Homework 1

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1 Maxwell's equations in dielectrics, Lorentz oscillators, and complex notation

1.1 Time-Average Quantities in Complex Notation

It is often important to be able to compute time-averaged quantities, such as the potential energy of a harmonic oscillator $U_{pe} = \frac{k}{2}\left\langle x^2 \right\rangle$ or the electric field energy density $U_{\mathrm{el}} = \frac{\epsilon_0}{2}\left\langle \mathbf{E}^2 \right\rangle$. Here, the time-average of a function, $f(t)$, is defined as, $\langle f(t) \rangle = (1/T)\int_{t-T/2}^{t+T/2} dt' f(t')$, where T is defined as either the characteristic period of the oscillating system (i.e., $T = 2\pi/\omega$) or infinity. Such time averaging is drastically simplified by using complex notation.

To see this, suppose that we have any two functions $A(t)$ and $B(t)$, both of which take on a time harmonic form. Without loss of generality, we assume that $A(t) = A_0\cos(\omega t + \phi)$, and $B(t) = B_0\cos(\omega t + \theta)$, where ϕ and θ are arbitrary phase factors.

1.1.1

We have

$$\begin{aligned}
\langle A(t)B(t) \rangle &= \frac{1}{T}\int_{t-T/2}^{t+T/2} \mathrm{d}t'\, A_0\cos(\omega t' + \phi) B_0 \cos(\omega t' + \theta) \\
&= A_0 B_0 \frac{1}{T}\int_{t-T/2}^{t+T/2} \mathrm{d}t'\, \frac{1}{2}(\cos(\omega t' + \phi + \omega t' + \theta) + \cos(\omega t' + \phi - \omega t' - \theta)) \\
&= \frac{1}{2} A_0 B_0 \cos(\phi - \theta).
\end{aligned} \tag{1}$$

Here we have used the condition that $T = 2\pi/\omega$ so that the first term vanishes.

1.1.2

We have

$$A(t) = \tilde{A}_0 \mathrm{e}^{-\mathrm{i}\omega t}, \quad B(t) = \tilde{B}_0 \mathrm{e}^{-\mathrm{i}\omega t}, \quad \tilde{A}_0 = A_0 \mathrm{e}^{-\mathrm{i}\phi}, \quad \tilde{B}_0 = B_0 \mathrm{e}^{-\mathrm{i}\theta}, \tag{2}$$

and therefore

$$\operatorname{Re} \tilde{A}_0 B_0 = \operatorname{Re} A_0 \tilde{B}_0 = \operatorname{Re} A_0 B_0 \mathrm{e}^{\mathrm{i}(\phi-\theta)} = A_0 B_0 \cos(\phi - \theta), \tag{3}$$

and hence

$$\langle A(t)B(t) \rangle = \frac{1}{2}\operatorname{Re} \tilde{A}_0 B_0 = \frac{1}{2}\operatorname{Re} A_0 \tilde{B}_0. \tag{4}$$

We can also straightforwardly do the follows. We have

$$\begin{aligned}
\langle A(t)B(t) \rangle &= \left\langle \frac{1}{2}(\tilde{A}(t) + \tilde{A}^*(t)) \cdot \frac{1}{2}(\tilde{B}(t) + \tilde{B}^*(t)) \right\rangle \\
&= \frac{1}{4}\left\langle \tilde{A}_0 \tilde{B}_0 \mathrm{e}^{-2\mathrm{i}\omega t} + \tilde{A}_0 \tilde{B}_0^* + \tilde{A}_0{}^* \tilde{B}_0^* \mathrm{e}^{2\mathrm{i}\omega t} + \tilde{A}_0^* \tilde{B}_0 \right\rangle \\
&= \frac{1}{4}\left\langle A_0^* B_0 + \text{c.c.} \right\rangle \\
&= \frac{1}{2} A_0^* B_0 = \frac{1}{2} A_0 B_0^*.
\end{aligned} \tag{5}$$

