Asymptotic Methods II

1 Basic Definitions

- Big/little O notation, asymptotic equality, asymptotic sequence, asymptotic expansion, uniqueness

2 Approximation of Integrals

Watson's Lemma

Suppose

$$f(t) \sim \sum_{n=0}^{\infty} a_n t^{\alpha+\beta n} \text{ as } t \to 0^+$$

$\alpha > -1$

1. $|f(t)| < Ke^{bt}$ for all $t > 0$, or
2. $\int_0^T |f(t)|\,\mathrm{d}t < \infty$

Then

$$\int_0^T e^{-xt} f(t)\,\mathrm{d}t \sim \sum_{n=0}^{\infty} a_n \frac{\Gamma(\alpha+\beta n+1)}{x^{\alpha+\beta n+1}} \text{ for } x \to +\infty$$

Laplace's Method

$$F(x) = \int_a^b f(t) e^{x\phi(t)}\,\mathrm{d}t \text{ as } x \to \infty$$

- Expand ϕ about global maximum, then evaluate $\int e^{-s^p}\,\mathrm{d}s$: become gamma function

Method of Stationary Phase

$$F(x) = \int_a^b f(t) e^{ix\phi(t)}\,\mathrm{d}t \text{ as } x \to \infty$$

- Riemann Lebesgue Lemma: the above integral tends to 0 if f integrable
- If ϕ monotonic, only have contributions from two ends, decay as $O(1/x)$ (by parts)
- If ϕ has stationary points, have slower decay than $O(1/x)$: expand near stationary point, get $\int e^{-is^2}\,\mathrm{d}s$
- All stationary points contribute to asymptotic behaviour

Method of Steepest Descent

- Deform contour in complex plane, can approximate better (e.g. by Watson/ Laplace) on lines of constant $\mathrm{Im}(\phi)$ (lines of steepest descent contours)
- On saddle points $\phi'(c) = 0$, choose contour where $\mathrm{Re}(\phi(z))$ decreases away from c.
- Evaluate integrals with constant phase using Laplace (approximating contour as straight line locally)

3 Airy function

Airy equation $y'' = xy$: solutions are

$$\int_C \exp\left(\frac{1}{3}t^3 + xt\right) \mathrm{d}t$$

for contour C starting and ending where integral is defined

$$\begin{aligned}
\mathrm{Ai}(z) &= \frac{1}{\pi}\int_0^\infty \cos\left(\frac{1}{3}t^3 + xt\right) \mathrm{d}t \\
&\sim \frac{1}{2\sqrt{\pi}} x^{-1/4} e^{-2x^{3/2}/3} && \text{, as } x \to +\infty \\
&= \frac{1}{\sqrt{\pi}}(-x)^{-1/4} \cos\left(\frac{2}{3}(-x)^{3/2} - \frac{\pi}{4}\right) && \text{, as } x \to -\infty
\end{aligned}$$

4 Solution to 2nd order ODE

4.1 Liouville Green Method

- Solve $y'' = Q(x)y$ about an irregular singular point
- Let $y = e^{S(X)}$, then
$$S'' + (S')^2 = Q(x)$$
Suppose Q slowly varying, then first approximation is $S_0' = \pm\sqrt{Q}$, i.e. $S_0 = \pm \int \sqrt{Q}\,\mathrm{d}x$.
- Adding another order: $S_1 \sim \frac{-1}{4}\log Q$
- Can write recurrent relation for higher orders
- Liouville Green approximation is sum of $\pm$ solutions for S

WKBJ Method

- Solve $\varepsilon^2 y'' = q(x)y$ with small ε
- Asymptotic solution
$$y = \sum_{\pm} A_{\pm} q^{-1/4} \exp\left(\pm\frac{1}{\varepsilon}\int \sqrt{q(x)}\,\mathrm{d}x\right)$$

Turning point

- Method fails near $z = a$ if $q(a) = 0$, called a turning point
- Use local approximation: let $q'(a) = \mu$ (wlog $\mu > 0$,so $q > 0$ for $x > a$), then near a,
$$\varepsilon^2 y'' \approx \mu(x-a)y,$$
Substitute $z = \left(\frac{\mu}{\varepsilon^2}\right)^{1/3}(x-a)$ gives Airy equation
$$\frac{\mathrm{d}^2 y}{\mathrm{d}z^2} = zy$$

