

ODEs

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1 First order ODEs

1.1 Linear ODEs

An ODE in the form of

$$y'(x) + p(x)y(x) = q(x) \quad (1)$$

is considered **linear**. All linear ODEs can be solved by the following procedure. First we have

$$(y' + py)e^{\int p dx} = qe^{\int p dx}, \quad (2)$$

and now the LHS is a derivative:

$$\frac{d}{dx} \left(ye^{\int p dx} \right) = qe^{\int p dx}, \quad (3)$$

and now we can integrate over x and get

$$ye^{\int p dx} = \int qe^{\int p dx} dx, \quad (4)$$

$$y = e^{-\int p dx} \int qe^{\int p dx} dx. \quad (5)$$

1.2 “Energy-conservation lines” and exact equations

Another way to represent the solution of an ODE is the form $\phi(x, y) = \text{const}$. Note that the RHS contains no variables, and we have

$$0 = \frac{d\phi}{dx} = \frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y} \frac{dy}{dx}, \quad (6)$$

and thus if

$$y' = f(x, y) \quad (7)$$

is algebraically equivalent to (6), the equation is already solved: We should find M, N such that

$$y' = -\frac{M}{N}, \quad M = \frac{\partial \phi}{\partial x}, \quad N = \frac{\partial \phi}{\partial y}, \quad (8)$$

and then $\phi(x, y)$ solves the equation. In this case we say $y' = -M/N$ is **exact**.

To test for exactness, we only have to test whether

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}, \quad (9)$$

and if so, the existence of ϕ is guaranteed. (Since we work on a topological trivial space, things like cohomology group will not bother us.) We can now use “partial integral” to find ϕ .

Example: suppose in a calculation we find

$$\frac{\partial \phi}{\partial x} = 2y^2 + ye^{xy}, \quad \frac{\partial \phi}{\partial y} = 4xy + xe^{xy} + 2y. \quad (10)$$

After partial integration, we find

$$\phi(x, y) = \underbrace{2xy^2 + e^{xy} + h(y)}_{\int \frac{\partial \phi}{\partial x} dx} = \underbrace{2xy^2 + e^{xy} + y^2 + g(x)}_{\int \frac{\partial \phi}{\partial y} dy}, \quad (11)$$

and we have to choose

$$h(y) = y^2, \quad g(x) = \text{const}, \quad (12)$$

and the solution is

$$\phi(x, y) = 2xy^2 + e^{xy} + y^2 + \text{const}. \quad (13)$$

Note that even when the decomposition $f = -M/N$ doesn't give an exact equation for us, we can still use the method of exact equations: we can multiply a factor μ to both M and N , and try to guess the form of μ so that

$$\frac{\partial(\mu M)}{\partial y} = \frac{\partial(\mu N)}{\partial x}. \quad (14)$$

An example can be found in solving

$$y' = -\frac{1}{3x - e^{-2y}}. \quad (15)$$

We have

$$\frac{\partial 1}{\partial y} = 0, \quad \frac{\partial(3x - e^{-2y})}{\partial x} = 3,$$

so the equation is not exact if we choose $M = 1$ and $N = 3x - e^{-2y}$. However, (14) can be fulfilled now: it's now

$$\frac{\partial \mu}{\partial y} = 3\mu + (3x - e^{-2y}) \frac{\partial \mu}{\partial x},$$

and the most convenient way to solve it (we *don't* need to find all solutions of this equation!) is to let μ contain y only, so the tricky term on the RHS disappears, and thus we choose $\mu = e^{3y}$, and we get

$$\phi(x, y) = \int \mu M \, dx = \int e^{3y} \, dx = xe^{3y} + u(y),$$

$$\phi(x, y) = \int \mu N \, dy = \int (3xe^{3y} - e^y) \, dy = xe^{3y} - e^y + v(x),$$

so

$$\phi(x, y) = xe^{3y} - e^y + \text{const}. \quad (16)$$

1.3 Bernoulli equation

Consider the following **Bernoulli equation**

$$y' + P(x)y = R(x)y^\alpha. \quad (17)$$

When $\alpha = 0, 1$, the equation can be solved by the standard methods for linear first order ODEs. When this is not the case, we may do the substitution

$$v = y^\beta, \quad (18)$$

and then the equation becomes

$$\begin{aligned} \frac{1}{\beta} v^{1/\beta-1} v' + P(x)v^{1/\beta} &= R(x)v^{\alpha/\beta}, \\ v' + P(x)v &= R(x)v^{1+\frac{\alpha-1}{\beta}}. \end{aligned} \quad (19)$$

