Photon transferring

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1 Heuristic derivation of the light transportation equation

In this section we derive the light transportation equation in a way inspired by the quantum Boltzmann equation (QBE). We know in an isotropic medium, the dispersion relation of light is always in the form of

$$\omega = ck, \tag{1}$$

where c is the effective speed of light in the medium, and therefore

$$\boldsymbol{\nabla}_{\boldsymbol{k}} = c\hat{\boldsymbol{k}} =: c\hat{\boldsymbol{s}}, \tag{2}$$

where $\hat{\boldsymbol{s}}$ is the direction of the photon momentum, i.e. the direction of light propagation. Thus, for photons in an isotropic and uniform medium, In the LHS of QBE, we have

$$\begin{aligned} \text{LHS} &= \frac{\partial f}{\partial t} + \frac{\partial \omega}{\partial \boldsymbol{k}} \cdot \frac{\partial f}{\partial \boldsymbol{r}} - \frac{\partial \omega}{\partial \boldsymbol{r}} \cdot \frac{\partial f}{\partial \boldsymbol{k}} \\ &= \frac{\partial f}{\partial t} + c\hat{\boldsymbol{s}} \cdot \frac{\partial f}{\partial \boldsymbol{r}}. \end{aligned} \tag{3}$$

Ignoring non-linear processes, due to conservation of energy, the magnitude of the $\boldsymbol{k}$ vector is conserved, and therefore we may confine ourselves to a small segment of $f(\boldsymbol{r}, \boldsymbol{k}, t)$ where $|\boldsymbol{k}|$ is a given constant. We name the resulting distribution function $L(\boldsymbol{r}, \hat{\boldsymbol{s}}, t)$ **radiance**, where the degree of freedom $\boldsymbol{k}$ is reduced to its direction. We choose the following normalization scheme:

$$I(\boldsymbol{r}, t) = \int \mathrm{d}\Omega' \, L(\boldsymbol{r}, \hat{\boldsymbol{s}}, t), \tag{4}$$

where I is the radiation intensity at $\boldsymbol{r}, t$. There of course has to be some kind of normalization constant between f and L; currently, however, since we are only working with single-photon processes, and all scattering events can be thought of as scattering with some kind of "disorders" in the medium, the collision integral in QBE, obtained from Fermi golden rule, is linear with respect to f: it always takes the form of

$$-2\pi \sum_{\boldsymbol{k}'} |M(\boldsymbol{k} \to \boldsymbol{k}')|^2 (f(\boldsymbol{r}, \boldsymbol{k}, t) - f(\boldsymbol{r}, \boldsymbol{k}', t))\delta(\omega_{\boldsymbol{k}} - \omega_{\boldsymbol{k}'}),$$

where the quantum corrections to the incoming and the outgoing terms cancel each other. Applying this fact to the isotropic, uniform and linear case mentioned above, we find the collision integral in the Boltzmann of photon is

$$\begin{aligned} \text{RHS} &= c\mu_{\mathrm{s}} \int \mathrm{d}\Omega' \, (L(\boldsymbol{r}, \hat{\boldsymbol{s}}', t) - L(\boldsymbol{r}, \hat{\boldsymbol{s}}, t)) P(\hat{\boldsymbol{s}}, \hat{\boldsymbol{s}}') \\ &= c\mu_{\mathrm{s}} \int \mathrm{d}\Omega' \, L(\boldsymbol{r}, \hat{\boldsymbol{s}}', t) P(\hat{\boldsymbol{s}}, \hat{\boldsymbol{s}}') - c\mu_{\mathrm{s}} L(\boldsymbol{r}, \hat{\boldsymbol{s}}, t), \end{aligned} \tag{5}$$

where we have decomposed the scattering matrix into an overall strength $c\mu_{\mathrm{s}}$ (the c factor will cancel with the c factor in the second term of LHS), and a function $P(\hat{\boldsymbol{s}}, \hat{\boldsymbol{s}}')$ measuring how anisotropic the scattering process is, and the normalization condition is