1.1.3

When

$$\boldsymbol{E}=\hat{\boldsymbol{x}}\,\mathrm{Re}\,\tilde{E}_0\mathrm{e}^{-\mathrm{i}(\omega t-kz)}, \tag{6}$$

from

$$\boldsymbol{\nabla}\times\boldsymbol{E}=-\,\partial\boldsymbol{B}/\partial t \tag{7}$$

we obtain

$$\begin{aligned}&\mathrm{i}\boldsymbol{k}\times\boldsymbol{E}=-(-\mathrm{i}\omega)\boldsymbol{B}\\ \Rightarrow&\boldsymbol{B}=\frac{1}{\omega}k\hat{\boldsymbol{z}}\times\boldsymbol{E}=\frac{k}{\omega}\hat{\boldsymbol{y}}\,\mathrm{Re}\,\tilde{E}_0\mathrm{e}^{-\mathrm{i}(\omega t-kz)},\end{aligned} \tag{8}$$

and therefore

$$\langle\boldsymbol{S}\rangle=\frac{1}{\mu_0}\langle\boldsymbol{E}\times\boldsymbol{B}\rangle=\frac{1}{\mu_0}\cdot\frac{1}{2}\,\mathrm{Re}\,\underbrace{\hat{\boldsymbol{x}}\tilde{E}_0\mathrm{e}^{\mathrm{i}kz}}_{\tilde{\boldsymbol{E}}_0}\times\underbrace{\frac{k}{\omega}\hat{\boldsymbol{y}}\tilde{E}_0^*\mathrm{e}^{-\mathrm{i}kz}}_{\tilde{\boldsymbol{B}}_0}=\frac{k}{2\mu_0\omega}|\tilde{E}_0|^2\hat{\boldsymbol{z}}, \tag{9}$$

and since the refraction index is n, we eventually get

$$\omega=k\cdot\frac{c}{n} \tag{10}$$

and therefore

$$\langle\boldsymbol{S}\rangle=\frac{n}{2}\sqrt{\frac{\epsilon_0}{\mu_0}}|\tilde{E}_0|^2\hat{\boldsymbol{z}}. \tag{11}$$

The direction of the energy flow is parallel to the z axis.

1.1.4

The expected value of the electric energy density is

$$\langle u_e\rangle=\frac{1}{2}\epsilon_0\epsilon_\mathrm{r}\left\langle\boldsymbol{E}^2\right\rangle=\frac{1}{2}\epsilon_0 n^2\cdot\frac{1}{2}|\tilde{E}_0|^2=\frac{1}{4}\epsilon_0 n^2|\tilde{E}|^2, \tag{12}$$

and the expected value of the magnetic energy density is

$$\langle u_m\rangle=\frac{1}{2\mu_0}\left\langle\boldsymbol{B}^2\right\rangle=\frac{1}{2\mu_0}\cdot\frac{1}{2}\frac{k^2}{\omega^2}|\tilde{E}_0|^2=\frac{1}{4}\frac{n^2}{c^2\mu_0}\left|\tilde{E}_0\right|^2=\frac{1}{4}\epsilon_0 n^2\left|\tilde{E}_0\right|^2. \tag{13}$$

So we find

$$\frac{\langle u_e\rangle}{\langle u_m\rangle}=1. \tag{14}$$

2 Lorentz oscillator in an AC field and optical forces

2.1 Optical response of an ensemble of Lorentz oscillators

Consider a dilute ensemble of Lorentz oscillators, uniformly distributed over space with number density N, in an AC electric field given by $\mathbf{E}=\mathrm{Re}\left[\tilde{\mathbf{E}}_0e^{-i\omega t}\right]$. Each oscillator is driven by the local electric field according to the equation of motion given by

$$\ddot{\mathbf{p}}+\gamma\dot{\mathbf{p}}+\Omega^2\mathbf{p}=\frac{q^2}{m}\mathbf{E}(\mathbf{r}),$$

where $\mathbf{r}, m$, and q are the respective oscillator position, reduced mass, and charge.

