Bosonic modes in Fermi liquid

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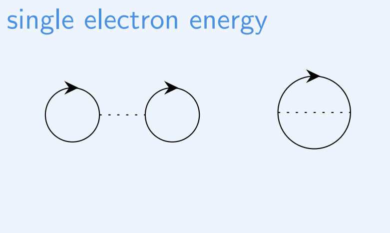
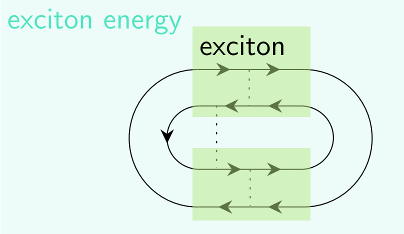
Background

In a Fermi liquid we have . . .

- Quasiparticles (electron/hole) with Σ-correction
- Any anything else?





. . . and more

Question

What to do
Finding modes other than the corrected single electron/hole

Why it's important
Usually not for C_V but for optical response: ϵ, $\chi^{(3)}$, etc.

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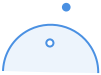
Today's topic

Electron-hole bosonic modes in Fermi liquid (with *some* scattering picked up back, i.e. beyond $\delta E \sim \varepsilon\, \delta n + f\, \delta n\, \delta n$), i.e.

$$|\text{single excitation}\rangle = \sum_{\boldsymbol{k}_1, \boldsymbol{k}_2} c_{\boldsymbol{k}_1 \boldsymbol{k}_2} \left| \quad \right\rangle \tag{1}$$



No trion, higher order correlation, or even more exotic spinons, etc. beyond Fermi liquid

Methodology

Serious quantitative prediction Bethe-Salpeter eq. (BSE)
Problem: no picture about "how the electron moves"

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Single electron linear response singularity = bosonic modes
$c^\dagger_{\boldsymbol{k}_1} c_{\boldsymbol{k}_2}$ = single-electron distribution = electron-hole pair annihilation operator

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Quantum Boltzmann eq. (QBE) Easiest kinetic theory for single-electron distribution to external field perturbation.
Conditions of QBE

- low (external and inherent) ω
- long wave length
- well-defined quasiparticles; high order correlation not important

Overview

What to investigate

Stable oscillation modes of QBE ($\Leftrightarrow$ infinite response to external field $\Leftrightarrow$ bosonic mode): for $n_{\boldsymbol{p}\sigma\sigma'}(\boldsymbol{r})$, $\varepsilon_{\boldsymbol{p}\sigma\sigma'} = \varepsilon[\delta n]$,

$$\frac{\partial n_{\boldsymbol{p}}}{\partial t} + \underbrace{\frac{\partial \varepsilon_{\boldsymbol{p}}}{\partial \boldsymbol{p}} \cdot \frac{\partial n_{\boldsymbol{p}}}{\partial \boldsymbol{r}}}_{\text{diffusion}} \underbrace{- \frac{\partial \varepsilon_{\boldsymbol{p}}}{\partial \boldsymbol{r}} \cdot \frac{\partial n_{\boldsymbol{p}}}{\partial \boldsymbol{p}}}_{\text{force}} + \underbrace{\mathrm{i}\left[\varepsilon_{\boldsymbol{p}}, n_{\boldsymbol{p}}\right]}_{\text{multi-band}} = \underbrace{I_{\text{Fermi golden rule}}}_{\text{collision}} . \tag{2}$$

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What to expect

Three types of important bosonic modes:

- Zero sound in uncharged Fermi liquid: collective density fluctuation in $\boldsymbol{k}$- (but not $\boldsymbol{r}$-) space
- Plasmon in charged Fermi liquid = zero sound + long range interaction
- Exciton in charged multi-band Fermi liquid

