

Homework 2

Jinyuan Wu

February 22, 2024

1

1.1

Comparison between the approximate and exact susceptibilities. We consider a driven damped simple harmonic oscillator (following L8.N3). We assume that $\dot{p} = \sum_i f_i = -kq - D\dot{q} + f(t)$, where k, D , and $f(t)$ are the spring constant, damping, and driving force, respectively. In what follows, we assume that $f(t) = f_0 \cos(\omega t + \phi)$.

(a) *Starting from Newton's equations, find the first order equation of motion in complex form. [Hint: you should find something like $\dot{a} = -i\omega_0 a + (\dots)$, and you're looking to identify all of the (...) terms.]*

When there is no damping the EOMs are

$$\dot{p} = -kq, \quad \dot{q} = \frac{p}{m}, \quad (1)$$

and the following definition of the modes

$$q = \sqrt{\frac{1}{2m\omega_0}}(a^* + a), \quad p = i\sqrt{\frac{m\omega_0}{2}}(a^* - a), \quad \omega_0^2 = \frac{k}{m} \quad (2)$$

turns the Hamiltonian into $H = \omega_0 a^* a$. Under this definition of a, a^* , the EOM with damping and driving therefore becomes

$$\begin{aligned} \dot{a}^* - \dot{a} &= i\omega_0(a^* + a) - \frac{D}{m}(a^* - a) - i\sqrt{\frac{2}{m\omega_0}}f(t), \\ \dot{a}^* + \dot{a} &= i\omega_0(a^* - a), \end{aligned} \quad (3)$$

from which we find

$$\dot{a} = -i\omega_0 a + \frac{D}{2m}(a^* - a) + i\sqrt{\frac{2}{m\omega_0}}f(t) \quad (4)$$

and its complex conjugate

$$\dot{a}^* = i\omega_0 a^* - \frac{D}{2m}(a^* - a) - i\sqrt{\frac{2}{m\omega_0}}f(t). \quad (5)$$

(b) *Perform the rotating wave approximation to the solution you obtained in part (a). Identify the terms that are eliminated by this approximation.*

Now we do RWA and let $a = \bar{a}e^{-i\omega_0 t}$, and the EOM of a becomes

$$\dot{\bar{a}}e^{-i\omega_0 t} + \bar{a}(-i\omega_0)e^{-i\omega_0 t} = -i\omega_0 \bar{a}e^{-i\omega_0 t} + \frac{D}{2m}(\bar{a}^*e^{i\omega_0 t} - \bar{a}e^{-i\omega_0 t}) + i\sqrt{\frac{2}{m\omega_0}}f(t),$$

or

$$\dot{\bar{a}} = -\frac{D}{2m}(\bar{a} - \bar{a}^*e^{i2\omega_0 t}) + i\sqrt{\frac{2}{m\omega_0}}e^{i\omega_0 t}f(t). \quad (6)$$

The fast oscillating terms include the $\frac{D}{2m}\bar{a}^*e^{i2\omega_0 t}$ term and the $e^{i\omega t}$ part of $f(t)$, assuming that ω is close to ω_0 . If these terms are removed, the resulting EOM is

$$\dot{\bar{a}} = -\frac{D}{2m}\bar{a} + i\sqrt{\frac{1}{2m\omega_0}} \cdot \frac{1}{2}f_0 e^{-i(\omega - \omega_0)t}. \quad (7)$$

(c) Use the approximate equation of motion from part (b) to find the complex susceptibility, $\alpha_a[\omega]$, that relates the complex wave amplitude and force as $a[\omega] = \alpha_a[\omega]f[\omega]$.

Noting that in the stationary solution $\bar{a} \propto e^{-i(\omega - \omega_0)t}$, we have

$$a(t) = e^{-i\omega_0 t} \bar{a} = \frac{1}{-i(\omega - \omega_0) + \frac{D}{2m}} \frac{i}{\sqrt{2m\omega_0}} \cdot \frac{1}{2} f_0 e^{-i\omega t}, \quad (8)$$

and therefore the response of a is (we divide $a(t)$ by $f_0 e^{-i\omega t}/2$)

$$\alpha_a[\omega] = -\frac{1}{\sqrt{2m\omega_0}} \frac{1}{\omega - \omega_0 + i\frac{D}{2m}}. \quad (9)$$

1.2

(a) Starting from the second order equation, $m\ddot{q} = \sum_i f_i = -kq - D\dot{q} + f(t)$, find the exact susceptibility $q[\omega] = \alpha_q[\omega]f[\omega]$ of the driven-damped SHO.

