Elasticity

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1 The theoretical framework

1.1 The displacement field

For now, we use $\boldsymbol{r}$ to refer to the position vector of a position in a continuum, and $\boldsymbol{r}'(\boldsymbol{r}, t)$ its corresponding position at time t – but note that in this note sometimes we will need to use the symbol $\boldsymbol{r}$ to refer to $\boldsymbol{r}'$ (Section 1.3.2). The displacement field is therefore

$$\boldsymbol{u}(\boldsymbol{r}, t) = \boldsymbol{r}'(\boldsymbol{r}, t) - \boldsymbol{r}. \tag{1}$$

A rough estimation of the number of degrees of freedom implies that all information about the material – the position of each atom – has already been stored in $\boldsymbol{u}(\boldsymbol{r}, t)$. To see why, note that we can do Fourier transform to $\boldsymbol{u}(\boldsymbol{r}, t)$ in variable $\boldsymbol{r}$. Suppose the size of the system is $\sim L^d$, where d is the dimension of the system, and the microscopic length scale of the system is $\sim a$. The wave vector components of $k_{x,y,z}$ therefore are confined to the sequence that starts with 0 and ends with $2\pi/a$ (the microscopic cutoff), with a step of $2\pi/L$; the total length of this sequence is $\simeq L/a$, and thus the total number of possible wave vectors is $(L/a)^d$. Thus the number of real number variables included in the field variable $\boldsymbol{u}(\boldsymbol{r}, t)$ at a given time step is $d \cdot (L/a)^d$: the prefactor d comes from the d components of $\boldsymbol{u}$.

On the other hand, there are $\simeq (L/a)^d$ atoms in the system, and each of them has d directions of motion, and therefore, we find that number of real number variables included in $\boldsymbol{u}(\boldsymbol{r}, t)$ is the same as the number of real number variables of the atoms, and thus $\boldsymbol{u}(\boldsymbol{r}, t)$ contains all the information contained in the system. This should not be surprising: that we have a wave vector cutoff $\simeq 2\pi/a$ means we have a real space resolution of $\simeq a$, so at each time step t, $\boldsymbol{u}(\boldsymbol{r}, t)$ can be completely described by a real space grid with the separation between the sample points being $\simeq a$ – and the points in the grid is just equivalent to the initial positions of the atoms.

There are however some subtleties in the above argument. If the material is not a crystal, when $k \simeq 2\pi/a$, the translational symmetry is already broken, and therefore the wave vector is not well-defined any more; thus the wave vector cutoff should be set to a much lower value. If the system is an insulator, we can take a to be the length scale of the primitive unit cell, and $2\pi/a$ is the magnitude of the width of the first Brillouin zone. If there is only one kind of atom, then indeed $\boldsymbol{u}(\boldsymbol{r}, t)$ contains all information in the system; but often we have more than one kind of atoms (two atoms that are not connected by any symmetry operations should be considered to be two kinds of atoms, even when they are of the same species), and then $\boldsymbol{u}(\boldsymbol{r}, t)$ only contains the acoustic phonon modes.

So in conclusion, the displacement field $\boldsymbol{u}(\boldsymbol{r}, t)$ only contains the acoustic modes, both in crystals and non-crystals; in both cases there may be additional hidden microscopic degrees of freedom. These hidden degrees of freedom, like dislocation, can be treated by introducing various discontinuities to $\boldsymbol{u}(\boldsymbol{r})$.

Below we often use u_i to represent $\boldsymbol{u}$, since we are going to deal with tensors frequently and it's a good idea to keep the notation the same as that of higher order tensors.

1.2 General guidelines of elastic dynamics

Let's now move to the following question of what the dynamics of an elastic solid system looks like. In this section we progressively introduce assumptions on this topic.

1.2.1 Dynamic variables: u_i only

The first assumption we introduce is *the dynamics of the system should be able to be cast into a form that is only about u_i and its conjugate momentum*. Note that even when more dynamic variables are involved, we still can use, say, Mori-Zwanzig formalism to get a theory about u_i only, and thus in order for this approximation to be truly restrictive, we need to impose some implicit constraints on the formalism used when modeling the dynamics. If we constrain the formalism to be Hamiltonian dynamics plus some dissipation terms, then essentially we are assuming that the material is elastic (Section 1.2.2).

