Homework 2

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Problem 1 Suppose we have a quantum rotor (or particle on a circle) with the following imaginary-time action:

$$S[\varphi] = \int_0^T \mathrm{d}\tau \left(\frac{1}{2}\dot{\varphi}^2 - i\frac{\theta}{2\pi}\dot{\varphi} + V(\varphi) \right).$$

Here the potential $V(\varphi) = V_0(1 - \cos\varphi)$. We will use semi-classical approximation to study the θ dependence of the ground state energy. For this purpose it is useful to consider the boundary condition $\varphi_i = \varphi_f = 0$, so the path integral computes the imaginary-time propagator $\left\langle 0 \middle| e^{-T\hat{H}} \middle| 0 \right\rangle$. 1. Show that the equation of motion admits instanton solutions corresponding to the particle going around the circle. Calculate the action of a single instanton $S_0(\theta)$. Hint: there are two distinct single instaton solutions. 2. Calculate the path integral using semi-classical approximation, by summing over all instanton solutions under the "dilute gas" approximation. Following what we did in class for double well potential, you may introduce a parameter K for the contributions of Gaussian fluctuations and assume that K does not depend on θ. You do not need to compute K. 3. Determine the θ dependence of the ground state energy E_0.

Solution

1. The EOM is

$$\frac{\mathrm{d}}{\mathrm{d}t}(\dot{\varphi} - \mathrm{i}\theta/2\pi) = \frac{\mathrm{d}}{\mathrm{d}\varphi} V_0(1 - \cos\varphi),$$
$$\ddot{\varphi} = V_0 \sin\varphi. \tag{1}$$

Integrating over φ, we have

$$\frac{1}{2}\dot{\varphi}^2 = -V_0\cos\varphi + C. \tag{2}$$

The range of C is between $\pm V_0$, because it corresponds to the φ when $\dot{\varphi} = 0$. The boundary condition that when $\tau \to \pm\infty$, φ stays zero, so we have $\varphi = 0$ and $\dot{\varphi} = 0$ in the two limits, so $C = V_0$, and therefore

$$\frac{1}{2}\dot{\varphi}^2 = \underbrace{V_0(1 - \cos\varphi)}_{V(\varphi)}, \tag{3}$$

$$\pm\sqrt{2V_0}\tau = \int \frac{\mathrm{d}\varphi}{\sqrt{1 - \cos\varphi}} = \sqrt{2}\ln\tan\frac{\varphi}{4},$$
$$\varphi = 4\arctan \mathrm{e}^{\pm\sqrt{V_0}(\tau - \tau_0)}.$$

By checking continuity and the $\tau \to \pm\infty$ limits, we find

$$\varphi_+ = 4\arctan \mathrm{e}^{\sqrt{V_0}(\tau-\tau_0)}, \quad \varphi_- = 4\arctan \mathrm{e}^{-\sqrt{V_0}(\tau-\tau_0)} \tag{4}$$

are exactly the two single instanton solutions we need – there is no need "cut and connect" branches of solutions.

We have

$$\begin{aligned} S[\varphi] &= \int \mathrm{d}\tau \left(\frac{1}{2}\dot{\varphi}^2 - \mathrm{i}\frac{\theta}{2\pi}\dot{\varphi} + V(\varphi) \right) \\ &= -\mathrm{i}\frac{\theta}{2\pi}(\varphi(\infty) - \varphi(-\infty)) + 2\int \mathrm{d}\tau\, V(\varphi). \end{aligned}$$

For φ_+, the first term is $-\mathrm{i}\theta$, while for φ_-, the first term is $\mathrm{i}\theta$. For φ_+, the second term is

$$\begin{aligned} & 2V_0 \int_{-\infty}^{\infty} \mathrm{d}\tau \left(1 - \cos\left(4\arctan \mathrm{e}^{\sqrt{V_0}(\tau-\tau_0)}\right)\right) \\ &= 2\sqrt{V_0}\int_{-\infty}^{\infty} \mathrm{d}x\,(1 - \cos(4\arctan \mathrm{e}^x)) \\ &= 8\sqrt{V_0}. \end{aligned}$$

The same is true for φ_- because of the time reversal symmetry. So we have

$$S_{0,+}(\theta) = -\mathrm{i}\theta + 8\sqrt{V_0}, \quad S_{0,-}(\theta) = \mathrm{i}\theta + 8\sqrt{V_0}. \tag{5}$$

2. The saddle point approximation, without considering the instantons, gives

$$U(0,T;0,0) = \sqrt{\frac{m\omega}{2\pi \sinh \omega T}}, \tag{6}$$

where the oscillation frequency is just

$$\omega = \sqrt{V_0}. \tag{7}$$

Now we insert instantons into the paths taken into consideration. We make the dilute instanton gas approximation, assuming that the total time T and the distances between instantons are largely enough compared with the temporal size of each instanton ($\sim 1/\sqrt{V_0}$), and in this case, action has additivity, and the contribution to the action of each instanton is approximately the same as the action of the instanton with the $-\infty < \tau < \infty$ time span, which we just evaluated in (5). So for a configuration with n_+ φ_+ instantons and n_- φ_- instantons, the total saddle-point action is

$$K^{n_1+n_2}\mathrm{e}^{-n_+S_{0,+}-n_-S_{0,-}}.$$

The number of the possible orders of the instantons is $\binom{n_1+n_2}{n_1}$, so the path integral is[1]

