Homework 1

Jinyuan Wu

September 13, 2023

1 Maxwell's equations in dielectrics, Lorentz oscillators, and complex notation

1.1 Time-Average Quantities in Complex Notation

It is often important to be able to compute time-averaged quantities, such as the potential energy of a harmonic oscillator $U_{pe} = \frac{k}{2}\left\langle x^2 \right\rangle$ or the electric field energy density $U_{\mathrm{el}} = \frac{\varepsilon_0}{2}\left\langle \mathbf{E}^2 \right\rangle$. Here, the time-average of a function, $f(t)$, is defined as, $\langle f(t) \rangle = (1/T)\int_{t-T/2}^{t+T/2} dt' f(t')$, where T is defined as either the characteristic period of the oscillating system (i.e., $T = 2\pi/\omega$) or infinity. Such time averaging is drastically simplified by using complex notation.

To see this, suppose that we have any two functions $A(t)$ and $B(t)$, both of which take on a time harmonic form. Without loss of generality, we assume that $A(t) = A_0 \cos(\omega t + \phi)$, and $B(t) = B_0 \cos(\omega t + \theta)$, where ϕ and θ are arbitrary phase factors.

1.1.1

We have

$$\begin{aligned} \langle A(t)B(t) \rangle &= \frac{1}{T}\int_{t-T/2}^{t+T/2} \mathrm{d}t' \, A_0 \cos(\omega t' + \phi) B_0 \cos(\omega t' + \theta) \\ &= A_0 B_0 \frac{1}{T}\int_{t-T/2}^{t+T/2} \mathrm{d}t' \, \frac{1}{2}(\cos(\omega t' + \phi + \omega t' + \theta) + \cos(\omega t' + \phi - \omega t' - \theta)) \\ &= \frac{1}{2} A_0 B_0 \cos(\phi - \theta). \end{aligned} \tag{1}$$

Here we have used the condition that $T = 2\pi/\omega$ so that the first term vanishes.

1.1.2

We have

$$A(t) = \tilde{A}_0 \mathrm{e}^{-\mathrm{i}\omega t}, \quad B(t) = \tilde{B}_0 \mathrm{e}^{-\mathrm{i}\omega t}, \quad \tilde{A}_0 = A_0 \mathrm{e}^{-\mathrm{i}\phi}, \quad \tilde{B}_0 = B_0 \mathrm{e}^{-\mathrm{i}\theta}, \tag{2}$$

and therefore

$$\operatorname{Re} \tilde{A}_0 B_0 = \operatorname{Re} A_0 \tilde{B}_0 = \operatorname{Re} A_0 B_0 \mathrm{e}^{\mathrm{i}(\phi-\theta)} = A_0 B_0 \cos(\phi - \theta), \tag{3}$$

and hence

$$\langle A(t)B(t) \rangle = \frac{1}{2} \operatorname{Re} \tilde{A}_0 B_0 = \frac{1}{2} \operatorname{Re} A_0 \tilde{B}_0. \tag{4}$$

We can also straightforwardly do the follows. We have

$$\begin{aligned} \langle A(t)B(t) \rangle &= \left\langle \frac{1}{2}(\tilde{A}(t) + \tilde{A}^*(t)) \cdot \frac{1}{2}(\tilde{B}(t) + \tilde{B}^*(t)) \right\rangle \\ &= \frac{1}{4}\left\langle \tilde{A}_0 \tilde{B}_0 \mathrm{e}^{-2\mathrm{i}\omega t} + \tilde{A}_0 \tilde{B}_0^* + \tilde{A}_0{}^* \tilde{B}_0^* \mathrm{e}^{2\mathrm{i}\omega t} + \tilde{A}_0^* \tilde{B}_0 \right\rangle \\ &= \frac{1}{4}\left\langle A_0^* B_0 + \text{c.c.} \right\rangle \\ &= \frac{1}{2} A_0^* B_0 = \frac{1}{2} A_0 B_0^*. \end{aligned} \tag{5}$$

1.1.3

When

$$\boldsymbol{E} = \hat{\boldsymbol{x}} \operatorname{Re} \tilde{E}_0 \mathrm{e}^{-\mathrm{i}(\omega t - kz)}, \tag{6}$$

from

$$\boldsymbol{\nabla} \times \boldsymbol{E} = -\, \partial \boldsymbol{B}/\partial t \tag{7}$$

we obtain

$$\begin{aligned} &\mathrm{i}\boldsymbol{k} \times \boldsymbol{E} = -(-\mathrm{i}\omega)\boldsymbol{B} \\ &\Rightarrow \boldsymbol{B} = \frac{1}{\omega} k \hat{\boldsymbol{z}} \times \boldsymbol{E} = \frac{k}{\omega} \hat{\boldsymbol{y}} \operatorname{Re} \tilde{E}_0 \mathrm{e}^{-\mathrm{i}(\omega t - kz)}, \end{aligned} \tag{8}$$

and therefore

$$\langle \boldsymbol{S} \rangle = \frac{1}{\mu_0} \langle \boldsymbol{E} \times \boldsymbol{B} \rangle = \frac{1}{\mu_0} \cdot \frac{1}{2} \operatorname{Re} \underbrace{\hat{\boldsymbol{x}} \tilde{E}_0 \mathrm{e}^{\mathrm{i}kz}}_{\tilde{\boldsymbol{E}}_0} \times \underbrace{\frac{k}{\omega} \hat{\boldsymbol{y}} \tilde{E}_0^* \mathrm{e}^{-\mathrm{i}kz}}_{\tilde{\boldsymbol{B}}_0} = \frac{k}{2\mu_0 \omega} |\tilde{E}_0|^2 \hat{\boldsymbol{z}}, \tag{9}$$

and since the refraction index is n, we eventually get

$$\omega = k \cdot \frac{c}{n} \tag{10}$$

and therefore

$$\langle \boldsymbol{S} \rangle = \frac{n}{2} \sqrt{\frac{\epsilon_0}{\mu_0}} |\tilde{E}_0|^2 \hat{\boldsymbol{z}}. \tag{11}$$

The direction of the energy flow is parallel to the z axis.