This assumption means the system we are investigating into doesn't have typical fluid behavior: in Navier-Stokes equation, the density $\rho(\boldsymbol{r}, t)$ is also a dynamic variable which can be used to decide p on the RHS of the equation;[1] this isn't necessary in solid mechanics: even when we do have ρ dependence in the EOMs, it should be able to be decided explicitly by, say, $\boldsymbol{\nabla} \cdot \boldsymbol{u}$.[2]

In other words, when $T = 0$, the Hamiltonian of the system is basically a Hamiltonian of phonons. The quantum Hamiltonian of fluid, on the other hand, involves ρ and $\boldsymbol{v}$ (and a strange commutation relation between them) [1].

Sometimes we do need corrections to this assumption: it's possible that long-range electromagnetic fields are created, and in this case $\boldsymbol{E}$ and $\boldsymbol{B}$ should also be included into the dynamic variables (or otherwise we have memory effects, retardation effects, etc.). The approximation therefore may be loosened to *no degree of freedom other that u_i that is about the atomic positions in the system is needed in the dynamics*.

1.2.2 Elasticity: statelessness and immediate response

Elasticity is sometimes defined as the follows: *when we stop applying force to the system, it always goes back to one equilibrium shape*. In other words, if we push the system and then stop, the system bounces back.

This definition is too weak: it may be possible that after a loading-unloading process, although the system goes back to its original shape, its inner structure has changed and its reaction to another round of loading and unloading is different from that of the previous round. So we need to impose a stronger formulation of elasticity: *the deformation of the system has nothing to do with its history and is completely decided by the force applied to it*. This formulation implies the weaker version: when no load is present, the system has one and only one configuration.

The above formulation however has some ambiguity. In experiments usually the force applied to the system is changed very slowly, and the system is quasi-static. In dynamics, however, we may be interested in oscillation, where the force applied changes quickly, and now it's possible that the response of the system is retarded;[3] if we go to frequency space, we may find that the response of the system has frequency dependence. This is of course possible, and we may still say that the deformation of the system is decided by the force applied to it. Often, we make a even stronger assumption: *the deformation of the system in a particular moment is complete decided by the force applied to it in that very moment*. This means the system has immediate response to the force applied to it.

1.2.3 Dissipation and finite temperature effects

When $T = 0$, we may want to use a Hamiltonian formalism (a quantum many-body theory for phonons, actually) to describe the elastic system; this however is usually not sufficient since we have various dissipations: it's inevitable that some weight flows into the hidden degrees of freedom that we lose track of.[4] An accurate formalism for dissipation is hard since it involves quantum master equation of the density matrix and dissipative quantum jump channels; but in ordinary elastic theory, usually we don't care about what happens to the quantum many-body

[1] When the fluid is incompressible, p needs to be used in place of ρ as a dynamic variable.

[2] Note that solid-like dynamics is not the logical opposition of fluid-like dynamics: it's possible to have systems with both solid-like and fluid-like behaviors. These materials are studied in rheology.

[3] Note that as is known in effective field theory, the response of the system to external driving force and internal force should be the same, as long as the external driving force is coupled to the deformation of the system in the same way internal forces are coupled to the deformation system, and since we call external forces "force" this should be true.

[4] It's still possible that even the system does have a dissipation-less Hamiltonian description, there is still damping behavior for a single mode, when the mixing between modes is too severe (as in, say, Landau damping). Since all degrees of freedom are well kept track of, this is "dissipation-less damping".

wave function, and all we want to know is $\langle u_i \rangle$, ignoring its higher order correlations. Thus usually dissipation is just modeled by adding a term like $\partial_t u_i$ to the EOM.

Another issue is finite temperature effects. Here we do *local equilibrium approximation*: we assume that when $T \neq 0$, there can be T gradience in the system, but *at each given point, we have local equilibrium with the local temperature, the force and the deformation being thermodynamic coordinates, and the relaxation time is ignored*. Local equilibrium approximation is equivalent to the condition that $\omega\tau \ll 1$, *where ω is the characteristic frequency scale of the dynamic and τ the relaxation time*. This corresponds to the $\omega\tau \ll 1$ limit in Fermi liquid kinetics, where we have ordinary hydrodynamics and ordinary sound mode [2]. Again, the most accurate treatment of finite temperature effects involves quantum kinetic theory, but in the $\omega\tau \ll 1$ limit the quantum kinetic theory is reduced to local thermodynamics.