$$\begin{aligned}
&\sum_{n_-,n_+}\int_0^T \mathrm{d}\tau_1 \int_{\tau_1}^T \mathrm{d}\tau_2 \cdots \int_{\tau_{n-1}}^T \mathrm{d}\tau_n \binom{n_+ + n_-}{n_+} K^{n_+ + n_-}\mathrm{e}^{-n_+S_{0,+}-n_-S_{0,-}}U(0,T;0,0)\\
&= U(0,T;0,0)\sum_{n_-,n_+}\frac{T^{n_+ + n_-}}{(n_+ + n_-)!}\frac{(n_+ + n_-)!}{n_+!n_-!}K^{n_+ + n_-}\mathrm{e}^{-n_+S_{0,+}-n_-S_{0,-}}\\
&= U(0,T;0,0)\sum_{n_+}\frac{(TK\mathrm{e}^{-S_{0,+}})^{n_+}}{n_+!}\sum_{n-}\frac{(TK\mathrm{e}^{-S_{0,-}})^{n_-}}{n_-!}\\
&= U(0,T;0,0)\mathrm{e}^{TK\mathrm{e}^{-S_{0,+}}}\mathrm{e}^{TK\mathrm{e}^{-S_{0,-}}}.
\end{aligned}$$

So we get

$$\langle 0|\mathrm{e}^{-HT}|0\rangle = U(0,T;0,0)\mathrm{e}^{TK\mathrm{e}^{-S_{0,+}}}\mathrm{e}^{TK\mathrm{e}^{-S_{0,-}}}. \tag{8}$$

3. When $T\to\infty$, we know (in the last homework)

$$U(0,T;0,0) \sim \mathrm{e}^{-\frac{1}{2}\omega T}, \tag{9}$$

Since in the long run

$$\langle 0|\mathrm{e}^{-HT}|0\rangle \sim \mathrm{e}^{-\frac{1}{2}\omega T}\mathrm{e}^{TK(\mathrm{e}^{-S_{0,+}}+\mathrm{e}^{-S_{0,-}})} =: \mathrm{e}^{-ET}, \tag{10}$$

we have

$$\begin{aligned}
E &= \frac{1}{2}\omega - K(\mathrm{e}^{-S_{0,+}} + \mathrm{e}^{-S_{0,-}})\\
&= \frac{1}{2}\omega - K\mathrm{e}^{-8\sqrt{V_0}}(\mathrm{e}^{\mathrm{i}\theta} + \mathrm{e}^{-\mathrm{i}\theta})\\
&= \frac{1}{2}\sqrt{V_0} - 2K\mathrm{e}^{-8\sqrt{V_0}}\cos\theta.
\end{aligned} \tag{11}$$

So the ground state energy oscillates with respect to θ.

[1] Note that here both K and $U(0,T;0,0)$ have something to do with harmonic oscillators. $U(0,T;0,0)$ is the "global" harmonic fluctuation, while K is the fluctuation of an instanton, which can also be seen as a normalization coefficient linking

$$\int \mathrm{d}\tau_1 \int \mathrm{d}\tau_2 \cdots \int \mathrm{d}\tau_n$$

and $\mathcal{D}x$.

Problem 2 Consider a harmonic oscillator $\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega_0^2\hat{x}^2$, and let $j = e\dot{x}$ be the current operator. 1. Use the path integral formalism to calculate the (time-ordered) current correlation function $iG_{jj}(t) = \langle 0|\mathcal{T}(j(t)j(0))|0\rangle$, where $|0\rangle$ is the ground state. Find its form in frequency space $G_{jj}(\omega) = \int dt G_{jj}(t)e^{i\omega t}$. 2. Let $\sigma(\omega)$ be the finite frequency conductance, defined by $j(\omega) = \sigma(\omega)E(\omega)$, where $E(t)$ is a perturbing electric field: $H \to H - eE(t)x$. Compute $\sigma(\omega)$ for the classical harmonic oscillator, adding in a small amount of friction by hand. 3. Show that for $\omega > 0, \sigma(\omega)$ is of the form $\sigma(\omega) = C\frac{G_{jj}(\omega)}{\omega}$, and find the constant C. 4. Show that $\sigma(\omega)$ has a nonzero real part only when $\omega = \omega_0$. At that frequency the oscillator can absorb energies by jumping to higher excited states. This example demonstrates that a finite real DC conductance requires gapless excitations.

Solution

1. We have

$$\mathcal{T}\,\langle 0|j(t)j(0)|0\rangle = \langle j(t)j(0)\rangle\,,$$

where

$$\langle\cdots\rangle := \frac{\int \mathcal{D}x(\cdots)\mathrm{e}^{\mathrm{i}\int \mathrm{d}tL}}{\int \mathcal{D}x\mathrm{e}^{\mathrm{i}\int \mathrm{d}tL}}.$$

So we can do Fourier expansion to $j(t)$ without fears of details of normal ordering. We have

$$\begin{aligned} j(t) &= \frac{\mathrm{d}}{\mathrm{d}t}\int \frac{\mathrm{d}\omega}{2\pi}\mathrm{e}^{-\mathrm{i}\omega t}ex(\omega) \\ &= \int \frac{\mathrm{d}\omega}{2\pi}\mathrm{e}^{-\mathrm{i}\omega t}(-\mathrm{i}\omega)ex(\omega), \end{aligned}$$