The EOM of q is

$$m\ddot{q} + D\dot{q} + kq = \frac{f_0}{2}(e^{i\omega t} + e^{-i\omega t}), \quad (10)$$

and since the system is linear, we can only consider the response of q to the $e^{-i\omega t}$ component of f , and get

$$\alpha_q[\omega] = \frac{1}{-m\omega^2 - iD\omega + k} = -\frac{1}{m} \frac{1}{\omega^2 - \omega_0^2 + i\frac{D}{m}\omega}. \quad (11)$$

(b) Compare the approximate response you obtained in part (1.1c) with the exact response you found in (1.2a). [Remember, $q(t)$ is contained in $a(t)$.]

Since $p = m\dot{q}$, we have

$$\alpha_p[\omega] = \frac{i\omega}{\omega^2 - \omega_0^2 + i\frac{D}{m}\omega}, \quad (12)$$

and this means

$$\begin{aligned} \alpha_a^{\text{accurate}} &= \frac{1}{2} \left(\sqrt{2m\omega_0} \alpha_q + i\sqrt{\frac{2}{m\omega_0}} \alpha_p \right) \\ &= -\frac{1}{\sqrt{2m\omega_0}} \frac{\omega + \omega_0}{\omega^2 - \omega_0^2 + i\frac{D}{m}\omega}. \end{aligned} \quad (13)$$

(c) Can you find a set of approximations that will make your exact susceptibility converge with the susceptibility you found from the RWA?

The validity of RWA is equivalent to $\omega \approx \omega_0$, which then means

$$\alpha_a = -\frac{1}{\sqrt{2m\omega_0}} \frac{1}{\frac{\omega^2 - \omega_0^2}{\omega + \omega_0} + i\frac{D}{m} \frac{\omega}{\omega + \omega_0}} \approx -\frac{1}{\sqrt{2m\omega_0}} \frac{1}{\omega - \omega_0 + i\frac{D}{2m}}. \quad (14)$$

This is expected since when RWA was made, it was already assumed that $\omega \approx \omega_0$.

Verifying this condition by calculating α_q seems to be not viable, because q is to be calculated from a and a^* , the response function of the latter being $\alpha[-\omega^*]$; but then when RWA around $\omega \approx \omega_0$ works for a , it doesn't work for a^* . RWA for q 's response seems to be ill-defined.

2

Time Modulation of Dielectric Constant: Next, we consider the impact of a time-modulated dielectric constant on the modes of a resonator. In this case, we assume that a standing-wave mode of the form $\mathbf{E}(r, t) = (a - a^*) \mathbf{E}^o(r)$ with real-valued field distribution $\mathbf{E}^o(r)$ is localized within a structured dielectric, having dielectric distribution $\varepsilon_r(r)$. For example, you could imagine that this is an eigenmode of a dielectric photonic crystal resonator of the type seen as Fig 8 in Chapter 7 of [JJSJ]. In the absence of a dielectric perturbation, the Hamiltonian takes the form $H_a = \omega_o a^* a$, where ω_o is the natural frequency of this time-harmonic mode.

2.1

(a) As a starting point, find an expression for δH associated with an arbitrary dielectric perturbation $\Delta\epsilon_r(r)$.

Using the normalization

$$\mathbf{E} = i(a - a^*)\mathbf{E}^0(\mathbf{r}), \quad (15)$$

we have

$$\delta H = \int d^3\mathbf{r} \frac{1}{2}\epsilon_0\Delta\epsilon_r\mathbf{E}^2 = \frac{\epsilon_0}{2}(a - a^*)^2 \langle \mathbf{E}^0 | \Delta\epsilon_r | \mathbf{E}_0 \rangle. \quad (16)$$

(b) Using your result from part (a), find an equations of motion for a and a^* .

