

# Topics in optics and quantum electronics

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## 1 Mode description of electromagnetic fields

In this section we mainly focus on the mode description of classical electrodynamics. Modes are seen in resonators, waveguides, photonic crystals, and more. Loss can then be included perturbatively and we get leaky modes.

In the case of electrodynamics in vacuum, in the Coulomb gauge (i.e.  $\varphi = 0$ ), we have

$$\nabla \times \nabla \times \mathbf{A} = \left(\frac{\omega}{c}\right)^2 \mathbf{A}, \quad (1)$$

and under the inner product definition

$$\langle \mathbf{A} | \mathbf{B} \rangle := \int d\mathbf{r} \mathbf{A}^* \cdot \mathbf{B}, \quad (2)$$

the LHS of the equation on  $\mathbf{A}$  is Hermitian, guaranteeing that  $\omega$  is real.

From the well-known relation  $[a, a^\dagger] = 1$ , it follows that the classical counterparts of creation and annihilation operators are complex variables which satisfies the relation

$$\{a, a^*\} = -i. \quad (3)$$

This means if the Poisson brackets are to be defined in terms of  $a$  and  $a^*$ , it should be defined as

$$\{A, B\} = \frac{1}{i} \left( \frac{\partial A}{\partial a} \frac{\partial B}{\partial a^*} - \frac{\partial A}{\partial a^*} \frac{\partial B}{\partial a} \right), \quad (4)$$

where  $a$  and  $a^*$  are to be regarded as independent variables. The general relation between  $a, a^*$  and  $x, p$  is

$$a = \alpha q + i\beta p, \quad q = \frac{a + a^*}{2\alpha}, \quad p = \frac{a - a^*}{2i\beta}, \quad 2\alpha\beta = 1. \quad (5)$$

If  $q$  and  $p$  follow the canonical commutation relation, then  $a$  and  $a^*$  follow the commutation relation between a creation operator and an annihilation operator, and vice versa.

We can also justify (3) by Hamiltonian's equations. From (5) and the chain rule, we find

$$\begin{aligned} \dot{a} &= \frac{\partial a}{\partial q} \dot{q} + \frac{\partial a}{\partial p} \dot{p} = \frac{\partial a}{\partial q} \frac{\partial H}{\partial p} - \frac{\partial a}{\partial p} \frac{\partial H}{\partial q} \\ &= \frac{\partial a}{\partial q} \left( \frac{\partial H}{\partial a} \frac{\partial a}{\partial p} + \frac{\partial H}{\partial a^*} \frac{\partial a^*}{\partial p} \right) - \frac{\partial a}{\partial p} \left( \frac{\partial H}{\partial a} \frac{\partial a}{\partial q} + \frac{\partial H}{\partial a^*} \frac{\partial a^*}{\partial q} \right) \\ &= -2i\alpha\beta \frac{\partial H}{\partial a^*}, \end{aligned} \quad (6)$$

and similarly

$$\dot{a}^* = 2i\alpha\beta \frac{\partial H}{\partial a}. \quad (7)$$

Now that means we have to define the Poisson bracket as (4) to keep the form

$$\frac{dA}{dt} = \{A, H\}. \quad (8)$$

The  $2\alpha\beta = 1$  constraint can be explained by conservation of area in phase space: roughly speaking the area inside the trajectory of  $a(t)$  or  $(q(t), p(t))$  is the “strength of oscillation”, like the number of photons.

Now going back to the problem of electromagnetic fields. Consider the one dimensional periodic boundary condition problem. We have (in phaser form)

$$\tilde{E}_\pm(x, t) = \tilde{E}_\pm^0 e^{ikx - i\omega t}, \quad (9)$$

and we consider  $A$  to be the coordinate and hence  $-E = \partial A / \partial t$  is the momentum. To decompose  $A$  into modes, we write  $A$  as static, spatially varying factors weighted by time-dependent real values, and thus  $A = q(t)A_0 e^{ikz}$ ,

## 2 Perturbation theory

Stationary electrodynamics in terms of  $\mathbf{E}$  is not a Hermitian problem, but a generalized Hermitian one: the problem is

$$\nabla \times \nabla \times \mathbf{E} = \left(\frac{\omega}{c}\right)^2 \epsilon_r(\mathbf{r}) \mathbf{E}(\mathbf{r}), \quad (10)$$

and although the operators on both LHS and RHS are Hermitian, the LHS multiplied with the inverse of the RHS is not. The time-independent perturbation theory for this problem is therefore slightly more complicated, and the “energy” being solved is

$$\alpha := \frac{\omega^2}{c^2}, \quad (11)$$

instead of the frequency.