(a) The polarization density is

$$\boldsymbol{P} = N\boldsymbol{p}. \tag{15}$$

The EOM for $\boldsymbol{P}$ is

$$\ddot{\boldsymbol{P}} + \gamma\dot{\boldsymbol{P}} + \Omega^2\boldsymbol{P} = \frac{Nq^2}{m}\boldsymbol{E}. \tag{16}$$

We can switch to the Fourier representation. Thus we have

$$((-\mathrm{i}\omega)^2 + \gamma(-\mathrm{i}\omega) + \Omega^2)\tilde{\boldsymbol{P}} = \frac{Nq^2}{m}\tilde{\boldsymbol{E}}, \tag{17}$$

and from

$$\boldsymbol{D} = \epsilon_0\boldsymbol{E} + \boldsymbol{P} \tag{18}$$

we get

$$\tilde{\boldsymbol{D}} = \epsilon_0\epsilon_\mathrm{r}\tilde{\boldsymbol{E}}, \quad \epsilon_\mathrm{r}(\boldsymbol{k},\omega) = 1 + \frac{\omega_\mathrm{p}^2}{-\omega^2 - \mathrm{i}\gamma\omega + \Omega^2}, \quad \omega_\mathrm{p}^2 = \frac{Nq^2}{m\epsilon_0}. \tag{19}$$

So we already get ϵ_r; it has explicit dependence on ω, but not $\boldsymbol{k}$.
(b) The form of a propagating plane wave in the free space is

$$\boldsymbol{E} = \boldsymbol{E}_0 \mathrm{e}^{\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{r} - \omega t)} + \text{c.c.}, \quad \boldsymbol{k} = k\hat{\boldsymbol{k}}, \quad k = \frac{\sqrt{\epsilon_\mathrm{r}}\omega}{c} = \frac{n\omega}{c}. \tag{20}$$

Note that it's possible that k has an imaginary part and n is the complex refraction index. The direction of $\operatorname{Re}\boldsymbol{k}$ and $\operatorname{Im}\boldsymbol{k}$ is assumed to be the same. The phase velocity is given by

$$v = \frac{\omega}{\operatorname{Re} k} = \frac{c}{\operatorname{Re} n} = \frac{c}{\operatorname{Re}\sqrt{\epsilon_\mathrm{r}}} = \frac{c}{\operatorname{Re}\sqrt{1 + \frac{\omega_\mathrm{p}^2}{-\omega^2 - \mathrm{i}\gamma\omega + \Omega^2}}}. \tag{21}$$

The group velocity is

$$v_\mathrm{g} = \frac{\mathrm{d}\omega}{\mathrm{d}\operatorname{Re} k} = c\left(\frac{\mathrm{d}\operatorname{Re}\sqrt{1 + \frac{\omega_\mathrm{p}^2}{-\omega^2 - \mathrm{i}\gamma\omega + \Omega^2}}}{\mathrm{d}\omega}\right)^{-1}. \tag{22}$$

(c) I'm confused by what the problem is about. If it's about the evaluating $\langle S\rangle$ and $\langle u\rangle$ in a plane wave by definition, then the answer is simply the phase velocity; but we need to assume that damping isn't huge or otherwise the first term in (66) can't be ignored and v_E has time/space dependence.

If, as the name *energy velocity* more frequently means, the problem is about the speed of energy of a wave packet, and $\langle u_{EM}\rangle$ is understood as

$$\int \mathrm{d}t \left\langle \frac{\partial u_{EM}}{\partial t} \right\rangle,$$

then when damping is small, by considering a wave packet whose frequency dependence is around ω we will find

$$\langle u_e\rangle = \frac{1}{4}\varepsilon_0 \frac{\partial(\omega\epsilon_\mathrm{r})}{\partial\omega} \tag{23}$$

and the resulting energy velocity will be the group velocity; but again this only works when the damping is small; when this is not the case, $\langle \partial u_e/\partial t\rangle$ is not a total derivative of t, and a neat expression of $\langle u_e\rangle$ can't be found.
(d) The extinction coefficient is

$$\kappa = \operatorname{Im} n = \operatorname{Im}\sqrt{\epsilon_\mathrm{r}}. \tag{24}$$

From the definition we find

$$\kappa = \sqrt{\frac{-\operatorname{Re}\epsilon_\mathrm{r} + |\epsilon_\mathrm{r}|}{2}}. \tag{25}$$

The resulting expression is too long to display here. When $\gamma \to 0$ ϵ_r has real part only, and we get $\kappa = 0$, which is expected since when there is no source of damping in the system there is no dissipation.

