Homework 2

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February 21, 2024

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1.1

Comparison between the approximate and exact susceptibilities. We consider a driven damped simple harmonic oscillator (following L8.N3). We assume that $\dot{p} = \sum_i f_i = -kq - D\dot{q} + f(t)$, where k, D, and $f(t)$ are the spring constant, damping, and driving force, respectively. In what follows, we assume that $f(t) = f_o \cos(\omega t + \phi)$.

(a) *Starting from Newton's equations, find the first order equation of motion in complex form. [Hint: you should find something like $\dot{a} = -\mathrm{i}\omega_o a + (\ldots)$, and you're looking to identify all of the (...) terms.]*

When there is no damping the EOMs are

$$\dot{p} = -kq, \quad \dot{q} = \frac{p}{m}, \tag{1}$$

and the following definition of the modes

$$q = \sqrt{\frac{1}{2m\omega_0}}(a^* + a), \quad p = \mathrm{i}\sqrt{\frac{m\omega_0}{2}}(a^* - a), \quad \omega_0^2 = \frac{k}{m} \tag{2}$$

turns the Hamiltonian into $H = \omega_0 a^* a$. Under this definition of a, a^*, the EOM with damping and driving therefore becomes

$$\begin{aligned} \dot{a}^* - \dot{a} &= \mathrm{i}\omega_0(a^* + a) - \frac{D}{m}(a^* - a) - \mathrm{i}\sqrt{\frac{2}{m\omega_0}} f(t), \\ \dot{a}^* + \dot{a} &= \mathrm{i}\omega_0(a^* - a), \end{aligned} \tag{3}$$

from which we find

$$\dot{a} = -\mathrm{i}\omega_0 a + \frac{D}{2m}(a^* - a) + \mathrm{i}\sqrt{\frac{2}{m\omega_0}} f(t) \tag{4}$$

and its complex conjugate

$$\dot{a}^* = \mathrm{i}\omega_0 a^* - \frac{D}{2m}(a^* - a) - \mathrm{i}\sqrt{\frac{2}{m\omega_0}} f(t). \tag{5}$$

(b) *Perform the rotating wave approximation to the solution you obtained in part (a). Identify the terms that are eliminated by this approximation.*

Now we do RWA and let $a = \bar{a}\mathrm{e}^{-\mathrm{i}\omega_0 t}$, and the EOM of a becomes

$$\dot{\bar{a}}\mathrm{e}^{-\mathrm{i}\omega_0 t} + \bar{a}(-\mathrm{i}\omega_0)\mathrm{e}^{-\mathrm{i}\omega_0 t} = -\mathrm{i}\omega_0 \bar{a}\mathrm{e}^{-\mathrm{i}\omega_0 t} + \frac{D}{2m}(\bar{a}^* \mathrm{e}^{\mathrm{i}\omega_0 t} - \bar{a}\mathrm{e}^{-\mathrm{i}\omega_0 t}) + \mathrm{i}\sqrt{\frac{2}{m\omega_0}} f(t),$$

or

$$\dot{\bar{a}} = -\frac{D}{2m}(\bar{a} - \bar{a}^* \mathrm{e}^{\mathrm{i}2\omega_0 t}) + \mathrm{i}\sqrt{\frac{2}{m\omega_0}} \mathrm{e}^{\mathrm{i}\omega_0 t} f(t). \tag{6}$$

The fast oscillating terms include the $\frac{D}{2m}\bar{a}^* \mathrm{e}^{\mathrm{i}2\omega_0 t}$ term and the $\mathrm{e}^{\mathrm{i}\omega t}$ part of $f(t)$, assuming that ω is close to ω_0. If these terms are removed, the resulting EOM is

$$\dot{\bar{a}} = -\frac{D}{2m}\bar{a} + \mathrm{i}\sqrt{\frac{1}{2m\omega_0}} \cdot \frac{1}{2} f_0 \mathrm{e}^{-\mathrm{i}(\omega - \omega_0)t}. \tag{7}$$

(c) *Use the approximate equation of motion from part (b) to find the complex susceptibility, $\alpha_a[\omega]$, that relates the complex wave amplitude and force as $a[\omega] = \alpha_a[\omega] f[\omega]$.*

