Metric and Topological Spaces IB Mich

1 Metric Spaces

- Metric Space = (X, d), a set X, a metric d, satisfying the axioms:
  1. $d(a, b) \geq 0$, with equality iff $a = b$ (Positive definite)
  2. $d(a, b) = d(b, a)$ (Symmetric)
  3. $d(a, c) \leq d(a, b) + d(b, c)$ (Triangle inequality)
- Examples:
  - On $\mathbb{R}^n$: discrete metric (1 if equal, 0 otherwise), Euclidean metric, Manhattan metric, British railway metric
  - On $\mathbb{Z}$: p-adic metric
  - On $C[0, 1]$: uniform metric ($d(f, g) = \max_{x \in [0,1]} |f(x) - g(x)|$)
  - Counting (Hamming metric)
- Metric subspace: subset with same metric
- Convergence and continuity can be defined via metric:
  (see below for alternative definitions of continuity)
- Notions in vector space naturally give metric space:
  - Norm: (positive definite, triangle inequality, scalable), e.g. L^1, L^2, uniform norm L^∞
  - Inner product: (positive definite, symmetric, linear), e.g. dot product, integral etc. Cauchy-Schwarz give triangle inequality

Open and closeness

- Open ball: $B_r(x) = \{y \in X : d(x, y) < r\}$
  Closed ball: $\bar{B}_r(x) = \{y \in X : d(x, y) \leq r\}$
- Open subset: $U \subseteq X$ is open if $\forall x \in U$, $\exists \delta > 0$ s.t. $B_\delta(x) \subseteq U$
  (Every point is interior point)
  Closed subset: C is closed in X if $X \backslash C$ is open
  (Open or close is a property of the *subset*, depends on the parent set as well)

- Open neighbourhood of x in X: an open subset in X containing x
- Limit point: any point x s.t. there exists sequence $x_n \to x$
  (can be approached using a sequence)
- $C \subseteq X$ is closed iff every limit point of C belongs to C
- Properties:
  - $\emptyset$ and X are open subsets of X
  - Union (finite or infinite, both countable and uncountable) of open sets is an open set
  - Finite intersection of open sets is open

Alternate characterisation of continuity

Using open/close sets, or $\varepsilon - \delta$ definition:
Given metric spaces (X, d_x) and (Y, d_y), $f : X \to Y$, f is continuous

- $\forall x_n \to x$, $f(x_n) \to f(x)$
- $U \subseteq Y$ open $\implies f^{-1}(U) \subseteq X$ open
- $C \subseteq Y$ open $\implies f^{-1}(C) \subseteq X$ closed
- $\forall x \in X$, $\forall \varepsilon > 0$, $\exists\, \delta > 0$ such that $f(B_\delta(x)) \subseteq B_\varepsilon(f(x))$

Composition of continuous functions is continuous

- A sequence x_n is Cauchy if for all $\varepsilon > 0, \exists N(\varepsilon)$ s.t.

  $$d(x_m, x_n) < \varepsilon \text{ whenever } n, m \geq N(\varepsilon)$$

  (same as IA Analysis)
- A metric space is complete if every Cauchy sequences converges
  (not a topological property)

2 Topological Spaces

- Topological Space: a set X (the space) and $\tau \subseteq \mathbb{P}(X)$, (the topology) s.t.
  - $\emptyset, X \in \tau$
  - $V_\alpha \in \tau$ for all $\alpha \in A \implies \bigcup_{\alpha \in A} V_\alpha \in \tau$ (finite or ctbly/unctbly infinite)
  - $V_1, V_2, \dots V_n \in \tau \implies \bigcap_{i=1}^n V_i \in \tau$ (must be finite)

  (X is the collection of points, and τ is all subsets we *designate* to be open)

- Induced topology:
  From a given metric d, set τ to be the open subsets of X under d

- Examples:
  - Coarse/indiscrete Topology: $\tau = \{\emptyset, X\}$
  - Discrete Topology: all subsets $\tau = \mathbb{P}(X)$ (from discrete metric)
  - Cofinite Topology: $\tau = \{A \subseteq X : X \backslash A$ is is finite or $A = \emptyset\}$
  - Right Order Topology on $\mathbb{R}$: $\tau = \{(a, \infty) : a \in \mathbb{R}\} \cup \{\mathbb{R} = (-\infty, \infty), \emptyset\}$

- Continuity, close subsets can be defined only using open sets given by topology

Interior and Closure

Given a topological space (X, τ), and $A \subseteq X$,

- The interior of A is the union of all open sets, i.e. the largest open set contained in A:
$$\text{Int } (A) = \bigcup \{U \in \tau : U \subseteq A\}$$

- The closure of A is the intersection of all closed sets, i.e. the smallest closed set containing A:
$$\text{Cl } (A) = \bigcap \{F \text{ closed} : F \supseteq A\}$$

- $$(\text{Cl }(A^c))^c = \text{Int }(A), \quad (\text{Int }(A^c))^c = \text{Cl }(A)$$

- In a metric space (X, d), the closure of $A \subseteq X$ adds in all the limit points of A in X

