

Fermi liquid

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1 Free electron gas

A free electron gas is described by¹

$$H = \sum_i \frac{\mathbf{p}_i^2}{2m}, \quad (1)$$

and we get a Fermi sphere in the momentum space containing all occupied states, and a Fermi momentum p_F . After we add a periodic potential $V(\mathbf{r})$ to the model, we get **bands**: parabolic bands are displaced periodically according to the \mathbf{G} -grid, and the crossing points of these curves undergo degeneracy breaking, and the spectrum, eventually, becomes several bands confined in the first Brillouin zone. Despite this correction, the number of filled electrons never changes.

For a free electron gas, we have well-known observables like the density of states at E_F (here $\omega = \varepsilon - E_F$)

$$N(\omega = 0) = \frac{m p_F}{\pi^2 \hbar^3}, \quad (2)$$

the electronic specific heat

$$C_v = \underbrace{\frac{\pi^2 k_B^2}{3} N(\omega = 0)}_{\text{Sommerfield } \gamma} \cdot T, \quad (3)$$

and the magnetic susceptibility

$$\chi = \mu_0 \mu_B^2 \cdot N(\omega = 0). \quad (4)$$

^3He atoms are fermions: we have two electrons, two protons, and only one neutron in one ^3He atom, and therefore there are odd fermions inside and the composite state is a fermion. In a ^3He liquid, since the distance between the atoms is drastically reduced, we can expect the effect of repulsion is much stronger than the case in the gas phase. This is a cold atom model of electrons in solids: we can easily measure the specific heat, etc. of the liquid ^3He .

The C_V - T curve measured looks like TODO: fig 1 As is expected, when $T \rightarrow 0$, we see a $C_V \propto T$ behavior, qualitatively agreeing with (3). The coefficient however is not what is predicted by (2) and (3). Now if we apply pressure to liquid ^3He , C_v/T clearly increases.

For a fermion gas, we can measure the mass according to (3) and (2). Now if we calculate the “mass” of ^3He according to the two equations, we get a pressure-dependent (quite surprising because the density of liquids almost stays the same when we press it) “effective mass” that is never the same as the bare mass of ^3He .

The next question, then, is whether liquid ^3He — a system with known strong interaction — can be equivalently described by something with the effective mass.

2 Assumptions of Fermi liquid theory

2.1 Motivation: what happens after interaction is turned on

The general form of Hamiltonian of fermionic systems that is found in condensed matter physics looks like

$$H = \sum_i \frac{\mathbf{p}_i^2}{2m} + \sum_{i,j} V(|\mathbf{r}_i - \mathbf{r}_j|). \quad (5)$$

When we deal with a two-body problem with this dynamics, the collision behavior is completely described by the scattering cross-section. If we extend this picture to a many-body system, we immediately see scattering changes the momentum distribution of particles, because the final

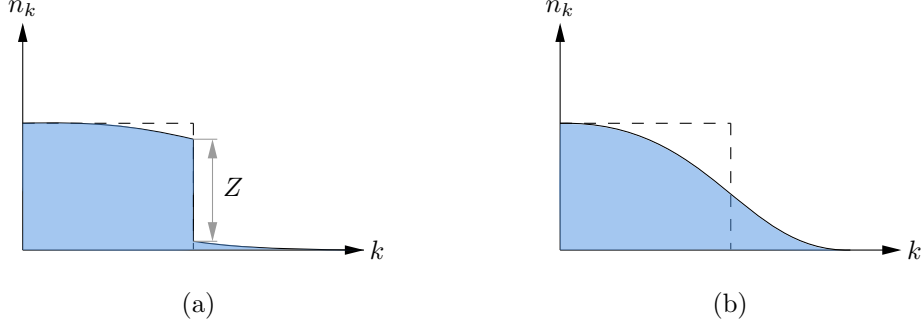


Figure 1: Two types of interaction correction to the electron distribution. (a) The Fermi liquid case: there is still a stepwise change of n with respect to the energy. (b) The non-Fermi liquid case: the change of n is smooth.

states of scattering can obviously be over the Fermi surface, and therefore there is still some distribution above p_F now (Fig. 1).

