

# Homework 3

Jinyuan Wu

November 2, 2022

## Lecture 8, Exercise 3

**Solution** We have

$$U = I_1 R = \frac{1}{C} \int dt I_2 = L \frac{dI_3}{dt},$$

and

$$I = I_1 + I_2 + I_3.$$

In the frequency domain, we have

$$U[\omega] = I_1[\omega] R = \frac{1}{C} \frac{1}{-i\omega} I_2 = L(-i\omega) I_3,$$

and therefore

$$Z[\omega] = \frac{U[\omega]}{I_1[\omega] + I_2[\omega] + I_3[\omega]} = \frac{1}{\frac{1}{R} - i\omega C - \frac{1}{i\omega L}}. \quad (1)$$

So we have

$$\text{Re } Z[\omega] = \frac{1}{R} \frac{1}{\frac{1}{R^2} + \left(\frac{1}{\omega L} - \omega C\right)^2}, \quad (2)$$

and

$$\text{Im } Z[\omega] = -\left(\frac{1}{\omega L} - \omega C\right) \frac{1}{\frac{1}{R^2} + \left(\frac{1}{\omega L} - \omega C\right)^2}. \quad (3)$$

The zero points of the denominator are given by

$$\omega = \omega_{1,2} := \frac{-\frac{i}{R} \pm \sqrt{-\frac{1}{R^2} + \frac{4C}{L}}}{2C}. \quad (4)$$

## Lecture 9, Exercise 3

**Solution** The energy density is (see (33))

$$\mathcal{E} = \frac{1}{2} c V^2 + \frac{1}{2} l I^2. \quad (5)$$

Therefore since

$$\frac{\partial V}{\partial x} = -l \frac{\partial I}{\partial t}, \quad \frac{\partial I}{\partial x} = -c \frac{\partial V}{\partial t}, \quad (6)$$

we have

$$\begin{aligned} \frac{\partial J}{\partial x} &= -\frac{\partial \mathcal{E}}{\partial t} = -cV \frac{\partial V}{\partial t} - lI \frac{\partial I}{\partial t} \\ &= V \frac{\partial I}{\partial x} + I \frac{\partial V}{\partial x} = \frac{\partial}{\partial x} (VI), \end{aligned}$$

and  $J$  is therefore ( $J$  can't be uniquely decided purely from the continuity equation, but we know  $V \sim E$ ,  $I \sim \Phi \sim B$ , and in electrodynamics we already know  $J \sim E \times B$ , so the expression here agrees with the general definition)

$$J = VI = \sqrt{Z}(A^\rightarrow + A^\leftarrow) \frac{1}{\sqrt{Z}}(A^\rightarrow - A^\leftarrow) = (A^\rightarrow)^2 - (A^\leftarrow)^2. \quad (7)$$

Since both  $A^\rightarrow$  and  $A^\leftarrow$  are in harmonic oscillation, we have

$$|\bar{J}| = \frac{1}{2} \left| |A^\rightarrow|^2 - |A^\leftarrow|^2 \right|. \quad (8)$$

## Lecture 10, Exercise 1

## Lecture 11, Exercise 2

**Solution** The PDE problem in question is the follows: For the bulk equations,

$$\begin{aligned}\frac{\partial A^{\rightarrow}}{\partial x} + \frac{1}{v_p} \frac{\partial A^{\rightarrow}}{\partial t} &= 0, & \frac{\partial A^{\leftarrow}}{\partial x} - \frac{1}{v_p} \frac{\partial A^{\leftarrow}}{\partial t} &= 0, \\ A^{\rightarrow} &= \frac{V}{2\sqrt{Z}} + \frac{\sqrt{Z}}{2} I, & A^{\leftarrow} &= \frac{V}{2\sqrt{Z}} - \frac{\sqrt{Z}}{2} I.\end{aligned}\tag{9}$$

The boundary conditions are

$$A^{\rightarrow}(d, t) = -A^{\leftarrow}(d, t), \quad I(0, t) = \Theta(t)I_0.\tag{10}$$

The input current is given by a current source, so we don't need to analyze the complicated detail about how the current goes back – all we need to know is somehow  $I(0, t)$  is in the given form, regardless of how this is possible. By time domain Fourier transformation, we get

$$\frac{\partial A^{\rightarrow}}{\partial x} - \frac{i\omega}{v_p} A^{\rightarrow} = 0, \quad \frac{\partial A^{\leftarrow}}{\partial x} + \frac{i\omega}{v_p} A^{\leftarrow} = 0,\tag{11}$$

and therefore

$$A^{\rightarrow}[d, \omega] = e^{\frac{i\omega d}{v_p}} A^{\rightarrow}[0, \omega],$$

and

$$A^{\leftarrow}[d, \omega] = e^{-\frac{i\omega d}{v_p}} A^{\leftarrow}[0, \omega],$$

and by the boundary condition we find

$$A^{\leftarrow}[0, \omega] = -e^{\frac{i2\omega d}{v_p}} A^{\rightarrow}[0, \omega].\tag{12}$$

