Homework 3

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1

By convolution theorem we have

$$\begin{aligned}\mathcal{F}^{-1}\left[\frac{1}{(1+\mathrm{i}\omega)(2+\mathrm{i}\omega)}\right] &= \mathcal{F}^{-1}\left[\frac{1}{1+\mathrm{i}\omega}\right]\otimes\mathcal{F}^{-1}\left[\frac{1}{2+\mathrm{i}\omega}\right] = \mathrm{e}^{-x}H(x)\otimes\mathrm{e}^{-2x}H(x) \\ &= \int_{-\infty}^{\infty}\mathrm{e}^{-x'}H(x')\mathrm{e}^{-2(x-x')}H(x-x')\,\mathrm{d}x' \\ &= \mathrm{e}^{-2x}\int_0^x H(x)\mathrm{e}^{x'}\,\mathrm{d}x' = (\mathrm{e}^{-x}-\mathrm{e}^{-2x})H(x).\end{aligned} \tag{1}$$

2

We need to solve

$$\begin{aligned}u_t = 4u_{xx} \text{ for } 0<x<L, t>0 \\ u(0,t) = u(L,t) = 0, \\ u(x,0) = x^2(L-x).\end{aligned} \tag{2}$$

This can be done by standard separation of variables. Suppose $u = XT$. We have

$$XT' = 4TX'' \Rightarrow \frac{X''}{X} = \frac{T'}{4T} = \lambda.$$

Since the boundary conditions require that

$$X(0) = X(L) = 0,$$

X can't be exponential, and we have $\lambda = -k^2$, and the x part of the problem is then

$$X'' + k^2 X = 0, \quad X(0) = X(L) = 0.$$

This just gives us an odd Fourier series: we have

$$X = A\cos(kx) + B\sin(kx),$$

and since $X(0) = 0$, $A = 0$, and since $X(L) = 0$, we have

$$kL = \pi n, \quad n \in \mathbb{Z}.$$

The set of independent Xs is therefore

$$X_n = \sin(k_n x), \quad k_n = \frac{\pi n}{L}, \quad n = 1, 2, \dots \tag{3}$$

The t part of the problem is then

$$\begin{aligned}T_n' + 4k_n^2 T_n = 0, \\ T_n = T_n(0)\mathrm{e}^{-4k_n^2 t}.\end{aligned} \tag{4}$$

The general solution is therefore

$$u = \sum_{n=1}^{\infty} c_n \sin(k_n x)\mathrm{e}^{-4k_n^2 t}, \quad k_n = \frac{\pi n}{L}. \tag{5}$$

Now we apply the initial condition. We have

$$\sum_{n=1}^{\infty} c_n \sin\Big(\frac{\pi n}{L}x\Big) = x^2(L-x),$$

and therefore

$$\frac{L}{2}\cdot c_n = \int_0^L \sin\Big(\frac{\pi n}{L}x\Big)x^2(L-x)\,\mathrm{d}x = -\frac{L^4}{(n\pi)^4}(2\pi n + 4\pi n(-1)^n), \tag{6}$$

and

$$u(x,t) = -\sum_{n=1}^{\infty} \frac{4L^3(1+2(-1)^n)}{(n\pi)^3} \sin\Big(\frac{\pi n}{L}x\Big)\mathrm{e}^{-4(\pi n/L)^2 t}. \tag{7}$$

3

Since

$$(a\cos\omega x + b\sin\omega x)\mathrm{e}^{-\omega^2 kt}$$

is a specific solution of

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}, \tag{8}$$

the general solution is

$$u(x,t) = \int_0^{\infty} (a_\omega \cos\omega x + b_\omega \sin\omega x)\mathrm{e}^{-\omega^2 kt}\,\mathrm{d}\omega\,. \tag{9}$$

