Homework 4

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Problem 1 In this problem we will introduce another way to think about Hall conductance. As discussed in class, the edge of a Chern insulator with Chern number n should have n chiral edge modes. Suppose that the n edge modes move one way near the $y=0$ edge, and the opposite way at the $y=L_y$ edge. 1. First consider the $y=L_y$ edge. Each of the n edge modes near $y=L_y$ can be modeled as a non-interacting free fermion model $H=\sum_k\left(\epsilon_{1k}-\mu_1\right)c_k^{\dagger}c_k$, where ϵ_{1k} is a monotonically increasing function of k, μ_1 corresponds to the chemical potential at the $y=L_y$ edge. We assume that the infinitely many states with $\epsilon_{1k}<\mu_1$ are all filled (in reality, the limit $k \rightarrow -\infty$ corresponds to moving into the bulk, and k is cut off accordingly). Suppose one changes the chemical potential $\mu_1 \rightarrow \mu_1+\Delta\mu_1$. What is the corresponding change ΔI_1 in the $y=L_y$ edge current? (Include all n edge modes). 2. Now consider the $y=0$ edge mode, modeled by n chiral modes of the form $H=\sum_k\left(\epsilon_{2k}-\mu_2\right)c_k^{\dagger}c_k$, where ϵ_{2k} is a monotonically decreasing function of k. What is the corresponding change ΔI_2 in the $y=0$ edge current if one changes the chemical potential $\mu_2 \rightarrow \mu_2+\Delta\mu_2$? 3. Find the total current $I=I_1+I_2$ induced by a chemical potential difference $\mu_1-\mu_2=\Delta\mu$ between the two edges, assuming that $I=0$ when $\mu_1=\mu_2$. Interpret your result in terms of the Hall conductance σ_H. 4. Now repeat the calculation at finite temperature T. Show that the edge currents and the Hall conductance do not change. (In reality, there is a temperature dependence coming from thermal activation above the bulk gap Δ, but it is exponentially small $\sim e^{-\Delta/T}$).

Solution The problem is illustrated in Figure 1.

(a) The current is

$$
\begin{aligned}
I_1 &= -e\cdot\text{density of states}\cdot\text{cross section}\cdot v \\
&= -e\frac{1}{L}\sum_{\text{ocupied states}}\frac{1}{\hbar}\frac{\partial\epsilon_{1k}}{\partial k} = -\frac{e}{\hbar}\frac{n}{L}\sum_{\Lambda_1\leq k\leq k_{1\mathrm{F}}}\frac{\partial\epsilon_{1k}}{\partial k} = -n\frac{e}{\hbar}\int_{\Lambda_1}^{k_{1\mathrm{F}}}\frac{\mathrm{d}k}{2\pi}\frac{\partial\epsilon_{1k}}{\partial k} \\
&= -n\frac{e}{h}(\epsilon_{1\mathrm{F}}-\epsilon_{1,k=\Lambda_1}) = -n\frac{e}{h}(\mu_1-\epsilon_{1,k=\Lambda_1}).
\end{aligned} \tag{1}
$$

So we have

$$
\Delta I_1 = -n\frac{e}{h}\Delta\mu_1. \tag{2}
$$

(b) Similarly, we have

$$
\begin{aligned}
I_2 &= -e\cdot\text{density of states}\cdot\text{cross section}\cdot v \\
&= -e\frac{1}{L}\sum_{\text{ocupied states}}\frac{1}{\hbar}\frac{\partial\epsilon_{1k}}{\partial k} = -\frac{e}{\hbar}\frac{n}{L}\sum_{k_{2\mathrm{F}}\leq k\leq\Lambda_2}\frac{\partial\epsilon_{2k}}{\partial k} = -n\frac{e}{\hbar}\int_{k_{2\mathrm{F}}}^{\Lambda_2}\frac{\mathrm{d}k}{2\pi}\frac{\partial\epsilon_{2k}}{\partial k} \\
&= -n\frac{e}{h}(\epsilon_{2,k=\Lambda_2}-\epsilon_{2\mathrm{F}}) = -n\frac{e}{h}(\epsilon_{2,k=\Lambda_2}-\mu),
\end{aligned} \tag{3}
$$

and

$$
\Delta I_2 = n\frac{e}{h}\Delta\mu_2. \tag{4}
$$

(c) We have

$$
I = -n\frac{e}{h}(\mu_1-\mu_2+\epsilon_{2,k=\Lambda_2}-\epsilon_{1,k=\Lambda_1}). \tag{5}
$$