We have

$$\frac{da}{dt} = -i\frac{\partial H}{\partial a^*} = -i\omega_0 a + i\epsilon_0(a - a^*) \langle \mathbf{E}^0 | \Delta\epsilon_r | \mathbf{E}_0 \rangle, \quad (17)$$

and therefore

$$\frac{da^*}{dt} = i\omega_0 a^* + i\epsilon_0(a - a^*) \langle \mathbf{E}^0 | \Delta\epsilon_r | \mathbf{E}_0 \rangle. \quad (18)$$

If we formally solve the equation system, we will find the first order correction is

$$\Delta\omega^{(1)} = -\epsilon_0 \langle \mathbf{E}^0 | \Delta\epsilon_r | \mathbf{E}_0 \rangle. \quad (19)$$

Considering the normalization scheme here that leads to $H_0 = \omega a^* a$ is

$$2\epsilon_0 \langle \mathbf{E}^0 | \epsilon_r | \mathbf{E}^0 \rangle = \omega_0 \quad (20)$$

and not

$$\epsilon_0 \langle \mathbf{E}^0 | \epsilon_r | \mathbf{E}^0 \rangle = 1, \quad (21)$$

this result agrees with the first order perturbation in Homework 1.

(c) Next, we assume that perturbation to the dielectric constant takes the form $\Delta\epsilon_r(t) = \beta\epsilon_r(r) \cos(2\omega_0 t + \phi)$, where β is a small ($\beta \ll 1$) unitless constant. Find the new form of our equations of motion from part (b). [Remember, our field normalization permits us to reduce $\langle \mathbf{E}^0 | \epsilon_r(r) | \mathbf{E}^0 \rangle$ to a constant already defined above.]

The EOM of a now is

$$\begin{aligned} \frac{da}{dt} &= -i\omega_0 a + i\epsilon_0(a - a^*) \langle \mathbf{E}^0 | \epsilon_r | \mathbf{E}_0 \rangle \beta \cos(2\omega_0 t + \phi) \\ &= -i\omega_0 a + \frac{i\omega_0}{2}\beta(a - a^*) \cos(2\omega_0 t + \phi). \end{aligned} \quad (22)$$

(d) Using a trial solution in the form $a(t) = \bar{a}(t)e^{-i\omega_0 t}$, apply the rotating wave approximation to the equations of motion from part (c). [Hint: your results should closely resemble the problem involving a child on a swing.]

Under the ansatz $a(t) = \bar{a}(t)e^{-i\omega_0 t}$ we have

$$\begin{aligned} e^{-i\omega_0 t} \frac{d\bar{a}}{dt} &= \frac{i\omega_0}{2}\beta(\bar{a}e^{-i\omega_0 t} - \bar{a}^*e^{i\omega_0 t}) \cdot \frac{1}{2}(e^{i(2\omega_0 t + \phi)} + e^{-i(2\omega_0 t + \phi)}) \\ \Rightarrow \frac{d\bar{a}}{dt} &\approx -\frac{i\omega_0}{4}\beta\bar{a}^*e^{-i\phi}, \end{aligned} \quad (23)$$

and therefore under RWA,

$$\frac{d\bar{a}^*}{dt} = \frac{i\omega_0}{4}\beta\bar{a}e^{i\phi}. \quad (24)$$

Therefore, we get

$$\frac{d^2\bar{a}}{dt^2} = \frac{\omega_0^2\beta^2}{16}\bar{a}, \quad (25)$$

and the exponential growing solution is

$$a = \bar{a}e^{-i\omega_0 t} = \bar{a}(t=0)e^{-i\omega_0 t} e^{\frac{\omega_0\beta}{4}t}. \quad (26)$$

This expression contains no phase ϕ , but ϕ does appear in the growth of H , which is given by

$$\dot{H} = \omega_0(\dot{a}^*a + a^*\dot{a}) = \omega_0(\dot{\bar{a}}^*\bar{a} + \bar{a}^*\dot{\bar{a}}) = \frac{\omega_0^2}{2}\beta|\bar{a}|^2 \sin \phi. \quad (27)$$

(e) Treating $\Delta\epsilon_r(r)$ as a perturbation on our original mode, solve for the first order correction to our mode amplitude.

