Homework 3

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1

By convolution theorem we have

$$
\mathcal{F}^{-1}\left[\frac{1}{(1+\mathrm{i}\omega)(2+\mathrm{i}\omega)}\right] = \mathcal{F}^{-1}\left[\frac{1}{1+\mathrm{i}\omega}\right] \otimes \mathcal{F}^{-1}\left[\frac{1}{2+\mathrm{i}\omega}\right] = \mathrm{e}^{-x}H(x) \otimes \mathrm{e}^{-2x}H(x)
$$

$$
= \int_{-\infty}^{\infty} \mathrm{e}^{-x'}H(x')\mathrm{e}^{-2(x-x')}H(x-x')\,\mathrm{d}x' \tag{1}
$$

$$
= \mathrm{e}^{-2x}\int_0^x H(x)\mathrm{e}^{x'}\,\mathrm{d}x' = (\mathrm{e}^{-x}-\mathrm{e}^{-2x})H(x).
$$

2

We need to solve

$$
\begin{aligned}
u_t &= 4u_{xx} \text{ for } 0 < x < L, t > 0 \\
u(0,t) &= u(L,t) = 0, \\
u(x,0) &= x^2(L-x).
\end{aligned} \tag{2}
$$

This can be done by standard separation of variables. Suppose $u = XT$. We have

$$
XT' = 4TX'' \Rightarrow \frac{X''}{X} = \frac{T'}{4T} = \lambda.
$$

Since the boundary conditions require that

$$
X(0) = X(L) = 0,
$$

X can't be exponential, and we have $\lambda = -k^2$, and the x part of the problem is then

$$
X'' + k^2 X = 0, \quad X(0) = X(L) = 0.
$$

This just gives us an odd Fourier series: we have

$$
X = A\cos(kx) + B\sin(kx),
$$

and since $X(0) = 0$, $A = 0$, and since $X(L) = 0$, we have

$$
kL = \pi n, \quad n \in \mathbb{Z}.
$$

The set of independent Xs is therefore

$$
X_n = \sin(k_n x), \quad k_n = \frac{\pi n}{L}, \quad n = 1, 2, \ldots \tag{3}
$$

The t part of the problem is then

$$
\begin{aligned}
T'_n + 4k_n^2 T_n &= 0, \\
T_n &= T_n(0)\mathrm{e}^{-4k_n^2 t}.
\end{aligned} \tag{4}
$$

The general solution is therefore

$$
u = \sum_{n=1}^{\infty} c_n \sin(k_n x)\mathrm{e}^{-4k_n^2 t}, \quad k_n = \frac{\pi n}{L}. \tag{5}
$$

Now we apply the initial condition. We have

$$\sum_{n=1}^{\infty} c_n \sin\left(\frac{\pi n}{L}x\right) = x^2(L-x),$$

and therefore

$$\frac{L}{2} \cdot c_n = \int_0^L \sin\left(\frac{\pi n}{L}x\right) x^2(L-x)\,\mathrm{d}x = -\frac{L^4}{(n\pi)^4}(2\pi n + 4\pi n(-1)^n), \tag{6}$$

and

$$u(x,t) = -\sum_{n=1}^{\infty} \frac{4L^3(1 + 2(-1)^n)}{(n\pi)^3} \sin\left(\frac{\pi n}{L}x\right) e^{-4(\pi n/L)^2 t}. \tag{7}$$

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Since

$$(a\cos\omega x + b\sin\omega x)e^{-\omega^2 kt}$$

is a specific solution of

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}, \tag{8}$$

the general solution is

$$u(x,t) = \int_0^{\infty} (a_\omega \cos\omega x + b_\omega \sin\omega x)e^{-\omega^2 kt}\,\mathrm{d}\omega. \tag{9}$$