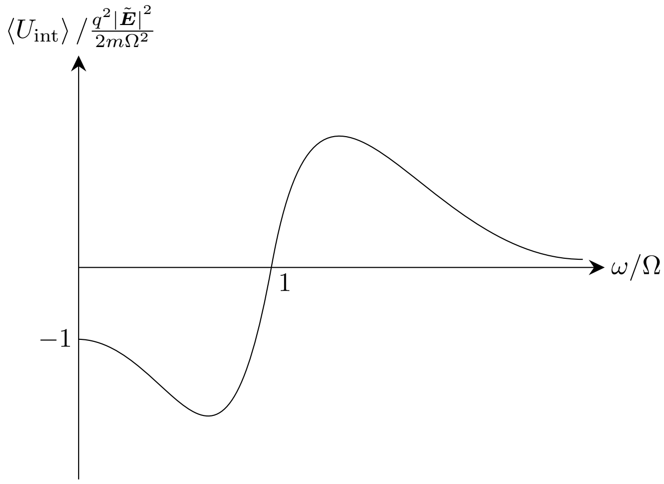

Figure 1: Plot of (27)

2.2 Optical Tweezers

(a) From

$$\tilde{\boldsymbol{p}} = -\frac{q^2}{m}\frac{1}{\omega^2 - \Omega^2 + \mathrm{i}\gamma\omega}\tilde{\boldsymbol{E}} \tag{26}$$

and $U_{\text{int}} = -\boldsymbol{p}\cdot\boldsymbol{E}$, we get

$$\begin{aligned}\langle U_{\text{int}}\rangle &= -\frac{1}{2}\operatorname{Re}\tilde{\boldsymbol{p}}^*\cdot\tilde{\boldsymbol{E}} = \frac{1}{2}\frac{q^2}{m}\left|\tilde{\boldsymbol{E}}\right|^2\operatorname{Re}\frac{1}{\omega^2-\Omega^2+\mathrm{i}\gamma\omega}\\ &= \frac{q^2}{2m}\left|\tilde{\boldsymbol{E}}\right|^2\frac{\omega^2-\Omega^2}{(\omega^2-\Omega^2)^2+\gamma^2\omega^2}.\end{aligned} \tag{27}$$

(b) The relation can be found in Fig. 1. When $\omega \gg \Omega$, the coupling energy vanishes; when $\omega \ll \Omega$, we have

$$\langle U_{\text{int}}\rangle \approx -\frac{q^2\left|\tilde{\boldsymbol{E}}\right|^2}{2m\Omega^2}. \tag{28}$$

(c) The force acting on the dipole is

$$\boldsymbol{F} = -\boldsymbol{\nabla}\langle U_{\text{int}}\rangle = \frac{q^2}{2m\Omega^2}\boldsymbol{\nabla}\left|\tilde{\boldsymbol{E}}\right|^2. \tag{29}$$

Therefore to trap the dipole in an optical trap, $\boldsymbol{E}(\boldsymbol{r})$ must follow the following condition: the intensity around a particular point is the highest, and away from the point $\left|\boldsymbol{E}(\boldsymbol{r})\right|^2$ should rapidly go down, so the gradient of the intensity to the center is large enough to give the dipole a large force towards the center, where it is confined.