The initial condition

$$u(x,t=0) = f(x) = \begin{cases} \mathrm{e}^{-x}, & |x|\le 1, \\ 0, & |x|>1 \end{cases} \tag{10}$$

means

$$\int_0^{\infty} (a_\omega \cos\omega x + b_\omega \sin\omega x)\,\mathrm{d}\omega = f(x),$$

and since

$$\int_{-\infty}^{\infty} \cos\omega x\cos\omega' x\,\mathrm{d}x = \pi\delta(\omega+\omega')+\pi\delta(\omega-\omega'), \qquad \int_{-\infty}^{\infty} \sin\omega x\sin\omega' x\,\mathrm{d}x = \pi\delta(\omega-\omega')-\pi\delta(\omega+\omega'), \tag{11}$$

we have

$$\pi a_\omega = \int_{-\infty}^{\infty} f(x)\cos(\omega x)\,\mathrm{d}x = \int_{-1}^{1} \mathrm{e}^{-x}\cos(\omega x)\,\mathrm{d}x = \frac{\left(1+\mathrm{e}^2\right)\omega\sin(\omega) + \left(\mathrm{e}^2-1\right)\cos(\omega)}{\mathrm{e}\left(\omega^2+1\right)}, \tag{12}$$

$$\pi b_\omega = \int_{-\infty}^{\infty} f(x)\sin(\omega x)\,\mathrm{d}x = \int_{-1}^{1} \mathrm{e}^{-x}\sin(\omega x)\,\mathrm{d}x = \frac{\left(\mathrm{e}^2-1\right)\omega\cos(\omega) - \left(1+\mathrm{e}^2\right)\sin(\omega)}{\mathrm{e}\left(\omega^2+1\right)}. \tag{13}$$

So we have

$$\begin{aligned} u(x,t) = \frac{1}{\pi}\int_0^{\infty} \mathrm{e}^{-\omega^2 kt}\Bigg(&\frac{\left(1+\mathrm{e}^2\right)\omega\sin(\omega) + \left(\mathrm{e}^2-1\right)\cos(\omega)}{\mathrm{e}\left(\omega^2+1\right)}\cos(\omega x) \\ &+ \frac{\left(\mathrm{e}^2-1\right)\omega\cos(\omega) - \left(1+\mathrm{e}^2\right)\sin(\omega)}{\mathrm{e}\left(\omega^2+1\right)}\sin(\omega x)\Bigg)\,\mathrm{d}\omega\,. \end{aligned} \tag{14}$$

We can also write the solution using the heat kernel

$$u(x,t) = \frac{1}{2\sqrt{\pi kt}}\int_{-\infty}^{\infty} f(\xi)\mathrm{e}^{-(x-\xi)^2/4kt}\mathrm{d}\xi = \frac{1}{2\sqrt{\pi kt}}\int_{-1}^{1} \mathrm{e}^{-x}\mathrm{e}^{-(x-\xi)^2/4kt}\,\mathrm{d}\xi\,. \tag{15}$$

4

The problem is

$$\begin{aligned} y_{tt} &= c^2 y_{xx} \text{ for } 0 < x < L, t > 0 \\ y(0,t) &= y(L,t) = 0, \\ y(x,0) &= f(x), \\ y_t(x,0) &= g(x) \end{aligned} \tag{16}$$

Again we use separation of variables. Suppose

$$y = XT.$$

We have

$$XT'' = c^2 TX'' \Rightarrow \frac{X''}{X} = \frac{T''}{c^2 T} = \lambda,$$

and again because the boundary conditions, we can only have $\lambda < 0$, and therefore $\lambda = -k^2$, and

$$X'' + k^2 X = 0, \tag{17}$$

and repeating the procedure in Section 2, we get

$$X_n = \sin k_n x, \quad k_n = \frac{n\pi}{L}. \tag{18}$$

The corresponding solution for the T equation is

$$T_n'' + k_n^2 c^2 T_n = 0 \Rightarrow T_n = a_n \cos(k_n ct) + b_n \sin(k_n ct). \tag{19}$$

The general solution is therefore

$$y(x,t) = \sum_{n=0}^{\infty} (a_n \cos(k_n ct) + b_n \sin(k_n ct)) \sin k_n x, \quad k_n = \frac{n\pi}{L}. \tag{20}$$