$$\Delta\omega_n^{(1)} = -\frac{\omega_n^{(0)}}{2} \langle \tilde{\mathbf{E}}_n | \epsilon_0 \Delta\epsilon_r | \tilde{\mathbf{E}}_n \rangle. \quad (12)$$

The minus sign can be understood using the following line of argumentation: if a piece of dielectric is inserted into an isolated system, the total energy is conserved but some energy is stored within the internal degree of freedom dielectric, so  $\omega(a^\dagger a + 1/2)$ , the vacuum part of the Hamiltonian has to decrease. One important caveat is that the leading order of energy correction *can't* be evaluated by  $\langle \mathbf{E}_n | \Delta\epsilon_r | \mathbf{E}_n \rangle$ : since the mapping from the electric field to the magnetic field involves at least one time derivative, the electromagnetic Hamiltonian in the form of  $\mathbf{E}^2 + \mathbf{B}^2$  can't be seen as an ordinary Hamiltonian whose corresponding wave functions are electric fields, and therefore ordinary perturbation techniques fail. One way to work around this is doing everything in terms of modes. This is done in homework 2.

## 3 Loss as perturbation

Now we study the influence of  $\epsilon_2$ , the imaginary part of  $\epsilon_r$ , as a perturbation. We can just inject  $\Delta\epsilon_r = i\epsilon_2$  into (12), and we get

$$\Delta\omega_n^{(1)} = -i\frac{1}{\tau_n}, \quad \frac{1}{\tau_n} = \frac{\omega_n}{2} \langle \tilde{\mathbf{E}}_n | \epsilon_0 \epsilon_2 | \tilde{\mathbf{E}}_n \rangle. \quad (13)$$

This means the time evolution of the corresponding mode  $a_n$  contains a  $e^{-t/\tau_n}$  factor, and the EOM seems to be

$$\dot{a} = -i\omega a - \frac{1}{\tau} a. \quad (14)$$

We can add a driving field to the RHS of this equation of motion; without considering any nonlinear effects,<sup>1</sup> the lifetime remains the same regardless of the possible external driving. The question is whether (14) is truly the correct equation for a damped system without external driving. It is of course possible to have a different kind of damping: if we choose to not start from the imaginary part of  $\epsilon_r$  and instead insert a  $-kv$  type of damping into the EOM of  $p$  and  $q$ , then the damping term for  $a$  should assume the form of

$$\dot{a} + i\omega_0 a = -\frac{a}{\tau} + \frac{a^*}{\tau}. \quad (15)$$

Interestingly, from an equally intuitive starting point, now we get a set of coupled EOMs for  $a$  and  $a^*$ , while in (14), the EOMs for  $a, a^*$  are not coupled.

The “real” form of the damping term depends on how the system couples to the bath; if we are studying a harmonic oscillator then (15) is exact, while (14) is not; the most natural way to include loss in Maxwell's equations also leads to something similar to (15). So indeed, although (14) seems quite natural, it's not exact, and the question is how it's related to the exact equations.

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<sup>1</sup>Damping always comes from coupling of the system with a much larger, external bath. The first order correction from the bath adds a finite imaginary part to the self energy of the system; the interaction with the bath of course also modifies the coupling between the bath and the system, but this is not first-ordered, and can be ignored when damping is small.

The answer is rotating wave approximation. From the usual procedure we will find the coupled terms in the loss are related to fast oscillation terms, which can be ignored for the low frequency part of  $a, a^*$ . This is valid as long as there is separation between the time scales of damping and of oscillation. This is the condition of the validity of RWA of an external driving field, which, in the latter case, means the frequency of driving should be close to  $\omega_0$  while the driving *amplitude* should be small. Since the damping term is not oscillating, the validity condition of RWA for damping is simpler.

TODO: does rotating wave approximation change the way the system couple to external fields? (c.f. general relativity and the  $f(R)$  theory)

## 4 Mode coupling

Consider a ring resonator. We have a right mover mode and a left mover mode, which we refer to as

$$\tilde{E}_k = a_k E_k^0 e^{ikz} \quad (16)$$

and

$$\tilde{E}_{-k} = a_{-k} E_{-k}^0 e^{-ikz}, \quad (17)$$

respectively.