2.3 A simple derivation of electrostrictive pressure

(a) The interaction energy between electromagnetic field and the matter is

$$u_{\text{int}} = -\boldsymbol{P}\cdot\boldsymbol{E} = -\epsilon_0(\epsilon_{\text{r}}-1)\boldsymbol{E}\cdot\boldsymbol{E}, \tag{30}$$

and therefore

$$\langle u_{\text{int}}\rangle = -\epsilon_0(\epsilon_{\text{r}}-1)\frac{1}{2}\left|\tilde{\boldsymbol{E}}\right|^2. \tag{31}$$

The interaction energy of a small volume with the electric field is

$$\langle U_{\text{int}}\rangle = -\epsilon_0(\epsilon_{\text{r}}-1)\frac{1}{2}\left|\tilde{\boldsymbol{E}}\right|^2 V. \tag{32}$$

(b) We have

$$\begin{aligned} \mathrm{d}U_{\mathrm{int}} &= -\epsilon_0 \frac{\partial \epsilon_{\mathrm{r}}}{\partial \rho} \frac{\mathrm{d}\rho}{\mathrm{d}V} \,\mathrm{d}V \cdot \frac{1}{2} |\tilde{\boldsymbol{E}}|^2 V - \epsilon_0 (\epsilon_{\mathrm{r}} - 1) \frac{1}{2} |\tilde{\boldsymbol{E}}|^2 \,\mathrm{d}V \\ &= \epsilon_0 \rho \frac{\partial \epsilon_{\mathrm{r}}}{\partial \rho} \frac{1}{2} |\tilde{\boldsymbol{E}}|^2 \,\mathrm{d}V - \epsilon_0 (\epsilon_{\mathrm{r}} - 1) \frac{1}{2} |\tilde{\boldsymbol{E}}|^2 \,\mathrm{d}V \,, \end{aligned} \tag{33}$$

where we have used

$$\rho = \frac{m}{V} \Rightarrow V \,\mathrm{d}V \,\frac{\mathrm{d}\rho}{\mathrm{d}V} = V \,\mathrm{d}V \cdot - \frac{m \,\mathrm{d}V}{V^2} = -\rho \,\mathrm{d}V \,. \tag{34}$$

The electrostriction pressure is therefore

$$P = -\epsilon_0 \rho \frac{\partial \epsilon_{\mathrm{r}}}{\partial \rho} \frac{1}{2} |\tilde{\boldsymbol{E}}|^2 + \epsilon_0 (\epsilon_{\mathrm{r}} - 1) \frac{1}{2} |\tilde{\boldsymbol{E}}|^2 . \tag{35}$$

2.4 Fourier-Domain Treatment of the Wave Equation

(a) Separation of variables fails whenever formally carrying out the separation of variable procedure *doesn't* lead to good Sturm-Liouville eigenvalue problems. This happens in several cases. Maybe the structure of the equation is not good, e.g. when the problem is to find how the density change with a given but complicated velocity distribution:

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot \rho \boldsymbol{v} = 0, \tag{36}$$

from which separation of variable simply can't proceed. Another case is when the boundary condition is very complicated or simply has a weird geometrical shape.
(b) From the 1D wave equation

$$\partial_x^2 E(x,t) - \frac{\varepsilon}{c^2} \partial_t^2 E(x,t) = 0, \tag{37}$$

and the ansatz

$$E(x,t) = \bar{E}(x) f(t), \quad \bar{E}(x) = \int \mathrm{d}k \,\tilde{E}(k) \mathrm{e}^{\mathrm{i}kx}, \quad f(t) = \int \mathrm{d}\omega \,\tilde{f}(\omega) \mathrm{e}^{-\mathrm{i}\omega t} \tag{38}$$

we get

$$\int \mathrm{d}\omega \int \mathrm{d}k \left((-\mathrm{i}k)^2 - \frac{\varepsilon}{c^2} (-\mathrm{i}\omega)^2 \right) \mathrm{e}^{\mathrm{i}(kx - \omega t)} \tilde{f}(\omega) \tilde{E}(k) = 0, \tag{39}$$

and therefore the equation in frequency domain is

$$\left(k^2 - \frac{\varepsilon \omega^2}{c^2} \right) \tilde{f}(\omega) \tilde{E}(k) = 0. \tag{40}$$

