Homework 1

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1 Problem 1: The Beam Splitter

Since $|t|^2 = |r|^2 = 1/2$, we have

$$\begin{pmatrix} E_c \\ E_d \end{pmatrix} = \underbrace{\begin{pmatrix} \mathrm{e}^{\mathrm{i}\phi_{ta}} & \mathrm{e}^{\mathrm{i}\phi_{rb}} \\ \mathrm{e}^{\mathrm{i}\phi_{ra}} & \mathrm{e}^{\mathrm{i}\phi_{tb}} \end{pmatrix}}_{M} \begin{pmatrix} E_a \\ E_b \end{pmatrix}. \tag{1}$$

The unitary condition means

$$M^\dagger M = I, \tag{2}$$

which in turns means

$$\begin{aligned} I &= \frac{1}{2} \begin{pmatrix} \mathrm{e}^{-\mathrm{i}\phi_{ta}} & \mathrm{e}^{-\mathrm{i}\phi_{ra}} \\ \mathrm{e}^{-\mathrm{i}\phi_{rb}} & \mathrm{e}^{-\mathrm{i}\phi_{tb}} \end{pmatrix} \begin{pmatrix} \mathrm{e}^{\mathrm{i}\phi_{ta}} & \mathrm{e}^{\mathrm{i}\phi_{rb}} \\ \mathrm{e}^{\mathrm{i}\phi_{ra}} & \mathrm{e}^{\mathrm{i}\phi_{tb}} \end{pmatrix} \\ &= \frac{1}{2} \begin{pmatrix} 2 & \mathrm{e}^{\mathrm{i}(\phi_{rb}-\phi_{ta})} + \mathrm{e}^{\mathrm{i}(\phi_{tb}-\phi_{ra})} \\ \mathrm{e}^{\mathrm{i}(\phi_{ta}-\phi_{rb})} + \mathrm{e}^{\mathrm{i}(\phi_{ra}-\phi_{tb})} & 2 \end{pmatrix}, \end{aligned}$$

and this is equivalent to

$$\mathrm{e}^{\mathrm{i}(\phi_{rb}-\phi_{ta})} + \mathrm{e}^{\mathrm{i}(\phi_{tb}-\phi_{ra})} = 0,$$

or in other words

$$\phi_{rb} - \phi_{ta} = \phi_{tb} - \phi_{ra} + \pi n, \quad n \text{ odd}. \tag{3}$$

2 Problem 2: Interferometers

Consider a Michelson interferometer, and rotate the beam splitter with an angle of θ, and also rotate one mirror with an angle of 2θ, and we get Figure 1. The change of the optical path of the green ray is

$$\Delta L_{\text{green}} = \frac{l_1 + d}{\cos 2\theta} - (l_1 + d) = (l_1 + d)\left(1 + \frac{1}{2}(2\theta)^2 + \cdots - 1\right) = 2(l_1 + d)\theta^2 + \cdots, \tag{4}$$

and the change of the optical path of the orange ray is

$$\Delta L_{\text{orange}} = l_2 + \frac{d}{\cos 2\theta} - (l_2 + d) = d\left(1 + \frac{1}{2}(2\theta)^2 + \cdots - 1\right) = 2d\theta^2 + \cdots. \tag{5}$$

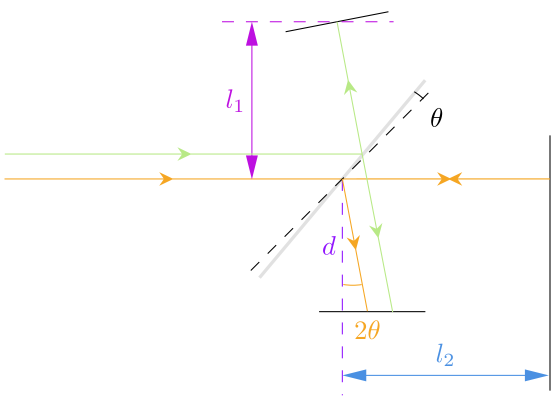

Figure 1: Michelson interferometer with tilted mirrors

Thus the changes of both paths are $\mathcal{O}(\theta^2)$.