Now we take

$$c = 5, \quad L = \pi, \quad f(x) = \sin 2x, \quad g(x) = \pi - x. \tag{21}$$

We have

$$\sum_{n=0}^{\infty} a_n \sin nx = \sin 2x,$$

and

$$a_2 = 1, \quad a_n = 0, \quad n \neq 2. \tag{22}$$

The $g(x)$ condition means

$$\sum_{n=0}^{\infty} b_n \cdot 5n \cdot \sin nx = \pi - x,$$

$$\frac{L}{2} \cdot b_n \cdot nc = \int_0^{\pi} (\pi - x) \sin nx \, \mathrm{d}x = \frac{\pi}{n} \Rightarrow b_n = \frac{2}{5n^2}.$$

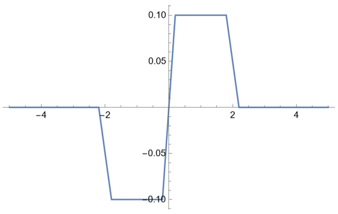
So we have

$$y(x,t) = \sum_{n=0}^{\infty} \frac{2}{5n^2} \sin(5nt) \sin(nx) + \cos(5nt) \sin(2x). \tag{23}$$

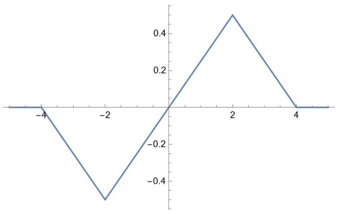
5

The wave function with velocity c has the following general solution:

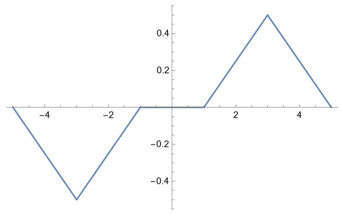
$$y(x,t) = \frac{1}{2} f(x - ct) + \frac{1}{2} f(x + ct) + \frac{1}{2c} \int_{x-ct}^{x+ct} g(x') \, \mathrm{d}x'. \tag{24}$$



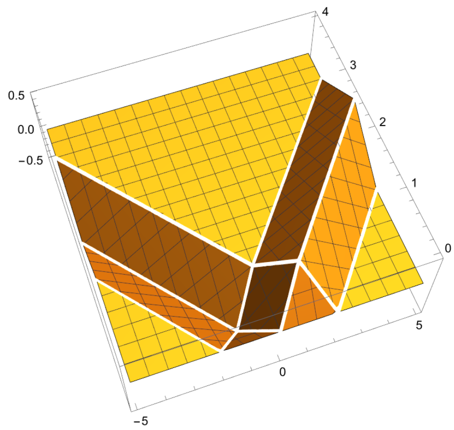
(a) $t = 0.1$



(b) $t = 1$



(c) $t = 1.5$



(d) 3D plot; the y axis represents the t variable

Figure 1: Plot of $y(x, t)$ with different t

When $c = 2$, $f(x) = 0$, and

$$g(x) = \begin{cases} 1 \text{ for} & 0 \leqslant x \leqslant 2 \\ -1 \quad \text{for} & -2 \leqslant x < 0 \\ 0 \text{ for} & |x| > 2 \end{cases}, \tag{25}$$

we have

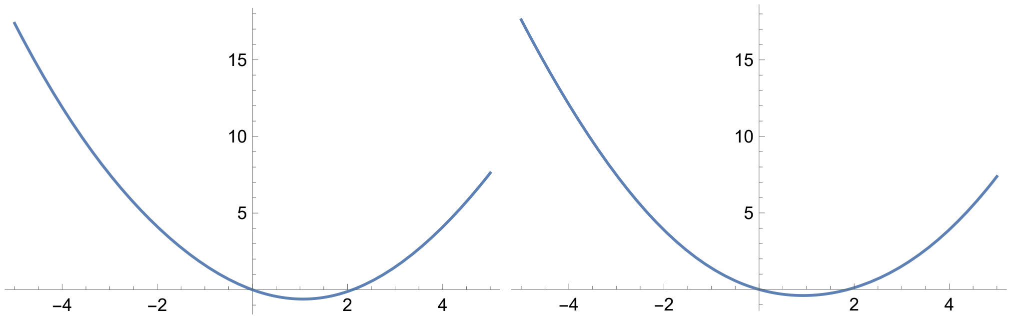
$$\begin{aligned} y(x,t) &= \frac{1}{4}\int_{\max(x-2t,-2)}^{\min(x+2t,2)} g(x')\,\mathrm{d}x' \\ &= \frac{1}{4}\begin{cases} 0, & \max(x-2t,-2) \geq 2, \\ \min(x+2t,2) - \max(x-2t,-2), & 0 < \max(x-2t,-2) < 2, \\ \min(x+2t,2) + \max(x-2t,-2), & \max(x-2t,-2) \leq 0 \leq \min(x+2t,2), \\ -\min(x+2t,2) + \max(x-2t,-2), & \min(x+2t,2) < 0, \\ 0, & \min(x+2t,2) \leq -2. \end{cases} \end{aligned} \tag{26}$$

