High Density Electron Gas with Coulomb Interaction

Jinyuan Wu

February 16, 2022

It is said that for a simple demonstration of what happens in a metal, we can only work with the **jellium model**, ignoring the details of the lattice, which means we can just work with a non-relativistic electron gas with Coulomb interaction. This article is an extension of Section 6.2 in this solid state physics note. We will discuss some early development of RPA. Zhengzhong Li, Sec. 4.2

1 Notations and basic facts about the jellium model

We define

$$\rho(\boldsymbol{r}) = \sum_{\sigma} \psi_{\sigma}^{\dagger}(\boldsymbol{r})\psi_{\sigma}(\boldsymbol{r}) = \frac{1}{V}\sum_{\boldsymbol{k},\boldsymbol{k}',\sigma} c_{\boldsymbol{k}\sigma}^{\dagger} c_{\boldsymbol{k}'\sigma} \mathrm{e}^{-\mathrm{i}(\boldsymbol{k}-\boldsymbol{k}')\cdot\boldsymbol{r}} = \frac{1}{V}\sum_{\boldsymbol{q}}\sum_{\boldsymbol{k},\sigma} c_{\boldsymbol{k}\sigma}^{\dagger} c_{\boldsymbol{k}+\boldsymbol{q},\sigma} \mathrm{e}^{\mathrm{i}\boldsymbol{q}\cdot\boldsymbol{r}}, \tag{1}$$

and the commutation relations are

$$[c_{\boldsymbol{k}}, c_{\boldsymbol{k}'}^{\dagger}] = \delta_{\boldsymbol{k}\boldsymbol{k}'}. \tag{2}$$

We can also write down $\rho(\boldsymbol{r})$ in the first quantization formulation, i.e.

$$\rho(\boldsymbol{r}) = \sum_{i} \delta(\boldsymbol{r}-\boldsymbol{r}_i) = \sum_{i}\frac{1}{V}\sum_{\boldsymbol{q}} \mathrm{e}^{\mathrm{i}\boldsymbol{q}\cdot(\boldsymbol{r}-\boldsymbol{r}_i)} = \frac{1}{V}\sum_{\boldsymbol{q}}\sum_{i} \mathrm{e}^{\mathrm{i}\boldsymbol{q}\cdot(\boldsymbol{r}-\boldsymbol{r}_i)}. \tag{3}$$

We have

$$\rho_{\boldsymbol{q}} := \sum_{i} \mathrm{e}^{-\mathrm{i}\boldsymbol{q}\cdot\boldsymbol{r}_i} = \sum_{\boldsymbol{k},\sigma} c_{\boldsymbol{k}\sigma}^{\dagger} c_{\boldsymbol{k}+\boldsymbol{q},\sigma}, \quad \rho(\boldsymbol{r}) = \frac{1}{V}\sum_{\boldsymbol{q}} \rho_{\boldsymbol{q}} \mathrm{e}^{\mathrm{i}\boldsymbol{q}\cdot\boldsymbol{r}}. \tag{4}$$

Easily, we find

$$\rho_{\boldsymbol{q}=0} = \int \mathrm{d}^3\boldsymbol{r}\, \rho(\boldsymbol{r}) = N. \tag{5}$$

The Hamiltonian of electrons is

$$H_{\text{electron}} = \sum_{\boldsymbol{k},\sigma} \epsilon_{\boldsymbol{k}} c_{\boldsymbol{k}\sigma}^{\dagger} c_{\boldsymbol{k}\sigma} + \frac{1}{2V}\sum_{\boldsymbol{k},\boldsymbol{k}',\boldsymbol{q},\sigma,\sigma'} c_{\boldsymbol{k}+\boldsymbol{q},\sigma}^{\dagger} c_{\boldsymbol{k}'-\boldsymbol{q},\sigma'}^{\dagger} V(q) c_{\boldsymbol{k}'\sigma'} c_{\boldsymbol{k}\sigma},$$

where

$$V(q) = \int \mathrm{e}^{-\mathrm{i}\boldsymbol{q}\cdot\boldsymbol{r}} \frac{e^2}{r} = \frac{4\pi e^2}{\boldsymbol{q}^2}. \tag{6}$$

The Hamiltonian of the lattice and the electron-lattice coupling is

$$H_{\text{lattice}} = \frac{1}{2V}\sum_{\boldsymbol{q}} \rho_{\text{ion},-\boldsymbol{q}} V(q) \rho_{\text{ion},\boldsymbol{q}} - \frac{1}{V}\sum_{\boldsymbol{q}} \rho_{\text{ion},\boldsymbol{q}} V(q) \rho_{\boldsymbol{q}}.$$

When $|\boldsymbol{q}| = 0$, $V(q)$ is divergent, but this does not matter. In the jellium model, the only non-zero Fourier component of $\rho_{\text{ion}}(\boldsymbol{r})$ is $\rho_{\text{ion},\boldsymbol{q}=0} = N$ (the solid is neutral so the number of positive charges must be equal to the number of negative charges), and we soon find

$$H_{\text{lattice}} + \frac{1}{2V}\sum_{\boldsymbol{q}=0,\boldsymbol{k},\boldsymbol{k}',\sigma,\sigma'} c_{\boldsymbol{k}+\boldsymbol{q},\sigma}^{\dagger} c_{\boldsymbol{k}'-\boldsymbol{q},\sigma'}^{\dagger} V(q) c_{\boldsymbol{k}'\sigma'} c_{\boldsymbol{k}\sigma} = 0.$$

