Chern-Simons Theory for Topological States of Matter

Jinyuan Wu

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1 Chern-Simons theory

A type of well-studied *topological quantum field theory (TQFT)* – which has no non-trivial particle dynamics – is **Chern-Simons theories**. We consider, for example, the following theory where Schrödinger bosons are coupled to a single gauge field, and there is a Chern-Simons term for the gauge field itself:

$$\mathcal{L} = \mathrm{i}\bar{\phi}D_0\phi + \frac{1}{2m}\bar{\phi}\boldsymbol{D}^2\phi + \underbrace{\frac{1}{4\theta}\epsilon^{\mu\nu\lambda}a_\mu\partial_\nu a_\lambda}_{\mathcal{L}_{\text{CS}}}, \tag{1}$$

where

$$D_\mu = \partial_\mu + \mathrm{i}a_\mu. \tag{2}$$

We have not proved that (1) is indeed gauge invariant yet. Under a local transformation

$$a_\mu \longrightarrow a_\mu + \partial_\mu h,$$

we have

$$\begin{aligned}\frac{1}{4\theta}\epsilon^{\mu\nu\lambda}a_\mu\partial_\nu a_\lambda \longrightarrow & \frac{1}{4\theta}\epsilon^{\mu\nu\lambda}(a_\mu + \partial_\mu h)\partial_\nu(a_\lambda + \partial_\lambda h) \\ &= \frac{1}{4\theta}(\epsilon^{\mu\nu\lambda}a_\mu\partial_\nu a_\lambda + \epsilon^{\mu\nu\lambda}\partial_\mu h\partial_\nu a_\lambda + \epsilon^{\mu\nu\lambda}a_\mu\partial_\nu\partial_\lambda h + \epsilon^{\mu\nu\lambda}\partial_\mu h\partial_\nu\partial_\lambda h) \\ &= \frac{1}{4\theta}(\epsilon^{\mu\nu\lambda}a_\mu\partial_\nu a_\lambda + \epsilon^{\mu\nu\lambda}\partial_\mu(h\partial_\nu a_\lambda) - \epsilon^{\mu\nu\lambda}h\partial_\nu\partial_\mu a_\lambda) \\ &= \frac{1}{4\theta}(\epsilon^{\mu\nu\lambda}a_\mu\partial_\nu a_\lambda + \epsilon^{\mu\nu\lambda}\partial_\mu(h\partial_\nu a_\lambda)),\end{aligned}$$

where the last two terms in the second line vanish because $\epsilon^{\mu\nu\lambda}a_\mu a_\nu = 0$, and this also how we get the last line. Note that though after the transformation the Lagrangian *density* changes, after integrating over the whole space the total Lagrangian does *not* change because the second term in the last line above is a boundary term. So we find that in (1), the Chern-Simons part is indeed gauge invariant, though it cannot be written in terms of $\boldsymbol{E}$ and $\boldsymbol{B}$. The coupling term in (1) transforms in the same way of QED, where the bosonic field undergoes a local $U(1)$ transformation. Thus, (1) is a $U(1)$ *Chern-Simons theory*.

Now we evaluate the EOM of (1). The Chern-Simons terms gives

$$\begin{aligned}\partial_\rho\frac{\partial\mathcal{L}}{\partial\partial_\rho a_\sigma} - \frac{\partial\mathcal{L}}{\partial a_\sigma} &= \frac{1}{4\theta}\partial_\rho\epsilon^{\mu\rho\sigma}a_\mu - \frac{1}{4\theta}\epsilon^{\sigma\nu\lambda}\partial_\nu a_\lambda \\ &= -\frac{1}{2\theta}\epsilon^{\sigma\nu\lambda}\partial_\nu a_\lambda.\end{aligned}$$

Another way to find the above equation is to use integration by parts to rephrase all terms containing $a_\mu\partial_\rho a_\sigma$ into $-\partial_\rho a_\mu a_\sigma$, so the first term in the LHS of the first line vanishes, and we get a 2 factor. In the Schrödinger field part, terms involving a_μ are

$$\begin{aligned}&-\bar{\phi}a_0\phi + \frac{\mathrm{i}}{2m}\bar{\phi}\partial_i(a_i\phi) + \frac{\mathrm{i}}{2m}\bar{\phi}a_i\partial_i\phi - \frac{1}{2m}a_i^2\bar{\phi}\phi \\ &= -\bar{\phi}a_0\phi - \frac{\mathrm{i}}{2m}(\partial_i\bar{\phi})a_i\phi + \frac{\mathrm{i}}{2m}\bar{\phi}a_i\partial_i\phi - \frac{1}{2m}a_i^2\bar{\phi}\phi,\end{aligned}$$

and we have

$$\partial_\rho\frac{\partial\mathcal{L}}{\partial\partial_\rho a_0} - \frac{\partial\mathcal{L}}{\partial a_0} = \bar{\phi}\phi,$$

and

$$\partial_\rho \frac{\partial \mathcal{L}}{\partial \partial_\rho a_i} - \frac{\partial \mathcal{L}}{\partial a_i} = \frac{a_i}{m}\bar{\phi}\phi + \frac{\mathrm{i}}{2m}(\phi \partial_i \bar{\phi} - \bar{\phi}\partial_i \phi).$$

We soon find that the RHSs of the above two equations are just conservation flows of a Schrödinger field coupled to a gauge field, as is the case in QED. So the EOM of a_μ is just

$$j^\sigma + \frac{1}{2\theta}\epsilon^{\sigma\nu\lambda}\partial_\nu a_\lambda = 0. \tag{3}$$

Specifically, the temporal part of the EOM is

$$\epsilon^{ij}\partial_i a_j = -2\theta\rho, \quad \rho = \bar{\phi}\phi. \tag{4}$$

Actually, (3) can also be derived by simply adding a

$$S_{\text{coupling}} = \int \mathrm{d}^3 x \, j^\mu a_\mu \tag{5}$$

term into the action.

