ODEs

Jinyuan Wu

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1 First order ODEs

1.1 Linear ODEs

An ODE in the form of

$$y'(x) + p(x)y(x) = q(x) \tag{1}$$

is considered **linear**. All linear ODEs can be solved by the following procedure. First we have

$$(y' + py)\mathrm{e}^{\int p\mathrm{d}x} = q\mathrm{e}^{\int p\mathrm{d}x}, \tag{2}$$

and now the LHS is a derivative:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(y\mathrm{e}^{\int p\mathrm{d}x}\right) = q\mathrm{e}^{\int p\mathrm{d}x}, \tag{3}$$

and now we can integrate over x and get

$$y\mathrm{e}^{\int p\mathrm{d}x} = \int q\mathrm{e}^{\int p\mathrm{d}x}\,\mathrm{d}x\,, \tag{4}$$

$$y = \mathrm{e}^{-\int p\mathrm{d}x}\int q\mathrm{e}^{\int p\mathrm{d}x}\,\mathrm{d}x\,. \tag{5}$$

1.2 "Energy-conservation lines" and exact equations

Another way to represent the solution of an ODE is the form $\phi(x,y) = \text{const}$. Note that the RHS contains no variables, and we have

$$0 = \frac{\mathrm{d}\phi}{\mathrm{d}x} = \frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}x}, \tag{6}$$

and thus if

$$y' = f(x,y) \tag{7}$$

is algebraically equivalent to (6), the equation is already solved: We should find M, N such that

$$y' = -\frac{M}{N}, \quad M = \frac{\partial\phi}{\partial x}, \quad N = \frac{\partial\phi}{\partial y}, \tag{8}$$

and then $\phi(x,y)$ solves the equation. In this case we say $y' = -M/N$ is **exact**.

To test for exactness, we only have to test whether

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}, \tag{9}$$

and if so, the existence of ϕ is guaranteed. (Since we work on a topological trivial space, things like cohomology group will not bother us.) We can now use "partial integral" to find ϕ.

Example: suppose in a calculation we find

$$\frac{\partial\phi}{\partial x} = 2y^2 + y\mathrm{e}^{xy}, \quad \frac{\partial\phi}{\partial y} = 4xy + x\mathrm{e}^{xy} + 2y. \tag{10}$$

After partial integration, we find

$$\phi(x,y) = \underbrace{2xy^2 + \mathrm{e}^{xy} + h(y)}_{\int\frac{\partial\phi}{\partial x}\mathrm{d}x} = \underbrace{2xy^2 + \mathrm{e}^{xy} + y^2 + g(x)}_{\int\frac{\partial\phi}{\partial y}\mathrm{d}y}, \tag{11}$$

and we have to choose

$$h(y) = y^2, \quad g(x) = \text{const}, \tag{12}$$

and the solution is

$$\phi(x, y) = 2xy^2 + \mathrm{e}^{xy} + y^2 + \text{const}. \tag{13}$$

Note that even when the decomposition $f = -M/N$ doesn't give an exact equation for us, we can still use the method of exact equations: we can multiply a factor μ to both M and N, and try to guess the form of μ so that

$$\frac{\partial(\mu M)}{\partial y} = \frac{\partial(\mu N)}{\partial x}. \tag{14}$$

An example can be found in solving

$$y' = -\frac{1}{3x - \mathrm{e}^{-2y}}. \tag{15}$$

We have

$$\frac{\partial 1}{\partial y} = 0, \quad \frac{\partial(3x - \mathrm{e}^{-2y})}{\partial x} = 3,$$

so the equation is not exact if we choose $M = 1$ and $N = 3x - \mathrm{e}^{-2y}$. However, (14) can be fulfilled now: it's now

$$\frac{\partial \mu}{\partial y} = 3\mu + \left(3x - \mathrm{e}^{-2y}\right) \frac{\partial \mu}{\partial x},$$

and the most convenient way to solve it (we *don't* need to find all solutions of this equation!) is to let μ contain y only, so the tricky term on the RHS disappears, and thus we choose $\mu = \mathrm{e}^{3y}$, and we get

$$\phi(x, y) = \int \mu M \,\mathrm{d}x = \int \mathrm{e}^{3y} \,\mathrm{d}x = x\mathrm{e}^{3y} + u(y),$$

$$\phi(x, y) = \int \mu N \,\mathrm{d}y = \int (3x\mathrm{e}^{3y} - \mathrm{e}^{y}) \,\mathrm{d}y = x\mathrm{e}^{3y} - \mathrm{e}^{y} + v(x),$$

so

$$\phi(x, y) = x\mathrm{e}^{3y} - \mathrm{e}^{y} + \text{const}. \tag{16}$$

1.3 Bernoulli equation

Consider the following **Bernoulli equation**

$$y' + P(x)y = R(x)y^{\alpha}. \tag{17}$$

When $\alpha = 0, 1$, the equation can be solved by the standard methods for linear first order ODEs. When this is not the case, we may do the substitution

$$v = y^{\beta}, \tag{18}$$

and then the equation becomes

$$\frac{1}{\beta} v^{1/\beta - 1} v' + P(x) v^{1/\beta} = R(x) v^{\alpha/\beta},$$

$$v' + P(x)v = R(x) v^{1 + \frac{\alpha - 1}{\beta}}. \tag{19}$$

The next step is to choose a good beta so that the equation gets simplified. We may want to make to exponent to be zero, and this means we should choose

$$\beta = 1 - \alpha, \tag{20}$$

and the ODE is now

$$v' + Pu = R, \tag{21}$$

which can then be solved by the method in Section 1.1.

