Homework 1

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1 Maxwell's equations in dielectrics, Lorentz oscillators, and complex notation

1.1 Time-Average Quantities in Complex Notation

It is often important to be able to compute time-averaged quantities, such as the potential energy of a harmonic oscillator $U_{pe} = \frac{k}{2}\left\langle x^2 \right\rangle$ or the electric field energy density $U_{\mathrm{el}} = \frac{\varepsilon_0}{2}\left\langle \mathbf{E}^2 \right\rangle$. Here, the time-average of a function, $f(t)$, is defined as, $\langle f(t) \rangle = (1/T)\int_{t-T/2}^{t+T/2} dt' f(t')$, where T is defined as either the characteristic period of the oscillating system (i.e., $T = 2\pi/\omega$) or infinity. Such time averaging is drastically simplified by using complex notation.

To see this, suppose that we have any two functions $A(t)$ and $B(t)$, both of which take on a time harmonic form. Without loss of generality, we assume that $A(t) = A_0 \cos(\omega t + \phi)$, and $B(t) = B_0 \cos(\omega t + \theta)$, where ϕ and θ are arbitrary phase factors.

1.1.1

We have

$$\begin{aligned}
\langle A(t)B(t) \rangle &= \frac{1}{T}\int_{t-T/2}^{t+T/2} \mathrm{d}t'\, A_0 \cos(\omega t' + \phi) B_0 \cos(\omega t' + \theta) \\
&= A_0 B_0 \frac{1}{T}\int_{t-T/2}^{t+T/2} \mathrm{d}t'\, \frac{1}{2}(\cos(\omega t' + \phi + \omega t' + \theta) + \cos(\omega t' + \phi - \omega t' - \theta)) \\
&= \frac{1}{2} A_0 B_0 \cos(\phi - \theta).
\end{aligned} \tag{1}$$

Here we have used the condition that $T = 2\pi/\omega$ so that the first term vanishes.

1.1.2

We have

$$A(t) = \tilde{A}_0 \mathrm{e}^{-\mathrm{i}\omega t}, \quad B(t) = \tilde{B}_0 \mathrm{e}^{-\mathrm{i}\omega t}, \quad \tilde{A}_0 = A_0 \mathrm{e}^{-\mathrm{i}\phi}, \quad \tilde{B}_0 = B_0 \mathrm{e}^{-\mathrm{i}\theta}, \tag{2}$$

and therefore

$$\operatorname{Re} \tilde{A}_0 B_0 = \operatorname{Re} A_0 \tilde{B}_0 = \operatorname{Re} A_0 B_0 \mathrm{e}^{\mathrm{i}(\phi-\theta)} = A_0 B_0 \cos(\phi - \theta), \tag{3}$$

and hence

$$\langle A(t)B(t) \rangle = \frac{1}{2}\operatorname{Re} \tilde{A}_0 B_0 = \frac{1}{2}\operatorname{Re} A_0 \tilde{B}_0. \tag{4}$$

We can also straightforwardly do the follows. We have

$$\begin{aligned}
\langle A(t)B(t) \rangle &= \left\langle \frac{1}{2}(\tilde{A}(t) + \tilde{A}^*(t)) \cdot \frac{1}{2}(\tilde{B}(t) + \tilde{B}^*(t)) \right\rangle \\
&= \frac{1}{4}\left\langle \tilde{A}_0 \tilde{B}_0 \mathrm{e}^{-2\mathrm{i}\omega t} + \tilde{A}_0 \tilde{B}_0^* + \tilde{A}_0{}^* \tilde{B}_0^* \mathrm{e}^{2\mathrm{i}\omega t} + \tilde{A}_0^* \tilde{B}_0 \right\rangle \\
&= \frac{1}{4}\left\langle A_0^* B_0 + \text{c.c.} \right\rangle \\
&= \frac{1}{2} A_0^* B_0 = \frac{1}{2} A_0 B_0^*.
\end{aligned} \tag{5}$$