1.2.4 The form of EOM with local equilibrium

Local equilibrium gives us well-defined local thermodynamic variables like the entropy density or the temperature; still, we need to find some way to incorporate them into a dynamic theory. The variables u_i and the internal force density f_i are not thermalized and are dynamic variables; if we focus on a small volume element of the system and use a coordinate frame attached to that volume, the $\rho\, \partial^2 u_i / \partial t^2$ term can be seen as an external field, and thus from balance of forces we have

$$\frac{\delta \Phi[\boldsymbol{u}]}{\delta u_i} = -\rho \frac{\partial^2 u_i}{\partial t^2} + f_i^{\text{dissipation}}(\partial_t \boldsymbol{u}, \partial_t^2 \boldsymbol{u}, \ldots),$$

and thus

$$\rho \frac{\partial^2 u_i}{\partial t^2} = -\frac{\delta \Phi[\boldsymbol{u}]}{\delta u_i} + f_i^{\text{dissipation}}(\partial_t \boldsymbol{u}, \partial_t^2 \boldsymbol{u}, \ldots), \tag{2}$$

where Φ is the thermodynamic potential, and the functional derivative is taken by assuming the conditions that make Φ the appropriate thermodynamic potential (like, when Φ is U, we need to assume an adiabatic process, while when Φ is F, we are working with an isothermal process). Choosing the appropriate thermodynamic potential is important: for statics, we usually assume local isothermal evolution, while for things like sound propagation, we usually assume local adiabatic evolution (although a serious treatment of the process also needs modeling of, say, heat transportation). When $T \to 0$, F becomes the total energy of the system and we go back to Hamiltonian dynamics.

Formally, here we insert the free energy into the Lagrangian as the potential energy; of course this raises the doubt about mixing mixed state and pure state formalisms, and a microscopic justification of it should be done. From a microscopic perspective, (2) can also be derived from a kinetic theory; as [2] says, for example, conservation laws can be obtained from the quantum Boltzmann equation of Fermi liquid, from which we get Navier-Stokes equations or similar equations, where the internal forces of the system can be obtained by taking the derivatives of certain thermodynamic potentials; actually this line of reasoning is exactly the opposite of the equilibrium-to-near-equilibrium line of reasoning above: if we start from a microscopic kinetic theory, we derive the whole thermodynamics at the end of our derivation, where we ignore all time evolution.

It's kind of silly to spend so much time to get (2), but we have seen that for many frequently seen materials, we need to go beyond (2): when the system has strong fluid-like behavior, when we have plastic deformation, and when there is retardation.

1.3 Stress

1.3.1 There are only surface forces

For real elastic systems, the form of the EOM often can be further constrained. First, $F[\boldsymbol{u}]$ *is usually local*, and thus we have

$$\frac{\delta F[\boldsymbol{u}]}{\delta u_i} = \frac{\partial f}{\partial u_i}, \quad F = \int \mathrm{d}^d \boldsymbol{r}\, f, \tag{3}$$

where f is the volume density of free energy; in the rest of this note we simply use F to refer to the volume density of free energy, because f may be confused with "force". Moreover, we

can assume that the length scale of the interaction between atoms is very small – if we have an interaction channel with a large length scale then we just introduce the relevant part of electromagnetic field into dynamic variables – and since the spatial resolution of $\boldsymbol{u}$ is taken to be much larger than a, effectively we can assume that *the internal forces are all surface forces*, and therefore we write down[5]

$$\rho \frac{\partial^2 u_i}{\partial t^2} = \partial_j \sigma_{ji} + f_i^{\text{dissipation}}, \tag{4}$$

where we have replaced the internal force with the divergence of a second order tensor σ_{ji}, so that

$$\int_V \mathrm{d}^d \boldsymbol{r}\, \partial_j \sigma_{ji} = \int_{\partial V} \mathrm{d}^{d-1} s_j\, \sigma_{ji}. \tag{5}$$

