Many-body Physics Homework 2

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Problem 1
Solution

1. We make the Trotter decomposition:

$$\begin{aligned}
\langle\alpha_f|\mathrm{e}^{-\mathrm{i}H(t_f-t_i)}|\alpha_i\rangle &= \lim_{N\to\infty}\prod_{j=1}^{N-1}\int\frac{\mathrm{d}\alpha_j}{\pi}\prod_{j=1}^{N}\langle\alpha_j|\mathrm{e}^{-\mathrm{i}\Delta tH}|\alpha_{j-1}\rangle \\
&= \lim_{N\to\infty}\prod_{j=1}^{N-1}\int\frac{\mathrm{d}\alpha_j}{\pi}\prod_{j=1}^{N}\mathrm{e}^{-\mathrm{i}\Delta t\omega\alpha_{j-1}^*\alpha_{j-1}}\langle\alpha_j|\alpha_{j-1}\rangle \\
&= \lim_{N\to\infty}\prod_{j=1}^{N-1}\int\frac{\mathrm{d}\alpha_j}{\pi}\prod_{j=1}^{N}\mathrm{e}^{-\mathrm{i}\Delta t\omega\alpha_{j-1}^*\alpha_{j-1}}\mathrm{e}^{-\frac{1}{2}(|\alpha_j|^2+|\alpha_{j-1}|^2)+\alpha_j^*\alpha_{j-1}},
\end{aligned} \tag{1}$$

where $\Delta\tau=(t_f-t_i)/N$, $\alpha_N=\alpha_f$, and $\alpha_0=\alpha_i$.

2. To continue, we can use the condition that α_j and α_{j-1} is close to each other and make the following derivation:

$$\begin{aligned}
\langle\alpha_f|\mathrm{e}^{-\mathrm{i}H(t_f-t_i)}|\alpha_i\rangle &= \lim_{N\to\infty}\prod_{j=1}^{N-1}\int\frac{\mathrm{d}\alpha_j}{\pi}\prod_{j=1}^{N}\mathrm{e}^{-\mathrm{i}\Delta t\omega\alpha_{j-1}^*\alpha_{j-1}}\mathrm{e}^{-\alpha_j^*\alpha_j+\alpha_j^*\alpha_{j-1}} \\
&= \lim_{N\to\infty}\prod_{j=1}^{N-1}\int\frac{\mathrm{d}\alpha_j}{\pi}\prod_{j=1}^{N}\mathrm{e}^{-\mathrm{i}\Delta t\omega\alpha_{j-1}^*\alpha_{j-1}-\alpha_j^*(\alpha_j-\alpha_{j-1})} \\
&= \lim_{N\to\infty}\prod_{j=1}^{N-1}\int\frac{\mathrm{d}\alpha_j}{\pi}\prod_{j=1}^{N}\mathrm{e}^{\mathrm{i}\Delta t(\mathrm{i}\alpha_j^*(\alpha_j-\alpha_{j-1})/\Delta t-\omega\alpha_{j-1}^*\alpha_{j-1})} \\
&= \lim_{N\to\infty}\left(\prod_{j=1}^{N-1}\int\frac{\mathrm{d}\alpha_j}{\pi}\right)\mathrm{e}^{\mathrm{i}\sum_j\Delta t(\mathrm{i}\alpha_j^*(\alpha_j-\alpha_{j-1})/\Delta t-\omega\alpha_{j-1}^*\alpha_{j-1})},
\end{aligned}$$

so after taking the continuous limit, we get

$$\langle\alpha_f|\mathrm{e}^{-\mathrm{i}H(t_f-t_i)}|\alpha_i\rangle = \int\mathcal{D}\alpha\,\mathrm{e}^{\mathrm{i}\int_{t_i}^{t_f}\mathrm{d}t(\mathrm{i}\alpha^*\dot{\alpha}-\omega|\alpha|^2)}. \tag{2}$$