Now suppose we have such a field configuration:

$$E = a_k \tilde{E}_k + \text{c.c.} + a_{-k} \tilde{E}_{-k}, \quad (18)$$

and there is a  $\Delta\epsilon_r$  perturbation in the system. The change of the frequency is therefore given by

$$\Delta\omega^{(1)} = -\frac{\omega^{(0)}}{2} \langle \Delta\epsilon_r \rangle \propto (a_k^* + a_{-k})(a_k^* + a_{-k}) + \text{c.c.}, \quad (19)$$

and therefore the contribution of  $\Delta\epsilon_r$  to the energy of the system looks like

$$\Delta E = -\tilde{\gamma}(a_k^* + a_{-k})(a_k^* + a_{-k}) + \text{c.c.} \quad (20)$$

The EOM of  $a_k$  therefore is (TODO: formalism)

$$\dot{a}_k = -i\omega_k a_k + 2i\gamma(a_k^* + a_{-k}). \quad (21)$$

Again we can apply RWA and remove the “fast”  $a_k^*$  term, so eventually the EOMs of the two modes are (TODO: RWA without external driving)

$$\dot{a}_k = -i\omega_0 a_k + i\tilde{g}a_{-k}, \quad \dot{a}_{-k} = -i\omega_0 a_{-k} + i\tilde{g}^* a_k, \quad \tilde{g} = 2\gamma. \quad (22)$$

The EOMs are still within the Hamiltonian dynamics framework, so we can diagonalize them and find that when

$$\Delta\epsilon_r = \Delta\epsilon \cos(2kz), \quad (23)$$

the new mode  $a_k + a_{-k}$  has a lower frequency  $\omega_0 - |\tilde{g}|$  while the new mode  $a_k - a_{-k}$  has a higher frequency  $\omega_0 + |\tilde{g}|$ . But the above formalism is not restricted to this perturbation: whatever  $\Delta\epsilon_r$  is, the way it affects the two modes we consider here is by influencing  $\tilde{g}$ . Specifically,  $|\tilde{g}|$  can be used to estimate the roughness of a ring resonator.

Of course, we can also perturb the system by modulation in *time*. Consider a classic  $LC$ -circuit, for example: it can be verified that an oscillating  $C$  results in

$$H = \omega_0 a^* a + \frac{\gamma(t)}{2} (a + a^*)^2, \quad (24)$$

because there is no  $L^2$  term that cancels the  $aa^*$  terms. The resulting EOMs are

$$\dot{a} = -i\omega_0 a - i\gamma(t)(a + a^*), \quad (25)$$

and the perturbation  $\gamma(t)$  usually takes the form like

$$\gamma(t) = \frac{\tilde{\gamma}_0}{2} e^{-i\Omega t} + \text{c.c.} \quad (26)$$

where  $\Omega = 2\omega_0$ . After RWA the only term that survives is

$$\dot{a} = -i\omega_0 a - i\frac{\tilde{\gamma}_0}{2} e^{-i\Omega t} a^* \quad (27)$$

and hence we find

$$\frac{d}{dt} \begin{bmatrix} a e^{-i\omega_0 t} \\ a^* e^{i\omega t} \end{bmatrix} = \begin{bmatrix} 0 & -i\frac{\tilde{\gamma}_0}{2} \\ i\frac{\tilde{\gamma}_0^*}{2} & 0 \end{bmatrix} \begin{bmatrix} a e^{-i\omega_0 t} \\ a^* e^{i\omega t} \end{bmatrix} \quad (28)$$

We can easily see that the “source” of the  $e^{-i\Omega t}$  term is  $a^*$ , which in turn comes from the  $e^{i\Omega t}$  term and  $a$ . This eventually gives us a feedback (usually *positive* but not always – see below) to  $a$ , leading to the increase of the total energy. The derivative of the “free” energy is

$$\dot{H}_{\text{SHO}} = \text{Re}(-i|\tilde{\gamma}_0| e^{i\phi} a_0), \quad (29)$$

where  $\phi$  is the phase of  $\tilde{\gamma}_0$ , and  $a_0$  is the slow oscillating part of  $a$ . This expression tells us two things. First, the phase of the pumping matters: it can be chosen in a specific way to *reduce* the total energy. Second, the growth or damping of the total energy is exponential, and therefore if at first you don’t have a finite amplitude, there is no growth at all. To start the growth, noise is important; classically this comes from thermal fluctuation; in the quantum case, the above EOM involves some terms from non-trivial commutation relations and we may say the growth is started by the zero-point energy.