$$\int \mathrm{d}\Omega' \, P(\hat{\boldsymbol{s}}, \hat{\boldsymbol{s}}') = 1, \tag{6}$$

and when the scattering process is also completely isotropic, we have

$$P(\hat{\boldsymbol{s}}, \hat{\boldsymbol{s}}') = \frac{1}{4\pi}. \tag{7}$$

The QBE of photons therefore becomes the following **radiation transfer equation (RTE)**:

$$\frac{1}{c}\frac{\partial L}{\partial t} + \hat{\boldsymbol{s}} \cdot \boldsymbol{\nabla}_{\boldsymbol{r}} L = -\mu_{\mathrm{s}} L + \mu_{\mathrm{s}} \int \mathrm{d}\Omega' \, L(\boldsymbol{r}, \hat{\boldsymbol{s}}', t) P(\hat{\boldsymbol{s}}, \hat{\boldsymbol{s}}'). \tag{8}$$

The conditions of the validity of the above equation includes the aforementioned assumptions that the medium is uniform and isotropic, and there is no nonlinear process in the system, and also the fundamental validity condition of QBE: the validity of gradient expansion of the effective density matrix in the Wigner representation. Physically speaking, this means the system should allow the formation of defined wave pockets, which has well-defined positions but also looks like a plane wave when we zoom in on it.

The next step is to find the true meaning of μ_{s}. Assuming a clean background and a stationary configuration, the second term in the RHS of (8) vanishes, and we know $I \propto L$, and we find

$$\frac{\mathrm{d}I}{\mathrm{d}z} = -\mu_{\mathrm{s}} I,$$

where z is the propagation length. So we find $1/\mu_{\mathrm{s}}$ is roughly the mean path l_{s} between two scattering incidents, and (8) can also be alternatively written as

$$\frac{1}{c}\frac{\partial L}{\partial t} + \hat{\boldsymbol{s}} \cdot \boldsymbol{\nabla}_{\boldsymbol{r}} L = \frac{1}{l_{\mathrm{s}}} \int \mathrm{d}\Omega' \, (L(\boldsymbol{r}, \hat{\boldsymbol{s}}', t) - L(\boldsymbol{r}, \hat{\boldsymbol{s}}, t)) P(\hat{\boldsymbol{s}}, \hat{\boldsymbol{s}}'), \tag{9}$$

from which we have

$$\frac{\mathrm{d}I}{\mathrm{d}z} + \frac{1}{l_{\mathrm{s}}} I = 0 \tag{10}$$

when the system reaches the stationary state.

Some scattering processes turn the photons into other degrees of freedom, and in the foreseeable future they will not come back; these processes are just absorption processes, and we can just add them to (8) using the relaxation time approximation, and get

$$\begin{aligned} \frac{1}{c}\frac{\partial L}{\partial t} + \hat{\boldsymbol{s}} \cdot \boldsymbol{\nabla}_{\boldsymbol{r}} L &= -\mu_{\mathrm{a}} L + \underbrace{\frac{1}{l_{\mathrm{s}}}}_{\mu_{\mathrm{s}}} \int \mathrm{d}\Omega' \, (L(\boldsymbol{r}, \hat{\boldsymbol{s}}', t) - L(\boldsymbol{r}, \hat{\boldsymbol{s}}, t)) P(\hat{\boldsymbol{s}}, \hat{\boldsymbol{s}}') \\ &= -\underbrace{(\mu_{\mathrm{a}} + \mu_{\mathrm{s}})}_{\mu_{\mathrm{t}}} L + \mu_{\mathrm{s}} \int \mathrm{d}\Omega' \, L(\boldsymbol{r}, \hat{\boldsymbol{s}}', t) P(\hat{\boldsymbol{s}}, \hat{\boldsymbol{s}}'), \end{aligned} \tag{11}$$

where μ_{t} is known as the extinction coefficient.