So the Lagrangian is

$$L=\mathrm{i}\alpha^*\dot{\alpha}-\omega|\alpha|^2. \tag{3}$$

3. By putting

$$\alpha=\sqrt{\frac{m\omega}{2}}(x+\mathrm{i}p/m\omega) \tag{4}$$

into (3), we get

$$\begin{aligned}
L &= \mathrm{i}\frac{m\omega}{2}\left(x-\frac{\mathrm{i}p}{m\omega}\right)\left(\dot{x}+\frac{\mathrm{i}\dot{p}}{m\omega}\right)-\omega\frac{m\omega}{2}\left(x-\frac{\mathrm{i}p}{m\omega}\right)\left(x+\frac{\mathrm{i}p}{m\omega}\right) \\
&= \mathrm{i}\frac{m\omega}{2}\left(x-\frac{\mathrm{i}p}{m\omega}\right)\left(\dot{x}+\frac{\mathrm{i}\dot{p}}{m\omega}\right)-\frac{p^2}{2m}-\frac{1}{2}m\omega^2x^2.
\end{aligned}$$

By integration by parts, we have

$$\dot{p}\left(x-\frac{\mathrm{i}p}{m\omega}\right)=\frac{\mathrm{d}}{\mathrm{d}t}\left(px-\frac{\mathrm{i}p^2}{m\omega}\right)-p\left(\dot{x}-\frac{\mathrm{i}\dot{p}}{m\omega}\right),$$

and thus
$$
\begin{aligned}
&\mathrm{i}\frac{m\omega}{2}\left(x-\frac{\mathrm{i}p}{m\omega}\right)\left(\dot{x}+\frac{\mathrm{i}\dot{p}}{m\omega}\right)\\
&=\mathrm{i}\frac{m\omega}{2}\left(x\dot{x}-\frac{\mathrm{i}}{m\omega}p\dot{x}+\frac{\mathrm{i}}{m\omega}\left(\frac{\mathrm{d}}{\mathrm{d}t}\left(px-\frac{\mathrm{i}p^2}{m\omega}\right)-p\left(\dot{x}-\frac{\mathrm{i}\dot{p}}{m\omega}\right)\right)\right)\\
&=\frac{\mathrm{i}m\omega}{2}\left(-\frac{2\mathrm{i}}{m\omega}p\dot{x}+\frac{\mathrm{d}}{\mathrm{d}t}\left(x^2-\frac{p^2}{m^2\omega^2}\right)\right)\\
&=p\dot{x}+\text{total time derivative.}
\end{aligned}
$$
So we have
$$
L=p\dot{x}-\frac{p^2}{2m}-\frac{1}{2}m\omega^2x^2+\text{total time derivative.} \tag{5}
$$

Problem 2
Solution

1. The discrete path integral is

$$
\begin{aligned}
\langle \boldsymbol{x}_f|\mathrm{e}^{-\mathrm{i}Ht}|\boldsymbol{x}_i\rangle &= \lim_{N\to\infty}\prod_{j=1}^{N-1}\int \mathrm{d}^3\boldsymbol{x}_j\prod_{j=1}^{N}\langle \boldsymbol{x}_j|\mathrm{e}^{-\mathrm{i}\Delta tH}|\boldsymbol{x}_{j-1}\rangle\\
&=\lim_{N\to\infty}\prod_{j=1}^{N-1}\int \mathrm{d}^3\boldsymbol{x}_j\prod_{j=1}^{N}\langle \boldsymbol{x}_j|\mathrm{e}^{-\mathrm{i}\Delta t(\hat{\boldsymbol{p}}^2-\hat{\boldsymbol{p}}\cdot\boldsymbol{A}(\boldsymbol{x}_{j-1})-\boldsymbol{A}(\boldsymbol{x}_j)\cdot\hat{\boldsymbol{p}}+\boldsymbol{A}(\boldsymbol{x}_{j-1})^2)/2m}|\boldsymbol{x}_{j-1}\rangle\,.
\end{aligned}
$$

