Homework 1

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September 17, 2022

Exercise 9 in chapter 1 (**) Let us show now one simple way to produce the realizations of B_m knowing the realizations of one example of $B_{1/2}$. From the set of the realizations of $B_{1/2}$, which we can view as a real number $r \in [0,1[$ by forming the binary digit number $0.\beta^{(1)}\beta^{(2)}\beta^{(3)}\beta^{(4)}\beta^{(5)}\ldots$ (example: $0.011010110001111\ldots$), we can obtain the real numbers $r^{(1)}, r^{(2)}, r^{(3)}$ etc... from the formula:

$$r^{(n)} = (2^n r) \bmod 1$$

The realisations $\beta_m^{(n)}$ will be obtained from $r^{(n)}$ by the expression

$$\beta_m^{(n)} = 0 \text{ if } r^{(n)} \geqslant m$$
$$\beta_m^{(n)} = 1 \text{ if } r^{(n)} < m$$

Prove that this last expression yields the expected properties for B_m.

Solution Since

$$r^{(n)} = \beta^{(1)}\beta^{(2)}\cdots\beta^{(n)}.\beta^{(n+1)}\beta^{(n+2)}\cdots,$$

we know

$$r^{(n)} = 0.\beta^{(n+1)}\beta^{(n+2)}\cdots. \tag{1}$$

Each digit of $r^{(n)}$ is 0 or 1, and thus the possible range of $r^{(n)}$ is [0, 1].[1] Suppose

$$x = 0.x^{(1)}x^{(2)}\cdots \in [0,1],$$

we have

$$\begin{aligned}
P(r^{(n)} < x) &= P(\beta^{(n+1)} < x^{(1)}) + P(\beta^{(n+1)} = x^{(1)})P(\beta^{(n+2)} < x^{(2)}) + \cdots \\
&= \frac{1}{2}\delta_{x^{(1)},1} + \frac{1}{2} \times \frac{1}{2}\delta_{x^{(2)},1} + \cdots \\
&= 0.x^{(1)}x^{(2)}\cdots = x,
\end{aligned}$$

so $r^{(n)}$ has a uniform probabilistic distribution on $[0,1]$. So the probability of $r^{(n)} < m$ i.e. $\beta_m^{(n)} = 1$ is exactly m, and therefore $\beta_m^{(n)}$ is a realization of B_m, regardless of what n is.

Exercise 14 in chapter 1 (**) Explain the link between the binomial distribution and the expansion of $(a+b)^N$.

Solution The binomial distribution can be derived by an intermediate step used to derive the expansion of $(a+b)^N$.

The binomial coefficient $\binom{N}{n}$ gives the number of ways to pick n points in N different points. Without invoking the commutative property of multiplication, there are 2^N terms in the expansion of $(a+b)^N$, each of which is like

$$aabbabba\cdots.$$

Now by the definition of the binomial coefficient, there are $\binom{N}{n}$ terms that have n a's and $(N-n)$ b's.

From this conclusion we can derive the expansion of $(a+b)^N$: there are $\binom{N}{n}$ terms in the total 2^N terms which has n a's and $(N-n)$ b's, and we have

$$(a+b)^N = \sum_{n=0}^{N} \binom{N}{n} a^n b^{N-n}. \tag{2}$$

[1] It's actually possible to have $r^{(n)} = 1$, because the binary $0.11111\cdots$ is actually 1, in the same way $0.9999\cdots = 1$ in the decimal case. But the probability to have such a $r^{(n)}$ is $1/2 \times 1/2 \times \cdots = 0$. That is, the event that $r^{(n)} = 1$ is possible but is a null set.