2 Second order ODEs

2.1 Linear 2nd order ODE with initial values

A linear second order ODE has the following form:

$$y'' + p(x)y' + q(x)y = f(x). \tag{22}$$

It usually comes with initial value conditions

$$y(x_0) = A, \quad y'(x_0) = B. \tag{23}$$

This course is about concrete calculations, but knowing what we are doing makes sense is important. Here is an existence and uniqueness theorem: if $p(x), q(x)$, and $f(x)$ are continuous over an interval I, and $x_0 \in I$, then a unique solution exists for (22) with the initial conditions given above.

Usually, we start by looking at the **homogeneous** second order ODE

$$y'' + p(x)y' + q(x)y = 0. \tag{24}$$

The influence of $f(x)$ can be included as the "response" of the LHS. The full solution of (24) takes the form

$$y = c_1 y_1 + c_2 y_2, \tag{25}$$

where c_1, c_2 are constants to be decided by initial conditions, and y_1 and y_2 are linearly independent solutions of (24). The **Wronskian** is defined as

$$W(x) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}. \tag{26}$$

By checking if it is non-zero at most points, we can find whether y_1 and y_2 are truly linearly independent to each other.

There is a method to arrive at y_2 from y_1: we can always take the ansatz

$$y_2 = y_1 u, \tag{27}$$

and therefore we get

$$(u'' y_1 + 2u' y_1' + u y_1'') + p(u' y_1 + u y_1') + q u y_1 = 0,$$

and the condition that y_1 is a solution to (24) means

$$u'' + \underbrace{\frac{2y_1' + p y_1}{y_1}}_{g(x)} u' = 0, \tag{28}$$

which is essentially a first order ODE, because we can replace u' by v, and then we find

$$\ln v = -\int g(x)\,\mathrm{d}x\,,$$

and

$$u(x) = \int \mathrm{e}^{-\int g(x)\mathrm{d}x}\,\mathrm{d}x\,. \tag{29}$$

2.2 Constant coefficients

The equation

$$y'' + Ay + By = 0 \tag{30}$$

can be solved directly by the following construction:

$$y = c_1 \mathrm{e}^{\lambda_1 x} + c_2 \mathrm{e}^{\lambda_2 x}, \tag{31}$$

where λ_1, λ_2 are solutions of

$$\lambda^2 + A\lambda + B = 0. \tag{32}$$

For example, to solve the equation

$$y'' - 2y' + 10y = 0, \tag{33}$$

we just solve

$$\lambda^2 - 2\lambda + 10 = 0,$$

which gives us

$$\lambda = 1 \pm 3\mathrm{i}, \tag{34}$$

and therefore a general solution is

$$y = \mathrm{e}^x (c_1 \mathrm{e}^{3\mathrm{i}x} + c_2 \mathrm{e}^{-3\mathrm{i}x}). \tag{35}$$

It should be noted that c_1, c_2 can be complex, even when we restrict y in $\mathbb{R}$: we can let the imaginary part of y vanish as long as we impose some constraints over c_1, c_2. If we are determined to work in the real space, two alternative linearly independent solutions can be used:

$$y_1(x) = \mathrm{e}^x \cos(3x), \quad y_2(x) = \mathrm{e}^x \sin(3x). \tag{36}$$

Although we can immediately say they are linearly independent, we can use them as a demonstration of the Wronskian method: now we have

$$W(x) = y_1(x)y_2'(x) - y_2(x)y_1'(x) = 3\mathrm{e}^{2x}, \tag{37}$$

which of course isn't constantly zero.

(32) is faced with the problem of having only one solution when $A^2 - 4B = 0$. In this case we need to go back to the standard procedure to get y_2 from y_1. An example is

$$y'' + 6y + 9 = 0, \tag{38}$$

for which (32) only gives

$$y_1 = \mathrm{e}^{-3x}. \tag{39}$$

Suppose $y_2 = u\mathrm{e}^{-3x}$, we have TODO

2.3 Euler equation

A **Euler equation** has the following form:

$$x^2 y'' + Axy' + By = 0, \tag{40}$$

where A, B are constants. One solution can be immediate found: it always looks like

$$y = x^a. \tag{41}$$

We then find

$$a(a-1) + Aa + B = 0. \tag{42}$$

If there are two solutions of the equation, (40) has already been solved. If not, we can use the trick (27).

An example: let's solve

$$x^2 y'' + 3xy' + y = 0. \tag{43}$$

The equation about a is now

$$a(a-1) + 3a + 1 = 0,$$

and it only has one solution $a = -1$. Therefore we have

$$y_1 = \frac{1}{x}.$$

Suppose

$$y_2 = u y_1,$$

we get

$$x^2\left(\frac{u''}{x^2} - \frac{2u'}{x} + \frac{2u}{x^2}\right) + 3x\left(\frac{u'}{x} - \frac{u}{x^2}\right) + \frac{u}{x} = 0,$$

which is equivalent to

$$v'x + v = 0, \quad v = u'$$

the solution of which is

$$\ln v + \ln x = \mathrm{const},$$

and therefore

$$v = \frac{C'}{x}, \quad u = C' \ln x + C,$$

$$y_2 = \frac{1}{x}(C' \ln x + C).$$

This essentially gives *all* solutions we need: for y_1, we just have $u = 1$, which corresponds to $C = 1$. So now the equation is completely solved.