1.1.4

The expected value of the electric energy density is

$$\langle u_e \rangle = \frac{1}{2} \epsilon_0 \epsilon_\mathrm{r} \langle \boldsymbol{E}^2 \rangle = \frac{1}{2} \epsilon_0 n^2 \cdot \frac{1}{2} |\tilde{E}_0|^2 = \frac{1}{4} \epsilon_0 n^2 |\tilde{E}|^2, \tag{12}$$

and the expected value of the magnetic energy density is

$$\langle u_m \rangle = \frac{1}{2\mu_0} \langle \boldsymbol{B}^2 \rangle = \frac{1}{2\mu_0} \cdot \frac{1}{2} \frac{k^2}{\omega^2} |\tilde{E}_0|^2 = \frac{1}{4} \frac{n^2}{c^2 \mu_0} \left| \tilde{E}_0 \right|^2 = \frac{1}{4} \epsilon_0 n^2 \left| \tilde{E}_0 \right|^2. \tag{13}$$

So we find

$$\frac{\langle u_e \rangle}{\langle u_m \rangle} = 1. \tag{14}$$

2 Lorentz oscillator in an AC field and optical forces

2.1 Optical response of an ensemble of Lorentz oscillators

Consider a dilute ensemble of Lorentz oscillators, uniformly distributed over space with number density N, in an AC electric field given by $\mathbf{E} = \operatorname{Re}\left[\tilde{\mathbf{E}}_0 e^{-i\omega t}\right]$. Each oscillator is driven by the local electric field according to the equation of motion given by

$$\ddot{\mathbf{p}} + \gamma \dot{\mathbf{p}} + \Omega^2 \mathbf{p} = \frac{q^2}{m} \mathbf{E}(\mathbf{r}),$$

where $\mathbf{r}, m,$ and q are the respective oscillator position, reduced mass, and charge.

2.1.1

The polarization density is

$$\boldsymbol{P} = N\boldsymbol{p}. \tag{15}$$

The EOM for $\boldsymbol{P}$ is

$$\ddot{\boldsymbol{P}} + \gamma\dot{\boldsymbol{P}} + \Omega^2\boldsymbol{P} = \frac{Nq^2}{m}\boldsymbol{E}. \tag{16}$$

We can switch to the Fourier representation. Thus we have

$$((-\mathrm{i}\omega)^2 + \gamma(-\mathrm{i}\omega) + \Omega^2)\tilde{\boldsymbol{P}} = \frac{Nq^2}{m}\tilde{\boldsymbol{E}}, \tag{17}$$

and from

$$\boldsymbol{D} = \epsilon_0\boldsymbol{E} + \boldsymbol{P} \tag{18}$$

we get

$$\tilde{\boldsymbol{D}} = \epsilon_0 \underbrace{\left(1 + \frac{Nq^2}{m\epsilon_0}\frac{1}{-\omega^2 - \mathrm{i}\gamma\omega + \Omega^2}\right)}_{\epsilon_\mathrm{r}} \tilde{\boldsymbol{E}}. \tag{19}$$

So we already get ϵ_r; it has explicit dependence on ω, but not $\boldsymbol{k}$.

2.1.2

The phase velocity is given by

$$v = \frac{c}{\sqrt{\epsilon_\mathrm{r}}} = \frac{c}{\sqrt{1 + \frac{Nq^2}{m\epsilon_0}\frac{1}{-\omega^2 - \mathrm{i}\gamma\omega + \Omega^2}}}. \tag{20}$$

As for the group velocity, we have

$$\begin{aligned} \omega^2 &= \frac{c^2k^2}{\epsilon_\mathrm{r}} \\ \Rightarrow 2\omega\,\mathrm{d}\omega &= \frac{2c^2k\,\mathrm{d}k}{\epsilon_\mathrm{r}} - c^2k^2\frac{\mathrm{d}\epsilon_\mathrm{r}}{\epsilon_\mathrm{r}^2} \\ \Rightarrow v_\mathrm{g} &= \frac{2c^2k}{\epsilon_\mathrm{r}}\frac{1}{2\omega + \frac{c^2k^2}{\epsilon_\mathrm{r}^2}\frac{\mathrm{d}\epsilon_\mathrm{r}}{\mathrm{d}\omega}}, \end{aligned} \tag{21}$$

where

$$\frac{\mathrm{d}\epsilon_\mathrm{r}}{\mathrm{d}\omega} = \frac{Nq^2}{m\epsilon_0}\frac{2\omega + \mathrm{i}\gamma}{(-\omega^2 - \mathrm{i}\gamma\omega + \Omega^2)^2}. \tag{22}$$

2.1.3

General relation between energy velocity and group velocity and phase velocity

2.2 Optical Tweezers