Now we introduce a $\boldsymbol{p}$ variable to eliminate the momentum operator:

$$
\begin{aligned}
&\langle \boldsymbol{x}_j|\mathrm{e}^{-\mathrm{i}\Delta t(\hat{\boldsymbol{p}}^2-\hat{\boldsymbol{p}}\cdot\boldsymbol{A}(\boldsymbol{x}_{j-1})-\boldsymbol{A}(\boldsymbol{x}_j)\cdot\hat{\boldsymbol{p}}+\boldsymbol{A}(\boldsymbol{x}_{j-1})^2)/2m}|\boldsymbol{x}_{j-1}\rangle\\
&=\int \mathrm{d}^3\boldsymbol{p}\,\langle \boldsymbol{x}_j|\boldsymbol{p}\rangle\,\langle \boldsymbol{p}|\mathrm{e}^{-\mathrm{i}\Delta t(\hat{\boldsymbol{p}}^2-\hat{\boldsymbol{p}}\cdot\boldsymbol{A}(\boldsymbol{x}_{j-1})-\boldsymbol{A}(\boldsymbol{x}_j)\cdot\hat{\boldsymbol{p}}+\boldsymbol{A}(\boldsymbol{x}_{j-1})^2)/2m}|\boldsymbol{x}_{j-1}\rangle\\
&=\int\frac{\mathrm{d}^3\boldsymbol{p}}{(2\pi)^3}\mathrm{e}^{\mathrm{i}\boldsymbol{p}\cdot\boldsymbol{x}_j}\mathrm{e}^{-\mathrm{i}\Delta t(\boldsymbol{p}^2-\boldsymbol{p}\cdot\boldsymbol{A}(\boldsymbol{x}_{j-1})-\boldsymbol{A}(\boldsymbol{x}_j)\cdot\boldsymbol{p}+\boldsymbol{A}(\boldsymbol{x}_{j-1})^2)/2m}\mathrm{e}^{-\mathrm{i}\boldsymbol{p}\cdot\boldsymbol{x}_{j-1}}\\
&=\mathrm{e}^{-\mathrm{i}\Delta t\boldsymbol{A}(\boldsymbol{x}_{j-1})^2/2m}\int\frac{\mathrm{d}^3\boldsymbol{p}}{(2\pi)^3}\mathrm{e}^{-\frac{1}{2}\frac{\mathrm{i}\Delta t}{m}\boldsymbol{p}^2}\mathrm{e}^{\mathrm{i}\boldsymbol{p}\cdot\left(\boldsymbol{x}_j-\boldsymbol{x}_{j-1}+\frac{\Delta t}{2m}(\boldsymbol{A}(\boldsymbol{x}_j)+\boldsymbol{A}(\boldsymbol{x}_{j-1}))\right)}\\
&=\mathrm{e}^{-\mathrm{i}\Delta t\boldsymbol{A}(\boldsymbol{x}_{j-1})^2/2m}\frac{1}{(2\pi)^3}\sqrt{\frac{(2\pi)^3}{(\mathrm{i}\Delta t/m)^3}}\mathrm{e}^{-\frac{1}{2}\frac{m}{\mathrm{i}\Delta t}\left(\boldsymbol{x}_j-\boldsymbol{x}_{j-1}+\frac{\Delta t}{2m}(\boldsymbol{A}(\boldsymbol{x}_j)+\boldsymbol{A}(\boldsymbol{x}_{j-1}))\right)^2}\\
&\approx\sqrt{\frac{-\mathrm{i}m^3}{(2\pi)^3\Delta t^3}}\mathrm{e}^{\mathrm{i}\Delta t\frac{m}{2}\left(\frac{(\boldsymbol{x}_j-\boldsymbol{x}_{j-1})^2}{\Delta t^2}+\frac{\boldsymbol{x}_j-\boldsymbol{x}_{j-1}}{\Delta t}\cdot\frac{\boldsymbol{A}(\boldsymbol{x}_j)+\boldsymbol{A}(\boldsymbol{x}_{j-1})}{2}\right)}.
\end{aligned}
$$