so

$$\begin{aligned} \int \mathrm{d}t\,\mathrm{e}^{\mathrm{i}\omega t}\langle j(t)j(0)\rangle &= \frac{1}{2\pi\delta(0)}\int \mathrm{d}t\,\mathrm{e}^{\mathrm{i}\omega t}\int \mathrm{d}t_2\,\langle j(t+t_2)j(t_2)\rangle \\ &= \frac{1}{2\pi\delta(0)}\int \mathrm{d}t\,\mathrm{e}^{\mathrm{i}\omega t}\int \mathrm{d}t_2\int\frac{\mathrm{d}\omega_1}{2\pi}(-e\mathrm{i}\omega_1)\mathrm{e}^{-\mathrm{i}\omega_1(t+t_2)} \\ &\quad\cdot\int\frac{\mathrm{d}\omega_2}{2\pi}(-\mathrm{i}e\omega_2)\mathrm{e}^{-\mathrm{i}\omega_2 t_2}\langle j(\omega_1)j(\omega_2)\rangle \\ &= \frac{1}{2\pi\delta(0)}\int\frac{\mathrm{d}\omega_1}{2\pi}\int\frac{\mathrm{d}\omega_2}{2\pi}(-\mathrm{i}e\omega_1)(-\mathrm{i}e\omega_2)2\pi\delta(\omega-\omega_1)2\pi\delta(\omega_1+\omega_2)\langle j(\omega_1)j(\omega_2)\rangle \\ &= e^2\frac{1}{2\pi}\omega^2\langle j(\omega)j(-\omega)\rangle\,. \end{aligned}$$

On the other hand, we have

$$\int\frac{\mathrm{d}\omega}{2\pi}\langle x(t)x(0)\rangle = \frac{1}{2\pi}\langle x(\omega)x(-\omega)\rangle\,,$$

and thus

$$\mathrm{i}G_{jj}(\omega) = \int \mathrm{d}t\,\mathrm{e}^{\mathrm{i}\omega t}\langle j(t)j(0)\rangle = e^2\omega^2\int \mathrm{d}t\,\mathrm{e}^{\mathrm{i}\omega t}\langle x(t)x(0)\rangle = e^2\omega^2\frac{\mathrm{i}}{m(\omega^2-\omega_0^2+\mathrm{i}\epsilon)}, \tag{12}$$

$$G_{jj}(\omega) = \frac{e^2\omega^2}{m(\omega^2-\omega_0^2+\mathrm{i}\epsilon)}. \tag{13}$$

2. The EOMs are

$$\dot{x} = \frac{\partial H}{\partial p} = \frac{p}{m},\quad \dot{p} = -\frac{\partial H}{\partial x} = -m\omega_0^2 + eE,$$
$$m\ddot{x} + m\omega_0^2 x = eE.$$

After adding a small friction we get

$$m\ddot{x} + m\epsilon\dot{x} + m\omega_0^2 x = eE. \tag{14}$$

Again by Fourier transformation

$$x(t) = \int\frac{\mathrm{d}\omega}{2\pi}\mathrm{e}^{-\mathrm{i}\omega t}x(\omega),\quad E(t) = \int\frac{\mathrm{d}\omega}{2\pi}\mathrm{e}^{-\mathrm{i}\omega t}E(\omega),$$

we have

$$(-m\omega^2 + m\omega_0^2 - \mathrm{i}m\epsilon\omega)x(\omega) = eE(\omega),$$

$$\sigma(\omega) = \frac{j(\omega)}{E(\omega)} = -\mathrm{i}\omega e\frac{x(\omega)}{E(\omega)} = \mathrm{i}\frac{e^2\omega}{m(\omega^2 - \omega_0^2 + \mathrm{i}\,\mathrm{sgn}(\omega)\epsilon)}. \tag{15}$$

3. So when $\omega > 0$, $\mathrm{sgn}(\omega)\epsilon$ is just 0^+, and we get

$$\sigma(\omega) = C\frac{G_{jj}(\omega)}{\omega}, \quad C = \mathrm{i}. \tag{16}$$

This is expected: the correlation function corresponding to $\sigma(\omega)$ is $G_{j,ex}$, not G_{jj}. The two all contain a e^2 factor but they differ with a time derivative, which is the origin of the $-\mathrm{i}\omega$ in the denominator.

4. We have

$$\begin{aligned}\sigma(\omega) &= \mathrm{i}\frac{e^2}{m}\omega\left(\mathrm{P}\frac{1}{\omega^2 - \omega_0^2} - \pi\mathrm{i}\,\mathrm{sgn}(\omega)\delta(\omega^2 - \omega_0^2)\right) \\ &= \frac{\pi e^2}{m}\omega\delta(\omega^2 - \omega_0^2) + \mathrm{i}\frac{e^2\omega}{m}\mathrm{P}\frac{1}{\omega^2 - \omega_0^2}.\end{aligned} \tag{17}$$

So the real part is non-zero only when $\omega = \omega_0$.

Problem 3 Consider a system of two coupled harmonic oscillators:

$$L[x, X] = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}m\omega_0^2 x^2 + \frac{1}{2}M\dot{X}^2 - \frac{1}{2}M\Omega_0^2 X^2 - gxX.$$

Assume that $\Omega_0 \gg \omega_0$. In this regime, X is a high energy degree of freedom while x is a low energy "soft" degree of freedom. The two are coupled by the gxX term. 1. Find the low energy effective theory L_{eff} that describes the low energy dynamics of the soft degree of freedom x by writing down the path integral for the system and integrating out X (You should include at least the leading $1/\Omega_0$ corrections). In particular, find the effective mass m^* and effective spring constant $m^*\left(\omega_0^*\right)^2$. What happens when g becomes large? 2. Express X and $\dot{X}$ in terms of the variables in the effective theory. (Hint: introduce a coupling EX in the theory). 3. At what energy scale is the effective theory a good description of the system?

