Homework 2

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1 Polarization of electromagnetic field

1.1 The general form of a pure state

We have (assuming $\hat{\boldsymbol{k}} = \hat{\boldsymbol{z}}$)

$$\boldsymbol{E} = E_x \begin{pmatrix} 1 \\ 0 \end{pmatrix} + E_y \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \sqrt{|E_x|^2 + |E_y|^2} \mathrm{e}^{\mathrm{i}\varphi_x} \begin{pmatrix} \frac{|E_x|}{\sqrt{|E_x|^2+|E_y|^2}} \\ \mathrm{e}^{\mathrm{i}(\varphi_y - \varphi_x)} \frac{|E_y|}{\sqrt{|E_x|^2+|E_y|^2}} \end{pmatrix},$$

and by defining

$$E_0 = \sqrt{|E_x|^2 + |E_y|^2} \mathrm{e}^{\mathrm{i}\varphi_x}, \tag{1}$$

$$\cos\theta = \frac{|E_x|}{\sqrt{|E_x|^2 + |E_y|^2}}, \tag{2}$$

and

$$\phi = \varphi_y - \varphi_x, \tag{3}$$

we find

$$\boldsymbol{E} = E_0 \begin{pmatrix} \cos\theta \\ \mathrm{e}^{\mathrm{i}\phi} \sin\theta \end{pmatrix}. \tag{4}$$

We have

$$|H\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |V\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \tag{5}$$

and therefore after normalization, we have

$$\rho = (\cos\theta \, |H\rangle + \mathrm{e}^{\mathrm{i}\phi} \sin\theta \, |V\rangle)(\cos\theta \, \langle H| + \mathrm{e}^{-\mathrm{i}\phi} \sin\theta \, \langle V|) = \begin{pmatrix} \cos^2\theta & \mathrm{e}^{-\mathrm{i}\phi} \sin\theta\cos\theta \\ \mathrm{e}^{\mathrm{i}\phi} \sin\theta\cos\theta & \sin^2\theta \end{pmatrix}. \tag{6}$$

1.2 The pure state $\rho^2 = \rho$ condition

We can prove the pure state condition $\rho^2 = \rho$ explicitly:

$$\begin{aligned} \rho^2 &= \begin{pmatrix} \cos^4\theta + \sin^2\theta\cos^2\theta & \mathrm{e}^{-\mathrm{i}\phi} \sin\theta\cos^3\theta + \mathrm{e}^{-\mathrm{i}\phi} \sin^3\theta\cos\theta \\ \mathrm{e}^{\mathrm{i}\phi} \sin\theta\cos^3\theta + \mathrm{e}^{\mathrm{i}\phi} \sin^3\theta\cos\theta & \sin^2\theta\cos^2\theta + \sin^4\theta \end{pmatrix} \\ &= \begin{pmatrix} \cos^2\theta & \mathrm{e}^{-\mathrm{i}\phi} \sin\theta\cos\theta \\ \mathrm{e}^{\mathrm{i}\phi} \sin\theta\cos\theta & \sin^2\theta \end{pmatrix} = \rho. \end{aligned} \tag{7}$$

1.3 Mixed state

The condition that ρ is Hermite means it can be written as

$$\rho = R(\sigma^0 + x\sigma^x + y\sigma^y + z\sigma^z),$$

where $R, x, y, z \in \mathbb{R}$, because the σ matrices constitute a basis for all Hermite matrices in $\mathbb{C}^{2\times 2}$. Since $\sigma^{x,y,z}$ are traceless, from the condition $\operatorname{tr}\rho = 1$, we have

$$1 = \operatorname{tr}\rho = R \operatorname{tr}\sigma^0 = 2R \Rightarrow R = \frac{1}{2},$$