When $\Delta\mu=0$, we have $I=0$, so we find the two cutoff terms cancel each other, and

$$
I = -n\frac{e}{h}\Delta\mu. \tag{6}
$$

In a Hall effect setting, the difference of the chemical potentials arises from the external electric field: conservation of energy tells us

$$
\mu_1 - eU = \mu_2, \tag{7}
$$

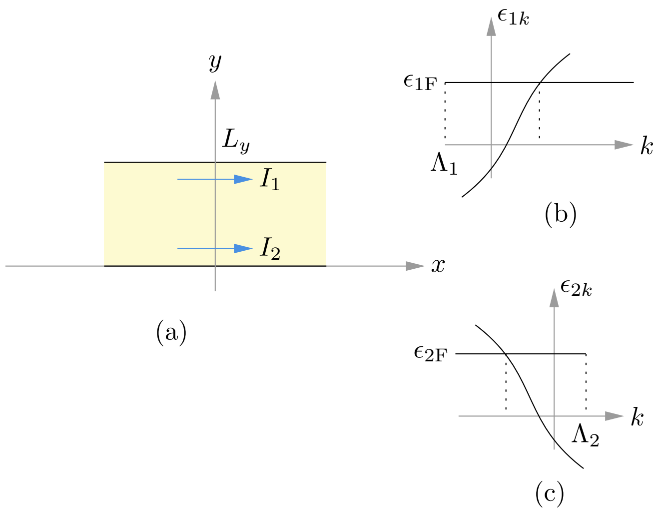

Figure 1: Boundary modes of a 2D Chern insulator (a) The device and sign conventions (b) The spectrum of the boundary modes at $y = L_y$, where Λ_1 is a momentum cutoff (c) The spectrum of the boundary modes at $y = 0$, where Λ_2 is a momentum cutoff

and therefore

$$I = -n\frac{e}{h} \cdot eU = \underbrace{-n\frac{e^2}{h}}_{1/R_{\mathrm{H}}} U. \tag{8}$$

So we get the expected quantized conductance. Since we are working in a 2D material, by definition

$$\sigma_{\mathrm{H}} = \frac{1}{R_{\mathrm{H}}} = -n\frac{e^2}{h}. \tag{9}$$

(d) Ignoring excited bulk modes, we have

$$\begin{aligned} I_1 &= -n\frac{e}{\hbar}\int_{\Lambda_1}^{\infty} \frac{\mathrm{d}k}{2\pi} \frac{1}{1+\mathrm{e}^{(\epsilon_{1k}-\mu_1)/k_{\mathrm{B}}T}} \frac{\partial \epsilon_{1k}}{\partial k} \\ &= -n\frac{e}{h}\int_{\epsilon_{1,k=\Lambda_1}-\mu_1}^{\infty} \mathrm{d}\xi\, \frac{1}{1+\mathrm{e}^{\xi/k_{\mathrm{B}}T}}, \end{aligned} \tag{10}$$

and similarly

$$\begin{aligned} I_2 &= -n\frac{e}{\hbar}\int_{-\infty}^{\Lambda_2} \frac{\mathrm{d}k}{2\pi} \frac{1}{1+\mathrm{e}^{(\epsilon_{2k}-\mu_2)/k_{\mathrm{B}}T}} \frac{\partial \epsilon_{2k}}{\partial k} \\ &= -n\frac{e}{h}\int_{\infty}^{\epsilon_{2,k=\Lambda_2}-\mu_2} \mathrm{d}\xi\, \frac{1}{1+\mathrm{e}^{\xi/k_{\mathrm{B}}T}} = n\frac{e}{h}\int_{\epsilon_{2,k=\Lambda_2}-\mu_2}^{\infty} \mathrm{d}\xi\, \frac{1}{1+\mathrm{e}^{\xi/k_{\mathrm{B}}T}}, \end{aligned} \tag{11}$$

and therefore

$$I = -n\frac{e}{h}\int_{\epsilon_{1,k=\Lambda_1}-\mu_1}^{\epsilon_{2,k=\Lambda_2}-\mu_2} \frac{\mathrm{d}\xi}{1+\mathrm{e}^{\xi/k_{\mathrm{B}}T}}. \tag{12}$$

Since the Fermi-Dirac distribution function is always positive, the condition that when $\Delta\mu = 0$, $I = 0$ implies $\epsilon_{1,k=\Lambda_1} = \epsilon_{2,k=\Lambda_2} =: \epsilon_\Lambda$, so we have

$$I = -n\frac{e}{h}\int_{\epsilon_\Lambda-\mu_1}^{\epsilon_\Lambda-\mu_2} \frac{\mathrm{d}\xi}{1+\mathrm{e}^{\xi/k_{\mathrm{B}}T}}. \tag{13}$$

Since Λ must be large, a very good approximation is

$$I = -n\frac{e}{h}\Delta\mu = -n\frac{e^2}{h}U, \tag{14}$$

so the edge current and the Hall conductance are not affected much by a finite T.