## 5 Nonlinearity

We know the free  $\epsilon_0 \epsilon_r \mathbf{E}^2/2 + \mathbf{B}^2/2\mu_0$  Hamiltonian can be written as  $\sum \omega a^* a$ ; similarly a nonlinear Hamiltonian can be written as  $\sum a a^* + \text{c.c.}$ ; in principle all modes can be mixed together by nonlinearity; usually however only transitions satisfy the phase matching condition are strong enough; the definition of “phase matching” is somehow unclear: for plain waves, it means  $\sum \mathbf{k} = 0$  and has to be satisfied *strictly*; but in other cases, like in a photonic crystal, if  $\mathbf{k}$  means Bloch wave vectors, then the TODO: again, if a transition channel doesn’t satisfy the phase matching condition, does it mean this channel vanish in reciprocal space?

Phase matching in the time domain has to be exactly satisfied only when we are dealing with a scattering problem, in which we integrate from  $t = -\infty$  to  $t = \infty$  and get a  $\sum \omega = 0$  condition; for a finite time problem, this condition doesn’t need to be exactly satisfied, although a transition channel satisfying the  $\sum \omega = 0$  condition still likely to be the most influential one. In Feynman diagrams we do have strict frequency conservation conditions at each vertex but then we don’t have the on-shell condition for internal lines; by the correspondence between Feynman diagrams and Goldstone diagrams, we can let the internal lines to satisfy the on-shell conditions but let the total energy of the system to be off-shell, therefore arriving at results obtained from the EOM approach.

The EOM of one mode looks like (here NL means “nonlinear”)

$$\dot{a}_1 = -i\omega_0 a_1 - i\frac{\partial H_{\text{NL}}}{\partial a_1^*}. \quad (30)$$

For SHG, for example, we have

$$\begin{aligned} \dot{a}_1 &= -i\omega_1 a_1 + i\tilde{\gamma} a_1^* a_2, \\ \dot{a}_2 &= -i\omega_2 a_2 + i\tilde{\gamma}^* a_1 a_1. \end{aligned} \quad (31)$$

We can also study a nonlinear system using nonlinear Maxwell’s equations. This can be seen as a cross-verification of the correctness of the mode-based, usually somehow simplified formalism; when the dielectric function has a strong “dispersion” – i.e. frequency dependence – the Hamiltonian approach is also hard to use.

## 6 Temporal module coupling

Now we consider the coupling of modes with the external world. The formalism is almost identical to the so-called input-output formalism in quantum optics. As a motivation, consider

the following model: imagine we have a one-dimensional photonic crystal, with one defect (for example a missing cavity in a string of cavities). Now if a pulse is injected from one end of the photonic crystal we expect to see reflectivity and radiation to the 3D space around the defect. Whatever the defect is, it can be modeled as a mode (or several modes) coupled with “ports” around it. In the absence of the ports, the EOM of the system always takes the form  $\dot{a} = -i\omega_0 a$ , or possibly with a decay. Now we consider the structure of a port. We represent the waves going “into” the defect as  $S_i$  and the waves going “out” of the defect as  $S_o$ . Both variables may be linear combination of modes, or even contain some nonlinear effects. The EOM of the defect then is (TODO: Peter says it only holds under RWA)

$$\dot{a} = -i\omega_0 a - \frac{a}{\tau} + K S_i. \quad (32)$$

Here we do a Langevin-like approximation and assume that the dissipation term faithfully captures the coupling of the system with the output port. Of course it’s possible to have “dissipative loss”, which means we have modes that are not considered as a part of the port but are nevertheless coupled to  $a$  and damp it; here we ignore these dissipation channels and assume that dissipation mainly comes from the output modes of the port. Now if we somehow manage to integrate out  $a$  and consider the theory of the port. Ignoring all nonlinear effects and retardation effects, the relation between  $S_o$  and  $S_i, a$  can only take the following form:

$$S_o = \beta S_i + \gamma a, \quad (33)$$

where  $\beta$  and  $\gamma$  have to be related to  $\tau$ .