1.1.3

When

$$\boldsymbol{E} = \hat{\boldsymbol{x}}\,\mathrm{Re}\,\tilde{E}_0 \mathrm{e}^{-\mathrm{i}(\omega t - kz)}, \tag{6}$$

from

$$\boldsymbol{\nabla} \times \boldsymbol{E} = -\,\partial \boldsymbol{B}/\partial t \tag{7}$$

we obtain

$$\begin{aligned} \mathrm{i}\boldsymbol{k} \times \boldsymbol{E} &= -(-\mathrm{i}\omega)\boldsymbol{B} \\ \Rightarrow \boldsymbol{B} &= \frac{1}{\omega} k\hat{\boldsymbol{z}} \times \boldsymbol{E} = \frac{k}{\omega}\hat{\boldsymbol{y}}\,\mathrm{Re}\,\tilde{E}_0 \mathrm{e}^{-\mathrm{i}(\omega t - kz)}, \end{aligned} \tag{8}$$

and therefore

$$\langle \boldsymbol{S} \rangle = \frac{1}{\mu_0}\langle \boldsymbol{E} \times \boldsymbol{B} \rangle = \frac{1}{\mu_0} \cdot \frac{1}{2}\,\mathrm{Re}\,\underbrace{\hat{\boldsymbol{x}}\tilde{E}_0 \mathrm{e}^{\mathrm{i}kz}}_{\tilde{\boldsymbol{E}}_0} \times \underbrace{\frac{k}{\omega}\hat{\boldsymbol{y}}\tilde{E}_0^* \mathrm{e}^{-\mathrm{i}kz}}_{\tilde{\boldsymbol{B}}_0} = \frac{k}{2\mu_0 \omega}\left|\tilde{E}_0\right|^2 \hat{\boldsymbol{z}}, \tag{9}$$

and since the refraction index is n, we eventually get

$$\omega = k \cdot \frac{c}{n} \tag{10}$$

and therefore

$$\langle \boldsymbol{S} \rangle = \frac{n}{2}\sqrt{\frac{\epsilon_0}{\mu_0}}\left|\tilde{E}_0\right|^2 \hat{\boldsymbol{z}}. \tag{11}$$

The direction of the energy flow is parallel to the z axis.

1.1.4

The expected value of the electric energy density is

$$\langle u_e \rangle = \frac{1}{2}\epsilon_0 \epsilon_{\mathrm{r}} \left\langle \boldsymbol{E}^2 \right\rangle = \frac{1}{2}\epsilon_0 n^2 \cdot \frac{1}{2}\left|\tilde{E}_0\right|^2 = \frac{1}{4}\epsilon_0 n^2 \left|\tilde{E}\right|^2, \tag{12}$$

and the expected value of the magnetic energy density is

$$\langle u_m \rangle = \frac{1}{2\mu_0}\left\langle \boldsymbol{B}^2 \right\rangle = \frac{1}{2\mu_0} \cdot \frac{1}{2}\frac{k^2}{\omega^2}\left|\tilde{E}_0\right|^2 = \frac{1}{4}\frac{n^2}{c^2 \mu_0}\left|\tilde{E}_0\right|^2 = \frac{1}{4}\epsilon_0 n^2 \left|\tilde{E}_0\right|^2. \tag{13}$$

So we find

$$\frac{\langle u_e \rangle}{\langle u_m \rangle} = 1. \tag{14}$$

2 Lorentz oscillator in an AC field and optical forces

2.1 Optical response of an ensemble of Lorentz oscillators

Consider a dilute ensemble of Lorentz oscillators, uniformly distributed over space with number density N, in an AC electric field given by $\mathbf{E} = \mathrm{Re}\left[\tilde{\mathbf{E}}_0 e^{-i\omega t}\right]$. Each oscillator is driven by the local electric field according to the equation of motion given by

$$\ddot{\mathbf{p}} + \gamma\dot{\mathbf{p}} + \Omega^2 \mathbf{p} = \frac{q^2}{m}\mathbf{E}(\mathbf{r}),$$

where $\mathbf{r}, m$, and q are the respective oscillator position, reduced mass, and charge.

