Vector analysis

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1 Trajectory

Suppose we have a vector function $\boldsymbol{F}(t)$, which maps the time to the position in a space, usually $\mathbb{R}^3$. The length element of the trajectory of $\boldsymbol{F}$ is

$$\mathrm{d}s = \mathrm{d}|\boldsymbol{F}| = \left|\frac{\mathrm{d}\boldsymbol{F}}{\mathrm{d}t}\right| \mathrm{d}t\,. \tag{1}$$

We define

$$\boldsymbol{T} = \frac{\mathrm{d}\boldsymbol{F}}{\mathrm{d}t}, \quad \hat{\boldsymbol{T}} = \frac{\mathrm{d}\boldsymbol{F}/\mathrm{d}t}{|\mathrm{d}\boldsymbol{F}/\mathrm{d}t|}. \tag{2}$$

It's easy to find $\boldsymbol{T}$ is a tangent vector of the trajectory of $\boldsymbol{F}$. Since the length of $\boldsymbol{T}$ is fixed to unity, from the fact that

$$\frac{\mathrm{d}\hat{\boldsymbol{T}} \cdot \hat{\boldsymbol{T}}}{\mathrm{d}t} = 0,$$

we find $\hat{\boldsymbol{T}}$ – and therefore $\boldsymbol{T}$ – is orthogonal to

$$\boldsymbol{N} = \hat{\boldsymbol{T}}', \quad \hat{\boldsymbol{N}} = \frac{\hat{\boldsymbol{T}}'}{|\hat{\boldsymbol{T}}'|}. \tag{3}$$

We define

$$\kappa = |\boldsymbol{F}''| = |\boldsymbol{T}'| = |\boldsymbol{N}|\,. \tag{4}$$

Now the acceleration of $\boldsymbol{F}$ can be decomposed into a clear form:

$$\frac{\mathrm{d}^2\boldsymbol{F}}{\mathrm{d}t^2} = \frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{\mathrm{d}\boldsymbol{F}}{\mathrm{d}s}\frac{\mathrm{d}s}{\mathrm{d}t}\right)\frac{\mathrm{d}s}{\mathrm{d}t} = \frac{\mathrm{d}^2\boldsymbol{F}}{\mathrm{d}s^2}\left(\frac{\mathrm{d}s}{\mathrm{d}t}\right)^2 + \frac{\mathrm{d}\boldsymbol{F}}{\mathrm{d}s}\frac{\mathrm{d}^2 s}{\mathrm{d}t^2} = \kappa v^2 \hat{\boldsymbol{N}} + \frac{\mathrm{d}v}{\mathrm{d}t}\hat{\boldsymbol{T}}. \tag{5}$$

In the above derivation we have used the facts

$$\frac{\mathrm{d}\boldsymbol{F}}{\mathrm{d}s} = \frac{\mathrm{d}\boldsymbol{F}}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}s} = \frac{\mathrm{d}\boldsymbol{F}}{\mathrm{d}t}\frac{1}{|\frac{\mathrm{d}\boldsymbol{F}}{\mathrm{d}t}|} = \hat{\boldsymbol{T}}, \tag{6}$$

and

$$\frac{\mathrm{d}^2\boldsymbol{F}}{\mathrm{d}s^2} = \frac{\mathrm{d}t}{\mathrm{d}s}\frac{\mathrm{d}\hat{\boldsymbol{T}}}{\mathrm{d}t} = \tag{7}$$

We can also define a vector orthogonal to both TODO

$$\boldsymbol{B} = \tag{8}$$

2 Flow

Now we consider a vector field $\boldsymbol{F}(x, y, z)$ defined *everywhere* in space, which may be understood as a flow field. The trajectory of a particle $\boldsymbol{R}(t)$ follows $\boldsymbol{F}$. This means

$$\frac{\mathrm{d}\boldsymbol{R}}{\mathrm{d}t} \propto \boldsymbol{F} \Rightarrow \frac{\mathrm{d}x}{f} = \frac{\mathrm{d}y}{g} = \frac{\mathrm{d}z}{h}. \tag{9}$$