Advanced Electrodynamics by Prof. Kun Din

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Some contents of the first several lectures are covered in this optics note, but some technical details are worth a separate note.

1   Guiding vectors and eigenmodes

We use the guiding vector method to find eigenmodes in several systems. Here we summarize the procedure: what we want to do is to solve the equation

$$(\nabla^2 + k^2)\boldsymbol{M} = 0, \quad \boldsymbol{\nabla} \cdot \boldsymbol{M} = 0. \tag{1}$$

By doing so we can make the ansatz

$$\boldsymbol{M} = \boldsymbol{\nabla} \times (\psi \boldsymbol{c}), \quad \boldsymbol{\nabla} \cdot \boldsymbol{c} = \text{const}, \quad \boldsymbol{\nabla} \times \boldsymbol{c} = 0, \tag{2}$$

which is always possible. (see (11.2) in this optics note) Now (1) becomes

$$\nabla^2 \boldsymbol{\nabla} \times (\psi \boldsymbol{c}) + k^2 \boldsymbol{\nabla} \times (\psi \boldsymbol{c}) = 0.$$

The LHS is

$$
\begin{aligned}
&\nabla^2 \boldsymbol{\nabla} \times (\psi \boldsymbol{c}) + k^2 \boldsymbol{\nabla} \times (\psi \boldsymbol{c}) \\
&= -\boldsymbol{\nabla} \times \boldsymbol{\nabla} \times \boldsymbol{\nabla} \times (\psi \boldsymbol{c}) + k^2 \boldsymbol{\nabla} \times (\psi \boldsymbol{c}) \\
&= -\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \boldsymbol{\nabla} \cdot (\psi \boldsymbol{c}) - \nabla^2 (\psi \boldsymbol{c})) + k^2 \boldsymbol{\nabla} \times (\psi \boldsymbol{c}) \\
&= \boldsymbol{\nabla} \times (\nabla^2 (\psi \boldsymbol{c})) + k^2 \boldsymbol{\nabla} \times (\psi \boldsymbol{c}) \\
&= (\nabla^2 + k^2) \boldsymbol{\nabla} \times (\psi \boldsymbol{c}) \\
&= (\nabla^2 + k^2)
\end{aligned}
$$

TODO So what should be done is to find a certain $\boldsymbol{c}$ and then solve

$$\nabla^2 \psi + k^2 \psi = 0. \tag{3}$$

Note that we are sure that $\boldsymbol{c}$ is perpendicular to $\boldsymbol{M}$, but not necessarily $\boldsymbol{N}$. A good example is reflection: $\boldsymbol{c}$ is the normal vector of the surface, $\boldsymbol{\nabla} \psi$ is the wave vector, and it can be easily found that only one of the electric field and the magnetic field can be perpendicular to $\boldsymbol{c}$. Therefore, we call the solution $\boldsymbol{E} = \boldsymbol{M}, \boldsymbol{B} = \boldsymbol{N}$ the **TE mode**, the solution $\boldsymbol{B} = \boldsymbol{M}, \boldsymbol{E} = \boldsymbol{N}$ the **TM mode**. There is yet a third mode

$$\boldsymbol{L} = \boldsymbol{\nabla} \psi, \tag{4}$$

which is a longitude mode.

Note that it is possible that $\boldsymbol{\nabla} \psi \parallel \boldsymbol{c}$ everywhere, and in this case we need to find another vector field $\boldsymbol{a}$ such that TODO: always scalar?

> **Info: Bessel functions**
>
> We consider the Bessel equation
>
> $$x^2 \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + x \frac{\mathrm{d}y}{\mathrm{d}x} + \left(x^2 - \alpha^2\right) y = 0. \tag{5}$$
>
> Here is a list of its solutions:
>
> | α | type | first type | second type |
> |---------|------|------------|-------------|
> | integer | Bessel functions | J_α | Y_α or N_α |
> | integer | Hankel functions | $H_\alpha^{(1)} = J_\alpha + \mathrm{i}Y_\alpha$ | $H_\alpha^{(2)} = J_\alpha - \mathrm{i}Y_\alpha$ |

J_α is well defined everywhere. Y_α diverges when $x \to 0$.