When  $S_i, S_o$  are appropriately normalized we expect to see that  $|S_i|^2$  is the power incident while  $|S_o|^2$  is the output power. For example, suppose we have a ring resonator, and we place a defect in it. Since a ring resonator is usually thin, it can be modeled as a pipe with periodic boundary condition, and the power going through an arbitrary point in the ring resonator takes the form of

$$P = v_g \frac{\omega_0 a^* a}{L}, \quad (34)$$

where  $a$  is a mode in the ring resonator (not the mode around the defect), and therefore  $S$  should be defined such that

$$P = S S^* = v_g \frac{\omega_0 a^* a}{L}. \quad (35)$$

Moreover we normalize  $a$  so that  $|a|^2$  is the mode energy.

Now we seriously consider how to derive the relation between  $\tau$  and  $\gamma, K$ . The most famous example for coupling with outgoing modes leading to dissipation is arguably the fact that an infinite network of LC circuits is equivalent to a resistor. Similar systems include a mass point connected to an infinite spring or a leaky cavity. When there is no input to the defect mode, the loss of energy in the mode equals the outgoing flux  $S_o$ , and therefore

$$\frac{dU_{\text{defect}}}{dt} = -\frac{2}{\tau} U_{\text{defect}} = -|S_o|^2, \quad (36)$$

and therefore

$$S_o = \underbrace{\sqrt{\frac{2}{\tau}} e^{i\alpha}}_{\gamma} a. \quad (37)$$

The phase factor is a variable that depends on the exact definition of the output of the system; consider for example a defect embedded into a wave guide: we need to choose a point at which we decide “we are no longer near the defect”, and the definition – hence the phase – of the outgoing mode depends on this point. TODO: what happens when the input is turned on? We will get something like  $2 \text{Re} K S_i^* a = |S_i|^2$ . Whether this is always true can be verified using a harmonic oscillator; essentially that’s about what is the definition of “outgoing”. Using MBPT this can be easily justified: when everything is linear, couplings between the  $S_i$  and  $S_o$  modes do not influence each other.

We still have  $K$  and  $\beta$ . Since  $a, S_i, S_o$  are assumed to form a closed system, we expect they form a system with time reversal symmetry. After time reversal operation, hence the EOM of  $a$  becomes

$$-\frac{d}{dt} a^*(-t) = -i\omega_0 a^*(-t) - \frac{a^*(-t)}{\tau} + \mathcal{T} K S_i, \quad (38)$$

where we don't immediately write what's the time reverse of  $S_i$ : time reversal operation also changes the mode label, and the correct transform is  $\mathcal{T}S_i = S_o^*(-t)$ , so eventually

$$-\frac{d}{dt}a^*(-t) = -i\omega_0 a^*(-t) - \frac{a^*(-t)}{\tau} + KS_o^*(-t), \quad (39)$$

and therefore

$$\frac{d}{dt}a(-t) = -i\omega_0 a(-t) + \frac{a(-t)}{\tau} - K^*S_o^*(-t). \quad (40)$$

Similarly

$$S_i(-t) = \beta^*S_o(-t) + \gamma^*a(-t). \quad (41)$$

TODO: interestingly here  $\mathcal{T}$  doesn't change the coefficients but does turn  $a$  to  $a^*$ . Now the EOMs should be equivalent to the EOMs before time reversal operation, and when the input modes are turned off, from

$$S_o(-t) = -\frac{\gamma^*}{\beta^*}a(-t) = -\frac{e^{-i\alpha}}{\beta^*}\sqrt{\frac{2}{\tau}}a(-t)$$

we find

$$\beta = -e^{2i\alpha}. \quad (42)$$

Similarly by comparing the EOMs of  $a$ , when the input is turned off, we have

$$\frac{d}{dt}a(-t) = -i\omega_0 a(-t) + \frac{a(-t)}{\tau} - K^*e^{i\alpha}\sqrt{\frac{2}{\tau}}a(-t),$$

and since we also expect to see

$$\frac{d}{dt}a(-t) = \frac{1}{\tau}a(-t),$$

we have

$$K = e^{i\alpha}\sqrt{\frac{2}{\tau}}. \quad (43)$$

Since  $\alpha$  can be shifted by redefining the modes, usually either we choose  $e^{i\alpha} = 1$  or  $e^{i\alpha} = -1$ . In the former convention we have

$$\dot{a} = -i\omega_0 a - \frac{a}{\tau} + \sqrt{\frac{2}{\tau}}S_i, \quad S_o = -S_i + \sqrt{\frac{2}{\tau}}a. \quad (44)$$