Problem 2 In general, one can construct a Hall conductance for every conserved quantity. The usual electric Hall conductance σ_H originates from conservation of charge. Of course, most systems have another conserved quantity, which is even more fundamental: energy. Correspondingly, there is another important Hall conductance - the "thermal Hall conductance" κ_H. Just as the electric Hall conductance is defined by the electric current induced by a chemical potential difference between the top and bottom edges of a Hall bar, the thermal Hall conductance is defined by the energy or heat current I_Q induced by a temperature difference ΔT between the two edges:

$$\kappa_H = \left.\frac{\partial I_Q}{\partial \Delta T}\right|_{\Delta T \to 0}.$$

In this problem we compute the thermal Hall conductance for a Chern insulator with Chern number n. 1. As above, we begin with the $y = L_y$ edge. Each of the n edge modes near $y = L_y$ can be modeled as a noninteracting free fermion model $H = \sum_k (\epsilon_{1k} - \mu_1) c_k^\dagger c_k$, where ϵ_{1k} is a monotonically increasing function of k. What is the energy/heat current as a function of temperature, $I_{Q1}(T)$? You may write your answer in terms of a dimensionless integral. 2. Compute κ_H. You should find $\kappa_H = n\frac{\pi^2}{3h}T$ if you evaluate the integral explicitly. Remark: The thermal Hall conductance κ_H is quantized in multiples of $\frac{\pi^2}{3h}T$, just like the electric Hall conductance is quantized in multiples of $\frac{e^2}{h}$, and it is equally robust against interactions and disorder. The two quantities are almost perfect analogues (though it is unclear what the analogue of the Laughlin argument is for the thermal case). 3. Can one construct a "mixed" thermo-electric Hall conductance that describes the electric current induced by a temperature difference between the two edges, or the energy/heat current induced by a chemical potential difference?

Solution

(a) The only change we need to do is to replace the electric charge $-e$ by the "energy charge" ϵ_k, and following the line of reasoning in Problem 1 (d), we have

$$\begin{aligned} I_{Q1} &= n\frac{1}{\hbar}\int_{\Lambda_1}^{\infty} \epsilon_{1k}\frac{\mathrm{d}k}{2\pi}\frac{1}{1+\mathrm{e}^{(\epsilon_{1k}-\mu_1)/k_\mathrm{B}T}}\frac{\partial \epsilon_{1k}}{\partial k} \\ &= \frac{n}{h}\int_{\epsilon_{1,k=\Lambda_1}}^{\infty} \mathrm{d}\epsilon\, \frac{1}{1+\mathrm{e}^{(\epsilon-\mu_1)/k_\mathrm{B}T_1}}(\epsilon-\mu_1), \end{aligned} \tag{15}$$

and similarly

$$I_{Q2} = -\frac{n}{h}\int_{-\infty}^{\epsilon_{2,k=\Lambda_2}} \frac{1}{1+\mathrm{e}^{(\epsilon-\mu_2)/k_\mathrm{B}T_2}}(\epsilon-\mu_2). \tag{16}$$

Because of the chemical potential term appears in the Hamiltonian, we need to use $\epsilon - \mu$ as the energy carried by an electron. Here since there is no external field to establish a chemical potential gradience, we have $\mu_1 = \mu_2 =: \mu$. The sign difference between I_{Q1} and I_{Q2} comes in the same way of (11). Again, when $T_1 = T_2$, we need $I_Q = I_{Q1} + I_{Q2} = 0$, so $\epsilon_{1,k=\Lambda_1} = \epsilon_{2,k=\Lambda_2} =: \epsilon_\Lambda$.

Another step to simply the expressions is to notice that although the cutoff ϵ_Λ is important to have well-defined I_{Q1} and I_{Q2}, since only $I_{Q1} + I_{Q2}$ matters – which seems to be well defined if we take the difference between the two integrated functions in I_{Q1} and I_{Q2} and integrate it from $-\infty$ to ∞ – formally we can just extend $\epsilon_\Lambda \to 0$ before we calculate $I_{Q1} + I_{Q2}$. So finally we get (since what appears in the integral is always $\epsilon - \mu$, we have already replaced $\epsilon - \mu$ by ϵ)

$$I_{Q1} = \frac{n}{h}\int_{-\infty}^{\infty} \mathrm{d}\epsilon\, \epsilon f_{T_1}(\epsilon), \tag{17}$$

and

$$I_{Q2} = -\frac{n}{h}\int_{-\infty}^{\infty} \mathrm{d}\epsilon\, \epsilon f_{T_2}(\epsilon). \tag{18}$$

