Homework 1

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Exercise 9 in chapter 1 (**) Let us show now one simple way to produce the realizations of B_m knowing the realizations of one example of $B_{1/2}$. From the set of the realizations of $B_{1/2}$, which we can view as a real number $r \in [0,1[$ by forming the binary digit number $0.\beta^{(1)}\beta^{(2)}\beta^{(3)}\beta^{(4)}\beta^{(5)}\dots$ (example: $0.011010110001111\dots$), we can obtain the real numbers $r^{(1)}, r^{(2)}, r^{(3)}$ etc... from the formula:

$$r^{(n)} = (2^n r) \bmod 1$$

The realisations $\beta_m^{(n)}$ will be obtained from $r^{(n)}$ by the expression

$$\beta_m^{(n)} = 0 \text{ if } r^{(n)} \geqslant m$$
$$\beta_m^{(n)} = 1 \text{ if } r^{(n)} < m$$

Prove that this last expression yields the expected properties for B_m.

Solution Since

$$r^{(n)} = \beta^{(1)}\beta^{(2)}\cdots\beta^{(n)}.\beta^{(n+1)}\beta^{(n+2)}\cdots,$$

we know

$$r^{(n)} = 0.\beta^{(n+1)}\beta^{(n+2)}\cdots. \tag{1}$$

Each digit of $r^{(n)}$ is 0 or 1, and thus the possible range of $r^{(n)}$ is [0, 1].[1] Suppose

$$x = 0.x^{(1)}x^{(2)}\cdots \in [0,1],$$

we have

$$\begin{aligned} P(r^{(n)} < x) &= P(\beta^{(n+1)} < x^{(1)}) + P(\beta^{(n+1)} = x^{(1)})P(\beta^{(n+2)} < x^{(2)}) + \cdots \\ &= \frac{1}{2}\delta_{x^{(1)},1} + \frac{1}{2} \times \frac{1}{2}\delta_{x^{(2)},1} + \cdots \\ &= 0.x^{(1)}x^{(2)}\cdots = x, \end{aligned}$$

so $r^{(n)}$ has a uniform probabilistic distribution on $[0,1]$. So the probability of $r^{(n)} < m$ i.e. $\beta_m^{(n)} = 1$ is exactly m, and therefore $\beta_m^{(n)}$ is a realization of B_m, regardless of what n is.

Exercise 14 in chapter 1 (**) Explain the link between the binomial distribution and the expansion of $(a+b)^N$.

Solution The binomial distribution can be derived by an intermediate step used to derive the expansion of $(a+b)^N$.

The binomial coefficient $\binom{N}{n}$ gives the number of ways to pick n points in N different points. Without invoking the commutative property of multiplication, there are 2^N terms in the expansion of $(a+b)^N$, each of which is like

$$aabbabba\cdots.$$

Now by the definition of the binomial coefficient, there are $\binom{N}{n}$ terms that have n a's and $(N-n)$ b's.

From this conclusion we can derive the expansion of $(a+b)^N$: there are $\binom{N}{n}$ terms in the total 2^N terms which has n a's and $(N-n)$ b's, and we have

$$(a+b)^N = \sum_{n=0}^{N} \binom{N}{n} a^n b^{N-n}. \tag{2}$$

[1] It's actually possible to have $r^{(n)} = 1$, because the binary $0.11111\cdots$ is actually 1, in the same way $0.9999\cdots = 1$ in the decimal case. But the probability to have such a $r^{(n)}$ is $1/2 \times 1/2 \times \cdots = 0$. That is, the event that $r^{(n)} = 1$ is possible but is a null set.

Similarly, if we consider the probabilistic distribution of

$$X_{m,N} = \sum_{k=1}^{N} B_{m,k}, \tag{3}$$

we will find the probability of the event that $X_{m,N} = x$ is the sum of the probability of all outputs of $\{B_{m,k}\}$ in which there are x 1 outputs and $N - x$ 0 outputs, and for each possible output, the probability is

$$p(1)^x p(0)^{N-x} = m^x (1-m)^{N-x},$$

and we have

$$p_{m,N}(x) = \binom{N}{x} m^x (1-m)^{N-x}. \tag{4}$$

So the relation between the binomial distribution and the $(a+b)^N$ expansion is they both involve the notion of "picking x points from N points". Indeed, by considering the normalization condition of (4), which is

$$1 = \sum_x p_{m,N}(x) = \sum_{x=0}^{N} \binom{N}{x} m^x (1-m)^{N-x}, \tag{5}$$

we rediscover the expansion of $(a+b)^N$, where we set $a = m$ and $b = 1 - m$.