The next step is to choose a good beta so that the equation gets simplified. We may want to make to exponent to be zero, and this means we should choose

$$\beta = 1 - \alpha, \quad (20)$$

and the ODE is now

$$v' + Pv = R, \quad (21)$$

which can then be solved by the method in Section 1.1.

2 Second order ODEs

2.1 Linear 2nd order ODE with initial values

A linear second order ODE has the following form:

$$y'' + p(x)y' + q(x)y = f(x). \quad (22)$$

It usually comes with initial value conditions

$$y(x_0) = A, \quad y'(x_0) = B. \quad (23)$$

This course is about concrete calculations, but knowing what we are doing makes sense is important. Here is an existence and uniqueness theorem: if $p(x)$, $q(x)$, and $f(x)$ are continuous over an interval I , and $x_0 \in I$, then a unique solution exists for (22) with the initial conditions given above.

Usually, we start by looking at the **homogeneous** second order ODE

$$y'' + p(x)y' + q(x)y = 0. \quad (24)$$

The influence of $f(x)$ can be included as the “response” of the LHS. The full solution of (24) takes the form

$$y = c_1 y_1 + c_2 y_2, \quad (25)$$

where c_1, c_2 are constants to be decided by initial conditions, and y_1 and y_2 are linearly independent solutions of (24). The **Wronskian** is defined as

$$W(x) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}. \quad (26)$$

By checking if it is non-zero at most points, we can find whether y_1 and y_2 are truly linearly independent to each other.

There is a method to arrive at y_2 from y_1 : we can always take the ansatz

$$y_2 = y_1 u, \quad (27)$$

and therefore we get

$$(u'' y_1 + 2u' y_1' + u y_1'') + p(u' y_1 + u y_1') + q u y_1 = 0,$$

and the condition that y_1 is a solution to (24) means

$$u'' + \underbrace{\frac{2y_1' + p y_1}{y_1}}_{g(x)} u' = 0, \quad (28)$$

which is essentially a first order ODE, because we can replace u' by v , and then we find

$$\ln v = - \int g(x) dx,$$

and

$$u(x) = \int e^{-\int g(x) dx} dx. \quad (29)$$

2.2 Constant coefficients

The equation

$$y'' + Ay + By = 0 \quad (30)$$

can be solved directly by the following construction:

$$y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}, \quad (31)$$

where λ_1, λ_2 are solutions of

$$\lambda^2 + A\lambda + B = 0. \quad (32)$$

For example, to solve the equation

$$y'' - 2y' + 10y = 0, \quad (33)$$

we just solve

$$\lambda^2 - 2\lambda + 10 = 0,$$

which gives us

$$\lambda = 1 \pm 3i, \quad (34)$$

and therefore a general solution is

$$y = e^x(c_1 e^{3ix} + c_2 e^{-3ix}). \quad (35)$$

It should be noted that c_1, c_2 can be complex, even when we restrict y in \mathbb{R} : we can let the imaginary part of y vanish as long as we impose some constraints over c_1, c_2 . If we are determined to work in the real space, two alternative linearly independent solutions can be used:

$$y_1(x) = e^x \cos(3x), \quad y_2(x) = e^x \sin(3x). \quad (36)$$

Although we can immediately say they are linearly independent, we can use them as a demonstration of the Wronskian method: now we have

$$W(x) = y_1(x)y_2'(x) - y_2(x)y_1'(x) = 3e^{2x}, \quad (37)$$

which of course isn't constantly zero.

