Bosonization by Prof. Yue Yu

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1 An overview of CFTs

This is a note about a lecture given by Prof. Yue Yu as a part of his course on CFT.

To not start from the abstract mathematics, we start from bosonization of (1+1)-dimensional interacting fermion systems. This results in a CFT with $c = 1$ in (1+1)-dimensions. We are especially interested in the (1+1)-dimensional case, because in higher dimensional systems, the conformal symmetry is a global symmetry, something like $\mathrm{SL}(2, \mathbb{C})$, while in the (1+1)-dimensional case, the conformal symmetry is a *local* symmetry, and therefore it is actually a *gauge symmetry*.

CFTs often describe critical points of 2-dimensional classical models and (1+1)-dimensional quantum models. From CFT we can calculate correlation functions and critical exponents *exactly*. A classification of a certain group of CFTs is also a classification of the corresponding physical models.

Though $c = 1$ CFTs are directly related to the Luttinger liquid, they contain infinite primary fields and therefore is too "large" to be handled. The case when $c < 1$ is even worse: the number of primary fields can be infinite and the primary fields can be not well-defined. Fortunately, we still have **minimal models** where we have a finite number of primary fields, and the whole Hilbert space is well-defined. (On the other hand, the Hilbert space of a generic gauge theory may contain states with negative norms and therefore is not well-defined before gauge fixing.) Some of minimal models corresponds to well-known exactly solvable statistical models, e.g. Ising model, tricritical Ising model, Potts model. The primary fields in a minimal model generally obeys anyon statistics, Abelian and non-Abelian.

CFTs may also describe systems with a finite characteristic length, if we place them on a finite Riemann surface (for example, a torus). This reduces the conformal symmetry to a smaller symmetry. For example, on a torus, the conformal invariance reduces into modular invariance. A finite size system can also be described by a CFT with finite size scaling, and the CFT can predict boundary effects.

For $c > 1$, we have the **Wess-Zumino-Witten (WZW) models** with Lie group symmetry. The most interesting WZW models are those with simple Lie group symmetry.

CFTs with different central charges are connected by RG flows, which is shown by the **c-theorem**. CFTs with definite central charges, therefore, are fixed points.

In higher dimensions, the naive generalization of CFTs is not interesting, but we also have some results. There are higher dimensional generalization of the c-theorem. AdS/CFT is a correspondence between gravity and CFTs and was once thought to be hopeful to solve some strongly correlated problems. Conformal bootstrap can also be generalized to higher dimensions, though we are less confident about whether we take the constraints correctly there.

Probably the most intriguing application of CFTs in condensed matter physics is about topological quantum computation, where non-Abelian anyons are thought as qubits, and the braiding of anyons corresponds to the unitary transformations. We can build quantum gates with braiding. The simplest example is Ising non-Abelian anyon, which is the bound state of the spin and Majorana fermion. Braidings of Ising non-Abelian anyons cannot represent all unitary transformations. The Fibonacci anyon can achieve universal topological quantum computing.

Note that fermions can also be non-Abelian. For example, a charge fermion can be delocalized into two types of Majorana fermions. Majorana fermions can also achieve universal topological quantum computing. Note, however, they can not be generated within a CFT. The theory describing them is a unitary tensor category, a generalization of CFTs.

2 Bosonization of free fields

This section is mainly based on [1, 2]. [1] is an introduction. [2] is a work by Prof. Yu about some details.

To make things easier, we redefine the unit of energy, and the Hamiltonian of a one-dimensional spinless fermion gas is

$$H = \int \mathrm{d}x \, |\boldsymbol{\nabla}\psi|^2 + \frac{1}{2}\int \mathrm{d}x \, \mathrm{d}y \, V(x-y)\rho(x)\rho(y). \tag{1}$$

The free theory case is

$$H_{\text{eff}} = H - \mu N = \sum_k \xi_k c_k^\dagger c_k, \quad \xi_k = \begin{cases} (k^2 - k_{\text{F}}^2)/\lambda, & |k| < k_{\text{F}}, \\ k^2 - k_{\text{F}}^2, & |k| > k_{\text{F}}. \end{cases} \tag{2}$$

Here we take a more generalized form. For (1), of course $\lambda = 1$. The theory with $\lambda = 3$ is actually about edge states of $\nu = 1/3$ FQHE and is related to anyon statistics in the sense of exclusion.