Exercise 3 in chapter 2 (**) (a) Show that the above expression (2.15) for $w(x,t)$ with $t > 0$ satisfies this equation. (b) By using a double Fourier transform in x and t show that the Green's function of the Smoluchowsky equation (2.26) is indeed the above expression (2.15) for $w(x,t)$ with $t \geq 0$.

Solution

(a) From (2.15) we have

$$\begin{aligned}\frac{\partial}{\partial t} w(x,t) &= -\frac{1}{2}\sqrt{\frac{1}{4\pi D t^3}} \mathrm{e}^{-\frac{(x-v_d t)^2}{4Dt}} - \sqrt{\frac{1}{4\pi D t}} \mathrm{e}^{-\frac{(x-v_d t)^2}{4Dt}} \frac{1}{4Dt^2}(2v_d(v_d t - x)t - (x - v_d t)^2) \\ &= -\sqrt{\frac{1}{4\pi D t}} \mathrm{e}^{-\frac{(x-v_d t)^2}{4Dt}} \left(\frac{1}{2t} + \frac{(v_d t - x)(v_d t + x)}{4Dt^2}\right),\end{aligned}$$

$$\frac{\partial}{\partial x} w(x,t) = -\sqrt{\frac{1}{4\pi D t}} \mathrm{e}^{-\frac{(x-v_d t)^2}{4Dt}} \frac{x - v_d t}{2Dt},$$

and

$$\frac{\partial^2}{\partial x^2} w(x,t) = -\sqrt{\frac{1}{4\pi D t}} \mathrm{e}^{-\frac{(x-v_d t)^2}{4Dt}} \left(\frac{1}{2Dt} - \left(\frac{x - v_d t}{2Dt}\right)^2\right),$$

The RHS of the Smoluchowski equation is

$$\begin{aligned} D\frac{\partial^2 w}{\partial x^2} - v_d \frac{\partial w}{\partial x} &= -\sqrt{\frac{1}{4\pi D t}} \mathrm{e}^{-\frac{(x-v_d t)^2}{4Dt}} \left(\frac{1}{2t} - \frac{(x - v_d t)^2}{4Dt^2} - v_d \frac{x - v_d t}{2Dt}\right) \\ &= -\sqrt{\frac{1}{4\pi D t}} \mathrm{e}^{-\frac{(x-v_d t)^2}{4Dt}} \left(\frac{1}{2t} - \frac{x^2 - v_d^2 t^2}{4Dt^2}\right),\end{aligned}$$

so we have

$$\frac{\partial}{\partial x} w(x,t) = D\frac{\partial^2 w}{\partial x^2} - v_d \frac{\partial w}{\partial x}.$$

(b) The initial condition is

$$\lim_{t \to 0} w = \delta(x),$$

which can be imposed to (2.26) by adding an "impact":

$$\frac{\partial w}{\partial t} = D\frac{\partial^2 w}{\partial x^2} - v_d \frac{\partial w}{\partial x} + \delta(x)\delta(t). \tag{6}$$

Now by Fourier transformation we have

$$w(x,t) = \int \frac{\mathrm{d}k\,\mathrm{d}\omega}{(2\pi)^2} \mathrm{e}^{-\mathrm{i}(\omega t - kx)} \tilde{w}(k,\omega),$$
$$-\mathrm{i}\omega\tilde{w} = D(\mathrm{i}k)^2\tilde{w} - \mathrm{i}kv_d\tilde{w} + 1.$$

