Homework 2

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1 Quasiparticle weight in Landau Fermi liquid

1.1 Quasiparticle weight in Na

From Table 2.3 in A&M, in Na, $m^*/m = 1.3$, so $Z = m/m^* = 0.77$. Here m^* is obtained by thermodynamic measurement: the Sommerfield model tells us

$$C_V = \frac{\pi^2}{2}\frac{k_\mathrm{B}T}{E_\mathrm{F}} n k_\mathrm{B}, \quad E_\mathrm{F} = \frac{\hbar^2}{2m}\left(\frac{3\pi^2 N}{V}\right)^{3/2}, \tag{1}$$

and therefore

$$\frac{C_{V,\text{measured}}}{C_{V,\text{free electron}}} = \frac{m^*}{m}. \tag{2}$$

1.2 Direct observation of occupation discontinuity

From [3], a direct measurement of the electron occupation tells us the experimental result for Na: $Z = 0.58$. This is much smaller than the value found in the thermodynamic measurement.

The difference can be explained by the fact that the electron liquid in sodium is not strictly homogeneous, and therefore we simply can't have a *single* effective mass that both satisfies $Z = m/m^*$ in the electron Green function and (2). The proof of $Z = m/m^*$ is done by noticing

$$\begin{aligned}\frac{1}{\omega - \frac{\boldsymbol{p}^2}{2m} + \mu - \Sigma(\omega, \boldsymbol{p})} &= \frac{1}{\omega - \frac{\boldsymbol{p}^2}{2m} + \mu - \Sigma(\omega = 0, \boldsymbol{p}) - \frac{\partial\Sigma}{\partial\omega}\omega} \\ &= \frac{Z}{\omega - \frac{\boldsymbol{p}^2}{2m^*} + \mu^*} + \text{satellite terms},\end{aligned} \tag{3}$$

and therefore

$$Z = \frac{1}{1 - \frac{\partial\Sigma}{\partial\omega}}, \quad \left(1 - \frac{\partial\Sigma}{\partial\omega}\right)\frac{1}{m^*} = \frac{1}{m} \Rightarrow \frac{m^*}{m} = \frac{1}{Z}. \tag{4}$$

This derivation however has assumed that all $\boldsymbol{p}$ dependence in Σ can be seen as an indirect ω dependence, which is not true for an inhomogeneous system. Also, for an inhomogeneous system, we don't have an m^* that can be inserted into the free-electron DOS in place of m, and the expression of the heat capacity isn't the free-electron heat capacity with m replaced by m^*.

2 Exotic phenomena in a Landau Fermi liquid

2.1 Heavy fermion systems

A heavy fermion system is a system in which the bands that come across the Fermi energy are very flat, resulting in a very high effective mass of the electrons. Since now Coulomb interaction energy exceeds the "kinetic energy" (kinetic energy plus crystal potential, to be exact), the material is on the vicinity of a magnetic transition.

Signatures of heavy fermion materials include a low-temperature heat capacity that is, say, 1000 times of the free electron heat capacity, low conductivity, and other facts that correspond to a large m^*.

Example of heavy fermion systems include $CeAl_3$ [1] and YbBiPt [2].

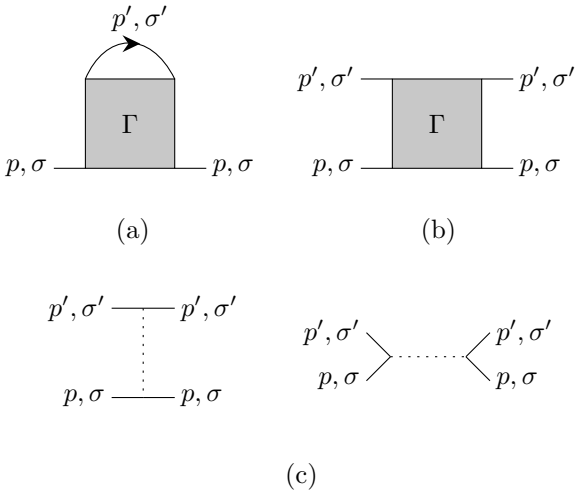

Figure 1: Skeleton diagrams that decide Landau parameters (a) The vertex function decides the single electron self-energy; in other words, f appears in ε. (b) The vertex function appears as the effective interaction channel; in other words, f appears in the energy functional i.e. the effective Hamiltonian. (c) Hartree-Fock approximation in the vertex function. The left one is the Hartree term ($\propto V(\boldsymbol{q}=0)$). The right one is the Fock term ($\propto V(\boldsymbol{q}=\boldsymbol{p}-\boldsymbol{p}')$).

