Waves II

1 Sound Waves

- Governing equations for inviscid, compressible fluid:

  Conservation of mass

  $$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{u}) = 0, \text{ or}$$
  $$\frac{\mathrm{D}\rho}{\mathrm{D}t} = \frac{\partial \rho}{\partial t} + \boldsymbol{u} \cdot \nabla \rho = -\rho(\nabla \cdot \boldsymbol{u})$$

  KBC: only normal component, cannot impose no-slip

  Momentum conservation: Euler's equation

  $$\rho \frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} = \rho \left(\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} \right) = -\nabla p + \boldsymbol{F}$$

  DBC: e.g. free surface, surface tension

  Equation of state: assume homentropic ideal gas: entropy of fluid parcel same everywhere and unchanged over timescale of motion

  $$p = p_0 \left(\frac{\rho}{\rho_0} \right)^{\gamma}$$

  Internal energy change $\mathrm{d}e = -p\,\mathrm{d}V = \frac{p}{\rho^2}\,\mathrm{d}\rho$ and enthalpy $H = e + \frac{p}{\rho}$

Linear sound Waves

- Perturb from base state ($\boldsymbol{u} = 0, \rho = \rho_0, p = p_0$), get linearised equations: (equation of state, mass and momentum)

  $$\tilde{p} = c_0^2 \tilde{\rho}$$
  $$\frac{\partial \tilde{\rho}}{\partial t} = -\rho_0 (\nabla \cdot \boldsymbol{u})$$
  $$\rho_0 \frac{\partial \boldsymbol{u}}{\partial t} = -\nabla \tilde{p}$$

  where $c_0^2 = \left(\frac{\partial p}{\partial \rho} \right)_{\rho_0, S}$ is the sound speed.

  Combining gives wave equation

  $$\frac{\partial^2 \tilde{\rho}}{\partial t^2} = c_0^2 \nabla^2 \tilde{\rho}$$

- Condition for small perturbation: Mach number $M = \frac{U}{c_0} \ll 1$
- Vorticity independent of time, so can use acoustic velocity potential for irrotational flow:

  $$\boldsymbol{u} = \nabla\phi$$

  Then taking time derivatives gives

  $$\frac{\partial\phi}{\partial t} = -\frac{\tilde{p}}{\rho_0}$$
  $$\frac{\partial^2\phi}{\partial t^2} = c_0^2\nabla^2\phi$$

- Plane wave solutions to wave equation for any unit vector $\hat{\boldsymbol{k}}$ (c.f. D'Alembert): :

  $$f(\hat{\boldsymbol{k}} \cdot \boldsymbol{x} - c_0 t)$$

- Harmonic waves satisfy dispersion relation (non-dispersive):

  $$\omega = \pm c_0 \left|\boldsymbol{k}\right|$$

Energetics

- Total energy density is KE + internal energy due to pressure

  $$E = \rho\left[\frac{1}{2}\left|\boldsymbol{u}\right|^2 + \int^{\rho} \frac{p(\hat{\rho}, S)}{\hat{\rho}^2}\,\mathrm{d}\hat{\rho}\right]$$

- Conservation of energy follows from mass and momentum:

  $$\frac{\partial E}{\partial t} + \nabla \cdot (\boldsymbol{u}E + p\boldsymbol{u}) = 0$$

  Integral form: change in E = advection of E into volume + rate of working of p on surface

- Linear waves:

  $$\frac{\partial}{\partial t}\left[\underbrace{\frac{1}{2}\rho_0\left|\boldsymbol{u}\right|^2}_{K} + \underbrace{\frac{1}{2}\frac{c_0^2\tilde{\rho}^2}{\rho_0}}_{W}\right] + \nabla \cdot (\tilde{p}\boldsymbol{u}) = 0$$

  Rate of change of KE + PE (compression) = energy flux

- Have acoustic density $\boldsymbol{I} = \tilde{p}\boldsymbol{u}$, instantaneous equipartition of K and W, group velocity = phase velocity
- Taking time average:

$$\left\langle \mathrm{Re}(Ae^{i\omega t})\,\mathrm{Re}(Be^{i\omega t})\right\rangle = \frac{1}{2}\,\mathrm{Re}(AB^*)$$

- Example: transmission and reflection, radiation condition

2 Elastic Wave in solids

- Here u is displacement, v velocity, a acceleration

$$\rho a_i = F_i + \frac{\partial \sigma_{ij}}{\partial x_j}$$

- Stress tensor, symmetry, and energy equation

$$\frac{d}{dt}\int_V \frac{1}{2}\rho\,|v|^2\,dV = \oint_S v_i\sigma_{ij}n_j dS + \int_V F_i v_i dV \\ - \int_V \sigma_{ij}\frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right)dV$$