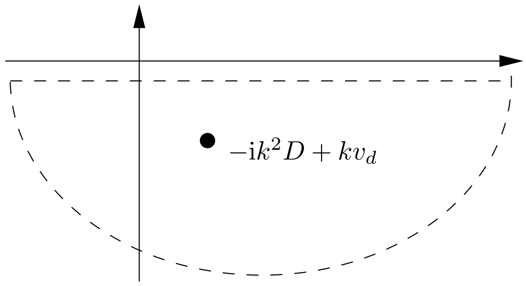
We find

$$\tilde{w} = \frac{1}{-\mathrm{i}\omega + k^2D + \mathrm{i}kv_d},$$

and thus

$$w(x,t) = \int \frac{\mathrm{d}k\,\mathrm{d}\omega}{(2\pi)^2} \mathrm{e}^{-\mathrm{i}(\omega t - kx)} \frac{1}{-\mathrm{i}\omega + k^2D + \mathrm{i}kv_d}.$$

We first complete the integral over ω, with the following contour:



$$\int \mathrm{d}\omega\, \mathrm{e}^{-\mathrm{i}(\omega t - kx)} \frac{1}{\omega + \mathrm{i}Dk^2 - kv_d} = -2\pi\mathrm{i}\mathrm{e}^{-\mathrm{i}(-\mathrm{i}k^2Dt + kv_dt - kx)}.$$

Thus

$$\begin{aligned} w(x,t) &= \frac{\mathrm{i}}{(2\pi)^2} \int \mathrm{d}k\,(-2\pi\mathrm{i})\mathrm{e}^{-\mathrm{i}(-\mathrm{i}k^2Dt + kv_dt - kx)} \\ &= \frac{1}{2\pi} \int \mathrm{d}k\, \mathrm{e}^{-k^2Dt - \mathrm{i}k(v_dt - x)} \\ &= \frac{1}{2\pi} \cdot \sqrt{\frac{2\pi}{2Dt}} \mathrm{e}^{\frac{1}{2}\frac{1}{2Dt}(-\mathrm{i}(v_dt - x))^2} \\ &= \sqrt{\frac{1}{4\pi Dt}} \mathrm{e}^{-\frac{(x - v_dt)^2}{4Dt}}. \end{aligned}$$

This is exactly (2.15).

Exercise 5 in chapter 2 (**) Explain in detail how, by measuring for the first time the position diffusion constant of a small Brownian sphere immerged in water, the physicist Jean Perrin, using the Einstein relation, was able to measure Avogadro's Number N_A, thereby confirming the existence of atoms (Jean Perrin received the Nobel prize for this work in 1926, see his Nobel lecture on the Nobel website). Use Stokes' law stating that a sphere of radius R moving at a velocity V feels in a fluid with viscosity η a frictional force

$$F = 6\pi R\eta V$$

Remember that Avogadro's Number N_A is involved in the ideal gas constant, defined by the relation

$$\frac{\text{pressure volume}}{\text{temperature}} = nR_{\text{ig}}$$

where n is the number of moles of the volume of gas considered. In the kinetic theory of gases, R_{ig} is given by

$$R_{\text{ig}} = N_Ak_B$$

Solution The Stokes' law

$$F = 6\pi R\eta v \tag{7}$$

connects two physical quantities arising from the same dissipation process in the fluid: the viscosity μ and the response coefficient

$$\mu = \frac{v}{F}. \tag{8}$$

The relation between the two is imposed by the Navier-Stokes equation. Since we also have

$$\mu = \frac{D}{k_{\mathrm{B}}T}, \tag{9}$$

we have

$$\frac{1}{6\pi R\eta} = \frac{D}{k_{\mathrm{B}}T}. \tag{10}$$

This equation can be used to measure k_{B}: each quantities involved in the equation can be measured separately. The viscosity η can be measured by standard fluid dynamic methods. The radius R can be measured by letting the particles fall in the fluid and recording its terminal velocity, and then we have

$$R = \frac{mg}{6\pi\eta v_{\mathrm{terminal}}}. \tag{11}$$

The diffusion coefficient D can be measured by looking at the trajectory of a Brownian particle. The temperature is measured by a thermometer. Now we find k_{B}, and by the ideal gas equation

$$pV = nR_{\mathrm{ig}}T \tag{12}$$

we can measure R_{ig}, so finally, by

$$R_{\mathrm{ig}} = N_A k_{\mathrm{B}}, \tag{13}$$

the Avogadro constant is found.

Exercise 4 in lecture 3 Treat the case of the Shrapnell process in dimension 2.

