Homework 1

Jinyuan Wu

January 28, 2023

1

Problem

$$1 + e^{y/x} - \frac{y}{x} e^{y/x} + e^{y/x} y' = 0, \quad y(1) = -5. \tag{1}$$

Solution The equation is equivalent to

$$\mathrm{e}^{y/x} \,\mathrm{d}y + \left(1 + \mathrm{e}^{y/x} - \frac{y}{x} \mathrm{e}^{y/x} \right) \mathrm{d}x = 0,$$

and we have

$$\frac{\partial}{\partial x} \mathrm{e}^{y/x} = \frac{\partial}{\partial y} \left(1 + \mathrm{e}^{y/x} - \frac{y}{x} \mathrm{e}^{y/x} \right) = -\frac{y}{x^2} \mathrm{e}^{y/x},$$

so the equation is exact. Suppose the solution is $\phi(x, y) = C$, and we have

$$\phi(x, y) = \int \mathrm{d}y \, \mathrm{e}^{y/x} = x\mathrm{e}^{y/x} + f(x),$$

and therefore

$$1 + \mathrm{e}^{y/x} - \frac{y}{x} \mathrm{e}^{y/x} = \frac{\partial \phi}{\partial x} = \mathrm{e}^{y/x} + x \cdot \frac{-y}{x^2} \mathrm{e}^{y/x} + \frac{\partial f}{\partial x} \Rightarrow f(x) = x + \text{const}.$$

So the solution is

$$\phi(x, y) = x\mathrm{e}^{y/x} + x = C. \tag{2}$$

When $x = 1$, $y = -5$, so we find $C = 1 + \mathrm{e}^{-5}$, and the final solution is

$$y = x \ln \left(\frac{1 + \mathrm{e}^{-5}}{x} - 1 \right). \tag{3}$$

2

Problem

$$y' = \frac{1}{2x} y^2 - \frac{1}{x} y - \frac{4}{x}. \tag{4}$$

Solution We have

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y^2 - 2y - 8}{2x},$$

and therefore

$$\int \frac{\mathrm{d}x}{2x} = \int \frac{\mathrm{d}y}{y^2 - 2y - 8} = \frac{1}{6} \int \mathrm{d}y \left(\frac{1}{y-4} - \frac{1}{y+2} \right),$$

$$\ln x = \frac{1}{3} (\ln(y - 4) - \ln(y + 2)) + C.$$

Solving out y, we get

$$y = -2 + \frac{6}{1 - Cx^3}. \tag{5}$$

3

Problem

$$y''(x) - 6y'(x) + 13y = -e^x, \quad y(0) = -1, \quad y'(0) = 1. \tag{6}$$

Solution The solutions of the homogeneous equation is given by

$$y = \mathrm{e}^{\lambda x},$$

$$\lambda^2 - 6\lambda + 13 = 0 \Rightarrow \lambda = 3 \pm 2\mathrm{i},$$

and therefore a linear combination of the solutions corresponding to the two λ's gives

$$y_1 = \mathrm{e}^{3x}\cos(2x), \quad y_2 = \mathrm{e}^{3x}\sin(2x). \tag{7}$$

Now we need to find a particular solution. Using the ansatz $y = A\mathrm{e}^x$, we have

$$A - 6A + 13A = -1 \Rightarrow A = -\frac{1}{8}.$$

So the general solution is

$$y = A\mathrm{e}^{3x}\cos(2x) + B\mathrm{e}^{3x}\sin(2x) - \frac{1}{8}\mathrm{e}^x. \tag{8}$$

The initial conditions mean

$$A - \frac{1}{8} = -1, \quad 3A + 2B - \frac{1}{8} = 1 \Rightarrow A = -\frac{7}{8}, \quad B = \frac{15}{8},$$

so

$$y = -\frac{7}{8}\mathrm{e}^{3x}\cos(2x) + \frac{15}{8}\mathrm{e}^{3x}\sin(2x) - \frac{1}{8}\mathrm{e}^x. \tag{9}$$

4

Problem

$$y'' + 2y' - 3y = 0, \quad y(0) = 6, \quad y'(0) = -2. \tag{10}$$

Solution Following the same recipe above, we try to solve

$$\lambda^2 + 2\lambda - 3 = 0,$$

and we find $\lambda = 1, -3$, so

$$y = A\mathrm{e}^x + B\mathrm{e}^{-3x}, \tag{11}$$

and

$$A + B = 6, \quad A - 3B = -2 \Rightarrow A = 4, \quad B = 2.$$

So we get

$$y = 4\mathrm{e}^x + 2\mathrm{e}^{-3x}. \tag{12}$$

5

Problem

$$y'' - y' - 6y = 8e^{2x}. \tag{13}$$

Solution Again following the same procedure: the general solutions of the homogeneous equation is

$$y = A\mathrm{e}^{3x} + B\mathrm{e}^{-2x},$$

and taking the ansatz $y = A\mathrm{e}^{2x}$ to find a particular solution, we have

$$A - A - 6A = 8 \Rightarrow A = -\frac{4}{3},$$

so the general solution is

$$y = A\mathrm{e}^{3x} + B\mathrm{e}^{-2x} - \frac{4}{3}\mathrm{e}^{2x}. \tag{14}$$

6

Problem

$$x^2y'' + xy' - 4y = 0. \tag{15}$$

Solution This is a Euler equation. We use the ansatz $y = x^a$ to find a particular solution: we have

$$a(a-1) + a - 4 = 0 \Rightarrow a = \pm 2.$$

So the general solution is

$$y = Ax^2 + \frac{B}{x^2}. \tag{16}$$

7

Problem

$$y'' + xy' + xy = 0. \tag{17}$$

Solution Since the polynomials before y, y', y'' are all analytic, the general form of a solution is

$$y = \sum_{n=0}^{\infty} c_n x^n, \quad xy = \sum_{n=1}^{\infty} c_{n-1} x^n,$$

and therefore

$$y' = \sum_{n=1}^{\infty} c_n n x^{n-1}, \quad xy' = \sum_{n=1}^{\infty} c_n n x^n,$$

and

$$y'' = \sum_{n=2}^{\infty} c_n n(n-1) x^{n-2} = \sum_{n=0}^{\infty} c_{n+2}(n+2)(n+1) x^n.$$

So the equation becomes

$$\begin{aligned} 0 &= \sum_{n=0}^{\infty} c_{n+2}(n+2)(n+1)x^n + \sum_{n=1}^{\infty} c_n n x^n + \sum_{n=1}^{\infty} c_{n-1} x^n \\ &= c_2 + \sum_{n=1}^{\infty} x^n \left(c_{n+2}(n+2)(n+1) + c_n n + c_{n-1} \right) \end{aligned},$$

and we get

$$c_2 = 0, \quad (n+2)(n+1)c_{n+2} + nc_n + c_{n-1} = 0. \tag{18}$$

To pick up two particular solutions, we can set $c_0 = 1, c_1 = 0$, and vice versa. When $c_1 = 0, c_0 = 1$, we have

$$6c_3 + c_1 + c_0 = 0 \Rightarrow c_3 = -\frac{1}{6},$$

and

$$12c_4 + 2c_2 + c_1 = 0 \Rightarrow c_4 = 0.$$

So

8

Problem

$$4xy'' + 2y' + 2y = 0. \tag{19}$$

Solution