- Decompose deformation gradient, constitutive equation (linear, isotropic and uniform material)

$$\sigma_{ij} = \lambda\delta_{ij}e_{kk} + 2\mu e_{ij}$$
$$e_{ij} = \frac{1}{2\mu}\left[\sigma_{ij} - \frac{\lambda}{3\lambda+2\mu}\delta_{ij}\sigma_{kk}\right]$$

- Simple deformation: hydrostatic stress/pressure, simple shear, coaxial extension
- Pressure

$$p = -\frac{1}{3}\sigma_{ii} = -\left(\lambda + \frac{2\mu}{3}\right)e_{kk} = -\kappa\theta$$

  - Dilitation $\theta = e_{kk}$
  - Bulk modulus $\kappa = \lambda + \frac{2\mu}{3}$
  - Shear modulus μ
  - Young's modulus $E = \frac{(3\lambda+2\mu)\mu}{\lambda+\mu}$
  - Poisson's ratio $\nu = \frac{\lambda}{2(\lambda+\mu)}$

Wave equation for elastic solids

- Substitute linear constitutive equation into momentum equation (linearise $a_i = \frac{\partial^2 \boldsymbol{u}}{\partial t^2}$):

$$\begin{aligned}\rho_0 \frac{\partial^2 u_i}{\partial t^2} &= (\lambda + \mu)\nabla(\nabla \cdot \boldsymbol{u}) + \mu \nabla^2 \boldsymbol{u} \\ &= (\lambda + 2\mu)\nabla(\nabla \cdot \boldsymbol{u}) - \mu \nabla \times (\nabla \times \boldsymbol{u})\end{aligned}$$

Take curl and div respectively: $\theta = \nabla \cdot \boldsymbol{u}$ and $\boldsymbol{\omega} = \nabla \times \boldsymbol{u}$ satisfy

$$\begin{aligned}\frac{\partial^2 \boldsymbol{\omega}}{\partial t^2} &= \frac{\mu}{\rho_0}\nabla^2 \boldsymbol{\omega} = c_s^2 \nabla^2 \boldsymbol{\omega} \\ \frac{\partial^2 \theta}{\partial t^2} &= \frac{\lambda + 2\mu}{\rho_0}\nabla^2 \theta = c_p^2 \nabla^2 \theta\end{aligned}$$

P waves have phase speed $c_p = \left(\frac{\lambda+2\mu}{\rho_0}\right)^{\frac{1}{2}}$

S waves have phase speed $c_s = \left(\frac{\mu}{\rho_0}\right)^{\frac{1}{2}}$

Energy (when $F_i = 0$):

$$\frac{d}{dt}\int_V \left(\frac{1}{2}\rho_0 \left|\dot{\boldsymbol{u}}\right|^2 + \frac{1}{2}\sigma_{ij} e_{ij}\right) dV = \oint_S \dot{\boldsymbol{u}} \cdot \boldsymbol{\sigma} \cdot \boldsymbol{n} dS$$

Rate of change of KE + elastic PE = integral of energy flux $\boldsymbol{I} = -\dot{\boldsymbol{u}} \cdot \boldsymbol{\sigma}$

Plane waves

Form of plane wave

$$\boldsymbol{f}(\hat{\boldsymbol{k}} \cdot \boldsymbol{x} - c_0 t)$$

- P-waves at speed c_p parallel to $\hat{\boldsymbol{k}}$, longitudinal waves
- S-waves at speed c_s perpendicular to $\hat{\boldsymbol{k}}$, transverse waves, have polarisation
- Energy: equipartition of K and W, group velocity same as phase velocity

Harmonic waves

- Have P, SH (horizontal), SV (same plane as $\hat{\boldsymbol{k}}$ and $\boldsymbol{e_z}$)

- Reflection transmission: Snell's Law evanescence, total internal reflection
- SH decoupled, P and SV coupled
- Rayleigh waves: P and SV coupled have self sustained evanescence at a boundary

3 Dispersive Wave

- BC impose dispersion relation, cut-off frequency, group velocity
- Love waves: waveguide (2 boundaries with TIR) and SH wave can self-sustain
- Superposition of waves of nearly equal frequencies: modulated waves with beating
- Using FT, get superposition of harmonic solutions
- Method of stationary phase: waves disperse, if one travels at some velocity V wrt wave, might have stationary phase, then wave disperse less quickly (Ch.12)
- Example: Linear water waves, dispersion relation in the limit of deep and shallow waves
- In stratified fluid, have internal gravity waves:

  Density $\rho = \rho_0(z) + \tilde{\rho}$, $p = p_0 + \tilde{p}$, $\boldsymbol{u} = (u, v, w)$ have equation

  $$\rho_0 \frac{\partial^2}{\partial t^2} \left[\nabla^2 w + \frac{1}{\rho_0} \frac{\mathrm{d}\rho_0}{\mathrm{d}z} \frac{\partial w}{\partial z} \right] = g \frac{\mathrm{d}\rho_0}{\mathrm{d}z} \nabla_h^2 w$$