Plots of $y(x, t)$ are shown in Figure 1.

6

We repeat the procedure in last section, but now with

$$\begin{aligned} c &= 4, \\ f(x) &= x^2 - 2x \\ g(x) &= \cos x \end{aligned} \tag{27}$$



(a) (b)

Figure 2: (a) $F(x)$ (b) $G(x)$

Now

$$\begin{aligned} y(x,t) &= \frac{1}{2}((x-4t)^2 - 2(x-4t)) + \frac{1}{2}((x+4t)^2 - 2(x+4t)) + \frac{1}{8}(\sin(x+4t) - \sin(x-4t)) \\ &= \underbrace{\frac{1}{2}((x-4t)^2 - 2(x-4t)) - \frac{1}{8}\sin(x-4t)}_{F(x-4t)} + \underbrace{\frac{1}{2}((x+4t)^2 - 2(x+4t)) + \frac{1}{8}\sin(x+4t)}_{G(x+4t)}. \end{aligned} \tag{28}$$

The plots of F and G are shown in Figure 2. It can be seen that the behavior of the solution is dominated by the $x^2 - 2x$ term. As time goes by, the parts of F and G with large $|x|$ moves to the vicinity of $x = 0$, making $y(x,t)$ larger and larger. Ignoring the contribution of $g(x)$, we get

$$y(x,t) = \frac{1}{2}((x-4t)^2 - 2(x-4t)) + \frac{1}{2}((x+4t)^2 - 2(x+4t)) = x^2 - 2x + 4t^2, \tag{29}$$

so $y(x,t)$ is roughly a parabolic being moved upwards.

7

The general solution of $\nabla^2 u = 0$ in a circle (with the implicit boundary condition that $u(r=0)$ should be well-defined) in polar coordinates, is

$$u(r,\theta) = a_0 + \sum_{n=1}^{\infty} \left[a_n r^n \cos(n\theta) + b_n r^n \sin(n\theta)\right]. \tag{30}$$

The boundary condition

$$u(r=5,\theta) = \theta\cos\theta \tag{31}$$

then means

$$2\pi a_0 = \int_0^{2\pi} \theta\cos\theta \,\mathrm{d}\theta = 0, \tag{32}$$

$$\pi \cdot a_1 \cdot 5 = \int_0^{2\pi} \theta\cos^2 \mathrm{d}\theta = \pi^2, \tag{33}$$

$$\pi \cdot a_n \cdot 5^n = \int_0^{2\pi} \cos(n\theta)\cdot\theta\cos\theta\,\mathrm{d}\theta = 0, \quad n \geq 2, \tag{34}$$

and

$$\pi \cdot b_n \cdot 5^n = \int_0^{2\pi} \sin(n\theta)\cdot\theta\cos\theta\,\mathrm{d}\theta = \begin{cases} -\frac{\pi}{2}, & n = 2, \\ -\frac{2n\pi}{n^2-1}, & \text{otherwise.} \end{cases} \tag{35}$$

So we have

$$u(r,\theta) = \frac{1}{5}r\cos\theta - \frac{1}{10}r\sin\theta - \sum_{n=2}^{\infty}\frac{r^n}{5^n}\frac{2n}{n^2-1}\sin n\theta. \tag{36}$$