We can alternatively get σ_{ji} by writing down the conservation equation of the momentum and then σ_{ji} is the "current of momentum". We call σ_{ij} the **Cauchy stress tensor**. From (5) we immediately get the boundary condition about force passing: if the force acting on a surface element $\mathrm{d}^{d-1} s_i$ is $p_i\, \mathrm{d}^{d-1} s_i$, then at the position of the surface element, we have

$$p_i = n_j \sigma_{ji}, \tag{6}$$

where n_j is the jth component of the normal vector $\boldsymbol{n}$ of the surface element $\mathrm{d}^{d-1} s_i$, and we have

$$\mathrm{d}^{d-1} s_i = n_i\, \mathrm{d}s\,. \tag{7}$$

There is however an important subtlety in (4). When we take the time derivative on the LHS, we are doing so with $\boldsymbol{u} = \boldsymbol{u}(\boldsymbol{r}, t)$, and $\boldsymbol{r}$ is kept unchanged when we change t (recall that $\boldsymbol{r}$ is actually the continuous version of the index of atoms in the system). On the RHS, however, what I call $\boldsymbol{r}$ is actually $\boldsymbol{r}'$! This can be easily found by looking at (6): definitely the direction of $\boldsymbol{u}$ is decided by the surface *after* deformation, not before deformation. Now since $\boldsymbol{n}$ is defined in terms of $\boldsymbol{r}'$, then so is $\mathrm{d}^{d-1} s_j$ and hence $\mathrm{d}^d \boldsymbol{r}$: the last is just $\mathrm{d}^d \boldsymbol{r}'$.

In order to keep the meaning of ∂_t and $\partial_{\boldsymbol{r}}$ the same on both sides of (2), some remarks on the relation between $\boldsymbol{u}(\boldsymbol{r}, t)$ and $\boldsymbol{u}(\boldsymbol{r}', t)$ is needed. This is done in the next section.

1.3.2 Interlude: Lagrangian and Eulerian description

We say we are working with **Lagrangian**[6] description of the system if we work with $\boldsymbol{r}$ (i.e. the coordinates that are attached to the unstrained system) and t, and **Eulerian** description of the system if we work with $\boldsymbol{r}'$ and t.

The time derivative in the Lagrangian framework (it appears on the LHS of (2) and (4)) can be written in terms of Eulerian derivatives as follows:

$$\begin{aligned} \left(\frac{\partial}{\partial t}\right)_{\boldsymbol{r}} &= \left(\frac{\partial}{\partial t}\right)_{\boldsymbol{r}'} + \left(\frac{\partial \boldsymbol{r}'}{\partial t}\right)_{\boldsymbol{r}} \cdot \left(\frac{\partial}{\partial \boldsymbol{r}'}\right)_t \\ &= \left(\frac{\partial}{\partial t}\right)_{\boldsymbol{r}'} + \underbrace{\left(\frac{\partial \boldsymbol{u}}{\partial t}\right)_{\boldsymbol{r}}}_{=:\boldsymbol{v}} \cdot \left(\frac{\partial}{\partial \boldsymbol{r}'}\right)_t, \end{aligned} \tag{8}$$

where $\boldsymbol{v}$ can be evaluated directly from $\boldsymbol{u}$ in the Lagrangian description but is better regarded as a dynamic variable in the Eulerian description (as in Navier-Stokes equation). People sometimes call $(\partial_t)_{\boldsymbol{r}}$ the **material derivative** and use a distinct symbol to represent it, sometimes $\mathrm{D}/\mathrm{D}t$; I will just use $\mathrm{d}/\mathrm{d}t$, the meaning of which is self-evident if we regard $\boldsymbol{r}$ as a discrete index.

So the correct form of (4), in the Eulerian framework, is

$$\rho \left(\frac{\partial \boldsymbol{v}}{\partial t} + \boldsymbol{v} \cdot \boldsymbol{\nabla}' \boldsymbol{v}\right) = \frac{\partial \sigma_{ji}}{\partial r'_j} + f_i^{\text{dissipation}}, \tag{9}$$

or, if we want to consistently work in the Eulerian framework, we can just rename $\boldsymbol{r}'$ as $\boldsymbol{r}$ and ignore $\boldsymbol{r}$, and now the equation is

$$\rho \left(\frac{\partial \boldsymbol{v}}{\partial t} + \boldsymbol{v} \cdot \boldsymbol{\nabla} \boldsymbol{v}\right) = \frac{\partial \sigma_{ji}}{\partial r_j} + f_i^{\text{dissipation}}, \tag{10}$$

[5] The meaning of ∂_t on LHS has a subtlety discussed immediately below.