Solution Now the damage is

$$X = \frac{\Omega}{r}, \tag{14}$$

and the probability per unit surface is

$$n = \frac{1}{\pi R^2}.$$

The condition $X < x$ is equivalent to

$$r > \frac{\Omega}{x}. \tag{15}$$

We have

$$p(r > \Omega/x) = \frac{R^2 - (\Omega/x)^2}{R^2}, \tag{16}$$

so the probability density is

$$w(x) = \frac{\mathrm{d}p(r > \Omega/x)}{\mathrm{d}x} = \frac{2\Omega^2}{R^2x^3}. \tag{17}$$

There is a minimum of X: it's Ω/R, because explosion doesn't happen outside the circle. Now the first momentum is

$$\langle X\rangle = \int_{\Omega/R}^{\infty} \mathrm{d}x\, x \frac{2\Omega^2}{R^2x^3} = \frac{2\Omega}{R} < \infty, \tag{18}$$

and the second momentum is

$$\langle X\rangle = \int_{\Omega/R}^{\infty} \mathrm{d}x\, x^2 \frac{2\Omega^2}{R^2x^3} = \infty. \tag{19}$$

So still the high order momenta of the variable diverges, and thus the central limit theorem fails.

Problem 2

Solution

(a) For a single bit we have

$$H(\mathrm{B}_{1/2}) = -\frac{1}{2}\log_2\frac{1}{2} - \frac{1}{2}\log_2\frac{1}{2} = 1. \tag{20}$$

For N independent bits, we have

$$H(\otimes_{j=1}^{N}\mathrm{B}_{1/2,j}) = -\sum_i \left(\frac{1}{2}\right)^N \log_2\left(\frac{1}{2}\right)^N = -2^N \times \frac{1}{2^N} \times (-N) = N. \tag{21}$$

(b) We have

$$S = -\frac{\partial F}{\partial T}, \quad F = -k_{\mathrm{B}}T\ln Z.$$

Now with the definition of the partition function

$$Z = \sum_i \mathrm{e}^{-E_i/k_{\mathrm{B}}T}, \tag{22}$$

we have

$$\begin{aligned}
\frac{\partial}{\partial T}T\ln Z &= \ln Z + \frac{T}{Z}\frac{\partial Z}{\partial T} \\
&= \ln Z + \frac{T}{Z}\sum_i \frac{E_i}{k_{\mathrm{B}}T^2}\mathrm{e}^{-E_i/k_{\mathrm{B}}T} \\
&= \ln Z + \frac{1}{k_{\mathrm{B}}T}\sum_i p_i E_i.
\end{aligned}$$

Thus

$$S = -k_{\mathrm{B}}\ln Z - \frac{1}{T}\sum_i p_i E_i = -k_{\mathrm{B}}\ln Z - \frac{\langle E\rangle}{T}. \tag{23}$$

(c) In the high temperature limit $E_i/k_{\mathrm{B}}T \to 0$ for every E_i, so energy is no longer important in determining the probabilistic distribution and each configuration has the same probability. The energy of N indistinguishable random bits, in this case, is therefore

$$E = \sum_{j=1}^{N}\mathrm{B}_{1/2,j}. \tag{24}$$

The probability of $E = \epsilon$ is

$$p(E=\epsilon) = \binom{N}{\epsilon} \times \frac{1}{2^N}, \tag{25}$$

which reaches its peak when $\epsilon = N/2$, so

$$\mathcal{E} = N/2. \tag{26}$$

There are $\binom{N}{\epsilon}$ microstates in the macrostate $(N, \mathcal{C})$, so

$$\begin{aligned}
\lim_{N\to\infty}\frac{1}{N}S(N,\mathcal{E}) &= k_{\mathrm{B}}\frac{1}{N}\ln\binom{N}{N/2} \\
&= \frac{k_{\mathrm{B}}}{N}(\ln N! - 2\ln(N/2)!) \\
&\approx \frac{k_{\mathrm{B}}}{N}(N\ln N - 2(N/2)\ln N/2) \\
&= k_{\mathrm{B}}\ln 2.
\end{aligned}$$