2.4 Non-homogeneous cases or how to find the linear response

Now we discuss how to solve

$$y'' + p(x)y' + q(x)y = f(x). \tag{44}$$

A general solution is

$$y(x) = y_\mathrm{p}(x) + y_\mathrm{h}(x), \tag{45}$$

where the subscript p means a particular solution, and the subscript h means the general solution of the corresponding homogeneous equation.

We need some common sense to find a particular solution. To solve

$$y'' - y' - 2y = 2x^2 + 5, \tag{46}$$

we don't expect y to be, say, $\cos(2x)$: instead, it's usually the case that y is a polynomial. An ansatz is

$$y = Ax^2 + Bx + C.$$

We don't want a x^3 term because it doesn't appear on RHS. The equation then becomes

$$2A - (2Ax + Bx) - 2(Ax^2 + Bx + C) = 2x^2 + 5,$$

$$-2A = 2, \quad -2A - 2B = 0, \quad 2A - B - 2C = 5,$$

and therefore $A = -1, B = 1, C = -4$. Therefore we get a particular solution:

$$y_\mathrm{p} = -x^2 + x - 4. \tag{47}$$

A particular solution may also be determined as a "linear combination" of homogeneous solutions, although now the coefficients have temporal variation. That's to say, we take the ansatz

$$y_\mathrm{p}(x) = u(x)y_1(x) + v(x)y_2(x). \tag{48}$$

This is quite similar to the procedure introduced in Section 1.1. After substituting y with (48) in (44), we get

$$u'y_1' + v'y_2' = f. \tag{49}$$

Introducing the constraint

$$u'y_1 + v'y_2 = 0, \tag{50}$$

we have

$$\begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix} \begin{pmatrix} u' \\ v' \end{pmatrix} = \begin{pmatrix} 0 \\ f \end{pmatrix}, \tag{51}$$

from which we find u', v' and hence u, v. The Wronskian – the determinant of the matrix on LHS – is non-zero, so the equation always has a solution.

Example: let's solve

$$y'' + y = \tan x. \tag{52}$$

We have

$$y_1 = \cos x, \quad y_2 = \sin x,$$

and therefore

$$W(x) = y_1 y_2' - y_2 y_1' = 1.$$

So

$$\begin{aligned} u' &= -\int \frac{y_2 f}{W(x)} \, \mathrm{d}x = -\int \frac{\sin^2 x}{\cos x} \, \mathrm{d}x = -\int \frac{1 - \cos^2 x}{\cos x} \, \mathrm{d}x \\ &= -\frac{1}{2} \ln \left| \frac{1 + \sin x}{1 - \sin x} \right| + \sin x, \end{aligned}$$

and similarly we have

$$v =$$

2.5 Analyticity

The stimulus can be non-analytic. f is analytic at x_0, if we can expand it into a power series around x_0:

$$f(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n \tag{53}$$

in a interval around x_0. The function $f(x) = \ln x$, then, is not analytic at $x = 0$ – but $f(x) = \ln(x + 1)$ is analytic at $x = 0$, though not at $x = -1$.

There is a theorem: if p, q, f are all analytic at x_0, then (44) together with conditions $y(x_0) = A$ and $y'(x_0) = B$ has a unique solution that is analytic at x_0.

An example:

$$x'' + y' - xy = 0, \quad y(0) = -2, y'(0) = 0. \tag{54}$$

Suppose

$$y(x) = \sum_{n=0}^{\infty} a_n x^n, \quad y'(x) = \sum_{n=0}^{\infty} n a_n x^{n-1}, \quad y''(x) = \sum_{n=0}^{\infty} n(n+1) a_n x^{n-2},$$

we have

$$\begin{aligned} 0 &= \sum_{n=0}^{\infty} n(n-1) a_n x^{n-2} + \sum_{n=0}^{\infty} n a_n x^{n-1} - \sum_{n=0}^{\infty} a_n x^{n+1} \\ &= a_1 + 2a_2 + \sum_{n=1}^{\infty} x^n ((n+2)(n+1) a_{n+2} + (n+1) a_{n+1} - a_{n-1}), \end{aligned}$$

and therefore

$$a_1 + 2a_2 = 0, \quad a_{n+2} = -\frac{a_{n+1}}{n+2} + \frac{a_{n-1}}{(n+1)(n+2)}.$$

The conditions $y(0) = -2$ and $y'(0) = 0$ means

$$a_0 = -2, \quad a_1 = 0,$$

and then we can in principle find all a_n's – although it's often hard to see a pattern and write down a closed-form expression for a_n.

Now we consider the equation

$$P(x) y'' + Q(x) y' + R(x) y = F(x). \tag{55}$$

Of course we can divide the equation with $P(x)$ and go back to (44), but if $P(x)$ is zero at some points, p, q, f in (44) are no longer always analytic. Thus the solution isn't guaranteed to be analytic everywhere. In other words, we no longer have a power series solution – or do we? We can still try a generalized power series, like

$$y(x) = \sum_{n=0}^{\infty} c_n (x - x_0)^{n+r}, \tag{56}$$

where r can be a fraction. This is guaranteed with

$$(x - x_0)y'' + Q(x)y' + R(x)y = F(x). \tag{57}$$

To demonstrate this, consider

$$y'' + \frac{1}{2x}y' - \frac{1}{4x}y = 0. \tag{58}$$

We plug

$$y(x) = \sum_{n=0}^{\infty} c_n x^{n+r}$$

into

$$4xy'' + 2y' - y = 0,$$

and get

$$\sum_{n=0}^{\infty} \left(4(n+r)(n+r-1)c_n x^{n+r-1} + 2(n+r)c_n x^{n+r-1} - c_n x^{n+r}\right) = 0.$$

