Homework 2

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1

Problem

$$y' - 9y = t, \quad y(0) = 5. \tag{1}$$

Solution After Laplace transformation we get

$$sY(s) - 5 - 9Y(s) = \frac{1}{s^2},$$

$$Y(s) = \frac{5}{s-9} + \frac{1}{s^2(s-9)}.$$

The second term can be decomposed (by multiplying x, x^2 or $(x-9)$ and taking the limit $x \to 0$ and $x \to 9$) as

$$\frac{1}{s^2(s-9)} = \frac{1}{81}\frac{1}{s-9} - \frac{1}{81}\frac{1}{s} + \frac{1}{9}\frac{1}{s^2},$$

and

$$Y(s) = \frac{406}{81}\frac{1}{s-9} - \frac{1}{81}\frac{1}{s} - \frac{1}{9}\frac{1}{s^2}.$$

The inverse Laplace transform is

$$y(t) = \frac{406}{81}\mathrm{e}^{9t} - \frac{1}{81} + \frac{1}{9}t. \tag{2}$$

2

Problem

$$y'' - 4y' + 4y = f(t); y(0) = -2, y'(0) = 1 \text{ with}$$
$$f(t) = \begin{cases} t & \text{for } 0 \leq t < 3 \\ t+2 & \text{for } t \geq 3 \end{cases}$$

Solution The Laplace transform of the LHS is

$$s^2Y(s) - sy(0) - y'(0) - 4(sY(s) - y(0)) + 4Y(s) = (s-2)^2Y(s) + 2s - 9.$$

The RHS is

$$f(t) = t(H(t) - H(t-3)) + (t+2)H(t-3) = tH(t) + 2H(t-3) \xrightarrow{\mathcal{L}} \frac{1}{s^2} + 2\cdot\mathrm{e}^{-3s}\cdot\frac{1}{s}.$$

So the equation is equivalent to

$$(s-2)^2Y(s) + 2s - 9 = \frac{1}{s^2} + \frac{2\mathrm{e}^{-3s}}{s},$$

$$Y(s) = -\frac{2}{s-2} + \frac{5}{(s-2)^2} + \frac{1}{s^2(s-2)^2} + \frac{2}{s(s-2)^2}\mathrm{e}^{-3s}. \tag{3}$$

We immediately get

$$\mathcal{L}^{-1}\frac{1}{s-2} = \mathrm{e}^{2t}, \quad \mathcal{L}^{-1}\frac{1}{(s-2)^2} = \mathrm{e}^{2t}t.$$

From the decomposition

$$\frac{1}{s(s-2)^2} = \frac{1}{4s} - \frac{1}{4(s-2)} + \frac{1}{2}\frac{1}{(s-2)^2},$$

we find

$$\mathcal{L}^{-1}\frac{2}{s(s-2)^2} = \frac{1}{2} - \frac{1}{2}\mathrm{e}^{2t} + t\mathrm{e}^{2t},$$

and therefore

$$\mathcal{L}^{-1}\frac{2}{s(s-2)^2}\mathrm{e}^{-3s} = (\frac{1}{2} - \frac{1}{2}\mathrm{e}^{2(t-3)} + (t-3)\mathrm{e}^{2(t-3)})H(t-3).$$

The above decomposition in turn means

$$\begin{aligned}
\frac{1}{s^2(s-2)^2} &= \frac{1}{4s^2} - \frac{1}{4s(s-2)} + \frac{1}{2}\frac{1}{s(s-2)^2} \\
&= \frac{1}{4s^2} - \frac{1}{8}\left(\frac{1}{s-2} - \frac{1}{s}\right) + \frac{1}{2}\left(\frac{1}{4s} - \frac{1}{4(s-2)} + \frac{1}{2}\frac{1}{(s-2)^2}\right) \\
&= \frac{1}{4s^2} + \frac{1}{4s} - \frac{1}{4(s-2)} + \frac{1}{4(s-2)^2},
\end{aligned}$$

and the inverse Laplace transform is

$$\mathcal{L}^{-1}\frac{1}{s^2(s-2)^2} = \frac{1}{4}t + \frac{1}{4} - \frac{1}{4}\mathrm{e}^{2t} + \frac{1}{4}\mathrm{e}^{2t}t.$$