Noting that in the stationary solution $\bar{a} \propto \mathrm{e}^{-\mathrm{i}(\omega-\omega_0)t}$, we have

$$a(t) = \mathrm{e}^{-\mathrm{i}\omega_0 t}\bar{a} = \frac{1}{-\mathrm{i}(\omega-\omega_0) + \frac{D}{2m}} \frac{\mathrm{i}}{\sqrt{2m\omega_0}} \cdot \frac{1}{2} f_0 \mathrm{e}^{-\mathrm{i}\omega t}, \tag{8}$$

and therefore the response of a is (we divide $a(t)$ by $f_0 \mathrm{e}^{-\mathrm{i}\omega t}/2$)

$$\alpha_a[\omega] = -\frac{1}{\sqrt{2m\omega_0}} \frac{1}{\omega - \omega_0 + \mathrm{i}\frac{D}{2m}}. \tag{9}$$

1.2

(a) *Starting from the second order equation, $m\ddot{q} = \sum_i f_i = -kq - D\dot{q} + f(t)$, find the exact susceptibility $q[\omega] = \alpha_q[\omega] f[\omega]$ of the driven-damped SHO.*

The EOM of q is

$$m\ddot{q} + D\dot{q} + kq = \frac{f_0}{2}(\mathrm{e}^{\mathrm{i}\omega t} + \mathrm{e}^{-\mathrm{i}\omega t}), \tag{10}$$

and since the system is linear, we can only consider the response of q to the $\mathrm{e}^{-\mathrm{i}\omega t}$ component of f, and get

$$\alpha_q[\omega] = \frac{1}{-m\omega^2 - \mathrm{i}D\omega + k} = -\frac{1}{m} \frac{1}{\omega^2 - \omega_0^2 + \mathrm{i}\frac{D}{m}\omega}. \tag{11}$$

(b) *Compare the approximate response you obtained in part (1.1c) with the exact response you found in (1.2a). [Remember, $q(t)$ is contained in $a(t)$.]*

Since $p = m\dot{q}$, we have

$$\alpha_p[\omega] = \frac{\mathrm{i}\omega}{\omega^2 - \omega_0^2 + \mathrm{i}\frac{D}{m}\omega}, \tag{12}$$

and this means

$$\begin{aligned} \alpha_a^{\text{accurate}} &= \frac{1}{2}\left(\sqrt{2m\omega_0}\alpha_q + \mathrm{i}\sqrt{\frac{2}{m\omega_0}}\alpha_p\right) \\ &= -\frac{1}{\sqrt{2m\omega_0}} \frac{\omega + \omega_0}{\omega^2 - \omega_0^2 + \mathrm{i}\frac{D}{m}\omega}. \end{aligned} \tag{13}$$

(c) *Can you find a set of approximations that will make your exact susceptibility converge with the susceptibility you found from the RWA?*

The validity of RWA is equivalent to $\omega \approx \omega_0$, which then means

$$\alpha_a = -\frac{1}{\sqrt{2m\omega_0}} \frac{1}{\frac{\omega^2-\omega_0^2}{\omega+\omega_0} + \mathrm{i}\frac{D}{m}\frac{\omega}{\omega+\omega_0}} \approx -\frac{1}{\sqrt{2m\omega_0}} \frac{1}{\omega - \omega_0 + \mathrm{i}\frac{D}{2m}}. \tag{14}$$

This is expected since when RWA was made, it was already assumed that $\omega \approx \omega_0$.

Verifying this condition by calculating α_q seems to be not viable, because q is to be calculated from a and a^*, the response function of the latter being $\alpha[-\omega^*]$; but then when RWA around $\omega \approx \omega_0$ works for a, it doesn't work for a^*. RWA for q's response seems to be ill-defined.

2

Time Modulation of Dielectric Constant: Next, we consider the impact of a time-modulated dielectric constant on the modes of a resonator. In this case, we assume that a standing-wave mode of the form $\mathbf{E}(r,t) = (a - a^*)\,\mathbf{E}^o(r)$ with real-valued field distribution $\mathbf{E}^o(r)$ is localized within a structured dielectric, having dielectric distribution $\varepsilon_r(r)$. For example, you could imagine that this is an eigenmode of a dielectric photonic crystal resonator of the type seen as Fig 8 in Chapter 7 of [JJSJ]. In the absence of a dielectric perturbation, the Hamiltonian takes the form $H_a = \omega_o a^* a$, where ω_o is the natural frequency of this time-harmonic mode.