It should be noted that Fig. 1 is the distribution function concerning the momentum, not the energy: we still have a very clear energy upper bound in the energy distribution function. The reason of electron distribution over p_F comes from the fact that $n(\mathbf{p})$ is no longer a good quantum number after the interaction is turned on and scattering occurs. The electrons with $|\mathbf{p}| > p_F$ still have energies below the corrected Fermi energy. Sometimes people say that because scattering changes \mathbf{p} , the momentum is no longer a good quantum number; what we want to point out here is it may also be possible that the single-electron picture completely breaks down (and we can't even talk about the momentum of a single electron). If the interaction is strong enough, we may find $Z = 0$, and in the spectrum of the system we no longer have well-defined electron-like quasiparticles: when an electron is injected into the system, it quickly excites many electron-hole pairs because of scattering and decays.

The next question is when we can get something close enough to the Fermi gas model. It's hard to know when the single-body picture works (we have some general clues from studies in strongly correlated phases, like the band structure should be dispersive, etc., but it's hard to give an explicit criterion). But when the single-body picture does hold, and the only thing we need is well-defined momentum, the condition is quite clear: low temperature. There is another origin of blurriness in electronic distribution: the thermal fluctuation. When we calculate expected heat capacity, magnetic response, etc. using the Fermi gas model, we take the Fermi-Dirac distribution into account. Thus, the condition under which we have a well-defined momentum degree of freedom seems to be that the blurriness caused by *quantum* fluctuation should be much weaker than the blurriness caused by *thermal* fluctuation. Suppose τ is the lifetime of an electron with a well-defined momentum \mathbf{p} (after which the $|\mathbf{p}\rangle$ state evolves into a cloud of unrecognizable excitations), what is needed is then

$$\frac{\hbar}{\tau} \ll k_B T. \quad (6)$$

Thinking for a while about the collision integral in Boltzmann equation, we find (this is essentially Fermi golden rule)

$$\frac{1}{\tau} \propto (\Delta p)^2 \propto \frac{1}{T^2}, \quad (7)$$

where Δp is the width of the region in which particle occupation goes smoothly from one to zero, and is proportion to T , as can be shown using the Fermi-Dirac distribution function. So we find as long as $T \rightarrow 0$, if we have a well-defined single-electron picture, then for each electron, we have a well-defined momentum degree of freedom. Then we say we get a **Fermi liquid**.

The exact condition concerning how low the temperature should be is linked to energy scales in the system. Usually the energy scale is just E_F (recall that that's the condition under which the Fermi-Dirac distribution doesn't look very different from a stepwise function, which is exactly what we want to reduce collision) and therefore the Fermi liquid theory is safe in the energy scales on which condensed matter physicists work. For cold atom systems, however, it may be

¹For some reason, when talking about Fermi liquid, people like to use \mathbf{p} instead of \mathbf{k} .

the case that T raises above E_F while the system is still liquid; in this case maybe classical electrodynamics is a good choice? TODO

We need to emphasize here again that the condition $T \ll E_F$ is about when a system with known single-electron behaviors is a Fermi liquid; failing to meet this condition doesn't result in a strongly correlated system, but a classical fluid (TODO: really?). Indeed, strongly correlated behaviors usually break down when the temperature is high, and therefore the high-temperature behavior of a condensed matter system is always well described by a band electron model (as long as the temperature isn't too high so that the crystal melts).

2.2 Assumption: correspondence between interactive corrected states and uncorrected states

We assume that we still have “states labeled by \mathbf{p} , band index n , etc.” in a Fermi liquid, and the ground state and the first several excited states can be totally labeled by $\delta n(\mathbf{p})$, which is $n(\mathbf{p})$ – some sort of “filling” on these states – minus $n_0(\mathbf{p})$ – the Fermi gas filling. These states labeled by \mathbf{p} , n , etc. are essentially the “single-electron wave function” in the single-electron Green function, and the ground state $n(\mathbf{p})$ is essentially the spectral function. We further assume δn is small, so that what we get is Fig. 1(a), instead of Fig. 1(b). This agrees with the observed fact that the $C_v \propto T$ relation is still true, so Fermi liquid should be somehow similar to the Fermi gas. Also, we assume the total number of electrons should change considerably:

$$\delta N = \sum_{\mathbf{p}} \delta n(\mathbf{p}) \ll \sum_{\mathbf{p}} n(\mathbf{p}). \quad (8)$$

Note that although we are talking about “partial occupation” or even “quasiparticle decaying” (see the follows), Fermi liquid theory is still a *pure state* theory; as is shown by Green function theory, partial occupation and even decaying can happen in the zero-temperature case, due to quantum fluctuation. A single-electron state can evolve into a multiple-electron state, and therefore the ground state is a mixture of single-electron, double-electron, etc. states, and if we insist on a single-electron theory, certain dissipation channels appear. The Fermi liquid theory, essentially, is assuming that the single-particle picture still works well enough.