The initial condition

$$u(x, t = 0) = f(x) = \begin{cases} e^{-x}, & |x| \le 1, \\ 0, & |x| > 1 \end{cases} \tag{10}$$

means

$$\int_0^{\infty} (a_\omega \cos\omega x + b_\omega \sin\omega x)\,\mathrm{d}\omega = f(x),$$

and since

$$\int_{-\infty}^{\infty} \cos\omega x \cos\omega' x\,\mathrm{d}x = \pi\delta(\omega+\omega')+\pi\delta(\omega-\omega'), \quad \int_{-\infty}^{\infty} \sin\omega x \sin\omega' x\,\mathrm{d}x = \pi\delta(\omega-\omega')-\pi\delta(\omega+\omega'), \tag{11}$$

we have

$$\pi a_\omega = \int_{-\infty}^{\infty} f(x)\cos(\omega x)\,\mathrm{d}x = \int_{-1}^{1} e^{-x}\cos(\omega x)\,\mathrm{d}x = \frac{\left(1+e^2\right)\omega\sin(\omega) + \left(e^2-1\right)\cos(\omega)}{e\left(\omega^2+1\right)}, \tag{12}$$

$$\pi b_\omega = \int_{-\infty}^{\infty} f(x)\sin(\omega x)\,\mathrm{d}x = \int_{-1}^{1} e^{-x}\sin(\omega x)\,\mathrm{d}x = \frac{\left(e^2-1\right)\omega\cos(\omega) - \left(1+e^2\right)\sin(\omega)}{e\left(\omega^2+1\right)}. \tag{13}$$

So we have

$$u(x,t) = \frac{1}{\pi}\int_0^{\infty} e^{-\omega^2 kt}\left(\frac{\left(1+e^2\right)\omega\sin(\omega) + \left(e^2-1\right)\cos(\omega)}{e\left(\omega^2+1\right)}\cos(\omega x)\right.$$
$$\left. + \frac{\left(e^2-1\right)\omega\cos(\omega) - \left(1+e^2\right)\sin(\omega)}{e\left(\omega^2+1\right)}\sin(\omega x)\right)\mathrm{d}\omega. \tag{14}$$

We can also write the solution using the heat kernel

$$u(x,t) = \frac{1}{2\sqrt{\pi kt}}\int_{-\infty}^{\infty} f(\xi)e^{-(x-\xi)^2/4kt}\mathrm{d}\xi = \frac{1}{2\sqrt{\pi kt}}\int_{-1}^{1} e^{-x}e^{-(x-\xi)^2/4kt}\,\mathrm{d}\xi. \tag{15}$$

4

The problem is

$$y_{tt} = c^2 y_{xx} \text{ for } 0 < x < L, t > 0$$
$$y(0,t) = y(L,t) = 0,$$
$$y(x,0) = f(x),$$
$$y_t(x,0) = g(x)$$

(16)

Again we use separation of variables. Suppose

$$y = XT.$$

We have

$$XT'' = c^2 T X'' \Rightarrow \frac{X''}{X} = \frac{T''}{c^2 T} = \lambda,$$

and again because the boundary conditions, we can only have $\lambda < 0$, and therefore $\lambda = -k^2$, and

$$X'' + k^2 X = 0,$$

(17)

and repeating the procedure in Section 2, we get

$$X_n = \sin k_n x, \quad k_n = \frac{n\pi}{L}.$$

(18)

The corresponding solution for the T equation is

$$T_n'' + k_n^2 c^2 T_n = 0 \Rightarrow T_n = a_n \cos(k_n ct) + b_n \sin(k_n ct).$$

(19)

The general solution is therefore

$$y(x,t) = \sum_{n=0}^{\infty} (a_n \cos(k_n ct) + b_n \sin(k_n ct)) \sin k_n x, \quad k_n = \frac{n\pi}{L}.$$

(20)

Now we take

$$c = 5, \quad L = \pi, \quad f(x) = \sin 2x, \quad g(x) = \pi - x.$$

(21)

We have

$$\sum_{n=0}^{\infty} a_n \sin nx = \sin 2x,$$

and

$$a_2 = 1, \quad a_n = 0, \quad n \neq 2.$$

(22)

