Homework 4

Jinyuan Wu

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1

The original matrix is

$$
\mathbf{A} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \\ 1 & 3 \\ 0 & 1 \end{pmatrix}. \tag{1}
$$

Following these steps:

1. Move the third line to the top, and
2. Subtract the second line from the fourth line,

we get the row reduced form

$$
\mathbf{A}_{\mathrm{R}} = \begin{pmatrix} 1 & 3 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, \tag{2}
$$

and applying the same procedure to $\mathbf{I}_{4\times 4}$ we get

$$
\mathbf{\Omega}_{\mathrm{R}} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix}, \tag{3}
$$

such that $\mathbf{\Omega}_{\mathrm{R}}\mathbf{A} = \mathbf{A}_{\mathrm{R}}$.

2

The equations are

$$
\begin{aligned} 6x_1 - x_2 + x_3 = 0 \\ x_1 - x_4 + 2x_5 = 0, \\ x_1 - 2x_5 = 0 \end{aligned} \tag{4}
$$

which is equivalent to

$$
\begin{pmatrix} 6 & -1 & 1 & 0 & 0 \\ 1 & 0 & 0 & -1 & 2 \\ 1 & 0 & 0 & 0 & -2 \end{pmatrix} \mathbf{x} = \mathbf{0}. \tag{5}
$$

The row reduced form of the matrix in the LHS is

$$
\begin{pmatrix} 1 & 0 & 0 & 0 & -2 \\ 0 & 1 & -1 & 0 & -12 \\ 0 & 0 & 0 & 1 & -4 \end{pmatrix}
$$

If we switch the third and the fourth coulomb, we immediate find two independent solutions

$$
\begin{pmatrix} 0 \\ -1 \\ 0 \\ -1 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} -2 \\ -12 \\ -4 \\ 0 \\ -1 \end{pmatrix},
$$

and if we switch the coulombs back we find a basis of the solution space is

$$\begin{pmatrix} 0 \\ -1 \\ -1 \\ 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} -2 \\ -12 \\ 0 \\ -4 \\ -1 \end{pmatrix}, \tag{6}$$

and the solution space is 2-dimensional, and the general solution looks like

$$x_1 = 2t_2, \quad x_2 = t_1 + 12t_2, \quad x_3 = t_1, \quad x_4 = 4t_2, \quad x_5 = t_2. \tag{7}$$

3

The equation system

$$\begin{aligned} 2x_1 - 3x_2 + x_4 = 1 \\ 3x_1 + x_3 - x_4 = 0 \\ 2x_1 - 3x_2 + 10x_3 = 0 \end{aligned} \tag{8}$$

is equivalent to

$$\begin{pmatrix} 2 & -3 & 0 & 1 \\ 3 & 0 & 1 & -1 \\ 2 & -3 & 10 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}. \tag{9}$$

The reduced matrix of the LHS is

$$\begin{pmatrix} 1 & 0 & 0 & -\frac{3}{10} \\ 0 & 1 & 0 & -\frac{8}{15} \\ 0 & 0 & 1 & -\frac{1}{10} \end{pmatrix},$$

and the general solution of the homogeneous version of the equation is therefore

$$t \begin{pmatrix} \frac{3}{10} \\ \frac{8}{15} \\ \frac{1}{10} \\ 1 \end{pmatrix}.$$

A specific solution of the equation system can be easily found by setting $x_4 = 0$:

$$\begin{pmatrix} \frac{1}{30} \\ -\frac{14}{45} \\ -\frac{1}{10} \end{pmatrix}.$$

So the general solution is

$$x_1 = \frac{3}{10}t + \frac{1}{30}, \quad x_2 = \frac{8}{15}t - \frac{14}{45}, \quad x_3 = \frac{1}{10}t - \frac{1}{10}, \quad x_4 = t. \tag{10}$$