[6] Not to be confused with Lagrangian $\mathcal{L}$.

and we also have the additional equation about the definition of $\boldsymbol{v}$:

$$\boldsymbol{v} = \frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{v} \cdot \boldsymbol{\nabla} \boldsymbol{u}. \tag{11}$$

Note that the density is always the density changed by possible deformation and not the original density: we need the third equation – just the mass conservation equation

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot \rho \boldsymbol{v} = 0 \tag{12}$$

to complete the equation system about $\boldsymbol{u}, \boldsymbol{v}$ and ρ.

Note that the equation system also works for fluids; we still need to assume that σ_{ij} and $f_i^{\text{dissipation}}$ are only dependent to $\boldsymbol{u}$ and its derivatives to confine the equation system to the regime of solids.

(10), (11) and (12) contain an additional variable $\boldsymbol{v}$. We can alternatively describe everything within the Lagrangian framework and get rid of the $\boldsymbol{v}$ variable, but this means we need another stress tensor defined in the Lagrangian framework that's different from the Cauchy stress tensor (which is defined in the Eulerian framework – see the comments at the end of the last section) which is hard to conceptualize intuitively, so I just skip this part. In practice, however, the above formalisms are almost useless for elastic theory, because $\boldsymbol{u}$ almost will always be small; when $\boldsymbol{u}$ is not that small, usually we are just dealing with very complicated static problem and minimizing the free energy should be enough.

From now on, most of the time we will work in the Eulerian framework.

1.3.3 Is the Cauchy stress tensor symmetric?

An additional constraint on σ_{ij} that is not necessary but works for most elastic bodies is that *σ_{ij} is symmetric*. This constraint is equivalent to the condition that *there is no body torsion*: the torsion on a volume is

$$\begin{aligned} M_{ij} &= \int \mathrm{d}^d \boldsymbol{r} \, (r_i \partial_k \sigma_{kj} - \partial_k \sigma_{ki} r_j) \\ &= \int \mathrm{d}^d \boldsymbol{r} \, (\partial_k (r_i \sigma_{kj}) - \sigma_{kj} \delta_{ik} - \partial_k (r_j \sigma_{ki}) + \sigma_{ki} \delta_{jk}) \\ &= \int \mathrm{d}^{d-1} s_k \, (r_i \sigma_{kj} - r_j \sigma_{ki}) - \int \mathrm{d}^d \boldsymbol{r} \, (\sigma_{ij} - \sigma_{ji}), \end{aligned} \tag{13}$$

and it can be seen that the first term is just the torsion caused by the forces applied to the surface of this volume; if we assume that there is no body torsion, then the second term should always be zero regardless of the exact shape and position of the volume, and this condition is equivalent to

$$\sigma_{ij} = \sigma_{ji}. \tag{14}$$

Of course, we don't have body torsion in an equilibrium system, since if we have, then a part of the system will begin to rotate. However we do have body torsion sometimes: in a liquid crystal system, for example, if a group of rod-like molecules get an angle difference from the rods near them, they will be pulled back, and in this *transient* process a torsion is created. Another case is when macroscopic electromagnetic field is present: it may be able to apply a torque to a volume element, and in order to have equilibrium, σ_{ij} has to be different with σ_{ji}.

Essentially, whether we have (14) is equivalent to whether a volume element in the system can be seen as a point mass, i.e. a degree of freedom with only $\boldsymbol{r}, \boldsymbol{p}$ variables and without any internal state. In the cases above where (14) breaks down, a volume element has internal angular momentum and can no longer be treated as a point mass; we can of course go to smaller length scales so that the internal degrees of freedom are reduced, but then it's possible that we can no longer treat interaction between volume elements as surface forces. Of course, the internal angular momentum always exists in reality, so by saying that the internal angular momentum of a volume element is not important, we are hinting at decoupling between macroscopic dynamics and the dynamics of the internal angular momentum: the latter always reaches equilibrium quickly enough.