The coefficient of the x^{r-1} term is

$$4r(r-1) + 2r = 0,$$

from which we find $r = 0, 1/2$. For the rest of the terms, we have

$$4(n+r)(n+r-1)c_n + 2(n+r)c_n - c_{n-1} = 0,$$

$$c_n = \frac{c_{n-1}}{2(n+r)(2n+2r-1)}.$$

When $r = 0$, this gives

$$c_n = \frac{c_{n-1}}{2n(2n-1)}, \quad c_n = \frac{c_0}{(2n)!},$$

and we get

$$y(x) = \sum_{n=0}^{\infty} \frac{x^n}{(2n)!}. \tag{59}$$

When $r = 1/2$, this gives

$$c_n = \frac{c_{n-1}}{(2n+1) \cdot 2n}, \quad c_n = \frac{c_0}{(2n+1)!},$$

and we get

$$y(x) = \sum_{n=0}^{\infty} \frac{x^{n+1/2}}{(2n+1)!}. \tag{60}$$

So we have already obtained two independent solutions.

3 The Laplace transformation

The **Laplace transformation** is defined for a function $f(t)$ as

$$\mathcal{L}[f(t)] = F(s) = \int_0^{\infty} f(t)\mathrm{e}^{-st}\,\mathrm{d}t\,, \tag{61}$$

for every s where the integral converges. It's easy to see that $\mathcal{L}$ is linear,

3.1 Laplace transforms of basic elementary functions

The simplest Laplace transform is

$$\mathcal{L}[1] = \int_0^\infty \mathrm{e}^{-st}\,\mathrm{d}t = \frac{1}{s}, \quad s > 0, \tag{62}$$

and by integration by parts, we also have

$$\mathcal{L}[t] = \int_0^\infty t\mathrm{e}^{-st}\,\mathrm{d}t = \frac{1}{s^2}. \tag{63}$$

By multiple rounds of integration by parts, we have

$$\mathcal{L}[t^2] = \frac{2}{s^3}, \tag{64}$$

and by iteratively using integration by parts we get

$$\mathcal{L}[t^n] = \frac{n!}{s^{n+1}}. \tag{65}$$

Aside from polynomials, we have

$$\mathcal{L}[\mathrm{e}^{at}] = \int_0^\infty \mathrm{e}^{-(s-a)t}\,\mathrm{d}t = \frac{1}{s-a}, \quad s > a. \tag{66}$$

Since we have

$$\cos x = \frac{\mathrm{e}^{\mathrm{i}x} + \mathrm{e}^{-\mathrm{i}x}}{2}, \quad \sin x = \frac{\mathrm{e}^{\mathrm{i}x} - \mathrm{e}^{-\mathrm{i}x}}{2\mathrm{i}},$$

we have

$$\mathcal{L}[\cos(at)] = \frac{1}{2}\left(\frac{1}{s - \mathrm{i}a} + \frac{1}{s + \mathrm{i}a}\right) = \frac{s}{s^2 + a^2}. \tag{67}$$

Similarly we have

$$\mathcal{L}[\sin(at)] = \frac{a}{s^2 + a^2}. \tag{68}$$

Consider a pulse signal

$$f(t) = \begin{cases} 0, & t < a \text{ or } t > b, \\ 1, & a \le t \le b. \end{cases} \tag{69}$$

This can be easily implemented by the Heaviside function: we have

$$f(t) = H(t - a) - H(t - b). \tag{70}$$

So the Laplace transform is

$$\mathcal{L}[f(t)] = \mathrm{e}^{-sa}\mathcal{L}[1] - \mathrm{e}^{-sb}\mathcal{L}[1] = \frac{1}{s}(\mathrm{e}^{-sa} - \mathrm{e}^{-sb}). \tag{71}$$

3.2 Laplace transform of differential equations

We have

$$\begin{aligned} \mathcal{L}[f'(t)] &= \int_0^\infty \frac{\mathrm{d}f}{\mathrm{d}t}\mathrm{e}^{-st}\,\mathrm{d}t = f\mathrm{e}^{-st}\Big|_{s=0}^{\infty} - \int_0^\infty f\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{e}^{-st}\,\mathrm{d}t \\ &= -f(0) + s\int_0^\infty f(t)\mathrm{e}^{-st}\,\mathrm{d}t = s\mathcal{L}[f(t)] - f(0). \end{aligned} \tag{72}$$

Applying this twice, we get

$$\mathcal{L}[f''(t)] = s\mathcal{L}[f'(t)] - f'(0) = s^2\mathcal{L}[f(t)] - sf(0) - f'(0). \tag{73}$$

The general formula is therefore

$$\mathcal{L}[f^{(n)}(t)] = s^n\mathcal{L}[f(t)] - s^{n-1}f(0) - s^{n-2}f'(0) - \cdots - f^{(n-1)}(0). \tag{74}$$