4

Since

$$\mathbf{A} = \begin{pmatrix} -1 & 0 \\ 4 & 4 \end{pmatrix}, \tag{11}$$

we have $\det \mathbf{A} = -4$, and therefore it's not singular. The inverse is given by

$$\mathbf{A}^{-1} = \frac{1}{\det \mathbf{A}} \begin{pmatrix} 4 & -4 \\ 0 & -1 \end{pmatrix}^\top = \begin{pmatrix} -1 & 0 \\ 1 & 1/4 \end{pmatrix}. \tag{12}$$

5

The equation system

$$\begin{aligned} 8x_1 - 4x_2 + 3x_3 &= 0 \\ x_1 + 5x_2 - x_3 &= -5 \\ -2x_1 + 6x_2 + x_3 &= -4 \end{aligned} \tag{13}$$

is equivalent to

$$\underbrace{\begin{pmatrix} 8 & -4 & 3 \\ 1 & 5 & -1 \\ -2 & 6 & 1 \end{pmatrix}}_{\mathbf{A}} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ -5 \\ -4 \end{pmatrix}. \tag{14}$$

We have

$$\det \mathbf{A} = 132, \tag{15}$$

and by Cramer's rule we have

$$x_1 = \frac{1}{132} \cdot -66 = -\frac{1}{2}, \quad x_2 = \frac{1}{132} \cdot -114 = -\frac{19}{22}, \quad x_3 = \frac{1}{132} \cdot 24 = \frac{2}{11}. \tag{16}$$

6

$$\mathbf{A} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 2 \\ 0 & 1 & 3 \end{pmatrix}, \tag{17}$$

so

$$\det(\mathbf{A} - \lambda \mathbf{I}) = 0 \Rightarrow \lambda = \frac{3 \pm \sqrt{17}}{2}, 0. \tag{18}$$

$\lambda = 0$ corresponds to

$$\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 2 \\ 0 & 1 & 3 \end{pmatrix} \mathbf{v} = 0,$$

a solution of which is

$$\begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix}.$$

$\lambda = (3 - \sqrt{17})/2$ corresponds to

$$\begin{pmatrix} \frac{\sqrt{17}-3}{2} & 0 & 0 \\ 1 & \frac{\sqrt{17}-3}{2} & 2 \\ 0 & 1 & \frac{3+\sqrt{17}}{2} \end{pmatrix} \mathbf{v} = \mathbf{0},$$

a solution of which is

$$\begin{pmatrix} 0 \\ \frac{\sqrt{17}+3}{2} \\ -1 \end{pmatrix}.$$

$\lambda = (3 + \sqrt{17})/2$ corresponds to

$$\begin{pmatrix} -\frac{\sqrt{17}+3}{2} & 0 & 0 \\ 1 & -\frac{\sqrt{17}+3}{2} & 2 \\ 0 & 1 & \frac{3-\sqrt{17}}{2} \end{pmatrix} \mathbf{v} = \mathbf{0},$$

a solution of which is

$$\begin{pmatrix} 0 \\ \frac{3-\sqrt{17}}{2} \\ -1 \end{pmatrix}.$$

So we have

$$\mathbf{P} = \begin{pmatrix} 0 & 0 & 2 \\ \frac{3-\sqrt{17}}{2} & \frac{3+\sqrt{17}}{2} & 3 \\ -1 & -1 & -1 \end{pmatrix}, \tag{19}$$

and

$$\mathbf{A} = \mathbf{P} \begin{pmatrix} \frac{3+\sqrt{17}}{2} & & \\ & \frac{3-\sqrt{17}}{2} & \\ & & 0 \end{pmatrix} \mathbf{P}^{-1}. \tag{20}$$