A final correction to RTE is the spatial non-uniformity of optical properties, including scattering and change of light speed. This means both $\mu_{\mathrm{a/s}}$ and $P(\hat{\boldsymbol{s}}, \hat{\boldsymbol{s}}')$ may have spatial variance, and so does c; of course, this variance shouldn't break the validity of QBE. In this case even though the correct generalization of RTE contains, say, $\boldsymbol{\nabla}_{\boldsymbol{r}}(cL)$, due to the slow spatial variance of c, we can still rewritten it as $c\boldsymbol{\nabla}_{\boldsymbol{r}} L$, and get back to (11).

2 From RTE to the diffusion equation

Under the following assumptions, RTE is reduced to the diffusion equation:

- The radiance is nearly isotropic – and therefore if we do spherical harmonic expansion to it, only the $l = 0, 1$ components need to be kept. This happens when scattering is strong enough so that the direction of light propagation is randomized; to be exact, this means scattering is much stronger than absorption: $1/c\mu_{\mathrm{a}}$ is the time scale of how long radiation lasts in the system before being dampened by absorption, and scattering should randomized the directions of light propagation within this period of time.

- The change of current density is much slower than the speed photons pass the mean free path.

- The scattering property has rotational symmetry (although not completely isotropic): thus $P(\hat{\boldsymbol{s}}, \hat{\boldsymbol{s}}') = P(\hat{\boldsymbol{s}} \cdot \hat{\boldsymbol{s}}')$. This can be seen as included in the second approximation, since otherwise it's not likely that the radiance finally converges to an isotropic solution.

If we only keep the first two spherical harmonic components of L (here the first approximation is used), i.e. the $\hat{\boldsymbol{s}}$ dependence of L is either trivial or is proportional to something dot $\hat{\boldsymbol{s}}$, then we have

$$L(\boldsymbol{r}, \hat{\boldsymbol{s}}, t) = \frac{1}{4\pi}\Phi(\boldsymbol{r}, t) + \frac{3}{4\pi}\boldsymbol{J}\cdot\hat{\boldsymbol{s}}, \tag{12}$$

where

$$\boldsymbol{J}(\boldsymbol{r}, t) = \int \mathrm{d}\Omega\,\hat{\boldsymbol{s}}L(\boldsymbol{r}, \hat{\boldsymbol{s}}, t) \tag{13}$$

is recognized as the current density; this seems intuitive from the microscopic meaning of L, and we are also going to explicitly show that $\boldsymbol{J}$ appears in its expected position in the diffusion equation. We can explicitly verify that the normalization conditions are correct: for example, we have

$$\int \mathrm{d}\Omega\,\frac{3}{4\pi}\boldsymbol{J}\cdot\hat{\boldsymbol{s}}\cdot\underbrace{\cos\theta}_{\hat{s}_z} = J_z, \tag{14}$$

where the $J_{x,y}$ terms, since they contain a $\sin\varphi$ or $\cos\varphi$ factor, vanish under the integral over $\mathrm{d}\Omega$.

Now, by applying $\int\mathrm{d}\Omega$ to (11), the LHS becomes

$$\mathrm{LHS} = \frac{1}{c}\frac{\partial\Phi}{\partial t} + \int\mathrm{d}\Omega\,\hat{\boldsymbol{s}}\cdot\boldsymbol{\nabla}_{\boldsymbol{r}}L(\boldsymbol{r}, \hat{\boldsymbol{s}}, t) = \frac{1}{c}\frac{\partial\Phi}{\partial t} + \boldsymbol{\nabla}\cdot\underbrace{\int\mathrm{d}\Omega\,\hat{\boldsymbol{s}}L(\boldsymbol{r}, \hat{\boldsymbol{s}}, t)}_{\boldsymbol{J}}, \tag{15}$$

where we have used the condition

$$\boldsymbol{\nabla}\cdot(\hat{\boldsymbol{s}}L) = \hat{\boldsymbol{s}}\cdot\boldsymbol{\nabla}L \tag{16}$$