Its time domain version is

$$A^{\leftarrow}(0, t) = -A^{\rightarrow}\left(0, t - \frac{2\omega d}{v_p}\right).\tag{13}$$

In principle, we have

$$\frac{V[0, \omega]}{I[0, \omega]} = Z \frac{A^{\rightarrow} + A^{\leftarrow}}{A^{\rightarrow} - A^{\leftarrow}} = Z \frac{1 - e^{i2\omega d/v_p}}{1 + e^{i2\omega d/v_p}} = -iZ \tan\left(\frac{\omega d}{v_p}\right),\tag{14}$$

and by Fourier transformation

$$I[0, \omega] = \int_{-\infty}^{\infty} e^{i\omega t} dt \Theta(t) I_0 = I_0 \frac{i}{\omega + i0^+},\tag{15}$$

and therefore

$$V(0, t) = \int \frac{d\omega}{2\pi} e^{-i\omega t} V[0, \omega] = Z I_0 \int \frac{d\omega}{2\pi} e^{-i\omega t} \tan\left(\frac{\omega d}{v_p}\right) \frac{1}{\omega + i0^+}.\tag{16}$$

But this method has convergence problem which I don't know how to solve. For this specific case, it's possible to solve the problem in the time domain. During the  $t < 0$  period,  $A^{\leftarrow, \rightarrow}$ ,  $U$  and  $I$  are all zero, so in the  $0 < t < 2\omega d/v_p$  period,  $A^{\leftarrow}(0, t)$  is zero, and therefore

$$V(0, t) = \sqrt{Z}(A^{\leftarrow}(0, t) + A^{\rightarrow}(0, t)) = 2 \cdot \sqrt{Z} \cdot \frac{\sqrt{Z}}{2} \cdot I_0 \Theta(t) = Z I_0.\tag{17}$$

When  $2\omega d/v_p < t < 4\omega d/v_p$ , however, we have

$$\begin{aligned}A^{\leftarrow}(0, t) &= -A^{\rightarrow}\left(0, t - \frac{2\omega d}{v_p}\right) = -\frac{V\left(0, t - \frac{2\omega d}{v_p}\right)}{2\sqrt{Z}} - \frac{\sqrt{Z}}{2} I\left(0, t - \frac{2\omega d}{v_p}\right) \\ &= -\sqrt{Z} I_0,\end{aligned}\tag{18}$$

and therefore

$$V(0, t) = \sqrt{Z}(A^{\leftarrow}(0, t) + A^{\rightarrow}(0, t)) = 0.\tag{19}$$

Repeating this procedure, we get

$$V(0, t) = \begin{cases} Z I_0, & 2n \frac{2\omega d}{v_p} < t < (2n+1) \frac{2\omega d}{v_p}, \\ 0, & (2n+1) \frac{2\omega d}{v_p} < t < (2n+2) \frac{2\omega d}{v_p}, \end{cases} \quad n = 0, 1, 2, \dots\tag{20}$$

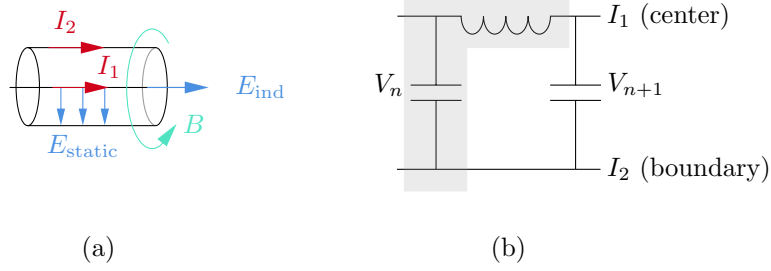


Figure 1: Transmission line

### Problem

### Solution

(a) The transmission line model is still a quasi-magnetostatic model. Ignoring all resistance, and assuming that the only conducting objects in the system are the line at  $r = 0$  and the boundary of the transmission line (or otherwise we need several “vertical layers” in the effective circuit model), that the inner structures of which can be ignored (or otherwise there may be multiple modes of the system), and that the boundary is grounded everywhere so there is no spatial variation of voltage on it (or otherwise there has to be an inductor on the top line in Figure 1(b)), the physics of a line element is shown in Figure 1(a): The current along the  $x$  direction creates a magnetic field, which induces an electric field component along the  $x$  axis because  $\nabla \times \mathbf{E} = -\partial \mathbf{B} / \partial t$ . We also have a radial component of  $\mathbf{E}$  because of the electrostatic field established between the surface and the inner conducting line.