This can be used to solve ODEs. Consider, for example,

$$y'' + 4y' + 3y = \mathrm{e}^t, \quad y(0) = 0, \quad y'(0) = 2. \tag{75}$$

We have (below we follow the convention to use big letters to refer to functions in the Laplace space)

$$\mathcal{L}[\text{LHS}] = (s^2 Y(s) - sy(0) - y'(0)) + 4(sY(s) - y(0)) + 3Y(s) = (s^2 + 4s + 3)Y(s) - 2,$$

where we have already applied the initial conditions, and

$$\mathcal{L}[\text{RHS}] = \frac{1}{s-1}.$$

So what need to be done is to solve

$$(s^2 + 4s + 3)Y(s) - 2 = \frac{1}{s-1},$$

and we get

$$Y(s) = \frac{2s-1}{(s-1)(s+1)(s+3)}. \tag{76}$$

Thus, once we do the inverse Laplace transformation, we get $y(t)$. It's possible to do an inverse integral transformation, but in this case, what's more convenient is to make the decomposition

$$Y(s) = \frac{A}{s-1} + \frac{B}{s+1} + \frac{C}{s+3},$$

and we can find

$$A = \frac{1}{8}, \quad B = \frac{3}{4}, \quad C = -\frac{7}{8}.$$

Then we can read the Laplace transformation table in the inverse direction: from (66), we find

$$y(t) = \frac{1}{8}\mathrm{e}^t + \frac{3}{4}\mathrm{e}^{-t} - \frac{7}{8}\mathrm{e}^{-3t}. \tag{77}$$

So the linear 2nd order ODE is completely solved.

TODO: whether we will encounter things like

$$\mathcal{L}[f] = \frac{1}{s}, \quad s > 100. \tag{78}$$

It seems in most real world problems, we don't really need to worry about the region of allowed s: we just extend the region of allowed s as large as we can, solve the algebraic equation in the Laplace space, and then go back.

We can repeat the procedure in the last example for a system of ODEs. Consider for example

$$x' - 2y' = 1, \quad x' - x + y = 0, \quad x(0) = y(0) = 0. \tag{79}$$

For the first equation, Laplace transform gives

$$sX(s) - x(0) - 2(sY(s) - y(0)) = \mathcal{L}[1] = \frac{1}{s},$$

and for the second equation we have

$$sX(s) - x(0) - X(s) + Y(s) = 0.$$

Solving this linear equation system, we get

$$X(s) = \frac{1}{s^2(2s-1)}, \quad Y(s) = -\frac{s-1}{s^2(2s-1)}.$$

Since

$$X(s) = -\frac{2}{s} - \frac{1}{s^2} + 2\frac{1}{s-1/2},$$

we have

$$x(t) = -2 - t + 2\mathrm{e}^{t/2}. \tag{80}$$

Similarly,

$$y(t) = -1 + \mathrm{e}^{t/2} - t. \tag{81}$$

3.3 Shifting of s and t

Laplace transform of an integral can also be found. Here we investigate into things like

$$\int_0^t f(t')\,\mathrm{d}t',$$

and we need to pay attention to what variable is inside the integration and what variable is exposed to the Laplace operator. We have

$$\begin{aligned}\mathcal{L}\left[\int_0^t f(t')\,\mathrm{d}t'\right] &= -\frac{1}{s}\int_{t'=0}^{\infty}\int_0^t f(t')\,\mathrm{d}t'\,\mathrm{d}\mathrm{e}^{-st} \\ &= -\frac{1}{s}\left(\mathrm{e}^{-st}\int_0^t f(t')\,\mathrm{d}t'\Big|_0^\infty - \int_0^\infty f(t)\mathrm{e}^{-st}\,\mathrm{d}t\right) \\ &= \frac{1}{s}\mathcal{L}[f(t)].\end{aligned} \tag{82}$$

This is the inverse of the rule of derivatives above, which is expected.

Another theorem is the **s-shifting theorem**: we have

$$\mathcal{L}[\mathrm{e}^{at}f(t)] = F(s-a) = \mathcal{L}[f(t)]_{s\to s-a}. \tag{83}$$

This is exactly what leads to (66). Correspondingly we have the t-shifting theorem:

$$\mathcal{L}[f(t-a)H(t-a)] = \mathrm{e}^{-sa}\mathcal{L}[f(t)], \tag{84}$$

where $H(t-a)$ is 1 when $t \geq 0$ and zero otherwise, and $a > 0$.

The theorems can be used to evaluate Laplace transforms of complex functions. We have

$$\mathcal{L}[t^2\mathrm{e}^{-t}] = \mathcal{L}[t^2]|_{s\to s+1} = \frac{2}{(s+1)^3}. \tag{85}$$

3.4 Convolution

The **convolution integral** of $f(t)$ and $g(t)$ is defined as

$$f \otimes g = \int_0^t f(t')g(t-t')\,\mathrm{d}t'. \tag{86}$$

This can be found very frequently in science: it appears when we deal with interaction: for example, the $t-t'$ time may come from indirect interaction (where t' is the time an intermediate step happens).

A way around the reasonable but hard to calculate convolution integral is taking its Laplace transform. We have

$$\begin{aligned}\mathcal{L}[f\otimes g] &= \int_0^\infty \mathrm{e}^{-st}\int_0^t f(t')g(t-t')\,\mathrm{d}t'\,\mathrm{d}t \\ &= \int_0^\infty f(t')\int_{t'}^\infty g(t-t')\mathrm{e}^{-st}\,\mathrm{d}t\,\mathrm{d}t' \\ &= \int_0^\infty f(t')\int_0^\infty g(t'')\mathrm{e}^{-s(t''+t')}\,\mathrm{d}t''\,\mathrm{d}t' \\ &= \int_0^\infty f(t')\mathrm{e}^{-st'}\int_0^\infty g(t'')\mathrm{e}^{-st''}\,\mathrm{d}t'' \\ &= \mathcal{L}[f]\mathcal{L}[g].\end{aligned} \tag{87}$$

The second line uses another way to see the integration region: by saying $0 < t < \infty$, $0 < t' < t$, we also mean $0 < t' < \infty$, $t' < t < \infty$. The third line replaces $t - t'$ with t''. The final result no longer contains convolution.

