Homework 4

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November 16, 2022

Exercise 1 in lecture 13 We have the following transmission model: the sent signal X has 3 possible values $x = 0, \pm 1$ with $p_0 = p_1 = p_{-1} = 1/3$. The noise Ξ has two possible values $\zeta = \pm 1$. We suppose the channel function F has the following expression $Y = \operatorname{Im}\left[\mathrm{e}^{i2\pi(X+\Xi)/}\right]$. What are all the possible values of Y ? Give the expression of the distributions $w(x), w(y), w(x, y)$ and $w(y \mid x)$.

Solution The possible results are listed in Table 1. The possible values of Y are 0 and $\pm\sqrt{3}/2$. From Table 1 we have

$$w(x) = \begin{cases} \frac{1}{3}, & x = 0, \\ \frac{1}{3}, & x = 1, \\ \frac{1}{3}, & x = -1, \end{cases} \tag{1}$$

$$w(y) = \begin{cases} \frac{1}{3}, & y = 0, \\ \frac{1}{3}, & y = \sqrt{3}/2, \\ \frac{1}{3}, & y = -\sqrt{3}/2, \end{cases} \tag{2}$$

and

$$w(x, y) = \begin{cases} \frac{1}{6}, & (x, y) = (0, \sqrt{3}/2), \\ \frac{1}{6}, & (x, y) = (0, -\sqrt{3}/2), \\ \frac{1}{6}, & (x, y) = (1, -\sqrt{3}/2), \\ \frac{1}{6}, & (x, y) = (1, 0), \\ \frac{1}{6}, & (x, y) = (-1, 0), \\ \frac{1}{6}, & (x, y) = (-1, \sqrt{3}/2), \\ 0, & \text{otherwise}. \end{cases} \tag{3}$$

So

$$w(y|x) = \frac{w(x, y)}{w(x)} = \begin{cases} \frac{1}{2}, & (x, y) = (0, \sqrt{3}/2), \\ \frac{1}{2}, & (x, y) = (0, -\sqrt{3}/2), \\ \frac{1}{2}, & (x, y) = (1, -\sqrt{3}/2), \\ \frac{1}{2}, & (x, y) = (1, 0), \\ \frac{1}{2}, & (x, y) = (-1, 0), \\ \frac{1}{2}, & (x, y) = (-1, \sqrt{3}/2), \\ 0, & \text{otherwise}. \end{cases} \tag{4}$$

Exercise

Table 1: Probabilistic distribution of Y; the probability of each row is $1/6$.

X	Ξ	$Y = \operatorname{Im} \mathrm{e}^{\mathrm{i}2\pi(X+\Xi)/3}$
0	1	$\frac{\sqrt{3}}{2}$
	-1	$-\frac{\sqrt{3}}{2}$
1	1	$-\frac{\sqrt{3}}{2}$
	-1	0
-1	1	0
	-1	$\frac{\sqrt{3}}{2}$

Solution Suppose p is the probability of $X = -1$, and we have

$$m = \langle X \rangle = -p + (1 - p) = 1 - 2p, \tag{5}$$

and therefore

$$p = \frac{1 - m}{2}, \tag{6}$$

and

$$\begin{aligned} H_\alpha(X) &= \frac{1}{1 - \alpha} \log_2 (p^\alpha + (1 - p)^\alpha) \\ &= \frac{1}{1 - \alpha} \log_2 \left(\left(\frac{1 - m}{2} \right)^\alpha + \left(\frac{1 + m}{2} \right)^\alpha \right). \end{aligned} \tag{7}$$