(c) The dispersion relation is

$$k = \pm \sqrt{\varepsilon} \frac{\omega}{c}. \tag{41}$$

Since ε is a constant, this means the equation has no dispersion. The solution of the equation therefore can be rewritten as (we here redefine $\tilde{E}$)

$$E(x,t) = \int \frac{\mathrm{d}\omega}{2\pi} \left(\tilde{E}_1(\omega) \mathrm{e}^{\mathrm{i}\left(\sqrt{\varepsilon} \frac{\omega}{c} x - \omega t\right)} + \tilde{E}_2(\omega) \mathrm{e}^{\mathrm{i}\left(-\sqrt{\varepsilon} \frac{\omega}{c} x - \omega t\right)} \right) . \tag{42}$$

So there are two solutions corresponding to each ω; by linear recombination of the solutions, the two are left-going and right-going, correspondingly.

(d) We define the complex refraction index as

$$\tilde{n} = \sqrt{\varepsilon} = n + \mathrm{i}\kappa, \tag{43}$$

and

$$k = \frac{\omega}{c}(n + \mathrm{i}\kappa). \tag{44}$$

If we let k be complex, we have

$$\begin{aligned} E(x,t) &= E_0 \mathrm{e}^{\mathrm{i}(kx-\omega t)} + \text{c.c.} \\ &= E_0 \mathrm{e}^{-\kappa \frac{\omega}{c} x} \mathrm{e}^{\mathrm{i}\left(n \frac{\omega}{c} x - \omega t\right)} + \text{c.c.}, \end{aligned} \tag{45}$$

and the decay length is

$$l = \frac{1}{\kappa \frac{\omega}{c}} = \frac{c}{\omega \kappa}. \tag{46}$$

On the other hand, if we let ω be complex, we have

$$\omega = \frac{ck}{n} = \frac{ck}{n^2 + \kappa^2}(n - \mathrm{i}\kappa), \tag{47}$$

and the wave looks like

$$E(x,t) = E_0 \mathrm{e}^{-kc \frac{\kappa}{n^2+\kappa^2} t} \mathrm{e}^{\mathrm{i}\left(kx - kc\frac{n}{n^2+\kappa^2} t\right)} + \text{c.c.} \tag{48}$$

This solution can be achieved by injecting a plane wave into the medium and removing the pumping source, letting the wave "cooling down" within the medium, while the first solution (45), where k is complex, describes the field configuration where at the boundary of the medium the field strength is fixed, probably because of a strong and stable source outside. Now the time scale of damping is

$$\tau = \frac{n^2 + \kappa^2}{\kappa} \frac{1}{kc}, \tag{49}$$

and in this "homogeneously cooling down" case, the speed of the light becomes

$$v = \frac{n}{n^2 + \kappa^2} c, \tag{50}$$

not just the c/n in (45). Now the decay length can be found from τ as

$$l = v\tau = \frac{n}{\kappa k}. \tag{51}$$

If we make identification $c/n = \omega/k$, we find (46) and (51) are the same.

But I have no idea why we don't make identification $\omega = kcn/(n^2 + \kappa^2)$

2.4.1 Non-instantaneous material response

The most general relation between D and E is

$$D(x,t) = \epsilon_0 \int_{-\infty}^{\infty} \mathrm{d}t' \int_{-\infty}^{\infty} \mathrm{d}x' \, \epsilon_\mathrm{r}(x - x', t - t') E(x', t'). \tag{52}$$

(a) When we have locality and causality, $\epsilon_\mathrm{r}(x - x', t - t')$ should be proportional to $\delta(x - x')$ and $\theta(t - t')$, and thus

$$D(x,t) = \epsilon_0 \int_{-\infty}^{t} \mathrm{d}t' \, \epsilon_\mathrm{r}(t - t') E(x, t'), \tag{53}$$

$$\epsilon_\mathrm{r}(t - t') = \epsilon_\mathrm{r}(t - t')\theta(t - t'). \tag{54}$$

Here $\epsilon_\mathrm{r}(x - x', t - t')$ shouldn't have any space dependence other than $\delta(x - x')$ or it's not a function of $x - x'$.

