

Bosonization in Electronic Systems

Jinyuan Wu

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1 The physical picture

We have already discussed bosonization in Section. 11.1 in [this solid state physics note](#). Facts like that particle-hole pairs in 1D are much simpler than the case in higher dimensions and that the density-density correlation function have two clear poles, which means we have two bosonic density modes all hint the possibility of bosonization. The first fact is the argument used in [this solid state physics note](#). Here we briefly discuss the second argument. It is well known that the retarded density-density Green function is

$$D^R(\mathbf{q}, \omega) = \int \frac{d^d \mathbf{p}}{(2\pi)^d} \frac{n_{\mathbf{p}} - n_{\mathbf{p}+\mathbf{q}}}{\omega - \xi_{\mathbf{p}+\mathbf{q}} + \xi_{\mathbf{p}} + i0^+}, \quad (1) \quad \text{Fradkin Eq. (6.7)}$$

and when \mathbf{q} is small, we have

$$\xi_{\mathbf{p}+\mathbf{q}} - \xi_{\mathbf{p}} \approx \mathbf{q} \cdot \nabla_{\mathbf{p}} \xi_{\mathbf{p}} = \mathbf{q} \cdot \mathbf{v},$$

and

$$n_{\mathbf{p}} - n_{\mathbf{p}+\mathbf{q}} \approx -\mathbf{q} \cdot \nabla_{\mathbf{p}} n_{\mathbf{p}} = -\mathbf{q} \cdot \nabla_{\mathbf{p}} \theta(\mu - \epsilon_{\mathbf{p}}) \approx \mathbf{q} \cdot \mathbf{v}_F \delta(-\xi_{\mathbf{p}}),$$

and therefore

$$\begin{aligned} D^R(\mathbf{q}, \omega) &= \int \frac{d^d \mathbf{p}}{(2\pi)^d} \delta(-\xi_{\mathbf{p}}) \frac{\mathbf{q} \cdot \mathbf{v}_F}{\omega - \mathbf{q} \cdot \mathbf{v} + i0^+} \\ &= \int_{\text{FS}} \frac{d^{d-1} S}{(2\pi)^d} \frac{1}{v_F} \frac{\mathbf{q} \cdot \mathbf{v}_F}{\omega - \mathbf{q} \cdot \mathbf{v}_F + i0^+} \\ &= \int_{\text{FS}} \frac{d^{d-1} S}{(2\pi)^d} \frac{\mathbf{q} \cdot \hat{\mathbf{v}}_F}{\omega - \mathbf{q} \cdot \mathbf{v}_F + i0^+} \\ &\sim \frac{p_F^{d-1}}{(2\pi)^d} \int_{\text{FS}} d^{d-1} \Omega \frac{\mathbf{q} \cdot \hat{\mathbf{v}}_F}{\omega - \mathbf{q} \cdot \mathbf{v}_F + i0^+}, \end{aligned} \quad (2) \quad \begin{array}{l} \text{Fradkin} \\ \text{Eq. (6.10),} \\ \text{Wen below} \\ \text{Eq. (4.3.1)} \end{array}$$

where FS means the Fermi surface. For a 1D electron gas we therefore get

$$D^R \sim \frac{q}{2\pi} \left(\frac{1}{\omega - qv_F + i0^+} - \frac{1}{\omega + qv_F + i0^+} \right). \quad (3) \quad \begin{array}{l} \text{Fradkin} \\ \text{Eq. (6.10),} \\ \text{Wen} \\ \text{Eq. (4.3.2)} \end{array}$$

So we get two poles, instead of branch cuts.

This article discusses some details in bosonization, summarizing important facts in famous textbooks and papers. Bosonization can be viewed as one way to exactly solve a system, which we will see in Luttinger liquid and bosonization of spin systems. A more “physical” motivation is to capture the bosonic modes in the system, i.e. density and current fluctuations, or since we usually only deal with low-energy perturbations in condensed matter systems, to capture *sound waves* [1,2]. This is why the fields after bosonization are all density modes: We really do not know what to do other than this. Sometimes, bosonization are known as the *hydrodynamical approach* (for example in Wen’s famous textbook), which reveals its physical nature. This name is slightly misleading, as *hydrodynamical approaches* often denote more “kinetic” approaches, which is more about deriving EOMs of expectation values instead of quantum field operators. This physical picture can also be justified using the following line of thinking: one motivation to study the electronic structure of a condensed matter system is to find its electromagnetic response, and A_μ is always coupled to a bilinear of the fermionic fields. What can be revealed directly by electromagnetic responses, therefore, are only bosonic modes (for example see Section 6 here).

