ODEs

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1 First order ODEs

1.1 Linear ODEs

An ODE in the form of

$$y'(x) + p(x)y(x) = q(x) \tag{1}$$

is considered **linear**. All linear ODEs can be solved by the following procedure. First we have

$$(y' + py)\mathrm{e}^{\int p \mathrm{d}x} = q\mathrm{e}^{\int p \mathrm{d}x}, \tag{2}$$

and now the LHS is a derivative:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(y\mathrm{e}^{\int p \mathrm{d}x}\right) = q\mathrm{e}^{\int p \mathrm{d}x}, \tag{3}$$

and now we can integrate over x and get

$$y\mathrm{e}^{\int p \mathrm{d}x} = \int q\mathrm{e}^{\int p \mathrm{d}x} \,\mathrm{d}x\,, \tag{4}$$

$$y = \mathrm{e}^{-\int p \mathrm{d}x} \int q\mathrm{e}^{\int p \mathrm{d}x} \,\mathrm{d}x\,. \tag{5}$$

1.2 "Energy-conservation lines" and exact equations

Another way to represent the solution of an ODE is the form $\phi(x, y) = \text{const}$. Note that the RHS contains no variables, and we have

$$0 = \frac{\mathrm{d}\phi}{\mathrm{d}x} = \frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}x}, \tag{6}$$

and thus if

$$y' = f(x, y) \tag{7}$$

is algebraically equivalent to (6), the equation is already solved: We should find M, N such that

$$y' = -\frac{M}{N}, \quad M = \frac{\partial \phi}{\partial x}, \quad N = \frac{\partial \phi}{\partial y}, \tag{8}$$

and then $\phi(x, y)$ solves the equation. In this case we say $y' = -M/N$ is **exact**.

To test for exactness, we only have to test whether

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}, \tag{9}$$

and if so, the existence of ϕ is guaranteed. (Since we work on a topological trivial space, things like cohomology group will not bother us.) We can now use "partial integral" to find ϕ.

Example: suppose in a calculation we find

$$\frac{\partial \phi}{\partial x} = 2y^2 + y\mathrm{e}^{xy}, \quad \frac{\partial \phi}{\partial y} = 4xy + x\mathrm{e}^{xy} + 2y. \tag{10}$$

After partial integration, we find

$$\phi(x, y) = \underbrace{2xy^2 + \mathrm{e}^{xy} + h(y)}_{\int \frac{\partial \phi}{\partial x}\mathrm{d}x} = \underbrace{2xy^2 + \mathrm{e}^{xy} + y^2 + g(x)}_{\int \frac{\partial \phi}{\partial y}\mathrm{d}y}, \tag{11}$$

and we have to choose

$$h(y) = y^2, \quad g(x) = \text{const}, \tag{12}$$

and the solution is

$$\phi(x, y) = 2xy^2 + \mathrm{e}^{xy} + y^2 + \text{const}. \tag{13}$$

Note that even when the decomposition $f = -M/N$ doesn't give an exact equation for us, we can still use the method of exact equations: we can multiply a factor μ to both M and N, and try to guess the form of μ so that

$$\frac{\partial(\mu M)}{\partial y} = \frac{\partial(\mu N)}{\partial x}. \tag{14}$$

An example can be found in solving

$$y' = -\frac{1}{3x - \mathrm{e}^{-2y}}. \tag{15}$$

We have

$$\frac{\partial 1}{\partial y} = 0, \quad \frac{\partial(3x - \mathrm{e}^{-2y})}{\partial x} = 3,$$

so the equation is not exact if we choose $M = 1$ and $N = 3x - \mathrm{e}^{-2y}$. However, (14) can be fulfilled now: it's now

$$\frac{\partial \mu}{\partial y} = 3\mu + \left(3x - \mathrm{e}^{-2y}\right) \frac{\partial \mu}{\partial x},$$

and the most convenient way to solve it (we *don't* need to find all solutions of this equation!) is to let μ contain y only, so the tricky term on the RHS disappears, and thus we choose $\mu = \mathrm{e}^{3y}$, and we get

$$\phi(x, y) = \int \mu M \, \mathrm{d}x = \int \mathrm{e}^{3y} \, \mathrm{d}x = x\mathrm{e}^{3y} + u(y),$$

$$\phi(x, y) = \int \mu N \, \mathrm{d}y = \int (3x\mathrm{e}^{3y} - \mathrm{e}^{y}) \, \mathrm{d}y = x\mathrm{e}^{3y} - \mathrm{e}^{y} + v(x),$$

