Chern-Simons Theory for Topological States of Matter

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1 The simplest Chern-Simons theory

A type of well-studied *topological quantum field theory (TQFT)* – which has no non-trivial particle dynamics – is **Chern-Simons theories**. We consider, for example, the following theory where Schrödinger bosons are coupled to a single gauge field, and there is a Chern-Simons term for the gauge field itself:

$$\mathcal{L} = \mathrm{i}\bar{\phi} D_0 \phi + \frac{1}{2m}\bar{\phi}\boldsymbol{D}^2\phi + \underbrace{\frac{1}{4\theta}\epsilon^{\mu\nu\lambda} a_\mu \partial_\nu a_\lambda}_{\mathcal{L}_{\mathrm{CS}}}, \tag{1}$$

where

$$D_\mu = \partial_\mu + \mathrm{i} a_\mu. \tag{2}$$

We have not proved that (1) is indeed gauge invariant yet. Under a local transformation

$$a_\mu \longrightarrow a_\mu + \partial_\mu h,$$

we have

$$\begin{aligned}
\frac{1}{4\theta}\epsilon^{\mu\nu\lambda} a_\mu \partial_\nu a_\lambda \longrightarrow & \frac{1}{4\theta}\epsilon^{\mu\nu\lambda}(a_\mu + \partial_\mu h)\partial_\nu(a_\lambda + \partial_\lambda h) \\
&= \frac{1}{4\theta}(\epsilon^{\mu\nu\lambda} a_\mu \partial_\nu a_\lambda + \epsilon^{\mu\nu\lambda}\partial_\mu h \partial_\nu a_\lambda + \epsilon^{\mu\nu\lambda} a_\mu \partial_\nu \partial_\lambda h + \epsilon^{\mu\nu\lambda}\partial_\mu h \partial_\nu \partial_\lambda h) \\
&= \frac{1}{4\theta}(\epsilon^{\mu\nu\lambda} a_\mu \partial_\nu a_\lambda + \epsilon^{\mu\nu\lambda}\partial_\mu (h \partial_\nu a_\lambda) - \epsilon^{\mu\nu\lambda} h \partial_\nu \partial_\mu a_\lambda) \\
&= \frac{1}{4\theta}(\epsilon^{\mu\nu\lambda} a_\mu \partial_\nu a_\lambda + \epsilon^{\mu\nu\lambda}\partial_\mu (h \partial_\nu a_\lambda)),
\end{aligned}$$

where the last two terms in the second line vanish because $\epsilon^{\mu\nu\lambda} a_\mu a_\nu = 0$, and this also how we get the last line. Note that though after the transformation the Lagrangian *density* changes, after integrating over the whole space the total Lagrangian does *not* change because the second term in the last line above is a boundary term. So we find that in (1), the Chern-Simons part is indeed gauge invariant, though it cannot be written in terms of $\boldsymbol{E}$ and $\boldsymbol{B}$. The coupling term in (1) transforms in the same way of QED, where the bosonic field undergoes a local $U(1)$ transformation. Thus, (1) is a *$U(1)$ Chern-Simons theory*.

Now we evaluate the EOM of (1). The Chern-Simons terms gives

$$\begin{aligned}
\partial_\rho \frac{\partial \mathcal{L}}{\partial \partial_\rho a_\sigma} - \frac{\partial \mathcal{L}}{\partial a_\sigma} &= \frac{1}{4\theta}\partial_\rho \epsilon^{\mu\rho\sigma} a_\mu - \frac{1}{4\theta}\epsilon^{\sigma\nu\lambda}\partial_\nu a_\lambda \\
&= -\frac{1}{2\theta}\epsilon^{\sigma\nu\lambda}\partial_\nu a_\lambda.
\end{aligned}$$