The difference between a Fermi liquid and a Luttinger liquid can be illustrated in Figure 1. The distribution function of a free Fermi gas is plotted in dotted lines. After the introduction of

Philip
Phillips
Chapter 10

Wen Section
5.1.3, 5.3.3
and 5.3.4

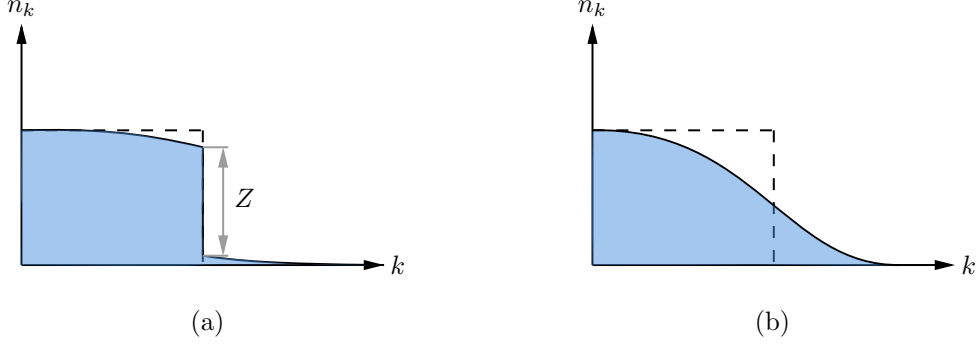


Figure 1: The electron distribution function of (a) a Fermi liquid and (b) of a Luttinger liquid. There is no discontinuity (i.e. no Fermi surface) in a Luttinger liquid.

interaction, if we still have well-defined electron-like excitations, then the electron Green function will be something like

$$\frac{Z_{\mathbf{k}}}{\omega - \xi_{\mathbf{k}}^{\text{renor}} + i0^+} + \dots,$$

where the superscript means “renormalized” (i.e. corrected by self-energy), and a renormalization constant $Z_{\mathbf{k}}$ appears. The behavior of the contribution of the first term into the density distribution function is a step function times $Z_{\mathbf{k}}$, which is shown in Figure 1(a) and defined the Fermi surface. Since the total number of electrons is conserved before and after introducing the interaction and $Z_{\mathbf{k}} < 1$, additional terms must appear in the Green function to make up for the loss of $(1 - Z_{\mathbf{k}})N_e$ electrons caused by $Z_{\mathbf{k}}$, which explains why Figure 1(a) is not a perfect step function. When in the Luttinger liquid, however, there is no step function behavior in the distribution function, so $Z_{\mathbf{k}} \rightarrow 0$ and the $Z/(\omega - \xi)$ term in the electron Green function disappears. This means that there is no clear pole in the electron Green function, and that there are no electron-like excitations. Actually, electrons are unstable, non-perturbative *solitons* in the Luttinger liquid, the fermionic behavior arising from string construction similar to the one in Jordan–Wigner transformation.

2 Luttinger liquid from a Hubbard model

In this section, we repeat the calculation done in Section 10.1 in Philip Phillips.

Since most modes we are interested in are near the two Fermi points, we can only keep the two $k = \pm k_F$ component in the spacial electron operator, i.e.

Philip
Phillips
Section 10.1

$$\Psi_{n\sigma} = e^{ik_F na} \Psi_{n\sigma+} + e^{-ik_F na} \Psi_{n\sigma-}, \quad (4)$$

Philip
Phillips
Eq. (10.4)

where the $-$ mode is the left-moving mode and the $+$ mode is the right-moving mode. We can also make the following first-order Taylor expansion:

$$\Psi_{n+1,\sigma,\pm} = \Psi_{n\sigma\pm} + a\partial_x \Psi_{n\sigma\pm}. \quad (5)$$