We only consider (2) near the two Fermi points, and the spectrum is now

$$\xi_{k,\pm} = \begin{cases} \pm v_{\text{F}}(k \mp k_{\text{F}}), & |k| > k_{\text{F}}, \\ \pm v_{\text{F}}(k \mp k_{\text{F}})/\lambda, & |k| < k_{\text{F}}, \end{cases} \tag{3}$$

where $\pm$ in the subscript of $\xi_{k,\pm}$ distinguishes the mode around $\pm k_{\text{F}}$. Since we are only interested in the density fluctuation, around $k = k_{\text{F}}$, we define

$$\rho_q^+ = \sum_{k > k_{\text{F}}} : c_{k+q}^\dagger c_k : + \sum_{k < k_{\text{F}} - \lambda q} : c_{k+\lambda q}^\dagger c_k : + \sum_{k_{\text{F}} - \lambda q < k < k_{\text{F}}} : c_{\frac{k-k_{\text{F}}}{\lambda} + k_{\text{F}} + q}^\dagger c_k : \tag{4}$$

for $0 < q \ll k_{\text{F}}$. When $\lambda = 1$, the definition goes back to the ordinary

$$\rho_q^+ = \sum_k : c_{k+q}^\dagger c_k : . \tag{5}$$

Similarly, the density fluctuation around $k = -k_{\text{F}}$ is defined as

$$\rho_q^- = \sum_{k < -k_{\text{F}}} : c_{k-q}^\dagger c_k : + \sum_{k > -k_{\text{F}} + \lambda q} : c_{k-\lambda q}^\dagger c_k : + \sum_{-k_{\text{F}} + \lambda q > k > -k_{\text{F}}} : c_{\frac{k+k_{\text{F}}}{\lambda} - k_{\text{F}} - q}^\dagger c_k : . \tag{6}$$

Now we make the **Tomonaga approximaton**: we replace $[\rho_q^\pm, \rho_{q'}^\pm]$ by its ground state expectation value. The procedure we are taking here is similar to how we simply (15) in this note. This gives

$$[\rho_q^\pm, \rho_{q'}^\pm] \approx \langle 0|[\rho_q^\pm, \rho_{q'}^\pm]|0\rangle = \sum_{k_{\text{F}} - \lambda q < k < k_{\text{F}}} \langle 0|c_{k+\lambda q} c_{k+\lambda q'}^\dagger|0\rangle = \frac{L}{2\pi} q\delta_{qq'}. \tag{7}$$

Since there is no interaction between c_+ modes and c_- modes, the Hamiltonian in (2) is

$$[H, \rho_q^\pm] = [H_\pm, \rho_q^\pm] \approx \pm v_{\text{F}} q \rho_q^\pm. \tag{8}$$

under the Tomonaga approximation. (8) is the same as the commutation relation in a free phonon system. Defining

$$b_q = \sqrt{\frac{2\pi}{q}}\rho_q^+, \quad \tilde{b}_q = \sqrt{\frac{2\pi}{q}}(\rho_q^-)^\dagger, \tag{9}$$

we find that the Hamiltonian

$$H_{\text{B}} = v_{\text{s}} \sum_{q>0} \left(q(b_q^\dagger b_q + \tilde{b}_q^\dagger \tilde{b}_q) + \frac{\pi}{2L}(\lambda M^2 + \frac{1}{\lambda}J^2) \right) \tag{10}$$

has the same commutation relation with density fluctuations with H_{eff}. If only density fluctuations are important, (10) is a good effective theory of the free fermion theory, which is usually called the **Luttinger liquid theory**.

