

# Homework 2

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## 1 Polarization of electromagnetic field

### 1.1 The general form of a pure state

We have (assuming  $\hat{\mathbf{k}} = \hat{\mathbf{z}}$ )

$$\mathbf{E} = E_x \begin{pmatrix} 1 \\ 0 \end{pmatrix} + E_y \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \sqrt{|E_x|^2 + |E_y|^2} e^{i\varphi_x} \begin{pmatrix} \frac{|E_x|}{\sqrt{|E_x|^2 + |E_y|^2}} \\ e^{i(\varphi_y - \varphi_x)} \frac{|E_y|}{\sqrt{|E_x|^2 + |E_y|^2}} \end{pmatrix},$$

and by defining

$$E_0 = \sqrt{|E_x|^2 + |E_y|^2} e^{i\varphi_x}, \quad (1)$$

$$\cos \theta = \frac{|E_x|}{\sqrt{|E_x|^2 + |E_y|^2}}, \quad (2)$$

and

$$\phi = \varphi_y - \varphi_x, \quad (3)$$

we find

$$\mathbf{E} = E_0 \begin{pmatrix} \cos \theta \\ e^{i\phi} \sin \theta \end{pmatrix}. \quad (4)$$

We have

$$|H\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |V\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad (5)$$

and therefore after normalization, we have

$$\rho = (\cos \theta |H\rangle + e^{i\phi} \sin \theta |V\rangle)(\cos \theta \langle H| + e^{-i\phi} \sin \theta \langle V|) = \begin{pmatrix} \cos^2 \theta & e^{-i\phi} \sin \theta \cos \theta \\ e^{i\phi} \sin \theta \cos \theta & \sin^2 \theta \end{pmatrix}. \quad (6)$$

### 1.2 The pure state $\rho^2 = \rho$ condition

We can prove the pure state condition  $\rho^2 = \rho$  explicitly:

$$\begin{aligned} \rho^2 &= \begin{pmatrix} \cos^4 \theta + \sin^2 \theta \cos^2 \theta & e^{-i\phi} \sin \theta \cos^3 \theta + e^{-i\phi} \sin^3 \theta \cos \theta \\ e^{i\phi} \sin \theta \cos^3 \theta + e^{i\phi} \sin^3 \theta \cos \theta & \sin^2 \theta \cos^2 \theta + \sin^4 \theta \end{pmatrix} \\ &= \begin{pmatrix} \cos^2 \theta & e^{-i\phi} \sin \theta \cos \theta \\ e^{i\phi} \sin \theta \cos \theta & \sin^2 \theta \end{pmatrix} = \rho. \end{aligned} \quad (7)$$

### 1.3 Mixed state

The condition that  $\rho$  is Hermite means it can be written as

$$\rho = R(\sigma^0 + x\sigma^x + y\sigma^y + z\sigma^z),$$

where  $R, x, y, z \in \mathbb{R}$ , because the  $\sigma$  matrices constitute a basis for all Hermite matrices in  $\mathbb{C}^{2 \times 2}$ . Since  $\sigma^{x,y,z}$  are traceless, from the condition  $\text{tr } \rho = 1$ , we have

$$1 = \text{tr } \rho = R \text{tr } \sigma^0 = 2R \Rightarrow R = \frac{1}{2},$$

so

$$\rho = \frac{1}{2}(\sigma^0 + x\sigma^x + y\sigma^y + z\sigma^z). \quad (8)$$

In the matrix form, we have

$$\rho = \begin{pmatrix} \frac{1+z}{2} & \frac{x-iy}{2} \\ \frac{x+iy}{2} & \frac{1-z}{2} \end{pmatrix},$$

and by substitution of variables (this is a three variables to three variables mapping, and therefore is valid)

$$\frac{1}{2}(1-p) + p\cos^2\theta = \frac{1+z}{2}, \quad x = p\cos\phi\sin 2\theta, \quad y = p\sin\phi\sin 2\theta,$$

we get

$$\frac{1-z}{2} = \frac{1}{2}(1-p) + p\sin^2\theta,$$

and therefore

$$\rho = (1-p) \begin{pmatrix} \frac{1}{2} & \\ & \frac{1}{2} \end{pmatrix} + p \begin{pmatrix} \cos^2\theta & e^{-i\phi}\sin\theta\cos\theta \\ e^{i\phi}\sin\theta\cos\theta & \sin^2\theta \end{pmatrix}. \quad (9)$$