Solution

1. The path integral is

$$\begin{aligned}Z &= \int \mathcal{D}X\mathcal{D}x \mathrm{e}^{\mathrm{i}\int \mathrm{d}t\left(\frac{1}{2}m\dot{x}^2 - \frac{1}{2}m\omega_0^2x^2 + \frac{1}{2}M\dot{X}^2 - \frac{1}{2}M\Omega_0^2X^2 - gxX\right)} \\ &= \int \mathcal{D}x\mathrm{e}^{\mathrm{i}\int\mathrm{d}t\left(\frac{1}{2}m\dot{x}^2 - \frac{1}{2}m\omega_0^2x^2\right)}\int\mathcal{D}X\mathrm{e}^{\mathrm{i}\int\mathrm{d}t\left(\frac{1}{2}M\dot{X}^2 - \frac{1}{2}M\Omega_0^2X^2 - gxX\right)}.\end{aligned} \tag{18}$$

We need to integrate out the X variable to obtain an effective theory for x. We have

$$\begin{aligned}&\int\mathcal{D}X\mathrm{e}^{\mathrm{i}\int\mathrm{d}t\left(\frac{1}{2}M\dot{X}^2 - \frac{1}{2}M\Omega_0^2X^2 - gxX\right)} \\ &= \int\mathcal{D}X\mathrm{e}^{\mathrm{i}\int\mathrm{d}t\left(-\frac{1}{2}MX(\partial_t^2 + \Omega_0^2)X - gxX\right)} \\ &= \mathrm{const}\cdot\exp\left(\mathrm{i}\int\,\mathrm{d}t\,\frac{1}{2}g^2x\frac{1}{M(\partial_t^2 + \Omega_0^2)}x\right) \\ &= \mathrm{const}\cdot\exp\left(\mathrm{i}\int\,\mathrm{d}t\,\frac{1}{2}g^2x\frac{1}{M\Omega_0^2}\left(1 - \frac{1}{\Omega_0^2}\frac{\mathrm{d}^2}{\mathrm{d}t^2} + \cdots\right)x\right) \\ &= \mathrm{const}\cdot\exp\left(\mathrm{i}\int\,\mathrm{d}t\left(\frac{g^2}{2M\Omega_0^2}x^2 + \frac{1}{2}\frac{g^2}{M\Omega_0^4}\dot{x}^2 + \cdots\right)\right).\end{aligned}$$

Only keeping the first-order correction, we have

$$m^* = m + \frac{g^2}{M\Omega_0^4}, \tag{19}$$

$$m^*(\omega_0^*)^2 = m\omega_0^2 - \frac{g^2}{M\Omega_0^2}. \tag{20}$$

When g is large, (20) becomes negative, and the theory about x becomes instable. This is faithful to the original theory, because when g is large, the potential

$$\begin{pmatrix} -\frac{1}{2}m\omega_0 & -\frac{g}{2} \\ -\frac{g}{2} & -\frac{1}{2}M\Omega_0^2 \end{pmatrix}$$

is also not stable.

2. We just need to replace gx by $gx - E$ in (18). Now after integrating out X, we get

$$\begin{aligned}
&\text{const} \cdot \exp\left(\mathrm{i}\int \mathrm{d}t\, \frac{1}{2}(gx-E)\frac{1}{M(\partial_t^2+\Omega_0^2)}(gx-E)\right) \\
&= \text{const} \cdot \exp\left(\mathrm{i}\int \mathrm{d}t\, \frac{1}{2}(gx-E)\frac{1}{M\Omega_0^2}\left(1 - \frac{1}{\Omega_0^2}\frac{\mathrm{d}^2}{\mathrm{d}t^2} + \cdots\right)(gx-E)\right) \\
&= \text{const} \cdot \exp\left(\mathrm{i}\int \mathrm{d}t\left(\frac{1}{2M\Omega_0^2}(gx-E)^2 + \frac{1}{2}\frac{1}{M\Omega_0^4}\left(\frac{\mathrm{d}(gx-E)}{\mathrm{d}t}\right)^2 + \cdots\right)\right) \\
&= \text{const} \cdot \exp\left(\mathrm{i}\int \mathrm{d}t\left(\frac{1}{2M\Omega_0^2}(gx-E)^2 + \frac{1}{2}\frac{g^2}{M\Omega_0^4}\dot{x}^2 + \cdots\right)\right).
\end{aligned} \tag{21}$$

So now the effective theory is

$$L_{\text{eff}} = \frac{1}{2}\left(m + \frac{g^2}{M\Omega_0^4}\right)\dot{x}^2 - \frac{1}{2}m\omega_0^2 x^2 + \frac{1}{2M\Omega_0^2}(gx-E)^2. \tag{22}$$

To find the expression of X, we just need to take the derivative of L_{eff} with respect to E, because to find an n-order correlation function of X, we just find the n-th derivative of Z, and if this is done with L_{eff}, then what is averaged over is just $\partial L_{\text{eff}}/\partial E$ to the n. So

$$X = \left.\frac{\partial L_{\text{eff}}}{\partial E}\right|_{E=0} = -\frac{g}{M\Omega_0^2}x, \tag{23}$$

and

$$\dot{X} = -\frac{g}{M\Omega_0^2}\dot{x}. \tag{24}$$

3. Note that integrating out X in this problem is *exact*: The place requiring approximation is not integrating out X, but only taking the first-order correction in (21). So in order for the effective theory to make sense, we just need $\Omega_0 \gg \partial_t$, or in other words

$$\Omega_0 \gg \omega_0. \tag{25}$$

Problem 4 In this problem, we will compute various correlation functions for non-interacting and interacting superfluids. For these calculations, you may find Wick's theorem useful. Let A_i be operators which are linear combinations of $\hat{a}, \hat{a}^\dagger$, and let $\langle\cdot\rangle$ denote the expectation value in the zero-boson state. Also let $:\hat{O}:$ denote the normal ordered form of $\hat{O}$ (i.e. with all the $\hat{a}^\dagger$'s to the left of $\hat{a}$'s). Then Wick's theorem for 4 operators says:

$$\begin{aligned}
T(A_1A_2A_3A_4) =&: A_1A_2A_3A_4 : + : A_1A_2 : \langle T(A_3A_4)\rangle + : A_1A_3 : \langle T(A_2A_4)\rangle + : A_1A_4 : \langle T(A_1A_4)\rangle \\
&+ : A_2A_3 : \langle T(A_2A_3)\rangle + : A_2A_4 : \langle T(A_2A_3)\rangle + : A_3A_4 : \langle T(A_3A_4)\rangle + \\
&+ \langle T(A_1A_2)\rangle\langle T(A_3A_4)\rangle + \langle T(A_1A_3)\rangle\langle T(A_2A_4)\rangle + \langle T(A_1A_4)\rangle\langle T(A_2A_3)\rangle.
\end{aligned}$$