One application of this fact is shown in the following example. Consider

$$\frac{1}{s^2-a^2} = \underbrace{\frac{1}{s-a}}_{G(s)}\cdot\underbrace{\frac{1}{s+a}}_{F(s)}.$$

What's its inverse Laplace transform? By shifting theorem, we have

$$f(t) = \mathrm{e}^{-at}, \quad g(t) = \mathrm{e}^{at},$$

and therefore

$$\mathcal{L}^{-1}\left[\frac{1}{s^2 - a^2}\right] = \int_0^t \mathrm{e}^{-at'}\mathrm{e}^{a(t-t')}\,\mathrm{d}t' = \mathrm{e}^{at}\int_0^t \mathrm{e}^{-2at'}\,\mathrm{d}t' = \frac{1}{a}\sinh(at). \tag{88}$$

We can also use the convolution theorem to give solutions to very generic equations. Consider the following ODE problem:

$$y'' - 5y' + 6y = f(t), \quad f(0) = f'(0) = 0. \tag{89}$$

The point here is we *don't* know what is $f(t)$, but still want to give a template of the solution. So we just do Laplace transform:

$$(s^2 - 5s + 6)Y(s) = F(s).$$

To find the Laplace transform of $1/(s^2 - 5s + 6)$, we have

$$\mathcal{L}^{-1}\left[\frac{1}{(s^2 - 5s + 6)}\right] = \mathcal{L}^{-1}\left[\frac{1}{s-3} - \frac{1}{s-2}\right] = \mathrm{e}^{3t} - \mathrm{e}^{2t},$$

and

$$\begin{aligned} y(t) = \mathcal{L}^{-1}[Y(s)] = \mathcal{L}^{-1}&\left[\frac{1}{(s^2 - 5s + 6)}F(s)\right] \\ = \mathcal{L}^{-1}&\left[\frac{1}{(s^2 - 5s + 6)}\right] \otimes f(t) \\ = \int_0^t& f(t)(\mathrm{e}^{3(t-t')} - \mathrm{e}^{2(t-t')})\,\mathrm{d}t'\,. \end{aligned} \tag{90}$$

4 Fourier analysis

4.1 Definition of Fourier series

Consider a real, integrable function $f(x)$ on $-L \leq x \leq L$. The Fourier series of it is

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)\right). \tag{91}$$

This form has a very intuitive physical picture: any sound can be obtained by mixing a series of simple harmonic oscillations.

The coefficients can be obtained by integrating from $-L$ to L. Since

$$\int_{-L}^{L} \sin\left(\frac{n\pi x}{L}\right)\mathrm{d}x = \int_{-L}^{L} \cos\left(\frac{n\pi x}{L}\right)\mathrm{d}x = 0, \tag{92}$$

we have

$$\frac{1}{L}\int_{-L}^{L} f(x)\,\mathrm{d}x = a_0, \tag{93}$$

and similarly since

$$\int_{L}^{-L} \cos\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi x}{L}\right) = \int_{L}^{-L} \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right) = \begin{cases} 0, & n \neq m, \\ L, & n = m, \end{cases} \tag{94}$$

and the cross term

$$\int_{L}^{-L} \sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi x}{L}\right) = 0, \tag{95}$$

we find

$$a_m = \frac{1}{L}\int_{-L}^{L} f(x)\cos\Big(\frac{m\pi x}{L}\Big)\,\mathrm{d}x\,, \tag{96}$$

$$b_m = \frac{1}{L}\int_{-L}^{L} f(x)\sin\Big(\frac{m\pi x}{L}\Big)\,\mathrm{d}x\,. \tag{97}$$

Note that the first equation also covers the $m = 0$ case. The $1/L$ normalization factor can be found by considering the $f(x) = 1$ case.

Strictly speaking, the above equations only make sense when we can define the integrals. And another question – whether the Fourier series really *converges* to the original function $f(x)$ – is a question needing investigation. Another question is whether we can have Fourier series for discontinuous functions, or in other words, how can a stepwise function be simulated by a series of sine waves. Another problem is it seems at $x = \pm L$, the Fourier series have to converge to the *save* value, because every term has the same value at $x = \pm L$.

To lay a rigorous foundation for Fourier series, we state the following theorem: if $f(x)$ is piecewise smooth on $[-L, L]$, then the Fourier series of $f(x)$ converges to $(f(x^+) + f(x^-))/2$ at each $x \in (-L, L)$, and to $(f(-L) + f(L))/2$ at $x = \pm L$. Here the term **piecewise smooth** means that the function may have jumps in $[-L, L]$, but there may be one or more point at which $f(x^+)$ and $f(x^-)$ are different. At such points of confusion, we just take an average of the two options.