Here in the last line we make the approximation that $\boldsymbol{A}(\boldsymbol{x}_j)$ and $\boldsymbol{A}(\boldsymbol{x}_{j-1})$ are close to each other, so the two $\boldsymbol{A}^2$ terms cancel with each other. So the final discrete path integral is

$$
\begin{aligned}
\langle \boldsymbol{x}_f|\mathrm{e}^{-\mathrm{i}Ht}|\boldsymbol{x}_i\rangle =& \lim_{N\to\infty}\prod_{j=1}^{N-1}\left(\frac{-\mathrm{i}m^3}{(2\pi)^3\Delta t^3}\right)^{N/2}\int \mathrm{d}^3\boldsymbol{x}_j\\
&\cdot\mathrm{e}^{\sum_{j=1}^{N}\mathrm{i}\Delta t\left(\frac{m}{2}\frac{(\boldsymbol{x}_j-\boldsymbol{x}_{j-1})^2}{\Delta t^2}+\frac{\boldsymbol{x}_j-\boldsymbol{x}_{j-1}}{\Delta t}\cdot\frac{\boldsymbol{A}(\boldsymbol{x}_j)+\boldsymbol{A}(\boldsymbol{x}_{j-1})}{2}\right)}.
\end{aligned}\tag{6}
$$

2. The derivation is largely the same, but now in each time stp, the $-2\hat{\boldsymbol{p}}\cdot\boldsymbol{A}(\boldsymbol{x})$ term results in $-2\hat{\boldsymbol{p}}\cdot\boldsymbol{A}(\boldsymbol{x}_{j-1})$, and the result is

$$
\begin{aligned}
\langle \boldsymbol{x}_f|\mathrm{e}^{-\mathrm{i}Ht}|\boldsymbol{x}_i\rangle =& \lim_{N\to\infty}\prod_{j=1}^{N-1}\left(\frac{-\mathrm{i}m^3}{(2\pi)^3\Delta t^3}\right)^{N/2}\int \mathrm{d}^3\boldsymbol{x}_j\\
&\cdot\mathrm{e}^{\sum_{j=1}^{N}\mathrm{i}\Delta t\left(\frac{m}{2}\frac{(\boldsymbol{x}_j-\boldsymbol{x}_{j-1})^2}{\Delta t^2}+\frac{\boldsymbol{x}_j-\boldsymbol{x}_{j-1}}{\Delta t}\cdot\boldsymbol{A}(\boldsymbol{x}_{j-1})+\frac{\mathrm{i}}{2m}\boldsymbol{\nabla}\cdot\boldsymbol{A}(\boldsymbol{x}_{j-1})\right)}.
\end{aligned}\tag{7}
$$

3. We make the following replacements:

$$
\frac{(\boldsymbol{x}_j-\boldsymbol{x}_{j-1})^2}{\Delta t^2}\longrightarrow\dot{\boldsymbol{x}}^2,\quad \sum_j\Delta t=\int\mathrm{d}t\,,
$$

and from (6) we get

$$\langle \boldsymbol{x}_f|\mathrm{e}^{-\mathrm{i}Ht}|\boldsymbol{x}_i\rangle = \int \mathcal{D}\boldsymbol{x}\mathrm{e}^{\mathrm{i}\int_{t_i}^{t_f}\mathrm{d}t\left(\frac{1}{2}m\dot{\boldsymbol{x}}^2+\dot{\boldsymbol{x}}\cdot\boldsymbol{A}\right)}, \tag{8}$$

and from (7) we get

$$\langle \boldsymbol{x}_f|\mathrm{e}^{-\mathrm{i}Ht}|\boldsymbol{x}_i\rangle = \int \mathcal{D}\boldsymbol{x}\mathrm{e}^{\mathrm{i}\int_{t_i}^{t_f}\mathrm{d}t\left(\frac{1}{2}m\dot{\boldsymbol{x}}^2+\dot{\boldsymbol{x}}\cdot\boldsymbol{A}+\frac{\mathrm{i}}{2m}\boldsymbol{\nabla}\cdot\boldsymbol{A}\right)}. \tag{9}$$