Thus, in Figure 1(a), we have two electric current degrees of freedom (one at  $r = 0$ , the other on the boundary), one electric field degrees of freedom between the central line and the boundary, one magnetic field degree of freedom which gives  $\mathbf{E}_{\text{ind}}$  along the  $x$  direction. The distribution of the magnetic field is completely decided by the currents so we only need one scalar – the magnetic flux – to represent the magnetic field, and similarly we only need two scalars – the voltage on the central conducting line and the voltage between the  $r = 0$  line and the surface – to represent  $\mathbf{E}_{\text{ind}}$  and  $\mathbf{E}_{\text{static}}$ . The resulting theory for the line element in Figure 1(a) has the same equations of motion with the shaded area in Figure 1(b).

Suppose the parameters of the capacitor and the inductor in Figure 1(b) are  $C$  and  $L$ , respectively. Note that  $\varphi(x, t)$  is defined on the capacitors, not the inductors, and the variable doesn’t represent the magnetic flux.

By basic circuit analysis we know the electromotive force on the inductor in Figure 1(b) is

$$\mathcal{E} = V_{n+1} - V_n,$$

and according to Faraday’s law of induction, the electromotive force on the inductor is (here the “inductor” is single-turned, which can be found by looking at Figure 1(a))

$$\mathcal{E} = -\frac{d\Phi}{dt},$$

so the current is

$$I = \frac{1}{L} \Phi = -\frac{1}{L} \int dt \mathcal{E} = \frac{1}{L} (\varphi_n - \varphi_{n+1}),$$

so the energy is

$$\frac{1}{2} L I^2 = \frac{1}{2L} (\varphi_n - \varphi_{n+1})^2. \quad (21)$$

The energy of  $C_n$  is by definition

$$\frac{1}{2} C V_n^2 = \frac{1}{2} C \dot{\varphi}_n^2. \quad (22)$$

Tracing the origin of (21) and (22), we find they eventually come from the magnetic part and the electric part of electromagnetic energy, and since in the electromagnetism Lagrangian, they are recognized as the kinetic energy part and the potential energy part, respectively, in our effective model the same should be done, so the Lagrangian of the shaded part in Figure 1(b) is

$$L_n = \frac{1}{2} C \dot{\varphi}_n^2 - \frac{1}{2L} (\varphi_n - \varphi_{n+1})^2. \quad (23)$$

So the total Lagrangian is

$$\begin{aligned}
L &= \sum_n L_n = \sum_n \left( \frac{1}{2} C \dot{\varphi}_n^2 - \frac{1}{2L} (\varphi_n - \varphi_{n+1})^2 \right) = \sum \frac{\Delta x}{a} \left( \frac{1}{2} C \dot{\varphi}_n^2 - \frac{1}{2L} (\varphi_n - \varphi_{n+1})^2 \right) \\
&= \sum \Delta x \left( \frac{1}{2} \frac{C}{a} \dot{\varphi}_n^2 - \frac{1}{2L/a} \frac{(\varphi_n - \varphi_{n+1})^2}{a^2} \right) \\
&= \int dx \left( \frac{1}{2} c (\partial_t \varphi)^2 - \frac{1}{2l} (\nabla \varphi)^2 \right).
\end{aligned} \tag{24}$$

The dynamic equation is

$$\begin{aligned}
\partial_t \frac{\partial \mathcal{L}}{\partial \partial_t \varphi} + \nabla \cdot \frac{\partial \mathcal{L}}{\partial \nabla \varphi} &= 0, \\
c \frac{\partial^2 \varphi}{\partial t^2} &= \frac{1}{l} \nabla^2 \varphi, \quad \frac{1}{v_p^2} \frac{\partial^2 \varphi}{\partial t^2} = \frac{\partial^2 \varphi}{\partial x^2}, \quad v_p = \frac{1}{\sqrt{lc}}.
\end{aligned} \tag{25}$$

(b) By definition,

$$\pi(x, t) = \frac{\partial \mathcal{L}}{\partial \dot{\varphi}} = c \dot{\varphi}, \tag{26}$$

and the Hamiltonian is

$$\begin{aligned}
H &= \int dx (\pi \dot{\varphi} - \mathcal{L}) = \int dx \left( \frac{1}{2} c \dot{\varphi}^2 + \frac{1}{2l} (\nabla \varphi)^2 \right) \\
&= \int dx \left( \frac{1}{2c} \pi^2 + \frac{1}{2l} (\nabla \varphi)^2 \right).
\end{aligned} \tag{27}$$