Thus

$$\lim_{N\to\infty}\frac{1}{N}S(N,\mathcal{E}) = k_{\mathrm{B}}\ln 2 = k_{\mathrm{B}}\ln 2 \times S_{\mathrm{Shannon}}(\mathrm{B}_{1/2}). \tag{27}$$

(d)

(f) The terms in the RHS are

$$H(p_1+p_2,\dots,p_M) = -(p_1+p_2)\log_2(p_1+p_2) - \sum_{i\geq 3} p_i \log_2 p_i,$$

and

$$\begin{aligned}
&(p_1+p_2)H\left(\frac{p_1}{p_1+p_2}, \frac{p_2}{p_1+p_2}\right) \\
&= -(p_1+p_2)\left(\frac{p_1}{p_1+p_2}\log_2\frac{p_1}{p_1+p_2} + \frac{p_2}{p_1+p_2}\log\frac{p_2}{p_1+p_2}\right) \\
&= -p_1\log_2 p_1 + p_1\log(p_1+p_2) - p_2\log p_2 + p_2\log(p_1+p_2),
\end{aligned}$$

so the RHS is

$$-p_1\log_2 p_1 - p_2\log_2 p_2 - \sum_{i\geq 3} p_i\log_2 p_i,$$

which is just the LHS. So

$$H(p_1,p_2,\dots,p_M) = H(p_1+p_2,\dots,p_M) + (p_1+p_2)H\left(\frac{p_1}{p_1+p_2}, \frac{p_2}{p_1+p_2}\right). \tag{28}$$

For example we have

$$H(1/2,1/4,1/4) = H(1/2,1/2) + \frac{1}{2}H(1/2,1/2) = 1 + \frac{1}{2} = 1.5. \tag{29}$$

(28) means the Shannon entropy is additive to some extent: if we first decide to ignore the difference between ξ_1 and ξ_1, then the resulting entropy is $H(p_1+p_2,\dots,p_M)$. To recover the original entropy, we just need to calculate the "inner" entropy of the ξ_1-or-ξ_2 possibility – that is, to calculate $H(p_1/(p_1+p_2), p_2/(p_1+p_2))$ – and then multiply a (p_1+p_2) weight to it, and after putting the two parts of entropies together, we get the original entropy corresponding to the full amount of information.

(g) We need to take the $\alpha\to 1$ limit of

$$H_\alpha(X) = \frac{1}{1-\alpha}\log_2\left(\sum_i p_i^\alpha\right). \tag{30}$$

When $\alpha=1$, both the numerator ($\log_2 1 = 0$) and the denominator are zero, so we can use the L'Hospital's rule:

$$\lim_{\alpha\to 1} H_\alpha = \lim_{\alpha\to 1}\frac{\frac{\sum_i \ln p_i p_i^\alpha}{\ln 2\sum_i p_i^\alpha}}{-1} = -\sum_i p_i\log p_i = H(X).$$

(h) The eigenvalues of ρ are $(1\pm|\boldsymbol{a}|)/2$. Thus

$$\begin{aligned}
H_1(\rho) &= -\rho_1\log_2\rho_1 - \rho_2\log_2\rho_2 \\
&= 1 - \frac{1+|\boldsymbol{a}|}{2}\log_2(1+|\boldsymbol{a}|) - \frac{1-|\boldsymbol{a}|}{2}\log_2(1-|\boldsymbol{a}|),
\end{aligned} \tag{31}$$

and

$$\begin{aligned}
H_2(\rho) &= -\log_2\left(\frac{(1+|\boldsymbol{a}|)^2}{4} + \frac{(1-|\boldsymbol{a}|)^2}{4}\right) \\
&= -\log_2\frac{1+|\boldsymbol{a}|^2}{2}.
\end{aligned} \tag{32}$$

The maximum 1 is reached when $|\boldsymbol{a}|=0$, and the minimum 0 is reached when $|\boldsymbol{a}|=1$. When $|\boldsymbol{a}|=0$, ρ is essentially a classical 50 %-50 % probabilistic distribution, so the entropy is the same as the entropy of a random bit, which is 1. This is the most "noisy" case and indeed we get a maximal entropy here. When $|\boldsymbol{a}|=1$, ρ is a pure state, and there is nothing uncertain about it, so its entropy is 0, which is the minimum.