Another way to find the above equation is to use integration by parts to rephrase all terms containing $a_\mu \partial_\rho a_\sigma$ into $-\partial_\rho a_\mu a_\sigma$, so the first term in the LHS of the first line vanishes, and we get a 2 factor. In the Schrödinger field part, terms involving a_μ are

$$\begin{aligned}
& -\bar{\phi} a_0 \phi + \frac{\mathrm{i}}{2m}\bar{\phi}\partial_i(a_i\phi) + \frac{\mathrm{i}}{2m}\bar{\phi} a_i \partial_i \phi - \frac{1}{2m} a_i^2 \bar{\phi}\phi \\
= & -\bar{\phi} a_0 \phi - \frac{\mathrm{i}}{2m}(\partial_i \bar{\phi}) a_i \phi + \frac{\mathrm{i}}{2m}\bar{\phi} a_i \partial_i \phi - \frac{1}{2m} a_i^2 \bar{\phi}\phi,
\end{aligned}$$

and we have

$$\partial_\rho \frac{\partial \mathcal{L}}{\partial \partial_\rho a_0} - \frac{\partial \mathcal{L}}{\partial a_0} = \bar{\phi}\phi,$$

and

$$\partial_\rho \frac{\partial \mathcal{L}}{\partial \partial_\rho a_i} - \frac{\partial \mathcal{L}}{\partial a_i} = \frac{a_i}{m} \bar{\phi}\phi + \frac{\mathrm{i}}{2m}(\phi \partial_i \bar{\phi} - \bar{\phi}\partial_i \phi).$$

We soon find that the RHSs of the above two equations are just conservation flows of a Schrödinger field coupled to a gauge field, as is the case in QED. So the EOM of a_μ is just

$$j^\sigma + \frac{1}{2\theta} \epsilon^{\sigma\nu\lambda} \partial_\nu a_\lambda = 0. \tag{3}$$

Specifically, the temporal part of the EOM is

$$\epsilon^{ij} \partial_i a_j = -2\theta\rho, \quad \rho = \bar{\phi}\phi. \tag{4}$$

Actually, (3) can also be derived by simply adding a

$$S_{\text{coupling}} = \int \mathrm{d}^3 x \, j^\mu a_\mu \tag{5}$$

term into the action.

Note

Note that we are working in a 2+1 dimensional Minkowski spacetime, and $a^i = -a_i$. Opposite signs in (3) can be seen in different sources, which comes from the fact that when deriving (3), these sources switch to the Euclidean space, where

$$a_i = a^i = a^i_{\text{Minkowski}} = -a_{i,\text{Minkowski}}.$$

Often, we introduce a parameter K such that

$$\theta = \frac{\pi}{K}. \tag{6}$$

Also, the field a_μ has its own electric field and magnetic field, which are (note that we are still in the Minkowski spacetime and $a_i = -a^i$, so the signs may seem different with those in the ordinary vector calculus forms)

$$e^i = -\partial_0 a^i + \partial^i a^0, \quad b = -\epsilon^{ij} \partial_i a_j. \tag{7}$$

Therefore, we have

$$j^0 = \rho = \frac{K}{2\pi} b. \tag{8}$$ David Tong QHE Eq. (5.9)

Since we have

$$\begin{aligned} \epsilon^{i\nu\lambda} \partial_\nu a_\lambda &= \epsilon^{i0\lambda} \partial_0 a_\lambda + \epsilon^{ij\lambda} \partial_j a_\lambda \\ &= \epsilon^{i0j} \partial_0 a_j + \epsilon^{ij0} \partial_j a_0 \\ &= -\epsilon^{ij} \partial_0 a_j + \epsilon^{ij} \partial_j a_0 = \epsilon^{ij} e_j, \end{aligned}$$

we have

$$j^i = -\frac{K}{2\pi} \epsilon^{ij} e_j = \frac{K}{2\pi} \epsilon^{ij} e^j. \tag{9}$$ David Tong QHE Eq. (5.8)

There is no photon in the Chern-Simons theory. Or, if we introduce a small Maxwell term into $\mathcal{L}_{\text{CS}}$ to manually add some photons into the theory, we will find that the mass of the pure Chern-Simons theory is infinity. This means Chern-Simons theory may describe a large class of systems with gapped bulk where there is a gauge field with no dynamics itself and primarily gives A-B phases to particles immersed in it. Eq. (3.32) in [2]