2.1.1

The polarization density is

$$\boldsymbol{P} = N\boldsymbol{p}. \tag{15}$$

The EOM for $\boldsymbol{P}$ is

$$\ddot{\boldsymbol{P}} + \gamma\dot{\boldsymbol{P}} + \Omega^2\boldsymbol{P} = \frac{Nq^2}{m}\boldsymbol{E}. \tag{16}$$

We can switch to the Fourier representation. Thus we have

$$((-\mathrm{i}\omega)^2 + \gamma(-\mathrm{i}\omega) + \Omega^2)\tilde{\boldsymbol{P}} = \frac{Nq^2}{m}\tilde{\boldsymbol{E}}, \tag{17}$$

and from

$$\boldsymbol{D} = \epsilon_0\boldsymbol{E} + \boldsymbol{P} \tag{18}$$

we get

$$\tilde{\boldsymbol{D}} = \epsilon_0 \underbrace{\left(1 + \frac{Nq^2}{m\epsilon_0}\frac{1}{-\omega^2 - \mathrm{i}\gamma\omega + \Omega^2}\right)}_{\epsilon_\mathrm{r}}\tilde{\boldsymbol{E}}. \tag{19}$$

So we already get ϵ_r; it has explicit dependence on ω, but not $\boldsymbol{k}$.

2.1.2

The phase velocity is given by

$$v = \frac{c}{\sqrt{\epsilon_\mathrm{r}}} = \frac{c}{\sqrt{1 + \frac{Nq^2}{m\epsilon_0}\frac{1}{-\omega^2 - \mathrm{i}\gamma\omega + \Omega^2}}}. \tag{20}$$

As for the group velocity, we have

$$\begin{aligned} \omega^2 &= \frac{c^2k^2}{\epsilon_\mathrm{r}} \\ \Rightarrow 2\omega\,\mathrm{d}\omega &= \frac{2c^2k\,\mathrm{d}k}{\epsilon_\mathrm{r}} - c^2k^2\frac{\mathrm{d}\epsilon_\mathrm{r}}{\epsilon_\mathrm{r}^2} \\ \Rightarrow v_\mathrm{g} &= \frac{2c^2k}{\epsilon_\mathrm{r}}\frac{1}{2\omega + \frac{c^2k^2}{\epsilon_\mathrm{r}^2}\frac{\mathrm{d}\epsilon_\mathrm{r}}{\mathrm{d}\omega}}, \end{aligned} \tag{21}$$

where

$$\frac{\mathrm{d}\epsilon_\mathrm{r}}{\mathrm{d}\omega} = \frac{Nq^2}{m\epsilon_0}\frac{2\omega + \mathrm{i}\gamma}{(-\omega^2 - \mathrm{i}\gamma\omega + \Omega^2)^2}. \tag{22}$$

2.1.3

Since ϵ_r has frequency dependence, the relation between $\boldsymbol{E}(t)$ and $\boldsymbol{D}(t)$ is not localized in the time domain, and therefore although we still know that the energy would be a quadratic form of $\boldsymbol{E}$ or $\boldsymbol{D}$, since

$$\frac{\partial\boldsymbol{E}}{\partial t}\cdot\boldsymbol{D} \neq \frac{\partial\boldsymbol{D}}{\partial t}\cdot\boldsymbol{E}, \tag{23}$$

the simple relation

$$u_e = \frac{1}{2}\boldsymbol{D}\cdot\boldsymbol{E}$$

no longer holds. Instead, we should start from the most generic theory and utilize

$$\frac{\partial u_e}{\partial t} = \boldsymbol{E}\cdot\frac{\partial\boldsymbol{D}}{\partial t}. \tag{24}$$

