Homework 2

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1.1

Comparison between the approximate and exact susceptibilities. We consider a driven damped simple harmonic oscillator (following L8.N3). We assume that $\dot{p} = \sum_i f_i = -kq - D\dot{q} + f(t)$, where k, D, and $f(t)$ are the spring constant, damping, and driving force, respectively. In what follows, we assume that $f(t) = f_o \cos(\omega t + \phi)$.

(a) *Starting from Newton's equations, find the first order equation of motion in complex form. [Hint: you should find something like $\dot{a} = -\mathrm{i}\omega_o a + (\ldots)$, and you're looking to identify all of the (...) terms.]*

When there is no damping the EOMs are

$$\dot{p} = -kq, \quad \dot{q} = \frac{p}{m}, \tag{1}$$

and the following definition of the modes

$$q = \sqrt{\frac{1}{2m\omega_0}}(a^* + a), \quad p = \mathrm{i}\sqrt{\frac{m\omega_0}{2}}(a^* - a), \quad \omega_0^2 = \frac{k}{m} \tag{2}$$

turns the Hamiltonian into $H = \omega_0 a^* a$. Under this definition of a, a^*, the EOM with damping and driving therefore becomes

$$\begin{aligned} \dot{a}^* - \dot{a} &= \mathrm{i}\omega_0(a^* + a) - \frac{D}{m}(a^* - a) - \mathrm{i}\sqrt{\frac{2}{m\omega_0}}f(t), \\ \dot{a}^* + \dot{a} &= \mathrm{i}\omega_0(a^* - a), \end{aligned} \tag{3}$$

from which we find

$$\dot{a} = -\mathrm{i}\omega_0 a + \frac{D}{2m}(a^* - a) + \mathrm{i}\sqrt{\frac{2}{m\omega_0}}f(t) \tag{4}$$

and its complex conjugate

$$\dot{a}^* = \mathrm{i}\omega_0 a^* - \frac{D}{2m}(a^* - a) - \mathrm{i}\sqrt{\frac{2}{m\omega_0}}f(t). \tag{5}$$

(b) *Perform the rotating wave approximation to the solution you obtained in part (a). Identify the terms that are eliminated by this approximation.*

Now we do RWA and let $a = \bar{a}\mathrm{e}^{-\mathrm{i}\omega_0 t}$, and the EOM of a becomes

$$\dot{\bar{a}}\mathrm{e}^{-\mathrm{i}\omega_0 t} + \bar{a}(-\mathrm{i}\omega_0)\mathrm{e}^{-\mathrm{i}\omega_0 t} = -\mathrm{i}\omega_0\bar{a}\mathrm{e}^{-\mathrm{i}\omega_0 t} + \frac{D}{2m}(\bar{a}^*\mathrm{e}^{\mathrm{i}\omega_0 t} - \bar{a}\mathrm{e}^{-\mathrm{i}\omega_0 t}) + \mathrm{i}\sqrt{\frac{2}{m\omega_0}}f(t),$$

or

$$\dot{\bar{a}} = -\frac{D}{2m}(\bar{a} - \bar{a}^*\mathrm{e}^{\mathrm{i}2\omega_0 t}) + \mathrm{i}\sqrt{\frac{2}{m\omega_0}}\mathrm{e}^{\mathrm{i}\omega_0 t}f(t). \tag{6}$$

The fast oscillating terms include the $\frac{D}{2m}\bar{a}^*\mathrm{e}^{\mathrm{i}2\omega_0 t}$ term and the $\mathrm{e}^{\mathrm{i}\omega t}$ part of $f(t)$, assuming that ω is close to ω_0. If these terms are removed, the resulting EOM is

$$\dot{\bar{a}} = -\frac{D}{2m}\bar{a} + \mathrm{i}\sqrt{\frac{1}{2m\omega_0}} \cdot \frac{1}{2}f_0\mathrm{e}^{-\mathrm{i}(\omega - \omega_0)t}. \tag{7}$$

(c) *Use the approximate equation of motion from part (b) to find the complex susceptibility, $\alpha_a[\omega]$, that relates the complex wave amplitude and force as $a[\omega] = \alpha_a[\omega]f[\omega]$.*

Noting that in the stationary solution $\bar{a} \propto \mathrm{e}^{-\mathrm{i}(\omega-\omega_0)t}$, we have

$$a(t) = \mathrm{e}^{-\mathrm{i}\omega_0 t}\bar{a} = \frac{1}{-\mathrm{i}(\omega-\omega_0) + \frac{D}{2m}} \frac{\mathrm{i}}{\sqrt{2m\omega_0}} \cdot \frac{1}{2} f_0 \mathrm{e}^{-\mathrm{i}\omega t}, \tag{8}$$

and therefore the response of a is (we divide $a(t)$ by $f_0\mathrm{e}^{-\mathrm{i}\omega t}/2$)

$$\alpha_a[\omega] = -\frac{1}{\sqrt{2m\omega_0}} \frac{1}{\omega - \omega_0 + \mathrm{i}\frac{D}{2m}}. \tag{9}$$

1.2

(a) *Starting from the second order equation, $m\ddot{q} = \sum_i f_i = -kq - D\dot{q} + f(t)$, find the exact susceptibility $q[\omega] = \alpha_q[\omega]f[\omega]$ of the driven-damped SHO.*