(a) *As a starting point, find an expression for δH associated with an arbitrary dielectric perturbation $\Delta\varepsilon_r(r)$.*

Using the normalization

$$\boldsymbol{E} = \mathrm{i}(a - a^*)\boldsymbol{E}^0(\boldsymbol{r}), \tag{15}$$

we have

$$\delta H = \int \mathrm{d}^3\boldsymbol{r}\, \frac{1}{2}\epsilon_0 \Delta\epsilon_\mathrm{r} \boldsymbol{E}^2 = \frac{\epsilon_0}{2}(a - a^*)^2 \left\langle \boldsymbol{E}^0 \middle| \Delta\epsilon_\mathrm{r} \middle| \boldsymbol{E}_0 \right\rangle. \tag{16}$$

(b) *Using your result from part (a), find an equations of motion for a and a^*.*

We have

$$\frac{\mathrm{d}a}{\mathrm{d}t} = -\mathrm{i}\frac{\partial H}{\partial a^*} = -\mathrm{i}\omega_0 a + \mathrm{i}\epsilon_0 (a - a^*) \left\langle \boldsymbol{E}^0 \middle| \Delta\epsilon_\mathrm{r} \middle| \boldsymbol{E}_0 \right\rangle, \tag{17}$$

and therefore

$$\frac{\mathrm{d}a^*}{\mathrm{d}t} = \mathrm{i}\omega_0 a^* + \mathrm{i}\epsilon_0 (a - a^*) \left\langle \boldsymbol{E}^0 \middle| \Delta\epsilon_\mathrm{r} \middle| \boldsymbol{E}_0 \right\rangle. \tag{18}$$

If we formally solve the equation system, we will find the first order correction is

$$\Delta\omega^{(1)} = -\epsilon_0 \left\langle \boldsymbol{E}^0 \middle| \Delta\epsilon_\mathrm{r} \middle| \boldsymbol{E}_0 \right\rangle. \tag{19}$$

Considering the normalization scheme here that leads to $H_0 = \omega a^* a$ is

$$2\epsilon_0 \left\langle \boldsymbol{E}^0 \middle| \Delta\epsilon_\mathrm{r} \middle| \boldsymbol{E}^0 \right\rangle = \omega_0 \tag{20}$$

and not

$$\epsilon_0 \left\langle \boldsymbol{E}^0 \middle| \Delta\epsilon_\mathrm{r} \middle| \boldsymbol{E}^0 \right\rangle = 1, \tag{21}$$

this result agrees with the first order perturbation in Homework 1.

(c) *Next, we assume that perturbation to the dielectric constant takes the form $\Delta\varepsilon_r(t) = \beta\varepsilon_r(r)\cos(2\omega_o t + \phi)$, where β is a small ($\beta \ll 1$) unitless constant. Find the new form of our equations of motion from part (b). [Remember, our field normalization permits us to reduce $\langle \mathbf{E}^o | \varepsilon_o \varepsilon_r(r) | \mathbf{E}^o \rangle$ to a constant already defined above.]*

The EOM of a now is

$$\begin{aligned} \frac{\mathrm{d}a}{\mathrm{d}t} &= -\mathrm{i}\omega_0 a + \mathrm{i}\epsilon_0 (a - a^*) \left\langle \boldsymbol{E}^0 \middle| \epsilon_\mathrm{r} \middle| \boldsymbol{E}_0 \right\rangle \beta \cos(2\omega_0 t + \phi) \\ &= -\mathrm{i}\omega_0 a + \frac{\mathrm{i}\omega_0}{2}\beta(a - a^*)\cos(2\omega_0 t + \phi). \end{aligned} \tag{22}$$

(d) *Using a trial solution in the form $a(t) = \bar{a}(t)e^{-i\omega_o t}$, apply the rotating wave approximation to the equations of motion from part (c). [Hint: your results should closely resemble the problem involving a child on a swing.]*