Equation governing zero sound

System Single-band Fermi liquid with spin ignored

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Kinetics of uncharged Fermi liquid *Landau equation* = QBE +

$$\varepsilon_{\boldsymbol{p}}(\boldsymbol{r}) = \varepsilon_{\boldsymbol{p}}^0 + \frac{1}{V} \sum_{\boldsymbol{p}'} f_{\boldsymbol{p}\boldsymbol{p}'} \, \delta n_{\boldsymbol{p}} \, (\boldsymbol{r}) \tag{3}$$

(assumption: $\boldsymbol{q} \to 0$ in $c^\dagger_{\boldsymbol{p}+\boldsymbol{q}} c_{\boldsymbol{p}}$, i.e. $\delta n_{\boldsymbol{p}} \, (\boldsymbol{r})$ being smooth in $\boldsymbol{r}$)

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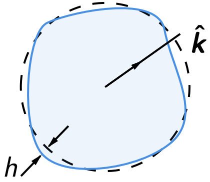
EOM governing zero sound No nonlinearity, no dissipation:

$$\frac{\partial \, \delta n_{\boldsymbol{p}}}{\partial t} + \frac{\partial \varepsilon_{\boldsymbol{p}}^{\text{static}}}{\partial \boldsymbol{p}} \cdot \frac{\partial \, \delta n_{\boldsymbol{p}}}{\partial \boldsymbol{r}} - \frac{\partial n_{\boldsymbol{p}}^{\text{static}}}{\partial \boldsymbol{p}} \cdot \underbrace{\frac{1}{V} \sum_{\boldsymbol{p}'} f_{\boldsymbol{p}\boldsymbol{p}'} \frac{\partial \, \delta n_{\boldsymbol{p}}}{\partial \boldsymbol{r}}}_{\partial \delta \varepsilon_{\boldsymbol{p}} / \partial \boldsymbol{r}} = 0 \tag{4}$$

Fermi surface vibration

Ansatz Disturbance as small as possible ...

$$n_{\boldsymbol{p}}(\boldsymbol{r}, t) = \mathrm{e}^{\mathrm{i}(\boldsymbol{q}\cdot\boldsymbol{r} - \mathrm{i}\,\omega t)}\,\theta(\mu - \varepsilon_{\boldsymbol{p}}^{\text{stable}} - h(\hat{\boldsymbol{p}})) \tag{5}$$



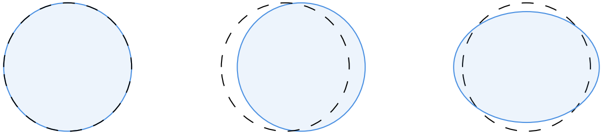
Eigenvalue problem

$$(\omega - \boldsymbol{q}\cdot\boldsymbol{v})h(\hat{\boldsymbol{k}}) = \boldsymbol{q}\cdot\boldsymbol{v}\int \frac{\mathrm{d}\Omega'}{4\pi} F(\vartheta) h(\hat{\boldsymbol{k}}'). \tag{6}$$

where $\boldsymbol{v}$ is single-electron velocity. $\Rightarrow$ zero sound has linear dispersion.
Non-trivial zero sound requires $F \neq 0$ ($F = 0 \Rightarrow$ bare electron-hole pair)

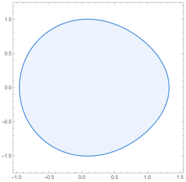
Modes

Shape of Fermi surface when $\boldsymbol{q} = 0$ In the $d = 2$ case:



$l = 0$ $l = 1$ $l = 2$

Distortion when $\boldsymbol{q} \neq 0$ More electrons in $\hat{\boldsymbol{q}}$; less electrons in $-\hat{\boldsymbol{q}}$



Zero sound is not real space density wave In zero sound $V_{\text{Fermi sea}} = \text{const.} \Rightarrow$ zero sound is not ordinary sound