As an example, let's try Fourier series on

$$f(x) = \begin{cases} 1, & 0 \le x \le L \\ -1, & -L \le x < 0. \end{cases} \tag{98}$$

So

$$a_0 = \frac{1}{L}(-L + L) = 0,$$

and indeed $a_m = 0$ for all m's: the function in question is odd, and all the a terms are even, so automatically they all vanish. For the odd terms, we have

$$\begin{aligned} b_n &= \frac{1}{L}\int_{-L}^{L} f(x)\sin\Big(\frac{n\pi x}{L}\Big)\,\mathrm{d}x \\ &= \frac{1}{L}\left(-\int_{-L}^{0} \sin\Big(\frac{n\pi x}{L}\Big)\,\mathrm{d}x + \int_{0}^{L} \sin\Big(\frac{n\pi x}{L}\Big)\,\mathrm{d}x\right) \\ &= \frac{1}{n\pi}(2 - 2\cos(n\pi)), \end{aligned}$$

and therefore

$$b_n = \begin{cases} \frac{4}{n\pi}, & n = 1, 3, 5, \ldots, \\ 0, & \text{otherwise}, \end{cases} \tag{99}$$

and

$$f(x) = \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{1}{2n-1}\sin\left(\frac{(2n-1)\pi x}{L}\right). \tag{100}$$

4.2 Parseval's theorem

We have

$$\int_{-L}^{L} f(x)^2\,\mathrm{d}x = \frac{1}{2}a_0\int_{-L}^{L} f(x)\,\mathrm{d}x + \sum_{n=1}^{\infty}\left(a_n\int_{-L}^{L}\cos\Big(\frac{n\pi x}{L}\Big)f(x)\,\mathrm{d}x + b_n\int_{-L}^{L}\sin\Big(\frac{n\pi x}{L}\Big)f(x)\,\mathrm{d}x\right),$$

and the integral in each term again gives a Fourier coefficient, and in the end we get

$$\frac{1}{L}\int_{-L}^{L} f(x)^2\,\mathrm{d}x = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n^2 + b_n^2). \tag{101}$$

This is called **Parseval's theorem**.

4.3 Complex Fourier series

We can also change the way we show the Fourier series into the follows:

$$f(x) = \sum_{n=-\infty}^{\infty} d_n \mathrm{e}^{\mathrm{i}n\pi x/L}. \tag{102}$$

This is merely a linear combination of the original definition in terms of sin and cos. Repeating the integral over $[-L, L]$, we get

$$\begin{aligned}\int_{-L}^{L} f(x)\mathrm{e}^{-\mathrm{i}m\pi x/L}\,\mathrm{d}x &= \sum_{n=-\infty}^{\infty} d_n \int_{-L}^{L} \mathrm{e}^{\mathrm{i}(n-m)\pi x/L}\,\mathrm{d}x \\ &= \sum_{n=-\infty}^{\infty} d_n \frac{2L}{(n-m)\pi}\sin(n-m)\pi.\end{aligned}$$

When $n \neq m$, the RHS is zero; when $n = m$, we need to take the $n - m \to 0$ limit and get

$$\frac{2L}{(n-m)\pi}\sin(n-m)\pi = 2L.$$

So the conclusion is

$$d_m = \frac{1}{2L}\int_{-L}^{L} f(x)\mathrm{e}^{-\mathrm{i}m\pi x/L}\,\mathrm{d}x\,. \tag{103}$$

This version of Fourier series immediately leads to Fourier transform when $L \to \infty$: We define

$$\omega_n = \frac{n\pi}{L}, \tag{104}$$

and we get

$$\Delta\omega = \frac{\pi}{L}. \tag{105}$$

Therefore

$$\begin{aligned}f(x) &= \sum_{n=-\infty}^{\infty} \mathrm{e}^{\mathrm{i}n\pi x/L} \cdot \frac{1}{2L}\int_{-L}^{L} f(x')\mathrm{e}^{-\mathrm{i}n\pi x'/L}\,\mathrm{d}x' \\ &= \sum_{n=-\infty}^{\infty} \frac{\Delta\omega}{2\pi}\mathrm{e}^{\mathrm{i}\omega_n x} \cdot \int_{-L}^{L} f(x')\mathrm{e}^{-\mathrm{i}\omega_n x'}\,\mathrm{d}x' \\ &= \int_{-\infty}^{\infty} \frac{\mathrm{d}\omega}{2\pi}\mathrm{e}^{\mathrm{i}\omega x} \cdot \int_{-\infty}^{\infty} f(x')\mathrm{e}^{-\mathrm{i}\omega x'}\,\mathrm{d}x'\,.\end{aligned}$$

So we can just define

$$F(\omega) = \int_{-\infty}^{\infty} f(x')\mathrm{e}^{-\mathrm{i}\omega x'}\,\mathrm{d}x'\,, \tag{106}$$

and

$$f(x) = \int_{-\infty}^{\infty} \frac{\mathrm{d}\omega}{2\pi}\mathrm{e}^{\mathrm{i}\omega x}F(\omega). \tag{107}$$

As an example, we evaluate the Fourier transform of

$$f(x) = x\mathrm{e}^{-|x|}. \tag{108}$$

5 Sturm-Liouville equations

A **Sturm-Liouville** equation has the form

$$(ry')' + (q + \lambda p)y = 0 \tag{109}$$

on an interval $[a, b]$. r, q, p are known functions and λ is a constant which is not known. The goal is to find values of λ for which the equation has non-trivial solutions. This is essentially an eigenvalue problem.