When the potential – in optics, the refractive index – is changed, the path of the beam may be changed, but as is outlined above, slight change of the angle of propagation only causes a $\mathcal{O}(\theta^2)$ change on the optical path, so the main contribution of the change of the refractive index is the correction factor to terms like l_1 or d in ΔL_{green} or ΔL_{orange}. If, for example, a sample is placed on l_1, then we have

$$\Delta L_{\text{green}} = n\frac{l_1+d}{\cos 2\theta} - (l_1+d) = (l_1+d)\left(n\left(1+\frac{1}{2}(2\theta)^2+\cdots\right)-1\right) = (l_1+d)(n-1+2n\theta^2+\cdots), \tag{6}$$

and the first order variance of ΔL_{green} comes from the n factor in the $n(l_1+d)/\cos 2\theta$ term.

3 Problem 3: Correlation function and Other Properties of the Blackbody Field

3.1 Energy at ω; Total Energy

3.1.1 Energy of an electromagnetic mode

From

$$\boldsymbol{\nabla}\times\boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t},$$

we have

$$\mathrm{i}\boldsymbol{k}\times\boldsymbol{E}_\omega = \mathrm{i}\omega\boldsymbol{B}_\omega,$$

and therefore

$$|\boldsymbol{B}_\omega| = \frac{k}{\omega}|\boldsymbol{E}_\omega| = \frac{1}{c}|\boldsymbol{E}_\omega|,$$

so

$$\begin{aligned} u_\omega &= \frac{\epsilon_0}{2}|\boldsymbol{E}_\omega^2| + \frac{1}{2\mu_0}|\boldsymbol{B}_\omega|^2 \\ &= \frac{\epsilon_0}{2}|\boldsymbol{E}_\omega^2| + \frac{1}{2\mu_0}\underbrace{\frac{1}{c^2}}_{\mu_0\epsilon_0}|\boldsymbol{E}_\omega|^2 \\ &= \epsilon_0|\boldsymbol{E}_\omega|^2. \end{aligned} \tag{7}$$

Here the notation u_ω may be slightly confusing. What we want is

$$u = \int \mathrm{d}\omega\, u_\omega. \tag{8}$$

If we interpret it as the energy density (spatial density) of *one* photon mode with frequency ω, and the energy density contributed by *all* photon modes with the frequency being between ω and $\omega+\mathrm{d}\omega$ is $n(\omega)\,\mathrm{d}\omega\cdot u_\omega$, where $n(\omega)$ is the density of states. In this way, we get the expressions in the beginning of Section 3.1.2.

We can also define $\boldsymbol{E}_\omega$ according to the standard time-domain Fourier transformation:

$$\boldsymbol{E}_\omega = \int \mathrm{e}^{\mathrm{i}\omega t}\boldsymbol{E}(\boldsymbol{r},t)\,\mathrm{d}t\,, \quad \boldsymbol{E}(\boldsymbol{r},t) = \sum_{\boldsymbol{k},\sigma=1,2}\mathrm{i}\sqrt{\frac{\hbar\omega_{\boldsymbol{k}}}{2\epsilon_0 V}}a_{\boldsymbol{k}\sigma}\hat{\boldsymbol{e}}_\sigma\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}-\mathrm{i}\omega_{\boldsymbol{k}}t} + \text{h.c.}, \tag{9}$$

and we have

$$\langle\boldsymbol{E}(\boldsymbol{r},0)\cdot\boldsymbol{E}(\boldsymbol{r},t)\rangle = \int\frac{\mathrm{d}\omega'}{2\pi}\int\frac{\mathrm{d}\omega}{2\pi}\langle\boldsymbol{E}(\boldsymbol{r},\omega')\cdot\boldsymbol{E}(\boldsymbol{r},\omega)\rangle\,\mathrm{e}^{-\mathrm{i}\omega t}. \tag{10}$$