Here T is time ordering. While the expression is long, there is a clear pattern: the first term is the product $T(A_1A_2A_3A_4)$ normal ordered, then all possible ways to normal order two of them, while the other two are replaced by their expectation value, etc. The generalization to any even number of $\hat{A}_i$'s should be straightforward. 1. First consider a system of N non-interacting bosons $\hat{H} = \sum_{\mathbf{k}} \frac{\mathbf{k}^2}{2m}\hat{a}_{\mathbf{k}}^\dagger \hat{a}_{\mathbf{k}}$ in (3D) volume $\mathcal{V}$. Let $|\Psi_0\rangle$ denote the ground state. Calculate the time-ordered correlation function

$$iG(x,t) = \left\langle \Psi_0 \left| T\left(\hat{a}(\mathbf{x},t)\hat{a}^\dagger(\mathbf{0},0)\right)\right| \Psi_0 \right\rangle.$$

For $N = 0$ and for finite N. Take the thermodynamic limit $N \to \infty, \mathcal{V} \to \infty$ but with $N/\mathcal{V} = \rho_0$ a constant. Show that $iG(x,t) \to \text{const}$ as $x \to \infty$ at a fixed t. This shows that the Bose-Einstein condensation has long range order. 2. Calculate the time ordered density-density correlation function $\langle \Psi_0 | T\left(\hat{\rho}(\mathbf{x},t)\hat{\rho}(\mathbf{0},0)\right)|\Psi_0\rangle$. 3. Now consider the 3D interacting superfluid, with the low-energy effective theory discussed in class:

$$L = \int \mathrm{d}^3\mathrm{x}\left[\frac{1}{2U_0}\left(\partial_t\theta\right)^2 - \frac{\rho_0}{2m}(\nabla\theta)^2\right].$$

Denote by $|0\rangle$ the ground state of the interacting superfluid. Calculate $\langle 0|T(\hat{\rho}(\mathbf{x},t)\hat{\rho}(\mathbf{0},0))|0\rangle$, in ω, k space and in real space. 4. Compare the asymptotic behavior of the density-density correlation functions for the (non-interacting) Bose condensate and the (interacting) superfluid state in the limit $x \to \infty, t$ fixed. Both approach the constant $\langle\hat{\rho}\rangle^2$, but is the approach described by a power law, exponential or some other function?

Solution

1. We have

$$a(\boldsymbol{x},t) = \frac{1}{\sqrt{V}}\sum_{\boldsymbol{k}} \mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x} - \mathrm{i}\omega_{\boldsymbol{k}}t} a_{\boldsymbol{k}}. \tag{26}$$

So

$$\mathrm{i}G(\boldsymbol{x},t) = \frac{1}{V}\sum_{\boldsymbol{k}}\sum_{\boldsymbol{k}'} \mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}} \langle 0|\mathcal{T}\,\mathrm{e}^{-\mathrm{i}\omega_{\boldsymbol{k}}t} a_{\boldsymbol{k}} a^\dagger_{\boldsymbol{k}'}|0\rangle\,.$$

When $t > 0$, we have

$$\mathcal{T}\,\mathrm{e}^{-\mathrm{i}\omega_{\boldsymbol{k}}t} a_{\boldsymbol{k}} a^\dagger_{\boldsymbol{k}'} = \mathrm{e}^{-\mathrm{i}\omega_{\boldsymbol{k}}t} a_{\boldsymbol{k}} a^\dagger_{\boldsymbol{k}'} = \mathrm{e}^{-\mathrm{i}\omega_{\boldsymbol{k}}t}(a^\dagger_{\boldsymbol{k}'} a_{\boldsymbol{k}} + \delta_{\boldsymbol{k}\boldsymbol{k}'}),$$

and when $t < 0$, we have

$$\mathcal{T}\,\mathrm{e}^{-\mathrm{i}\omega_{\boldsymbol{k}}t} a_{\boldsymbol{k}} a^\dagger_{\boldsymbol{k}'} = \mathrm{e}^{-\mathrm{i}\omega_{\boldsymbol{k}}t} a^\dagger_{\boldsymbol{k}'} a_{\boldsymbol{k}}.$$

The momentum correlation function is then evaluated as follows:

$$\begin{aligned}
\langle\Psi_0|\mathcal{T}\,\mathrm{e}^{-\mathrm{i}\omega_{\boldsymbol{k}}t} a_{\boldsymbol{k}} a^\dagger_{\boldsymbol{k}'}|\Psi_0\rangle &= \theta(t)\delta_{\boldsymbol{k}\boldsymbol{k}'}\mathrm{e}^{-\mathrm{i}\omega_{\boldsymbol{k}}t} + \mathrm{e}^{-\mathrm{i}\omega_{\boldsymbol{k}}t}\langle\Psi_0|a^\dagger_{\boldsymbol{k}'}a_{\boldsymbol{k}}|\Psi_0\rangle \\
&= \theta(t)\delta_{\boldsymbol{k}\boldsymbol{k}'}\mathrm{e}^{-\mathrm{i}\omega_{\boldsymbol{k}}t} + \mathrm{e}^{-\mathrm{i}\omega_{\boldsymbol{k}}t}\delta_{\boldsymbol{k},0}\delta_{\boldsymbol{k}',0}\langle\Psi_0|a^\dagger_0 a_0|\Psi_0\rangle \\
&= \theta(t)\delta_{\boldsymbol{k}\boldsymbol{k}'}\mathrm{e}^{-\mathrm{i}\omega_{\boldsymbol{k}}t} + \mathrm{e}^{-\mathrm{i}\omega_{\boldsymbol{k}}t}\delta_{\boldsymbol{k},0}\delta_{\boldsymbol{k}',0}N.
\end{aligned}$$