Since this note is about quasi-static behaviors of ordinary elastic materials, without any electromagnetic coupling, we will just adopt the assumption (14). It should be noted that the

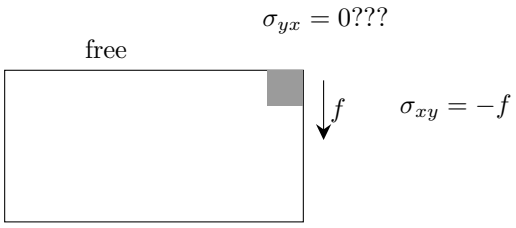

Figure 1: Counterexample of (14)

absence of *local (or inherent) angular momentum* doesn't mean *orbital angular momentum* is not considered: we know in a multiple point mass system, we can divide $\sum_{\text{all points}} \boldsymbol{r} \times \boldsymbol{p}$ into inherent angular momenta of its subsystems and orbital angular momenta of them; by ignoring the internal angular momentum, we are still able to treat the orbital angular momentum.

At the first glance, we can construct a counterexample of (14): consider the case in Fig. 1, where we load surface force to the right edge of a brick, but do nothing to its top side. Then at the top right corner, it seems $\sigma_{xy} = -f$, while σ_{yx} vanishes, and (14) is wrong at that point. But then we find it's actually a singularity: it's permissible that the condition $\boldsymbol{n} \cdot \sigma = \sigma_{yx,yy,yz} = 0$ breaks at the boundary of the upper side of the brick, and at the top right corner, we can just assign $\sigma_{yz} = \sigma_{xy}$ so that (14) is still correct everywhere in the system.

1.3.4 Stress and free energy

Now we explicitly derive the relation between the free energy density F and the stress σ_{ij}. In order to "mold" the configuration of the elastic body (without causing any real plastic deformation, of course), we need to introduce a static external force f_i such that

$$f_i + \partial_k \sigma_{ki} = 0. \tag{15}$$

Essentially f_i is the Lagrange multiplier to fix the deformation configuration of the system to what we want. Thus when we change the deformation of the system, work is done by f_i, and we have (below we use δ to refer to infinitesimal change caused by variance of the deformation, and d to refer to things like volume element that come from the coordinate system background)

$$\delta U = T\,\delta S + f_i\,\delta u_i\,, \quad \delta F = \delta U - \delta TS = -S\,\delta T + f_i\,\delta u_i\,. \tag{16}$$

The total free energy changes as

$$\begin{aligned}
\delta \int F\,\mathrm{d}^d\boldsymbol{r} &= -\int S\,\delta T\,\mathrm{d}^d\boldsymbol{r} + \int \mathrm{d}^d\boldsymbol{r}\, f_i\,\delta u_i \\
&= -\int S\,\delta T\,\mathrm{d}^d\boldsymbol{r} - \int \mathrm{d}^d\boldsymbol{r}\,\partial_k\sigma_{ki}\,\delta u_i \\
&= -\int S\,\delta T\,\mathrm{d}^d\boldsymbol{r} - \oint \mathrm{d}^{d-1}s_k\,\sigma_{ki}\,\delta u_i + \int \mathrm{d}^d\boldsymbol{r}\,\sigma_{ki}\,\delta(\partial_k u_i)\,.
\end{aligned} \tag{17}$$

The three variables appearing after δ – the temperature in the bulk, the displacement on the boundary, and the gradient of the displacement in the bulk – are all independent variables. Thus, we find

$$\sigma_{ki} = \frac{\delta}{\delta\partial_k u_i}\int F\,\mathrm{d}^d\boldsymbol{r} = \frac{\partial F}{\partial(\partial_k u_i)}. \tag{18}$$

This is a very important finding: it seems we need to focus on the gradient of $\boldsymbol{u}$, instead of $\boldsymbol{u}$ itself; and indeed this is true, since we can trivially add a global translation to the system and have a very large $\boldsymbol{u}$; besides the global translation, the global rotation component of $\partial_k u_i$, if any, is also trivial; to get rid of these we can simply require that $\boldsymbol{u} = 0$ at ∞.

1.4 Deformation and strain

Above we have assumed that $\boldsymbol{u}$ is the only dynamic variable; the actual number of variables may even be smaller: when there is no electric field in the space, for example, the absolute value of $\boldsymbol{u}$ is never important; what is important is its gradient.

