Elasticity in structural mechanics

Jinyuan Wu

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1 Rigid body analysis

2 Elastic medium

Definition The deformation $\boldsymbol{u}(t)$ of the system is completely decided by the external loading at t. Notable counterparts:

- *Fluid.* $\boldsymbol{u} \Leftarrow \boldsymbol{v} \Leftarrow \boldsymbol{F}$: not elastic.
- *Plastic.* $\boldsymbol{u}$ depends on history: not elastic.

Degrees of freedom, with infinitesimal deformation We deal with two sets of variables:

- *Stress* σ_{ij}. $\mathrm{d}F_i = \sigma_{ij}\,\mathrm{d}A_j$. Moment is not needed for bulk equation of equilibrium; but it's needed to capture the spatially fast-varying internal force in low-dimensional systems.
- *Strain* u_{ij}. For small deformation

$$u_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right). \tag{1}$$

- *Constitutive relations.* $\sigma_{ij} = \sigma_{ij}[u_{ij}]$.

Uniform isotropic linear medium Constitutive relation

$$\sigma_{ik} = K u_{ll}\delta_{ik} + 2\mu\left(u_{ik} - \frac{1}{3}\delta_{ik}u_{ll}\right). \tag{2}$$

Temperature expansion The strain induced by temperature change:

$$\frac{\mathrm{d}u}{\mathrm{d}x} = \alpha(T - T_0), \tag{3}$$

where T_0 is the "overall" temperature.

3 Uniform isotropic linear medium, in experiments

Two modes of strain

- *Compression/tension.* Along one direction (for example z):

$$\epsilon = \frac{\delta}{L} = u_{zz}. \tag{4}$$

- *Shear.* On the xy plane:

$$\gamma = \theta_{xx'} + \theta_{yy'} = 2u_{xy}. \tag{5}$$

Young's modulus Relation between tension and force:

$$E = \frac{P}{\epsilon} = \frac{PL}{\delta} \Rightarrow F = PA = \frac{\delta}{L}\cdot EA. \tag{6}$$

Poisson's ratio Relation between transverse strain and axial strain (in Young's modulus experiment):

$$\sigma = \nu = -\frac{\mathrm{d}\epsilon_{\text{transverse}}}{\mathrm{d}\epsilon_{\text{axial}}}. \tag{7}$$

This is how the material becomes thinner when stretched.

Volume modulus Relation between pressure and volume:

$$K = -V\frac{\mathrm{d}P}{\mathrm{d}V}. \tag{8}$$

Here K is that parameter in (2).

Shear modulus Relation between shear stress and shear strain:

$$\mu = G = \frac{\tau}{\gamma}. \tag{9}$$

Here τ is σ_{xy} (or yz or zx); γ is the shear strain.

How many independent parameters? In isothermal process:

$$E = \frac{9K\mu}{3K+\mu}, \quad \sigma = \frac{1}{2}\frac{3K-2\mu}{3K+\mu}. \tag{10}$$

When is the linear elasticity condition broken?

1. Linear region.
2. Proportional limit.
3. Elastic limit.
4. Yield point.
5. Ultimate tensile point.
6. Breaking point.

4 Low dimension system: torsion of cylinder-like rod

Reaction of φ to torque Here T is the torque:

$$\frac{\mathrm{d}\varphi}{\mathrm{d}z} = \frac{T(z)}{JG}, \quad T(z) = \int_0^z \mathrm{d}z' \, \frac{\mathrm{d}\text{ torque}}{\mathrm{d}z'}. \tag{11}$$

Relation between torque and stress

$$\gamma = \gamma_{xz} = \frac{\mathrm{d}\varphi}{\mathrm{d}z}r, \quad \tau = G\gamma, \tag{12}$$

$$\tau_{\max} = G\frac{\mathrm{d}\varphi}{\mathrm{d}z}R = \frac{TR}{J}. \tag{13}$$

Here R may also be written as c.

Note on J It's actually not moment of inertia!

5 Low dimension system: beam, or rod predominantly bended

Below x, y, z are measured from the neutral axis.

Important degrees of freedom Suppose the beam is in direction z. Sign convention: for force, deflection: downwards = +; for moment: counterclockwise = +.

- *Deflection.* Referred to as w.
- *Shear force.* The internal force, averaged:

$$\boldsymbol{F}_\perp = \boldsymbol{\sigma} \cdot \boldsymbol{A} = \sigma_{xz}A\hat{\boldsymbol{x}} + \sigma_{yz}A\hat{\boldsymbol{y}}, \tag{14}$$

and usually we only consider one direction (say x), and $\boldsymbol{F}_\perp = V\hat{\boldsymbol{x}}$. Below we change z to x.