To use this equation to get an expression of u_e, we should no longer work with plane waves, or otherwise u_e is a constant and we don't see any change of u_e at all. Below we work with a wave packet centered at $\pm\omega_0$. For the wave packet, the electric field is

$$\boldsymbol{E}(t) = \mathrm{e}^{-\mathrm{i}\omega_0 t}\cdot\underbrace{\int\frac{\mathrm{d}\omega}{2\pi}\mathrm{e}^{-\mathrm{i}(\omega-\omega_0)t}\tilde{\boldsymbol{E}}(\omega),}_{=:\boldsymbol{E}_0(t)} \tag{25}$$

$$\boldsymbol{D}(t) = \mathrm{e}^{-\mathrm{i}\omega_0 t} \cdot \int \frac{\mathrm{d}\omega}{2\pi} \mathrm{e}^{-\mathrm{i}(\omega-\omega_0)t} \varepsilon(\omega) \tilde{\boldsymbol{E}}(\omega). \tag{26}$$

By Taylor expansion of ε we have

$$\begin{aligned}
\partial \boldsymbol{D}/\partial t &= \mathrm{e}^{-\mathrm{i}\omega_0 t} \cdot \int \frac{\mathrm{d}\omega}{2\pi} \mathrm{e}^{-\mathrm{i}(\omega-\omega_0)t} \tilde{\boldsymbol{E}}(\omega)(-\mathrm{i}\omega) \left(\varepsilon(\omega_0) + (\omega - \omega_0) \left.\frac{\mathrm{d}\varepsilon}{\mathrm{d}\omega}\right|_{\omega=\omega_0} + \cdots \right) \\
&\approx \mathrm{e}^{-\mathrm{i}\omega_0 t} \cdot \int \frac{\mathrm{d}\omega}{2\pi} \mathrm{e}^{-\mathrm{i}(\omega-\omega_0)t} \tilde{\boldsymbol{E}}(\omega) \left(-\mathrm{i}\omega_0 \varepsilon(\omega_0) - \mathrm{i}(\omega-\omega_0)\varepsilon(\omega)_0 - \mathrm{i}(\omega-\omega_0)\omega \left.\frac{\mathrm{d}\varepsilon}{\mathrm{d}\omega}\right|_{\omega=\omega_0} \right) \\
&\approx \mathrm{e}^{-\mathrm{i}\omega_0 t} \cdot \int \frac{\mathrm{d}\omega}{2\pi} \mathrm{e}^{-\mathrm{i}(\omega-\omega_0)t} \tilde{\boldsymbol{E}}(\omega) \Bigg(-\mathrm{i}\omega_0 \varepsilon(\omega_0) \underbrace{-\mathrm{i}(\omega-\omega_0)\varepsilon(\omega)_0 - \mathrm{i}(\omega-\omega_0)\omega_0 \left.\frac{\mathrm{d}\varepsilon}{\mathrm{d}\omega}\right|_{\omega=\omega_0}}_{=-\mathrm{i}(\omega-\omega_0)\left.\frac{\mathrm{d}(\omega\varepsilon)}{\mathrm{d}\omega}\right|_{\omega=\omega_0}} \Bigg) \\
&= \mathrm{e}^{-\mathrm{i}\omega_0 t} \underbrace{\left(-\mathrm{i}\omega_0 \varepsilon(\omega_0) \boldsymbol{E}_0(t) + \frac{\mathrm{d}(\omega\varepsilon)}{\mathrm{d}\omega} \frac{\partial \boldsymbol{E}_0}{\partial t} \right)}_{=:\boldsymbol{D}_0(t)}.
\end{aligned} \tag{27}$$

In the second line we throw away the higher order Taylor terms; in the third line we only keep terms linear to $(\omega - \omega_0)$. These approximations require the wave packet to be focused enough. We use $\langle \cdots \rangle$ to refer to averaging over the fast oscillations; thus, $\boldsymbol{E}_0(t)$ and $\boldsymbol{D}_0(t)$ above can be regarded as constants when applying $\langle \cdots \rangle$, and hence we find