The EOM in Hamiltonian language is

$$\dot{\varphi} = \frac{1}{c} \pi, \quad \dot{\pi} = -\frac{\delta H}{\delta \varphi} = \frac{1}{l} \nabla^2 \varphi. \tag{28}$$

(c) By definition

$$A^\rightarrow(x, t) = \frac{1}{\sqrt{L}} \sum_{k>0} (e^{ikx} A_k(t) + e^{-ikx} A_k^*(t)), \tag{29}$$

and

$$A^\leftarrow(x, t) = \frac{1}{\sqrt{L}} \sum_{k<0} (e^{ikx} A_k(t) + e^{-ikx} A_k^*(t)), \tag{30}$$

because  $A_k(t) \sim e^{-i\omega t}$ , and therefore for  $A^\rightarrow$ , we have to have  $k > 0$  so that  $\partial A^\rightarrow / \partial x$  and  $v_p \partial A^\rightarrow / \partial t$  cancel. Since  $A^\rightarrow$  and  $A^\leftarrow$  are defined in terms of  $V$  and  $I$ , we need to rewrite the Hamiltonian in terms of  $U$  and  $I$ . We have

$$-\frac{\partial V}{\partial x} = l \frac{\partial I}{\partial t} = -\frac{\partial}{\partial t} \frac{\partial \varphi}{\partial x}, \tag{31}$$

so

$$I = -\frac{1}{l} \nabla \varphi, \tag{32}$$

and the Hamiltonian is therefore

$$H = \int dx \left( \frac{1}{2} c V^2 + \frac{1}{2} l I^2 \right). \tag{33}$$

From

$$\begin{aligned}
A^\rightarrow(x, t) &= \frac{V(x, t)}{2\sqrt{Z}} + \frac{\sqrt{Z} I(x, t)}{2} \\
A^\leftarrow(x, t) &= \frac{V(x, t)}{2\sqrt{Z}} - \frac{\sqrt{Z} I(x, t)}{2},
\end{aligned} \tag{34}$$

we have

$$\begin{aligned}
H &= \int dx \left( \frac{1}{2} c Z (A^\rightarrow + A^\leftarrow)^2 + \frac{1}{2} \frac{l}{Z} (A^\rightarrow - A^\leftarrow)^2 \right) \\
&= \sqrt{lc} \int dx ((A^\rightarrow)^2 + (A^\leftarrow)^2).
\end{aligned} \tag{35}$$

We have

$$\begin{aligned}\int dx (A^\rightarrow(x, t))^2 &= \frac{1}{L} \sum_{k, k' > 0} \int dx (e^{ikx} A_k(t) + e^{-ikx} A_k^*(t)) (e^{ik'x} A_{k'}(t) + e^{-ik'x} A_{k'}^*(t)) \\ &= \frac{2}{L} \cdot L \sum_{k > 0} A_k(t) A_k^*(t),\end{aligned}$$

and similarly

$$\int dx (A^\leftarrow(x, t))^2 = 2 \sum_{k < 0} A_k(t) A_k^*(t).$$

So

$$H = 2\sqrt{lc} \sum_k A_k(t) A_k^*(t). \quad (36)$$

So the total energy of the transmission line is the sum of  $|A_k|^2$ , and that's why we call  $\{A_k\}$  normal mode amplitudes.

(d) We have

$$\begin{aligned}\varphi(x, t) &= \int_0^t dt' V(x, t') = \int_0^t dt' \sqrt{Z} (A^\leftarrow + A^\rightarrow) \\ &= \sqrt{Z} \frac{1}{\sqrt{L}} \sum_k \int_0^t dt' (e^{ikx} A_k(t') + e^{-ikx} A_k^*(t')) \\ &= \frac{\sqrt{Z}}{\sqrt{L}} \sum_k \frac{1}{-i\omega_k} e^{ikx} A_k(t)\end{aligned}$$

By (25) we have

$$\omega_k = \frac{1}{\sqrt{lc}} |k|, \quad (37)$$

so

(e) The variable  $\varphi$  appears in the Lagrangian, and from it we can find all other physical quantities, so it's the “vector potential”.  $V = \dot{\varphi}$ , so it has the status of the electric field, and since  $I = -\partial_x \varphi / l$ , it's the counterpart of the magnetic field.  $A_k(t)$  appears in the expansion of the variables mentioned above and  $H \sim \sum_k |A_k|^2$ , so its counterpart is the annihilation operator (and  $A_k^*$  is the creation operator).