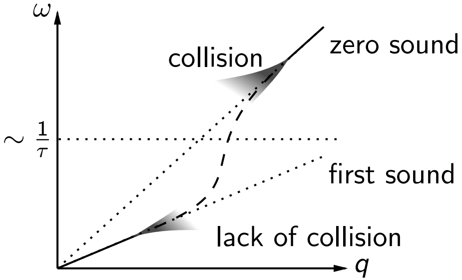
Comparison with ordinary sound

Ordinary sound Fermi liquid theory $\Rightarrow \partial\rho/\partial P \Rightarrow$ another sound mode ("first sound", ordinary sound, density mode) from hydrodynamics

Relation with zero sound

- First sound appears when $\omega\tau \ll 1$: ordinary hydrodynamics $\Leftrightarrow$ local equilibrium $\Leftrightarrow \tau \ll 1/\omega$
- zero sound appears when $\omega\tau \gg 1$: no collision integral $\Leftrightarrow \tau \gg 1/\omega$

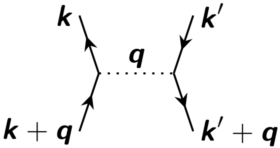
The two are connected: a radical finite-T correction



What happens with long-range interaction

The origin of $f_{\boldsymbol{pp}'}$

$$f_{\boldsymbol{kk}'} = \lim_{\boldsymbol{q}\to 0}$$



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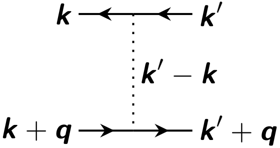

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Coulomb interaction $\Rightarrow$ first term divergent in $\boldsymbol{k}$ space $\Rightarrow$ it should be considered in $\boldsymbol{r}$ space

What happens with long-range interaction

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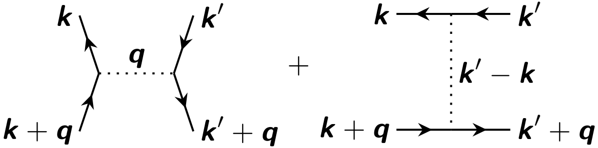
$$f_{\boldsymbol{k}\boldsymbol{k}'} = \lim_{\boldsymbol{q}\to 0} \qquad \tag{7}$$



Coulomb interaction $\Rightarrow$ first term divergent in $\boldsymbol{k}$ space $\Rightarrow$ it should be considered in $\boldsymbol{r}$ space

Landau-Silin eq.

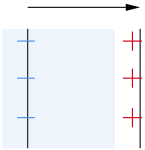
$$\frac{\partial n_{\boldsymbol{p}}}{\partial t} + \frac{\partial \varepsilon_{\boldsymbol{p}}}{\partial \boldsymbol{p}} \cdot \frac{\partial n_{\boldsymbol{p}}}{\partial \boldsymbol{r}} - \frac{\partial(\varepsilon_{\boldsymbol{p}} - e\varphi(\boldsymbol{r}))}{\partial \boldsymbol{r}} \cdot \frac{\partial n_{\boldsymbol{p}}}{\partial \boldsymbol{p}} = \underbrace{I_{\text{Fermi golden rule}}}_{\text{collision}}, \tag{8}$$

$$\varepsilon_{\boldsymbol{p}}(\boldsymbol{r}) = \varepsilon^0_{\boldsymbol{p}} + \frac{1}{V}\sum_{\boldsymbol{p}'} f_{\boldsymbol{p}\boldsymbol{p}'}\,\delta n_{\boldsymbol{p}}(\boldsymbol{r}), \quad \nabla^2\varphi = e\cdot\frac{1}{V}\sum_{\boldsymbol{p}}\delta n_{\boldsymbol{p}}(\boldsymbol{r}). \tag{9}$$

Plasmon mode

Plasmon is gapped When $\boldsymbol{q} \to 0$ we get to the elementary case

$$\boldsymbol{F} = -e\boldsymbol{E}$$



$$m\ddot{\boldsymbol{x}} = -m\omega^2 \boldsymbol{x} = (-e)\boldsymbol{E} = -e \cdot \frac{1}{\epsilon_0} en\boldsymbol{x} \Rightarrow \omega = \sqrt{\frac{ne^2}{\epsilon_0 m}}. \tag{10}$$