Another interesting equation is

$$x^2 \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + 2x \frac{\mathrm{d}y}{\mathrm{d}x} + \left(x^2 - n(n+1)\right) y = 0. \tag{6}$$

The solutions are

n	type	first type	second type
integer	spherical Bessel functions	j_n	h_n
integer	spherical Bessel functions	$h_n^{(1)} = j_n + \mathrm{i}y_n$	$h_n^{(2)} = j_n - \mathrm{i}y_n$

We also need to work with modified Bessel equation

$$x^2 \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + x \frac{\mathrm{d}y}{\mathrm{d}x} - \left(x^2 + \alpha^2\right) y = 0, \tag{7}$$

which can be derived from (5) with $x \to \mathrm{i}x$. The solutions are

α	type	first type	second type
integer	Modified Bessel functions	I_α	K_α

I_α is divergent as $x \to \infty$, and K_α is divergent when $x \to 0$.

We first consider the case of a cylinder. We set

$$\boldsymbol{c} = \boldsymbol{\rho} = (x, y, 0), \quad \boldsymbol{\nabla} \cdot \boldsymbol{c} = 2. \tag{8}$$

We need to solve (3) for a cylinder system. By separation of variables we have

Now we consider the case of a sphere. The equation is

$$\frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial \psi}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial \psi}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 \psi}{\partial \phi^2} + k^2 \psi = 0. \tag{9}$$

If we are dealing with a specific spherical cavity or something, k is quantized: The boundary condition will force k to take discrete values, just as is the case in the Hydrogen atom. We are, however, interested in a more generalized case, and therefore we just fix k, and solve the angular part of the problem. In the following derivation, we use n for l in quantum mechanics. The radius part is

$$\frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial \psi}{\partial r} \right) + \left(k^2 - \frac{n(n+1)}{r^2} \right) \psi = 0, \tag{10}$$

the solutions of which are the linear combination of

$$z_n(r) = j_n(r), j_n(r), h_n^{(1)}(r), h_n^{(2)}(r). \tag{11}$$

The solution of the angular part is just the spherical harmonics. Since we are not dealing with a specific system, we ignore the normalization and the solutions of (9) are

$$\begin{aligned}
\psi_{emn}(r, \theta, \phi) &= z_n(kr) P_n^m(\cos \theta) \cos(m\phi), \\
\psi_{omn}(r, \theta, \phi) &= z_n(kr) P_n^m(\cos \theta) \sin(m\phi).
\end{aligned} \tag{12}$$

Here the subscripts e and o represent the even and odd properties in ϕ. We called $\boldsymbol{M}$ and $\boldsymbol{N}$ functions **vector harmonics**, $\boldsymbol{M}$ **magnetic harmonics**, and $\boldsymbol{N}$ **electric harmonics**, because we usually make the following expansion

$$E = \sum_{m=0}^{\infty} \sum_{n=m}^{\infty} \left(B_{emn} M_{emn} + B_{omn} M_{omn} + A_{emn} N_{emn} + A_{omn} N_{omn} \right). \tag{13}$$

Note that when $n = 0$, we have non-trivial solution of ψ, but both $\boldsymbol{M}$ and $\boldsymbol{N}$ are zero. We can show this by a contradiction. Suppose $\boldsymbol{M}_0$ is obtained by ψ_{e00} or ψ_{o00} or a linear combination.

Since $\boldsymbol{M}_0$ is perpendicular to $\boldsymbol{c}$, we can see its values on a sphere a non-zero smooth vector field on S^2, without singularities (or otherwise due to the perfect rotational symmetry of the solution, singularities must be everywhere). The *Hairy ball theorem* tells us this is impossible.

Another noteworthy feature of the vector spherical functions are that they form a unambiguous definition of the electric and magnetic multipole expansion. The Taylor expansion definition, when the expansion order is high, has ambiguity whether the moments are defined as "primitive" tensors found in Taylor coefficients or traceless tensors. This gives rise to some "hidden" toroidal moments [1]. There are not toroidal moments in the multipole expansion defined with vector harmonics. For example, a magnetic field where the magnetic field lines form circles will be classified as a toroidal dipole term, but in the multipole expansion defined with vector harmonics, it is just $\boldsymbol{M}_{em1}$ or $\boldsymbol{M}_{om1}$.