We can't just make the integrals dimensionless, because

$$(k_\mathrm{B}T_1)^2\int_{-\infty}^{\infty} x\,\mathrm{d}x\,\frac{1}{\mathrm{e}^x+1}$$

is a ill-defined quantity. The total heat current is

$$
\begin{aligned}
I_Q = I_{Q1} + I_{Q2} &\approx \frac{n}{h} \underbrace{(T_1 - T_2)}_{\Delta T} \int_{-\infty}^{\infty} \frac{\partial f_T(\epsilon)}{\partial T} \epsilon \, \mathrm{d}\epsilon \\
&= \frac{n}{h} \Delta T \frac{1}{k_{\mathrm{B}} T^2} \int_{-\infty}^{\infty} \mathrm{d}\epsilon \, \epsilon^2 \frac{\mathrm{e}^{\epsilon / k_{\mathrm{B}} T}}{(\mathrm{e}^{\epsilon / k_{\mathrm{B}} T} + 1)^2} \\
&= \Delta T \cdot \frac{n}{h} k_{\mathrm{B}}^2 T \int_{-\infty}^{\infty} \mathrm{d}x \, \frac{x^2 \mathrm{e}^x}{(\mathrm{e}^x + 1)^2}.
\end{aligned} \tag{19}
$$

(b) Mathematica tells us that

$$
\int_{-\infty}^{\infty} \mathrm{d}x \, \frac{x^2 \mathrm{e}^x}{(\mathrm{e}^x + 1)^2} = \frac{\pi^2}{3}, \tag{20}
$$

so we get

$$
I_Q = \frac{\pi^2 T}{3h} n k_{\mathrm{B}}^2, \tag{21}
$$

and if we take the unit of temperature to be the same as the unit of energy, then

$$
\kappa_{\mathrm{H}} = \left. \frac{\partial I_Q}{\partial \Delta T} \right|_{\Delta T \to 0} = \frac{\pi^2 T}{3h} n. \tag{22}
$$

(c) For chemical potential difference-induced heat transportation, following the derivation of (19), we have

$$
\begin{aligned}
I_Q &= \frac{n}{h} \Delta\mu \frac{\partial}{\partial \mu} \int_{-\infty}^{\infty} (\epsilon - \mu) \, \mathrm{d}\epsilon \, \frac{1}{1 + \mathrm{e}^{(\epsilon - \mu)/k_{\mathrm{B}} T}} \\
&= \Delta\mu \cdot \frac{n}{h} \int_{-\infty}^{\infty} (\epsilon - \mu) \, \mathrm{d}\epsilon \, \frac{\mathrm{e}^{(\epsilon - \mu)/k_{\mathrm{B}} T}}{(1 + \mathrm{e}^{(\epsilon - \mu)/k_{\mathrm{B}} T})^2} \\
&= \Delta\mu \cdot \frac{n}{h} (k_{\mathrm{B}} T)^2 \int_{-\infty}^{\infty} \mathrm{d}x \, k_{\mathrm{B}} T x \frac{\mathrm{e}^x}{(1 + \mathrm{e}^x)^2} \\
&= 0,
\end{aligned} \tag{23}
$$

so the transverse thermal current induced by chemical potential difference is always zero.

Similarly

$$
\begin{aligned}
I &= -e \frac{n}{h} \Delta T \cdot \frac{\partial}{\partial T} \int_{-\infty}^{\infty} \mathrm{d}\epsilon \, \frac{1}{1 + \mathrm{e}^{(\epsilon - \mu)/k_{\mathrm{B}} T}} \\
&= \Delta T \cdot (-1) \cdot \frac{en}{h} \int_{-\infty}^{\infty} \mathrm{e}^{(\epsilon - \mu)/k_{\mathrm{B}} T} \frac{1}{(1 + \mathrm{e}^{(\epsilon - \mu)/k_{\mathrm{B}} T})^2} \frac{\epsilon - \mu}{k_{\mathrm{B}} T^2} \, \mathrm{d}\epsilon \\
&= \Delta T \cdot (-1) \cdot \frac{en}{h} k_{\mathrm{B}} \int_{-\infty}^{\infty} x \, \mathrm{d}x \, \frac{\mathrm{e}^x}{(1 + \mathrm{e}^x)^2} \\
&= 0,
\end{aligned} \tag{24}
$$

so the transverse electric current induced by temperature difference is also zero. Thus it makes no sense to define thermo-electric Hall conductance: it's always zero.