Homework 3

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Problem 1
Solution

(a) The conjugate momentum of θ is

$$p = \frac{\partial L}{\partial \dot{\theta}} = V \left(\frac{\dot{\theta}}{U_0} - \frac{\mu}{U_0} \right), \tag{1}$$

and therefore

$$\dot{\theta} = \frac{U_0}{V} p + \mu. \tag{2}$$

The Hamiltonian is

$$\begin{aligned} H &= p\dot{\theta} - L \\ &= p \left(\frac{U_0}{V} p + \mu \right) - V \left(\frac{1}{2U_0} \left(\frac{U_0}{V} p + \mu \right)^2 - \frac{\mu}{U_0} \left(\frac{U_0}{V} p + \mu \right) \right) \\ &= \frac{1}{2} \frac{U_0}{V} \left(p + \frac{\mu V}{U_0} \right)^2 . \end{aligned} \tag{3}$$

In Heisenberg's picture, the variance of θ can be evaluated in the follows. We know

$$\begin{aligned} \frac{\mathrm{d}\theta^2}{\mathrm{d}t} &= \frac{1}{\mathrm{i}} [\theta^2, H] \\ &= \frac{U_0}{2\mathrm{i}V} \left[\theta^2, \left(p + \frac{\mu V}{U_0} \right)^2 \right] \\ &= \frac{U_0}{V} \left(\theta \left(p + \frac{\mu V}{U_0} \right) + \left(p + \frac{\mu V}{U_0} \right) \theta \right), \end{aligned}$$

and therefore

$$\begin{aligned} \frac{\mathrm{d}^2\theta^2}{\mathrm{d}t^2} &= \frac{U_0}{V} \left(\dot{\theta} \left(p + \frac{\mu V}{U_0} \right) + \theta \dot{p} + \dot{p}\theta + \left(p + \frac{\mu V}{U_0} \right) \dot{\theta} \right) \\ &= \frac{2U_0^2}{V^2} \left(p + \frac{\mu V}{U_0} \right)^2 . \end{aligned} \tag{4}$$

Here we use the EOMs

$$\dot{\theta} = \frac{1}{\mathrm{i}} [\theta, H] = \frac{U_0}{V} \left(p + \frac{\mu V}{U_0} \right), \quad \dot{p} = 0. \tag{5}$$

To calculate σ_θ^2, we also need $\langle \theta \rangle^2$. Its time evolution is given by

$$\begin{aligned} \frac{\mathrm{d}^2 \langle \theta \rangle}{\mathrm{d}t^2} &= 2 \langle \theta \rangle \frac{\mathrm{d}^2 \langle \theta \rangle}{\mathrm{d}t^2} + 2 \left(\frac{\mathrm{d} \langle \theta \rangle}{\mathrm{d}t} \right)^2 \\ &= 2 \langle \theta \rangle \frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{U_0}{V} \langle p \rangle + \mu \right) + 2 \left(\frac{U_0}{V} \langle p \rangle + \mu \right)^2 \\ &= \frac{2U_0^2}{V^2} \left(\langle p \rangle + \frac{\mu V}{U_0} \right)^2 . \end{aligned} \tag{6}$$

From (4) and (6), we have

$$\begin{aligned} \frac{\mathrm{d}^2 \sigma_\theta^2}{\mathrm{d}t^2} &= \frac{\mathrm{d}^2}{\mathrm{d}t^2} \left(\langle \theta^2 \rangle - \langle \theta \rangle^2 \right) = \frac{2U_0^2}{V^2} \left(\left\langle \left(p + \frac{\mu V}{U_0} \right)^2 \right\rangle - \left(\langle p \rangle + \frac{\mu V}{U_0} \right)^2 \right) \\ &= \frac{2U_0^2}{V^2} \sigma_p^2 . \end{aligned} \tag{7}$$

From uncertainty relation we have

$$\sigma_p^2 \sigma_x^2 \simeq \frac{1}{4}, \tag{8}$$

and therefore

$$\frac{\mathrm{d}^2 \sigma_\theta^2}{\mathrm{d}t^2} \simeq \frac{2U_0^2}{V^2} \frac{1}{4\sigma^2}, \tag{9}$$

and therefore

$$\sigma_\theta \simeq \sqrt{\frac{U_0^2}{4V^2\sigma^2} t^2 + \sigma^2}. \tag{10}$$

(b) The speed sound is

$$v = \sqrt{\frac{\rho_0 U_0}{m}}, \tag{11}$$

so

$$\sigma_\theta = \sqrt{\frac{m^2 v^4}{4V^2 \rho_0^2 \sigma^2} t^2 + \sigma^2}, \tag{12}$$

and the time it takes to have $\sigma_\theta = 2\pi$ is

$$t = \sqrt{\frac{4V^2 \rho_0^2 \sigma^2}{m^2 v^4} (4\pi^2 - \sigma^2)}. \tag{13}$$

For ^{4}He, we have $m = 6.65 \times 10^{-27}\,\mathrm{kg}$, and the sound speed is around $\sim 20\,\mathrm{m/s}$ [1], and the mass density – the product of ρ_0 and m – is $\sim 125\,\mathrm{g/L}$, so $t \sim 5.6 \times 10^{45}\,\mathrm{s}$.