2 Effective continuous medium

Suppose $f(\boldsymbol{r})$ is a filter or "smoother". Suppose the electric charge is

$$\eta(\boldsymbol{r}) = \sum_n \sum_{i \in C_n} q_i \delta(\boldsymbol{r} - \boldsymbol{r}_i), \tag{14}$$

where n is the index of clusters of electrons, most frequently moleculars. The smoothened version is

$$
\begin{aligned}
\langle \eta \rangle (\boldsymbol{r}) &= \int \mathrm{d}^3\boldsymbol{r}'\, \eta(\boldsymbol{r} - \boldsymbol{r}')f(\boldsymbol{r}') \\
&= \int \mathrm{d}^3\boldsymbol{r}' \sum_n \sum_{i \in C_n} q_i \delta(\boldsymbol{r} - \boldsymbol{r}' - \boldsymbol{r}_i) \int \frac{\mathrm{d}^3\boldsymbol{k}}{(2\pi)^3} f(\boldsymbol{k}) \mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}'} \\
&= \sum_n \int \frac{\mathrm{d}^3\boldsymbol{k}}{(2\pi)^3} f(\boldsymbol{k}) \sum_{i \in C_n} \mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot(\boldsymbol{r}-\boldsymbol{r}_i)} \\
&= \sum_n \int \frac{\mathrm{d}^3\boldsymbol{k}}{(2\pi)^3} f(\boldsymbol{k}) \mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot(\boldsymbol{r}-\boldsymbol{r}_n)} \sum_{i \in C_n} \mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot(\boldsymbol{r}_n-\boldsymbol{r}_i)} \\
&= \sum_n \int \frac{\mathrm{d}^3\boldsymbol{k}}{(2\pi)^3} f(\boldsymbol{k}) \mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot(\boldsymbol{r}-\boldsymbol{r}_n)} \sum_{i \in C_n} (1 - \mathrm{i}\boldsymbol{k}\cdot(\boldsymbol{r}_i - \boldsymbol{r}_n) + \cdots),
\end{aligned}
$$

and if the perturbation of the charges is not strong, we can make a cutoff and get

$$
\begin{aligned}
&\sum_n \int \frac{\mathrm{d}^3\boldsymbol{k}}{(2\pi)^3} f(\boldsymbol{k}) \mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot(\boldsymbol{r}-\boldsymbol{r}_n)} \sum_{i \in C_n} (1 - \mathrm{i}\boldsymbol{k}\cdot(\boldsymbol{r}_i - \boldsymbol{r}_n)) \\
&= \sum_n \int \frac{\mathrm{d}^3\boldsymbol{k}}{(2\pi)^3} f(\boldsymbol{k}) \mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot(\boldsymbol{r}-\boldsymbol{r}_n)} (q_n - \mathrm{i}\boldsymbol{k}\cdot\boldsymbol{p}_n) \\
&= \sum_n \left(q_n \int \frac{\mathrm{d}^3\boldsymbol{k}}{(2\pi)^3} f(\boldsymbol{k}) \mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot(\boldsymbol{r}-\boldsymbol{r}_n)} - \boldsymbol{p}_n \cdot \boldsymbol{\nabla} \int \frac{\mathrm{d}^3\boldsymbol{k}}{(2\pi)^3} f(\boldsymbol{k}) \mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot(\boldsymbol{r}-\boldsymbol{r}_n)} \right) \\
&= \sum_n (q_n f(\boldsymbol{r} - \boldsymbol{r}_n) - \boldsymbol{p}_n \cdot \boldsymbol{\nabla} f(\boldsymbol{r} - \boldsymbol{r}_n)) \\
&= \sum_n (q_n f(\boldsymbol{r} - \boldsymbol{r}_n) - \boldsymbol{\nabla} \cdot (\boldsymbol{p}_n f(\boldsymbol{r} - \boldsymbol{r}_n))).
\end{aligned}
$$

Now we note that

$$\langle \delta(\boldsymbol{r} - \boldsymbol{r}_n) \rangle = f(\boldsymbol{r} - \boldsymbol{r}_n),$$

and we get

$$\langle \eta \rangle (\boldsymbol{r}) = \rho(\boldsymbol{r}) - \boldsymbol{\nabla} \cdot \boldsymbol{P}(\boldsymbol{r}), \tag{15}$$

where

$$\rho(\boldsymbol{r}) = \left\langle \sum_n q_n \delta(\boldsymbol{r} - \boldsymbol{r}_n) \right\rangle, \quad \boldsymbol{P}(\boldsymbol{r}) = \left\langle \sum_n \boldsymbol{p}_n \delta(\boldsymbol{r} - \boldsymbol{r}_n) \right\rangle. \tag{16}$$

