

# Quantum Optics by Prof. Saijun Wu

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## 1 Master equation

We consider a open quantum system, whose Hamiltonian is  $H$ , and at each time step, there is a probability of quantum jump to several given states. Suppose the probability of jumping to state  $|i\rangle$  is  $\Gamma_i$ , we define

$$C_i = \sqrt{\Gamma_i} |i\rangle\langle i|, \quad (1)$$

and the system can be described using a stochastic wave function method shown in previous lectures. If we use a density matrix formalism, we find

$$\rho(t + \Delta t) = \sum_i \Gamma_i \Delta t |i\rangle\langle i| + (1 - \sum_i \Gamma_i) |\psi_s(t + \Delta t)\rangle\langle\psi_s(t + \Delta t)|,$$

where  $|\psi_s\rangle$  evolves according to  $H$ . We therefore find

$$\dot{\rho} = \frac{1}{i\hbar} [H_{\text{eff}}, \rho] + \sum_i C_i \rho C_i^\dagger, \quad (2)$$

where

$$H_{\text{eff}} = H - \frac{i\hbar}{2} \sum_i C_i^\dagger C_i. \quad (3)$$

(2) is called **the master equation in Lindblad form**. It should be noted that the “commutator” in the equation is actually the anticommutator for the anti-Hermitian part of  $H_{\text{eff}}$ : the two terms in  $[H_{\text{eff}}, \rho]$  correspond to the time evolution of the bra- and ket-part of the density matrix, and since now  $H_{\text{eff}}$  is no longer a Hermitian operator, the time evolution of the bra-part is guided by  $H_{\text{eff}}^\dagger$ , not  $H_{\text{eff}}$ , and therefore actually

$$[H_{\text{eff}}, \rho] = [H, \rho] - \frac{i\hbar}{2} \sum_i \{C_i^\dagger C_i, \rho\}.$$

The consequence of this construction is the unitarity of (2). We have

$$\text{tr} \dot{\rho} = - \sum_j \text{tr} \left( \frac{1}{2} C_j^\dagger C_j \rho + \frac{1}{2} \rho C_j^\dagger C_j - C_j \rho C_j^\dagger \right). \quad (4)$$

The last term is called the **recycling term**, which makes the total probability increase, while the first two terms make the total probability decrease. With trace cyclic property, we find the total probability is conserved.

We consider a light-atom interacting system with RWA, where

$$H = \frac{\hbar}{2} \mathbf{\Omega} \cdot \boldsymbol{\sigma}, \quad C = \sqrt{\Gamma} |g\rangle\langle e|, \quad (5)$$

and we have

$$\begin{aligned} \dot{\rho}_{gg} &= \frac{i\Omega}{2} \rho_{ge} - \frac{i\Omega^*}{2} \rho_{eg} + \Gamma \rho_{ee}, \\ \dot{\rho}_{ee} &= -\dot{\rho}_{gg}, \\ \dot{\rho}_{ge} &= \left( -\frac{\Gamma}{2} + i\Delta \right) \rho_{ge} - \frac{i\Omega}{2} (\rho_{ee} - \rho_{gg}), \end{aligned} \quad (6)$$

where

$$\rho_{ee} + \rho_{gg} = 1, \quad \rho_{eg} = \rho_{ge}^*. \quad (7)$$

We define

$$\langle \sigma^x \rangle = \text{Re } \rho_{eg} =: u, \quad \langle \sigma^y \rangle = \text{Im } \rho_{eg} =: v, \quad \langle \sigma^z \rangle = \rho_{ee} - \rho_{gg} =: w, \quad (8)$$

and

$$\mathbf{n} = (\langle \sigma^x \rangle, \langle \sigma^y \rangle, \langle \sigma^z \rangle), \quad (9)$$

and we have

$$\dot{\mathbf{n}} = \mathbf{\Omega} \times \mathbf{n} - \begin{pmatrix} \gamma_T u \\ \gamma_T v \\ \gamma_L(w+1) \end{pmatrix}, \quad (10)$$

where

$$\gamma_T = \frac{\Gamma}{2} \quad (11)$$

is called the **transverse damping rate** and

$$\gamma_L = \Gamma \quad (12)$$

is called the **longitude relaxing rate**. (10) is called the **optical Bloch equation**.