Note that there is a constant term in (10) which does not affect the dynamics. It comes from the contribution of zero modes. We will skip the discussion about these terms currently, but actually it is these terms that *classifies* Luttinger liquids.

3   Interaction between density fluctuations

Now we take $\lambda = 1$ and add back the electron spins, which is merely to add an additional index to the density modes. Consider the Hamiltonian (often called **Tomonaga-Luttinger model**) introduced in [1]:

$$
\begin{aligned}
H &= H_0 + H_2 + H_4, \\
H_2 &= \frac{1}{L} \sum_{p,s,s'} \left(g_{2\parallel}(p)\delta_{ss'} + g_{2\perp}(p)\delta_{s,-s'} \right) \rho_{p+s}\rho_{-p,-,s'}, \\
H_4 &= \frac{1}{2L} \sum_{p,r,s,s'} \left(g_{4\parallel}(p)\delta_{ss'} + g_{4\perp}(p)\delta_{s,-s'} \right) : \rho_{prs}\rho_{-p,r,s'} :,
\end{aligned}
\tag{11}
$$

which only includes density-density interaction in the momentum space (or in other words, "forward scattering"; note that even forward scattering cannot be handled with perturbative calculation in 1D [1]). Here, $\parallel$ means interaction between spins with the same direction, while $\perp$ means interaction between spins with opposite directions. After bosonization of H_0, we find that the Hamiltonian is a bilinear of density fluctuations. Therefore, after bosonization, the Tomonaga-Luttinger model is exactly solvable. Despite being non-perturbative, (11) can be understood precisely.

The zero mode term is now

$$
\frac{\pi}{2L} \sum_s (v_N (\underbrace{N_{+s} + N_{-s}}_{N_s})^2 + v_J (\underbrace{N_{+s} - N_{-s}}_{J_s})^2),
\tag{12}
$$

where $N_{rs} = \rho_{0,r,s}$. When there is no interaction, we have $v_s = v_N = v_J = v_F$, but after introducing forward scattering, generally speaking there will be charge-spin separation, and the sound speed v_s, the charge excitation velocity v_N, the spin excitation velocity v_J and the Fermi velocity v_F are all different.

Now we can do diagonalization. We define charge and spin variables

$$
\begin{aligned}
\rho_r(p) &= \frac{1}{\sqrt{2}}(\rho_{p,r\uparrow} + \rho_{p,r,\downarrow}), & N_{r,\rho} &= \frac{1}{\sqrt{2}}(N_{r,\uparrow} + N_{r,\downarrow}), \\
\sigma_r(p) &= \frac{1}{\sqrt{2}}(\rho_{p,r\uparrow} - \rho_{p,r,\downarrow}), & N_{r,\sigma} &= \frac{1}{\sqrt{2}}(N_{r,\uparrow} - N_{r,\downarrow}).
\end{aligned}
\tag{13}
$$

This is actually a Bogoliubov transformation. We also redefine the interaction coefficients as

$$
g_{i\rho} = \frac{1}{2}(g_{i\parallel} + g_{i\perp}), \quad g_{i\sigma} = \frac{1}{2}(g_{i\parallel} - g_{i\perp}).
\tag{14}
$$

The Hamiltonian is, therefore,

$$
\begin{aligned}
H_0 &= \frac{\pi v_F}{L} \sum_{\nu,r,p\neq 0} : \nu_r(p)\nu_r(-p) : \\
&\quad + \frac{\pi}{2L} \left[v_{N\nu} \left(N_{+\nu} + N_{-\nu} \right)^2 + v_{J\nu} \left(N_{+\nu} - N_{-\nu} \right)^2 \right], \quad (v_{N\nu} = v_{J\nu} = v_F) \\
H_2 &= \frac{2}{L} \sum_{\nu,p} g_{2\nu}(p)\nu_+(p)\nu_-(-p), \\
H_4 &= \frac{1}{L} \sum_{\nu,r,p} g_{4\nu}(p) : \nu_r(p)\nu_r(-p) : .
\end{aligned}
\tag{15}
$$

where $\nu = \rho, \sigma$. We can see that H_2 is the interaction between density mods between different Fermi points.