2 Chern-Simons theory as an effective theory of IQHE

Now we follow the derivation of Section 4.4 in Wen's famous textbook and Section 5.1 in David Tong's lecture note on QHE. Since the bulk state of a IQHE system is an insulator, the only low energy degree of freedom is the electromagnetic field, and by integrating out the gapped Wen Section 4.4, David Tong QHE Section 5.1

bands, the effective theory of the electromagnetic field gets some correction, and the final effective theory can be any (2+1)-dimensional theory that has $U(1)$ gauge symmetry, translational symmetry, rotational symmetry, etc. The Chern-Simons term satisfies these conditions and therefore can appear in the effective theory of the electromagnetic field. We are confident that the Chern-Simons theory is indeed the effective theory of a IQHE system, because it can produce a transverse conductivity: since a_μ in (9) is the real electromagnetic field and j^μ the real current of electrons, we can rewrite (9) as

$$J^i = \frac{K}{2\pi}\epsilon^{ij}E^j, \tag{10}$$

where we use capitalized letters to denote quantities in the standard electrodynamics, and thus the Hall conductivity is

$$\sigma_{xy} = \frac{K}{2\pi}. \tag{11}$$

David Tong QHE Eq. (5.7)

Now if we are able to show that K can only take discrete values, than we can explain IQHE in a quite neat way without touching the dirty microscopic details.

This way to show that the Chern-Simons theory is the effective theory for IQHE is often called the **hydrodynamic method** because what we are doing is actually reverse engineering an action that can reproduce the linear response of density modes in IQHE, and our decision to keep only the Chern-Simons term is actually taking only the term with the lowest momentum order in the response function, i.e. the lowest term in the *gradient expansion* of the response function (the next term will give the familiar Maxwell action).

3 The microscopic derivation of Chern-Simons theory for IQHE

Beside the hydrodynamic method, we can use the so-called **Chern-Simons Ginzburg Landau theory** to prove that the final macroscopic description of IQHE is a Chern-Simons theory [3].

4 Anyons within Chern-Simons theory and an effective theory of FQHE

4.1 Chern-Simons theory describes anyons

This section reviews what is covered in [2]. Briefly speaking, our final goal is to encode the phases of anyons into a gauge theory. We expect a gauge theory will do the job because we can always mimic the exchange phases of anyons by considering the anyons as particles created by a bosonic field coupled to a gauge field. The particles created by the bosonic field carry both a flux and a charge, and thus the Aharonov–Bohm effect can reproduce the correct self and mutual phases. What we are going to talk about in this section are Abelian anyons, so things will not be too hard. We claim after the exchange phases are correctly encoded into a gauge theory, we can obtained a *complete* low energy effective theory of a topological order, since we are usually not interested in the scattering of anyons, and the low energy dynamics of an anyon system has nothing more than the exchange phases.

Why we use a gauge field theory instead of something like a Coulomb interaction is discussed in Section 3.1 in [2]. In summary, it is because with a Coulomb-like Hamiltonian we often run into duplicate phase factors. Another approach is to construct anyon field operators. It is possible, but the resulting theories are usually highly complicated and technical.

Since generally we are only interested in the exchange phases of anyons in a topological order, we just narrow down the candidates of gauge field theoretic effective descriptions of topological orders to TQFTs, and we will find that the Chern-Simons theory introduced in Section 1 is just the effective theory for FQHE.

Our next goal is to From (8) we can see that an anyon in a Chern-Simons theory is indeed a *charge-flux composite*. There is no separate charge excitation or flux excitation, which is the case in the $\mathbb{Z}_2$ gauge theory. We then calculate the exchange phase of anyons. Since (8), one may expect that the exchange phase is

$$\oint_{\partial S} \mathrm{d}\boldsymbol{r} \cdot \boldsymbol{a} = \int_S \mathrm{d}^2\boldsymbol{r}\, b = \frac{2\pi}{K} \int \mathrm{d}S\, \rho,$$

so the exchange phase per particle is $2\pi/K = 2\theta$. However, this is *not* the case. Here the subtle is that since (3), we have

$$\mathcal{L}_{\text{CS}} = \frac{1}{4\theta} \epsilon^{\mu\nu\lambda} a_\mu \partial_\nu a_\lambda = -\frac{1}{2} j^\mu a_\mu,$$

so the total Lagrangian is

$$\mathcal{L}_{\text{CS}} + \mathcal{L}_{\text{coupling}} = \frac{1}{2} j^\mu a_\mu. \tag{12}$$