(32) is faced with the problem of having only one solution when $A^2 - 4B = 0$. In this case we need to go back to the standard procedure to get y_2 from y_1 . An example is

$$y'' + 6y + 9 = 0, \quad (38)$$

for which (32) only gives

$$y_1 = e^{-3x}. \quad (39)$$

Suppose $y_2 = ue^{-3x}$, we have TODO

2.3 Euler equation

A **Euler equation** has the following form:

$$x^2 y'' + Axy' + By = 0, \quad (40)$$

where A, B are constants. One solution can be immediate found: it always looks like

$$y = x^a. \quad (41)$$

We then find

$$a(a-1) + Aa + B = 0. \quad (42)$$

If there are two solutions of the equation, (40) has already been solved. If not, we can use the trick (27).

An example: let's solve

$$x^2 y'' + 3xy' + y = 0. \quad (43)$$

The equation about a is now

$$a(a-1) + 3a + 1 = 0,$$

and it only has one solution $a = -1$. Therefore we have

$$y_1 = \frac{1}{x}.$$

Suppose

$$y_2 = uy_1,$$

we get

$$x^2 \left(\frac{u''}{x^2} - \frac{2u'}{x} + \frac{2u}{x^2} \right) + 3x \left(\frac{u'}{x} - \frac{u}{x^2} \right) + \frac{u}{x} = 0,$$

which is equivalent to

$$v'x + v = 0, \quad v = u'$$

the solution of which is

$$\ln v + \ln x = \text{const},$$

and therefore

$$v = \frac{C'}{x}, \quad u = C' \ln x + C,$$

$$y_2 = \frac{1}{x}(C' \ln x + C).$$

This essentially gives *all* solutions we need: for y_1 , we just have $u = 1$, which corresponds to $C = 1$. So now the equation is completely solved.

2.4 Non-homogeneous cases or how to find the linear response

Now we discuss how to solve

$$y'' + p(x)y' + q(x)y = f(x). \quad (44)$$

A general solution is

$$y(x) = y_p(x) + y_h(x), \quad (45)$$

where the subscript p means a particular solution, and the subscript h means the general solution of the corresponding homogeneous equation.

We need some common sense to find a particular solution. To solve

$$y'' - y' - 2y = 2x^2 + 5, \quad (46)$$

we don't expect y to be, say, $\cos(2x)$: instead, it's usually the case that y is a polynomial. An ansatz is

$$y = Ax^2 + Bx + C.$$

We don't want a x^3 term because it doesn't appear on RHS. The equation then becomes

$$2A - (2Ax + Bx) - 2(Ax^2 + Bx + C) = 2x^2 + 5,$$

$$-2A = 2, \quad -2A - 2B = 0, \quad 2A - B - 2C = 5,$$

and therefore $A = -1, B = 1, C = -4$. Therefore we get a particular solution:

$$y_p = -x^2 + x - 4. \quad (47)$$

A particular solution may also be determined as a "linear combination" of homogeneous solutions, although now the coefficients have temporal variation. That's to say, we take the ansatz

$$y_p(x) = u(x)y_1(x) + v(x)y_2(x). \quad (48)$$

This is quite similar to the procedure introduced in Section 1.1. After substituting y with (48) in (44), we get

$$u'y'_1 + v'y'_2 = f. \quad (49)$$

Introducing the constraint

$$u'y_1 + v'y_2 = 0, \quad (50)$$

we have

$$\begin{pmatrix} y_1 & y_2 \\ y'_1 & y'_2 \end{pmatrix} \begin{pmatrix} u' \\ v' \end{pmatrix} = \begin{pmatrix} 0 \\ f \end{pmatrix}, \quad (51)$$

from which we find u', v' and hence u, v . The Wronskian – the determinant of the matrix on LHS – is non-zero, so the equation always has a solution.

Example: let's solve

$$y'' + y = \tan x. \quad (52)$$

We have

$$y_1 = \cos x, \quad y_2 = \sin x,$$

and therefore

$$W(x) = y_1 y_2' - y_2 y_1' = 1.$$

So

$$\begin{aligned} u' &= - \int \frac{y_2 f}{W(x)} dx = - \int \frac{\sin^2 x}{\cos x} dx = - \int \frac{1 - \cos^2 x}{\cos x} dx \\ &= -\frac{1}{2} \ln \left| \frac{1 + \sin x}{1 - \sin x} \right| + \sin x, \end{aligned}$$

and similarly we have

$$v =$$

2.5 Analyticity

The stimulus can be non-analytic. f is analytic at x_0 , if we can expand it into a power series around x_0 :

$$f(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n \quad (53)$$

in a interval around x_0 . The function $f(x) = \ln x$, then, is not analytic at $x = 0$ – but $f(x) = \ln(x + 1)$ is analytic at $x = 0$, though not at $x = -1$.