So all divergences cancel with each other, and in the end, the total Hamiltonian is

$$\begin{aligned} H &= \sum_{\boldsymbol{k},\sigma} \epsilon_{\boldsymbol{k}} c_{\boldsymbol{k}\sigma}^{\dagger} c_{\boldsymbol{k}\sigma} + \frac{1}{2V}\sum_{\boldsymbol{q}\neq 0}\sum_{\boldsymbol{k},\boldsymbol{k}',\sigma,\sigma'} c_{\boldsymbol{k}+\boldsymbol{q},\sigma}^{\dagger} c_{\boldsymbol{k}'-\boldsymbol{q},\sigma'}^{\dagger} V(q) c_{\boldsymbol{k}'\sigma'} c_{\boldsymbol{k}\sigma} \\ &= \sum_{\boldsymbol{k},\sigma} \epsilon_{\boldsymbol{k}} c_{\boldsymbol{k}\sigma}^{\dagger} c_{\boldsymbol{k}\sigma} + \frac{1}{2V}\sum_{\boldsymbol{q}\neq 0} \rho_{\boldsymbol{q}}^{\dagger} V(q) \rho_{\boldsymbol{q}} = \sum_{\boldsymbol{k},\sigma} \epsilon_{\boldsymbol{k}} c_{\boldsymbol{k}\sigma}^{\dagger} c_{\boldsymbol{k}\sigma} + \frac{1}{2V}\sum_{\boldsymbol{q}\neq 0} \rho_{-\boldsymbol{q}} V(q) \rho_{\boldsymbol{q}}. \end{aligned} \tag{7}$$

The contribution of the positive ion "jell" both constrains the electrons in the solid (or in other words, give a chemical potential) and regularize the singularity of $V(\boldsymbol{q}=0)$.

Note that (7) differs with

$$H=\sum_i \frac{\boldsymbol{p}_i^2}{2m}+\frac{1}{2}\sum_{i\neq j}\frac{e^2}{|\boldsymbol{r}_i-\boldsymbol{r}_j|} \tag{8}$$

with the $i=j$ term. This is an infinite constant and does not matter.

2 Classical (or first quantized) theory of the collective oscillation of electrons

From (4), we have

Zhengzhong Li, Sec. 4.3

$$\begin{aligned}\dot{\rho}_{\boldsymbol{q}}&=\sum_j(-\mathrm{i}\boldsymbol{q})\cdot\boldsymbol{v}_j\mathrm{e}^{-\mathrm{i}\boldsymbol{q}\cdot\boldsymbol{r}_j},\\ \ddot{\rho}_{\boldsymbol{q}}&=\sum_j(-\mathrm{i}\boldsymbol{q}\cdot\dot{\boldsymbol{v}}_j-(\boldsymbol{q}\cdot\boldsymbol{v}_j)^2)\mathrm{e}^{-\mathrm{i}\boldsymbol{q}\cdot\boldsymbol{r}_j}.\end{aligned} \tag{9}$$

Now $\dot{\boldsymbol{v}}_j$ can be derived from (7):

$$\begin{aligned}m\dot{\boldsymbol{v}}_j&=-\boldsymbol{\nabla}_j\frac{1}{2V}\sum_{\boldsymbol{q}\neq 0}V(q)\rho_{\boldsymbol{q}}^\dagger\rho_{\boldsymbol{q}}\\ &=-\frac{1}{2V}\boldsymbol{\nabla}_j\sum_{\boldsymbol{q}\neq 0}V(q)\sum_{i,k}\mathrm{e}^{\mathrm{i}\boldsymbol{q}\cdot(\boldsymbol{r}_i-\boldsymbol{r}_k)}\\ &=-\frac{1}{2V}\sum_{\boldsymbol{q}\neq 0}V(q)\sum_k(\mathrm{i}\boldsymbol{q})\mathrm{e}^{\mathrm{i}\boldsymbol{q}\cdot(\boldsymbol{r}_j-\boldsymbol{r}_k)}+\sum_i(-\mathrm{i}\boldsymbol{q})\mathrm{e}^{\mathrm{i}\boldsymbol{q}\cdot(\boldsymbol{r}_j-\boldsymbol{r}_i)}\\ &=-\frac{1}{V}\sum_{\boldsymbol{q}\neq 0}\mathrm{i}\boldsymbol{q}V(q)\sum_i\mathrm{e}^{\mathrm{i}\boldsymbol{q}\cdot(\boldsymbol{r}_j-\boldsymbol{r}_i)}\\ &=-\frac{4\pi e^2}{V}\sum_{\boldsymbol{q}\neq 0}\frac{\mathrm{i}\boldsymbol{q}}{q^2}\mathrm{e}^{\mathrm{i}\boldsymbol{q}\cdot\boldsymbol{r}_j}\rho_{\boldsymbol{q}},\end{aligned}$$

and from (9), we have

$$\begin{aligned}\ddot{\rho}_{\boldsymbol{q}}&=-\sum_j\frac{4\pi e^2}{mV}\sum_{\boldsymbol{q}'\neq 0}\frac{\boldsymbol{q}\cdot\boldsymbol{q}'}{q'^2}\mathrm{e}^{\mathrm{i}\boldsymbol{q}'\cdot\boldsymbol{r}_j}\rho_{\boldsymbol{q}'}\mathrm{e}^{-\mathrm{i}\boldsymbol{q}\cdot\boldsymbol{r}_j}-\sum_j(\boldsymbol{q}\cdot\boldsymbol{v}_j)^2\mathrm{e}^{-\mathrm{i}\boldsymbol{q}\cdot\boldsymbol{r}_j}\\ &=-\frac{4\pi e^2}{mV}\sum_{\boldsymbol{q}'\neq 0}\rho_{\boldsymbol{q}'}\rho_{\boldsymbol{q}-\boldsymbol{q}'}-\sum_j(\boldsymbol{q}\cdot\boldsymbol{v}_j)^2\mathrm{e}^{-\mathrm{i}\boldsymbol{q}\cdot\boldsymbol{r}_j}.\end{aligned}$$

