Homology and Homotopy Groups

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March 16, 2022

This note is mainly based on [1].

Topological field theories often emerge from condensed matter systems, and the topological invariants they give are often connected to **homotopy groups**, which, intuitively speaking, classify possible field configurations into different topological sectors.

In practice homotopy groups are hard to calculate. That is why people often seek algebraic objects that are more easy to calculated and then connect them to homotopy groups (or other algebraic objects that we are interested in). This approach is called **algebraic topology**, and one most frequently used algebraic object is the **homology group**. They do not have very intuitive meaning, but they are easier to deal with.

1 Some basic facts about Abelian groups

We use $+$ to denote the group operation of an Abelian group. The expression $x - y$ is defined as $x^{-1} \circ y$. The unit is denoted as 0. The expression nx where $n \in \mathbb{N}$ and $g \in G$ means g^n. Sec. 3.1

A map between Abelian groups $f : G_1 \to G_2$ is said to be a **homomorphism** if $f(x + y) = f(x) + f(y)$, i.e. it keeps the multiplication relations. An **isomorphism** is a homomorphism that is also a bijection.

Suppose H is a subgroup of G. We have a equivalence relation $x \sim y$ if and only if $x - y \in H$. The equivalence class of x is denoted as $[x]$, and the set of all equivalence classes in G is denoted as G/H, which is the **the quotient space**. We can easily find that G/H is also a group and the group operation $+$ in G naturally induces the group operation $+$ in G/H. We have

$$G/G = \{[0]\} = \{[h]\}, \quad h \in H, \tag{1}$$

and

$$G/\{0\} = G. \tag{2}$$

Some examples of quotient spaces:

$$\mathbb{Z}/k\mathbb{Z} \simeq \mathbb{Z}_k. \tag{3}$$

If $f : G_1 \to G_2$ is a homomorphism, it can be found that $\ker f$ is a subgroup of G_1 and $\operatorname{im} f$ is a subgroup of G_2. This lemma can be proved almost directly by definition.

Now we state the **fundamental theorem of homomorphism**: for a homomorphism $f : G_1 \to G_2$, we have

$$G_1/\ker f \simeq \operatorname{im} f. \tag{4}$$

This is Theorem 3.1 in [1].

If group elements of an Abelian group G are all in the form of

$$n_1 x_1 + \cdots + n_r x_r, \quad n_i \in \mathbb{Z}, 1 \le i \le r, \tag{5}$$

we say that G is **(finitely) generated** with **generators** $x_1, \ldots, x_r$.

If G is finitely generated with r *linear independent* generators we say it is a **free Abelian group of rank** r. It should be noted that not every finitely generated Abelian group is a free Abelian group, because if $x_1, x_2, \ldots, x_n$ are not linear independent, i.e. there exist a integer sequence $\{n_r\}$ with some non-zero terms such that

$$n_1 x_1 + n_2 x_2 + \cdots n_r x_r = 0, \tag{6}$$

it is generally *insufficient* to obtain an explicit expression of x_r (assuming that $n_r \neq 0$) in terms of $x_1, \ldots, x_{r-1}$, since we *cannot* “divide” a group element with an integer. An example is that

$\mathbb{Z}_2$ is a finite generated group (the generator is 1) but is not a free group, because $\{1\}$ is not a linear independent set since $1+1=0$.

And now we want to analyze the structure of finitely generated groups and free groups. First we analyze the simple case of cyclic groups. **Cyclic groups**, or in other words groups generated with a single element, are important in group theory. It can be found using the fundamental theorem of homomorphism that a finite cyclic group is isomorphic to some $\mathbb{Z}_N$ while an infinite cyclic group is isomorphic to $\mathbb{Z}$. If G is a cyclic group generated by x and is finite, than there is a smallest positive integer N such that $Nx=0$. We consider the map $f:\mathbb{Z}\to G$,

$$f(n)=nx,$$

then clearly we have

$$\ker f=\{0,\pm N,\pm 2N,\ldots\}=N\mathbb{Z},$$

so we have

$$G=\operatorname{im} f\simeq \mathbb{Z}/\ker f=\mathbb{Z}/N\mathbb{Z}\simeq\mathbb{Z}_N.$$