Note

Note that we are working in a 2+1 dimensional Minkowski spacetime, and $a^i = -a_i$. Opposite signs in (3) can be seen in different sources, which comes from the fact that when deriving (3), these sources switch to the Euclidean space, where

$$a_i = a^i = a^i_{\text{Minkowski}} = -a_{i,\text{Minkowski}}.$$

Often, we introduce a parameter K such that

$$\theta = \frac{\pi}{K}. \tag{6}$$

Also, the field a_μ has its own electric field and magnetic field, which are (note that we are still in the Minkowski spacetime and $a_i = -a^i$, so the signs may seem different with those in the ordinary vector calculus forms)

$$e^i = -\partial_0 a^i + \partial^i a^0, \quad b = -\epsilon^{ij}\partial_i a_j. \tag{7}$$

Therefore, we have

$$j^0 = \rho = \frac{K}{2\pi} b. \tag{8}$$

David Tong QHE Eq. (5.9)

Since we have

$$\begin{aligned}
\epsilon^{i\nu\lambda}\partial_\nu a_\lambda &= \epsilon^{i0\lambda}\partial_0 a_\lambda + \epsilon^{ij\lambda}\partial_j a_\lambda \\
&= \epsilon^{i0j}\partial_0 a_j + \epsilon^{ij0}\partial_j a_0 \\
&= -\epsilon^{ij}\partial_0 a_j + \epsilon^{ij}\partial_j a_0 = \epsilon^{ij} e_j,
\end{aligned}$$

we have

$$j^i = -\frac{K}{2\pi}\epsilon^{ij} e_j = \frac{K}{2\pi}\epsilon^{ij} e^j. \tag{9}$$

David Tong QHE Eq. (5.8)

2 Chern-Simons theory as an effective theory of IQHE

Now we follow the derivation of Section 4.4 in Wen's famous textbook and Section 5.1 in David Tong's lecture note on QHE. Since the bulk state of a IQHE system is an insulator, the only low energy degree of freedom is the electromagnetic field, and by integrating out the gapped bands, the effective theory of the electromagnetic field gets some correction, and the final effective theory can be any (2+1)-dimensional theory that has $U(1)$ gauge symmetry, translational symmetry, rotational symmetry, etc. The Chern-Simons term satisfies these conditions and therefore can appear in the effective theory of the electromagnetic field. We are confident that the Chern-Simons theory is indeed the effective theory of a IQHE system, because it can produce

Wen Section 4.4, David Tong QHE Section 5.1

a transverse conductivity: since a_μ in (9) is the real electromagnetic field and j^μ the real current of electrons, we can rewrite (9) as

$$J^i = \frac{K}{2\pi}\epsilon^{ij}E^j, \tag{10}$$

where we use capitalized letters to denote quantities in the standard electrodynamics, and thus the Hall conductivity is

$$\sigma_{xy} = \frac{K}{2\pi}. \tag{11}$$

David Tong QHE Eq. (5.7)

Now if we are able to show that K can only take discrete values, than we can explain IQHE in a quite neat way without touching the dirty microscopic details.

3 Chern-Simons theory as an effective theory of FQHE

This section reviews what is covered in [1]. Briefly speaking, our final goal is to encode the phases of anyons into a gauge theory. We expect a gauge theory will do the job because we can always mimic the exchange phases of anyons by considering the anyons as particles created by a bosonic field coupled to a gauge field. The particles created by the bosonic field carry both a flux and a charge, and thus the Aharonov–Bohm effect can reproduce the correct self and mutual phases. What we are going to talk about in this section are Abelian anyons, so things will not be too hard. We claim after the exchange phases are correctly encoded into a gauge theory, we can obtained a *complete* low energy effective theory of a topological order, since we are usually not interested in the scattering of anyons, and the low energy dynamics of an anyon system has nothing more than the exchange phases.

Why we use a gauge field theory instead of something like a Coulomb interaction is discussed in Section 3.1 in [1]. In summary, it is because with a Coulomb-like Hamiltonian we often run into duplicate phase factors. Another approach is to construct anyon field operators. It is possible, but the resulting theories are usually highly complicated and technical.

Since generally we are only interested in the exchange phases of anyons in a topological order, we just narrow down the candidates of gauge field theoretic effective descriptions of topological orders to TQFTs, and we will find that the Chern-Simons theory introduced in Section 1 is just the effective theory for FQHE.

References

[1] Susanne F. Viefers. Field theory of anyons and the fractional quantum hall effect. https://inis.iaea.org/collection/NCLCollectionStore/_Public/30/061/30061237.pdf, 1997.