2.3 The energy functional

The energy of state $\{\delta n_{\mathbf{p}}\}$, E , reads

$$E = E_0 + \delta E, \quad \delta E = \sum_{\mathbf{p}} \varepsilon_0(\mathbf{p}) \delta n(\mathbf{p}) + \mathcal{O}(\delta n^2). \quad (9)$$

Here E_0 is the ground state energy of the free electron gas, (9) doesn't contain any many-body correction; a more accurate form of the energy of the system is

$$\delta E = \sum_{\mathbf{p}} \varepsilon_0(\mathbf{p}) \delta n(\mathbf{p}) + \frac{1}{2V} \sum_{\mathbf{p}, \mathbf{p}'} f_{\mathbf{p}\mathbf{p}'} \delta n(\mathbf{p}) \delta n(\mathbf{p}') + \mathcal{O}(\delta n^3). \quad (10)$$

This means an electron in a Fermi liquid now receives a (almost trivial) self-energy correction

$$\delta E = \sum_{\mathbf{p}} \varepsilon(\mathbf{p}) \delta n(\mathbf{p}), \quad \varepsilon(\mathbf{p}) = \varepsilon_0(\mathbf{p}) + \frac{1}{V} \sum_{\mathbf{p}'} f_{\mathbf{p}\mathbf{p}'} \delta n(\mathbf{p}'). \quad (11)$$

The chemical potential can be immediately decided:

$$\mu := \left. \frac{\delta E}{\delta n(\mathbf{p})} \right|_{\text{Fermi surface}} = \varepsilon(p_F). \quad (12)$$

For condensed matter systems, we have to make a further modification to the energy functional shown above: we need to introduce spins. So the full theory is

$$\delta E = \sum_{\mathbf{p}, \sigma} \varepsilon_{0\sigma}(\mathbf{p}) \delta n_{\sigma}(\mathbf{p}) + \frac{1}{2V} \sum_{\mathbf{p}, \mathbf{p}', \sigma, \sigma'} f_{\mathbf{p}\mathbf{p}'\sigma\sigma'} \delta n_{\sigma}(\mathbf{p}) \delta n_{\sigma'}(\mathbf{p}'). \quad (13)$$

When there is no outside spin polarizing factors like a magnetic field, we just have

$$f_{\uparrow\uparrow} = f_{\downarrow\downarrow}, \quad f_{\uparrow\downarrow} = f_{\downarrow\uparrow}, \quad \varepsilon_{0\uparrow} = \varepsilon_{0\downarrow} = \varepsilon_0, \quad (14)$$

and therefore we have

$$\begin{aligned} & \sum_{\mathbf{p}, \mathbf{p}', \sigma, \sigma'} f_{\mathbf{p}\mathbf{p}'\sigma\sigma'} \delta n_{\sigma}(\mathbf{p}) \delta n_{\sigma'}(\mathbf{p}') \\ &= \sum_{\mathbf{p}, \mathbf{p}'} \frac{1}{2} (f_{\uparrow\uparrow} + f_{\downarrow\downarrow}) \delta n(\mathbf{p}) \delta n(\mathbf{p}') + \sum_{\mathbf{p}, \mathbf{p}'} \frac{1}{2} (f_{\uparrow\uparrow} - f_{\downarrow\downarrow}) \delta \tilde{n}(\mathbf{p}) \delta \tilde{n}(\mathbf{p}'), \end{aligned}$$

where

$$\delta n(\mathbf{p}) = \delta n_{\uparrow}(\mathbf{p}) + \delta n_{\downarrow}(\mathbf{p}), \quad \delta \tilde{n}(\mathbf{p}) = \delta n_{\uparrow}(\mathbf{p}) - \delta n_{\downarrow}(\mathbf{p}). \quad (15)$$

We define

$$2f^S = f_{\uparrow\uparrow} + f_{\downarrow\downarrow}, \quad 2f^A = f_{\uparrow\uparrow} - f_{\downarrow\downarrow}, \quad (16)$$

and the interaction part of the energy functional then is

$$\sum_{\mathbf{p}, \mathbf{p}'} (f_{\mathbf{p}\mathbf{p}'}^S \delta n(\mathbf{p}) \delta n(\mathbf{p}') + f_{\mathbf{p}\mathbf{p}'}^A \delta \tilde{n}(\mathbf{p}) \delta \tilde{n}(\mathbf{p}')). \quad (17)$$