(f) The equation is (here  $C_{\text{ext}}$  and  $L_{\text{ext}}$  are used to avoid confusion with  $C$  and  $L$  in the transmission line; note that if we regard the inductor as a power source, then its *electromotive force* is  $-L\dot{I}$ , while if we regard it as an ordinary component, then its *voltage* is  $L\dot{I}$ )

$$\frac{q_{\text{ext}}}{C_{\text{ext}}} + L_{\text{ext}} \frac{dI_{\text{ext}}}{dt} = V_0(t) = V(0, t), \quad (38)$$

and (following the notation in the lecture notes, here  $I_0$  is the current on the first inductor in the transmission line)

$$\underbrace{I_0}_{I(0, t)} + \underbrace{\frac{dq_0}{dt}}_{C\dot{V}(0, t)} + I_{\text{ext}} = 0. \quad (39)$$

Since  $C = ca$ , in the continuous limit, the second term in the equation can be thrown away. (Which is equivalent to directly removing  $C_0$  and starting the transmission line with the inductor instead of the capacitor). So the EOM of the external circuit is

$$\frac{1}{C_{\text{ext}}} q_{\text{ext}} + L_{\text{ext}} \frac{dI_{\text{ext}}}{dt} = V(0, t), \quad I(0, t) + I_{\text{ext}} = 0. \quad (40)$$

Following the same procedure in the lecture note, we have

$$V(0, t) = ZI(0, t) + 2\sqrt{Z}A^\leftarrow(0, t) = -ZI^{\text{ext}}(t) + 2\sqrt{Z}A^\leftarrow(0, t), \quad (41)$$

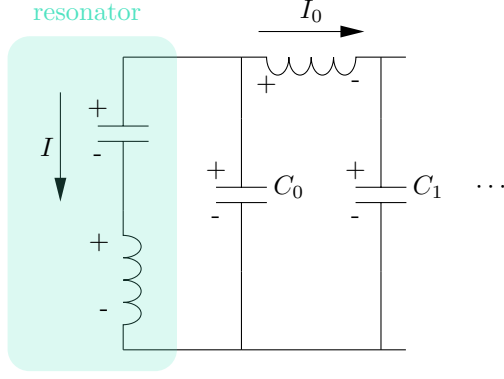


Figure 2: A serial  $LC$  resonator and a transmission line

and the EOM of the external circuit is

$$\frac{1}{C_{\text{ext}}}q_{\text{ext}} + L_{\text{ext}}\frac{dI_{\text{ext}}}{dt} + ZI_{\text{ext}} = \underbrace{2\sqrt{Z}A^{\leftarrow}(0,t)}_{U^{\text{noise}}}. \quad (42)$$

Here the  $ZI_{\text{ext}}$  term is the dissipative term introduced by energy loss into the transmission line, and the RHS is the fluctuation term introduced by random “radiation” of the transmission line into the  $LC$  resonator. The correlation function of the dissipative term is

$$\begin{aligned} \langle U^{\text{noise}}(t)U^{\text{noise}}(0) \rangle &= 4Z \langle A^{\leftarrow}(0,t)A^{\leftarrow}(0,0) \rangle \\ &= 4Z \frac{1}{L} \sum_{k,k' < 0} (A_k e^{-i\omega_k t} + A_k^* e^{i\omega_k t})(A_{k'} + A_{k'}^*). \end{aligned}$$

From (36) and the equipartition theorem we have

$$\begin{aligned} N_{\text{cell}} \times 2\sqrt{lc} \langle A_k A_k^* \rangle &= N_{\text{cell}} \times k_B T, \\ \langle A_k A_{k'}^* \rangle &= \delta_{kk'} \frac{1}{2\sqrt{lc}} k_B T. \end{aligned} \quad (43)$$

So

$$\begin{aligned} \langle U^{\text{noise}}(t)U^{\text{noise}}(0) \rangle &= 4Z \frac{1}{L} \sum_{k < 0} \frac{1}{2\sqrt{lc}} k_B T (e^{-i|k|t/\sqrt{lc}} + e^{i|k|t/\sqrt{lc}}) \\ &= 2Z k_B T \frac{1}{\sqrt{lc}} \frac{1}{L} \sum_k e^{-i|k|t/\sqrt{lc}} \\ &= 2Z k_B T \delta(t), \end{aligned} \quad (44)$$

which is the expected form in the Langevin equation.

$$r = \frac{A^{\leftarrow}(0,t)}{A^{\rightarrow}(0,t)} = \frac{V(0,t)-}{V(0,t)+} \quad (45)$$