Replacing $\Delta\epsilon_r$ by $\lambda\Delta\epsilon_r$, and doing the expansion $a = a^{(0)} + \lambda a^{(1)} + \dots$, we get

$$\left(\frac{d}{dt} + i\omega_0\right)(a^{(0)} + \lambda a^{(1)} + \dots) = i\epsilon_0(a^{(0)} - a^{(0)*} + \lambda(a^{(1)} - a^{(1)*})) \cdot \lambda \langle \mathbf{E}^0 | \Delta\epsilon_r | \mathbf{E}^0 \rangle,$$

and if we only consider the first order perturbation we get

$$\begin{aligned} \dot{a}^{(1)} &= -i\omega_0 a^{(1)} + i\epsilon_0(a^{(0)} - a^{(0)*}) \langle \mathbf{E}^0 | \Delta\epsilon_r | \mathbf{E}^0 \rangle \\ &= -i\omega_0 a^{(1)} - \omega_0 \beta \sin \omega_0 t \cos(2\omega_0 t + \phi). \end{aligned} \quad (28)$$

(f) Under what conditions (i.e., for what values of phase, ϕ) does the mode amplitude (and the energy of the system) grow? Under what conditions does the energy decay?

When $\sin \phi > 0$, i.e. $0 < \phi < \pi$, the amplitude grows, and otherwise it decays.

(g) Is this an elastic or inelastic scattering process?

Since there is energy change between the final state and the initial state, this is an inelastic scattering process.

2.2

Dynamical Bragg Scattering: Next, we consider mode coupling produced by a time-dependent dielectric perturbation. We build on the example presented in Lecture 9-10, which examines the coupling induced between modes of a ring resonator by a dielectric grating. In this case, however, we consider moving grating: $\Delta\epsilon(z, t) = \Delta\epsilon^0 \cos(\kappa z - \Omega t)$, where $\Delta\epsilon^0, \kappa$, and Ω are constants. We consider the impact of this dielectric perturbation on two counter-propagating modes $\tilde{\mathbf{E}}_1(r, t) = a_1 \tilde{\mathbf{E}}_1(z) = a_1 \tilde{\mathbf{E}}_1^0 e^{ik_1 z}$ and $\tilde{\mathbf{E}}_2(r, t) = a_2 \tilde{\mathbf{E}}_2(z) = a_2 \tilde{\mathbf{E}}_2^0 e^{-ik_2 z}$ with distinct natural frequencies ω_1 and ω_2 , respectively. In the presence of this perturbation, the Hamiltonian describing this system becomes $H = H_1 + H_2 + \delta H$, where $H_1 = \omega_1 a_1^ a_1$, $H_2 = \omega_2 a_2^* a_2$, and δH is the interaction term produced by $\Delta\epsilon(z, t)$. The ring has length L , so the wavevectors are $k_1 = (2\pi/L)n_1$, $k_2 = (2\pi/L)n_2$ and $\kappa = (2\pi/L)m$, where n_1, n_2 , and m are integers.*

(a) Find an expression for δH in terms of the quantities provided above.

By definition

$$\delta H = \frac{1}{2} \Delta\epsilon^0 \int dz (a_1^* \tilde{\mathbf{E}}_1^{0*} e^{-ik_1 z} + a_2^* \tilde{\mathbf{E}}_2^{0*} e^{ik_2 z}) \cdot (a_1 \tilde{\mathbf{E}}_1^0 e^{ik_1 z} + a_2 \tilde{\mathbf{E}}_2^0 e^{-ik_2 z}) \cos(\kappa z - \Omega t),$$

and since $\int_0^L dz \cos(\kappa z - \Omega t) = 0$, we have

$$\begin{aligned} \delta H &= \frac{1}{2} \Delta\epsilon^0 a_1^* a_2 \tilde{\mathbf{E}}_1^{0*} \tilde{\mathbf{E}}_2^0 \int_0^L dz e^{-i(k_1 + k_2)z} \cos(\kappa z - \Omega t) + \text{c.c.} \\ &= \frac{L}{4} \Delta\epsilon^0 a_1^* a_2 \tilde{\mathbf{E}}_1^{0*} \tilde{\mathbf{E}}_2^0 \sum_G (e^{-i\Omega t} \delta_{\kappa+G, k_1+k_2} + e^{i\Omega t} \delta_{-\kappa+G, k_1+k_2}) + \text{c.c.}, \end{aligned} \quad (29)$$

where

$$G = \frac{2\pi}{L} m, \quad (30)$$

m is an integer.

(b) Assuming that the two optical modes have distinct k -vector (i.e., $k_1 \neq k_2$), what value of κ is necessary to permit coupling between mode 1 and mode 2? This condition is known as phase matching.

From the Hamiltonian it can be seen that we need

$$\pm\kappa + G = k_1 + k_2 \quad (31)$$

for non-zero coupling between the two modes where G is an arbitrary “reciprocal lattice vector” with the form of $2\pi/L \cdot m$, m being an integer.