If For an infinite cyclic Abelian group, we have $\ker f=\{0\}$, so

$$G=\operatorname{im} f\simeq\mathbb{Z}/\ker f=\mathbb{Z}.$$

It can be seen that a finite cyclic Abelian group is not a free Abelian group because the fact that there exists a positive integer N such that $Nx=0$ means the generator set $\{x\}$ is not linear independent. On the other hand, an infinite free cyclic group $G\simeq\mathbb{Z}$ is free. Since we are discussing Abelian groups, we can imagine that an arbitrary finitely generated group G can be rephrase as a direct sum of cyclic groups, each of which is generated by one of the generators of G. This is indeed the case, and the fact is Theorem 3.2 in [1]: Let G be a finitely generated Abelian group (not necessarily free) with m generators. Then G is isomorphic to the direct sum of cyclic groups,

$$G\simeq\underbrace{\mathbb{Z}\oplus\cdots\oplus\mathbb{Z}}_{r}\oplus\mathbb{Z}_{k_1}\oplus\cdots\oplus\mathbb{Z}_{k_p},\tag{7}$$

where $m=r+p$. The number r is called the rank of G. This is called **fundamental theorem of finitely generated Abelian groups**. We can see that the rank defined in (7) is exactly the same as the rank defined for free Abelian groups.

2 Simplexes and simplicial complexes

We now consider how can a manifold be sliced into small "cells". What we are going to introduce in this section are simplexes and simplicial complexes. The former are "cells", and the latter are a certain type of topological objects assembled with simplexes. Sec. 3.2

The generalized definition for a r**-simplex** is

$$\langle p_0p_1\cdots p_r\rangle=\left\{x\in\mathbb{R}^m\mid x=\sum_{i=0}^r c_ip_i, c_i\geq 0,\sum_{i=0}^r c_i=1\right\}.\tag{8}$$

A 0-simplex is a point $\langle p_0\rangle$ or p_0 for short. A 1-simplex is a line $\langle p_0p_1\rangle$. A 2-simplex $\langle p_0p_1p_2\rangle$ is a triangle with its interior included and a 3-simplex $\langle p_0p_1p_2p_3\rangle$ is a solid tetrahedron. We require an r-simplex to be r-dimensional, or in other words $p_0,p_1,\ldots,p_r$ are required to be geometrically independent.

Let q be an integer such that $0\leq q\leq r$. If we choose $q+1$ points $p_{i_0},\ldots,p_{i_q}$ out of $p_0,\ldots,p_r$, these $q+1$ points define a q-simplex $\sigma_q=\langle p_{i_0},\ldots,p_{i_q}\rangle$, which is called a ***q*-face** of σ_r. We write $\sigma_q\leq\sigma_r$ if σ_q is a face of σ_r. If $\sigma_q\neq\sigma_r$, we say σ_q is a **proper face** of σ_r, denoted as $\sigma_q<\sigma_r$. The number of q-faces in an r-simplex is $\begin{pmatrix} r+1\\ q+1\end{pmatrix}$. A 0-simplex is defined to have no proper faces.

Let K be a set of finite number of simplexes in $\mathbb{R}^m$. If these simplexes are nicely fitted together, K is called a simplicial complex. By 'nicely' we mean:

(i) an arbitrary face of a simplex of K belongs to K, that is, if $\sigma\in K$ and $\sigma'\leq\sigma$ then $\sigma'\in K$, and

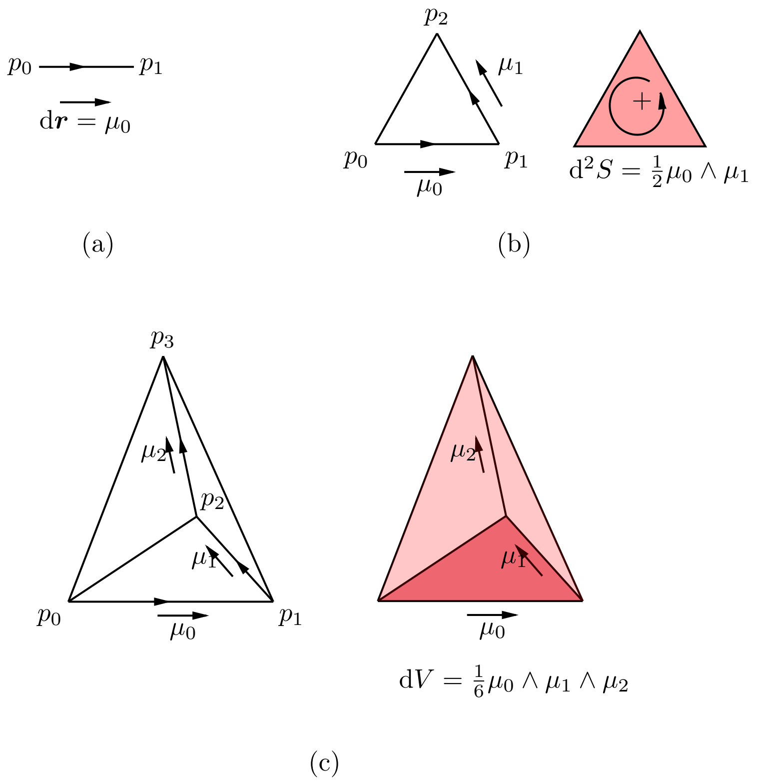

Figure 1: Orientations of a 1-simplex, a 2-simplex, and a 3-simplex, and the corresponding differential forms when the simplexes are small enough.