Comparison with zero sound When $\boldsymbol{q} \to 0$, $\varphi(\boldsymbol{r}) \Rightarrow$ oscillation: long-range interaction $\Rightarrow$ finite gap

Comparison with first sound $V_{\text{Fermi sea}} = \text{const.}$ in plasmon as well: plasmon is not a density mode in real space

Summary

Fermi liquid, uncharged: zero sound

- Linear, gapless
- From $f_{\boldsymbol{pp}'}$

Fermi liquid, charged: plasmon

- Divergent Hartree term $\Rightarrow$ self-energy correction in real space
- When $\boldsymbol{q} = 0$: $f_{\boldsymbol{pp}'}$ not important; gapped

Two bands: exciton

Determining $\boldsymbol{r}$ and $\boldsymbol{p}$ at the same time in QBE??

Why $n_{\boldsymbol{p}}(\boldsymbol{r})$ makes sense Because it's not a probabilistic distribution function: it's a Wigner function of $G^<(\boldsymbol{r}_1, \boldsymbol{r}_2)$.
By Fourier transform

$$\int \mathrm{d}^3\boldsymbol{r}\, n_{\boldsymbol{p}}(\boldsymbol{r})\, \mathrm{e}^{-\mathrm{i}\,\boldsymbol{q}\cdot\boldsymbol{r}} \simeq n_{\boldsymbol{p}\boldsymbol{q}} \simeq c^\dagger_{\boldsymbol{p}} c_{\boldsymbol{p}+\boldsymbol{q}} \tag{11}$$

we get the total momentum of the bosonic mode $\boldsymbol{q}$.

Why $\varepsilon_{\boldsymbol{p}}(\boldsymbol{r})$ makes sense Similarly, because of $\Sigma(\boldsymbol{r}_1, \boldsymbol{r}_2)$

Why $f_{\boldsymbol{p}\boldsymbol{p}'}$ has no spatial dependence

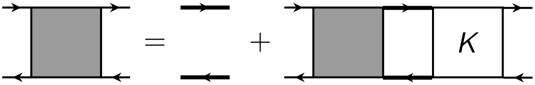
1. Interaction channel: $\delta n_{\boldsymbol{p}\boldsymbol{q}}\, f_{\boldsymbol{p}\boldsymbol{p}'\boldsymbol{q}\boldsymbol{q}'}\, \delta n_{\boldsymbol{p}'\boldsymbol{q}'}\, \delta_{\boldsymbol{q}\boldsymbol{q}'}$ ($\sum \boldsymbol{q} = \text{const.}$)
2. But we are working with QBE $\Rightarrow$ $\boldsymbol{q}$ small
3. So we take $\boldsymbol{q}, \boldsymbol{q}' \to 0$ limit:

$$\sum \delta n_{\boldsymbol{p}\boldsymbol{q}}\, f_{\boldsymbol{p}\boldsymbol{p}'}\, \delta n_{\boldsymbol{p}'\boldsymbol{q}'}\, \delta_{\boldsymbol{q}\boldsymbol{q}'} \xrightarrow{\text{Fourier}} \int \mathrm{d}^3\boldsymbol{r}\, \delta n_{\boldsymbol{p}}(\boldsymbol{r}) f_{\boldsymbol{p}\boldsymbol{p}'}\, \delta n_{\boldsymbol{p}'}(\boldsymbol{r}) \tag{12}$$

BSE and single-electron kinetic theory

Series calculation

Bethe–Salpeter equation (BSE) is for quantitative calculations.