so
$$\rho = \frac{1}{2}(\sigma^0 + x\sigma^x + y\sigma^y + z\sigma^z). \tag{8}$$
In the matrix form, we have
$$\rho = \begin{pmatrix} \frac{1+z}{2} & \frac{x-\mathrm{i}y}{2} \\ \frac{x+\mathrm{i}y}{2} & \frac{1-z}{2} \end{pmatrix},$$
and by substitution of variables (this is a three variables to three variables mapping, and therefore is valid)
$$\frac{1}{2}(1-p) + p\cos^2\theta = \frac{1+z}{2}, \quad x = p\cos\phi\sin 2\theta, \quad y = p\sin\phi\sin 2\theta,$$
we get
$$\frac{1-z}{2} = \frac{1}{2}(1-p) + p\sin^2\theta,$$
and therefore
$$\rho = (1-p)\begin{pmatrix} \frac{1}{2} & \\ & \frac{1}{2} \end{pmatrix} + p\begin{pmatrix} \cos^2\theta & \mathrm{e}^{-\mathrm{i}\phi}\sin\theta\cos\theta \\ \mathrm{e}^{\mathrm{i}\phi}\sin\theta\cos\theta & \sin^2\theta \end{pmatrix}. \tag{9}$$

1.4 Jones parameters and Stokes formalism

The definition of Stokes parameters are
$$\begin{aligned} I &= \langle E_x^2 \rangle + \langle E_y^2 \rangle, \\ Q &= \langle E_x^2 \rangle - \langle E_y^2 \rangle, \\ U &= \langle E_a^2 \rangle - \langle E_b^2 \rangle, \\ V &= \langle E_l^2 \rangle - \langle E_r^2 \rangle, \end{aligned} \tag{10}$$
where
$$\hat{\boldsymbol{a}} = \frac{1}{\sqrt{2}}(\hat{\boldsymbol{x}} + \hat{\boldsymbol{y}}), \quad \hat{\boldsymbol{b}} = \frac{1}{\sqrt{2}}(\hat{\boldsymbol{x}} - \hat{\boldsymbol{y}}), \tag{11}$$
and
$$\hat{\boldsymbol{l}} = \frac{1}{\sqrt{2}}(\hat{\boldsymbol{x}} + \mathrm{i}\hat{\boldsymbol{y}}), \quad \hat{\boldsymbol{r}} = \frac{1}{\sqrt{2}}(\hat{\boldsymbol{x}} - \mathrm{i}\hat{\boldsymbol{y}}). \tag{12}$$
Note that $E_{x,y}$ etc. above may be regarded as operators, and we have
$$\begin{aligned} E_x^2 + E_y^2 &= \sigma^0, \\ E_x^2 - E_y^2 &= \sigma^z, \\ E_a^2 - E_b^2 &= \sigma^x, \\ E_l^2 - E_r^2 &= \sigma^y. \end{aligned} \tag{13}$$
From these definitions and the fact that $(\sigma^i)^2 = \sigma^0$ and all other products of σ matrices are traceless, we find
$$I = \langle \sigma^0 \rangle = 1, \tag{14}$$
$$Q = \langle \sigma^z \rangle = \frac{1}{2}z \cdot 2 = 2p\cos^2\theta - p = p\cos 2\theta, \tag{15}$$
$$U = \frac{1}{2}x \cdot 2 = p\cos\phi\sin 2\theta, \tag{16}$$
and
$$V = \frac{1}{2}y \cdot 2 = p\sin\phi\sin 2\theta. \tag{17}$$
Here I is constantly 1, because we are working with the single-photon density matrix.

1.5 Mueller calculus

As is shown above, Mueller calculus actually works on the coefficients in (8), and therefore a Mueller matrix essentially gives the coefficients of $\rho \to U\rho U^\dagger$. It makes sense as long as after its application, the σ^0 component in ρ is still $1/2$.

1.6 Transformation and measurement

We have

$$
\begin{aligned}
|45\rangle &= \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \\
|\mathrm{rcp}\rangle &= \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -\mathrm{i} \end{pmatrix}
\end{aligned}. \tag{18}
$$

The correspond density matrices are

$$
\rho_{45} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \tag{19}
$$

and

$$
\rho_{\mathrm{rcp}} = \frac{1}{2}\begin{pmatrix} 1 & \mathrm{i} \\ -\mathrm{i} & 1 \end{pmatrix}. \tag{20}
$$

The operator

$$
U_{\mathrm{rcp}} = \begin{pmatrix} 1 & \\ & -\mathrm{i} \end{pmatrix} \tag{21}
$$

then turns ρ_{45} to ρ_{rcp}:

$$
U_{\mathrm{rcp}}\rho_{45}U_{\mathrm{rcp}}^\dagger = \frac{1}{2}\begin{pmatrix} 1 & \\ & -\mathrm{i} \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & \\ & \mathrm{i} \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & \mathrm{i} \\ -\mathrm{i} & 1 \end{pmatrix} = \rho_{\mathrm{rcp}}. \tag{22}
$$