2.2 Zero sound

2.2.1 Zero sound in neutral Fermi liquid

Zero sound is a bosonic branch of a Fermi liquid and therefore is known as "sound". The main difference between zero sound and ordinary sound is that zero sound is not simply a density mode in real space; instead, it's the fluctuation of the Fermi surface. Zero sound can be derived from quantum Boltzmann equation with the $\varepsilon(\boldsymbol{p})$ self-consistently depends on the distribution function.

Some explanation is needed when we insert $\delta n_{\boldsymbol{p}}(\boldsymbol{r})$ in place of $\delta n_{\boldsymbol{p}}$ in $\varepsilon(\boldsymbol{p})$ and put $\varepsilon(\boldsymbol{p})$ into the quantum Boltzmann equation about $\delta n_{\boldsymbol{p}}(\boldsymbol{r})$. The interaction between particles has a short characteristic length scale in the real space, which means the interaction vertex in the momentum space doesn't assume strong dependence on the exchanged momentum $\boldsymbol{q}$. Thus, after renormalization, the interaction between $\delta n_{\boldsymbol{p}}(\boldsymbol{r})$ and $\delta n_{\boldsymbol{p}'}(\boldsymbol{r}')$ – essentially Wigner transforms of $c^\dagger_{\boldsymbol{p}+\boldsymbol{q}} c_{\boldsymbol{p}}$ – has no large differences with $f_{\boldsymbol{p}\boldsymbol{p}'}\,\delta n_{\boldsymbol{p}}\,\delta n_{\boldsymbol{p}'}$, because $\boldsymbol{q}$ dependence is not important when $\boldsymbol{q}$ is small (or in other words, when the $\boldsymbol{r}$-dependence of $\delta n_{\boldsymbol{p}}(\boldsymbol{r})$ is smooth enough, which is always required if we want a reasonable Boltzmann equation). This explains why when deriving the kinetic equation of Fermi liquid, we just insert $\delta n_{\boldsymbol{p}}(\boldsymbol{r})$ in place of $\delta n_{\boldsymbol{p}}$: this works only when the interaction is short-range [5]. The resulting kinetic equation system – a fermionic quantum Boltzmann equation plus the relation between $\varepsilon_{\boldsymbol{p}}$, $\varepsilon^0_{\boldsymbol{p}\sigma}$ and $f_{\boldsymbol{p}\boldsymbol{p}'}$ – is known as the **Landau equation**. Zero sound is the linear oscillation mode of the Landau equation.

2.2.2 Zero sound (now plasmon) is gapped in metals

The speed of zero sound is usually $\sim v_{\mathrm{F}}$ (see discussion in Sec. 4 in [4]); thus it's $\sim 1\times 10^6\,\mathrm{m/s}$, much higher than the elastic sound velocity $\sim 1\times 10^3\,\mathrm{m/s}$ (and actually also higher than "ordinary sound in Fermi liquid" – also see the previous reference).

It however should be noted that we don't usually talk about the speed of zero sound in *metals*, because that kind of linear-dispersive zero sound described by the original theory of Landau only exists in charge-neutral systems, such as liquid ^{3}He; in electron systems, the zero sound is essentially the *plasmon*, which receives a gap ω_{p}. This originates from the long-range property of Coulomb interaction, which means at $\boldsymbol{q}=0$ we have a singularity, and this breaks the aforementioned condition that in the kinetic equation, the quasiparticle-interaction assumes no significant change when $\boldsymbol{q}$ is changed. Essentially, this means the $f_{\boldsymbol{p}\boldsymbol{p}'}$ function is also not well-defined. This can be explicitly checked by naively repeating the procedure to derive $f_{\boldsymbol{p}\boldsymbol{p}'}$ from the microscopic particle interaction potential: if we use the Hartree-Fock approximation

to find $f_{\boldsymbol{p}\boldsymbol{p}'}$, we get

$$f(\boldsymbol{p},\boldsymbol{p}') = V(\boldsymbol{q}=0) - \frac{1}{2}V(|\boldsymbol{p}-\boldsymbol{p}'|)(1+\boldsymbol{\sigma}\boldsymbol{\sigma}'), \tag{5}$$

with the first term being the Hartree term and the second term being the Fock term (Fig. 1). This expression gives an infinite result for Coulomb interaction, because $V(\boldsymbol{q}) = 4\pi e^2/\boldsymbol{q}^2$ now diverges at $\boldsymbol{q}=0$ [6].[1]