TODO: magnetic average

3 Constitutive relations

$$\boldsymbol{D} = \epsilon_0\boldsymbol{E} + \boldsymbol{P} = \epsilon_0(1 + \chi_{\mathrm{e}})\boldsymbol{E}, \quad \boldsymbol{H} = \frac{\boldsymbol{B}}{\mu_0} - \boldsymbol{M}, \tag{17}$$

$$\boldsymbol{\nabla} \cdot \boldsymbol{P} = -\rho, \tag{18}$$

4 Frequency dispersion and the Clausius-Mossotti equation

Now we work on a frequent mechanism of frequency dispersion, i.e. the response of the material depending on ω. We consider a oscillator model, where the electrons can be seen as oscillators, the EOM of each of which is

$$m\ddot{\boldsymbol{r}} = -m\omega_0^2\boldsymbol{r} - m\gamma\boldsymbol{r} + e\boldsymbol{E}_{\mathrm{driving}}. \tag{19}$$

Note that though the electrons has no interaction, but first, since the electrons are immersed in the radiation created by themselves, m, γ, and ω_0 will be modified, and second, an electron can feel the collective electric field of other electrons, and there must be a term for this in $\boldsymbol{E}_{\mathrm{drive}}$. Since $\boldsymbol{E}_{\mathrm{other\ electrons}} + \boldsymbol{E}_{\mathrm{this\ electron}}$ has no long scale effects and only modifies m, γ, and ω_0 (because they it modifies the lattice potential and therefore change ω_0, and it also modifies the phonon spectrum, thus modifying γ, and since we know the radiation of a moving charge has a term proportion to $\dddot{\boldsymbol{d}}$, m is also modified), we can say that

$$\boldsymbol{E}_{\mathrm{driving}} = \boldsymbol{E}_{\mathrm{external}} - \boldsymbol{E}_{\mathrm{self}}.$$

Here and after we write $\boldsymbol{E}_{\mathrm{external}}$ as $\boldsymbol{E}$, and we have

$$m\ddot{\boldsymbol{r}} = -m\omega_0^2\boldsymbol{r} - m\gamma\boldsymbol{r} + e(\boldsymbol{E} - \boldsymbol{E}_{\mathrm{self}}). \tag{20}$$

Note that this can also be seen as "using $\boldsymbol{E}_{\mathrm{self}}$ to modify m, ω_0 and γ and subtracting $\boldsymbol{E}_{\mathrm{sef}}$ from $\boldsymbol{E}$". A more generalized discussion of the modifications can be found in discussions around (16.12) in this optics note. What we are doing is actually a self-energy correction.

Suppose $\boldsymbol{r} \propto \mathrm{e}^{-\mathrm{i}\omega t}$, and we have

$$e\boldsymbol{r} =: \boldsymbol{d} = \alpha\boldsymbol{E}, \quad \alpha = \frac{e^2}{m(\omega_0^2 - \omega^2 - \mathrm{i}\gamma\omega)}. \tag{21}$$

The long range radiation is summarized as

$$\boldsymbol{P} = N\boldsymbol{d} = N\alpha(\boldsymbol{E} - \boldsymbol{E}_{\mathrm{self}}), \tag{22}$$

where N is the number of electrons per volume unit, and if we approximate the self field as

$$\boldsymbol{E}_{\mathrm{sef}} = -\frac{\boldsymbol{P}}{3\epsilon_0}, \tag{23}$$

which is the field strength in a empty cavity in a homogeneously polarized insulator, and more generally, is the self field of a large number of dipoles (see (**??**) in this note), then we get

$$\boldsymbol{P} = \frac{N\alpha}{1 - \frac{N\alpha}{3\epsilon_0}}\boldsymbol{E} =: \epsilon_0\chi_{\mathrm{e}}\boldsymbol{E}, \tag{24}$$

$$\chi_{\mathrm{e}} = \frac{N\alpha/\epsilon_0}{1 - N\alpha/(3\epsilon_0)}, \quad \frac{\epsilon_{\mathrm{r}} - 1}{\epsilon_{\mathrm{r}} + 2} = \frac{N\alpha}{3\epsilon_0}. \tag{25}$$

This is the so-called **Clausius-Mossotti equations**.