(ii) if σ and σ' are two simplexes of K, the intersection $\sigma \cap \sigma'$ is either empty or a common face of σ and σ', that is, if $\sigma, \sigma' \in K$ then either $\sigma \cap \sigma' = \varnothing$ or $\sigma \cap \sigma' \leq \sigma$ and $\sigma \cap \sigma' \leq \sigma'$.

A simplicial complex K is a set whose elements are simplexes. If each simplex is regarded as a subset of $\mathbb{R}^m (m \geq \dim K)$, the union of all the simplexes becomes a subset of $\mathbb{R}^m$. This subset is called the **polyhedron** $|K|$ of a simplicial complex K. The dimension of $|K|$ as a subset of $\mathbb{R}^m$ is the same as that of K; $\dim |K| = \dim K$.

Let X be a topological space. If there exists a simplicial complex K and a homeomorphism $f : |K| \to X, X$ is said to be triangulable and the pair (K, f) is called a triangulation of X. Given a topological space X, its triangulation is far from unique. We will be concerned with triangulable spaces only.

Now we assign *orientations* to an r-simplex for $r > 1$. A oriented simplex is labeled as $(p_0 p_1 \cdots p_r)$. Note that since any oriented manifold has only two orientations, for r points $p_0, p_1, \ldots, p_r$, we only have two types of oriented r-simplex. We can use $(p_0 p_1 \cdots p_r)$ and its even permutation to label the "positive" orientation, and use an odd permutation of $p_0, p_1, \ldots, p_r$ to label the "negative" orientation. If σ_r is a "positively" oriented r-simplex, we use $-\sigma_r$ to denote its negative version, and vice versa. In this way we have

$$(p_{i_0} p_{i_1} \ldots p_{i_r}) = \mathrm{sgn}(P)\,(p_0 p_1 \ldots p_r)\,, \quad P = \begin{pmatrix} 0 & 1 & \ldots & r \\ i_0 & i_1 & \ldots & i_r \end{pmatrix}. \tag{9}$$

We can have some intuitive understanding of the orientation of an r-simplex using pictures from differential geometry. For an oriented r-simplex $(p_0 p_1 \cdots p_r)$ that is small enough, we define

$$\mu_i = p_{i+1} - p_i,$$

and the corresponding differential form is

$$\frac{1}{r!}\mu_0 \wedge \mu_1 \wedge \cdots \wedge \mu_{r-1}.$$

Then the definition (9) starts to make sense. A 1-simplex can be viewed as a line from one point to the other point, inducing a line element (see Figure 1(a)). The direction of the line determines the sign of the corresponding differential form, so we have We have

$$(p_0p_1) = -(p_1p_0),$$

which is the $r = 1$ case of (9). The case of 2-complex (see Figure 1(b)) is almost the same, where the sign of the area element varies depending on whether the oriented boundary is clockwise or not, and all clockwise routines share the same sign, i.e.

$$(p_1p_2p_3) = (p_2p_3p_1) = (p_3p_1p_2),$$

and so do anticlockwise orientations, which is the $r = 2$ case of (9). In Figure 1(c), the bottom of the tetrahedron has the "positive orientation", and since the order of vertices is $p_0p_1p_2p_3$, the volume of the tetrahedron is also positive.

We formally define $(p_0) = p_0$.

3 Chain group, cycle group and boundary group

Note that we can regard the $-$ operation on simplexes as an *inverse* operation. We also know that differential forms may be added. Also, if two simplexes with the same orientation are sticked together, they just form a larger object (though not a simplex) which shares the orientation. Therefore, all oriented r-simplexes in a simplicial complex K *freely generate* an Abelian group, which is defined as the r**-chain group** $C_r(K)$. When $r > \dim K$ we define $C_r(K)$ to be the trivial group denoted as 0.

References

[1] Mikio Nakahara. *Geometry, Topology and Physics, Second Edition.* 06 2003.