One way to work around this singularity is to analyze the Hartree term in the real space, while still attributing other corrections in Fig. 1 to $f_{\boldsymbol{p}\boldsymbol{p}'}$. Thus, the kinetic equations for Fermi liquid in a metal now include three equations:

- A quantum Boltzmann equation coupled to a electrostatic field φ created by $\delta n_{\boldsymbol{p}\sigma}(\boldsymbol{r})$.
- The effective single-electron energy equation

$$\varepsilon_{\boldsymbol{p}\sigma}(\boldsymbol{r}) = \varepsilon^0_{\boldsymbol{p}\sigma} + \frac{1}{V}\sum_{\boldsymbol{p}',\sigma'} f_{\boldsymbol{p}\boldsymbol{p}'\sigma\sigma'}\,\delta n_{\boldsymbol{p}'\sigma'}(\boldsymbol{r}). \tag{6}$$

- The Poisson equation

$$\nabla^2\varphi = \frac{e}{\epsilon_0}\sum_{\boldsymbol{p},\sigma}\delta n_{\boldsymbol{p}\sigma}(\boldsymbol{r}). \tag{7}$$

This equation system is usually called **Landau-Silin equation**. The presence of the Poisson equation and the Hartree self-consistent field φ means when $\boldsymbol{q}\to 0$, we see plasma oscillations at frequency

$$\omega_{\mathrm{p}} = \sqrt{\frac{ne^2}{m\epsilon_0}}. \tag{8}$$

The physical picture on the electron side is still the same: distortion of the Fermi surface propagating around, creating a density mode in the momentum space instead of the real space.

3 Particle number conservation

3.1 Proof of particle number conservation

The Hamiltonian

$$H_N = \sum_{\mathbf{k}\sigma}\varepsilon_0(\mathbf{k})c^\dagger_{\mathbf{k}\sigma}c_{\mathbf{k}\sigma} + \frac{1}{2}\sum_{\substack{\mathbf{kpq}\\ \sigma\sigma'}} v_0(\mathbf{q})c^\dagger_{\mathbf{k}+\mathbf{q}\sigma}c^\dagger_{\mathbf{p}-\mathbf{q}\sigma'}c_{\mathbf{p}\sigma'}c_{\mathbf{k}\sigma} \tag{9}$$

commutes with

$$\widehat{N} = \sum_{\mathbf{k}\sigma} c^\dagger_{\mathbf{k}\sigma}c_{\mathbf{k}\sigma}. \tag{10}$$

The proof is shown as below. We have

$$\begin{aligned}[H,N] =& \sum_{\boldsymbol{k},\sigma,\boldsymbol{k}',\sigma'}\varepsilon_0(\boldsymbol{k})[c^\dagger_{\boldsymbol{k}\sigma}c_{\boldsymbol{k}\sigma}, c^\dagger_{\boldsymbol{k}'\sigma'}c_{\boldsymbol{k}'\sigma'}]\\ &+\frac{1}{2}\sum_{\substack{\boldsymbol{k},\boldsymbol{p},\boldsymbol{q},\sigma,\\ \sigma',\boldsymbol{k}'',\sigma''}} v_0(\boldsymbol{q})[c^\dagger_{\boldsymbol{k}+\boldsymbol{q},\sigma}c^\dagger_{\boldsymbol{p}-\boldsymbol{q},\sigma'}c_{\boldsymbol{p}\sigma'}c_{\boldsymbol{k}\sigma}, c^\dagger_{\boldsymbol{k}''\sigma''}c_{\boldsymbol{k}''\sigma''}]\,.\end{aligned}$$