Now we make the **random phase approximation (RPA)** in its original form: We assume that only the $\boldsymbol{q}=\boldsymbol{q}'$ term in the first term is important, because in the high density limit, there are no position preference of electrons (when the density is low, there might be a Wigner crystal, and RPA fails), and when $\boldsymbol{q}\neq 0$, both $\rho_{\boldsymbol{q}'}$ and $\rho_{\boldsymbol{q}-\boldsymbol{q}'}$ are sums of almost random phase factors $\mathrm{e}^{-\mathrm{i}\boldsymbol{q}\cdot\boldsymbol{r}_j}$, and therefore are both small enough. So we get the EOM after RPA:

$$\begin{aligned}\ddot{\rho}_{\boldsymbol{q}}&=-\frac{4\pi e^2}{mV}\rho_{\boldsymbol{q}}\rho_0-\sum_j(\boldsymbol{q}\cdot\boldsymbol{v}_j)^2\mathrm{e}^{-\mathrm{i}\boldsymbol{q}\cdot\boldsymbol{r}_j}\\ &=-\frac{4\pi e^2 n}{m}\rho_{\boldsymbol{q}}-\sum_j(\boldsymbol{q}\cdot\boldsymbol{v}_j)^2\mathrm{e}^{-\mathrm{i}\boldsymbol{q}\cdot\boldsymbol{r}_j},\end{aligned} \tag{10}$$

where $n=N/V$ is the jellium density. Section (5.4.2) in this solid state physics note tells us that the electron-hole pair excitations are gapless, but from the EOM above we soon find that in the $\boldsymbol{q}\to 0$ case there is a finite ω solution, which is given by

$$\ddot{\rho}_{\boldsymbol{q}}+\omega_{\mathrm{p}}^2\rho_{\boldsymbol{q}}=0,\quad \omega_{\mathrm{p}}^2=\frac{4\pi e^2 n}{m}. \tag{11}$$

We see that this term arises from the Coulomb interaction. In other words, long range interaction induces a collective modes in the metal, which is now known as **plasmon**. We know it is plasmon, or quantized plasma oscillation, because our derivation above also works for the oscillation of negative charges around positive charges in a plasma.

3 Second quantized EOM of density modes

Now we try to derive plasmon in the second quantization EOM framework. Generally speaking, a generalized hydrodynamic mode in an electron gas is made of a linear combination of Zhengzhong Li, Sec. 4.4

$$\rho^\dagger_{\boldsymbol{kq}} =: \sum_\sigma c^\dagger_{\boldsymbol{k}+\boldsymbol{q},\sigma} c_{\boldsymbol{k}\sigma}. \tag{12}$$

By evaluating the EOM, we may be able to identify stable density modes.

We start from the free theory. It can be verified that

$$[c^\dagger_1 c_2, c^\dagger_3 c_4] = \delta_{23} c^\dagger_1 c_4 - \delta_{14} c^\dagger_3 c_2, \tag{13}$$

and therefore we have

$$\begin{aligned}
\mathrm{i}\dot{\rho}^\dagger_{\boldsymbol{kq}} = [\rho^\dagger_{\boldsymbol{kq}}, H_0] &= \sum_{\boldsymbol{p},\alpha} \frac{\boldsymbol{p}^2}{2m} \sum_\sigma [c^\dagger_{\boldsymbol{k}+\boldsymbol{q},\sigma} c_{\boldsymbol{k}\sigma}, c^\dagger_{\boldsymbol{p}\alpha} c_{\boldsymbol{p}\alpha}] \\
&= \sum_{\boldsymbol{p},\alpha,\sigma} \frac{\boldsymbol{p}^2}{2m} (c^\dagger_{\boldsymbol{k}+\boldsymbol{q},\sigma} c_{\boldsymbol{p}\alpha} \delta_{\boldsymbol{kp}} \delta_{\sigma\alpha} - c^\dagger_{\boldsymbol{p}\alpha} c_{\boldsymbol{k}\sigma} \delta_{\boldsymbol{k}+\boldsymbol{q},\boldsymbol{p}} \delta_{\sigma\alpha}) \\
&= \sum_\sigma \left(\frac{\boldsymbol{k}^2}{2m} c^\dagger_{\boldsymbol{k}+\boldsymbol{q},\sigma} c_{\boldsymbol{k}\sigma} - \frac{(\boldsymbol{q}+\boldsymbol{k})^2}{2m} c^\dagger_{\boldsymbol{k}+\boldsymbol{q},\sigma} c_{\boldsymbol{k}\sigma} \right),
\end{aligned}$$

so

$$\mathrm{i}\dot{\rho}^\dagger_{\boldsymbol{kq}} = [\rho^\dagger_{\boldsymbol{kq}}, H_0] = -\omega^{\text{pair}}_{\boldsymbol{kq}} \rho^\dagger_{\boldsymbol{kq}}, \quad \omega^{\text{pair}}_{\boldsymbol{kq}} = \frac{(\boldsymbol{q}+\boldsymbol{k})^2}{2m} - \frac{\boldsymbol{k}^2}{2m}. \tag{14}$$

Therefore, we find there are electron-hole pairs in the free theory. In the last section we find that in a plasmon mode, we have $\rho_{\boldsymbol{k}} = \sum_{\boldsymbol{q}} \rho_{\boldsymbol{kq}} \sim \mathrm{e}^{-\mathrm{i}\omega_{\boldsymbol{p}} t}$, and $\omega_{\boldsymbol{q}=0} \neq 0$, but here when $\boldsymbol{q} = 0$, $\omega^{\text{pair}}_{\boldsymbol{kq}}$ vanishes, so there is no plasmon. This is expected, since in the last section we find the plasmon comes from the Coulomb interaction.