Similarly, if we consider the probabilistic distribution of

$$X_{m,N} = \sum_{k=1}^{N} B_{m,k}, \tag{3}$$

we will find the probability of the event that $X_{m,N} = x$ is the sum of the probability of all outputs of $\{B_{m,k}\}$ in which there are x 1 outputs and $N - x$ 0 outputs, and for each possible output, the probability is

$$p(1)^x p(0)^{N-x} = m^x (1-m)^{N-x},$$

and we have

$$p_{m,N}(x) = \binom{N}{x} m^x (1-m)^{N-x}. \tag{4}$$

So the relation between the binomial distribution and the $(a+b)^N$ expansion is they both involve the notion of "picking x points from N points". Indeed, by considering the normalization condition of (4), which is

$$1 = \sum_{x} p_{m,N}(x) = \sum_{x=0}^{N} \binom{N}{x} m^x (1-m)^{N-x}, \tag{5}$$

we rediscover the expansion of $(a+b)^N$, where we set $a = m$ and $b = 1 - m$.

Exercise 3 in chapter 2 (**) (a) Show that the above expression (2.15) for $w(x,t)$ with $t > 0$ satisfies this equation. (b) By using a double Fourier transform in x and t show that the Green's function of the Smoluchowsky equation (2.26) is indeed the above expression (2.15) for $w(x,t)$ with $t \geq 0$.

Solution

(a) From (2.15) we have

$$\begin{aligned}
\frac{\partial}{\partial t} w(x,t) &= -\frac{1}{2}\sqrt{\frac{1}{4\pi D t^3}} \mathrm{e}^{-\frac{(x-v_d t)^2}{4Dt}} - \sqrt{\frac{1}{4\pi D t}} \mathrm{e}^{-\frac{(x-v_d t)^2}{4Dt}} \frac{1}{4Dt^2} (2v_d(v_d t - x)t - (x - v_d t)^2) \\
&= -\sqrt{\frac{1}{4\pi D t}} \mathrm{e}^{-\frac{(x-v_d t)^2}{4Dt}} \left(\frac{1}{2t} + \frac{(v_d t - x)(v_d t + x)}{4Dt^2} \right),
\end{aligned}$$

$$\frac{\partial}{\partial x} w(x,t) = -\sqrt{\frac{1}{4\pi D t}} \mathrm{e}^{-\frac{(x-v_d t)^2}{4Dt}} \frac{x - v_d t}{2Dt},$$

and

$$\frac{\partial^2}{\partial x^2} w(x,t) = -\sqrt{\frac{1}{4\pi D t}} \mathrm{e}^{-\frac{(x-v_d t)^2}{4Dt}} \left(\frac{1}{2Dt} - \left(\frac{x - v_d t}{2Dt} \right)^2 \right),$$

The RHS of the Smoluchowski equation is

$$\begin{aligned}
D\frac{\partial^2 w}{\partial x^2} - v_d \frac{\partial w}{\partial x} &= -\sqrt{\frac{1}{4\pi D t}} \mathrm{e}^{-\frac{(x-v_d t)^2}{4Dt}} \left(\frac{1}{2t} - \frac{(x - v_d t)^2}{4Dt^2} - v_d \frac{x - v_d t}{2Dt} \right) \\
&= -\sqrt{\frac{1}{4\pi D t}} \mathrm{e}^{-\frac{(x-v_d t)^2}{4Dt}} \left(\frac{1}{2t} - \frac{x^2 - v_d^2 t^2}{4Dt^2} \right),
\end{aligned}$$

so we have

$$\frac{\partial}{\partial x} w(x,t) = D\frac{\partial^2 w}{\partial x^2} - v_d \frac{\partial w}{\partial x}.$$

(b) The initial condition is

$$\lim_{t \to 0} w = \delta(x),$$

which can be imposed to (2.26) by adding an "impact":

$$\frac{\partial w}{\partial t} = D\frac{\partial^2 w}{\partial x^2} - v_d \frac{\partial w}{\partial x} + \delta(x)\delta(t). \tag{6}$$

Now by Fourier transformation we have

$$w(x,t) = \int \frac{\mathrm{d}k\,\mathrm{d}\omega}{(2\pi)^2} \mathrm{e}^{-\mathrm{i}(\omega t - kx)} \tilde{w}(k,\omega),$$
$$-\mathrm{i}\omega\tilde{w} = D(\mathrm{i}k)^2\tilde{w} - \mathrm{i}kv_d\tilde{w} + 1.$$