## 1.4 Jones parameters and Stokes formalism

The definition of Stokes parameters are

$$\begin{aligned} I &= \langle E_x^2 \rangle + \langle E_y^2 \rangle, \\ Q &= \langle E_x^2 \rangle - \langle E_y^2 \rangle, \\ U &= \langle E_a^2 \rangle - \langle E_b^2 \rangle, \\ V &= \langle E_l^2 \rangle - \langle E_r^2 \rangle, \end{aligned} \quad (10)$$

where

$$\hat{\mathbf{a}} = \frac{1}{\sqrt{2}}(\hat{\mathbf{x}} + \hat{\mathbf{y}}), \quad \hat{\mathbf{b}} = \frac{1}{\sqrt{2}}(\hat{\mathbf{x}} - \hat{\mathbf{y}}), \quad (11)$$

and

$$\hat{\mathbf{l}} = \frac{1}{\sqrt{2}}(\hat{\mathbf{x}} + i\hat{\mathbf{y}}), \quad \hat{\mathbf{r}} = \frac{1}{\sqrt{2}}(\hat{\mathbf{x}} - i\hat{\mathbf{y}}). \quad (12)$$

Note that  $E_{x,y}$  etc. above may be regarded as operators, and we have

$$\begin{aligned} E_x^2 + E_y^2 &= \sigma^0, \\ E_x^2 - E_y^2 &= \sigma^z, \\ E_a^2 - E_b^2 &= \sigma^x, \\ E_l^2 - E_r^2 &= \sigma^y. \end{aligned} \quad (13)$$

From these definitions and the fact that  $(\sigma^i)^2 = \sigma^0$  and all other products of  $\sigma$  matrices are traceless, we find

$$I = \langle \sigma^0 \rangle = 1, \quad (14)$$

$$Q = \langle \sigma^z \rangle = \frac{1}{2}z \cdot 2 = 2p\cos^2\theta - p = p\cos 2\theta, \quad (15)$$

$$U = \frac{1}{2}x \cdot 2 = p\cos\phi\sin 2\theta, \quad (16)$$

and

$$V = \frac{1}{2}y \cdot 2 = p\sin\phi\sin 2\theta. \quad (17)$$

Here  $I$  is constantly 1, because we are working with the single-photon density matrix.

## 1.5 Mueller calculus

As is shown above, Mueller calculus actually works on the coefficients in (8), and therefore a Mueller matrix essentially gives the coefficients of  $\rho \rightarrow U\rho U^\dagger$ . It makes sense as long as after its application, the  $\sigma^0$  component in  $\rho$  is still 1/2.

## 1.6 Transformation and measurement

We have

$$\begin{aligned} |45\rangle &= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\ |\text{rcp}\rangle &= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -i \end{pmatrix}. \end{aligned} \quad (18)$$

The correspond density matrices are

$$\rho_{45} = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \quad (19)$$

and

$$\rho_{\text{rcp}} = \frac{1}{2} \begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix}. \quad (20)$$

The operator

$$U_{\text{rcp}} = \begin{pmatrix} 1 & \\ & -i \end{pmatrix} \quad (21)$$

then turns  $\rho_{45}$  to  $\rho_{\text{rcp}}$ :

$$U_{\text{rcp}} \rho_{45} U_{\text{rcp}}^\dagger = \frac{1}{2} \begin{pmatrix} 1 & \\ & -i \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & \\ & i \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix} = \rho_{\text{rcp}}. \quad (22)$$

The horizontal polarizer operator

$$\mathcal{O} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad (23)$$

is not unitary, because it has non-unitary eigenvalue 0. It is a projection operator: it takes in a beam polarized light and returns its  $x$  component. It also represents a measurement: we can use it in a projective measurement setting. In a projective measurement with operator  $\mathcal{O}$ ,  $\text{tr}(\rho\mathcal{O})$  is the probability that after measurement, the final state of the system falls into the subspace determined by  $\mathcal{O}$ . In our case, the subspace determined by  $\mathcal{O}$  is the subspace of horizontal polarization, so  $\text{tr}(\rho\mathcal{O})$  is the probability that after measurement, we find  $\rho$  to be a horizontally polarized state.

Application of  $\mathcal{O}$  on (9) is

$$\mathcal{O}\rho\mathcal{O}^\dagger = (1-p) \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 0 \end{pmatrix} + p \begin{pmatrix} \cos^2 \theta & 0 \\ 0 & 0 \end{pmatrix} = \left( \frac{1}{2} + \frac{p}{2} \cos 2\theta \right) \mathcal{O}. \quad (24)$$

So after the application of  $\mathcal{O}$ , we get a horizontally polarized state, as is expected. The result is not normalized; the factor before  $\mathcal{O}$  is just  $\text{tr}(\rho\mathcal{O})$ , which is the probability that after measurement, we find  $\rho$  to be horizontally polarized. When  $p = 0$ , it's  $1/2$ , which is expected for the unpolarized state; when  $p = 1$ , it's  $\cos^2 \theta$ , again the correct answer.