Therefore, the correct phase factor is $\mathrm{e}^{\frac{\mathrm{i}}{2} \oint \mathrm{d}\boldsymbol{r} \cdot \boldsymbol{a}}$, and the exchange phase per particle is $2\theta/2 = \theta$. Actually, this is why we we name the parameter as θ. Since the (alleged in Laughlin liquid) internal degrees of freedom are made up by anyons with exchange phase θ, we already get a satisfactory EFT of the internal degrees of freedom in the simplest $1/\nu$ FQHE.

4.2 "Where are the string operators?"

When studying topological orders defined with lattice models, i.e. $\mathbb{Z}_2$ gauge theory, we usually do not (explicitly) couple any matter field to the gauge field, because we can study the effect induced by the gauge theory to the matter field by simply examining the field configuration when a charge or flux is present. In other words, we can define anyons *within* the gauge theory. For example, we may define

$$Q_{\boldsymbol{i}} = \prod_{\boldsymbol{p} \in +_{\boldsymbol{i}}} \sigma^x_{\boldsymbol{p}}, \tag{13}$$

and

$$F_{\boldsymbol{i}} = \prod_{\boldsymbol{p} \in \square_{\boldsymbol{i}}} \sigma^z_{\boldsymbol{p}}. \tag{14}$$

On the other hand, the Chern-Simons theory is a continuum field theory, which means we can evaluate quantities like the Wilson loop more easily with some classical picture of the gauge field, but it is much harder to work directly with the many-body wavefunction and characterize the behavior of anyons within the gauge theory. In this section, we discuss the correspondence between objects in the Chern-Simons theory and objects in lattice models.

So we can just put a point charge at, say, $\boldsymbol{r} = 0$. The corresponding equation is

$$\boldsymbol{\nabla} \times \boldsymbol{a} = b = \frac{2\pi}{K} \rho = \frac{2\pi}{K} q \delta(\boldsymbol{r}).$$

With the Coulomb gauge condition $\boldsymbol{\nabla} \cdot \boldsymbol{a} = 0$, we have

$$a^i = \frac{2\pi}{K} q \cdot \frac{1}{2\pi} \epsilon^{ij} \frac{x^j}{x^2} = \frac{q}{K} \epsilon^{ij} \frac{x^j}{x^2}. \tag{15}$$

We do not seen any string structure in this expression, but this may be just illusion: strings can be viewed as field lines, and since we are working in the path integral formulation, the field (15) is a classical field contains *many* possible field lines, and we cannot distinguish a single, clear, well-defined field line. To find the string structure, we consider the following gauge transformation:

$$\boldsymbol{a}' = \boldsymbol{a} - \boldsymbol{\nabla}\chi, \quad \chi = \begin{cases} \arctan\left(\frac{y}{x}\right) + \pi, & x > 0, \\ \arctan\left(\frac{y}{x}\right), & x < 0. \end{cases} \tag{16}$$

We will find that $\boldsymbol{a} = 0$ except on the ray $x = 0, y > 0$, where there is a step discontinuity (see Figure 1 on page 5), and the value of $\boldsymbol{a}$ there is proportion to $\delta(x)$. This ray is just the string we expect. With a gauge transformation, we can distort the shape of the string, as is the case in string-net models.

One more thing we need to do is to find the EFT for the $1/\nu$ FQHE driven by the external electromagnetic field. The EOM of such a theory must result in something like (10),

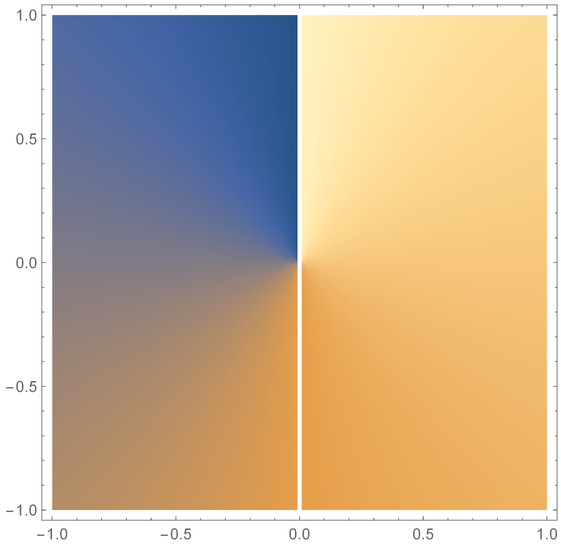

Figure 1: Density plot of χ on the x-y plane. Obtained by this Mathematica notebook.