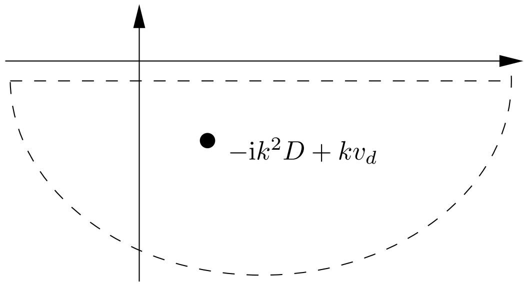
We find

$$\tilde{w} = \frac{1}{-\mathrm{i}\omega + k^2 D + \mathrm{i}kv_d},$$

and thus

$$w(x,t) = \int \frac{\mathrm{d}k\,\mathrm{d}\omega}{(2\pi)^2} \mathrm{e}^{-\mathrm{i}(\omega t - kx)} \frac{1}{-\mathrm{i}\omega + k^2 D + \mathrm{i}kv_d}.$$

We first complete the integral over ω, with the following contour:



$$\int \mathrm{d}\omega\, \mathrm{e}^{-\mathrm{i}(\omega t - kx)} \frac{1}{\omega + \mathrm{i}Dk^2 - kv_d} = -2\pi \mathrm{i} \mathrm{e}^{-\mathrm{i}(-\mathrm{i}k^2 Dt + kv_d t - kx)}.$$

Thus

$$\begin{aligned} w(x,t) &= \frac{\mathrm{i}}{(2\pi)^2} \int \mathrm{d}k\,(-2\pi\mathrm{i})\mathrm{e}^{-\mathrm{i}(-\mathrm{i}k^2 Dt + kv_d t - kx)} \\ &= \frac{1}{2\pi} \int \mathrm{d}k\, \mathrm{e}^{-k^2 Dt - \mathrm{i}k(v_d t - x)} \\ &= \frac{1}{2\pi} \cdot \sqrt{\frac{2\pi}{2Dt}} \mathrm{e}^{\frac{1}{2}\frac{1}{2Dt}(-\mathrm{i}(v_d t - x))^2} \\ &= \sqrt{\frac{1}{4\pi Dt}} \mathrm{e}^{-\frac{(x - v_d t)^2}{4Dt}}. \end{aligned}$$

This is exactly (2.15).

Exercise 5 in chapter 2 (**) Explain in detail how, by measuring for the first time the position diffusion constant of a small Brownian sphere immerged in water, the physicist Jean Perrin, using the Einstein relation, was able to measure Avogadro's Number N_A, thereby confirming the existence of atoms (Jean Perrin received the Nobel prize for this work in 1926, see his Nobel lecture on the Nobel website). Use Stokes' law stating that a sphere of radius R moving at a velocity V feels in a fluid with viscosity η a frictional force

$$F = 6\pi R\eta V$$

Remember that Avogadro's Number N_A is involved in the ideal gas constant, defined by the relation

$$\frac{\text{pressure volume}}{\text{temperature}} = nR_{\mathrm{ig}}$$

where n is the number of moles of the volume of gas considered. In the kinetic theory of gases, R_{ig} is given by

$$R_{\mathrm{ig}} = N_A k_B$$

Solution The Stokes' law

$$F = 6\pi R\eta v \tag{7}$$

connects two physical quantities arising from the same dissipation process in the fluid: the viscosity μ and the response coefficient

$$\mu = \frac{v}{F}. \tag{8}$$

The relation between the two is imposed by the Navier-Stokes equation. Since we also have

$$\mu = \frac{D}{k_\mathrm{B}T}, \tag{9}$$

we have

$$\frac{1}{6\pi R\eta} = \frac{D}{k_\mathrm{B}T}. \tag{10}$$

This equation can be used to measure k_B: each quantities involved in the equation can be measured separately. The viscosity η can be measured by standard fluid dynamic methods. The radius R can be measured by letting the particles fall in the fluid and recording its terminal velocity, and then we have

$$R = \frac{mg}{6\pi\eta v_\mathrm{terminal}}. \tag{11}$$

The diffusion coefficient D can be measured by looking at the trajectory of a Brownian particle. The temperature is measured by a thermometer. Now we find k_B, and by the ideal gas equation

$$pV = nR_\mathrm{ig}T \tag{12}$$

we can measure R_ig, so finally, by

$$R_\mathrm{ig} = N_A k_\mathrm{B}, \tag{13}$$

the Avogadro constant is found.