Now we move on to discuss the jellium model. We have

$$\begin{aligned}
[\rho^\dagger_{\boldsymbol{kq}}, H] &= [\rho^\dagger_{\boldsymbol{kq}}, H_0 + H_{\text{Coulomb}}] = -\omega^{\text{pair}}_{\boldsymbol{kq}} \rho^\dagger_{\boldsymbol{kq}} + \frac{1}{2V} \sum_{\boldsymbol{q}'\neq 0} V(q') [\rho^\dagger_{\boldsymbol{kq}}, \rho_{-\boldsymbol{q}'} \rho_{\boldsymbol{q}'}] \\
&= -\omega^{\text{pair}}_{\boldsymbol{kq}} \rho^\dagger_{\boldsymbol{kq}} + \frac{1}{2V} \sum_{\boldsymbol{q}'\neq 0} V(q') (\rho_{-\boldsymbol{q}'} [\rho^\dagger_{\boldsymbol{kq}}, \rho_{\boldsymbol{q}'}] + [\rho^\dagger_{\boldsymbol{kq}}, \rho_{-\boldsymbol{q}'}] \rho_{\boldsymbol{q}'}) \\
&= -\omega^{\text{pair}}_{\boldsymbol{kq}} \rho^\dagger_{\boldsymbol{kq}} + \frac{1}{2V} \sum_{\boldsymbol{q}'\neq 0} V(q') (\rho_{-\boldsymbol{q}'} [\rho^\dagger_{\boldsymbol{kq}}, \rho_{\boldsymbol{q}'}] + [\rho^\dagger_{\boldsymbol{kq}}, \rho_{\boldsymbol{q}'}] \rho_{-\boldsymbol{q}'}),
\end{aligned}$$

where

$$\begin{aligned}
[\rho^\dagger_{\boldsymbol{kq}}, \rho_{\boldsymbol{q}'}] &= \sum_\alpha \sum_{\boldsymbol{k}',\beta} [c^\dagger_{\boldsymbol{k}+\boldsymbol{q},\alpha} c_{\boldsymbol{k}\alpha}, c^\dagger_{\boldsymbol{k}'\beta} c_{\boldsymbol{k}'+\boldsymbol{q}',\beta}] \\
&= \sum_\alpha (c^\dagger_{\boldsymbol{k}+\boldsymbol{q},\alpha} c_{\boldsymbol{k}+\boldsymbol{q}',\alpha} - c^\dagger_{\boldsymbol{k}+\boldsymbol{q}-\boldsymbol{q}',\alpha} c_{\boldsymbol{k}\alpha}).
\end{aligned}$$

So the EOM of $\rho^\dagger_{\boldsymbol{kq}}$ is finally Zhengzhong Li (4.4.9)

$$\mathrm{i}\dot{\rho}^\dagger_{\boldsymbol{kq}} = -\omega^{\text{pair}}_{\boldsymbol{kq}} \rho^\dagger_{\boldsymbol{kq}} - \frac{1}{2V} \sum_{\boldsymbol{q}'\neq 0,\alpha} V(q') \{\rho_{-\boldsymbol{q}'}, c^\dagger_{\boldsymbol{k}+\boldsymbol{q}-\boldsymbol{q}',\alpha} c_{\boldsymbol{k}\alpha} - c^\dagger_{\boldsymbol{k}+\boldsymbol{q},\alpha} c_{\boldsymbol{k}+\boldsymbol{q}',\alpha}\}. \tag{15}$$

By taking all possible momenta in (15), we get a closed group of equations about density modes. In principle, this gives the (generalized) hydrodynamic modes in the jellium model

completely, though we all know it is highly difficult to solve EOM of operators. This can also be seen as a successful *bosonization* the theory, though unlike the case in a Luttinger liquid, this does not help us understand the behavior of the jellium model. A possible way to simplify (15) is to assume that $c^\dagger_{\boldsymbol{k}+\boldsymbol{q}-\boldsymbol{q}',\alpha}c_{\boldsymbol{k}\alpha} - c^\dagger_{\boldsymbol{k}+\boldsymbol{q},\alpha}c_{\boldsymbol{k}+\boldsymbol{q}',\alpha}$ does not fluctuate much, and we can replace it by the expectation value. if under this approximation we find a stable mode, then we have

$$\begin{aligned}(\omega - \omega^{\text{pair}}_{\boldsymbol{k}\boldsymbol{q}})\rho^\dagger_{\boldsymbol{k}\boldsymbol{q}} &= \frac{1}{2V}\sum_{\boldsymbol{q}'\neq 0,\alpha} V(q')\{\rho_{-\boldsymbol{q}'},\, \langle c^\dagger_{\boldsymbol{k}+\boldsymbol{q}-\boldsymbol{q}',\alpha}c_{\boldsymbol{k}\alpha} - c^\dagger_{\boldsymbol{k}+\boldsymbol{q},\alpha}c_{\boldsymbol{k}+\boldsymbol{q}',\alpha}\rangle\} \\ &= \frac{1}{2V}\sum_{\boldsymbol{q}'\neq 0,\alpha} 2V(q')\rho^\dagger_{\boldsymbol{q}'}(n_{\boldsymbol{k}}\delta_{\boldsymbol{q}\boldsymbol{q}'} - n_{\boldsymbol{k}+\boldsymbol{q}}\delta_{\boldsymbol{q}\boldsymbol{q}'}) \\ &= \frac{2}{V}\rho^\dagger_{\boldsymbol{q}}(n_{\boldsymbol{k}} - n_{\boldsymbol{k}+\boldsymbol{q}})V(q),\end{aligned}$$

where

$$n_{\boldsymbol{k}} \coloneqq \langle c^\dagger_{\boldsymbol{k}\sigma}c_{\boldsymbol{k}\sigma}\rangle\,. \qquad \text{(no summation over } \sigma) \tag{16}$$