Now according to (4) and (5), the tight-binding Hamiltonian can be expanded into

$$\begin{aligned} & -t \sum_{n,\sigma} \Psi_{n\sigma}^\dagger \Psi_{n+1,\sigma} \\ & = -t \sum_{n,\sigma} (e^{ik_F a} \Psi_{n\sigma+}^\dagger \Psi_{n\sigma+} + e^{-ik_F a} \Psi_{n\sigma-}^\dagger \Psi_{n\sigma-} + e^{ik_F a} a \Psi_{n\sigma+}^\dagger \partial_x \Psi_{n\sigma+} + e^{-ik_F a} a \Psi_{n\sigma-}^\dagger \partial_x \Psi_{n\sigma-}), \end{aligned}$$

where the non-zero-frequency terms vanish after summing up over n , i.e. after the inverse Fourier

transformation. Since the Hamiltonian is Hermitian, we have

$$\begin{aligned}
& -t \sum_{n,\sigma} \Psi_{n\sigma}^\dagger \Psi_{n+1,\sigma} \\
&= -\frac{t}{2} \sum_{n,\sigma} (e^{ik_F a} \Psi_{n\sigma+}^\dagger \Psi_{n\sigma+} + e^{-ik_F a} \Psi_{n\sigma-}^\dagger \Psi_{n\sigma-} + e^{ik_F a} a \Psi_{n\sigma+}^\dagger \partial_x \Psi_{n\sigma+} + e^{-ik_F a} a \Psi_{n\sigma-}^\dagger \partial_x \Psi_{n\sigma-} \\
&\quad + e^{-ik_F a} \Psi_{n\sigma+}^\dagger \Psi_{n\sigma+} + e^{ik_F a} \Psi_{n\sigma-}^\dagger \Psi_{n\sigma-} + e^{-ik_F a} a \underbrace{\partial_x \Psi_{n\sigma+}^\dagger \Psi_{n\sigma+}}_{-\Psi_{n\sigma+}^\dagger \partial_x \Psi_{n\sigma+}} + e^{ik_F a} a \underbrace{\partial_x \Psi_{n\sigma-}^\dagger \Psi_{n\sigma-}}_{-\Psi_{n\sigma-}^\dagger \partial_x \Psi_{n\sigma-}}),
\end{aligned}$$

so finally

$$\begin{aligned}
H_t = & -t \sum_{n,\sigma} (\cos(k_F a) \Psi_{n\sigma+}^\dagger \Psi_{n\sigma+} + \cos(k_F a) \Psi_{n\sigma-}^\dagger \Psi_{n\sigma-} + ia \sin(k_F a) \Psi_{n\sigma+}^\dagger \partial_x \Psi_{n\sigma+} \\
& - ia \sin(k_F a) \Psi_{n\sigma-}^\dagger \partial_x \Psi_{n\sigma-}).
\end{aligned} \tag{6}$$

We already know that the energy spectrum of the full H_t (i.e. taking into account not only the modes near $\pm k_F$, but also modes far from them) is

$$\epsilon_{k\sigma} = -2t \cos(ka), \tag{7}$$

so the chemical potential is

$$\mu = -2t \cos(k_F a), \tag{8}$$

and therefore the first two terms in (6) will be canceled by the term $-\mu N$. From (7) we also know that

$$v = \nabla_k \epsilon_k = 2at \sin(ka), \tag{9}$$

and therefore

$$v_F = 2at \sin(k_F a), \tag{10}$$

so (6) becomes

$$\begin{aligned}
H_t - \mu N &= -v_F \sum_{n,\sigma} (i \Psi_{n\sigma+}^\dagger \partial_x \Psi_{n\sigma+} - i \Psi_{n\sigma-}^\dagger \partial_x \Psi_{n\sigma-}) \\
&= -v_F \sum_{\sigma} \int dx (i \Psi_{\sigma+}^\dagger(x) \partial_x \Psi_{\sigma+}(x) - i \Psi_{\sigma-}^\dagger(x) \partial_x \Psi_{\sigma-}(x)), \quad \sqrt{a} \Psi_{\sigma\pm}(x) = \Psi_{n\sigma\pm}.
\end{aligned} \tag{11}$$

3 The Luttinger model

References

- [1] Sin itiro Tomonaga. Remarks on bloch's method of sound waves applied to many-fermion problems. *Progress of Theoretical Physics*, 5:544–569, 1950.
- [2] Norio Kawakami. Tomonaga's Theory for Collective Motion of Fermions: Basic Concept of One-Dimensional Correlated Electrons. *Progress of Theoretical Physics Supplement*, 170:185–197, 05 2007.