(b) The 1D electro-magnetic wave function then is the consequence of the following equation systems in the 1D case:

$$\nabla\times\boldsymbol{E}=-\frac{\partial\boldsymbol{B}}{\partial t},\quad \nabla\times\boldsymbol{H}=\frac{\partial\boldsymbol{D}}{\partial t},\quad \boldsymbol{B}=\mu_0\boldsymbol{H},\quad \boldsymbol{D}(x,t)=\epsilon_0\int_{-\infty}^{\infty}\mathrm{d}t'\,\epsilon_{\mathrm{r}}(t-t')\boldsymbol{E}(x,t'). \tag{55}$$

Here we have assumed that ϵ_{r} doesn't map E_x to D_y, etc., or otherwise the problem can't be restricted to the 1D case, and thus from $\nabla\cdot\boldsymbol{D}=0$ we get $\nabla\cdot\boldsymbol{E}=0$. Hence

$$-\nabla^2\boldsymbol{E}=\nabla\times(\nabla\times\boldsymbol{E})=-\mu_0\frac{\partial}{\partial t}\nabla\times\boldsymbol{H}=-\mu_0\frac{\partial^2}{\partial t^2}\boldsymbol{D}, \tag{56}$$

and therefore the final wave equation is

$$\partial_x^2E(x,t)=\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\int_{-\infty}^{\infty}\mathrm{d}t'\,\epsilon_{\mathrm{r}}(t-t')E(x,t'). \tag{57}$$

(c) We do Fourier transform

$$E(x,t)=\int\frac{\mathrm{d}\omega\,\mathrm{d}k}{(2\pi)^2}\mathrm{e}^{\mathrm{i}(kx-\omega t)}\tilde{E}(k,\omega),\quad \varepsilon(t-t')=\int\frac{\mathrm{d}\omega}{2\pi}\mathrm{e}^{-\mathrm{i}\omega t}\tilde{\varepsilon}_{\mathrm{r}}(\omega), \tag{58}$$

and LHS of (57) becomes

$$\int\frac{\mathrm{d}\omega\,\mathrm{d}k}{(2\pi)^2}(\mathrm{i}k)^2\mathrm{e}^{\mathrm{i}(kx-\omega t)}\tilde{E}(k,\omega), \tag{59}$$

and the RHS becomes

$$\begin{aligned}
&\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\int\mathrm{d}t'\int\frac{\mathrm{d}\omega'}{2\pi}\int\frac{\mathrm{d}\omega\,\mathrm{d}k}{(2\pi)^2}\tilde{\varepsilon}_{\mathrm{r}}(\omega')\mathrm{e}^{-\mathrm{i}\omega'(t-t')}\tilde{E}(k,\omega)\mathrm{e}^{\mathrm{i}(kx-\omega t')}\\
=&\frac{1}{c^2}\int\frac{\mathrm{d}\omega'}{2\pi}\frac{\mathrm{d}\omega}{2\pi}\frac{\mathrm{d}k}{2\pi}(-\mathrm{i}\omega)^2\tilde{\varepsilon}_{\mathrm{r}}(\omega')\tilde{E}(k,\omega)\mathrm{e}^{\mathrm{i}kx}\int\mathrm{d}t'\,\mathrm{e}^{\mathrm{i}(\omega'-\omega)t'}\\
=&\frac{1}{c^2}\int\frac{\mathrm{d}\omega'}{2\pi}\frac{\mathrm{d}\omega}{2\pi}\frac{\mathrm{d}k}{2\pi}(-\mathrm{i}\omega)^2\tilde{\varepsilon}_{\mathrm{r}}(\omega')\tilde{E}(k,\omega)\mathrm{e}^{\mathrm{i}kx-\mathrm{i}\omega' t}\cdot 2\pi\delta(\omega-\omega')\\
=&\frac{1}{c^2}\int\frac{\mathrm{d}\omega\,\mathrm{d}k}{(2\pi)^2}\mathrm{e}^{\mathrm{i}kx-\mathrm{i}\omega t}(-\mathrm{i}\omega)^2\tilde{\varepsilon}_{\mathrm{r}}(\omega)\tilde{E}(k,\omega).
\end{aligned} \tag{60}$$