since $\hat{\boldsymbol{s}}$ has no spatial dependence, while the RHS becomes

$$\begin{aligned}\mathrm{RHS} &= -\mu_t\Phi + \mu_{\mathrm{s}}\int\mathrm{d}\Omega'\,L(\boldsymbol{r}, \hat{\boldsymbol{s}}', t)\int\mathrm{d}\Omega\,P(\hat{\boldsymbol{s}}, \hat{\boldsymbol{s}}') \\ &= -\mu_t\Phi + \mu_{\mathrm{s}}\int\mathrm{d}\Omega'\,L(\boldsymbol{r}, \hat{\boldsymbol{s}}', t) \\ &= -\mu_t\Phi + \mu_{\mathrm{s}}\Phi = -\mu_{\mathrm{a}}\Phi,\end{aligned} \tag{17}$$

and thus we have

$$\frac{1}{c}\frac{\partial\Phi}{\partial t} + \boldsymbol{\nabla}\cdot\boldsymbol{J} + \mu_{\mathrm{a}}\Phi = 0. \tag{18}$$

Now we see $\boldsymbol{J}$ indeed is the current density.

Then, we apply $\int\mathrm{d}\Omega\,\hat{\boldsymbol{s}}$ to (11). This time the calculation will be slightly more non-trivial. The LHS now is

$$\mathrm{LHS} = \frac{1}{c}\frac{\partial\boldsymbol{J}}{\partial t} + \boldsymbol{\nabla}\cdot\int\mathrm{d}\Omega\,\hat{\boldsymbol{s}}\hat{\boldsymbol{s}}L, \tag{19}$$

where we again use the condition (16). A direct evaluation tells us

$$\frac{1}{4\pi}\int\mathrm{d}\Omega\,\hat{\boldsymbol{s}}\hat{\boldsymbol{s}}\Phi = \frac{1}{3}\Phi, \tag{20}$$

and the contribution from the $\boldsymbol{J}$ part is zero. The (z, z) component of $\int\mathrm{d}\Omega\,\hat{\boldsymbol{s}}\hat{\boldsymbol{s}}$, for example, is

$$\int\mathrm{d}\Omega\cos^2\theta = \int_0^\pi \sin\theta\,\mathrm{d}\theta\cos^2\theta\cdot 2\pi = \frac{2}{3}\cdot 2\pi,$$

and hence the $1/3$ factor. On the other hand, the (x, z) or (y, z) components suffer from the existence of a vanishing $\int\mathrm{d}\varphi\sin\varphi$ or $\cos\varphi$ factor, and for the (x, y) component, $\int\mathrm{d}\varphi\sin\varphi\cos\varphi$ is still zero. So now we find the full expression of the LHS:

$$\mathrm{LHS} = \frac{1}{c}\frac{\partial\boldsymbol{J}}{\partial t} + \frac{1}{3}\boldsymbol{\nabla}\Phi. \tag{21}$$

As for the RHS, we have

$$\mathrm{RHS} = -\mu_{\mathrm{t}}\boldsymbol{J} + \mu_{\mathrm{s}}\int\mathrm{d}\Omega'\,L(\boldsymbol{r}, \hat{\boldsymbol{s}}', t)\int\mathrm{d}\Omega\,\hat{\boldsymbol{s}}P(\hat{\boldsymbol{s}}\cdot\hat{\boldsymbol{s}}'). \tag{22}$$

We use $\hat{\boldsymbol{s}}'$ as the z axis, and then $P(\hat{\boldsymbol{s}} \cdot \hat{\boldsymbol{s}}') = P(\cos\theta)$, and therefore $\int \mathrm{d}\Omega\, \hat{\boldsymbol{s}} P(\hat{\boldsymbol{s}} \cdot \hat{\boldsymbol{s}}')$ is always parallel to $\hat{\boldsymbol{s}}'$, since the x and y components contain $\int \mathrm{d}\varphi \sin\varphi$ and $\int \mathrm{d}\varphi \cos\varphi$ and therefore vanish. We therefore write

$$\int \mathrm{d}\Omega\, \hat{\boldsymbol{s}} P(\hat{\boldsymbol{s}} \cdot \hat{\boldsymbol{s}}') = g\hat{\boldsymbol{s}}', \tag{23}$$

where g is an unknown factor. The RHS therefore becomes

$$\mathrm{RHS} = -\mu_\mathrm{t}\boldsymbol{J} + \mu_\mathrm{s} g \int \mathrm{d}\Omega'\, \hat{\boldsymbol{s}}' L(\boldsymbol{r}, \hat{\boldsymbol{s}}', t) = -\mu_\mathrm{t}\boldsymbol{J} + \mu_\mathrm{s} g \boldsymbol{J}. \tag{24}$$