so

$$\phi(x, y) = x\mathrm{e}^{3y} - \mathrm{e}^{y} + \text{const}. \tag{16}$$

1.3 Bernoulli equation

Consider the following **Bernoulli equation**

$$y' + P(x)y = R(x)y^{\alpha}. \tag{17}$$

When $\alpha = 0, 1$, the equation can be solved by the standard methods for linear first order ODEs. When this is not the case, we may do the substitution

$$v = y^{\beta}, \tag{18}$$

and then the equation becomes

$$\frac{1}{\beta} v^{1/\beta - 1} v' + P(x) v^{1/\beta} = R(x) v^{\alpha/\beta},$$

$$v' + P(x)v = R(x) v^{1 + \frac{\alpha - 1}{\beta}}. \tag{19}$$

The next step is to choose a good beta so that the equation gets simplified. We may want to make to exponent to be zero, and this means we should choose

$$\beta = 1 - \alpha, \tag{20}$$

and the ODE is now

$$v' + Pu = R, \tag{21}$$

which can then be solved by the method in Section 1.1.

2 Second order ODEs

2.1 Linear 2nd order ODE with initial values

A linear second order ODE has the following form:

$$y'' + p(x)y' + q(x)y = f(x). \tag{22}$$

It usually comes with initial value conditions

$$y(x_0) = A, \quad y'(x_0) = B. \tag{23}$$

This course is about concrete calculations, but knowing what we are doing makes sense is important. Here is an existence and uniqueness theorem: if $p(x), q(x)$, and $f(x)$ are continuous over an interval I, and $x_0 \in I$, then a unique solution exists for (22) with the initial conditions given above.

Usually, we start by looking at the **homogeneous** second order ODE

$$y'' + p(x)y' + q(x)y = 0. \tag{24}$$

The influence of $f(x)$ can be included as the "response" of the LHS. The full solution of (24) takes the form

$$y = c_1 y_1 + c_2 y_2, \tag{25}$$

where c_1, c_2 are constants to be decided by initial conditions, and y_1 and y_2 are linearly independent solutions of (24). The **Wronskian** is defined as

$$W(x) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}. \tag{26}$$

By checking if it is non-zero at most points, we can find whether y_1 and y_2 are truly linearly independent to each other.

There is a method to arrive at y_2 from y_1: we can always take the ansatz

$$y_2 = y_1 u, \tag{27}$$

and therefore we get

$$(u'' y_1 + 2u' y_1' + u y_1'') + p(u' y_1 + u y_1') + q u y_1 = 0,$$

and the condition that y_1 is a solution to (24) means

$$u'' + \underbrace{\frac{2y_1' + p y_1}{y_1}}_{g(x)} u' = 0, \tag{28}$$

which is essentially a first order ODE, because we can replace u' by v, and then we find

$$\ln v = -\int g(x) \, \mathrm{d}x \,,$$

and

$$u(x) = \int \mathrm{e}^{-\int g(x) \mathrm{d}x} \, \mathrm{d}x \,. \tag{29}$$

2.2 Constant coefficients

The equation

$$y'' + Ay + By = 0 \tag{30}$$

can be solved directly by the following construction:

$$y = c_1 \mathrm{e}^{\lambda_1 x} + c_2 \mathrm{e}^{\lambda_2 x}, \tag{31}$$

where λ_1, λ_2 are solutions of

$$\lambda^2 + A\lambda + B = 0. \tag{32}$$

For example, to solve the equation

$$y'' - 2y' + 10y = 0, \tag{33}$$

we just solve

$$\lambda^2 - 2\lambda + 10 = 0,$$

which gives us

$$\lambda = 1 \pm 3\mathrm{i}, \tag{34}$$

and therefore a general solution is

$$y = \mathrm{e}^x(c_1 \mathrm{e}^{3\mathrm{i}x} + c_2 \mathrm{e}^{-3\mathrm{i}x}). \tag{35}$$

It should be noted that c_1, c_2 can be complex, even when we restrict y in $\mathbb{R}$: we can let the imaginary part of y vanish as long as we impose some constraints over c_1, c_2. If we are determined to work in the real space, two alternative linearly independent solutions can be used:

$$y_1(x) = \mathrm{e}^x \cos(3x), \quad y_2(x) = \mathrm{e}^x \sin(3x). \tag{36}$$

Although we can immediately say they are linearly independent, we can use them as a demonstration of the Wronskian method: now we have

$$W(x) = y_1(x)y_2'(x) - y_2(x)y_1'(x) = 3\mathrm{e}^{2x}, \tag{37}$$

which of course isn't constantly zero.