The horizontal polarizer operator

$$
\mathcal{O} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \tag{23}
$$

is not unitary, because it has non-unitary eigenvalue 0. It is a projection operator: it takes in a beam polarized light and returns its x component. It also represents a measurement: we can use it in a projective measurement setting. In a projective measurement with operator $\mathcal{O}$, $\mathrm{tr}(\rho\mathcal{O})$ is the probability that after measurement, the final state of the system falls into the subspace determined by $\mathcal{O}$. In our case, the subspace determined by $\mathcal{O}$ is the subspace of horizontal polarization, so $\mathrm{tr}(\rho\mathcal{O})$ is the probability that after measurement, we find ρ to be a horizontally polarized state.

Application of $\mathcal{O}$ on (9) is

$$
\mathcal{O}\rho\mathcal{O}^\dagger = (1-p)\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 0 \end{pmatrix} + p\begin{pmatrix} \cos^2\theta & 0 \\ 0 & 0 \end{pmatrix} = \left(\frac{1}{2} + \frac{p}{2}\cos 2\theta\right)\mathcal{O}. \tag{24}
$$

So after the application of $\mathcal{O}$, we get a horizontally polarized state, as is expected. The result is not normalized; the factor before $\mathcal{O}$ is just $\mathrm{tr}(\rho\mathcal{O})$, which is the probability that after measurement, we find ρ to be horizontally polarized. When $p = 0$, it's $1/2$, which is expected for the unpolarized state; when $p = 1$, it's $\cos^2\theta$, again the correct answer.

2 The $\rho^2 = \rho$ condition for pure states

Suppose

$$
\rho = |\psi\rangle\langle\psi|, \quad |\psi\rangle = \sum_m a_m |m\rangle. \tag{25}
$$

We have

$$
\begin{aligned}
\rho^2 &= \sum_{m,n} a_m^* a_n \left|n\right\rangle \left\langle m\right| \sum_{j,k} a_j^* a_k \left|k\right\rangle \left\langle j\right| \\
&= \sum_{m,n,j,k} a_m^* a_n a_j^* \left|n\right\rangle \left\langle m|k\right\rangle \left\langle j\right| \\
&= \sum_{m,n,j,k} a_m^* a_n a_j^* a_k \left|n\right\rangle\!\left\langle j\right| \delta_{mk} \\
&= \underbrace{\sum_m a_m^* a_m}_{=\langle\psi|\psi\rangle=1} \sum_{n,j} a_n a_j^* \left|n\right\rangle\!\left\langle j\right| \\
&= \sum_{n,j} a_n a_j^* \left|n\right\rangle\!\left\langle j\right| = \rho.
\end{aligned}
\tag{26}
$$

3 Ammonia molecule

The Hamiltonian of the two low-energy states of ammonia is

$$
H = \begin{pmatrix} 0 & \Delta/2 \\ \Delta/2 & 0 \end{pmatrix}, \tag{27}
$$

where we set

$$
\left|L\right\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \left|R\right\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{28}
$$

This Hamiltonian is just a scaled σ^x matrix, and its eigenstates are straightforwardly given by

$$
\left|+\right\rangle = \frac{1}{\sqrt{2}}(\left|L\right\rangle + \left|R\right\rangle), \quad \left|-\right\rangle = \frac{1}{\sqrt{2}}(\left|L\right\rangle - \left|R\right\rangle), \tag{29}
$$

and the energies are

$$
E_+ = \Delta/2, \quad E_- = -\Delta/2. \tag{30}
$$

After an electric field is added, in $\left|L\right\rangle$ we have an additional energy contribution, and since the molecular configuration in $\left|R\right\rangle$ is the opposite of the one in $\left|L\right\rangle$, we have

$$
H = \begin{pmatrix} dE/2 & \Delta/2 \\ \Delta/2 & -dE/2 \end{pmatrix}. \tag{31}
$$

