Homework 4

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1

The original matrix is

$$\mathbf{A} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \\ 1 & 3 \\ 0 & 1 \end{pmatrix}. \tag{1}$$

Following these steps:

1. Move the third line to the top, and
2. Subtract the second line from the fourth line,

we get the row reduced form

$$\mathbf{A}_{\mathrm{R}} = \begin{pmatrix} 1 & 3 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, \tag{2}$$

and applying the same procedure to $\mathbf{I}_{4\times4}$ we get

$$\boldsymbol{\Omega}_{\mathrm{R}} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix}, \tag{3}$$

such that $\boldsymbol{\Omega}_{\mathrm{R}}\mathbf{A} = \mathbf{A}_{\mathrm{R}}$.

2

The equations are

$$\begin{aligned} 6x_1 - x_2 + x_3 &= 0 \\ x_1 - x_4 + 2x_5 &= 0, \\ x_1 - 2x_5 &= 0 \end{aligned} \tag{4}$$

which is equivalent to

$$\begin{pmatrix} 6 & -1 & 1 & 0 & 0 \\ 1 & 0 & 0 & -1 & 2 \\ 1 & 0 & 0 & 0 & -2 \end{pmatrix} \mathbf{x} = \mathbf{0}. \tag{5}$$

The row reduced form of the matrix in the LHS is

$$\begin{pmatrix} 1 & 0 & 0 & 0 & -2 \\ 0 & 1 & -1 & 0 & -12 \\ 0 & 0 & 0 & 1 & -4 \end{pmatrix}$$

If we switch the third and the fourth coulomb, we immediate find two independent solutions

$$\begin{pmatrix} 0 \\ -1 \\ 0 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ -12 \\ -4 \\ 0 \\ -1 \end{pmatrix},$$

and if we switch the coulombs back we find a basis of the solution space is

$$
\begin{pmatrix} 0 \\ -1 \\ -1 \\ 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} -2 \\ -12 \\ 0 \\ -4 \\ -1 \end{pmatrix}, \tag{6}
$$

and the solution space is 2-dimensional, and the general solution looks like

$$
x_1 = 2t_2, \quad x_2 = t_1 + 12t_2, \quad x_3 = t_1, \quad x_4 = 4t_2, \quad x_5 = t_2. \tag{7}
$$

3

The equation system

$$
\begin{aligned}
2x_1 - 3x_2 + x_4 &= 1 \\
3x_1 + x_3 - x_4 &= 0 \\
2x_1 - 3x_2 + 10x_3 &= 0
\end{aligned} \tag{8}
$$

is equivalent to

$$
\begin{pmatrix} 2 & -3 & 0 & 1 \\ 3 & 0 & 1 & -1 \\ 2 & -3 & 10 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}. \tag{9}
$$

The reduced matrix of the LHS is

$$
\begin{pmatrix} 1 & 0 & 0 & -\frac{3}{10} \\ 0 & 1 & 0 & -\frac{8}{15} \\ 0 & 0 & 1 & -\frac{1}{10} \end{pmatrix},
$$

and the general solution of the homogeneous version of the equation is therefore

$$
t \begin{pmatrix} \frac{3}{10} \\ \frac{8}{15} \\ \frac{1}{10} \\ 1 \end{pmatrix}.
$$

A specific solution of the equation system can be easily found by setting $x_4 = 0$:

$$
\begin{pmatrix} \frac{1}{30} \\ -\frac{14}{45} \\ -\frac{1}{10} \end{pmatrix}.
$$

So the general solution is

$$
x_1 = \frac{3}{10}t + \frac{1}{30}, \quad x_2 = \frac{8}{15}t - \frac{14}{45}, \quad x_3 = \frac{1}{10}t - \frac{1}{10}, \quad x_4 = t. \tag{10}
$$

4

Since

$$
\mathbf{A} = \begin{pmatrix} -1 & 0 \\ 4 & 4 \end{pmatrix}, \tag{11}
$$

we have $\det \mathbf{A} = -4$, and therefore it's not singular. The inverse is given by

$$
\mathbf{A}^{-1} = \frac{1}{\det \mathbf{A}} \begin{pmatrix} 4 & -4 \\ 0 & -1 \end{pmatrix}^{\top} = \begin{pmatrix} -1 & 0 \\ 1 & 1/4 \end{pmatrix}. \tag{12}
$$