(32) is faced with the problem of having only one solution when $A^2 - 4B = 0$. In this case we need to go back to the standard procedure to get y_2 from y_1. An example is

$$y'' + 6y + 9 = 0, \tag{38}$$

for which (32) only gives

$$y_1 = \mathrm{e}^{-3x}. \tag{39}$$

Suppose $y_2 = u\mathrm{e}^{-3x}$, we have TODO

2.3 Euler equation

A **Euler equation** has the following form:

$$x^2y'' + Axy' + By = 0, \tag{40}$$

where A, B are constants. One solution can be immediate found: it always looks like

$$y = x^a. \tag{41}$$

We then find

$$a(a-1) + Aa + B = 0. \tag{42}$$

If there are two solutions of the equation, (40) has already been solved. If not, we can use the trick (27).

An example: let's solve

$$x^2y'' + 3xy' + y = 0. \tag{43}$$

The equation about a is now

$$a(a-1) + 3a + 1 = 0,$$

and it only has one solution $a = -1$. Therefore we have

$$y_1 = \frac{1}{x}.$$

Suppose

$$y_2 = uy_1,$$

we get

$$x^2\left(\frac{u''}{x^2}-\frac{2u'}{x}+\frac{2u}{x^2}\right)+3x\left(\frac{u'}{x}-\frac{u}{x^2}\right)+\frac{u}{x}=0,$$

which is equivalent to

$$v'x+v=0, \quad v=u'$$

the solution of which is

$$\ln v+\ln x=\text{const},$$

and therefore

$$v=\frac{C'}{x}, \quad u=C'\ln x+C,$$

$$y_2=\frac{1}{x}(C'\ln x+C).$$

This essentially gives *all* solutions we need: for y_1, we just have $u=1$, which corresponds to $C=1$. So now the equation is completely solved.

2.4 Non-homogeneous cases or how to find the linear response

Now we discuss how to solve

$$y''+p(x)y'+q(x)y=f(x). \tag{44}$$

A general solution is

$$y(x)=y_{\mathrm{p}}(x)+y_{\mathrm{h}}(x), \tag{45}$$

where the subscript p means a particular solution, and the subscript h means the general solution of the corresponding homogeneous equation.

We need some common sense to find a particular solution. To solve

$$y''-y'-2y=2x^2+5, \tag{46}$$

we don't expect y to be, say, $\cos(2x)$: instead, it's usually the case that y is a polynomial. An ansatz is

$$y=Ax^2+Bx+C.$$

We don't want a x^3 term because it doesn't appear on RHS. The equation then becomes

$$2A-(2Ax+Bx)-2(Ax^2+Bx+C)=2x^2+5,$$

$$-2A=2, \quad -2A-2B=0, \quad 2A-B-2C=5,$$

and therefore $A=-1, B=1, C=-4$. Therefore we get a particular solution:

$$y_{\mathrm{p}}=-x^2+x-4. \tag{47}$$

A particular solution may also be determined as a "linear combination" of homogeneous solutions, although now the coefficients have temporal variation. That's to say, we take the ansatz

$$y_{\mathrm{p}}(x)=u(x)y_1(x)+v(x)y_2(x). \tag{48}$$

This is quite similar to the procedure introduced in Section 1.1. After substituting y with (48) in (44), we get

$$u'y_1'+v'y_2'=f. \tag{49}$$

Introducing the constraint

$$u'y_1+v'y_2=0, \tag{50}$$

we have

$$\begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix}\begin{pmatrix} u' \\ v' \end{pmatrix}=\begin{pmatrix} 0 \\ f \end{pmatrix}, \tag{51}$$

from which we find u', v' and hence u, v. The Wronskian – the determinant of the matrix on LHS – is non-zero, so the equation always has a solution.

Example: let's solve

$$y'' + y = \tan x. \tag{52}$$

We have

$$y_1 = \cos x, \quad y_2 = \sin x,$$

and therefore

$$W(x) = y_1 y_2' - y_2 y_1' = 1.$$

So

$$\begin{aligned} u' &= -\int \frac{y_2 f}{W(x)} \, \mathrm{d}x = -\int \frac{\sin^2 x}{\cos x} \, \mathrm{d}x = -\int \frac{1 - \cos^2 x}{\cos x} \, \mathrm{d}x \\ &= -\frac{1}{2} \ln \left| \frac{1 + \sin x}{1 - \sin x} \right| + \sin x, \end{aligned}$$

and similarly we have

$$v =$$