Solving

$$
\det \begin{pmatrix} dE/2 - \lambda & \Delta/2 \\ \Delta/2 & -dE/2 - \lambda \end{pmatrix} = 0,
$$

we get

$$
E_\pm = \pm \frac{1}{2}\sqrt{\Delta^2 + d^2E^2}, \tag{32}
$$

and hence

$$
\left|-\right\rangle = \frac{dE - \sqrt{d^2E^2 + \Delta^2}}{\sqrt{\Delta^2 + (\sqrt{d^2E^2 + \Delta^2} - dE)^2}} \left|L\right\rangle + \frac{\Delta}{\sqrt{\Delta^2 + (\sqrt{d^2E^2 + \Delta^2} - dE)^2}} \left|R\right\rangle, \tag{33}
$$

and

$$
\left|+\right\rangle = \frac{dE + \sqrt{d^2E^2 + \Delta^2}}{\sqrt{\Delta^2 + (\sqrt{d^2E^2 + \Delta^2} + dE)^2}} \left|L\right\rangle + \frac{\Delta}{\sqrt{\Delta^2 + (\sqrt{d^2E^2 + \Delta^2} + dE)^2}} \left|R\right\rangle. \tag{34}
$$

When E is large, we have

$$
\begin{aligned}
dE - \sqrt{d^2E^2 + \Delta^2} &\approx 0, \\
dE + \sqrt{d^2E^2 + \Delta^2} &\approx 2dE,
\end{aligned}
$$

and therefore

$$
E_+ = \frac{1}{2}dE, \quad \left|+\right\rangle = \left|L\right\rangle, \tag{35}
$$

and

$$E_- = -\frac{1}{2}dE, \quad |-\rangle = |R\rangle\,. \tag{36}$$

This is expected, because when $Ed \gg \Delta$, the non-diagonal terms in the Hamiltonian can be safely ignored.

4 Kaptiza's pendulum

4.1 Integrating out the fast variable

In the $\omega \to \infty$, $F_0 \to 0$ limit, the high-frequency part and the low-frequency part of the solution of

$$mR\ddot{\theta} = (-mg + F_0 \sin\omega t)\sin\theta \tag{37}$$

are not strongly coupled and the high-frequency degree of freedom can be integrated out to get an effective theory of the low-frequency part. We do the decomposition

$$\theta = \theta_f + \theta_s, \tag{38}$$

where θ_f is the fast variable. Observing (37), we find the EOM of θ_f should be

$$mR\ddot{\theta}_f = F_0 \sin\omega t \sin(\theta_f + \theta_s), \tag{39}$$

because the first term on the RHS of (37) has a much lower frequency magnitude compared with ω. We take the first order approximation of (39) and ignore the θ_f dependency on the RHS, and this gives

$$\theta_f = -\frac{F_0}{mR\omega^2}\sin\theta_s \sin\omega t. \tag{40}$$

Putting this back to (37), we get

$$\begin{aligned} mR\left(\frac{F_0}{mR}\sin\theta_s\sin\omega t + \ddot{\theta}_s\right) &= (-mg + F_0\sin\omega t)\sin\left(\theta_s - \frac{F_0}{mR\omega^2}\sin\theta_s\sin\omega t\right) \\ &= (-mg + F_0\sin\omega t)\left(\sin\theta_s - \cos\theta_s \cdot \frac{F_0}{mR\omega^2}\sin\theta_s\sin\omega t\right). \end{aligned}$$

Now we average over all high-frequency time dependencies. The first term on the LHS averages zero, and so do the $-mg\sin\omega t$ term and the $F_0 \sin\omega t \sin\theta_s$ term on the RHS. On the other hand, the $\sin^2\omega t$ term on the RHS averages

$$-\frac{F_0^2}{mR\omega^2}\sin\theta_s\cos\theta_s\left\langle\sin^2\omega t\right\rangle = -\frac{1}{2}\frac{F_0^2}{mR\omega^2}\sin\theta_s\cos\theta_s,$$

so the final EOM for θ_s is

$$mR\ddot{\theta}_s = -mg\sin\theta_s - \frac{1}{2}\frac{F_0^2}{mR\omega^2}\sin\theta_s\cos\theta_s. \tag{41}$$

4.2 Stable positions of θ_s

We let the LHS of (41) be zero, and the equation becomes

$$\sin\theta_s\left(mg + \frac{1}{2}\frac{F_0^2}{mR\omega^2}\cos\theta_s\right) = 0.$$

Since $F_0 \to 0$, the second factor on the LHS can't be zero, so the equation becomes

$$\sin\theta_s = 0 \Rightarrow \theta_s = 0, \pi. \tag{42}$$

Around $\theta_s = 0$, (41) is approximately

$$mR\ddot{\theta}_s = -mg\theta_s - \frac{1}{2}\frac{F_0^2}{mR\omega^2}\theta_s,$$

and therefore

$$\omega_{\theta=0} = \sqrt{\frac{g}{R} + \frac{F_0^2}{m^2R^2\omega^2}}. \tag{43}$$

This is always real, and therefore the $\theta_s = 0$ position is always stable.