  Drop second term on left by Boussinesq approximation:

  $$\frac{\partial^2}{\partial t^2} \nabla^2 w - \frac{g}{\rho_0} \frac{\mathrm{d}\rho_0}{\mathrm{d}z} \nabla_h^2 w = 0$$

  Buoyancy frequency $N^2 = -\frac{g}{\rho_0} \frac{\mathrm{d}\rho_0}{\mathrm{d}z}$

  Plane harmonic waves must point in particular directions: dispersion relation

  Internal gravity waves are transverse waves

- Wave with moving source, Doppler effect:

$$\omega' = \omega - \boldsymbol{k} \cdot \boldsymbol{U} = \Omega_0(\boldsymbol{k}) - \boldsymbol{k} \cdot \boldsymbol{U}$$

  Identify where steady waves $\omega = 0$ are: capillary wave, supersonic boom, Kelvin shop waves (duck)

4 Ray Theory

- Slowly varying media: varying over large length/time scale compared to wavelength/period
- Seek solution of form

$$A(\boldsymbol{x}, t)e^{i\theta(\boldsymbol{x},t)/\varepsilon}$$

  Phase $\theta(\boldsymbol{x}, t)/\varepsilon$ vary rapidly, get local wavevector $\boldsymbol{k}(\boldsymbol{x}, t)$ and frequency $\omega(\boldsymbol{x}, t)$ that satisfy a local dispersion relation

$$\omega = \Omega(\boldsymbol{k}(\boldsymbol{x}, t), \boldsymbol{x}, t)$$

- Use identities to get

$$\frac{\partial \boldsymbol{k}}{\partial t} + (\boldsymbol{c}_g \cdot \nabla)\boldsymbol{k} = -\nabla_{\boldsymbol{x}}\Omega$$
$$\frac{\partial \omega}{\partial t} + (\boldsymbol{c}_g \cdot \nabla)\omega = \frac{\partial \Omega}{\partial t}$$

  A ray $\boldsymbol{x}(t)$ 'moves' at group velocity

$$\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} = \boldsymbol{c}_g(\boldsymbol{k}(\boldsymbol{x}, t), \boldsymbol{x}, t)$$

5 Non-linear waves 1D

- Conservation of mass and momentum, homentropic flow

$$\frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial x} + \rho\frac{\partial u}{\partial x} = 0$$
$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = -\frac{1}{\rho}\frac{\partial p}{\partial x}$$
$$\frac{\partial p}{\partial x} = c^2\frac{\partial \rho}{\partial x}$$

So we get

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \frac{c^2}{\rho}\frac{\partial \rho}{\partial x} = 0$$
$$\left(\frac{\partial}{\partial t} + (u \pm c)\frac{\partial}{\partial x}\right)(u \pm Q) = 0$$

With

$$Q = \int_{\rho_0}^{\rho} \frac{c(\hat{\rho})}{\hat{\rho}}\,\mathrm{d}\hat{\rho}$$

Riemann invariants $R_\pm = u \pm Q$, constant on $C_\pm$ characteristics where $\frac{\mathrm{d}x}{\mathrm{d}t} = u \pm c$.

- For perfect gas:

$$c^2 = \frac{\gamma p}{\rho}$$

Reference sound speed $c^2 = \frac{\gamma p_0}{\rho_0}$, so

$$\frac{c}{c_0} = \left(\frac{\rho}{\rho_0}\right)\frac{\gamma - 1}{2}$$
$$Q = \frac{2}{\gamma - 1}(c - c_0)$$

- Simple waves: one set of $R_\pm$ is zero everywhere, so e.g. $u = Q$ everywhere, can relate u and c, get straight line characteristics

Shocks

- Shock formation when the same type of characteristics cross

- R-H Relations: across a shock, have mass, momentum and energy:

$$\rho_1 U_1 = \rho_0 U_0$$
$$p_1 + \rho_1 U_1^2 = p_0 + \rho_0 U_0^2$$
$$\frac{1}{2}U_1^2 + \frac{\gamma}{\gamma - 1}\frac{p_1}{\rho_1} = \frac{1}{2}U_0^2 + \frac{\gamma}{\gamma - 1}\frac{p_0}{\rho_0}$$

Using internal energy $e = \frac{1}{\gamma - 1}\frac{p}{\rho}$

Shallow water equations

$$\frac{\partial h}{\partial t} + u\frac{\partial h}{\partial x} + h\frac{\partial u}{\partial x} = 0$$
$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + g\frac{\partial \rho}{\partial x} = 0$$