Under this definition, we have two $2\pi\delta(\omega-\omega_{\boldsymbol{k}})$ factors in the RHS; one of them may be understood as imposing the energy conservation condition $\omega+\omega'=0$, which is then canceled by the integration $\int\mathrm{d}\omega'/2\pi$, and another of them becomes the density of states, because there are more than one $(\boldsymbol{k},\sigma)$ pair with which $\omega_{\boldsymbol{k}\sigma}=\omega$, and we sum over all $\boldsymbol{k}$'s and σ's. (Note that due to the momentum conservation condition and the orthogonal relation concerning $\hat{\boldsymbol{e}}_\sigma$, although in the RHS we have two sums over $\boldsymbol{k}$ and σ, only one of them is kept.) So the eventual expression of the correlation function looks like

$$
\begin{aligned}
\langle \boldsymbol{E}(\boldsymbol{r},0)\cdot\boldsymbol{E}(\boldsymbol{r},t)\rangle &= \int \frac{\mathrm{d}\omega}{2\pi}\mathrm{e}^{-\mathrm{i}\omega t}\int\frac{\mathrm{d}\omega'}{2\pi}\underbrace{S(\omega)2\pi\delta(\omega+\omega')}_{\langle \boldsymbol{E}(\boldsymbol{r},\omega')\cdot\boldsymbol{E}(\boldsymbol{r},\omega)\rangle} \\
&= \int\frac{\mathrm{d}\omega}{2\pi}\mathrm{e}^{-\mathrm{i}\omega t}S(\omega),
\end{aligned}
\tag{11}
$$

where

$$
\begin{aligned}
S(\omega) &= \frac{1}{\epsilon_0}\cdot\underbrace{\frac{1}{V}\sum_{\boldsymbol{k},\sigma}2\pi\delta(\omega-\omega_{\boldsymbol{k}\sigma})}_{\text{density of states per volume}}\;\hbar\omega_{\boldsymbol{k}}\cdot\underbrace{\frac{1}{2}\langle a_{\boldsymbol{k}\sigma}a^\dagger_{\boldsymbol{k}\sigma}+\text{h.c.}\rangle}_{n_{\boldsymbol{k}\sigma}+\frac{1}{2}} \\
&= \frac{1}{\epsilon_0}2\pi n(\omega)\hbar\omega\cdot\left(f(\omega)+\frac{1}{2}\right),
\end{aligned}
\tag{12}
$$

where $f(\omega)$ is the occupation on energy level ω, which is the Bose-Einstein distribution in an equilibrium state. Putting these together, we get

$$
\langle \boldsymbol{E}(\boldsymbol{r},0)\cdot\boldsymbol{E}(\boldsymbol{r},t)\rangle = \frac{1}{\epsilon_0}\int\mathrm{d}t\,\mathrm{e}^{-\mathrm{i}\omega t}\underbrace{n(\omega)\cdot\hbar\omega\left(f(\omega)+\frac{1}{2}\right)}_{=:u_\omega}. \tag{13}
$$

Multiplying ϵ_0 on both sides of the equation and take $t=0$, and we arrive at the desired expression of u_ω. Its relation with $\langle \boldsymbol{E}(\boldsymbol{r},\omega')\cdot\boldsymbol{E}(\boldsymbol{r},\omega)\rangle$ however involves some normalization factors: what we do have is

$$
\langle \boldsymbol{E}(\boldsymbol{r},\omega')\cdot\boldsymbol{E}(\boldsymbol{r},\omega)\rangle = 2\pi\delta(\omega+\omega')S(\omega),\quad S(\omega)=\frac{2\pi}{\epsilon_0}u_\omega. \tag{14}
$$

But this doesn't create much trouble: we can always find u_ω using the density of states and the occupation, and then the correlation find is known after a Fourier transformation.