So for the first path integral the Lagrangian is

$$L = \frac{1}{2}m\dot{\boldsymbol{x}}^2 + \dot{\boldsymbol{x}}\cdot\boldsymbol{A}, \tag{10}$$

while for the second, it is

$$L = \frac{1}{2}m\dot{\boldsymbol{x}}^2 + \dot{\boldsymbol{x}}\cdot\boldsymbol{A} + \frac{\mathrm{i}}{2m}\boldsymbol{\nabla}\cdot\boldsymbol{A}. \tag{11}$$

4.

Problem 3
Solution
Problem 4
Solution

1. We do the Trotter decomposition again:

$$\langle \boldsymbol{k}_f|\mathrm{e}^{-\mathrm{i}Ht}|\boldsymbol{k}_i\rangle = \lim_{N\to\infty}\prod_{j=1}^{N-1}\int\frac{V}{(2\pi)^3}\mathrm{d}^3\boldsymbol{k}_j\cdot\prod_{j=1}^{N}\langle\boldsymbol{k}_j|\mathrm{e}^{-\mathrm{i}\Delta tH}|\boldsymbol{k}_{j-1}\rangle\,,\quad \boldsymbol{k}_0=\boldsymbol{k}_i,\quad \boldsymbol{k}_N=\boldsymbol{k}_f.$$

Each time step is given by

$$\begin{aligned}
&\langle\boldsymbol{k}_j|\mathrm{e}^{-\mathrm{i}\Delta t(H_0+\hat{\boldsymbol{x}}^2/2\alpha-\boldsymbol{E}\cdot\hat{\boldsymbol{x}})}|\boldsymbol{k}_{j-1}\rangle\\
&=\langle\boldsymbol{k}_j|\mathrm{e}^{-\mathrm{i}\Delta t(\hat{\boldsymbol{x}}^2/2\alpha-\boldsymbol{E}\cdot\hat{\boldsymbol{x}})}|\boldsymbol{k}_{j-1}\rangle\,\mathrm{e}^{-\Delta t\epsilon_{\boldsymbol{k}_{j-1}}}\\
&=\mathrm{e}^{-\Delta t\epsilon_{\boldsymbol{k}_{j-1}}}\int\mathrm{d}^3\boldsymbol{r}\,u^*_{\boldsymbol{k}_j}(\boldsymbol{r})\mathrm{e}^{-\mathrm{i}\boldsymbol{k}_j\cdot\boldsymbol{r}}\mathrm{e}^{-\mathrm{i}\Delta t(\boldsymbol{r}^2/2\alpha-\boldsymbol{E}\cdot\boldsymbol{r})}u_{\boldsymbol{k}_{j-1}}(\boldsymbol{r})\mathrm{e}^{\mathrm{i}\boldsymbol{k}_{j-1}\cdot\boldsymbol{r}}\\
&=\mathrm{e}^{-\Delta t\epsilon_{\boldsymbol{k}_{j-1}}}\int\mathrm{d}^3\boldsymbol{r}\,u^*_{\boldsymbol{k}_j}(\boldsymbol{r})u_{\boldsymbol{k}_{j-1}}(\boldsymbol{r})\mathrm{e}^{-\frac{1}{2}\frac{\mathrm{i}\Delta t}{\alpha}\boldsymbol{r}^2+\mathrm{i}\boldsymbol{r}\cdot(\Delta t\boldsymbol{E}+\boldsymbol{k}_{j-1}-\boldsymbol{k}_j)}.
\end{aligned}$$