The commutator with the single-electron part is

$$\begin{aligned}[c^\dagger_{\boldsymbol{k}\sigma}c_{\boldsymbol{k}\sigma}, c^\dagger_{\boldsymbol{k}'\sigma'}c_{\boldsymbol{k}'\sigma'}] &= c^\dagger_{\boldsymbol{k}\sigma}c_{\boldsymbol{k}\sigma}c^\dagger_{\boldsymbol{k}'\sigma'}c_{\boldsymbol{k}'\sigma'} - c^\dagger_{\boldsymbol{k}'\sigma'}c_{\boldsymbol{k}'\sigma'}c^\dagger_{\boldsymbol{k}\sigma}c_{\boldsymbol{k}\sigma}\\ &= c^\dagger_{\boldsymbol{k}\sigma}(\delta_{\boldsymbol{k}\boldsymbol{k}'}\delta_{\sigma\sigma'} - c^\dagger_{\boldsymbol{k}'\sigma'}c_{\boldsymbol{k}\sigma})c_{\boldsymbol{k}'\sigma'} - c^\dagger_{\boldsymbol{k}'\sigma'}(\delta_{\boldsymbol{k}\boldsymbol{k}'}\delta_{\sigma\sigma'} - c^\dagger_{\boldsymbol{k}\sigma}c_{\boldsymbol{k}'\sigma'})c_{\boldsymbol{k}\sigma}\\ &= c^\dagger_{\boldsymbol{k}\sigma}c_{\boldsymbol{k}\sigma}\delta_{\boldsymbol{k}\boldsymbol{k}'}\delta_{\sigma\sigma'} - c^\dagger_{\boldsymbol{k}\sigma}c_{\boldsymbol{k}\sigma}\delta_{\boldsymbol{k}\boldsymbol{k}'}\delta_{\sigma\sigma'} = 0.\end{aligned}$$

[1] Note that we *can't* correct the Hartree term with screened Coulomb potential! The $\delta n_{\boldsymbol{p}\sigma}(\boldsymbol{r})$ variable used in the Hartree term comes from the renormalized Green function, which already contains, say, the ring diagrams that may appear in the middle of a Coulomb interaction line. If we correct the Coulomb interaction line in the Hartree term, double counting occurs. Indeed, the existence of plasmon modes is a piece of evidence for strong screening: emergence of plasmon is a byproduct of screening beside the screened Coulomb potential.

This can be expected, since fluctuation on one density mode of course doesn't inherently influence fluctuation on another. The commutator with the electron scattering part is

$$\begin{aligned}
&[c^\dagger_{\boldsymbol{k}+\boldsymbol{q},\sigma}c^\dagger_{\boldsymbol{p}-\boldsymbol{q},\sigma'}c_{\boldsymbol{p}\sigma'}c_{\boldsymbol{k}\sigma}, c^\dagger_{\boldsymbol{k}''\sigma''}c_{\boldsymbol{k}''\sigma''}]\\
&= c^\dagger_{\boldsymbol{k}+\boldsymbol{q},\sigma}c^\dagger_{\boldsymbol{p}-\boldsymbol{q},\sigma'}c_{\boldsymbol{p}\sigma'}c_{\boldsymbol{k}\sigma}c^\dagger_{\boldsymbol{k}''\sigma''}c_{\boldsymbol{k}''\sigma''} - c^\dagger_{\boldsymbol{k}''\sigma''}c_{\boldsymbol{k}''\sigma''}c^\dagger_{\boldsymbol{k}+\boldsymbol{q},\sigma}c^\dagger_{\boldsymbol{p}-\boldsymbol{q},\sigma'}c_{\boldsymbol{p}\sigma'}c_{\boldsymbol{k}\sigma}\\
&= c^\dagger_{\boldsymbol{k}+\boldsymbol{q},\sigma}c^\dagger_{\boldsymbol{p}-\boldsymbol{q},\sigma'}(\delta_{\boldsymbol{k}\boldsymbol{k}''}\delta_{\sigma\sigma''}c_{\boldsymbol{p}\sigma'} - \delta_{\boldsymbol{p}\boldsymbol{k}''}\delta_{\sigma'\sigma''}c_{\boldsymbol{k}\sigma} + c^\dagger_{\boldsymbol{k}''\sigma''}c_{\boldsymbol{p}\sigma'}c_{\boldsymbol{k}\sigma})c_{\boldsymbol{k}''\sigma''}\\
&\quad - c^\dagger_{\boldsymbol{k}''\sigma''}(c^\dagger_{\boldsymbol{k}+\boldsymbol{q},\sigma}c^\dagger_{\boldsymbol{p}-\boldsymbol{q},\sigma'}c_{\boldsymbol{k}''\sigma''} + \delta_{\boldsymbol{k}'',\boldsymbol{k}+\boldsymbol{q}}\delta_{\sigma''\sigma}c^\dagger_{\boldsymbol{p}-\boldsymbol{q},\sigma'} - \delta_{\boldsymbol{k}'',\boldsymbol{p}-\boldsymbol{q}}\delta_{\sigma''\sigma'}c^\dagger_{\boldsymbol{k}+\boldsymbol{q},\sigma})c_{\boldsymbol{p}\sigma'}c_{\boldsymbol{k}\sigma}\\
&= c^\dagger_{\boldsymbol{k}+\boldsymbol{q},\sigma}c^\dagger_{\boldsymbol{p}-\boldsymbol{q},\sigma'}c_{\boldsymbol{p}\sigma'}c_{\boldsymbol{k}\sigma}(\delta_{\boldsymbol{k}\boldsymbol{k}''} + \delta_{\sigma\sigma''}\delta_{\boldsymbol{p}\boldsymbol{k}''} - \delta_{\boldsymbol{k}'',\boldsymbol{k}+\boldsymbol{q}}\delta_{\sigma''\sigma} - \delta_{\boldsymbol{k}'',\boldsymbol{p}-\boldsymbol{q}}\delta_{\sigma''\sigma'}).
\end{aligned}$$