Now we can make some symmetric arguments. When we calculate physical quantities that only involves δn , like the specific heat capacity, there is no need to deal with $\delta \tilde{n}$; when calculating the spin response the case is the opposite. We can do the angular momentum expansion

$$f_{\mathbf{p}\mathbf{p}'}^{S/A} = \sum_{l=0}^{\infty} f_l^{S/A} P_l(\cos \theta), \quad (18)$$

TODO: only the angle θ between \mathbf{p} and \mathbf{p}' matters. When we talk about an isotropic excitation mode, only the $l = 0$ term matters, while if we impose a magnetic field along one axis, then usually the $l = 1$ term has the strongest contribution.

The expansion (18) also implies one way Landau Fermi liquid theory breaks: if high l terms have very strong contribution to δE , then even when we still have well-defined $\delta n(\mathbf{p})$, Landau Fermi liquid theory will be of limited use. Fortunately this is almost never the case.

We usually define dimensionless Landau parameters

$$F^{S/A} = N(\omega = 0) f^{S/A}, \quad (19)$$

where $N(\omega = 0)$ is the density of states at E_F .

3 Spinless phenomena

3.1 The specific heat capacity

Following the procedure in electron gas, we define

$$v_F = \nabla_{\mathbf{p}} \varepsilon(\mathbf{p})|_{p_F}, \quad (20)$$

and

$$m^* = \frac{p_F}{v_F}. \quad (21)$$

The density of states therefore is given by

$$N(\omega = 0) = m^* \frac{p_F}{\pi^2 \hbar^3}. \quad (22)$$

This is of course related to the specific heat; let's derive the latter. Suppose we put a Fermi liquid under a finite temperature T . The specific heat capacity can be found by

$$C_V = T \left. \frac{\partial S}{\partial T} \right|_V. \quad (23)$$

The entropy is

$$S = -k_B \ln \Omega = -k_B \frac{1}{V} \sum_{\mathbf{p}, \sigma} (n_\sigma(\mathbf{p}) \ln n_\sigma(\mathbf{p}) + (1 - n_\sigma(\mathbf{p})) \ln(1 - n_\sigma(\mathbf{p}))), \quad (24)$$

where now due to thermal fluctuation, we have TODO: ε has δn dependence, so what will happen in the Fermi-Dirac distribution? A tentative solution: by summing over $n_\sigma(\mathbf{p}')$ ($\mathbf{p} \neq \mathbf{p}'$) first, we have

$$n_\sigma(\mathbf{p}) = \frac{\langle e^{-(\varepsilon_\sigma(\mathbf{p}) - \mu)/k_B T} \rangle}{1 + \langle e^{-(\varepsilon_\sigma(\mathbf{p}) - \mu)/k_B T} \rangle},$$

where the expectation brackets mean to average over non- \mathbf{p} degrees of freedom, and since for fermions $n^2 = n$, we can move the expectation brackets into the exponent, and thus we have the self-consistent Fermi-Dirac distribution

$$n_\sigma(\mathbf{p}) = \frac{1}{1 + e^{(\varepsilon_\sigma(\mathbf{p}) - \mu)/k_B T}}, \quad (25)$$

where $\varepsilon_\sigma(\mathbf{p})$ is decided by (11), where the $\delta n_\sigma(\mathbf{p})$ in (11) is just in the Fermi-Dirac distribution. This can also be confirmed by thinking about the Hartree term in imaginary time field theory: $n_\sigma(\mathbf{p})$ can be obtained from the single-electron Green function in the same way in the free electron gas, and the loop electron line in the Hartree diagram gives exactly (the temperature-dependent) $n_\sigma(\mathbf{p})$.

$$\delta S = -\frac{1}{TV} \sum_{\mathbf{p}, \sigma} (\varepsilon_\sigma(\mathbf{p}) - \mu) \delta n_\sigma(\mathbf{p}) = -\frac{1}{TV} \sum_{\mathbf{p}, \sigma} (\varepsilon_\sigma(\mathbf{p}) - \mu) \left(\frac{\partial n_\sigma(\mathbf{p})}{\partial \varepsilon_\sigma(\mathbf{p})} \delta \varepsilon_\sigma(\mathbf{p}) + \frac{\partial n_\sigma(\mathbf{p})}{\partial T} \delta T \right). \quad (26)$$

TODO: eventually we find

$$\delta S = -\frac{1}{TV} \int d\varepsilon N(\omega) (\varepsilon - \mu) \frac{\varepsilon - \mu}{T} dT, \quad (27)$$

and

$$C_V = \frac{\pi^3}{3} N(\omega = 0) k_B^2 T. \quad (28)$$

Box 1: What is really $n_\sigma(\mathbf{p})$?

