\theta$ with no slip boundary at constant angle

  Try $\psi = Qf(\theta)$ by scaling grounds (purely radial solution), vorticity equation gives $f'''' + 4f'' = 0$

  General solution

$$f = A\cos 2\theta + B\sin 2\theta + C + D\theta$$

  Use symmetry and normalisation (volume flux Q) to fix solution

- Flow past a corner: driven by disturbances far from corner

  Seek solutions of the form $\psi = r^\lambda f(\theta)$, general solution

$$f = A\cos\lambda\theta + B\sin\lambda\theta + C\cos(\lambda-2)\theta + D\sin(\lambda-2)\theta$$

  Get Moffatt Eddies

  Use no slip BC

- Source flow fails when angle of corner too large: eigenvalues have $\mathrm{Re}\lambda > 0$, so disturbances to source flow gives modes of corner flow that are dominant, do not decay far away from corner

5 Thin-layer flow (lubrication theory)

- Almost unidirectional flow $\mathbf{u} = (u, v)$
- Vertical scale h, Horizontal scale L over which h changes and $h \ll L$, low reduced Reynolds number

$$0 = -\frac{\partial p}{\partial x} + \mu\frac{\partial^2 u}{\partial y^2}$$
$$0 = -\frac{\partial p}{\partial y}$$

  So $p = p(x)$, and integrate to find u directly, use BC

- Geometry of problem, solve unidirectional flow, find pressure gradient by mass conservation, and find forces
- Thrust-bearing flow, cylinder approaching wall, Hele-Shaw cell, gravitational spreading of a drop

6 Vorticity

- Take curl of N-S and use vector identity to get

$$\frac{\partial \omega}{\partial t} + (\mathbf{u} \cdot \nabla)\omega = (\omega \cdot \nabla)\mathbf{u} + \nu \nabla^2 \omega$$

  Vortex stretching and diffusion

- Balance of advection and diffusion: downward velocity (e.g. by suction) will trap vorticity in a boundary layer

- Burgers vortex (derive again)

7 Boundary Layer Flow

- Euler limit holds in most of the fluid except boundary layer (scaling $\sim \frac{L}{\sqrt{\mathrm{Re}}}$), regularise discontinuity in velocity/enforce no-slip

- First boundary layer approx: $\frac{\partial^2}{\partial x^2} \ll \frac{\partial^2}{\partial y^2}$

  Viscous term and inertial term balance gives scaling for δ;

  Outside BL (flow tends to $(U(x,t), v)$), drop viscous term and get

$$\rho \left(\frac{\partial U}{\partial t} + U \frac{\partial U}{\partial x} \right) = -\frac{\partial p}{\partial x}$$

- Second BL approx: Variation in pressure along and across the flow: (from x, y momentum equations)

$$\delta p_x \sim \rho U^2$$
$$\delta p_y \sim \rho U^2 \frac{\delta^2}{L^2}$$

  So variation of p across flow is negligible for thin BL (when $\mathrm{Re} \gg 1$)

  Approximate pressure gradient by

$$\left(\frac{\partial p}{\partial x} \right)_{\mathrm{BL}} \approx \left(\frac{\partial p}{\partial x} \right)_{\mathrm{outside}} = -\rho \left(\frac{\partial U}{\partial t} + U \frac{\partial U}{\partial x} \right)$$

- BL equations (flow (u, v), far field $U(x, y)$)
$$\rho\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right)=\rho\left(\frac{\partial U}{\partial t}+U\frac{\partial U}{\partial x}\right)+\mu\frac{\partial^2 u}{\partial y^2}$$
$$\nabla\cdot\mathbf{u}=0$$
$u = 0$ on surface
$u \to U(x, t)$ as $y/\delta \to \infty$

Examples

- BL near a flat plate: use stream function and scaling, get 1D ODE
- 2d momentum jet: balance of inertial and viscous terms, conservation of momentum flux
- acceleration/deceleration: acceleration keeps BL structure
- Wedge: have a generalisation of Stokes limit flow
- BL near a free surface: traction only has pressure, no tangential stress

  Boundary scaling: need to being velocity gradient to 0, use geometry to find scaling $\delta \sim \sqrt{\frac{\nu L}{U}}$

  Use scaling to find dissipation dominated by external flow, not in BL (can use this to find drag force by dissipation of irrotational flow)
- Compare drag coefficient of rigid sphere, bubble in Stokes flow vs high Re flow

8 Stability of unidirectional inviscid flow

- Add perturbation to steady flow, investigate linear stability
- Example: Kelvin-Helmholtz instability: add irrotational flow to two layers of flow
- Kinemtic BC at interface: $\frac{\mathrm{D}}{\mathrm{D}t}y - \eta(x, t) = 0$, so
$$v=\frac{\partial\eta}{\partial t}+u\frac{\partial\eta}{\partial x}$$
  Dynamic boundary: only have pressure continuous

  Linearise Bernoulli for unsteady potential flow