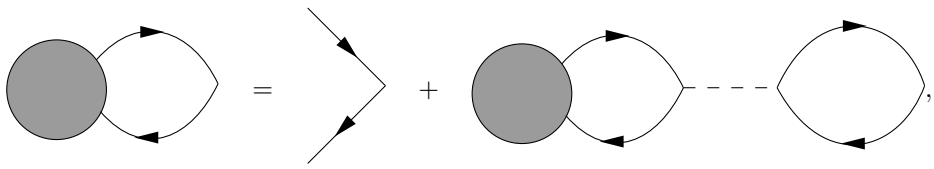
From the equation above we find

$$\rho^\dagger_{\boldsymbol{k}\boldsymbol{q}} = \frac{2V(q)}{V}\rho^\dagger_{\boldsymbol{q}}\frac{n_{\boldsymbol{k}} - n_{\boldsymbol{k}+\boldsymbol{q}}}{\omega - \omega^{\text{pair}}_{\boldsymbol{k}\boldsymbol{q}}}, \tag{17}$$

and summing over $\boldsymbol{k}$, we have

$$1 = \frac{2V(q)}{V}\sum_{\boldsymbol{k}}\frac{n_{\boldsymbol{k}} - n_{\boldsymbol{k}+\boldsymbol{q}}}{\omega - \omega^{\text{pair}}_{\boldsymbol{k}\boldsymbol{q}}}. \tag{18}$$

What we are doing here is actually a mean field approximation. It is natural to guess that the approximation corresponds to one Green function resummation strategy, which we will discuss in detail in Section 4. An intuitive way to see what is going on is to note that (17) is actually



(19)

where the $(n_{\boldsymbol{k}} - n_{\boldsymbol{k}+\boldsymbol{q}})/(\omega - \omega^{\text{pair}}_{\boldsymbol{k}\boldsymbol{q}})$ factor is "the propagator of an electron-hole pair". The approximation we have made is also a *random phase approximation*, because after the approximation, all $\boldsymbol{q}' \neq \boldsymbol{q}$ terms in (15) disappear.

Now we can easily find the frequencies and the states of all density modes. This also shows a benefit of our operator EOM based approach: in a Green function based approach, we can only easily find the spectrum of the density modes by checking the singularities of $\langle nn\rangle$, but it is not that easy to identify their actual "shape". We, however, will postpone the solution of (18) to Section 6, since (15) is actually just about the *real* part of the Green function: we can recognize factors like $1/(\omega - \omega^{\text{pair}}_{\boldsymbol{k}\boldsymbol{q}})$, which misses the important infinitesimal $\mathrm{i}0^+$ imaginary part in the denominator, which can be used to determine the damping rate.

4 Feynman diagrams for the jellium model

In this section we will discuss the jellium model as a field theory based on Huaiyu Wang's book.[1] The ultimate purpose for any condensed matter field theoretic calculation is to derive the response of the material to any external stimulation. We will discuss the linear response in Section 5, and in this section we just make some formal development about what to consider in a jellium model.

Huaiyu Wang Sec. 13.3.1

First we estimate the magnitude of the kinetic energy and the Coulomb repulsion energy. By definition we have

$$\frac{V}{N} = \frac{4}{3}\pi r_0^3,$$

[1] Which is accused of plagiarism of Eleftherios' book about Green functions – but since I have only read the former, I will go on with the section index of the former.

and

$$N = \frac{V}{(2\pi)^3} \cdot \frac{4}{3}\pi k_{\mathrm{F}}^3,$$

and therefore we have

$$r_{\mathrm{s}} =: \frac{r_0}{a_0} = \left(\frac{9}{4}\pi\right)^{1/3} \frac{1}{k_{\mathrm{F}} a_0}, \tag{20}$$

where

$$a_0 = \frac{1}{2me^2} \tag{21}$$

is the Bohr radius. The dimensionless r_{s} gives the ratio between the average distance between electrons and the Bohr radius. The high density condition $r_{\mathrm{s}} \ll 1$ is therefore equivalent to

$$k_{\mathrm{F}} a_0 \gg 1. \tag{22}$$

Huaiyu Wang Eq. (13.3.8)

The kinetic energy of a single electron is about

$$E_{\mathrm{K}} \sim \frac{k_{\mathrm{F}}^2}{2m}, \tag{23}$$

while the potential energy of a single electron is about

$$E_{\mathrm{V}} \sim \frac{1}{r_{\mathrm{s}} a_0}, \tag{24}$$

and the ratio between the two is

$$\frac{E_{\mathrm{V}}}{E_{\mathrm{K}}} \sim \frac{e^2}{r_{\mathrm{s}} a_0} \frac{2m}{k_{\mathrm{F}}^2} = \frac{1}{(k_{\mathrm{F}} a_0)^2 r_{\mathrm{s}}} \ll 1. \tag{25}$$

We reach a quite counter-intuitive result: in a *high* density electron gas, where the Coulomb repulsion is *strong*, the Coulomb energy is just a small perturbation! The explanation can be found in Section 6.1 in this solid state physics note.