When I first learned Fermi liquid theory, what puzzled me the most is what exactly is $n_\sigma(\mathbf{p})$: is it a quantum operator (and thus a label of the many-body wave function), or is it the single-electron Fermi-Dirac distribution (which is stored in the single-electron Matsubara Green function)? From the above discussion, we can find *both* views are correct. This is a trivial instance of the fact that Feynman diagram components, like the single-electron self-energy or the double-electron kernel, do have counterparts in the wave-function-plus-operator language (often referred to as second quantization, although technically second quantization is used everywhere, including Feynman diagram techniques), to which we can do everything we do for ordinary quantum operators.

The next question is what's m^* . There is actually a constraint imposed by the translational symmetry. Suppose we add a very small velocity \mathbf{v} globally. This means

$$\mathbf{p} \rightarrow \mathbf{p}' = \mathbf{p} - m\mathbf{v}, \quad E \rightarrow E' = E - \mathbf{p} \cdot \mathbf{v} + \frac{1}{2} m v^2. \quad (29)$$

Taylor expansion tells us

$$\varepsilon(\mathbf{p} - m\mathbf{v}) = \varepsilon(\mathbf{p}) - \left(\frac{m}{m^*} - 1 \right) \mathbf{p} \cdot \mathbf{v}, \quad (30)$$

and this has to agree with the TODO:

$$\frac{m^*}{m} = 1 + \frac{F_1^S}{3}. \quad (31)$$

This means the effective mass only carries information about the symmetric part of $f_{\mathbf{p}\mathbf{p}'\sigma\sigma'}$.

3.2 Electronic compressibility

The **electron compressibility** is defined as

$$\kappa = \frac{1}{n^2} \frac{\partial n}{\partial \mu}, \quad (32)$$

which is an analogy of the usual compressibility $-\frac{1}{V} \frac{\partial V}{\partial p}$ related to the pressure. It's about how easy it is to add an electron into the system by changing the chemical potential.

We have (note that here δ is not the δ -function)

$$\begin{aligned} \delta n_\sigma(\mathbf{p}) &= \sum_{\sigma', \mathbf{p}'} \frac{\partial n_\sigma(\mathbf{p})}{\partial \varepsilon_\sigma(\mathbf{p})} \delta(\varepsilon_\sigma(\mathbf{p}) - \mu) \\ &= \frac{\partial n_\sigma(\mathbf{p})}{\partial \varepsilon_\sigma(\mathbf{p})}. \end{aligned}$$

TODO: show this; note that

$$\delta \varepsilon_\sigma(\mathbf{p}) = \frac{1}{V} \sum_{\mathbf{p}, \sigma} f^S \delta n$$

So we find

$$\delta n = -N(\omega = 0)(f^S \delta n - \delta \mu), \quad (33)$$

and

$$\frac{\delta n}{\delta \mu} = \frac{N(\omega = 0)}{1 + N(\omega = 0)f^S}, \quad (34)$$

and therefore

$$\kappa = \frac{1}{n^2} \frac{N(\omega = 0)}{1 + F^S}. \quad (35)$$

The compressibility of a Fermi gas is

$$\kappa_0 = \frac{N(\omega = 0)}{n^2}, \quad (36)$$

and therefore

$$\frac{\kappa}{\kappa_0} = \frac{N(\omega = 0)}{N_0(\omega = 0)} \frac{1}{1 + F_0^S}. \quad (37)$$

The ratio between the interaction corrected and the free densities of states is determined by m^* , which in turn is determined by (31), and the final expression of the compressibility is

$$\frac{\kappa}{\kappa_0} = \left(1 + \frac{1}{3}F_1^S\right) \frac{1}{1 + F_0^S}. \quad (38)$$

In the band theory, a good metal has high electron density and high Fermi velocity. This means the electron compressibility is small: a change of μ only changes p_F slightly, and therefore the change of the occupied states, compared with existing electrons, isn't large. That's equivalent to say a not-so-good metal is very compressible. On the other hand, insulators are not compressible: adding one electron costs the energy gap. After (38) is applied, we find the interaction correction changes the compressibility in a complicated way.