3.1.2 Energy density

Now we derive the energy at ω. Between ω and $\omega+\mathrm{d}\omega$, we have

$$
\#\text{ of }\boldsymbol{k}\text{ per }\mathrm{d}\omega = \frac{V}{(2\pi)^3}4\pi k^2\,\mathrm{d}k\,,\quad k=\frac{\omega}{c}.
$$

Since there are two polarizations for each $\boldsymbol{k}$, the number of states per $\mathrm{d}\omega$ is

$$
\#\text{ of state per }\mathrm{d}\omega = 2\cdot\#\text{ of }\boldsymbol{k}\text{ per }\mathrm{d}\omega = \frac{V}{\pi^2c^3}\omega^2\,\mathrm{d}\omega\,. \tag{15}
$$

Now since the total energy in the cavity is

$$
U=\int\#\text{ of state per }\mathrm{d}\omega\cdot\hbar\omega\cdot\frac{1}{\mathrm{e}^{\hbar\omega/k_{\mathrm{B}}T}-1}, \tag{16}
$$

the total energy density – the amount of energy per $\mathrm{d}^3\boldsymbol{r}$ – is

$$
u=\int\mathrm{d}\omega\,\frac{\hbar\omega^3}{\pi^2c^3}\frac{1}{\mathrm{e}^{\hbar\omega/k_{\mathrm{B}}T}-1}. \tag{17}
$$

Using

$$
\int_0^\infty\frac{x^3\,\mathrm{d}x}{\mathrm{e}^x-1}=\frac{\pi^4}{15},
$$

we get

$$
u=\frac{\hbar}{\pi^2c^3}\left(\frac{k_{\mathrm{B}}T}{\hbar}\right)^4\cdot\frac{\pi^4}{15}. \tag{18}
$$

The intensity of radiation out of the cavity is

$$I = \sum_{m \text{ outgoing}} A\boldsymbol{n} \cdot \boldsymbol{S}_m, \quad \boldsymbol{S}_m = u_m c \hat{\boldsymbol{k}},$$

where $\boldsymbol{n}$ is the normal vector of the hole between the cavity and the outside word, m is the index of optical modes within the cavity, $\boldsymbol{S}_m$ is the Poynting vector of mode m. We can make use of the spherical symmetry of radiation: suppose $\mathrm{d}\Omega$ is the solid angle element of $\hat{\boldsymbol{k}}$, we have

$$\begin{aligned} J = \frac{I}{A} &= \underbrace{\frac{1}{4\pi}}_{\text{total solid angle}} \int_{\hat{\boldsymbol{k}} \text{ outgoing}} \mathrm{d}\Omega \, \boldsymbol{n} \cdot uc\hat{\boldsymbol{k}} \\ &= uc \cdot \frac{1}{4\pi} \int_{\theta \leq \pi/2} \sin\theta \, \mathrm{d}\theta \, \mathrm{d}\varphi \cos\theta \\ &= uc \cdot \frac{1}{4\pi} \cdot \frac{1}{2} \cdot 2\pi = \frac{1}{4} uc, \end{aligned}$$

and finally we get

$$J = \underbrace{\frac{\pi^2 k_\mathrm{B}^4}{60\hbar^3 c^2}}_{\sigma} T^4. \tag{19}$$

3.2 Correlation Function of the Black Body Field

The experimental definition of the correlation function is

$$R_{xx}(\tau) = \frac{1}{T} \int_{-T/2}^{T/2} \mathrm{d}t \, E_x(t+\tau) E_x(t), \tag{20}$$

and so on. Using the ergodic condition, this is equivalent to

$$R_{xx}(\tau) = \langle E_x(\tau) E_x(0) \rangle . \tag{21}$$

The same applies for R_{xy}, etc.

Now since we are dealing with linear optics, there is no SHG process, etc., and each state in the density matrix $\rho = \sum_n |n\rangle\langle n| \, \mathrm{e}^{-E_n/k_\mathrm{B}t}$ is a photon Fock state. We know E_x contains photon modes for which the polarization vector $\hat{\boldsymbol{e}}$ is in the x direction, while E_y contains photon modes for which the polarization vector $\hat{\boldsymbol{e}}$ is in the y direction. So for each $|n\rangle$ state that is an eigenstate of the density matrix, we have

$$\langle n|E_x E_y|n\rangle = C_1 \langle n|a_{\hat{\boldsymbol{x}}} a_{\hat{\boldsymbol{y}}}|n\rangle + C_2 \langle n|a_{\hat{\boldsymbol{x}}} a^\dagger_{\hat{\boldsymbol{y}}}|n\rangle + C_3 \langle n|a^\dagger_{\hat{\boldsymbol{x}}} a_{\hat{\boldsymbol{y}}}|n\rangle + C_4 \langle n|a^\dagger_{\hat{\boldsymbol{x}}} a^\dagger_{\hat{\boldsymbol{y}}}|n\rangle ,$$