- The Airy function gives the approximate local solution that decays for large x:

$$
\begin{aligned}
y_0 &\approx B\mathrm{Ai}(z) \\
&\approx \frac{B}{2\sqrt{\pi}} z^{-1/4} \exp\left(-\frac{2}{3}z^{3/2}\right) \\
&= \frac{B}{2\sqrt{\pi}\left(\left(\mu/\varepsilon^2\right)^{1/3}(x-a)\right)^{1/4}} \exp\left(-\frac{2}{3}\frac{\sqrt{\mu}}{\varepsilon}(x-a)^{3/2}\right) \qquad \text{, as } x \to +\infty \\
y_0 &\approx \frac{B}{\sqrt{\pi}}(-z)^{-1/4} \cos\left(\frac{2}{3}(-z)^{3/2} - \frac{\pi}{4}\right) \\
&= \frac{B}{\sqrt{\pi}\left(\left(\mu/\varepsilon^2\right)^{1/3}(a-x)\right)^{1/4}} \cos\left(\frac{2}{3}\frac{\sqrt{\mu}}{\varepsilon}(a-x))^{3/2} - \frac{\pi}{4}\right) \quad \text{, as } x \to -\infty
\end{aligned}
$$

- WKBJ solution is valid far away from $x = a$, and by bringing this close to $x = a$ where linear approximation of q is valid, have asymptotic behaviour:

$$
\begin{aligned}
y_+ &\approx Aq^{-1/4} \exp\left(-\frac{1}{\varepsilon}\int_a^x \sqrt{q(t)}\,\mathrm{d}t\right) \\
&\approx \frac{A}{(\mu(x-a))^{1/4}} \exp\left(-\frac{1}{\varepsilon}\int_a^x \sqrt{\mu(t-a)}\,\mathrm{d}t\right) \\
&= \frac{A}{(\mu(x-a))^{1/4}} \exp\left(-\frac{2}{3}\frac{\sqrt{\mu}}{\varepsilon}(x-a)^{3/2}\right) \qquad \text{, for } x > a, \\
y_- &\approx C(-q)^{-1/4} \cos\left(\frac{1}{\varepsilon}\int_x^a \sqrt{-q(t)}\,\mathrm{d}t - \gamma\right) \\
&\approx \frac{C}{(\mu(a-x))^{1/4}} \cos\left(\frac{1}{\varepsilon}\int_x^a \sqrt{\mu(a-t)}\,\mathrm{d}t - \gamma\right) \\
&= \frac{C}{(\mu(a-x))^{1/4}} \cos\left(\frac{2}{3}\frac{\sqrt{\mu}}{\varepsilon}(a-x)^{3/2} - \gamma\right) \qquad \text{, for } x < a,
\end{aligned}
$$

Matching asymptotics for both gives:

$$
\gamma = \frac{\pi}{4}, \quad B = 2\sqrt{\pi}(\mu\varepsilon)^{-1/6}, \quad C = 2A
$$

Connection formula:

$$
\begin{aligned}
y_+ &= Aq^{-1/4}\exp\left(-\frac{1}{\varepsilon}\int_a^x \sqrt{q(t)}\,\mathrm{d}t\right) && \text{, for } x > a, x - a \gg \varepsilon^{2/3} \\
y_0 &= 2\sqrt{\pi}(\mu\varepsilon)^{-1/6}\mathrm{Ai}(z) && \text{, for } |x-a| \ll 1 \\
y_- &= 2A(-q)^{-1/4}\cos\left(\frac{1}{\varepsilon}\int_a^x \sqrt{-q(t)}\,\mathrm{d}t - \frac{\pi}{4}\right) && \text{, for } x < a, a - x \gg \varepsilon^{2/3}
\end{aligned}
$$

Two turning points in q and bound state

- Suppose $a < b$ are the turning points and $q > 0$ outside them
- Want a solution that decays exponentially on either sides, so

$$
\begin{aligned}
y_1 &\sim Aq^{-1/4}\exp\left(-\frac{1}{\varepsilon}\int_x^a \sqrt{q(t)}\,\mathrm{d}t\right) && \text{, for } x < a \\
y_3 &\sim Bq^{-1/4}\exp\left(-\frac{1}{\varepsilon}\int_b^x \sqrt{q(t)}\,\mathrm{d}t\right) && \text{, for } x > b
\end{aligned}
$$

Then by connection formula, the region in the middle ($a < x < b$) has two asymptotic expansions that need to match

$$
\begin{aligned}
y_2 &\approx 2A(-q)^{-1/4}\cos\left(\frac{1}{\varepsilon}\int_a^x \sqrt{-q(t)}\,\mathrm{d}t - \frac{\pi}{4}\right) \\
y_2 &\approx 2B(-q)^{-1/4}\cos\left(\frac{1}{\varepsilon}\int_x^b \sqrt{-q(t)}\,\mathrm{d}t - \frac{\pi}{4}\right)
\end{aligned}
$$

So we need (by compound angle formula)

$$
\frac{1}{\varepsilon}\int_a^b \sqrt{-q(t)}\,\mathrm{d}t = \left(n + \frac{1}{2}\right)\pi,
$$

where $n = 0, 1, 2, \ldots$, and $A = (-1)^n B$.