Now we try to find a stable solution of (10). It is

$$\rho_{ee}^{\text{stable}} = \frac{(\Omega/\Gamma)^2}{1 + 2(\Omega/\Gamma)^2 + 4(\Delta/\Gamma)^2}, \quad (13)$$

and

$$\rho_{ge}^{\text{stable}} = -\frac{\Omega/2}{\Delta - i\Gamma/2}(2\rho_{ee} - 1). \quad (14)$$

We define

$$S = 2 \left( \frac{\Omega}{\Gamma} \right)^2. \quad (15)$$

We can also evaluate the response of the electric dipole. We have

$$\langle d \rangle = \rho_{eg} d_{ge} + \text{h.c.} = \alpha E + \text{c.c.}, \quad \alpha = \frac{1}{1 + S + 4(\Delta/\Gamma)^2} \left( \frac{2\Delta}{\Gamma} + i \right) \frac{3\lambda^3}{4\pi^2} \epsilon_0 =: \alpha(I). \quad (16)$$

saturated absorption, Saturated absorption spectroscopy **Lamb dip**

## 2 Rate equation

We choose a *adiabatic* basis, which are dressed states of  $H_{\text{eff}}$ . In this basis, assuming that the non-diagonal elements of the density matrix damp quickly enough, we have

$$\dot{\rho}_{nn} = -\gamma_n \rho_{nn} + \sum_{m \neq n} \gamma_{nm} \rho_{mm}, \quad \rho_{mn}|_{m \neq n} = 0, \quad (17)$$

which is called the **rate equation**.

Again for a two-level system where RWA works, we have

$$\gamma_{\tilde{g}} = \Gamma \sin^2 \theta, \quad \gamma_{\tilde{e}} = \Gamma \cos^2 \theta, \quad \gamma_{\tilde{g}\tilde{e}} = \Gamma \sin^4 \theta, \quad \gamma_{\tilde{e}\tilde{g}} = \Gamma \cos^4 \theta, \quad (18)$$

and the rate equation is

$$\dot{\rho}_{\tilde{g}\tilde{g}} = (-\sin^4 \theta \rho_{\tilde{g}\tilde{g}} + \cos^4 \theta \rho_{\tilde{e}\tilde{e}}) \Gamma. \quad (19)$$

The stable solution is

$$\rho_{\tilde{g}\tilde{g}}^{\text{stable}} = \frac{\cos^4 \theta}{\cos^4 \theta + \sin^4 \theta}, \quad \rho_{\tilde{e}\tilde{e}}^{\text{stable}} = \frac{\sin^4 \theta}{\cos^4 \theta + \sin^4 \theta}. \quad (20)$$

### 3 How atoms move in the space

All previous discussions were based on the assumption that atoms are somehow “fixed” or “trapped” at a given point. This is of course possible (using laser trap or something), but a more interesting case is when atoms are not that constrained. In this case, we need to take the spacial motion of atoms into account.

Consider a stationary mode in a cavity:

$$E = E_0 \cos(kx)e^{i\omega t} + \text{c.c.}, \quad (21)$$

and at each point the energies of the ground state and the excited state of a two-level atom are different. Since

$$H = \frac{\hbar}{2} \mathbf{\Omega} \cdot \boldsymbol{\sigma} \propto E, \quad (22)$$

we have

$$m \langle \ddot{\mathbf{r}} \rangle = -\nabla \langle H \rangle \propto -\nabla E. \quad (23)$$

Sisyphus cooling

Note: there are some subtleties here.

$$\langle F \rangle = \text{Re} \frac{\nabla \Omega}{\Omega} \alpha_r |E|^2 + \text{Im} \frac{\nabla \Omega}{\Omega} \alpha_i |E|^2 + \quad (24)$$

We find that the conservative force causes cooling, while the scattering force causes heating.

Doppler cooling is another cooling approach We will find

$$\Delta \rightarrow \Delta - kv, \quad (25)$$

$$F = -\beta v \quad (26)$$

What Doppler cooling fails to take into account is the Sisyphus cooling or heating mechanism above.

### 4 A four-level atom model, and when polarization is important

$$H = \hbar \Delta (|e+\rangle\langle e+| + |e-\rangle\langle e-|) + \frac{\hbar \Omega_T}{2} (|e+\rangle\langle g+| + |e-\rangle\langle g-|) + \text{h.c.} + \frac{\hbar \Omega_{\sigma-}}{2} (|e-\rangle\langle g+| + \text{h.c.}), \quad (27)$$

and the dressed states are

$$|\tilde{g}+\rangle = |g+\rangle + \frac{\Omega_{\sigma-}/2}{\Delta} |e-\rangle, \quad |\tilde{g}-\rangle = |g-\rangle. \quad (28)$$

We find  $|g-\rangle$  actually has no coupling with the optical field and therefore is not dressed at all. States like this are often called **dark states**, because if an atom falls on such a state, no radiation will be seen. It does not mean these states are completely irrelevant, because quantum hopping may drive an atom to such a state.

In such a system we can realize **sub-Doppler cooling**, again by Sisyphus cooling. This was first found by Steven Chu, who found his cooling device mysteriously worked better than the estimation of Doppler cooling.