7

The procedure is the same as the last problem, and we find the eigenvalues are $\pm\sqrt{17}$, and $\sqrt{17}$ corresponds to $(\sqrt{17}-4, 1)$, and $-\sqrt{17}$ corresponds to $(-\sqrt{17}-4, 1)$. Then the dot product of the two eigenvectors is

$$(\sqrt{17}-4)(-\sqrt{17}-4) + 1 = 0,$$

so the eigenvectors are orthogonal to each other. After normalization we get

$$\mathbf{P} = \begin{pmatrix} \frac{\sqrt{17}-4}{\sqrt{34-8\sqrt{17}}} & -\frac{4+\sqrt{17}}{\sqrt{34+8\sqrt{17}}} \\ \frac{1}{\sqrt{34-8\sqrt{17}}} & \frac{1}{\sqrt{34+8\sqrt{17}}} \end{pmatrix}, \tag{21}$$

and

$$\mathbf{A} = \mathbf{P} \begin{pmatrix} \sqrt{17} & \\ & -\sqrt{17} \end{pmatrix} \mathbf{P}^{-1}. \tag{22}$$

8

9

Following the same procedure used above, we do eigenvalue decomposition and find

$$\mathbf{A} = \begin{pmatrix} 4 & -2 \\ -2 & 1 \end{pmatrix} = \begin{pmatrix} -2 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 5 & \\ & 0 \end{pmatrix} \begin{pmatrix} -2 & 1 \\ 1 & 2 \end{pmatrix}^{-1}. \tag{23}$$

The transition matrix is therefore

$$\mathrm{e}^{\mathbf{A}t} = \begin{pmatrix} -2 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} \mathrm{e}^{5t} & \\ & 1 \end{pmatrix} \begin{pmatrix} -2 & 1 \\ 1 & 2 \end{pmatrix}^{-1} = \begin{pmatrix} \frac{4e^{5t}}{5} + \frac{1}{5} & \frac{2}{5} - \frac{2e^{5t}}{5} \\ \frac{2}{5} - \frac{2e^{5t}}{5} & \frac{e^{5t}}{5} + \frac{4}{5} \end{pmatrix}. \tag{24}$$

10

We do eigenvalue decomposition

$$\mathbf{A} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = \underbrace{\frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}}_{\mathbf{P}} \begin{pmatrix} 2 & \\ & 0 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}, \tag{25}$$

and the equation

$$\mathbf{X}' = \mathbf{A}\mathbf{X} + \mathbf{G} \tag{26}$$

is then equivalent to

$$(\mathbf{P}^{-1}\mathbf{X})' = \begin{pmatrix} 2 & \\ & 0 \end{pmatrix} (\mathbf{P}^{-1}\mathbf{X}) + \mathbf{P}^{-1}\mathbf{G}, \tag{27}$$

where

$$\mathbf{P}^{-1}\mathbf{G} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 6e^{3t} \\ 4 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 6\mathrm{e}^{3t} + 4 \\ -6\mathrm{e}^{3t} + 4 \end{pmatrix}. \tag{28}$$

The general solution of

$$y' = 2y + \frac{1}{\sqrt{2}}(6\mathrm{e}^{3t} + 4) \tag{29}$$

is

$$y = c_1\mathrm{e}^{2t} + \frac{1}{\sqrt{2}}(6\mathrm{e}^{3t} - 2), \tag{30}$$

and the general solution of

$$y' = \frac{1}{\sqrt{2}}(-6\mathrm{e}^{3t} + 4) \tag{31}$$

is

$$y = c_2 + \frac{1}{\sqrt{2}}(-2\mathrm{e}^{3t} + 4t) \tag{32}$$

The general solution of the original problem then is

$$\mathbf{X} = \mathbf{P}\begin{pmatrix} c_1\mathrm{e}^{2t} + \frac{1}{\sqrt{2}}(6\mathrm{e}^{3t} - 2) \\ c_2 + \frac{1}{\sqrt{2}}(-2\mathrm{e}^{3t} + 4t) \end{pmatrix} = \begin{pmatrix} C_1\mathrm{e}^{2t} - C_2 + 4\mathrm{e}^{3t} - 1 - 2t \\ C_1\mathrm{e}^{2t} + C_2 + 2\mathrm{e}^{3t} - 1 + 2t \end{pmatrix}. \tag{33}$$