1.4.1 Decomposition of deformation gradient

Let's first focus on the gradient of the displacement field; we choose to do so because

$$\mathrm{d}\boldsymbol{r}\cdot\boldsymbol{\nabla}\boldsymbol{u}\simeq\boldsymbol{u}_{i+1}-\boldsymbol{u}_i,$$

and if the interaction between particles in the system is localized enough, we should expect that the dynamics of the system is mainly about $\boldsymbol{\nabla}\boldsymbol{u}$. This is an approximation that is always adopted in the following discussion.

We can decompose the deformation gradient – the gradient of the displacement field – as follows:

$$E_{ij}=\frac{\partial u_i}{\partial x_j}=\underbrace{u_{ij}}_{\text{symm.}}+\underbrace{a_{ij}}_{\text{asymm.}}, \tag{19}$$

$$u_{ij}=\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right),\quad a_{ij}=\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j}-\frac{\partial u_j}{\partial x_i}\right). \tag{20}$$

When we work in the Lagrangian framework, the meaning of u_{ij} is quite clear: if the displacement is very small, we can easily verify that

$$\mathrm{d}V'=\mathrm{d}V\,(1+u_{ii}), \tag{21}$$

and we can also verify that a_{ij} gives us the local rotation of a volume element.

Intuitively, we may assume that u_{ij} contains the main components of E_{ij} that are relevant in elasticity. But there is no guarantee that the antisymmetric component of $\partial_i u_j$ is not involved in the expression of F. For example, if we analyze the deformation by analyzing the change of the metric, we may want to study the following tensor:

$$u_{ij}=\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}+\frac{\partial u_l}{\partial x_i}\frac{\partial u_l}{\partial x_j}\right), \tag{22}$$

and we can verify that when the displacement is small, we have

$$\mathrm{d}l'^2=\mathrm{d}l^2+2u_{ij}\,\mathrm{d}x_i\,\mathrm{d}x_j\,. \tag{23}$$

The third term of u_{ij} can only be obtained by using $a_{il}a_{jl}$ in (20).

The case can be even more complicated. When we are dealing with the most generic case, u_i is no longer always small, and now distinguishing between the Lagrangian and Eulerian descriptions becomes important. And indeed u_i can be large *without* a large external field: we can for example rotate the system globally, or we can add a load to a thin but large layer and have a rather large displacement at the loading position even when the force applied is very small.

The most generic theory is beyond the scope of this note; note that it requires a full theory of transportation, as is seen in fluid dynamics. In solid mechanics, usually it's same to assume that regardless of the absolute magnitude of u_i, the deformation gradient is always small. Note that if this is true in Lagrangian description, it's also true in Eulerian description, and the opposite is also true. Now (20) and (22) are essentially the same. Thus, we call u_{ij} the **strain** of the system; σ_{ij} now is a function of the strain.

When we have large deformation, the deformation gradient might appear large at the first glance: for

$$\boldsymbol{u}=\begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}-\begin{pmatrix}x\\ y\end{pmatrix}, \tag{24}$$

we have

$$\frac{\partial u_i}{\partial x_j}=\begin{pmatrix}\cos\theta-1 & -\sin\theta\\ \sin\theta & \cos\theta-1\end{pmatrix}, \tag{25}$$

which is by no means small when $\theta\sim\pi/2$. But if each segment of the system is "flat" enough so that we can attach a coordinate frame to it and then evaluate the energy of that fragment in the small strain limit, we are able to get the energy functional (or free energy functional, etc.) to derive all equations we need. Specifically, phenomena like a global displacement or rotation of the system are simply resolved by changing the coordinate frame (and adding terms like Coriolis

force into the EOMs as external fields). In the most generic theory, this means any reasonable definition of the strain should be able to remove "superficial" components of the deformation gradient, as is shown in (25); this note however will just skip this topic. Note that although the free energy is still linear in the strain, nonlinearity can enter the theory via geometric factors, as will be seen in the theory of, say, rod.

1.4.2 From a microscopic perspective

1.5 Summary

Now I list what we get in the discussion above, and their conditions.