So putting everything together, we get

$$y(t) = \frac{1}{4} + \frac{1}{4}t + \frac{21}{4}\mathrm{e}^{2t}t - \frac{9}{4}\mathrm{e}^{2t} + \frac{1}{2}(1 - \mathrm{e}^{2(t-3)} + 2(t-3)\mathrm{e}^{2(t-3)})H(t-3). \tag{4}$$

3

Problem Find the time domain correspondence of

$$\frac{5}{s^2(s^2+5)}.$$

Solution We have

$$\mathcal{L}\sin\sqrt{5}t = \frac{\sqrt{5}}{s^2+5},$$

so

$$\mathcal{L}^{-1}\frac{1}{s^2+5} = \frac{1}{\sqrt{5}}\sin\sqrt{5}t.$$

Also

$$\mathcal{L}\frac{1}{s^2} = t.$$

So by convolution theorem we find

$$y(t) = 2\int_0^t (t-t')\cdot\frac{1}{\sqrt{5}}\sin\sqrt{5}t'\,\mathrm{d}t'\,.$$

Evaluating the integral, we get

$$\begin{aligned}
y(t) &= \frac{2}{5}t(1-\cos\sqrt{5}t) - \frac{2}{5\sqrt{5}}\int_0^{\sqrt{5}t} x\sin x\,\mathrm{d}x \\
&= \frac{2}{5}t(1-\cos\sqrt{5}t) - \frac{2}{5\sqrt{5}}\left.(\sin x - x\cos x)\right|_0^{\sqrt{5}t} \\
&= \frac{2}{5}t - \frac{2\sqrt{5}}{25}\sin\sqrt{5}t.
\end{aligned} \tag{5}$$

4

Problem Solve the Sturm-Liouville problem

$$y'' + \lambda y = 0, \quad y(0) = y(\pi) = 0, \tag{6}$$

and expand

$$f(x) = x \tag{7}$$

with the eigenfunctions on $0 \le x \le 5$.

Solution Using the characteristic equation method, we know when $\lambda < 0$, the roots of the characteristic equations are real and we get exponential solutions of (6), while when $\lambda > 0$, the roots are imaginary and we get sin and cos functions. The boundary condition $y(0) = y(\pi)$ can't be achieved in the former case, and therefore we find $\lambda > 0$, and the general solution is

$$y(x) = A \sin \sqrt{\lambda} x + B \cos \sqrt{\lambda} x.$$

The condition $y(0) = 0$ requires that $B = 0$, while the condition $y(\pi) = 0$ requires

$$\sqrt{\lambda}\pi = n\pi, \quad n \in \mathbb{Z}.$$

So we find

$$y_n = \sin(nx), \quad \lambda_n = n^2, \quad n = 1, 2, 3, \ldots. \tag{8}$$

It's also possible to have $n < 0$, but this duplicates $n > 0$ solutions.

By the orthogonality condition we have

$$\int_0^\pi y_m y_n \, \mathrm{d}x = 0$$

when $m \neq n$. We also have

$$\int_0^\pi y_n^2 = \int_0^\pi \frac{1 - \cos 2nx}{2} \, \mathrm{d}x = \frac{\pi}{2}.$$

So suppose the expansion expression is

$$f(x) = \sum_{n=1}^{\infty} b_n \sin(nx), \tag{9}$$

we have

$$\begin{aligned} b_n &= \frac{2}{\pi} \int_0^\pi f(x) \sin nx \, \mathrm{d}x = \frac{2}{\pi n^2} \int_0^{n\pi} x \sin x \, \mathrm{d}x \\ &= \frac{2}{\pi n^2} \left(-x \cos x + \sin x\right)\Big|_0^{n\pi} \\ &= -\frac{2}{n}(-1)^n, \end{aligned} \tag{10}$$

and therefore the expansion is

$$f(x) = \sum_{n=1}^{\infty} \frac{-2}{n} (-1)^n \sin nx. \tag{11}$$

The partial sum of first five terms is shown in Figure 1.

5

Problem Find the Fourier series of

$$f(x) = \begin{cases} -x & \text{for } -5 \le x < 0 \\ 1 + x^2 & \text{for } 0 \le x \le 5 \end{cases} \tag{12}$$

and find what it converges to.