(c) From the total Hamiltonian, H , assuming that phase-matching is satisfied, find the equations of motion that describe coupling between our mode amplitudes a_1 and a_2 .

From δH we find

$$\begin{aligned}\dot{a}_1 &= -i\omega_1 a_1 - i \frac{\partial \delta H}{\partial a_1^*} \\ &= -i\omega_1 a_1 - i \frac{L}{4} \Delta \epsilon_0 \tilde{\mathbf{E}}_1^{0*} \tilde{\mathbf{E}}_2^0 \sum_G (e^{-i\Omega t} \delta_{\kappa+G, k_1+k_2} + e^{i\Omega t} \delta_{-\kappa+G, k_1+k_2}) a_2,\end{aligned}\quad (32)$$

and similarly

$$\begin{aligned}\dot{a}_2 &= -i\omega_2 a_2 - i \frac{\partial \delta H}{\partial a_2^*} \\ &= -i\omega_2 a_2 - i \frac{L}{4} \Delta \epsilon_0 \tilde{\mathbf{E}}_1^0 \tilde{\mathbf{E}}_2^{0*} \sum_G (e^{-i\Omega t} \delta_{\kappa+G, k_1+k_2} + e^{i\Omega t} \delta_{-\kappa+G, k_1+k_2}) a_1.\end{aligned}\quad (33)$$

(d) Assuming that our optical modes have a small frequency separation [i.e., $|\omega_2 - \omega_1| \ll (\omega_1, \omega_2)$] and that the modulation frequency, Ω , is much smaller than our optical frequencies [i.e., $\Omega \ll (\omega_1, \omega_2)$], apply the rotating wave approximation to the equations of motion from part (c).

We define $a_{1,2} := \bar{a}_{1,2} e^{-i\omega_{1,2}t}$. The EOMs of $a_{1,2}$ then becomes

$$\dot{\bar{a}}_1 = -i \frac{L}{4} \Delta \epsilon_0 \tilde{\mathbf{E}}_1^{0*} \tilde{\mathbf{E}}_2^0 \sum_G (e^{-i(\Omega+\omega_2-\omega_1)t} \delta_{\kappa+G, k_1+k_2} + e^{i(\Omega-\omega_2+\omega_1)t} \delta_{-\kappa+G, k_1+k_2}) \bar{a}_2, \quad (34)$$

$$\dot{\bar{a}}_2 = -i \frac{L}{4} \Delta \epsilon_0 \tilde{\mathbf{E}}_2^{0*} \tilde{\mathbf{E}}_1^0 \sum_G (e^{-i(\Omega+\omega_1-\omega_2)t} \delta_{\kappa+G, k_1+k_2} + e^{i(\Omega-\omega_1+\omega_2)t} \delta_{-\kappa+G, k_1+k_2}) \bar{a}_1. \quad (35)$$

Since the relation between Ω and $|\omega_1 - \omega_2|$ is not determined, we can't do RWA yet. RWA works when $\Omega \approx |\omega_1 - \omega_2|$, so one of the temporal Fourier components can be ignored. When $\Omega \approx \omega_1 - \omega_2$, the RWA EOMs are

$$\dot{\bar{a}}_1 = -i \frac{L}{4} \Delta \epsilon_0 \tilde{\mathbf{E}}_1^{0*} \tilde{\mathbf{E}}_2^0 \sum_G \delta_{\kappa+G, k_1+k_2} \bar{a}_2, \quad (36)$$

$$\dot{\bar{a}}_2 = -i \frac{L}{4} \Delta \epsilon_0 \tilde{\mathbf{E}}_2^{0*} \tilde{\mathbf{E}}_1^0 \sum_G \delta_{-\kappa+G, k_1+k_2} \bar{a}_1, \quad (37)$$

and when $\Omega \approx \omega_2 - \omega_1$, the RWA EOMs are

$$\dot{\bar{a}}_1 = -i \frac{L}{4} \Delta \epsilon_0 \tilde{\mathbf{E}}_1^{0*} \tilde{\mathbf{E}}_2^0 \sum_G \delta_{-\kappa+G, k_1+k_2} \bar{a}_2, \quad (38)$$

$$\dot{\bar{a}}_2 = -i \frac{L}{4} \Delta \epsilon_0 \tilde{\mathbf{E}}_2^{0*} \tilde{\mathbf{E}}_1^0 \sum_G \delta_{\kappa+G, k_1+k_2} \bar{a}_1. \quad (39)$$

(e) What value must Ω take to facilitate resonant coupling between a_1 and a_2 ? [Hint: Your answer should be consistent with phase matching conditions!] Find the reduced form of the coupled mode equations in the case of resonant coupling.