$$\begin{aligned}
\left\langle \frac{\partial u_e}{\partial t} \right\rangle &= \left\langle \boldsymbol{E} \cdot \frac{\partial \boldsymbol{D}}{\partial t} \right\rangle = \frac{1}{2} \cdot \frac{1}{4} \mathrm{Re}(\boldsymbol{D}_0^*(t) \cdot \boldsymbol{E}_0(t) + \boldsymbol{D}_0(t) \cdot \boldsymbol{E}_0^*(t)) \\
&\approx \frac{1}{4} \left(\left(\omega_0 \varepsilon_2(\omega_0) + \frac{\mathrm{d}(\omega\varepsilon_1)}{\mathrm{d}\omega} \right) \frac{\partial \boldsymbol{E}_0^*}{\partial t} \cdot \boldsymbol{E}_0 + \left(\omega_0 \varepsilon_2(\omega_0) + \frac{\mathrm{d}(\omega\varepsilon_1)}{\mathrm{d}\omega} \right) \boldsymbol{E}_0^* \cdot \frac{\partial \boldsymbol{E}_0}{\partial t} \right) \\
&= \frac{1}{4} \left(\omega_0 \varepsilon_2(\omega_0) + \frac{\mathrm{d}(\omega\varepsilon_1)}{\mathrm{d}\omega} \right) \frac{\partial |\boldsymbol{E}_0|^2}{\partial t}.
\end{aligned} \tag{28}$$

In the second line we have considered both the real and imaginary parts of ϵ.[1] Since u_e contains no fast oscillation, we have

$$\frac{\partial \langle u_e \rangle}{\partial t} = \left\langle \frac{\partial u_e}{\partial t} \right\rangle = \frac{1}{4} \left(\omega_0 \varepsilon_2(\omega_0) + \frac{\mathrm{d}(\omega\varepsilon_1)}{\mathrm{d}\omega} \right) \frac{\partial |\boldsymbol{E}_0|^2}{\partial t}. \tag{29}$$

Similarly we have

$$\frac{\partial \langle u_m \rangle}{\partial t} = \left\langle \frac{\partial u_m}{\partial t} \right\rangle = \frac{1}{4} \left(\omega_0 \mu_2(\omega_0) + \frac{\mathrm{d}(\omega\mu_1)}{\mathrm{d}\omega} \right) \frac{\partial |\boldsymbol{H}_0|^2}{\partial t}. \tag{30}$$

When the matter is modeled by harmonic oscillators, μ doesn't undergo any correction, but let's work with a slightly generalized case. Eventually we have

$$\langle u \rangle = \langle u_e + u_m \rangle = \frac{1}{4} \left(\omega_0 \varepsilon_2(\omega_0) + \frac{\mathrm{d}(\omega\varepsilon_1)}{\mathrm{d}\omega} \right) |\boldsymbol{E}_0|^2 + \frac{1}{4} \left(\omega_0 \mu_2(\omega_0) + \frac{\mathrm{d}(\omega\mu_1)}{\mathrm{d}\omega} \right) |\boldsymbol{H}_0|^2. \tag{31}$$

The evaluation of the time averaged Poynting vector is more straightforward: since

$$\boldsymbol{\nabla} \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t} \Rightarrow \mathrm{i}\boldsymbol{k} \cdot \boldsymbol{E} = -(-\mathrm{i}\omega)\boldsymbol{B}, \tag{32}$$

we just have

$$\begin{aligned}
\langle \boldsymbol{S} \rangle &= \frac{1}{\mu} \langle \boldsymbol{E} \times \boldsymbol{B} \rangle \\
&= \frac{1}{\mu} \cdot \frac{1}{4} \mathrm{Re} \left(\boldsymbol{E}_0^* \times \boldsymbol{B}_0 + \boldsymbol{E}_0 \times \boldsymbol{B}_0^* \right) \\
&= \frac{1}{2\mu} \frac{\boldsymbol{k}}{\omega} |\boldsymbol{E}_0|^2,
\end{aligned} \tag{33}$$

[1] Note that $\epsilon(\omega) = \epsilon(-\omega)^*$ comes from the fact that ϵ is real in the time domain; it says nothing about whether the system is Hermitian; the Hermitian condition is $\epsilon(\omega) = \epsilon(\omega)^*$.

where we have used the condition $\boldsymbol{k} \cdot \boldsymbol{E} = 0$. The energy velocity is therefore

$$v_E = \frac{|\langle \boldsymbol{S} \rangle|}{\langle u_e + u_m \rangle} = \tag{34}$$

2.2 Optical Tweezers