> **Note**
>
> Here we should use bosonic Bogoliubov transformation and *not* eigenvalue decomposition. See this link.

Here we briefly describe how to do bosonic Bogoliubov transformation. (According to this blog) Suppose we have a Hamiltonian

$$H = \epsilon_0 \left(a^\dagger a + b b^\dagger \right) + \Delta \left(a^\dagger b^\dagger + ab \right). \tag{16}$$

We take the following ansatz:

$$a = u\alpha - v\beta^\dagger, \quad b = u\beta - v\alpha^\dagger. \tag{17}$$

To keep α and β having the correct commutation relation, we require

$$1 = \left[a, a^\dagger \right] = |u|^2 [\alpha, \alpha^\dagger] + |v|^2 [\beta^\dagger, \beta]$$
$$= |u|^2 - |v|^2,$$

so we have the following normalization condition:

$$|u|^2 - |v|^2 = 1. \tag{18}$$

Note that since (17) is linear, (18) is *equivalent* to the condition of correct commutation relations. From (18) we find the bosonic commutation relation is *symplectic*, not unitary. Now we try to diagonalize (8). Since there is no complex number, we can safely assume that u and v are real. Then, by substituting (17) into (8), we get (see this Mathematica notebook)

$$H = -2uv\Delta + u^2\epsilon_0 + v^2\epsilon_0 + (-2uv\Delta + u^2\epsilon_0 + v^2\epsilon_0)(\alpha^\dagger\alpha + \beta^\dagger\beta)$$
$$+ (u^2\Delta + v^2\Delta - 2uv\epsilon_0)(\alpha\beta + \alpha^\dagger\beta^\dagger).$$

For diagonalization, we require

$$u^2\Delta + v^2\Delta - 2uv\epsilon_0 = 0. \tag{19}$$

Solving (18) and (19), we get

$$-2uv\Delta + u^2\epsilon_0 + v^2\epsilon_0 = \sqrt{\epsilon_0^2 - \Delta^2},$$

so the final Hamiltonian is

$$H = \sqrt{\epsilon_0^2 - \Delta^2}(1 + \alpha^\dagger\alpha + \beta^\dagger\beta). \tag{20}$$

Our final result is

$$H = \frac{\pi}{L} \sum_{r,\nu,p\neq 0} v_\nu(p) : \tilde{\nu}_r(p)\tilde{n}_r(-p) : + \frac{\pi}{L} \tag{21}$$

4 Relativistic field theoretic formulation

Now we can define a bosonic *field* from the creation and annihilation operators:

$$\Phi_\nu(x) = -\frac{i\pi}{L} \sum_{p\neq 0} \frac{e^{-\alpha|p|/2 - ipx}}{p} (\nu_+(p) + \nu_-(p)) - (N_{+\nu} + N_{-\nu})\frac{\pi}{L} \tag{22}$$

5 Fermion in the bosonized theory

Suppose $|N_{rs}, N_{-r,-s}\rangle$ is a state where states under p are filled and states over p are empty.

So the conclusion is, just like the fact that an ideal Fermi gas is a fixed point for Fermi liquids, an ideal exclusion anyon gas (??? TODO) is a fixed point of Luttinger liquids.

6 Umklapp scattering and sind-Gordon model

Therefore, the bosonization of fermions with Umklapp scattering cannot be diagonalized natively.

References

[1] J Voit. One-dimensional fermi liquids. *Reports on Progress in Physics*, 58(9):977–1116, Sep 1995.

[2] Yong-Shi Wu and Yue Yu. Bosonization of one-dimensional exclusons and characterization of luttinger liquids. *Phys. Rev. Lett.*, 75:890–893, Jul 1995.