5 Mutual Chern-Simons theory

We can extend the simplest Chern-Simons theory in Section 1 by adding more gauge fields, and get the following **mutal Chern-Simons theory**

$$\mathcal{L}_{\text{mutual CS}} = \frac{1}{4\pi} \epsilon^{\mu\nu\lambda} a^I_\mu \partial_\nu K_{IJ} a^J_\lambda \tag{17}$$

6 Chern-Simons description of the $\mathbb{Z}_2$ topological order

Note that the mutual exchange phase angle has an additional 2 factor compared to the self-statistical angle. (23-24) in [1]

We can say that we are in a "good" representation of the $\mathbb{Z}_2$ topological order, since there is no quantum fluctuation here. The microscopic model of a topological order, if in the language of the "real" degrees of freedom (i.e. degrees of freedom like spins or electrons that are "understandable"), often involves strong quantum fluctuation to generate an emergent gauge theory. A string-net model describes the same universality class yet can be solved exactly, and therefore has strong quantum fluctuation in the representation of "real" degrees of freedom but no strong quantum fluctuation in the representation of operators like A_i or B_p. The nonlocality and long-range entanglement is hidden in the definition of these seemingly simple operators. A TQFT description further hides the quantum fluctuation in the microscopic details.

$$K = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix} \tag{18}$$

To see so, we do an explicit derivation. Suppose $a^\mu = (a, b)$, we have

$$\begin{aligned} & \frac{\mathrm{i}}{4\pi} \int \mathrm{d}^3 x \, \epsilon^{\mu\nu\lambda} a^I_\mu K_{IJ} \partial_\nu a^J_\lambda \\ &= \frac{\mathrm{i}}{4\pi} \int \mathrm{d}^3 x \, (2 a_\mu \partial_\nu b_\lambda + 2 b_\mu \partial_\nu a_\lambda) \\ &= \frac{\mathrm{i}}{4\pi} \int \mathrm{d}^3 x \, (2 a_\mu \partial_\nu b_\lambda - 2 a_\lambda \partial_\nu b_\mu) \\ &= \frac{\mathrm{i}}{\pi} \int \mathrm{d}^3 x \, \epsilon^{\mu\nu\lambda} a_\mu \partial_\nu b_\lambda, \end{aligned}$$

and therefore the EOM can be obtained by calculating the variation of

$$\mathcal{L}_{\mathbb{Z}_2, \text{CS}} = \frac{1}{\pi} \epsilon^{\mu\nu\lambda} a_\mu \partial_\nu b_\lambda + j^\mu_a a_\mu + j^\mu_b b_\mu, \tag{19}$$

and we have

$$j_a^\mu + \frac{1}{\pi}\epsilon^{\mu\nu\lambda}\partial_\nu b_\lambda = 0, \quad j_b^\mu + \frac{1}{\pi}\epsilon^{\mu\nu\lambda}\partial_\nu a_\lambda = 0. \tag{20}$$

Substituting (20) into (19), we get

$$\mathrm{e}^{\mathrm{i}\int \mathrm{d}^3x\mathcal{L}} = \mathrm{e}^{\mathrm{i}\int \mathrm{d}^3x j_a^\mu a_\mu} = \mathrm{e}^{\mathrm{i}\int \mathrm{d}^3x j_b^\mu b_\mu}. \tag{21}$$

Therefore, the 1/2 factor in (12) is absent when the exchange phase angle is caused by the coupling between a_μ^I and j_J^μ, where $I \neq J$, and (19) is the Chern-Simons theory for $\mathbb{Z}_2$ topological order.

References

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