The above expression vanishes after we sum over $\boldsymbol{k}''$ and σ''. So we find the two terms in $[H, N]$ all vanish, and therefore $[H, N] = 0$. This means the total number of electrons is conserved in (9), which is obviously true because the scattering term in (9) only moves electrons from one state to another and does not really change the total number of them.

3.2

We have

$$\boldsymbol{S}_m = \frac{1}{2}c^\dagger_{m\alpha}\boldsymbol{\sigma}_{\alpha\beta}c_{m\beta}, \tag{11}$$

and therefore

$$\begin{aligned}
\sum_{m,n}U_{mnmn}\boldsymbol{S}_m\cdot\boldsymbol{S}_n &= \sum_{m,n,\alpha,\beta,\gamma,\delta}\frac{1}{4}U_{mnmn}c^\dagger_{m\alpha}c_{m\beta}c^\dagger_{n\gamma}c_{n\delta}\boldsymbol{\sigma}_{\alpha\beta}\cdot\boldsymbol{\sigma}_{\gamma\delta}\\
&= \frac{1}{4}\sum_{m,n,\alpha,\beta,\gamma,\delta}U_{mnmn}c^\dagger_{m\alpha}c_{m\beta}c^\dagger_{n\gamma}c_{n\delta}(2\delta_{\alpha\delta}\delta_{\beta\gamma} - \delta_{\alpha\beta}\delta_{\gamma\delta})\\
&= \frac{1}{2}\sum_{m,n,\alpha,\beta}U_{mnmn}c^\dagger_{m\alpha}c_{m\beta}c^\dagger_{n\beta}c_{n\alpha} - \frac{1}{4}\sum_{m,n,\alpha,\delta}U_{mnmn}\underbrace{c^\dagger_{m\alpha}c_{m\alpha}c^\dagger_{n\delta}c_{n\delta}}_{n_m n_n}.
\end{aligned}$$

Thus, we have

$$\begin{aligned}
\sum_{m\neq n,\sigma,\sigma'}U_{mnmn}c^\dagger_{m\sigma}c_{n\sigma'}c^\dagger_{m\sigma'}c_{n\sigma} &= -\sum_{m\neq n,\sigma,\sigma'}\underbrace{J^{\mathrm{F}}_{mn}}_{U_{mnmn}}c^\dagger_{m\sigma}c^\dagger_{m\sigma'}c_{n\sigma'}c_{n\sigma}\\
&= -2\sum_{m\neq n}J^{\mathrm{F}}_{mn}\left(\boldsymbol{S}_m\cdot\boldsymbol{S}_n + \frac{1}{4}n_m n_n\right).
\end{aligned} \tag{12}$$

References

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