The correlation function is therefore

$$\begin{aligned}
\mathrm{i}G(\boldsymbol{x},t) &= \frac{1}{V}\sum_{\boldsymbol{k},\boldsymbol{k}'} \mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}\left(\theta(t)\delta_{\boldsymbol{k}\boldsymbol{k}'}\mathrm{e}^{-\mathrm{i}\omega_{\boldsymbol{k}}t} + \mathrm{e}^{-\mathrm{i}\omega_{\boldsymbol{k}}t}\delta_{\boldsymbol{k},0}\delta_{\boldsymbol{k}',0}N\right) \\
&= \frac{N}{V} + \theta(t)\frac{1}{V}\sum_{\boldsymbol{k}}\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}\mathrm{e}^{-\mathrm{i}\frac{\boldsymbol{k}^2}{2m}t} \\
&= \rho_0 + \theta(t)\int\frac{\mathrm{d}^3\boldsymbol{k}}{(2\pi)^3}\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x} - \mathrm{i}\frac{\boldsymbol{k}^2}{2m}t}.
\end{aligned}$$

After completing the integral, we get

$$\mathrm{i}G(\boldsymbol{x},t) = \rho_0 + \theta(t)\sqrt{\frac{m^3}{(2\pi\mathrm{i}t)^3}}\mathrm{e}^{\frac{\mathrm{i}}{2}\frac{mx^2}{t}}. \tag{27}$$

When $|\boldsymbol{x}| \to \infty$, the second term oscillates fast, and its average is zero, so we have

$$\mathrm{i}G(\boldsymbol{x},t) \simeq \rho_0, \tag{28}$$

so there is indeed a long-range order in the ground state of BEC.

2. By Wick's theorem and the fact that $\langle 0|\mathcal{T}\,a_1 a_2|0\rangle = 0$, as well as $\langle 0|a^\dagger a|0\rangle$, we have

$$\begin{aligned}
&\mathcal{T}\,a^\dagger(\boldsymbol{x},t)a(\boldsymbol{x},t)a^\dagger(0,0)a(0,0) \\
=&: a^\dagger(\boldsymbol{x},t)a(\boldsymbol{x},t)a^\dagger(0,0)a(0,0): \\
&+ : a^\dagger(\boldsymbol{x},t)a(0,0): \langle 0|\mathcal{T}\,a(\boldsymbol{x},t)a^\dagger(0,0)|0\rangle \\
&+ : a(\boldsymbol{x},t)a^\dagger(0,0): \langle 0|\mathcal{T}\,a^\dagger(\boldsymbol{x},t)a(0,0)|0\rangle \\
&+ \langle 0|\mathcal{T}\,a(\boldsymbol{x},t)a^\dagger(0,0)|0\rangle\,\langle 0|\mathcal{T}\,a^\dagger(\boldsymbol{x},t)a(0,0)|0\rangle\,.
\end{aligned} \tag{29}$$

By space and time translational symmetry, we have

$$\begin{aligned}\langle 0|\mathcal{T}\, a^\dagger(\boldsymbol{x},t)a(0,0)|0\rangle &= \langle 0|\mathcal{T}\, a(0,0)a^\dagger(\boldsymbol{x},t)|0\rangle = \langle 0|\mathcal{T}\, a(-\boldsymbol{x},-t)a^\dagger(0,0)|0\rangle \\ &= \underbrace{\rho_0}_{=0 \text{ for } |0\rangle} + \theta(-t)\int \frac{\mathrm{d}^3\boldsymbol{k}}{(2\pi)^3}\mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}+\mathrm{i}\frac{\boldsymbol{k}^2}{2m}t}.\end{aligned}$$

Thus the last term in (29) vanishes, because it contains both $\theta(t)$ and $\theta(-t)$. So we get

$$\begin{aligned}&\langle \Psi_0|\mathcal{T}\, a^\dagger(\boldsymbol{x},t)a(\boldsymbol{x},t)a^\dagger(0,0)a(0,0)|\Psi_0\rangle \\ &= \langle \Psi_0|: a^\dagger(\boldsymbol{x},t)a(\boldsymbol{x},t)a^\dagger(0,0)a(0,0) :|\Psi_0\rangle \\ &\quad + \langle \Psi_0|: a^\dagger(\boldsymbol{x},t)a(0,0) :|\Psi_0\rangle\, \langle 0|\mathcal{T}\, a(\boldsymbol{x},t)a^\dagger(0,0)|0\rangle \\ &\quad + \langle \Psi_0|: a^\dagger(0,0)a(\boldsymbol{x},t) :|\Psi_0\rangle\, \langle 0|\mathcal{T}\, a(-\boldsymbol{x},-t)a^\dagger(0,0)|0\rangle\,.\end{aligned} \tag{30}$$