The semi-classical dynamics only works when $\psi_{\boldsymbol{k}}(\boldsymbol{r})$ is "concentrated" enough in the reciprocal space, which means $u_{\boldsymbol{k}}(\boldsymbol{r})$ should be very smooth compared with $\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}}$ (or otherwise the picture of an electron with a certain momentum traveling in the material is simply wrong). Thus, we have

$$\begin{aligned}
&\int\mathrm{d}^3\boldsymbol{r}\,u^*_{\boldsymbol{k}_j}(\boldsymbol{r})u_{\boldsymbol{k}_{j-1}}(\boldsymbol{r})\mathrm{e}^{-\frac{1}{2}\frac{\mathrm{i}\Delta t}{\alpha}\boldsymbol{r}^2+\mathrm{i}\boldsymbol{r}\cdot(\Delta t\boldsymbol{E}+\boldsymbol{k}_{j-1}-\boldsymbol{k}_j)}\\
&=\frac{1}{V_{\mathrm{u.c.}}}\int_{\mathrm{u.c.}}\mathrm{d}^3\boldsymbol{r}\,u^*_{\boldsymbol{k}_j}(\boldsymbol{r})u_{\boldsymbol{k}_{j-1}}(\boldsymbol{r})\int\mathrm{d}^3\boldsymbol{r}\,\mathrm{e}^{-\frac{1}{2}\frac{\mathrm{i}\Delta t}{\alpha}\boldsymbol{r}^2+\mathrm{i}\boldsymbol{r}\cdot(\Delta t\boldsymbol{E}+\boldsymbol{k}_{j-1}-\boldsymbol{k}_j)},
\end{aligned}$$

and the Gaussian integral on the RHS can be evaluated as

$$\begin{aligned}
&\int\mathrm{d}^3\boldsymbol{r}\,\mathrm{e}^{-\frac{1}{2}\frac{\mathrm{i}\Delta t}{\alpha}\boldsymbol{r}^2+\mathrm{i}\boldsymbol{r}\cdot(\Delta t\boldsymbol{E}+\boldsymbol{k}_{j-1}-\boldsymbol{k}_j)}\\
&=\sqrt{\frac{(2\pi)^3}{(\mathrm{i}\Delta t/\alpha)^3}}\mathrm{e}^{\frac{1}{2}\frac{\alpha}{\mathrm{i}\Delta t}(\mathrm{i}(\Delta t\boldsymbol{E}+\boldsymbol{k}_{j-1}-\boldsymbol{k}_j))^2}\\
&=\sqrt{\frac{(2\pi)^3}{(\mathrm{i}\Delta t/\alpha)^3}}\mathrm{e}^{\frac{\mathrm{i}\alpha}{2}(\boldsymbol{E}-\dot{\boldsymbol{k}})^2\Delta t}.
\end{aligned}$$