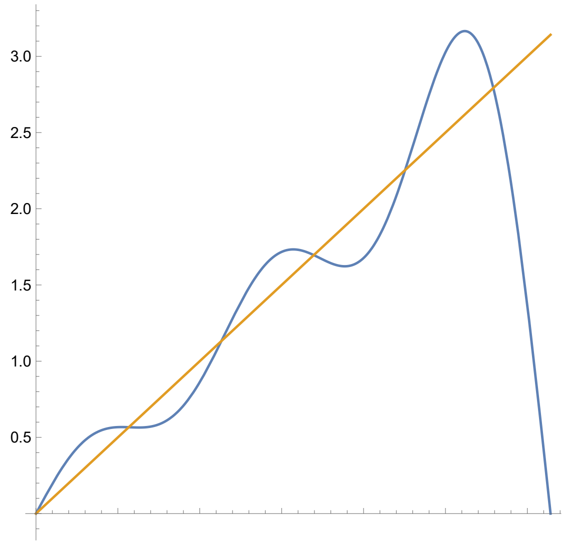

Figure 1: $N = 5$ partial sum of (11)

Solution The Fourier series is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \sin\left(\frac{n\pi x}{5}\right) + b_n \cos\left(\frac{n\pi x}{5}\right) \right). \tag{13}$$

The coefficients are found below. For a_0 we have

$$a_0 = \frac{1}{5} \int_{-5}^{5} f(x)\,\mathrm{d}x = \frac{71}{6}. \tag{14}$$

When $m \neq 0$,

$$\begin{aligned}
a_m &= \frac{1}{5} \int_{-5}^{5} f(x) \cos \frac{m\pi x}{5}\,\mathrm{d}x \\
&= \frac{1}{5} \int_{-5}^{0} (-x) \cos \frac{m\pi x}{5}\,\mathrm{d}x + \frac{1}{5} \int_{0}^{5} (x^2+1) \cos \frac{m\pi x}{5}\,\mathrm{d}x \\
&= \frac{1}{5} \left(\frac{5}{m\pi}\right)^2 \int_0^{m\pi} x \cos x\,\mathrm{d}x + \frac{1}{5} \left(\frac{5}{m\pi}\right)^3 \int_0^{m\pi} x^2 \cos x\,\mathrm{d}x \\
&= \frac{1}{5} \left(\frac{5}{m\pi}\right)^2 (x \sin x + \cos x)\Big|_0^{m\pi} + \frac{1}{5} \left(\frac{5}{m\pi}\right)^3 (x^2 \sin x + 2x \cos x - 2 \sin x)\Big|_0^{m\pi} \\
&= \frac{55(-1)^m - 5}{\pi^2 m^2}.
\end{aligned} \tag{15}$$

The sin coefficients are given by

$$\begin{aligned}
b_m &= \frac{1}{5} \int_{-5}^{0} (-x) \sin \frac{m\pi x}{5}\,\mathrm{d}x + \frac{1}{5} \int_{0}^{5} (x^2+1) \sin \frac{m\pi x}{5}\,\mathrm{d}x \\
&= -\frac{1}{5} \left(\frac{5}{m\pi}\right)^2 \int_0^{m\pi} x \sin x\,\mathrm{d}x + \frac{1}{5} \left(\frac{5}{m\pi}\right)^3 \int_0^{m\pi} x^2 \sin x\,\mathrm{d}x + \frac{1}{5} \frac{5}{m\pi} \int_0^{m\pi} \sin x\,\mathrm{d}x \\
&= -\frac{1}{5} \left(\frac{5}{m\pi}\right)^2 (-x \cos x + \sin x)\Big|_0^{m\pi} + \frac{1}{5} \left(\frac{5}{m\pi}\right)^3 (-x^2 \cos x + 2x \sin x + 2 \cos x)\Big|_0^{m\pi} \\
&\quad + \frac{1}{5} \frac{5}{m\pi} (-\cos(m\pi) + 1) \\
&= \frac{-21(-1)^m + 1}{m\pi} + \frac{50}{m^3\pi^3}((-1)^m - 1).
\end{aligned} \tag{16}$$

There is one discontinuity $x = 0$ in $(-5, 5)$. According to the convergence theorem, the Fourier series of $f(x)$ converges to $f(x)$ when $-5 < x < 0$ or $0 < x < 5$; it converges to $(f(0^-) + f(0^+))/2 = 1/2$ at $x = 0$, and to $(f(-5) + f(5))/2 = 31/2$ at $x = \pm 5$.