The EOM of q is

$$m\ddot{q} + D\dot{q} + kq = \frac{f_0}{2}(\mathrm{e}^{\mathrm{i}\omega t} + \mathrm{e}^{-\mathrm{i}\omega t}), \tag{10}$$

and since the system is linear, we can only consider the response of q to the $\mathrm{e}^{-\mathrm{i}\omega t}$ component of f, and get

$$\alpha_q[\omega] = \frac{1}{-m\omega^2 - \mathrm{i}D\omega + k} = -\frac{1}{m}\frac{1}{\omega^2 - \omega_0^2 + \mathrm{i}\frac{D}{m}\omega}. \tag{11}$$

(b) *Compare the approximate response you obtained in part (1.1c) with the exact response you found in (1.2a). [Remember, $q(t)$ is contained in $a(t)$.]*

Since $q = \sqrt{\frac{1}{2m\omega_0}}(a^* + a)$, and the response of a^* to $\mathrm{e}^{-\mathrm{i}\omega t}$ is the complex conjugate of the response of a to $\mathrm{e}^{\mathrm{i}\omega t}$, we find

$$\begin{aligned}
\alpha_q^{\mathrm{RWA}}[\omega] &= \frac{1}{\sqrt{2m\omega_0}}(\alpha_a^{\mathrm{RWA}}[\omega] + \alpha_a^{\mathrm{RWA}}[-\omega]^*) \\
&= -\frac{1}{2m\omega_0}\left(\frac{1}{\omega - \omega_0 + \mathrm{i}\frac{D}{2m}} + \frac{1}{-\omega - \omega_0 - \mathrm{i}\frac{D}{2m}}\right) \\
&= -\frac{1}{m}\frac{1}{\left(\omega + \frac{\mathrm{i}D}{2m}\right)^2 - \omega_0^2}.
\end{aligned} \tag{12}$$

It can be seen that the denominator of $\alpha^{\mathrm{RWA}}[\omega]$ has an additional term of $-\frac{D^2}{4m}$.

(c) *Can you find a set of approximations that will make your exact susceptibility converge with the susceptibility you found from the RWA?*

The validity of RWA is equivalent to

$$\frac{D^2}{4m^2} \ll |\omega^2 - \omega_0^2|. \tag{13}$$

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Time Modulation of Dielectric Constant: Next, we consider the impact of a time-modulated dielectric constant on the modes of a resonator. In this case, we assume that a standing-wave mode of the form $\mathbf{E}(r,t) = (a - a^*)\,\mathbf{E}^o(r)$ with real-valued field distribution $\mathbf{E}^o(r)$ is localized within a structured dielectric, having dielectric distribution $\varepsilon_r(r)$. For example, you could imagine that this is an eigenmode of a dielectric photonic crystal resonator of the type seen as Fig 8 in Chapter 7 of [JJSJ]. In the absence of a dielectric perturbation, the Hamiltonian takes the form $H_a = \omega_o a^* a$, where ω_o is the natural frequency of this time-harmonic mode.

(a) *As a starting point, find an expression for δH associated with an arbitrary dielectric perturbation $\Delta\varepsilon_r(r)$.*

(b) Using your result from part (a), find an equations of motion for a and a^*. (c) Next, we assume that perturbation to the dielectric constant takes the form $\Delta\varepsilon_r(t) = \beta\varepsilon_r(r)\cos(2\omega_o t + \phi)$, where β is a small ($\beta \ll 1$) unitless constant. Find the new form of our equations of motion from part (b). [Remember, our field normalization permits us to reduce $\langle \mathbf{E}^o \left| \varepsilon_o \varepsilon_r(r) \right| \mathbf{E}^o \rangle$ to a constant already defined above.] (d) Using a trial solution in the form $a(t) = \bar{a}(t)e^{-i\omega_o t}$, apply the rotating wave approximation to the equations of motion from part (c). [Hint: your results should closely resemble the problem involving a child on a swing.] (e) Treating $\Delta\varepsilon_r(r)$ as a perturbation on our original mode, solve for the first order correction to our mode amplitude.

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3.1

Third order nonlinear polarization: In this problem, we consider the effect of third order nonlinearity on a single time-harmonic mode. A single optical mode, $\mathbf{E}(\mathbf{r}, t)$, interacts with a polarizable medium. We assume that the medium produces a polarization field $\mathbf{P} = \chi^{(3)}|E|^2\mathbf{E}$, where $\chi^{(3)}$ is a constant. In the absence of the polarizable medium (i.e., $\chi^{(3)} = 0$), the Hamiltonian for the system is $H(a, a^*) = \omega a^* a$ where a and a^* are the complex mode amplitudes. As we have done in lecture, we assume that the mode amplitude is related to the field by $\mathbf{E}(\mathbf{r}, t) = (a - a^*)\,\mathbf{E}^o(\mathbf{r})$.

(a) *Find an expression for the interaction Hamiltonian, δH, in terms of the electric fields.*

(b) *Express your interaction Hamiltonian from part (a) as products of a, a^*; be sure to define all of the coefficients in this expression. [The coefficients will include integrals over space.]*

(c) *Find $\dot{a}$ using Poisson brackets.*

(d) *From part (c), show that the mode acquires intensity dependent frequency shift. Explain the significance of this term.*