Thus

$$\begin{aligned}
\langle \boldsymbol{k}_f|\mathrm{e}^{-\mathrm{i}Ht}|\boldsymbol{k}_i\rangle &= \lim_{N\to\infty} \mathcal{N} \prod_{j=1}^{N-1} \int \mathrm{d}^3\boldsymbol{k}_j\, \mathrm{e}^{-\mathrm{i}\Delta t \epsilon_{\boldsymbol{k}_{j-1}}} \mathrm{e}^{\frac{\mathrm{i}\Delta t\alpha}{2}(\dot{\boldsymbol{k}}-\boldsymbol{E})^2} \langle u_{\boldsymbol{k}_j}|u_{\boldsymbol{k}_{j-1}}\rangle \\
&= \lim_{N\to\infty} \mathcal{N} \prod_{j=1}^{N-1} \int \mathrm{d}^3\boldsymbol{k}_j\, \mathrm{e}^{-\mathrm{i}\Delta t \epsilon_{\boldsymbol{k}_{j-1}}} \mathrm{e}^{\frac{\mathrm{i}\Delta t\alpha}{2}(\dot{\boldsymbol{k}}-\boldsymbol{E})^2} (1 - \Delta t \dot{\boldsymbol{k}} \cdot \langle u_{\boldsymbol{k}_j}|\boldsymbol{\nabla}_{\boldsymbol{k}_j}|u_{\boldsymbol{k}_j}\rangle) \\
&= \lim_{N\to\infty} \mathcal{N} \prod_{j=1}^{N-1} \int \mathrm{d}^3\boldsymbol{k}_j\, \mathrm{e}^{-\mathrm{i}\Delta t \epsilon_{\boldsymbol{k}_{j-1}} + \frac{\mathrm{i}\Delta t\alpha}{2}(\dot{\boldsymbol{k}}-\boldsymbol{E})^2 - \Delta t \dot{\boldsymbol{k}} \cdot \langle u_{\boldsymbol{k}_j}|\boldsymbol{\nabla}_{\boldsymbol{k}_j}|u_{\boldsymbol{k}_j}\rangle} \\
&= \lim_{N\to\infty} \left(\mathcal{N} \prod_{j=1}^{N-1} \int \mathrm{d}^3\boldsymbol{k}_j \right) \mathrm{e}^{\mathrm{i}\Delta t (\sum_j - \epsilon_{\boldsymbol{k}_{j-1}} + \frac{\alpha}{2}(\dot{\boldsymbol{k}}-\boldsymbol{E})^2 + \mathrm{i}\dot{\boldsymbol{k}} \cdot \langle u_{\boldsymbol{k}_j}|\boldsymbol{\nabla}_{\boldsymbol{k}_j}|u_{\boldsymbol{k}_j}\rangle)}.
\end{aligned}$$

Putting all normalization factors into the measure, we get

$$\begin{aligned}
\langle \boldsymbol{k}_f|\mathrm{e}^{-\mathrm{i}Ht}|\boldsymbol{k}_i\rangle &= \int \mathcal{D}\boldsymbol{k} \mathrm{e}^{\mathrm{i}\int_i^f \mathrm{d}t L_{\text{eff}}}, \\
L_{\text{eff}} &= \dot{\boldsymbol{k}} \cdot \boldsymbol{\mathcal{A}} + \frac{\alpha}{2}(\dot{\boldsymbol{k}} - \boldsymbol{E})^2 - \epsilon_{\boldsymbol{k}}.
\end{aligned} \tag{12}$$

So we find

$$\mathcal{F}(\boldsymbol{k}, \dot{\boldsymbol{k}}) = \frac{\alpha}{2}(\dot{\boldsymbol{k}} - \boldsymbol{E})^2 - \epsilon_{\boldsymbol{k}}. \tag{13}$$

2. We have

$$\boldsymbol{\pi} = \frac{\partial L_{\text{eff}}}{\partial \dot{\boldsymbol{k}}} = \boldsymbol{\mathcal{A}} + \alpha(\dot{\boldsymbol{k}} - \boldsymbol{E}). \tag{14}$$

So

$$\begin{aligned}
H_{\text{eff}} &= \dot{\boldsymbol{k}} \cdot \boldsymbol{\pi} - L_{\text{eff}} \\
&= \frac{1}{2}\alpha \dot{\boldsymbol{k}}^2 - \frac{1}{2}\alpha \boldsymbol{E}^2 + \epsilon_{\boldsymbol{k}}.
\end{aligned}$$

Replacing $\dot{\boldsymbol{k}}$ by $\boldsymbol{\pi}$, we get

$$\begin{aligned}
H_{\text{eff}} &= \frac{1}{2}\alpha \left(\frac{\boldsymbol{\pi} - \boldsymbol{\mathcal{A}}}{\alpha} + \boldsymbol{E} \right)^2 - \frac{1}{2}\alpha \boldsymbol{E}^2 + \epsilon_{\boldsymbol{k}} \\
&= \frac{(\boldsymbol{\pi} - \boldsymbol{\mathcal{A}})^2}{2\alpha} + (\boldsymbol{\pi} - \boldsymbol{\mathcal{A}}) \cdot \boldsymbol{E} + \epsilon_{\boldsymbol{k}}.
\end{aligned}$$