- In a solid the dynamics of the system is mainly about the displacement field $\boldsymbol{u}$ and its time derivative, possibly with other variables like the electromagnetic field.
- When the deformation of a system at a moment is completely decided by the force applied to it at that very moment, without any retardation or memory effect, we say the system is elastic.
- When the thermodynamic relaxation time is much smaller than $1/\omega$, with ω being the frequency scale we are interested in, we can assume local thermodynamic equilibrium and use the derivative of the free energy to find the relation between deformation and internal force.
- When the interaction length scale is small enough, the internal interaction in a system is described by the Cauchy stress tensor σ_{ij}. Furthermore, when we don't consider any transient internal torsion, we have $\sigma_{ij} = \sigma_{ji}$. The Cauchy stress tensor can be obtained from the free energy by taking its derivative with respect to the deformation gradient.
- The Cauchy stress tensor σ_{ij} guides the time evolution of the system according to (10), (11) and (12). Note that the form of the equation system, without constraints over what σ_{ij} depends to, also works for fluids. When the deformation is small, terms like $\boldsymbol{v}$ can be removed in the equations.

All underlined sentences above have well-known counterexamples.

- In a liquid the density is a very important dynamic variable, so it's quite common that $\boldsymbol{u}$ isn't the only dynamic variable.
- Plastic deformation is common: the deformation history changes the internal degrees of freedom of the system.
- Non-equilibrium behaviors can be seen in, say, zero sound modes of Fermi liquid.
- We may work in a very small length scale where the internal forces in a system can no longer be described as surface forces.
- Large deformation does exist.

2 Linear elasticity for bulk system

Three-dimensional elastic systems generally have neither large $\boldsymbol{u}$ (with no global translation or rotation, of course) nor large strain. This sometimes may not be the case for 2D or 1D systems. Therefore, the elastic theory of three-dimensional systems is usually linear and therefore can be decided completely by symmetry.

3 Statics of boards: the small deflection theory

Landau chap. 2

3.1 Boundary conditions

One subtlety is we've never used the concept of moment of force when establishing a theory about the bulk state; the appearance of the moment in a board seems strange. The reason is, the force density f along the edge is not enough to characterize the small-scale force distribution at the edge: besides f, which should be multiplied by a δ function when understood as a *bulk*

force density, we can also have terms proportional to $\delta'(\boldsymbol{r}\cdot\hat{\boldsymbol{n}})$. The latter case just gives us the boundary condition about the moment. We can derive the boundary condition about the moment by putting the force density proportional to $\delta'(\boldsymbol{r}\cdot\hat{\boldsymbol{n}})$ into the bulk elastic equations; alternatively we can insert the $\delta'(\boldsymbol{r}\cdot\hat{\boldsymbol{n}})$ term into the free energy change, where we transfer the derivative to $\delta\zeta$, and therefore get a $\delta\ \partial\zeta/\partial n$ term in the variance of the free energy, which again gives the boundary condition about the moment.

We thus see why it's usually not necessary to talk about the moment density in a three dimensional theory: the appearance of $\delta(\boldsymbol{r}\cdot\hat{\boldsymbol{n}})$ and $\delta'(\boldsymbol{r}\cdot\hat{\boldsymbol{n}})$ is only possible when we are dealing with boundary conditions, and when they don't appear, talking about the force density is already enough.

3.2 Föppl–von Kármán equations for large deflections of thin flat plates

A brief overview of Föppl–von Kármán equations can be found in § 14 in [3]. The theory is based on the following assumptions, many of which are problematic physically:

- The von Kármán strain is linearly linked to the stress tensor. One may expect this to be true since the von Kármán strain is a natural generalization of the engineering strain; but then we can also say that since a high order term appears in F, similarly a high order term of the engineering strain can appear in F as well. The linear relation between the von Kármán strain is therefore not theoretically justified, and experimental demonstration of this constitutive relation is hard to carry out.

4 Statics of rod: the Euler–Bernoulli theory

Here is an overview of what we are going to do:

1. Finding a way to describe the deformation of rod; writing down the form of the free energy when the deformation is not too strong;
2. Analyzing two deformation modes of a rod; deciding the free energy under the small strain assumption.
3. Deciding the free energy by considering the two deformation modes as specific cases of the general theory; nonlinearity enters the free energy when we write the geometric properties of the deformed cross section as a function of the deformation of the rod.
4. Take the small deflection approximation.

4.1 Description of deformation

The equations concerning the distributed moment of course can be derived from the perspective of force distribution, or, in this specific case, from moment balance; the former actually directly implies the latter.

In principle we can now work on balance equations about $\int \mathrm{d}x\,\mathrm{d}y\,xyf_z$ or something like this, but in the specific case of a rod, only the moment appears in the free energy, so there is no such need.

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