Quantum Optics by Prof. Saijun Wu

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1 Spontaneous radiation of a two-level atom

We continue the discussion in the last lecture. For an atom coupled with an optical field (with its own dynamics), we have

$$\mathrm{i}\dot{c}_\mathrm{e} = \sum_k g_k \mathrm{e}^{-\mathrm{i}\Delta_k t} c_k, \quad \mathrm{i}\dot{c}_k = g_k^* \mathrm{e}^{\mathrm{i}\Delta_k t} c_\mathrm{e},$$

so we have

$$\dot{c}_\mathrm{e}(t) = -\int_0^t \mathrm{d}\tau\, K(t-\tau) c_\mathrm{e}(\tau), \tag{1}$$

where

$$K(t) = \sum_k |g_k|^2 \mathrm{e}^{-\mathrm{i}\Delta_k t}. \tag{2}$$

We make a Markov approximation, i.e. we assume $K(t-\tau)$ is significant only for τ close enough to t, and therefore we have approximately

$$\dot{c}_\mathrm{e} = -c_\mathrm{e}(t) \int_0^t \mathrm{d}\tau\, K(t-\tau).$$

Formally, we define

$$\int_0^t \mathrm{d}\tau\, K(t-\tau) =: \frac{\Gamma}{2} + \mathrm{i}\delta_\mathrm{L}, \tag{3}$$

so the time evolution equation is

$$\dot{c}_\mathrm{e} = -\left(\frac{\Gamma}{2} + \mathrm{i}\delta_\mathrm{L}\right) c_\mathrm{e}(t). \tag{4}$$

Note

The definition of Γ has a $1/2$ factor. This is to ensure that Γ is the damping rate of the probability that the atom is on the excited state, because if the time evolution of c_e is

$$c_\mathrm{e} \to (1 - \Gamma\Delta t/2) c_\mathrm{e},$$

then the time evolution of $|c_\mathrm{e}|^2$ is

$$|c_\mathrm{e}|^2 \to (1-\Gamma\Delta t/2)^2 |c_\mathrm{e}|^2 = (1-\Gamma\Delta t)|c_\mathrm{e}|^2.$$

We immediately find that the discussion in the previous lecture around (15) and (16) is a specific case of the above derivation, where $|c_\mathrm{e}(0)|^2$. In other words, the two definitions of Γ in the previous lecture and in this lecture are actually the same. In the language of QFT, we can call Γ the *scattering rate*.

Now we need to evaluate Γ and δ_L. We have

$$\begin{aligned}
\int_0^t \mathrm{d}\tau\, K(t-\tau) &= \sum_k |g_k|^2 \int_0^t \mathrm{d}\tau\, \mathrm{e}^{-\mathrm{i}\Delta_k(t-\tau)} \\
&= \sum_k |g_k|^2 \int_0^t \mathrm{d}\tau\, \mathrm{e}^{-\mathrm{i}(\omega_k - \omega_{eg})(t-\tau)} \\
&= \sum_k |g_k|^2 \int_0^t \mathrm{d}\tau'\, \mathrm{e}^{-\mathrm{i}(\omega_k - \omega_{eg})\tau'},
\end{aligned}$$

and since we are working in the region where $t \gg 1/\omega_{eg}$, we push the integral to the $t \to \infty$ limit, i.e. making the approximation that

$$\begin{aligned}\int_0^t \mathrm{d}\tau\, K(t-\tau) &\approx \sum_k |g_k|^2 \int_0^\infty \mathrm{d}\tau'\, \mathrm{e}^{-\mathrm{i}(\omega_k-\omega_{eg})\tau' - 0^+\tau'} \\ &= \sum_k |g_k|^2 \frac{-1}{-\mathrm{i}(\omega_k-\omega_{eg}) - 0^+} \\ &= \sum_k |g_k|^2 \frac{\mathrm{i}}{\omega_{eg}-\omega_k+\mathrm{i}0^+} \\ &= \sum_k |g_k|^2 \left(\mathrm{P}\frac{\mathrm{i}}{\omega_{eg}-\omega_k} + \pi\delta(\omega_{eg}-\omega_k)\right),\end{aligned}$$