So the answer is

$$H_{\text{eff}} = \frac{(\boldsymbol{\pi} - \boldsymbol{\mathcal{A}})^2}{2\alpha} + (\boldsymbol{\pi} - \boldsymbol{\mathcal{A}}) \cdot \boldsymbol{E} + \epsilon_{\boldsymbol{k}}. \tag{15}$$

3. We can interpret $-\boldsymbol{\pi}$ as some sort of "position", because

$$[\boldsymbol{k}, \boldsymbol{\pi}] = 1 \Leftrightarrow [-\boldsymbol{\pi}, \boldsymbol{k}] = 1,$$

so we replace $\boldsymbol{\pi}$ by $-\boldsymbol{x}$, and thus in the $\alpha \to \infty$ limit, we have

$$H_{\text{eff}} = -(\boldsymbol{x} + \boldsymbol{\mathcal{A}}) \cdot \boldsymbol{E} + \epsilon_{\boldsymbol{k}}. \tag{16}$$

4. We have

$$\dot{\boldsymbol{x}} = \frac{\partial H_{\text{eff}}}{\partial \boldsymbol{k}} = \boldsymbol{\nabla}_{\boldsymbol{k}} \epsilon_{\boldsymbol{k}} - \boldsymbol{\nabla}_{\boldsymbol{k}} (\boldsymbol{E} \cdot \boldsymbol{\mathcal{A}}).$$

By vector analysis formula, and by the condition that $\boldsymbol{E}$ is a constant, we have

$$\boldsymbol{\nabla}_{\boldsymbol{k}} (\boldsymbol{E} \cdot \boldsymbol{A}) = \boldsymbol{E} \times (\boldsymbol{\nabla}_{\boldsymbol{k}} \times \boldsymbol{\mathcal{A}}),$$

so finally we get

$$\dot{\boldsymbol{x}} = \boldsymbol{\nabla}_{\boldsymbol{k}} \epsilon_{\boldsymbol{k}} - \boldsymbol{E} \times \boldsymbol{\Omega}, \tag{17}$$

where

$$\boldsymbol{\Omega} = \boldsymbol{\nabla}_{\boldsymbol{k}} \times \boldsymbol{\mathcal{A}}. \tag{18}$$

Also

$$\dot{\boldsymbol{k}} = -\frac{\partial H_{\text{eff}}}{\partial \boldsymbol{x}} = \boldsymbol{E}. \tag{19}$$

5. From (17) and (19) we have[1]

$$\dot{\boldsymbol{x}} = -\boldsymbol{E} \times \boldsymbol{\Omega} = -\dot{\boldsymbol{k}} \times \boldsymbol{\Omega},$$

and therefore

$$\boldsymbol{\Omega} \times \dot{\boldsymbol{x}} = -\Omega^2 \dot{\boldsymbol{k}} + (\boldsymbol{\Omega} \cdot \dot{\boldsymbol{k}})\boldsymbol{\Omega}. \tag{20}$$

On the other hand, the classical EOM is (here $e = 1$)

$$\dot{\boldsymbol{p}} = -\dot{\boldsymbol{x}} \times \boldsymbol{B}. \tag{21}$$

So

$$\boldsymbol{\Omega} = \Omega_0 \hat{\boldsymbol{z}}, \quad \Omega_0 = \frac{1}{B}. \tag{22}$$

6. The size of the first Brillouin zone is

$$\frac{(2\pi)^2}{2\pi/B} = 2\pi B.$$

So

$$2\pi C = \int \mathrm{d}^2\boldsymbol{k}\, \Omega = 2\pi B \cdot \Omega_0 = 2\pi,$$

and thus the Chern number of the Landau level is 1.

[1] Here we assume there is a very weak electric field $\boldsymbol{E}$, so we can put (19) and (17) into one equation, and then we let $\boldsymbol{E} \to 0$.