## 2 The $\rho^2 = \rho$ condition for pure states

Suppose

$$\rho = |\psi\rangle\langle\psi|, \quad |\psi\rangle = \sum_m a_m |m\rangle. \quad (25)$$

We have

$$\begin{aligned}
\rho^2 &= \sum_{m,n} a_m^* a_n |n\rangle \langle m| \sum_{j,k} a_j^* a_k |k\rangle \langle j| \\
&= \sum_{m,n,j,k} a_m^* a_n a_j^* |n\rangle \langle m|k\rangle \langle j| \\
&= \sum_{m,n,j,k} a_m^* a_n a_j^* a_k |n\rangle \langle j| \delta_{mk} \\
&= \sum_m \underbrace{a_m^* a_m}_{=\langle \psi | \psi \rangle = 1} \sum_{n,j} a_n a_j^* |n\rangle \langle j| \\
&= \sum_{n,j} a_n a_j^* |n\rangle \langle j| = \rho.
\end{aligned} \tag{26}$$

### 3 Ammonia molecule

The Hamiltonian of the two low-energy states of ammonia is

$$H = \begin{pmatrix} 0 & \Delta/2 \\ \Delta/2 & 0 \end{pmatrix}, \tag{27}$$

where we set

$$|L\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |R\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{28}$$

This Hamiltonian is just a scaled  $\sigma^x$  matrix, and its eigenstates are straightforwardly given by

$$|+\rangle = \frac{1}{\sqrt{2}}(|L\rangle + |R\rangle), \quad |-\rangle = \frac{1}{\sqrt{2}}(|L\rangle - |R\rangle), \tag{29}$$

and the energies are

$$E_+ = \Delta/2, \quad E_- = -\Delta/2. \tag{30}$$

After an electric field is added, in  $|L\rangle$  we have an additional energy contribution, and since the molecular configuration in  $|R\rangle$  is the opposite of the one in  $|L\rangle$ , we have

$$H = \begin{pmatrix} dE/2 & \Delta/2 \\ \Delta/2 & -dE/2 \end{pmatrix}. \tag{31}$$

Solving

$$\det \begin{pmatrix} dE/2 - \lambda & \Delta/2 \\ \Delta/2 & -dE/2 - \lambda \end{pmatrix} = 0,$$

we get

$$E_{\pm} = \pm \frac{1}{2} \sqrt{\Delta^2 + d^2 E^2}, \tag{32}$$

and hence

$$|-\rangle = \frac{dE - \sqrt{d^2 E^2 + \Delta^2}}{\sqrt{\Delta^2 + (\sqrt{d^2 E^2 + \Delta^2} - dE)^2}} |L\rangle + \frac{\Delta}{\sqrt{\Delta^2 + (\sqrt{d^2 E^2 + \Delta^2} - dE)^2}} |R\rangle, \tag{33}$$

and

$$|+\rangle = \frac{dE + \sqrt{d^2 E^2 + \Delta^2}}{\sqrt{\Delta^2 + (\sqrt{d^2 E^2 + \Delta^2} + dE)^2}} |L\rangle + \frac{\Delta}{\sqrt{\Delta^2 + (\sqrt{d^2 E^2 + \Delta^2} + dE)^2}} |R\rangle. \tag{34}$$

When  $E$  is large, we have

$$\begin{aligned}
dE - \sqrt{d^2 E^2 + \Delta^2} &\approx 0, \\
dE + \sqrt{d^2 E^2 + \Delta^2} &\approx 2dE,
\end{aligned}$$

and therefore

$$E_+ = \frac{1}{2} dE, \quad |+\rangle = |L\rangle, \tag{35}$$

and

$$E_- = -\frac{1}{2}dE, \quad |-\rangle = |R\rangle. \quad (36)$$

This is expected, because when  $Ed \gg \Delta$ , the non-diagonal terms in the Hamiltonian can be safely ignored.