The $g(x)$ condition means

$$\sum_{n=0}^{\infty} b_n \cdot 5n \cdot \sin nx = \pi - x,$$

$$\frac{L}{2} \cdot b_n \cdot nc = \int_0^{\pi} (\pi - x) \sin nx \, dx = \frac{\pi}{n} \Rightarrow b_n = \frac{2}{5n^2}.$$

So we have

$$y(x,t) = \sum_{n=0}^{\infty} \frac{2}{5n^2} \sin(5nt) \sin(nx) + \cos(5nt) \sin(2x).$$

(23)

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The wave function with velocity c has the following general solution:

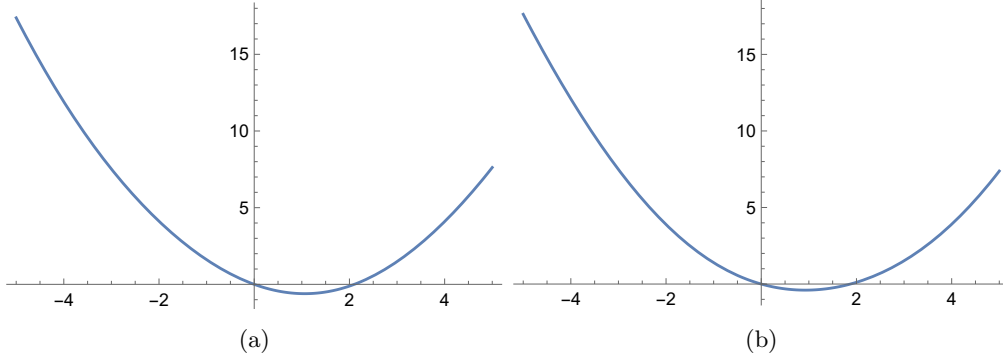$$y(x,t) = \frac{1}{2} f(x - ct) + \frac{1}{2} f(x + ct) + \frac{1}{2c} \int_{x-ct}^{x+ct} g(x') \, dx'.$$

(24)

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Figure 1: (a) $F(x)$ (b) $G(x)$

When $c = 2$, $f(x) = 0$, and

$$g(x) = \begin{cases} 1 \text{ for } & 0 \leqslant x \leqslant 2 \\ -1 & \text{for } -2 \leqslant x < 0 \\ 0 \text{ for } & |x| > 2 \end{cases}, \tag{25}$$

we have

$$y(x,t) = \frac{1}{4} \int_{\max(x-2t,-2)}^{\min(x+2t,2)} g(x')\,\mathrm{d}x' \tag{26}$$

$$=$$

Note that since $t > 0$,

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We repeat the procedure in last section, but now with

$$c = 4,$$
$$f(x) = x^2 - 2x \tag{27}$$
$$g(x) = \cos x$$

Now

$$y(x,t) = \frac{1}{2}((x-4t)^2 - 2(x-4t)) + \frac{1}{2}((x+4t)^2 - 2(x+4t)) + \frac{1}{8}(\sin(x+4t) - \sin(x-4t))$$

$$= \underbrace{\frac{1}{2}((x-4t)^2 - 2(x-4t)) - \frac{1}{8}\sin(x-4t)}_{F(x-4t)} + \underbrace{\frac{1}{2}((x+4t)^2 - 2(x+4t)) + \frac{1}{8}\sin(x+4t)}_{G(x+4t)}.$$
$$\tag{28}$$

The plots of F and G are shown in Figure 1. It can be seen that the behavior of the solution is dominated by the $x^2 - 2x$ term. As time goes by, the parts of F and G with large $|x|$ moves to the vicinity of $x = 0$, making $y(x,t)$ larger and larger. Ignoring the contribution of $g(x)$, we get

$$y(x,t) = \frac{1}{2}((x-4t)^2 - 2(x-4t)) + \frac{1}{2}((x+4t)^2 - 2(x+4t)) = x^2 - 2x + 4t^2. \tag{29}$$

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