There is a theorem: if p, q, f are all analytic at x_0 , then (44) together with conditions $y(x_0) = A$ and $y'(x_0) = B$ has a unique solution that is analytic at x_0 .

An example:

$$x'' + y' - xy = 0, \quad y(0) = -2, y'(0) = 0. \quad (54)$$

Suppose

$$y(x) = \sum_{n=0}^{\infty} a_n x^n, \quad y'(x) = \sum_{n=0}^{\infty} n a_n x^{n-1}, \quad y''(x) = \sum_{n=0}^{\infty} n(n+1) a_n x^{n-2},$$

we have

$$\begin{aligned} 0 &= \sum_{n=0}^{\infty} n(n-1) a_n x^{n-2} + \sum_{n=0}^{\infty} n a_n x^{n-1} - \sum_{n=0}^{\infty} a_n x^{n+1} \\ &= a_1 + 2a_2 + \sum_{n=1}^{\infty} x^n ((n+2)(n+1)a_{n+2} + (n+1)a_{n+1} - a_{n-1}), \end{aligned}$$

and therefore

$$a_1 + 2a_2 = 0, \quad a_{n+2} = -\frac{a_{n+1}}{n+2} + \frac{a_{n-1}}{(n+1)(n+2)}.$$

The conditions $y(0) = -2$ and $y'(0) = 0$ means

$$a_0 = -2, \quad a_1 = 0,$$

and then we can in principle find all a_n 's – although it's often hard to see a pattern and write down a closed-form expression for a_n .

Now we consider the equation

$$P(x)y'' + Q(x)y' + R(x)y = F(x). \quad (55)$$

Of course we can divide the equation with $P(x)$ and go back to (44), but if $P(x)$ is zero at some points, p, q, f in (44) are no longer always analytic. Thus the solution isn't guaranteed to be

analytic everywhere. In other words, we no longer have a power series solution – or do we? We can still try a generalized power series, like

$$y(x) = \sum_{n=0}^{\infty} c_n (x - x_0)^{n+r}, \quad (56)$$

where r can be a fraction. This is guaranteed with

$$(x - x_0)y'' + Q(x)y' + R(x)y = F(x). \quad (57)$$

To demonstrate this, consider

$$y'' + \frac{1}{2x}y' - \frac{1}{4x}y = 0. \quad (58)$$

We plug

$$y(x) = \sum_{n=0}^{\infty} c_n x^{n+r}$$

into

$$4xy'' + 2y' - y = 0,$$

and get

$$\sum_{n=0}^{\infty} (4(n+r)(n+r-1)c_n x^{n+r-1} + 2(n+r)c_n x^{n+r-1} - c_n x^{n+r}) = 0.$$

The coefficient of the x^{r-1} term is

$$4r(r-1) + 2r = 0,$$

from which we find $r = 0, 1/2$. For the rest of the terms, we have

$$4(n+r)(n+r-1)c_n + 2(n+r)c_n - c_{n-1} = 0,$$

$$c_n = \frac{c_{n-1}}{2(n+r)(2n+2r-1)}.$$

When $r = 0$, this gives

$$c_n = \frac{c_{n-1}}{2n(2n-1)}, \quad c_n = \frac{c_0}{(2n)!},$$

and we get

$$y(x) = \sum_{n=0}^{\infty} \frac{x^n}{(2n)!}. \quad (59)$$

When $r = 1/2$, this gives

$$c_n = \frac{c_{n-1}}{(2n+1) \cdot 2n}, \quad c_n = \frac{c_0}{(2n+1)!},$$

and we get

$$y(x) = \sum_{n=0}^{\infty} \frac{x^{n+1/2}}{(2n+1)!}. \quad (60)$$