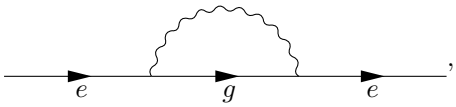
The final results are

$$\Gamma = 2\pi \sum_k |g_k|^2 \delta(\omega_k - \omega_{\mathrm{eg}}), \tag{5}$$

and

$$\delta_{\mathrm{L}} = \mathrm{P}\sum_k |g_k|^2 \rho(k) \frac{1}{\omega_{\mathrm{eg}} - \omega_k}. \tag{6}$$

We can see that Fermi golden rule can be derived from the Markov approximation. We also see that the approach we used in the last lecture fail to capture an imaginary factor. The imaginary part δ_{L} is a part of Lamb shift, which arises from the vacuum fluctuation of the optical field. The two parts can be obtained from the following Feynman diagram



, (7)

where Γ comes from half of the diagram. The above procedure – working in a subspace, regarding the optical field as in a large box and then applying the Markovian approximation and calculating Γ and δ_{L} – is called **Wigner-Weisskopf theory**.

Note

(5) and (6) are just two parts of a propagator in the form of $1/(E-H+\mathrm{i}0^+)$. What we are actually doing here is to "integrate out" the external optical field and get a non-Hermitian theory of the atom. We are assuming

- that the emitted photon leaves the atom quickly enough, so the Markovian approximation (the state of the system can be labeled completely by c_e and c_g), without any need to fear that the emitted photon may come back (i.e. the optical degrees of freedom can effectively be seen as a *bath*), and
- that the time scale we are interested is large enough so that we are able to take the $t \to \infty$ limit.

Formally, the first and the second approximation means the concepts in perturbative QFT work here, i.e. we have concepts like propagators and Feynman diagrams and we can just interpret the imaginary part of a propagator as decay. Only one diagram - (7) - is taken into account, because actually we are doing the self-energy correction to the atom, and (7) is the irreducible self energy. Therefore, the Wigner-Weisskopf theory is actually a non-perturbative theory.

However, we will soon find a huge problem in the Wigner-Weisskopf theory: the total probability is not conserved. What is happening is that the emitted photon gets observed by the environment from time to time, and therefore the wave function gets collapsed to certain states (for a two-level atom, the ground state) from time to time. We will see a detailed calculation in Section 3.

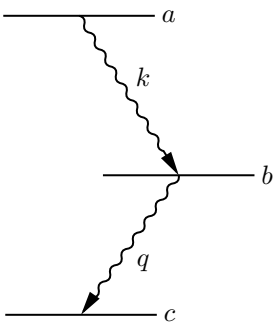


Figure 1: Energy levels of a three-level atom

Solving (4) and redefining the basis to absorb the Lamb shift, we get

$$c_e(t) = \mathrm{e}^{-\frac{\Gamma}{2}t}, \quad c_k(t) = \mathrm{i}g_k^* \frac{1 - \mathrm{e}^{-\mathrm{i}(\omega_{\mathrm{eg}} - \mathrm{i}\Gamma/2 - \omega_k)t}}{\omega_{\mathrm{eg}} - \mathrm{i}\Gamma/2 - \omega_k}. \tag{8}$$

Now we can calculate the optical correlation functions in the space. Since we are working in the single-photon subspace, we have

$$P^{(1)}(\boldsymbol{r}, t) = \eta |\mathcal{E}(\boldsymbol{r}, t)|^2, \tag{9}$$

where

$$\mathcal{E}(\boldsymbol{r}, t) = \langle 0 | E^+(\boldsymbol{r}, t) \sum_k c_k |1_k\rangle . \tag{10}$$

Again we consider the time region where $t \gg 1/\Gamma$ so spontaneous radiation is likely to start, while the time is not too large so that Markov approximation still works. We have

$$\begin{aligned}
\mathcal{E} &= \frac{\mathrm{i}}{(2\pi)^3} \int_0^\infty k^2 \,\mathrm{d}k \int_0^\pi \sin\theta \,\mathrm{d}\theta \int_0^{2\pi} \mathrm{d}\varphi \, \frac{\hbar\omega_k \boldsymbol{d}_{\mathrm{eg}}^2}{2\epsilon_0} \frac{\mathrm{e}^{-\mathrm{i}(\omega_k t - kr\cos\theta)}}{\omega_{\mathrm{eg}} - \frac{\mathrm{i}\Gamma}{2} - \omega_k} \\
&= \int_0^\infty \mathrm{d}k \, \frac{k^2 d^2}{\epsilon_0 r} \frac{\mathrm{e}^{-\mathrm{i}\omega_k(t - r/c)}}{\omega_{\mathrm{eg}} - \frac{\mathrm{i}\Gamma}{2} - \omega_k} - (r + t/c) \text{ terms} \\
&\approx -\int_{-\infty}^\infty \frac{\omega_k^2}{c^2},
\end{aligned}$$