- A subset $A \subseteq X$ is a dense in $F \subseteq X$ if Cl $(A) = F$

Homeomorphism

A function $f : X \to Y$ is a homeomorphism ($X \simeq Y$, homeomorphic, regarded as the same) if

- f is a bijection

- both f and f^{-1} are continuous

(Must require inverse to also be continuous, since continuous function might not continuous inverse)

- Homeomorphism is an equivalence relation

- Topological properties are preserved by homeomorphisms (things defined only using open sets)

Sequences

- Open neighbourhood can now be defined using the open sets given by the topology: an open subset $U \subseteq X$ s.t. $x \in U$

- Convergent sequence: $x_n \to x$ if for any open neighbourhood U of x, $\exists N$ s.t. $x_n \in U$ for all $n > N$

- A limit point $x \in A \subseteq X$ has sequence $x_n \to x$ where $x_n \in A$ for all n (i.e. for all open neighbourhood U of x, $U \cap A \neq \emptyset$)
(Alternative defn's exist)

- If U is closed, it contains all its limit points (converse not true for topological space)

In this definition, limit of sequence (if exists) may not be unique, but a special class, Hausdorff space:

- X is a Hausdorff space if for any $x_1, x_2 \in X$, there exist open neighbourhoods U_1 of x_1, and U_2 of x_2, s.t. $U_1 \cap U_2 = \emptyset$
(Can separate any two points by open neighbourhoods)

- Any sequence in a Hausdorff space has at most one limit

- Is a topological property

3 New Topological Spaces From Old

Subspace topology

If (X, τ_X) is a topology, $Y \subseteq X$, then the subspace topology on (Y, τ_Y) is given by $\tau_Y = \{Y \cap U \mid U \in \tau_X\}$
(open sets in Y are given by intersecting Y with open sets in X)

- If $Y \subseteq X$, with inclusion $\iota : Y \to X$, then if $f : Z \to Y$ is continuous iff $\iota \circ f : Z \to X$ is continuous

(Defining property: Topology on Y is the smallest topology on Y for which the inclusion $\iota Y \to X$ is continuous)

Product topology

- Basis: for a topological space (X, τ), a subset $\mathcal{B} \subset \tau$ is a basis for the topology if every $U \in \tau_X$ is a union of elements in $\mathcal{B}$

- If $(X, \tau_X), (Y, \tau_Y)$ are topological spaces, define product topology on $X \times Y$ using basis

$$\mathcal{B} = \{U_X \times U_Y \mid U_X \in \tau_X, U_Y \in \tau_y\}$$

  Alternatively, define by: $V_X \times V_Y \subseteq X \times Y$ is open if for all $(x, y) \in V_X \times V_Y$, there exist open neighbourhoods U_X of x, U_Y of y s.t. $(x, y) \in U_X \times U_Y \subseteq V_X \times V_Y$

  Projection maps are continuous

(Defining property: f is continuous iff $\pi_i \circ f$ are continuous)

Quotient topology

Given $\sim$ an equivalence relation on X, quotient map $q : X \to X/\sim$ by $q(x) = [x]$, define quotient topology by: U is open in $X/\sim$ if $q^{-1}(U)$ is open in X

Quotient maps are continuous by construction of $X/\sim$

(Defining property: $f : X \sim\to Y$ is continuous if and only if $f \circ q : X \to Y$ is continuous)

4 Connectivity

Topological space X is disconnected if $X = A \cup B$, union of two non-empty disjoint open sets
(topological property of the space)

- X is disconnected iff $\exists$ homeomorphism $X \to \{0, 1\}$ with discrete topology
  (alternative characterisation)
- $f : X \to Y$ cts, X connected, then $\operatorname{im} f$ connected.
- Path from x_0 to x_1: continuous $\gamma : [0, 1] \to X$ s.t. $\gamma(0) = x_0$ and $\gamma(1) = x_1$
- Path connected: there is a path between any 2 points
- Path connected $\implies$ connected
- $f : X \to Y$ homeomorphism, then restricting to any subset $A \subset X$, $f|_A : A \to f(A)$ is also homeomorphism
  Homeomorphic spaces stay homeomorphic after taking away points
- (n-connectedness)

Components: cut up disconnected space into components

- Path components: $x \sim y$ if there exists path from x to y, take $X/\sim$ the equivalence classes

- Connected components (regular connectivity):

$$C(x) = \bigcup \{\text{connected subsets of } X \text{ containing } x\}$$

  , also connected, equivalence classes
  (read notes again)
  maximal connected subspaces

5 Compactness

Open cover of X: a family of open sets $\{U_\alpha : \alpha \in A\}$ such that $\bigcup_{\alpha \in A} U_\alpha = X$
Compactness: every open cover of X has a finite subcover

- Finite subspace is compact, $[0, 1]$ is compact (read proof again)
- Closed subset of compact space is compact (as subspace topology)
- If X Hausdorff, C compact $\implies$ closed in X
- Compact metric space is bounded
- Heine-Borel
- Image of compact set under continuous map is compact
- Maximum value theorem
- If X and Y are compact, then $X \times Y$ is compact
- Compact metric space is complete

Sequential Compactness:
X is sequentially compact if any sequence (x_n) in X has convergent subsequence
Equivalent to compactness for metric spaces