The normal ordered operator factor in the second term is

$$\begin{aligned}\langle \Psi_0|: a^\dagger(\boldsymbol{x},t)a(0,0) :|\Psi_0\rangle &= \frac{1}{V}\sum_{\boldsymbol{k},\boldsymbol{k}'}\mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}\mathrm{e}^{\mathrm{i}\omega_{\boldsymbol{k}}t}\,\langle \Psi_0|a^\dagger_{\boldsymbol{k}}a_{\boldsymbol{k}'}|\Psi_0\rangle \\ &= \frac{1}{V}\sum_{\boldsymbol{k},\boldsymbol{k}'}\mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}\mathrm{e}^{\mathrm{i}\omega_{\boldsymbol{k}}t}N\delta_{\boldsymbol{k},0}\delta_{\boldsymbol{k}',0} = \frac{N}{V},\end{aligned}$$

and similarly the normal ordered operator factor in the third term is N/V. The first term is

$$\begin{aligned}&\langle \Psi_0|: a^\dagger(\boldsymbol{x},t)a(\boldsymbol{x},t)a^\dagger(0,0)a(0,0) :|\Psi_0\rangle \\ &= \frac{1}{V^2}\sum_{\boldsymbol{k}_1,\boldsymbol{k}_2,\boldsymbol{k}_3,\boldsymbol{k}_4}\mathrm{e}^{-\mathrm{i}\boldsymbol{k}_1\cdot\boldsymbol{x}+\mathrm{i}\omega_{\boldsymbol{k}_1}t}\mathrm{e}^{\mathrm{i}\boldsymbol{k}_2\cdot\boldsymbol{x}-\mathrm{i}\omega_{\boldsymbol{k}_2}t}\,\langle \Psi_0|a^\dagger_{\boldsymbol{k}_1}a^\dagger_{\boldsymbol{k}_3}a_{\boldsymbol{k}_2}a_{\boldsymbol{k}_4}|\Psi_0\rangle \\ &= \frac{1}{V^2}\sum_{\boldsymbol{k}_1,\boldsymbol{k}_2,\boldsymbol{k}_3,\boldsymbol{k}_4}\mathrm{e}^{-\mathrm{i}\boldsymbol{k}_1\cdot\boldsymbol{x}+\mathrm{i}\omega_{\boldsymbol{k}_1}t}\mathrm{e}^{\mathrm{i}\boldsymbol{k}_2\cdot\boldsymbol{x}-\mathrm{i}\omega_{\boldsymbol{k}_2}t}\delta_{\boldsymbol{k}_1,0}\delta_{\boldsymbol{k}_2,0}\delta_{\boldsymbol{k}_3,0}\delta_{\boldsymbol{k}_4,0}N(N-1) \\ &= \frac{N(N-1)}{V^2}.\end{aligned}$$

So the final result is

$$\begin{aligned}&\langle \Psi_0|\mathcal{T}\, \rho(\boldsymbol{x},t)a(\boldsymbol{x},t)\rho(0,0)|\Psi_0\rangle \\ &= \frac{N(N-1)}{V^2} + \frac{N}{V}\theta(t)\int\frac{\mathrm{d}^3\boldsymbol{k}}{(2\pi)^3}\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}-\mathrm{i}\frac{\boldsymbol{k}^2}{2m}t} + \frac{N}{V}\theta(-t)\int\frac{\mathrm{d}^3\boldsymbol{k}}{(2\pi)^3}\mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}+\mathrm{i}\frac{\boldsymbol{k}^2}{2m}t}.\end{aligned} \tag{31}$$

3. By the same procedure in (21), it can be found that (Eq. (3.3.12) in Xiao-gang Wen's textbook)

$$\rho = \rho_0 - \frac{1}{U_0}\partial_t\theta. \tag{32}$$

Since we are working on $|0\rangle$, all terms with odd θ's have vanishing expectation, so we have

$$\langle 0|\mathcal{T}\,\rho(\boldsymbol{x},t)\rho(0,0)|0\rangle = \rho_0^2 + \frac{1}{U_0^2}\,\langle 0|\mathcal{T}\,\partial_t\theta(\boldsymbol{x},t)\partial_t\theta(0,0)|0\rangle\,. \tag{33}$$

In the frequency domain we have (following the same procedure in (12))

$$\begin{aligned}&\int \mathrm{d}^3\boldsymbol{x}\,\mathrm{d}t\,\mathrm{e}^{\mathrm{i}(-\boldsymbol{k}\cdot\boldsymbol{x}+\omega t)}\,\langle 0|\mathcal{T}\,\rho(\boldsymbol{x},t)\rho(0,0)|0\rangle \\ &= (2\pi)^4\rho_0^2\delta^3(\boldsymbol{k})\delta(\omega) + \frac{1}{U_0^2}(-\mathrm{i}\omega)(\mathrm{i}\omega)\int \mathrm{d}^3\boldsymbol{x}\,\mathrm{d}t\,\mathrm{e}^{\mathrm{i}(-\boldsymbol{k}\cdot\boldsymbol{x}+\omega t)}\,\langle 0|\mathcal{T}\,\partial_t\theta(\boldsymbol{x},t)\partial_t\theta(0,0)|0\rangle \\ &= (2\pi)^4\rho_0^2\delta^3(\boldsymbol{k})\delta(\omega) + \frac{1}{U_0^2}\omega^2\frac{\mathrm{i}}{\frac{1}{U_0}\omega^2 - \frac{\rho_0}{m}\boldsymbol{k}^2 + \mathrm{i}0^+}.\end{aligned} \tag{34}$$

Here the propagator in the last line can be found in Section 3.3.8.1 in Wen's book or by directly taking the inverse of the Lagrangian. The inverse Fourier transformation gives

$$\langle 0|\mathcal{T}\,\rho(\boldsymbol{x},t)\rho(0,0)|0\rangle = \rho_0^2 + \frac{\mathrm{i}}{U_0}\int \frac{\mathrm{d}^3\boldsymbol{k}\,\mathrm{d}\omega}{(2\pi)^4}\mathrm{e}^{\mathrm{i}(-\omega t+\boldsymbol{k}\cdot\boldsymbol{x})}\frac{\omega^2}{\omega^2 - \underbrace{\frac{\rho_0 U_0}{m}\boldsymbol{k}^2}_{=:\omega_{\boldsymbol{k}}^2} + \mathrm{i}0^+}. \tag{35}$$