So we have

$$\frac{1}{c}\frac{\partial \boldsymbol{J}}{\partial t} + \frac{1}{3}\boldsymbol{\nabla}\Phi = -\mu_\mathrm{t}\boldsymbol{J} + \mu_\mathrm{s} g\boldsymbol{J} = -(\mu_\mathrm{a} + \mu_\mathrm{s}')\boldsymbol{J}, \quad \mu_\mathrm{s}' = (1-g)\mu_\mathrm{s}. \tag{25}$$

Now we invoke the condition that the change of the current is slow compared with $c\mu_\mathrm{t}$, and we approximately, we have

$$\frac{1}{3}\boldsymbol{\nabla}\Phi = -\mu_\mathrm{t}\boldsymbol{J} + \mu_\mathrm{s} g\boldsymbol{J} = -(\mu_\mathrm{a} + \mu_\mathrm{s}')\boldsymbol{J}, \quad \mu_\mathrm{s}' = (1-g)\mu_\mathrm{s}. \tag{26}$$

This gives the constitutive relation between the current density and the gradient of radiance.

Thus, we find with the three – or two – assumptions listed at the start of this section, we have

$$\frac{1}{c}\frac{\partial \Phi}{\partial t} + \boldsymbol{\nabla}\cdot\boldsymbol{J} + \mu_\mathrm{a}\Phi = 0, \quad \boldsymbol{J} = -D\boldsymbol{\nabla}\Phi, \tag{27}$$

where

$$D = \frac{1}{3(\mu_\mathrm{a} + \mu_\mathrm{s}')}, \quad \mu_\mathrm{s}' = (1-g)\mu_\mathrm{s}, \tag{28}$$

and

$$\int \mathrm{d}\Omega\, \hat{\boldsymbol{s}} P(\hat{\boldsymbol{s}} \cdot \hat{\boldsymbol{s}}') = g\hat{\boldsymbol{s}}' \Rightarrow 2\pi \int_0^\theta \sin\theta\, \mathrm{d}\theta \cdot \cos\theta P(\theta) = g. \tag{29}$$

Now we find

3 The rendering equation

In most everyday cases, the time evolution of L can be ignored: the system reaches equilibrium as soon as "the light is turned on", since the speed of light is very fast. It's also frequently assumed that c is <u>one hundred percent uniform in space and not just very slow in its spatial change</u>. In these cases, we get

$$\begin{aligned}\hat{\boldsymbol{s}} \cdot \boldsymbol{\nabla}_{\boldsymbol{r}} L &= -\mu_\mathrm{a} L + \underbrace{\frac{1}{l_\mathrm{s}}}_{\mu_\mathrm{s}} \int \mathrm{d}\Omega' \left(L(\boldsymbol{r}, \hat{\boldsymbol{s}}', t) - L(\boldsymbol{r}, \hat{\boldsymbol{s}}, t)\right) P(\hat{\boldsymbol{s}}, \hat{\boldsymbol{s}}') \\ &= -\underbrace{(\mu_\mathrm{a} + \mu_\mathrm{s})}_{\mu_\mathrm{t}} L + \mu_\mathrm{s} \int \mathrm{d}\Omega'\, L(\boldsymbol{r}, \hat{\boldsymbol{s}}', t) P(\hat{\boldsymbol{s}}, \hat{\boldsymbol{s}}').\end{aligned} \tag{30}$$

The time evolution term is thrown away. This equation is sometimes known as the **scattering equation**.

The main problem then becomes how to treat the boundaries appropriately.

The exact theory (under the aforementioned premises) of light propagation in this case turns out to be what is known as the **rendering equation**.