Homework 3

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Problem 1
Solution

(a) The conjugate momentum of θ is

$$p = \frac{\partial L}{\partial \dot{\theta}} = V \left(\frac{\dot{\theta}}{U_0} - \frac{\mu}{U_0} \right), \tag{1}$$

and therefore

$$\dot{\theta} = \frac{U_0}{V} p + \mu. \tag{2}$$

The Hamiltonian is

$$\begin{aligned} H &= p\dot{\theta} - L \\ &= p \left(\frac{U_0}{V} p + \mu \right) - V \left(\frac{1}{2U_0} \left(\frac{U_0}{V} p + \mu \right)^2 - \frac{\mu}{U_0} \left(\frac{U_0}{V} p + \mu \right) \right) \\ &= \frac{1}{2} \frac{U_0}{V} \left(p + \frac{\mu V}{U_0} \right)^2. \end{aligned} \tag{3}$$

The eigenstates of this Hamiltonian is the same as the "particle on a ring" model, and we have

$$\langle \theta | p \rangle = \frac{1}{\sqrt{2\pi}} \mathrm{e}^{\mathrm{i}p\theta}, \tag{4}$$

where $p \in \mathbb{Z}$ so that θ and $\theta + 2\pi$ are equivalent.

Suppose the initial state is

$$\langle \theta | \psi \rangle = \sum_p c_p \frac{1}{\sqrt{2\pi}} \mathrm{e}^{\mathrm{i}p\theta}. \tag{5}$$

Problem 2
Solution

(a) Repeating the procedure used in ordinary superfluid, we do the decomposition

$$\varphi = \sqrt{\rho} \mathrm{e}^{\mathrm{i}\theta} = \sqrt{\rho_0 + \delta\rho} \mathrm{e}^{\mathrm{i}\theta}, \tag{6}$$

and therefore

$$-\frac{\varphi^* \nabla^2 \varphi}{2m} = \frac{\rho}{2m} (\boldsymbol{\nabla}\theta)^2 + \frac{(\boldsymbol{\nabla}\rho)^2}{8\rho m}, \tag{7}$$

$$\varphi^* \partial_\tau \varphi = \underbrace{\frac{1}{2} \partial_\tau \rho}_{\text{time derivative, ignored}} + \mathrm{i}\rho \partial_\tau \theta, \tag{8}$$

$$|\varphi(\boldsymbol{x})| U(\boldsymbol{x} - \boldsymbol{y}) |\varphi(\boldsymbol{y})| = \rho(\boldsymbol{x}) U(\boldsymbol{x} - \boldsymbol{y}) \rho(\boldsymbol{y}), \tag{9}$$

the theory is now

$$S = \int \mathrm{d}\tau \left(\int \mathrm{d}^d \boldsymbol{x} \left(\mathrm{i}\rho \partial_\tau \theta + \frac{\rho}{2m} (\boldsymbol{\nabla}\theta)^2 + \frac{(\boldsymbol{\nabla}\rho)^2}{8\rho m} - \mu\rho \right) + \frac{1}{2} \int \mathrm{d}^d \boldsymbol{x} \int \mathrm{d}^d \boldsymbol{y} \, \rho(\boldsymbol{x}) U(\boldsymbol{x} - \boldsymbol{y}) \rho(\boldsymbol{y}) \right). \tag{10}$$

Around the ground state, we have (note that since we are around a saddle point, the sum of all terms containing $\delta\rho$ only is always zero; the resulting theory has the form of $c_1\,\delta\rho\,\partial_\tau\theta + c_2\,\delta\rho^2$; the chemical potential term is therefore missing in the theory around the saddle point)

$$\mathrm{i}\rho\partial_\tau\theta = \underbrace{\mathrm{i}\rho_0\partial_\tau\theta}_{\text{time derivative}} + \mathrm{i}\,\delta\rho\,\partial_\tau\theta,$$

and since $\boldsymbol{\nabla}\rho = \boldsymbol{\nabla}\,\delta\rho$, we have

$$\frac{(\boldsymbol{\nabla}\rho)^2}{8\rho m} \approx \frac{(\boldsymbol{\nabla}\,\delta\rho)^2}{8\rho_0 m},$$

ignoring the fluctuation of the ρ in the denominator. Similarly, since we are working on a low energy theory, the fluctuation of θ shouldn't be large, and we have

$$\frac{\rho}{2m}(\boldsymbol{\nabla}\theta)^2 \approx \frac{\rho_0}{2m}(\boldsymbol{\nabla}\theta)^2.$$

The theory is then

$$S = \int \mathrm{d}^{d+1}x \left(\frac{\rho_0}{2m}(\boldsymbol{\nabla}\theta)^2 + \mathrm{i}\,\delta\rho\,\partial_\tau\theta + \frac{(\boldsymbol{\nabla}\,\delta\rho)^2}{8\rho_0 m} + \frac{1}{2}\,\delta\rho\,(\boldsymbol{x}) \int \mathrm{d}^d\boldsymbol{y}\, U(\boldsymbol{x}-\boldsymbol{y})\,\delta\rho\,(\boldsymbol{y}) \right) + S_{\text{saddle}}. \tag{11}$$