Around $\theta_s = \pi$, we rewrite (41) in terms of $\theta_s' = \pi - \theta_s$, and get

$$-mR\ddot{\theta}_s' = mg\theta_s' - \frac{1}{2}\frac{F_0^2}{mR\omega^2}\theta_s' = 0,$$

and

$$\omega_{\theta_s=\pi} = \sqrt{\frac{1}{2}\frac{F_0^2}{m^2R^2\omega^2} - \frac{g}{R}}. \tag{44}$$

It can be seen that when $F_0 = 0$, the frequency is imaginary and therefore $\theta_s = \pi$ is not a stable position. However, when

$$\frac{1}{2}\frac{F_0^2}{m^2R^2\omega^2} \geq \frac{g}{R}, \tag{45}$$

we do have oscillation behavior around $\theta_s = \pi$.

5 Relaxation of a spin polarization due to an electric field

5.1 The spin-magnetic field coupling

We have

$$H = -\boldsymbol{\mu} \cdot \boldsymbol{B}, \quad \boldsymbol{\mu} = -g\mu_{\mathrm{B}}\boldsymbol{S}, \tag{46}$$

and this means

$$H = \frac{1}{2}g\mu_{\mathrm{B}}\boldsymbol{\sigma} \cdot \boldsymbol{B}. \tag{47}$$

This gives

$$\frac{\mathrm{d}\boldsymbol{S}}{\mathrm{d}t} = \frac{g\mu_{\mathrm{B}}}{\hbar}\boldsymbol{B} \times \boldsymbol{S}. \tag{48}$$

When $\boldsymbol{B}$ is fixed (instead of the motion magnetic field in (55)), we find the oscillation frequency is

$$\omega = \frac{g\mu_{\mathrm{B}}}{\hbar}B, \tag{49}$$

and therefore

$$\gamma = \frac{g\mu_{\mathrm{B}}}{\hbar}. \tag{50}$$

5.2 The $\sigma_\pm$ representation

We define

$$\sigma_\pm = \frac{\sigma_x \pm \mathrm{i}\sigma_y}{2}, \tag{51}$$

and the commutation relations are

$$[\sigma_+, \sigma_-] = \sigma_z, \quad [\sigma^z, \sigma_\pm] = 2\sigma_\pm. \tag{52}$$

The Hamiltonian is then

$$H = b^*\sigma_+ + b\sigma_- + b_z\sigma_z, \tag{53}$$

where

$$b = \frac{1}{2}g\mu_{\mathrm{B}}(B_x + \mathrm{i}B_y), \quad b_z = \frac{1}{2}g\mu_{\mathrm{B}}B_z. \tag{54}$$

5.3 Eliminating the static magnetic field

The magnetic field felt by an electron wit velocity $\boldsymbol{v}$ when an electric field $\boldsymbol{E}$ is present is

$$\boldsymbol{B} = -\boldsymbol{v} \times \boldsymbol{E}/c^2. \tag{55}$$

We also have a static magnetic field point towards $\hat{\boldsymbol{z}}$.

5.4 Time evolution

Now we use

$$\frac{\partial\rho(t)}{\partial t} = -\frac{1}{\hbar^2}\int_0^t \mathrm{d}t' \left[H(t), \left[H\left(t'\right), \rho(t)\right]\right] = -\frac{1}{\hbar^2}\int_0^t \mathrm{d}t' \left[H(t), \left[H(t-t'), \rho(t)\right]\right] \tag{56}$$

to find the time evolution of ρ, considering only second-order correlation in the random variable in H. We again do decomposition

$$\rho = \frac{1}{2}\sigma_0 + \rho_+\sigma_+ + \rho_-\sigma_- + \rho_z\sigma_z, \tag{57}$$

and