so finally we get the single photon wave function of the spontaneous radiation:

$$\mathcal{E}(\boldsymbol{r}, t) = \frac{\omega_{\mathrm{eg}}^2 d_{\mathrm{eg}}^2}{\epsilon_0 r c^2} \mathrm{e}^{-\mathrm{i}\omega_{\mathrm{eg}}(t - r/c)} \Theta(t - r/c). \tag{11}$$

2 Three-level atoms

Now we consider a three-level atom shown in Figure 1 on page 3. We work in the following subspace:

$$|\psi\rangle = c_a |a, 0\rangle + \sum_k c_{b_k} |b, 1_k\rangle + \sum_{k,q} c_{c,k,g} |c, 1_k, 1_q\rangle . \tag{12}$$

The Schrödinger equation now reads

$$\begin{aligned}
\mathrm{i}\dot{c}_a &= \sum_k g_{a,k}^* \mathrm{e}^{-\mathrm{i}\Delta t} c_{b,k}, \\
\mathrm{i}\dot{c}_{b,k} &= g_{a,k} \mathrm{e}^{\mathrm{i}\Delta_k t} c_a + \sum_q g_{b,q}^* \mathrm{e}^{-\mathrm{i}\Delta_q t} c_{c,k,q}, \\
\mathrm{i}\dot{c}_{c,k,q} &= g_{b,q} \mathrm{e}^{-\mathrm{i}\Delta_q t} c_{b,k},
\end{aligned} \tag{13}$$

and we can just repeat the procedure in the last section.

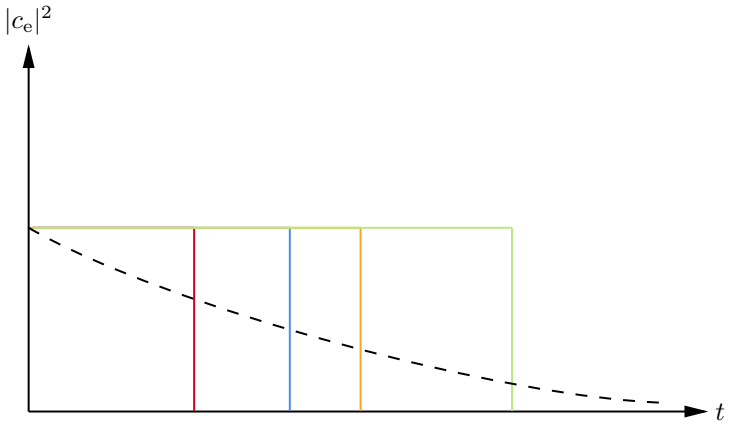


Figure 2: Random wave function calculating of a two-level atom. The lines with color are possible evolution path of the wave function, and after averaging over them we get the dashed exponential decay line.

Note

We should note that the so-called "wave function in a subspace" does not capture the high frequency details of the real wave function. What we are actually doing is *integrating out* high energy states and working with an effective theory. The renormalization of constants in the theory about the subspace can be calculated via path integral or adiabatic elimination or other approaches. The loss of high-frequency components of the wave function usually does not matter, though, because usually no detection means can be used to find the high-frequency behavior, and an effective theory is just enough.

3 Random wave function description of the two-level atom

We want to write down a theory on the two-level atom itself and ignore the optical field. This is usually hard because photons radiated may come back and interact with the atom again. If, however, we investigate the circumstance in a typical laboratory, where there are plenty of things that may absorb or "observe" the photon, we can assume that as soon as a photon is emitted, it gets observed, and so does the atom. If we observe a emitted photon then we are sure that the atom is currently on the ground state.