Now we put (21) into (25), and get

$$\epsilon_{\mathrm{r}} = 1 + \chi_{\mathrm{e}} = 1 + \frac{1}{\left(\frac{Ne^2}{m\epsilon_0}\right)^{-1}(\omega_0^2 - \omega^2 - \mathrm{i}\gamma\omega) - \frac{1}{3}},$$
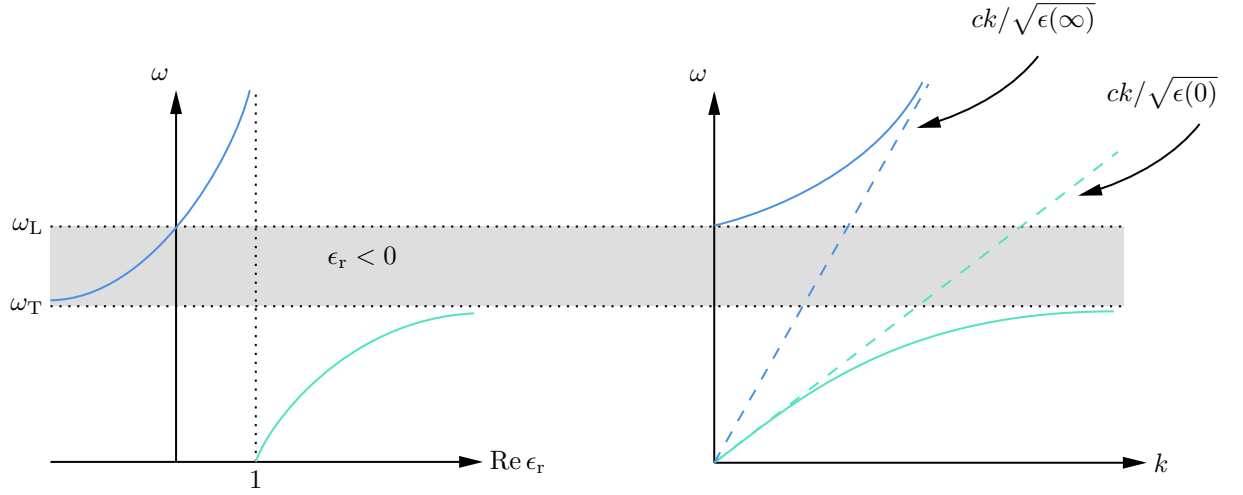
Figure 1: The photon band structure of (26)

and therefore after introducing the plasma frequency we get

$$\epsilon_{\mathrm{r}} = 1 + \chi_{\mathrm{e}} = 1 + \frac{\omega_{\mathrm{p}}^2}{\tilde{\omega}_0^2 - \omega^2 - \mathrm{i}\gamma\omega}, \quad \tilde{\omega}_0^2 = \omega_0^2 - \frac{1}{3}\omega_{\mathrm{p}}^2, \quad \omega_{\mathrm{p}}^2 = \frac{Ne^2}{m\epsilon_0}. \tag{26}$$

This equation means a non-trivial photon band structure. Now we assume $\gamma = 0$, and because we have

$$k^2 = \left(\frac{\omega}{c}\right)^2 \epsilon_{\mathrm{r}}(\omega), \tag{27}$$

it is possible that $\epsilon_{\mathrm{r}} < 0$ in some frequencies. The photon band structure, therefore, has a forbidden band. This is visualize in Figure 1 on page 5. It is easy to verify that

$$\omega_{\mathrm{L}} = \sqrt{\tilde{\omega}_0^2 + \omega_{\mathrm{p}}^2}, \quad \omega_{\mathrm{T}} = \tilde{\omega}_0. \tag{28}$$

The subscripts have meanings. When $\omega = \omega_{\mathrm{L}}$, $\epsilon_{\mathrm{r}} = 0$, and the equation

$$0 = \boldsymbol{\nabla} \cdot \boldsymbol{D} = \epsilon_{\mathrm{r}} \boldsymbol{\nabla} \cdot \boldsymbol{E}$$

is trivial, and it is possible that $\boldsymbol{\nabla} \cdot \boldsymbol{E} \neq 0$. So we see the subscript L means "longitude mode". When $\omega = \omega_{\mathrm{T}} = \tilde{\omega}_0$, the oscillators oscillate in its eigen mode and are coupled strongly to the electromagnetic field. Both of them vibrate transversely. Since there is no interaction between oscillators, the dispersion relation of density wave modes of these oscillators is a flat line, independent of $\boldsymbol{k}$, with the frequency ω_{T}. That is the meaning of the subscript T: the "transverse density mode".

References

[1] Stefan Nanz. *Toroidal Multipole Moments in Classical Electrodynamics*. Springer, 02 2016.