## 4 Kaptiza's pendulum

### 4.1 Integrating out the fast variable

In the  $\omega \rightarrow \infty$ ,  $F_0 \rightarrow 0$  limit, the high-frequency part and the low-frequency part of the solution of

$$mR\ddot{\theta} = (-mg + F_0 \sin \omega t) \sin \theta \quad (37)$$

are not strongly coupled and the high-frequency degree of freedom can be integrated out to get an effective theory of the low-frequency part. We do the decomposition

$$\theta = \theta_f + \theta_s, \quad (38)$$

where  $\theta_f$  is the fast variable. Observing (37), we find the EOM of  $\theta_f$  should be

$$mR\ddot{\theta}_f = F_0 \sin \omega t \sin(\theta_f + \theta_s), \quad (39)$$

because the first term on the RHS of (37) has a much lower frequency magnitude compared with  $\omega$ . We take the first order approximation of (39) and ignore the  $\theta_f$  dependency on the RHS, and this gives

$$\theta_f = -\frac{F_0}{mR\omega^2} \sin \theta_s \sin \omega t. \quad (40)$$

Putting this back to (37), we get

$$\begin{aligned} mR \left( \frac{F_0}{mR} \sin \theta_s \sin \omega t + \ddot{\theta}_s \right) &= (-mg + F_0 \sin \omega t) \sin \left( \theta_s - \frac{F_0}{mR\omega^2} \sin \theta_s \sin \omega t \right) \\ &= (-mg + F_0 \sin \omega t) \left( \sin \theta_s - \cos \theta_s \cdot \frac{F_0}{mR\omega^2} \sin \theta_s \sin \omega t \right). \end{aligned}$$

Now we average over all high-frequency time dependencies. The first term on the LHS averages zero, and so do the  $-mg \sin \omega t$  term and the  $F_0 \sin \omega t \sin \theta_s$  term on the RHS. On the other hand, the  $\sin^2 \omega t$  term on the RHS averages

$$-\frac{F_0^2}{mR\omega^2} \sin \theta_s \cos \theta_s \langle \sin^2 \omega t \rangle = -\frac{1}{2} \frac{F_0^2}{mR\omega^2} \sin \theta_s \cos \theta_s,$$

so the final EOM for  $\theta_s$  is

$$mR\ddot{\theta}_s = -mg \sin \theta_s - \frac{1}{2} \frac{F_0^2}{mR\omega^2} \sin \theta_s \cos \theta_s. \quad (41)$$

### 4.2 Stable positions of $\theta_s$

We let the LHS of (41) be zero, and the equation becomes

$$\sin \theta_s \left( mg + \frac{1}{2} \frac{F_0^2}{mR\omega^2} \cos \theta_s \right) = 0.$$

Since  $F_0 \rightarrow 0$ , the second factor on the LHS can't be zero, so the equation becomes

$$\sin \theta_s = 0 \Rightarrow \theta_s = 0, \pi. \quad (42)$$

Around  $\theta_s = 0$ , (41) is approximately

$$mR\ddot{\theta}_s = -mg\theta_s - \frac{1}{2} \frac{F_0^2}{mR\omega^2} \theta_s,$$

and therefore

$$\omega_{\theta=0} = \sqrt{\frac{g}{R} + \frac{F_0^2}{m^2 R^2 \omega^2}}. \quad (43)$$

This is always real, and therefore the  $\theta_s = 0$  position is always stable.

Around  $\theta_s = \pi$ , we rewrite (41) in terms of  $\theta'_s = \pi - \theta_s$ , and get

$$-mR\ddot{\theta}'_s = mg\theta'_s - \frac{1}{2} \frac{F_0^2}{mR\omega^2} \theta'_s = 0,$$

and

$$\omega_{\theta_s=\pi} = \sqrt{\frac{1}{2} \frac{F_0^2}{m^2 R^2 \omega^2} - \frac{g}{R}}. \quad (44)$$

It can be seen that when  $F_0 = 0$ , the frequency is imaginary and therefore  $\theta_s = \pi$  is not a stable position. However, when

$$\frac{1}{2} \frac{F_0^2}{m^2 R^2 \omega^2} \geq \frac{g}{R}, \quad (45)$$

we do have oscillation behavior around  $\theta_s = \pi$ .