Under the ansatz $a(t) = \bar{a}(t)\mathrm{e}^{-\mathrm{i}\omega_0 t}$ we have

$$\begin{aligned} \mathrm{e}^{-\mathrm{i}\omega_0 t}\frac{\mathrm{d}\bar{a}}{\mathrm{d}t} &= \frac{\mathrm{i}\omega_0}{2}\beta(\bar{a}\mathrm{e}^{-\mathrm{i}\omega_0 t} - \bar{a}^*\mathrm{e}^{\mathrm{i}\omega_0 t}) \cdot \frac{1}{2}(\mathrm{e}^{\mathrm{i}(2\omega_0 t + \phi)} + \mathrm{e}^{-\mathrm{i}(2\omega_0 t + \phi)}) \\ \Rightarrow \frac{\mathrm{d}\bar{a}}{\mathrm{d}t} &\approx -\frac{\mathrm{i}\omega_0}{4}\beta\bar{a}^*\mathrm{e}^{-\mathrm{i}\phi}, \end{aligned} \tag{23}$$

and therefore under RWA,

$$\frac{\mathrm{d}\bar{a}^*}{\mathrm{d}t} = \frac{\mathrm{i}\omega_0}{4}\beta\bar{a}\mathrm{e}^{\mathrm{i}\phi}. \tag{24}$$

Therefore, we get

$$\frac{\mathrm{d}^2\bar{a}}{\mathrm{d}t^2} = \frac{\omega_0^2\beta^2}{16}\bar{a}, \tag{25}$$

and the exponential growing solution is

$$a = \bar{a}\mathrm{e}^{-\mathrm{i}\omega_0 t} = \bar{a}(t=0)\mathrm{e}^{-\mathrm{i}\omega_0 t}\mathrm{e}^{\frac{\omega_0\beta}{4}t}. \tag{26}$$

This expression contains no phase ϕ, but ϕ does appear in the growth of H, which is given by

$$\dot{H} = \omega_0(\dot{a}^* a + a^*\dot{a}) = \omega_0(\dot{\bar{a}}^*\bar{a} + \bar{a}^*\dot{\bar{a}}) = \frac{\omega_0^2}{2}\beta|\bar{a}|^2\sin\phi. \tag{27}$$

(e) *Treating $\Delta\varepsilon_r(r)$ as a perturbation on our original mode, solve for the first order correction to our mode amplitude.*

3

3.1

Third order nonlinear polarization: In this problem, we consider the effect of third order nonlinearity on a single time-harmonic mode. A single optical mode, $\mathbf{E}(\mathbf{r},t)$, interacts with a polarizable medium. We assume that the medium produces a polarization field $\mathbf{P} = \chi^{(3)}|E|^2\mathbf{E}$, where $\chi^{(3)}$ is a constant. In the absence of the polarizable medium (i.e., $\chi^{(3)} = 0$), the Hamiltonian for the system is $H(a, a^*) = \omega a^* a$ where a and a^* are the complex mode amplitudes. As we have done in lecture, we assume that the mode amplitude is related to the field by $\mathbf{E}(\mathbf{r},t) = (a - a^*)\,\mathbf{E}^o(\mathbf{r})$.

(a) *Find an expression for the interaction Hamiltonian, δH, in terms of the electric fields.*

The nonlinear interaction Hamiltonian is

$$
\begin{aligned}
\delta H &= \int \boldsymbol{E}\cdot\delta\boldsymbol{D} = \int \boldsymbol{E}\cdot\chi^{(3)}(|\boldsymbol{E}|^2\,\delta\boldsymbol{E} + 2\boldsymbol{E}\boldsymbol{E}\cdot\delta\boldsymbol{E}) \\
&= \chi^{(3)}\int 3|\boldsymbol{E}|^2\boldsymbol{E}\cdot\delta\boldsymbol{E} \\
&= \frac{3}{4}\chi^{(3)}|\boldsymbol{E}|^4.
\end{aligned} \tag{28}
$$

(b) *Express your interaction Hamiltonian from part (a) as products of a, a^*; be sure to define all of the coefficients in this expression. [The coefficients will include integrals over space.]*

$$
\delta H = \frac{3}{4}\chi^{(3)}(a-a^*)^4\int \mathrm{d}^3\boldsymbol{r}\,|\boldsymbol{E}^0|^4. \tag{29}
$$

(c) *Find $\dot{a}$ using Poisson brackets.*

$$
\begin{aligned}
\frac{\mathrm{d}a}{\mathrm{d}t} &= -\mathrm{i}\omega_0 a - \mathrm{i}\frac{\partial\,\delta H}{\partial a^*} \\
&= -\mathrm{i}\omega_0 a + \mathrm{i}3\chi^{(3)}(a-a^*)^3\int \mathrm{d}^3\boldsymbol{r}\,|\boldsymbol{E}^0|^4.
\end{aligned} \tag{30}
$$

(d) *From part (c), show that the mode acquires intensity dependent frequency shift. Explain the significance of this term.*