Problem 2

Solution

(a) For a single bit we have

$$H(\mathrm{B}_{1/2}) = -\frac{1}{2}\log_2\frac{1}{2} - \frac{1}{2}\log_2\frac{1}{2} = 1. \tag{14}$$

For N independent bits, we have

$$H(\otimes_{j=1}^{N}\mathrm{B}_{1/2,j}) = -\sum_i \left(\frac{1}{2}\right)^N \log_2\left(\frac{1}{2}\right)^N = -2^N \times \frac{1}{2^N} \times (-N) = N. \tag{15}$$

(b) We have

$$S = -\frac{\partial F}{\partial T}, \quad F = -k_\mathrm{B}T\ln Z.$$

Now with the definition of the partition function

$$Z = \sum_i \mathrm{e}^{-E_i/k_\mathrm{B}T}, \tag{16}$$

we have

$$\begin{aligned}\frac{\partial}{\partial T}T\ln Z &= \ln Z + \frac{T}{Z}\frac{\partial Z}{\partial T}\\ &= \ln Z + \frac{T}{Z}\sum_i \frac{E_i}{k_\mathrm{B}T^2}\mathrm{e}^{-E_i/k_\mathrm{B}T}\\ &= \ln Z + \frac{1}{k_\mathrm{B}T}\sum_i p_i E_i.\end{aligned}$$

Thus

$$S = -k_\mathrm{B}\ln Z - \frac{1}{T}\sum_i p_i E_i = -k_\mathrm{B}\ln Z - \frac{\langle E\rangle}{T}. \tag{17}$$

(c) In the high temperature limit $E_i/k_\mathrm{B}T \to 0$ for every E_i, so energy is no longer important in determining the probabilistic distribution and each configuration has the same probability. The energy of N indistinguishable random bits, in this case, is therefore

$$E = \sum_{j=1}^{N} B_{1/2,j}. \tag{18}$$

The probability of $E = \epsilon$ is

$$p(E = \epsilon) = \binom{N}{\epsilon} \times \frac{1}{2^N}, \tag{19}$$

which reaches its peak when $\epsilon = N/2$, so

$$\mathcal{E} = N/2. \tag{20}$$

There are $\binom{N}{\epsilon}$ microstates in the macrostate $(N, \mathcal{C})$, so

$$\begin{aligned}
\lim_{N\to\infty} \frac{1}{N} S(N, \mathcal{E}) &= k_\mathrm{B} \frac{1}{N} \ln \binom{N}{N/2} \\
&= \frac{k_\mathrm{B}}{N} (\ln N! - 2\ln(N/2)!) \\
&\approx \frac{k_\mathrm{B}}{N} (N \ln N - 2(N/2) \ln N/2) \\
&= k_\mathrm{B} \ln 2.
\end{aligned}$$

Thus

$$\lim_{N\to\infty} \frac{1}{N} S(N, \mathcal{E}) = k_\mathrm{B} \ln 2. \tag{21}$$

(d)

(g) We need to take the $\alpha \to 1$ limit of

$$H_\alpha(X) = \frac{1}{1-\alpha} \log_2 \left(\sum_i p_i^\alpha \right). \tag{22}$$

When $\alpha = 1$, both the numerator ($\log_2 1 = 0$) and the denominator are zero, so we can use the L'Hospital's rule:

$$\lim_{\alpha\to 1} H_\alpha = \lim_{\alpha\to 1} \frac{\frac{\sum_i \ln p_i p_i^\alpha}{\ln 2 \sum_i p_i^\alpha}}{-1} = -\sum_i p_i \log p_i = H(X).$$