## 5 Relaxation of a spin polarization due to an electric field

### 5.1 The spin-magnetic field coupling

We have

$$H = -\boldsymbol{\mu} \cdot \mathbf{B}, \quad \boldsymbol{\mu} = -g\mu_B \mathbf{S}, \quad (46)$$

and this means

$$H = \frac{1}{2} g\mu_B \boldsymbol{\sigma} \cdot \mathbf{B}. \quad (47)$$

This gives

$$\frac{d\mathbf{S}}{dt} = \frac{g\mu_B}{\hbar} \mathbf{B} \times \mathbf{S}. \quad (48)$$

When  $\mathbf{B}$  is fixed (instead of the motion magnetic field in (55)), we find the oscillation frequency is

$$\omega = \frac{g\mu_B}{\hbar} B, \quad (49)$$

and therefore

$$\gamma = \frac{g\mu_B}{\hbar}. \quad (50)$$

### 5.2 The $\sigma_{\pm}$ representation

We define

$$\sigma_{\pm} = \frac{\sigma_x \pm i\sigma_y}{2}, \quad (51)$$

and the commutation relations are

$$[\sigma_+, \sigma_-] = \sigma_z, \quad [\sigma^z, \sigma_{\pm}] = \pm 2\sigma_{\pm}. \quad (52)$$

The Hamiltonian is then

$$H = b^* \sigma_+ + b \sigma_- + b_z \sigma_z, \quad (53)$$

where

$$b = \frac{1}{2} g\mu_B (B_x + iB_y), \quad b_z = \frac{1}{2} g\mu_B B_z. \quad (54)$$

### 5.3 Eliminating the static magnetic field

The magnetic field felt by an electron with velocity  $\mathbf{v}$  when an electric field  $\mathbf{E}$  is present is

$$\mathbf{B}_1 = -\mathbf{v} \times \mathbf{E}/c^2. \quad (55)$$

We also have a static magnetic field  $\mathbf{B}_0$  pointing towards  $\hat{\mathbf{z}}$ . Here we assume  $\mathbf{B}_0$  and the motion magnetic field are orthogonal to each other, and this means the Hamiltonian is

$$H = b^* \sigma_+ + b \sigma_- + \frac{1}{2} \hbar \gamma B_0 = b^* \sigma_+ + b \sigma_- + \underbrace{\frac{1}{2} \hbar \omega_0 \sigma_z}_{H_0}. \quad (56)$$

The time evolution corresponding to that  $\omega_0$  term is trivial, and using the interaction picture we can get rid of it. We have

$$e^{iH_0 t/\hbar} = \begin{pmatrix} e^{i\omega_0 t/2} & \\ & e^{-i\omega_0 t/2} \end{pmatrix}. \quad (57)$$

The new Hamiltonian is now

$$\begin{aligned} H &= e^{iH_0 t/\hbar} (b^* \sigma_+ + b \sigma_-) e^{-iH_0 t/\hbar} \\ &= e^{i\omega_0 t} b^* \sigma_+ + e^{-i\omega_0 t} b \sigma_-. \end{aligned} \quad (58)$$

We will work with this Hamiltonian later.

### 5.4 Time evolution

Now we use

$$\frac{\partial \rho(t)}{\partial t} = -\frac{1}{\hbar^2} \int_0^t dt' [H(t), [H(t'), \rho(t)]] = -\frac{1}{\hbar^2} \int_0^t dt' [H(t), [H(t-t'), \rho(t)]] \quad (59)$$

to find the time evolution of  $\rho$ , considering only second-order correlation in the random variable in  $H$ . We again do decomposition

$$\rho = \frac{1}{2} \sigma_0 + \rho_+ \sigma_+ + \rho_- \sigma_- + \rho_z \sigma_z, \quad (60)$$

and we have (here  $b'$  and  $b'^*$  are values taken at  $t-t'$ )

$$[H(t-t'), \rho(t)] = -\rho_+ e^{-i\omega_0(t-t')} b' \sigma_z + \rho_- e^{i\omega_0(t-t')} b'^* \sigma_z - 2\rho_z e^{i\omega_0(t-t')} b'^* \sigma_+ + 2\rho_z e^{-i\omega_0(t-t')} b' \sigma_-,$$

and by component, we get

$$\dot{\rho}_+ = -\frac{1}{\hbar^2} \int_0^t dt' \left( 2\rho_+ e^{i\omega_0 t'} b' b^* - 2\rho_- e^{2i\omega_0 t - i\omega_0 t'} b^* b'^* \right), \quad (61)$$

$$\dot{\rho}_- = -\frac{1}{\hbar^2} \int_0^t dt' \left( 2\rho_- e^{-i\omega_0 t'} b b'^* - 2\rho_+ e^{-2i\omega_0 t + i\omega_0 t'} b b' \right), \quad (62)$$

and

$$\dot{\rho}_z = -\frac{1}{\hbar^2} \int_0^t dt' \left( 2\rho_z e^{-i\omega_0 t'} b b'^* + 2\rho_z e^{i\omega_0 t'} b^* b' \right). \quad (63)$$