Note

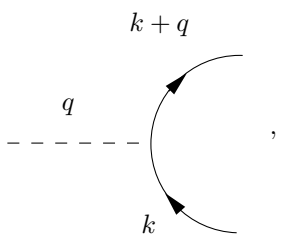
We need to be cautious when comparing energies, since we can add an arbitrary large or small constant to any term in the Hamiltonian. What really matters is the *range* in which an energy term can vary. For example, an electron can take any of the momenta in the Fermi sea, and therefore its kinetic energy ranges from 0 to $k_{\mathrm{F}}^2/2m$, and therefore we have (23).

Now we conduct loop diagram corrections. The first-order self-energy correction is done in (6.14) in this solid state physics note, and we find it is finite until we go close to the Fermi surface. This is not really concerning, because if we replace the bare interaction line ("photon line") by the *screened* version (which we are going to discuss soon), the divergence vanishes. Here we follow the approach in Section 13.3.2 and use a power counting method to estimate the divergence of corrections. It is often the case that a certain type of diagrams all diverges severely, but resummation of them removes the divergence. In our current study on the jellium model, divergence analysis is especially important. Suppose q is the four-momentum of an interaction line (i.e. a "virtual photon line" that carries the Coulomb repulsion), which of course we need to integrate over (since an interaction line cannot be an external line), we will show that only diagrams that are severely divergent in $|\boldsymbol{q}|$ are important with the argument in the discussion around Eq. (5.25) in Altland. We know an interaction line introduces the following component into the diagram:

Huaiyu Wang Eq. (13.3.10)

Huaiyu Wang Sec. 13.3.2

Altland Eq. (5.25)



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and the two electron propagators result in something like (5.58) in this solid state physics note. We find that when $\boldsymbol{q} \to 0$, $\boldsymbol{k}$ is near the Fermi surface, and q_0 is also approaching zero, $\sum_k G_k G_{q+k}$ reaches a peak because there is a step between $f(\xi_{\boldsymbol{k}})$ and $f(\xi_{\boldsymbol{k}+\boldsymbol{q}})$, just as is the case in (6.27) in this solid state physics note. Generally speaking, q_0 will not be zero because for example, we can assume that the two electron lines in the above diagram are external ones, and in this case q_0 is usually non-zero. However, if the diagram has algebraic divergence in $\boldsymbol{q}$, again we have something like

$$\frac{1}{|\boldsymbol{q}|}\sum_k G_k G_{q+k} = \frac{2}{V}\sum_{\boldsymbol{k}} \frac{1}{\omega + \xi_{\boldsymbol{k}} - \xi_{\boldsymbol{k}+\boldsymbol{q}}} \frac{f(\xi_{\boldsymbol{k}}) - f(\xi_{\boldsymbol{k}+\boldsymbol{q}})}{|\boldsymbol{q}|},$$

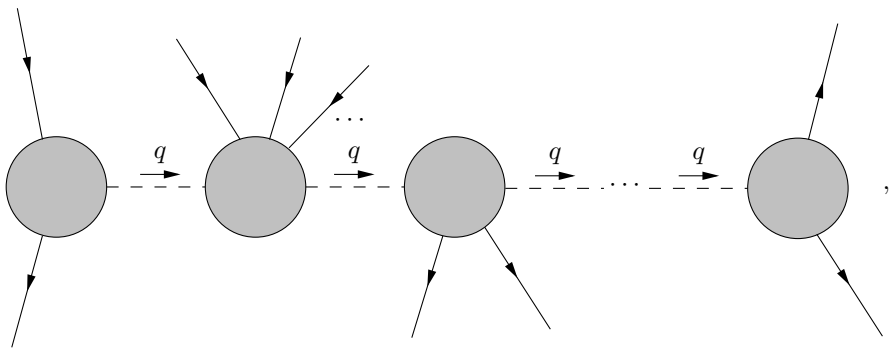
which evaluates to something proportion to the density of states at the Fermi surface. The denser the electron gas is, the larger this value is. We therefore conclude that in a high density electron system, the main contribution comes from Feynman diagrams with algebraic divergence in $|\boldsymbol{q}|$.

Note: Four momentum in condensed matter physics

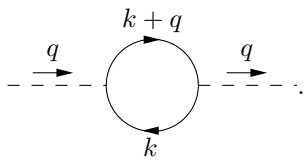
In condensed matter physics, though we do not have Lorentz invariance, the four-momentum notation is still used to avoiding using too many letters and to connect the frequency and the momentum of a particle.

Therefore, for a high-density electron system, we only need to sum all diagrams which are divergent in the momentum of the interaction lines. We should also note that since each interaction line brings in a $-\mathrm{i}e^2$ factor, we need to first classify the diagrams according to their order in e and then pick out the most divergent diagrams in q with the same order in e. In this way, among diagrams with the same degree of divergence, only those with the lowest order in e are evaluated, agreeing with our previous conclusion that the Coulomb repulsion is indeed a perturbation.

The problem, than, is how to find the most divergent diagrams. All interaction lines in the most divergent diagram should have as many interaction lines with the same momentum q as possible so that q has a large negative order. We find that this can only be achieved if the Feynman diagram is in the following form:

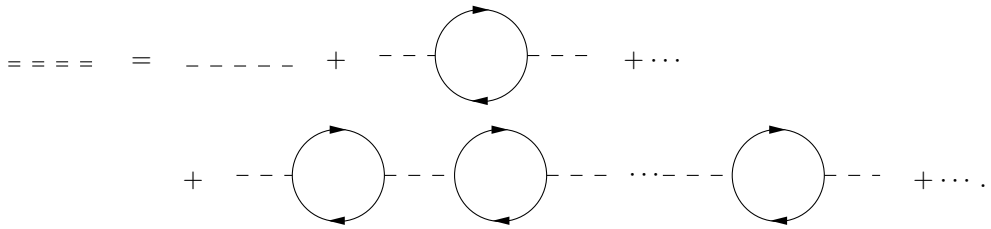


and there should be as less interaction lines in the grey blobs as possible. If a grey blob is not connected to an external line, we find it can only be in the following form:



The conclusion is that in a highly dense electron system, we only need to make the following

loop diagram correction: replacing the *bare* interaction line with the following:



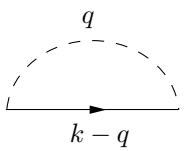
(26)

No other corrections are needed.