Integrating out $\delta\rho$, we get

$$\begin{aligned} S_{\text{eff}} &= \int \mathrm{d}^{d+1}x\,\frac{\rho_0}{2m}(\boldsymbol{\nabla}\theta)^2 - \frac{1}{2}\int \mathrm{d}\tau \int \mathrm{d}^d\boldsymbol{x}\,\mathrm{d}^d\boldsymbol{y}\,\mathrm{i}\partial_\tau\theta(\boldsymbol{x},\tau)\frac{1}{\int \mathrm{d}^d\boldsymbol{y}\,U(\boldsymbol{x}-\boldsymbol{y}) - \frac{1}{4\rho_0 m}\nabla^2}\mathrm{i}\partial_\tau\theta(\boldsymbol{y},\tau) \\ &= \int \mathrm{d}^{d+1}x\,\frac{\rho_0}{2m}(\boldsymbol{\nabla}\theta)^2 + \frac{1}{2}\int \mathrm{d}\tau \int \mathrm{d}^d\boldsymbol{x}\,\mathrm{d}^d\boldsymbol{y}\,\partial_\tau\theta(\boldsymbol{x},\tau)G(\boldsymbol{x}-\boldsymbol{y})\partial_\tau\theta(\boldsymbol{y}), \end{aligned} \tag{12}$$

where

$$\int \mathrm{d}^d\boldsymbol{y}\,U(\boldsymbol{x}-\boldsymbol{y})G(\boldsymbol{y}-\boldsymbol{z}) - \frac{1}{4\rho_0 m}\nabla^2_{\boldsymbol{x}}G(\boldsymbol{x}-\boldsymbol{z}) = \delta(\boldsymbol{x}-\boldsymbol{z}). \tag{13}$$

Similar to the procedure in dealing with ordinary superfluid, since we are only interested in the long wave length behaviors of θ, the ∇^2 term can be thrown away, and we have

$$\begin{aligned} \int \frac{\mathrm{d}^d\boldsymbol{p}}{(2\pi)^d}\mathrm{e}^{\mathrm{i}\boldsymbol{p}\cdot(\boldsymbol{x}-\boldsymbol{z})} &= \int \frac{\mathrm{d}^d\boldsymbol{p}}{(2\pi)^d}\int \mathrm{d}^d\boldsymbol{y}\,U(\boldsymbol{x}-\boldsymbol{y})G(\boldsymbol{p})\mathrm{e}^{\mathrm{i}\boldsymbol{p}\cdot(\boldsymbol{y}-\boldsymbol{z})} \\ &= \int \frac{\mathrm{d}^d\boldsymbol{p}}{(2\pi)^d}\int \mathrm{d}^d\boldsymbol{r}\,U(\boldsymbol{r})\mathrm{e}^{-\mathrm{i}\boldsymbol{p}\cdot\boldsymbol{r}}G(\boldsymbol{p})\mathrm{e}^{\mathrm{i}\boldsymbol{p}\cdot(\boldsymbol{x}-\boldsymbol{z})} \quad (\boldsymbol{r}=\boldsymbol{x}-\boldsymbol{y}), \end{aligned}$$

so

$$G(\boldsymbol{r}) = \int \frac{\mathrm{d}^d\boldsymbol{p}}{(2\pi)^d}\mathrm{e}^{\mathrm{i}\boldsymbol{p}\cdot\boldsymbol{r}}G(\boldsymbol{p}), \quad G(\boldsymbol{p}) = \frac{1}{U(\boldsymbol{p})} = \frac{1}{\int \mathrm{d}^d\boldsymbol{r}\,U(\boldsymbol{r})\mathrm{e}^{-\mathrm{i}\boldsymbol{p}\cdot\boldsymbol{r}}}. \tag{14}$$

To evaluate $G(\boldsymbol{p})$, we need to find

$$U(\boldsymbol{p}) = \int_0^\infty \mathrm{d}r\,\frac{2\pi^{d/2}}{\Gamma(d/2)}r^{d-1}\frac{U_0}{r^{d-\epsilon}} \tag{15}$$

Problem 3
Solution

(a) The energy now can be exactly evaluated (N is the number of sites):

$$E = \frac{UN}{2}(M^2 - M) - \mu N M = \frac{N}{2}(UM^2 - (U+2\mu)M). \tag{16}$$

At the ground state, E is minimized, and we have

$$M = \frac{U+2\mu}{2U}, \tag{17}$$

and the energy gap is

$$\Delta E = E|_{M+1} - E|_M = \frac{N}{2}(2UM - 2\mu) = \frac{1}{2}NU. \tag{18}$$