5

The equation system

$$
\begin{aligned}
8x_1 - 4x_2 + 3x_3 &= 0 \\
x_1 + 5x_2 - x_3 &= -5 \\
-2x_1 + 6x_2 + x_3 &= -4
\end{aligned}
\tag{13}
$$

is equivalent to

$$
\underbrace{\begin{pmatrix} 8 & -4 & 3 \\ 1 & 5 & -1 \\ -2 & 6 & 1 \end{pmatrix}}_{\mathbf{A}} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ -5 \\ -4 \end{pmatrix}.
\tag{14}
$$

We have

$$
\det \mathbf{A} = 132,
\tag{15}
$$

and by Cramer's rule we have

$$
x_1 = \frac{1}{132} \cdot -66 = -\frac{1}{2}, \quad x_2 = \frac{1}{132} \cdot -114 = -\frac{19}{22}, \quad x_3 = \frac{1}{132} \cdot 24 = \frac{2}{11}.
\tag{16}
$$

6

$$
\mathbf{A} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 2 \\ 0 & 1 & 3 \end{pmatrix},
\tag{17}
$$

so

$$
\det(\mathbf{A} - \lambda \mathbf{I}) = 0 \Rightarrow \lambda = \frac{3 \pm \sqrt{17}}{2}, 0.
\tag{18}
$$

$\lambda = 0$ corresponds to

$$
\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 2 \\ 0 & 1 & 3 \end{pmatrix} \mathbf{v} = 0,
$$

a solution of which is

$$
\begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix}.
$$

$\lambda = (3 - \sqrt{17})/2$ corresponds to

$$
\begin{pmatrix} \frac{\sqrt{17}-3}{2} & 0 & 0 \\ 1 & \frac{\sqrt{17}-3}{2} & 2 \\ 0 & 1 & \frac{3+\sqrt{17}}{2} \end{pmatrix} \mathbf{v} = \mathbf{0},
$$

a solution of which is

$$
\begin{pmatrix} 0 \\ \frac{\sqrt{17}+3}{2} \\ -1 \end{pmatrix}.
$$

$\lambda = (3 + \sqrt{17})/2$ corresponds to

$$
\begin{pmatrix} -\frac{\sqrt{17}+3}{2} & 0 & 0 \\ 1 & -\frac{\sqrt{17}+3}{2} & 2 \\ 0 & 1 & \frac{3-\sqrt{17}}{2} \end{pmatrix} \mathbf{v} = \mathbf{0},
$$

a solution of which is

$$
\begin{pmatrix} 0 \\ \frac{3-\sqrt{17}}{2} \\ -1 \end{pmatrix}.
$$

So we have
$$\mathbf{P} = \begin{pmatrix} 0 & 0 & 2 \\ \frac{3-\sqrt{17}}{2} & \frac{3+\sqrt{17}}{2} & 3 \\ -1 & -1 & -1 \end{pmatrix}, \tag{19}$$

and
$$\mathbf{A} = \mathbf{P} \begin{pmatrix} \frac{3+\sqrt{17}}{2} & & \\ & \frac{3-\sqrt{17}}{2} & \\ & & 0 \end{pmatrix} \mathbf{P}^{-1}. \tag{20}$$

7

The procedure is the same as the last problem, and we find the eigenvalues are $\pm\sqrt{17}$, and $\sqrt{17}$ corresponds to $(\sqrt{17}-4, 1)$, and $-\sqrt{17}$ corresponds to $(-\sqrt{17}-4, 1)$. Then the dot product of the two eigenvectors is
$$(\sqrt{17}-4)(-\sqrt{17}-4) + 1 = 0,$$
so the eigenvectors are orthogonal to each other. After normalization we get
$$\mathbf{P} = \begin{pmatrix} \frac{\sqrt{17}-4}{\sqrt{34-8\sqrt{17}}} & -\frac{4+\sqrt{17}}{\sqrt{34+8\sqrt{17}}} \\ \frac{1}{\sqrt{34-8\sqrt{17}}} & \frac{1}{\sqrt{34+8\sqrt{17}}} \end{pmatrix}, \tag{21}$$

and
$$\mathbf{A} = \mathbf{P} \begin{pmatrix} \sqrt{17} & \\ & -\sqrt{17} \end{pmatrix} \mathbf{P}^{-1}. \tag{22}$$

8

The system of equations is
$$\mathbf{X}' = \mathbf{AX}, \quad \mathbf{A} = \begin{pmatrix} 2 & 1 & -2 \\ 3 & -2 & 0 \\ 3 & -1 & -3 \end{pmatrix}. \tag{23}$$