and each term vanishes because after the operators $a_{\hat{\boldsymbol{x}}} a_{\hat{\boldsymbol{y}}}$ etc. are applied to the ket vectors, the photon occupation configurations on the right and the left are different. So for each $|n\rangle$ in ρ, $\langle E_x E_y \rangle = 0$, and therefore $\langle E_x E_y \rangle_\rho$ also vanishes. The same applies for R_{yz} or R_{zx}.

According to Section 3.1.1, we have

$$u = \epsilon_0 |\boldsymbol{E}|^2. \tag{22}$$

To relate $\langle E_x^2 \rangle$ to $\langle \boldsymbol{E}^2 \rangle$, note that

$$\left\langle \boldsymbol{E}^2 \right\rangle \propto \hat{\boldsymbol{e}}_1{}^2 + \hat{\boldsymbol{e}}_2{}^2 = 2,$$

and

$$\left\langle E_x^2 \right\rangle \propto (\hat{\boldsymbol{e}}_1 \cdot \hat{\boldsymbol{x}})^2 + (\hat{\boldsymbol{e}}_2 \cdot \hat{\boldsymbol{x}})^2 = 1 - (\hat{\boldsymbol{k}} \cdot \hat{\boldsymbol{x}})^2.$$

The coordinates needed to determine $\hat{\boldsymbol{k}}$, $\hat{\boldsymbol{e}}_1$, and $\hat{\boldsymbol{e}}_2$ are the polar angle θ and the azimuthal angle φ of $\hat{\boldsymbol{k}}$. To go over all possible polarizations, we need an additional parameter ψ specifying the direction of $\hat{\boldsymbol{e}}_1$, and once $\hat{\boldsymbol{k}}$ and $\hat{\boldsymbol{e}}_1$ are determined, we can get the orientation of $\hat{\boldsymbol{e}}_2$. To go over *independent* polarization modes, no further parameter is needed. So we have

$$\begin{aligned}
\frac{\langle E_x^2\rangle}{\langle \boldsymbol{E}^2\rangle} &= \frac{\int \mathrm{d}\Omega\,(1-(\hat{\boldsymbol{k}}\cdot\hat{\boldsymbol{x}})^2)}{\int \mathrm{d}\Omega \times 2} = \frac{1}{8\pi}\int \sin\theta\,\mathrm{d}\theta\,\mathrm{d}\varphi\,\left(1-\sin^2\theta\cos^2\varphi\right) \\
&= \frac{1}{8\pi}\left(4\pi - \int_0^{\pi}\sin^3\theta\,\mathrm{d}\theta\int_0^{2\pi}\cos^2\varphi\,\mathrm{d}\varphi\right) \\
&= \frac{1}{3},
\end{aligned}$$

and therefore

$$R_{xx}(0) = \left\langle E_x(0)^2\right\rangle = \frac{1}{3\epsilon_0}u, \tag{23}$$

and similarly

$$R_{xx}(0) = R_{yy}(0) = R_{zz}(0) = \frac{1}{3\epsilon_0}u. \tag{24}$$

Following (13) and the last equation, we have

$$R_{xx}(t) = \frac{1}{3\epsilon_0}\int \mathrm{d}t\,\mathrm{e}^{-\mathrm{i}\omega t}u_\omega, \tag{25}$$

3.3 Properties of 300 K black body field

Since

$$\langle I\rangle = \sigma T^4 = \frac{1}{4}c\epsilon_0\boldsymbol{E}^2, \tag{26}$$

when $T = 300\,\mathrm{K}$, we have $I = 459\,\mathrm{W/m^2}$, and $|\boldsymbol{E}| = 832\,\mathrm{V/m}$. Although SI100V/m can cause an electric shock, this "field strength" can't really be felt, because the strength and direction of $\boldsymbol{E}$ is constantly changing and a stable electric field toward a static direction is never established.