The condition of resonant coupling is the same as the condition of RWA, and we need $\Omega \approx |\omega_1 - \omega_2|$. From the RWA equations above, we also need to find two G 's such that

$$\kappa + G_1 = k_1 + k_2, \quad -\kappa + G_2 = k_1 + k_2, \quad (40)$$

which are equivalent to the condition that there exists one $G = 2\pi/Lm$ such that

$$k_1 + k_2 = \frac{G_1 + G_2}{2}, \quad \kappa = \frac{G_2 - G_1}{2}. \quad (41)$$

When all these conditions are satisfied we get

$$\dot{\bar{a}}_1 = -i \frac{L}{4} \Delta \epsilon_0 \tilde{\mathbf{E}}_1^{0*} \tilde{\mathbf{E}}_2^0 \bar{a}_2, \quad \dot{\bar{a}}_2 = -i \frac{L}{4} \Delta \epsilon_0 \tilde{\mathbf{E}}_2^{0*} \tilde{\mathbf{E}}_1^0 \bar{a}_1, \quad (42)$$

and therefore

$$\ddot{\bar{a}}_{1,2} = \frac{L^2}{16} \Delta \epsilon_0^2 |\tilde{\mathbf{E}}_1^0|^2 |\tilde{\mathbf{E}}_2^0|^2 \bar{a}_{1,2}. \quad (43)$$

(f) Is this an elastic or inelastic scattering process? Is energy conserved? If not, where is the energy coming from?

The energy is not conserved: it oscillates between $N\omega_1$ and $N\omega_2$, and therefore it's an inelastic scattering process. The energy comes from the field that induces the change of ϵ_r .

3

3.1

Third order nonlinear polarization: In this problem, we consider the effect of third order nonlinearity on a single time-harmonic mode. A single optical mode, $\mathbf{E}(\mathbf{r}, t)$, interacts with a polarizable medium. We assume that the medium produces a polarization field $\mathbf{P} = \chi^{(3)}|\mathbf{E}|^2\mathbf{E}$, where $\chi^{(3)}$ is a constant. In the absence of the polarizable medium (i.e., $\chi^{(3)} = 0$), the Hamiltonian for the system is $H(a, a^) = \omega a^*a$ where a and a^* are the complex mode amplitudes. As we have done in lecture, we assume that the mode amplitude is related to the field by $\mathbf{E}(\mathbf{r}, t) = (a - a^*)\mathbf{E}^o(\mathbf{r})$.*

(a) Find an expression for the interaction Hamiltonian, δH , in terms of the electric fields.

The nonlinear interaction Hamiltonian is

$$\begin{aligned}\delta H &= \int \mathbf{E} \cdot \delta \mathbf{D} = \int \mathbf{E} \cdot \chi^{(3)}(|\mathbf{E}|^2 \delta \mathbf{E} + 2\mathbf{E}\mathbf{E} \cdot \delta \mathbf{E}) \\ &= \chi^{(3)} \int 3|\mathbf{E}|^2 \mathbf{E} \cdot \delta \mathbf{E} \\ &= \frac{3}{4}\chi^{(3)}|\mathbf{E}|^4.\end{aligned}\tag{44}$$

(b) Express your interaction Hamiltonian from part (a) as products of a, a^* ; be sure to define all of the coefficients in this expression. [The coefficients will include integrals over space.]

$$\delta H = \frac{3}{4}\chi^{(3)}(a - a^*)^4 \int d^3\mathbf{r} |\mathbf{E}^o|^4.\tag{45}$$

(c) Find \dot{a} using Poisson brackets.

$$\begin{aligned}\frac{da}{dt} &= -i\omega_0 a - i \frac{\partial \delta H}{\partial a^*} \\ &= -i\omega_0 a + i3\chi^{(3)}(a - a^*)^3 \int d^3\mathbf{r} |\mathbf{E}^o|^4.\end{aligned}\tag{46}$$

(d) From part (c), show that the mode acquires intensity dependent frequency shift. Explain the significance of this term.