Random wave function is a formalism that is frequently used in quantum optics. We write down a non-Hermitian effective theory describing the atom, which is

$$H = H_0 - \frac{\mathrm{i}}{2} C^\dagger C, \tag{14}$$

where H_0 is the RWA Hamiltonian in this lecture, and

$$C = \sqrt{\Gamma}\,|\mathrm{g}\rangle\langle\mathrm{e}| \tag{15}$$

is called the **collapse operator**. The effective Hamiltonian does not keep the total probability. The correct way to use it is the following *random* Schrödinger evolution. For $|\psi(t)\rangle$, we have

$$P(\,|\psi(t+\Delta t)\rangle = \text{normalized } C\,|\psi(t)\rangle) = \langle\psi(t)|C^\dagger C|\psi(t)\rangle\,\Delta t, \tag{16}$$

and

$$P(\,|\psi(t+\Delta t)\rangle = \text{normalized } (1 + \Delta H_{\text{eff}}/\mathrm{i}\hbar)\,|\psi(t)\rangle) = 1 - \langle\psi(t)|C^\dagger C|\psi(t)\rangle\,\Delta t. \tag{17}$$

The former possibility is called a **quantum jump**: a photon is emitted and observed, and now we are sure that the atom is on the ground state.

Random wave function is an intuitive approach. If, for example, we start from a excited state, what we will get is Figure 2 on page 4.

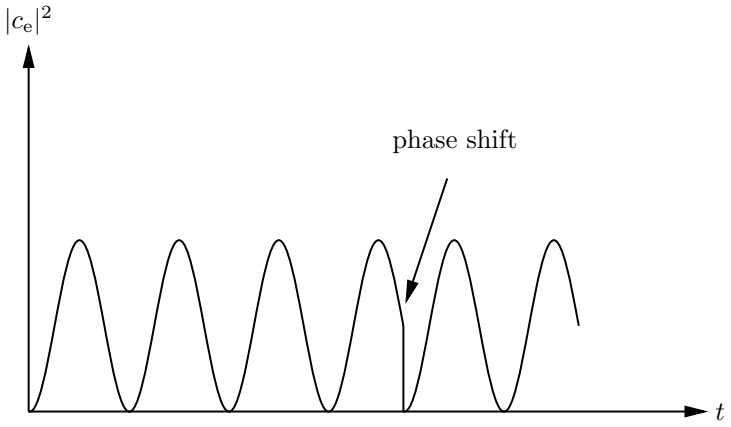


Figure 3: Time evolution in the strong coupling limit.

We consider another example. Suppose from 0 to t there is no quantum jump, and we want to know what is the probability that a quantum jump happens in the time duration t to $t+\Delta t$. We assume $\Omega \ll \Gamma$, and we have

$$c_g \approx 1, \quad c_e = \frac{\Omega}{2}\frac{1-e^{-i\Delta t-\Omega t/2}}{\Delta - i\Gamma/2}. \tag{18}$$

We find that

$$P(\text{jump during } t \text{ to } t+\Delta t) = \gamma \Delta t, \tag{19}$$

where

$$\gamma = \frac{|\Omega|^2}{4\Delta^2+\Gamma^2}\Gamma. \tag{20}$$

We can also use the random wave function method to calculate polarization in materials. We find the wave function

$$|\psi(t)\rangle \approx \frac{1}{\Delta - i\Gamma/2}\frac{\Omega}{2}|e\rangle + |g\rangle \tag{21}$$

is a meta-stable solution of the random evolution, and we have

$$\langle d\rangle = \frac{d_{eg}\Omega/2}{\Delta - i\Gamma/2}e^{-i\omega t} + \text{h.c.}. \tag{22}$$

We therefore have

$$\alpha = \frac{|d_{eg}|^2}{\omega_{eg} - i\Gamma/2 - \omega}. \tag{23}$$

The refractive index is now given by $n = \rho\alpha$. For glass, ω_{eg} is usually in the UV spectrum, so for the visible spectrum, $\omega_{eg} - \omega$ is large, and therefore absorption (or in other words random scattering) is not that important.

We now consider the $\Omega \gg \Gamma$ case. The wave function

$$|\psi(t)\rangle = \cos\frac{\Omega}{2}t\,|g\rangle - i\sin\frac{\Omega}{2}t\,|e\rangle \tag{24}$$

is a meta-stable state, but as time goes by, the probability of a quantum jump accumulates. A single time evolution trajectory is like

(4) is a quantum master equation. We can, actually, write down a non-Hermitian Hamiltonian

$$H_{\text{eff}} = -\frac{i\hbar\Gamma}{2}|e\rangle\langle e| + \hbar\sum_k g_k^* e^{i\Delta_k t}a_k^\dagger\,|g\rangle\langle e| + \text{h.c.}. \tag{25}$$

which reproduces (4). We can replace the second and the third term using rotational wave approximation in this lecture, and after averaging over all paths we get a damped oscillating curve.