The final equation therefore becomes

$$\left(k^2-\tilde{\varepsilon}_{\mathrm{r}}(\omega)\frac{\omega^2}{c^2}\right)\tilde{E}(k,\omega)=0. \tag{61}$$

(d) The dispersion relation now is

$$\omega=\frac{c|k|}{\sqrt{\tilde{\varepsilon}_{\mathrm{r}}(\omega)}}. \tag{62}$$

2.4.2 Energy absorption

(a) In the specific case of

$$E(x,t)=\bar{E}(x)\mathrm{e}^{-\mathrm{i}\omega t}+\bar{E}^*(x)\mathrm{e}^{\mathrm{i}\omega t}, \tag{63}$$

we have

$$\begin{aligned}
D(x,t)&=\int\mathrm{d}t'\,\varepsilon(t-t')(\bar{E}(x)\mathrm{e}^{-\mathrm{i}\omega t'}+\bar{E}^*(x)\mathrm{e}^{\mathrm{i}\omega t'})\\
&=\bar{E}(x)\mathrm{e}^{-\mathrm{i}\omega t}\int\mathrm{d}t'\,\mathrm{e}^{\mathrm{i}\omega(t-t')}\varepsilon(t-t')+\bar{E}^*(x)\mathrm{e}^{\mathrm{i}\omega t}\int\mathrm{d}t'\,\mathrm{e}^{-\mathrm{i}\omega(t-t')}\varepsilon(t-t')\\
&=\bar{E}(x)\mathrm{e}^{-\mathrm{i}\omega t}\tilde{\varepsilon}(\omega)+\bar{E}^*(x)\mathrm{e}^{\mathrm{i}\omega t}\tilde{\varepsilon}(-\omega).
\end{aligned} \tag{64}$$

Since $\varepsilon(t)$ is real, we can rewrite

$$\tilde{\varepsilon}(-\omega)=\tilde{\varepsilon}(\omega)^*. \tag{65}$$

(b) We have

$$\begin{aligned}\frac{\partial u_e}{\partial t} &= E \cdot \frac{\partial D}{\partial t} \\ &= (\bar{E}(x)\mathrm{e}^{-\mathrm{i}\omega t} + \bar{E}^*(x)\mathrm{e}^{\mathrm{i}\omega t}) \cdot ((-\mathrm{i}\omega)\bar{E}(x)\mathrm{e}^{-\mathrm{i}\omega t}\tilde{\varepsilon}(\omega) + (\mathrm{i}\omega)\bar{E}^*(x)\mathrm{e}^{\mathrm{i}\omega t}\tilde{\varepsilon}(-\omega)) \\ &= -2\omega|\bar{E}(\omega)|^2\varepsilon_2(\omega) + \bar{E}(x)^2(-\mathrm{i}\omega)\mathrm{e}^{-2\mathrm{i}\omega t}\tilde{\varepsilon}(\omega) + \text{c.c.}\end{aligned} \tag{66}$$

Here ε_1 and ε_2 are real and imaginary parts of ε, respectively. The sum of the second and third terms is a sine function; it oscillates and has zero average value. The first term gives the damping. Thus, the energy absorption rate is

$$\partial_t \langle u_e \rangle = -2\omega|\bar{E}(\omega)|^2\varepsilon_2(\omega). \tag{67}$$

(c) The energy absorbed is proportional to the imaginary part of the Fourier transform of the dielectric constant, which is expected since it's related to the imaginary part of the energy spectrum.