Diagrammatically, (26) can be justified by the intuition of "high density": the Coulomb field generated by an electron immediately excites an electron-hole pair in the dense electron gas nearby, and the process repeats over and over again to screen the bare Coulomb interaction.

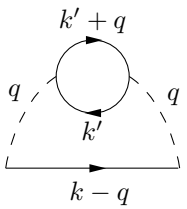
(26) is called the **ring diagram approximation** for obvious reasons. It is also called the RPA, since it only works in the high-density circumstance, and agree with (19) derived from the real random phase approximation. In modern days, (26) appears much more frequent than the original RPA, though they are equivalent. The advantage of the field theoretic version of RPA is that we know what we ignore and can improve the accuracy of our theory systematically.

We can check (26) for the electron self energy. In the jellium model since $V(0) = 0$, the Hartree term vanishes, and the first order self-energy diagram is just the Fock term: Huaiyu Wang Eq. (13.3.12) to Eq. (13.3.14)

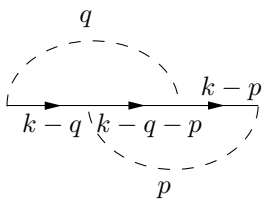


$$\sim e^2 \int \mathrm{d}^3\boldsymbol{q}\, \frac{1}{q^2} \sim \text{finite}, \tag{27}$$

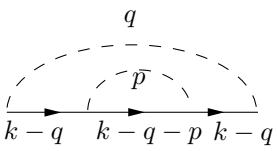
with no divergence in $\boldsymbol{q}$. The second order self-energy diagrams are



$$\sim e^4 \int \mathrm{d}^3\boldsymbol{q}\, \frac{1}{q^2}\frac{1}{q^2} \sim e^4 \frac{1}{q}, \tag{28}$$

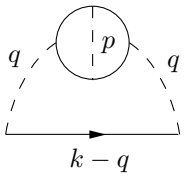


$$\sim e^4 \int \mathrm{d}^3\boldsymbol{q}\, \frac{1}{q^2} \int \mathrm{d}^3\boldsymbol{p}\, \frac{1}{p^2} \sim \text{finite value}, \tag{29}$$



$$\sim e^4 \int \mathrm{d}^3\boldsymbol{q}\, \frac{1}{q^2} \int \mathrm{d}^3\boldsymbol{p}\, \frac{1}{p^2} \sim \text{finite value}. \tag{30}$$

So we find only (28) is worth noticing. The following diagram



$$\sim e^6 \int \mathrm{d}^3\boldsymbol{q}\, \frac{1}{q^2}\frac{1}{q^2} \int \mathrm{d}^3\boldsymbol{p}\, \frac{1}{p^2} \sim \frac{e^6}{q}$$

has the same divergence degree with (28) but with a higher order e^n factor, so it is thrown away as well. Therefore, we have verified (26) to the second order.

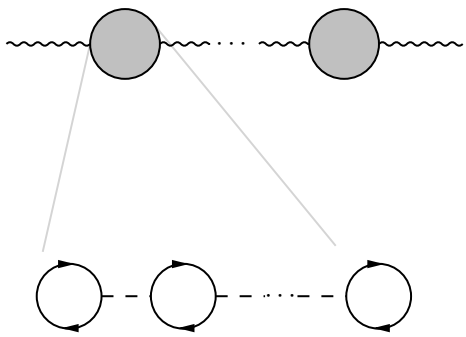


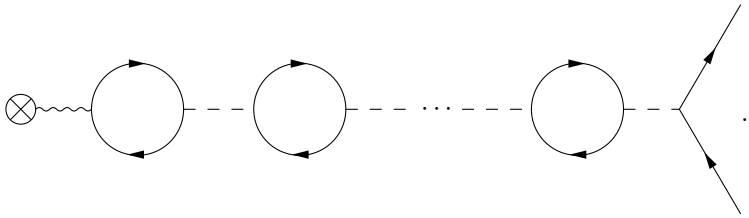
Figure 1: After obtaining (34) we need to take the self-energy correction of photons as well

5 The effective potential, linear response and the electric susceptibility

Now we go and see what (26) means to the electric susceptibility. We have already discussed this and the density-density Green function in the free theory in (5.58) in this solid state physics note.

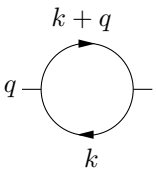
TODO: linear response theory

The LHS of (26) is often defined as the **effective (or screened) potential** V_q^{eff}. We can easily find that if we put an external charge into the electron gas, the potential actually felt by the electrons is also V_q^{eff}, because we can repeat the argument of the field theoretic version of RPA to the external potential field and conclude that what we need to calculate is just diagrams like this:



. (31)