The next step is to calculate the second term explicitly. When $t>0$, we should construct a contour surrounding the lower plane where $\mathrm{e}^{-\mathrm{i}\omega t}\to 0$ at the infinity, so we have

$$\int \mathrm{d}\omega\,\mathrm{e}^{-\mathrm{i}\omega t}\frac{\omega^2}{\omega^2-\omega_{\boldsymbol{k}}^2+\mathrm{i}0^+} = -2\pi\mathrm{i}\lim_{\omega\to\omega_{\boldsymbol{k}}}\frac{\omega^2}{\omega^2-\omega_{\boldsymbol{k}}^2}\mathrm{e}^{-\mathrm{i}\omega t}(\omega-\omega_{\boldsymbol{k}}) = -\pi\mathrm{i}\omega_{\boldsymbol{k}}\mathrm{e}^{-\mathrm{i}\omega_{\boldsymbol{k}}t},$$

and (below x is $|\boldsymbol{x}|$)

$$\begin{aligned}
&\int \frac{\mathrm{d}^3\boldsymbol{k}\,\mathrm{d}\omega}{(2\pi)^4}\frac{\omega^2}{\omega^2-\omega_{\boldsymbol{k}}^2+\mathrm{i}0^+}\mathrm{e}^{\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{x}-\omega t)}\\
&= \frac{1}{(2\pi)^4}\int \mathrm{d}^3\boldsymbol{k}\,(-\pi\mathrm{i})\omega_{\boldsymbol{k}}\mathrm{e}^{\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{x}-\omega_{\boldsymbol{k}}t)}\\
&= -\frac{\pi\mathrm{i}}{(2\pi)^4}\int k^2\mathrm{d}k\int\mathrm{d}\varphi\int\sin\theta\,\mathrm{d}\theta\sqrt{\frac{\rho_0U_0}{m}}k\mathrm{e}^{\mathrm{i}kx\cos\theta-\mathrm{i}kt\sqrt{\frac{\rho_0U_0}{m}}}\\
&= -\frac{\pi\mathrm{i}}{(2\pi)^4}\int k^2\mathrm{d}k\cdot 2\pi\cdot\sqrt{\frac{\rho_0U_0}{m}}k\mathrm{e}^{-\mathrm{i}kt\sqrt{\frac{\rho_0U_0}{m}}}\frac{1}{\mathrm{i}kx}(\mathrm{e}^{\mathrm{i}kx}-\mathrm{e}^{-\mathrm{i}kx})\\
&= -\frac{1}{8\pi^2}\frac{1}{x}v\int_0^\infty k^2(\mathrm{e}^{\mathrm{i}k(x-vt)}-\mathrm{e}^{-\mathrm{i}k(x+vt)})\,\mathrm{d}k\\
&= -\frac{1}{8\pi^2}\frac{1}{x}v\left(\frac{-2\mathrm{i}}{(x-vt)^3}-\frac{-2\mathrm{i}}{(-x-vt)^3}.\right)
\end{aligned}$$

Here we define

$$v=\sqrt{\frac{\rho_0U_0}{m}},\omega_{\boldsymbol{k}}=c|\boldsymbol{k}|. \tag{36}$$

Assuming a small damping in the spectrum of θ, i.e. a small negative imaginary part of $\omega_{\boldsymbol{k}}$ and hence v, we get

$$\int \frac{\mathrm{d}^3\boldsymbol{k}\,\mathrm{d}\omega}{(2\pi)^4}\frac{\omega^2}{\omega^2-\omega_{\boldsymbol{k}}^2+\mathrm{i}0^+} = \frac{\mathrm{i}}{4\pi^2}\frac{v}{x}\left(\frac{1}{(x-vt)^3}+\frac{1}{(x+vt)^3}\right).$$

When $t<0$, the only change is now the pole we integrate around becomes $\omega=-\omega_{\boldsymbol{k}}$, and

$$\int \mathrm{d}\omega\,\mathrm{e}^{-\mathrm{i}\omega t}\frac{\omega^2}{\omega^2-\omega_{\boldsymbol{k}}^2+\mathrm{i}0^+} = 2\pi\mathrm{i}\lim_{\omega\to-\omega_{\boldsymbol{k}}}\frac{\omega^2}{\omega^2-\omega_{\boldsymbol{k}}^2}\mathrm{e}^{-\mathrm{i}\omega t}(\omega+\omega_{\boldsymbol{k}}) = -\pi\mathrm{i}\omega_{\boldsymbol{k}}\mathrm{e}^{\mathrm{i}\omega_{\boldsymbol{k}}t},$$

so the only change is replacing t with $|t|$. So finally the correlation function is

$$\langle 0|\mathcal{T}\,\rho(\boldsymbol{x},t)\rho(0,0)|0\rangle = \rho_0^2 - \frac{1}{4\pi^2U_0}\frac{v}{x}\left(\frac{1}{(x-v|t|)^3}+\frac{1}{(x+v|t|)^3}\right). \tag{37}$$

4. The second and third terms of (31) are all rapidly oscillating in the same way we see in (27). so in BEC, we have

$$\langle\Psi_0|\mathcal{T}\,\rho(\boldsymbol{x},t)\rho(0,0)|\Psi_0\rangle \stackrel{|\boldsymbol{x}|\to\infty}{\longrightarrow} \rho_0^2. \tag{38}$$

This is also true for superfluid (37), but the decay is a x^{-4} power law.