4 Spin response

Now we discuss what happens after the two spins are treated differently. The coupling $-\boldsymbol{\mu} \cdot \mathbf{B}$ causes Zeeman splitting, which is how Pauli paramagnetism comes. What does Fermi liquid theory say about how Coulomb repulsion correct this behavior? Since we are going to study $\delta n_\downarrow - \delta n_\uparrow$, now it's F^A that determines the behavior of the system. The effective one-electron energy is

$$\varepsilon_\sigma(\mathbf{p}) = \varepsilon_0(\mathbf{p}) - \frac{1}{2}\mu_0\mu_B H + \frac{1}{V} \sum_{\mathbf{p}', \sigma'} f_{\mathbf{p}\mathbf{p}'\sigma\sigma'} \delta n_{\sigma'}(\mathbf{p}'). \quad (39)$$

Again we start from

$$\delta n_\sigma(\mathbf{p}) = \frac{\partial n_\sigma(\mathbf{p})}{\partial \varepsilon_\sigma(\mathbf{p})}.$$

Here it can be verified that the linear effects of H don't include $\delta\mu$, so we only need to take the first term into account. We then have

$$\begin{aligned}\delta\tilde{n} &= \frac{1}{2}N(\omega=0) \left(\frac{1}{2}\mu_0\mu_B H - 2f_0^A \delta\tilde{n} \right), \\ \delta\tilde{n} &= \frac{1}{2}N(\omega=0) \frac{1}{1+F_0^A} H.\end{aligned}\tag{40}$$

The total magnetization is therefore

$$M = \mu_0\mu_B^2 \frac{N(\omega=0)}{1+F_0^A} \cdot H,\tag{41}$$

$$\chi = \chi_P \cdot \frac{m^*}{m} \frac{1}{1+F_0^A} = \chi_P \cdot \frac{1+\frac{1}{3}F_1^S}{1+F_0^A},\tag{42}$$

where χ_P is the magnetic susceptibility predicted by Pauli paramagnetism of Fermi gas.

The question, then, is whether we really have non-zero F_0^A . Many mechanisms actually contribute to this, because we have magnetic dipole-dipole interaction or exchange interaction even in materials without magnetic order.

5 Stability of Fermi liquid

Many effects can slightly distort the Fermi surface. If a distortion costs energy, then nothing strange happens. If, however, a distortion *reduces* the energy of the system, then the Fermi surface is not the real ground state of the system. In this case we say it's not stable.

Box 2: ...But will the system go to the state with lower energy?

We may then ask why the system wants to go to a state with lower energy: it can just stay on an eigenstate with a slightly higher energy. Most of systems are sensitive to thermal fluctuation or quantum fluctuation (for example, the electron is always coupled to the electromagnetic field, and therefore we have spontaneous photon emission), so yes, they indeed tend to go to the ground state. But is it possible to have an electronic glass state?

The Fermi surface is given by the following stepwise distribution:

$$n_\sigma(\mathbf{p}) = \theta(p_F^0 - |\mathbf{p}|).\tag{43}$$

After a distortion of the Fermi surface, the distribution function is now

$$n_\sigma(\mathbf{p}) = \theta(p_F(\theta) - |\mathbf{p}|),\tag{44}$$

and therefore

$$\delta n_\sigma(\mathbf{p}) = \delta p_F \delta(p_F - |\mathbf{p}|)\tag{45}$$

TODO

$$\delta E - \mu \delta N = \sum_{l=0}^{\infty} \frac{N(\omega=0)}{8(2l+1)} \left((v_{l\uparrow} + v_{l\downarrow})^2 \left(1 + \frac{F_l^S}{2l+1} \right) + (v_{l\uparrow} - v_{l\downarrow})^2 \left(1 + \frac{F_l^A}{2l+1} \right) \right),\tag{46}$$

and in order for $\delta E - \mu \delta N \geq 0$, we need

$$\frac{F_l^{S/A}}{2l+1} + 1 \geq 0.\tag{47}$$

So we find as long as the Landau parameters are all positive – not surprising because a repulsion interaction should in general gives positive f – the Fermi surface is safe. But if in some l channels, somehow we get $F < 0$, then the Fermi surface tends to reduce its rotational symmetry and collapses into a more complicated shape.

TODO: to check whether it's δn or n in some formulae; where to add absolute value symbol