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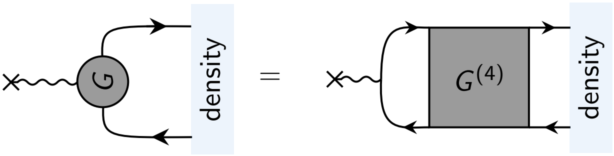
What we need Linear response of single-electron under external field = BSE (simplest single-electron theory: QBE)



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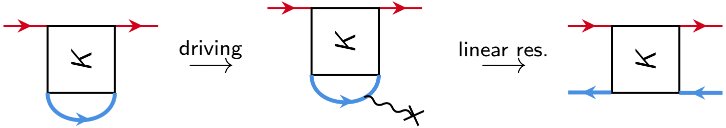
Next step: relation between K and Σ

Linking Σ with K

Linear response of a single self-energy diagram

$\Sigma =$



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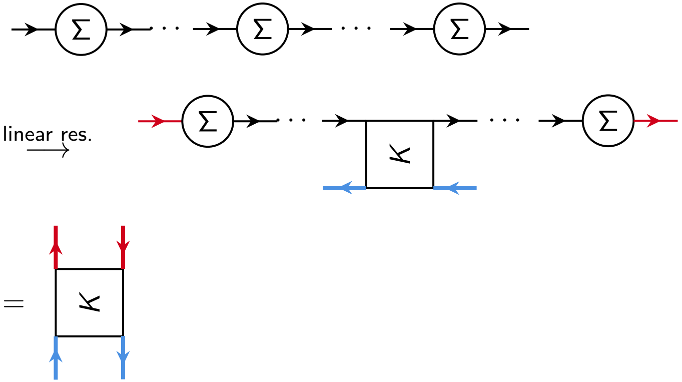
Whole picture



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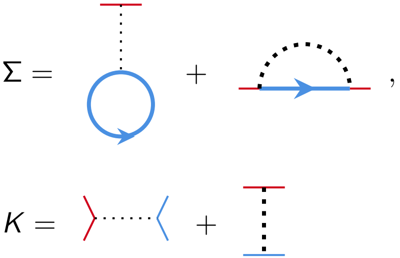
Linking Σ with K

Example: linear response from time-dependent GW = BSE



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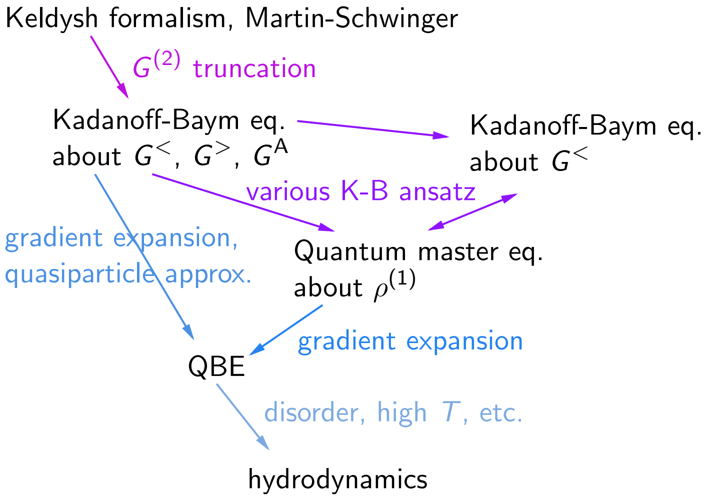
- First term = Electron Hartree term = Electron direct term = Exciton exchange term; $+1$ prefactor;
- Second term = Electron Fock term = Electron exchange term = Exciton direct term; (-1) prefactor.

Justifying quantum Boltzmann equation

Is QBE reliable?

Yes! When we intuitively expect it to work –



Fermi liquid is really liquid

In small T limit Infinite degrees of freedom; QBE still works but the system is not hydrodynamic in the normal sense; bosonization (note that $n_{\boldsymbol{p}}(\boldsymbol{r})$ may also be seen as the field operator or single-electron hole pair wave function)

In large T limit Only five hydrodynamic equations; described by Navier-Stokes equation.

Subtlety about plasmon

Classical derivation of plasmon involves a positive background; where does this background go in Landau-Silin equation?
Note that only when $\delta n \neq 0$ do we have non-zero φ: we have already assumed a positive jellium background.