There are three types of boundary conditions that frequently appear. A **regular** boundary condition looks like

$$A_1 y(a) + A_2 y'(a) = 0, \quad B_1 y(b) + B_2 y'(b) = 0. \tag{110}$$

TODO

Here we state the following theorem: if λ_n and λ_m are two eigenvalues with eigenfunctions ϕ_n and ϕ_m, then we have the orthogonal condition

$$\int_a^b p(x)\phi_n(x)\phi_m(x)\,\mathrm{d}x = 0. \tag{111}$$

To prove the theorem, we start with

$$(r\phi_n')'\phi_m + (q + \lambda_n p)\phi_n\phi_m = 0,$$

and

$$(r\phi_m')\phi_n + (q + \lambda_m p)\phi_m\phi_n = 0,$$

so we have

$$(r\phi_n')'\phi_m - (r\phi_m')'\phi_n + (\lambda_n - \lambda_m)p\phi_m\phi_n = 0.$$

So by integration by parts, we have

$$\begin{aligned}
&(\lambda_n - \lambda_m)\int_a^b \mathrm{d}x\, p\phi_m\phi_n \\
&= \int_a^b ((r\phi_n')'\phi_m - (r\phi_m')'\phi_n)\,\mathrm{d}x \\
&= ((r\phi_n')\phi_m - (r\phi_m')\phi_n)\Big|_a^b - \int_a^b ((r\phi_n')\phi_m' - (r\phi_m')\phi_n')\,\mathrm{d}x \\
&= ((r\phi_n')\phi_m - (r\phi_m')\phi_n)\Big|_a^b .
\end{aligned}$$

We can easily verify that the RHS vanishes at $x = a, b$ for all reasonable boundary conditions (listed above). Therefore the LHS has to be zero, and since $\lambda_n \neq \lambda_m$, the conclusion has to be that (111) is correct.

The orthogonality property means we can express functions on $[a, b]$ as series expansions in $\{\phi_n\}$: we have

$$f(x) = \sum_{n=1}^{\infty} c_n\phi_n(x), \tag{112}$$

where

$$c_m = \frac{\int_a^b f(x)\phi_m(x)\,\mathrm{d}x}{\int_a^b \phi_m(x)^2\,\mathrm{d}x}. \tag{113}$$

Note that here $f(x)$ should follow the boundary conditions used to decide $\{\phi_m\}$; however, there is no guarantee that the RHS of (112) is continuous at $x = a, b$: it's usually the case that the value of the RHS at $x = a^+$ equals to $f(a)$. Therefore, in practical calculation, we can disregard the boundary condition requirement of $f(x)$, and do (112) to *all* $f(x)$, as long as we only rely on the RHS in (a, b) where $f(x)$ is continuous. Here we encounter the same case in Fourier series. If we only take finite terms in (112), then the RHS may have wavy features around these discontinuous points, which is an indication of the subtleties concerning discontinuity.

Indeed, the Fourier series is just a specific case of the orthogonality theorem in Sturm-Liouville theory. Consider the problem

$$y'' + \lambda y = 0, \quad y(-L) = y(L) = 0. \tag{114}$$

The general solution of the differential equation is

$$y(x) = A\cos\sqrt{\lambda}x + B\sin\sqrt{\lambda}x.$$

When $B=0$, the boundary condition means

$$\sqrt{\lambda}L = \left(n - \frac{1}{2}\right)\pi, \tag{115}$$

and when $A=0$, we have

$$\sqrt{\lambda}L = n\pi. \tag{116}$$

There are other Sturm-Liouville problems that may be useful in physics. One is the **Laguerre equation**

$$xy'' + (1-x)y' + \lambda y = 0, \tag{117}$$

and another is the **Hermite equation**

$$y'' - 2xy' + \lambda y = 0. \tag{118}$$

Note that although (117) and (118) don't fit in the scheme of (109), they indeed are Sturm-Liouville problems. For (118), we can multiply a factor h to it and get

$$hy'' - 2xhy' + \lambda hy = 0,$$

and dictate that

$$h' = -2xh.$$

Then we see we can just take

$$h = \mathrm{e}^{-x^2},$$

so (118) is a Sturm-Liouville problem with

$$r(x) = \mathrm{e}^{-x^2}, \quad p(x) = \mathrm{e}^{-x^2}, \quad q(x) = 0. \tag{119}$$

For (117) we can do the same and get

$$r(x) = x\mathrm{e}^{-x}, \quad p(x) = \mathrm{e}^{-x}, \quad q(x) = 0. \tag{120}$$

Another important equation is the **Legendre** equation, which is

$$(1-x^2)y'' - 2xy + \lambda y = 0, \tag{121}$$

We can demonstrate how to solve the equation using series expansion. Suppose

$$y(x) = \sum_{n=0}^{\infty} a_n x^n, \tag{122}$$

we have

$$(1-x^2)\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - 2x\sum_{n=1}^{\infty} n a_n x^{n-1} + \lambda \sum_{n=0}^{\infty} a_n x^n = 0,$$

which is equivalent to

$$\begin{aligned} 2a_2 + \lambda a_0 &= 0, \\ 6a_3 - 2a_1 + \lambda a_1 &= 0, \end{aligned}$$

and

$$(n+2)(n+1)a_{n+2} - n(n-1)a_n - 2na_n + \lambda a_n = 0.$$

So we find

$$a_2 = -\frac{\lambda}{2}a_0, \quad a_3 = \frac{2-\lambda}{6}a_1,$$

and by repeating this we find

$$a_{n+2} = \frac{n(n+2)-\lambda}{(n+2)(n+1)}a_n. \tag{123}$$

So we find

$$a_{2k} = -\frac{\lambda(3\cdot 2-\lambda)\cdots((2k-1)(2k-2)-\lambda)}{(2k)!}a_0, \tag{124}$$

and

$$a_{2k+1} = \frac{(2-\lambda)(4\cdot 3-\lambda)\cdots(2k\cdot(2k-1)-\lambda)}{(2k+1)!}a_1. \tag{125}$$