We define



$$= 2 \times (-1) \times \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \mathrm{i}G_k^0 \times \mathrm{i}G_{k+q}^0 =: \mathrm{i}\Pi_q^0, \tag{32}$$

where the factor 2 comes from the spin-1/2 nature of the electron, (-1) comes from the closed fermionic line, and G^0 is the free electron Green function. In this way, (31) is just

$$-\mathrm{i}V_q^{\text{eff}} = -\mathrm{i}V_q^{\text{ext}} + (-\mathrm{i}V_q^{\text{ext}})(\mathrm{i}\Pi_q^0)(-\mathrm{i}V_q) + (-\mathrm{i}V_q^{\text{ext}})(\mathrm{i}\Pi_q^0)(-\mathrm{i}V_q)(\mathrm{i}\Pi_q^0)(-\mathrm{i}V_q) + \cdots ,$$

and the resummation gives

$$V_q^{\text{eff}} = V_q^{\text{ext}} \frac{1}{1 - V_q \Pi_q^0}. \tag{33}$$

Wen Sec. 5.4.3

For (26), we find that when $\boldsymbol{q} \to 0$, V_q^{eff} is no longer divergent. TODO: This also eliminates the diverging (6.15) in this solid state physics note. Also, now since we have

$$\boldsymbol{E} \sim V^{\text{eff}}, \quad \boldsymbol{D} \sim \epsilon_0 V^{\text{ext}},$$

it seem that

$$\epsilon_{\text{r}} = \frac{V_q^{\text{ext}}}{V_q^{\text{eff}}} = 1 - V_q \Pi_q^0. \tag{34}$$

Zhengzhong Li Eq. (4.6.14)

Note: So many Green functions

If we are working in the zero temperature real time field theory, G^0 in (32) is not the retarded Green function

$$G_k^{0,\mathrm{ret}} = \frac{1}{k^0 - \xi_{\boldsymbol{k}} + \mathrm{i}0^+} \tag{35}$$

Huaiyu Wu Eq. (9.4.21)

but the Feynman Green function

$$G_k^0 = \frac{1}{k^0 - \xi_{\boldsymbol{k}} + \mathrm{i}0^+ \operatorname{sgn}(k^0)} = \frac{\theta(\xi_{\boldsymbol{k}})}{k^0 - \xi_{\boldsymbol{k}} + \mathrm{i}0^+} + \frac{\theta(-\xi_{\boldsymbol{k}})}{k^0 - \xi_{\boldsymbol{k}} - \mathrm{i}0^+}. \tag{36}$$

Huaiyu Wu Eq. (9.4.23)

The Feynman Green function includes the contribution of holes (the antiparticle of the electron in condensed matter field theory). We can see that it is actually much easier to work in the Matsubara formalism.
The $\operatorname{sgn}(k^0)$ factor in (18) can be replaced by $\operatorname{sgn}(\xi_{\boldsymbol{k}})$, since only when we are close to the pole of (18) does the $\operatorname{sgn}(k^0)$ term become important. This explains the second equation of (36).
Note that (36) agrees with the Green function of fermions in HEP in its infinitesimal imaginary part, the denominator of which is $(k^0)^2 - \boldsymbol{p}^2 + \mathrm{i}0^+$, and we can see that there is one pole on the upper plane and another on the lower plane, as is the case in (36). This comes from the fact that both particles and antiparticles (in condensed matter physics, holes) are present.

Note that the definition of ϵ itself is ambiguous. For example, we may also consider the self-energy correction of *photons* (beside photons passing the Coulomb interaction) themselves, as is shown in Figure 1 on page 8. This gives rise to **local field enhancement** (see Section 4 in this note, for example).

6 Evaluating the electric susceptibility and bosonic modes

Now it is time to evaluate the analytic properties of the susceptibility. The dispersion relations of bosonic modes are given by $\epsilon = 0$ (not $\epsilon \to \infty$ – see Section 2.2.1 in this optics note). Note that the imaginary part of ϵ indicates damping (see Section 8.1.1 in this optics note), and therefore if $\operatorname{Im}\epsilon \neq 0$, even if on an $(\omega, \boldsymbol{k})$ point we have $\operatorname{Re}\epsilon = 0$, this is not an eigenmode. However, if $\operatorname{Im}\epsilon$ is not large, we can still say that "there is a mode at $(\omega, \boldsymbol{k})$ with a finite lifetime". So what we are going to do is to solve the equation $\operatorname{Re}\epsilon(\omega, \boldsymbol{k}) = 0$ and check the lifetime of each mode.

Zhengzhong Li, Sec. 4.6

Note

The occurrence of the plasmon is not accidental. What we are doing is to actually treat some of the interaction lines as actually (quasi)particles. The energy of the plasmon comes from the Coulomb repulsion energy. This is an example of the Hubbard-Stratonovich transformation, where a $\psi\psi\bar{\psi}\bar{\psi}$ interaction term is assumed to be carried by a kind of boson. RPA for electrons is just the self energy correction for the plasmon (or more generally, the bosonic field introduced by the Hubbard-Stratonovich transformation). Interaction between plasmons can be obtained by diagrams beyond RPA.

Now we check the lifetime. It can be seen that electron-hole pairs all have finite lifetimes and therefore are not really eigenmode. TODO When $q > q_{\mathrm{c}}$, since there are electron-hole modes with lower energies than the plasmon, it can be expected that a plasmon will soon decay into electron-hole pairs, which is verified by the imaginary part of ϵ.

7 The lifetime of electrons and the Fermi liquid

Here we follow the procedures in [1] and first calculate the lifetime of the quasiparticle using the unscreened, bare Coulomb potential and then the screened potential. We have argued in (6.32) of this solid state physics notethat as long as a system forms a Fermi liquid, we can always find a $1/\tau \sim \xi_{\boldsymbol{k}}^2$ relation, and we will explicitly verify the claim in this section.

TODO: unscreened

TODO: screened

Now that we have already calculated the lifetime of the quasiparticle, it is worth noting that we can also use a more *kinetic* way to find the same result, and to verify the optical theorem in this specific case. Zhengzhong Li Sec. 4.11

References

[1] Nicolas Dupuis. Notes on the Many-Body Problem. https://www.lptmc.jussieu.fr/users/dupuis, 2022.