- *Moment.* The "first-order moment" of internal force:

$$M + \mathrm{d}M = M + V\,\mathrm{d}x \Rightarrow V = \frac{\partial M}{\partial x}. \tag{15}$$

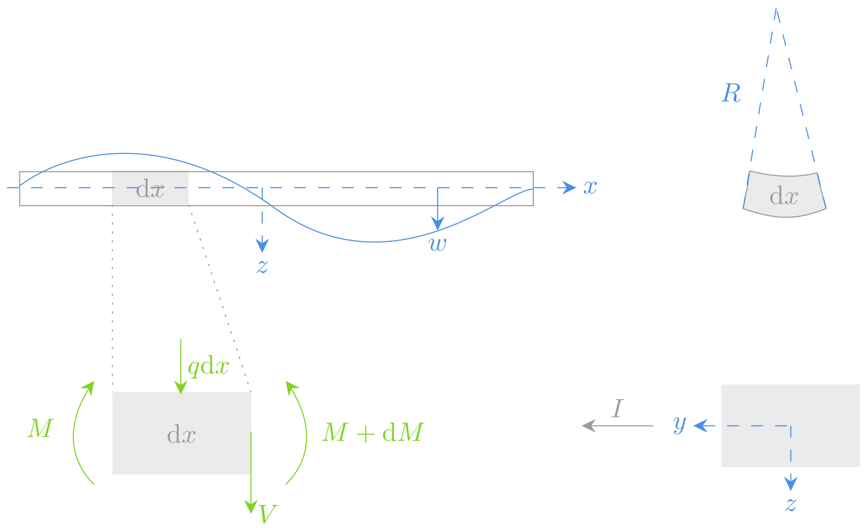

Figure 1: Analysis of a beam

Equation of equilibrium For determining V:

$$\frac{\partial V}{\partial x} + q = 0, \tag{16}$$

where q is force per unit length. The relation between moment and w:

$$M = -EI \frac{\partial^2 w}{\partial x^2}. \tag{17}$$

Here the axis of I is the same as the direction of M.

Details in bending stress Assuming R being large, each beam element can be seen as a beam element feeling stretching only, and thus

$$\frac{\mathrm{d}x'}{\mathrm{d}x} = \frac{R+z}{R}, \quad \sigma := \sigma_{xx} = E u_{xx} = \frac{z}{R} E, \tag{18}$$

while

$$M = \int \mathrm{d}z \, \mathrm{d}y \, \sigma \cdot z = \frac{E}{R} \underbrace{\int \mathrm{d}z \, \mathrm{d}y \, z^2}_{=:I}. \tag{19}$$

So after M is found from one of the equations above,

$$\sigma = \frac{z}{I} M, \tag{20}$$

and thus at a given point,

$$\sigma_{\max} = \frac{z_{\max}}{I} M, \tag{21}$$

which is needed to determine whether the beam fails.

Boundary condition The boundary condition of concentrated load can be determined if the following rules are followed:

- V is a downward force applied to the right end of a beam element by the beam element following it.
- No V is applied at the left boundary of the first beam element: thus if $F(x=0)$ is upward, then $V(x=0)$ is downward and is positive.
- No V is applied at the right boundary of the last beam element: thus if $F(x=L)$ is upward, then the shear force the $L - \mathrm{d}x$ beam element applies to the beam element at L is downward, and therefore the shear force applied to the beam element at $L - \mathrm{d}x$ is upward, and $V(x=L)$ is negative.

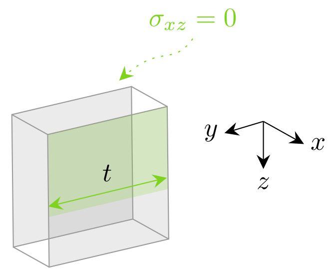

Figure 2: Analysis of shear stress

Details in shear stress Due to V we also have σ_{zx}, and

$$0 = \partial_x \sigma_{xx} + \partial_z \sigma_{zx} \Rightarrow 0 = \frac{1}{I}\frac{\partial M}{\partial x} \int \mathrm{d}z\, \mathrm{d}y\, z + \int \mathrm{d}y\, \sigma_{zx}, \tag{22}$$

and the average shear stress is (t is the width in y coordinate at z)

$$\tau := \bar{\sigma}_{zx} = \frac{1}{It}\frac{\partial M}{\partial x} \underbrace{\int_{\text{area above or below } z} z\, \mathrm{d}y\, \mathrm{d}z}_{Q} = \frac{Q}{It} V. \tag{23}$$

The integration range used in calculating Q is the green region in Fig. 2.
Determining the neutral axis Since we can start from the top of the beam as well as the bottom, to avoid subtleties in the definition of Q, we need to have

$$\int \mathrm{d}z\, \mathrm{d}u\, z = 0. \tag{24}$$

This is actually the criterion of the neutral axis. This means $\int \sigma\, \mathrm{d}A = 0$ which is reasonable (if bending happens at one position in the cross section, then stretching happens somewhere else).
Procedure

1. Finding all reaction forces from the loading.
2. Finding V.
3. Finding M.
4. Determining the neutral axis.
5. Finding σ and $\sigma_{\max}$.
6. Finding w if necessary.

6 Problems

- Using deformation to decide forces (that otherwise can't be determined).
- How fast a shaft can rotate: $P = T\omega$. Then $\tau_{\max}$ can be found.
- Beam analysis, and whether it fails because $\sigma_{\max}$ is too large.