We first work in $\mathbb{C}$, and find the following eigenvalue-eigenvector pairs:
$$1.86, \begin{pmatrix} 2.19 \\ 1.70 \\ 1 \end{pmatrix}; \quad -2.43 + 1.46\mathrm{i}, \begin{pmatrix} 0.295 + 0.255\mathrm{i} \\ 0.317 - 0.699\mathrm{i} \\ 1 \end{pmatrix}; \quad -2.43 - 1.46\mathrm{i}, \begin{pmatrix} 0.295 - 0.255\mathrm{i} \\ 0.317 + 0.699\mathrm{i} \\ 1 \end{pmatrix}.$$

We can do a linear combination of the last two solutions (which are complex conjugate of each other) to get the real part and the imaginary part, which are both real. The results are
$$\mathrm{e}^{-2.431t} \begin{pmatrix} 0.590(\cos(1.464t) - 0.864\sin(1.464t)) \\ 0.634(2.201\sin(1.464t) + 1.\cos(1.464t)) \\ 2\cos(1.464t) \end{pmatrix} \tag{24}$$

and
$$\mathrm{e}^{-2.431t} \begin{pmatrix} 0.510(1.157\sin(1.46437t) + 1.\cos(1.46437t)) \\ -1.39713(1.\cos(1.46437t) - 0.454204\sin(1.46437t)) \\ 2\sin(1.46437t) \end{pmatrix}. \tag{25}$$

The column vectors of the fundamental matrix is just the above two vectors and
$$\mathrm{e}^{1.86t} \begin{pmatrix} 2.19 \\ 1.70 \\ 1 \end{pmatrix}. \tag{26}$$

9

Following the same procedure used above, we do eigenvalue decomposition and find

$$\mathbf{A} = \begin{pmatrix} 4 & -2 \\ -2 & 1 \end{pmatrix} = \begin{pmatrix} -2 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 5 & \\ & 0 \end{pmatrix} \begin{pmatrix} -2 & 1 \\ 1 & 2 \end{pmatrix}^{-1}. \tag{27}$$

The transition matrix is therefore

$$e^{\mathbf{A}t} = \begin{pmatrix} -2 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} e^{5t} & \\ & 1 \end{pmatrix} \begin{pmatrix} -2 & 1 \\ 1 & 2 \end{pmatrix}^{-1} = \begin{pmatrix} \frac{4e^{5t}}{5} + \frac{1}{5} & \frac{2}{5} - \frac{2e^{5t}}{5} \\ \frac{2}{5} - \frac{2e^{5t}}{5} & \frac{e^{5t}}{5} + \frac{4}{5} \end{pmatrix}. \tag{28}$$

10

We do eigenvalue decomposition

$$\mathbf{A} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = \underbrace{\frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}}_{\mathbf{P}} \begin{pmatrix} 2 & \\ & 0 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}, \tag{29}$$

and the equation

$$\mathbf{X}' = \mathbf{A}\mathbf{X} + \mathbf{G} \tag{30}$$

is then equivalent to

$$(\mathbf{P}^{-1}\mathbf{X})' = \begin{pmatrix} 2 & \\ & 0 \end{pmatrix} (\mathbf{P}^{-1}\mathbf{X}) + \mathbf{P}^{-1}\mathbf{G}, \tag{31}$$

where

$$\mathbf{P}^{-1}\mathbf{G} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 6e^{3t} \\ 4 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 6e^{3t} + 4 \\ -6e^{3t} + 4 \end{pmatrix}. \tag{32}$$

The general solution of

$$y' = 2y + \frac{1}{\sqrt{2}}(6e^{3t} + 4) \tag{33}$$

is

$$y = c_1 e^{2t} + \frac{1}{\sqrt{2}}(6e^{3t} - 2), \tag{34}$$

and the general solution of

$$y' = \frac{1}{\sqrt{2}}(-6e^{3t} + 4) \tag{35}$$

is

$$y = c_2 + \frac{1}{\sqrt{2}}(-2e^{3t} + 4t) \tag{36}$$

The general solution of the original problem then is

$$\mathbf{X} = \mathbf{P} \begin{pmatrix} c_1 e^{2t} + \frac{1}{\sqrt{2}}(6e^{3t} - 2) \\ c_2 + \frac{1}{\sqrt{2}}(-2e^{3t} + 4t) \end{pmatrix} = \begin{pmatrix} C_1 e^{2t} - C_2 + 4e^{3t} - 1 - 2t \\ C_1 e^{2t} + C_2 + 2e^{3t} - 1 + 2t \end{pmatrix}. \tag{37}$$