(c) Ignoring density fluctuation, the perturbation is

$$-g \int \mathrm{d}^d \boldsymbol{x}\, (a^2 + (a^\dagger)^2) = -g \int \mathrm{d}^d \boldsymbol{x}\, n_0 (\mathrm{e}^{\mathrm{i}2\theta} + \mathrm{e}^{-\mathrm{i}2\theta}) = -2gn_0 \int \mathrm{d}^d \boldsymbol{x} \cos(2\theta). \tag{14}$$

In the uniform θ case, we further have

$$H = \frac{1}{2} \frac{U_0}{V} \left(p + \frac{\mu V}{U_0} \right)^2 - 2gn_0 V \cos(2\theta). \tag{15}$$

Problem 2
Solution

(a) Repeating the procedure used in ordinary superfluid, we do the decomposition

$$\varphi = \sqrt{\rho} \mathrm{e}^{\mathrm{i}\theta} = \sqrt{\rho_0 + \delta\rho} \mathrm{e}^{\mathrm{i}\theta}, \tag{16}$$

and therefore

$$-\frac{\varphi^* \nabla^2 \varphi}{2m} = \frac{\rho}{2m} (\boldsymbol{\nabla}\theta)^2 + \frac{(\boldsymbol{\nabla}\rho)^2}{8\rho m}, \tag{17}$$

$$\varphi^* \partial_\tau \varphi = \underbrace{\frac{1}{2} \partial_\tau \rho}_{\text{time derivative, ignored}} + \mathrm{i}\rho \partial_\tau \theta, \tag{18}$$

$$|\varphi(\boldsymbol{x})| U(\boldsymbol{x} - \boldsymbol{y}) |\varphi(\boldsymbol{y})| = \rho(\boldsymbol{x}) U(\boldsymbol{x} - \boldsymbol{y}) \rho(\boldsymbol{y}), \tag{19}$$

the theory is now

$$S = \int \mathrm{d}\tau \left(\int \mathrm{d}^d \boldsymbol{x} \left(\mathrm{i}\rho \partial_\tau \theta + \frac{\rho}{2m} (\boldsymbol{\nabla}\theta)^2 + \frac{(\boldsymbol{\nabla}\rho)^2}{8\rho m} - \mu\rho \right) + \frac{1}{2} \int \mathrm{d}^d \boldsymbol{x} \int \mathrm{d}^d \boldsymbol{y}\, \rho(\boldsymbol{x}) U(\boldsymbol{x} - \boldsymbol{y}) \rho(\boldsymbol{y}) \right). \tag{20}$$

Around the ground state, we have (note that since we are around a saddle point, the sum of all terms containing $\delta\rho$ only is always zero; the resulting theory has the form of $c_1\, \delta\rho\, \partial_\tau \theta + c_2\, \delta\rho^2$; the chemical potential term is therefore missing in the theory around the saddle point)

$$\mathrm{i}\rho \partial_\tau \theta = \underbrace{\mathrm{i}\rho_0 \partial_\tau \theta}_{\text{time derivative}} + \mathrm{i}\, \delta\rho\, \partial_\tau \theta,$$

and since $\boldsymbol{\nabla}\rho = \boldsymbol{\nabla}\,\delta\rho$, we have

$$\frac{(\boldsymbol{\nabla}\rho)^2}{8\rho m} \approx \frac{(\boldsymbol{\nabla}\,\delta\rho)^2}{8\rho_0 m},$$

ignoring the fluctuation of the ρ in the denominator. Similarly, since we are working on a low energy theory, the fluctuation of θ shouldn't be large, and we have

$$\frac{\rho}{2m}(\boldsymbol{\nabla}\theta)^2 \approx \frac{\rho_0}{2m}(\boldsymbol{\nabla}\theta)^2.$$

The theory is then

$$S = \int \mathrm{d}^{d+1}x \left(\frac{\rho_0}{2m}(\boldsymbol{\nabla}\theta)^2 + \mathrm{i}\,\delta\rho\,\partial_\tau\theta + \frac{(\boldsymbol{\nabla}\,\delta\rho)^2}{8\rho_0 m} + \frac{1}{2}\,\delta\rho\,(\boldsymbol{x}) \int \mathrm{d}^d\boldsymbol{y}\,U(\boldsymbol{x}-\boldsymbol{y})\,\delta\rho\,(\boldsymbol{y}) \right) + S_{\text{saddle}}. \tag{21}$$

Integrating out $\delta\rho$, we get

$$\begin{aligned} S_{\text{eff}} &= \int \mathrm{d}^{d+1}x\,\frac{\rho_0}{2m}(\boldsymbol{\nabla}\theta)^2 - \frac{1}{2}\int \mathrm{d}\tau \int \mathrm{d}^d\boldsymbol{x}\,\mathrm{d}^d\boldsymbol{y}\,\mathrm{i}\partial_\tau\theta(\boldsymbol{x},\tau)\frac{1}{\int \mathrm{d}^d\boldsymbol{y}\,U(\boldsymbol{x}-\boldsymbol{y}) - \frac{1}{4\rho_0 m}\nabla^2}\mathrm{i}\partial_\tau\theta(\boldsymbol{y},\tau) \\ &= \int \mathrm{d}^{d+1}x\,\frac{\rho_0}{2m}(\boldsymbol{\nabla}\theta)^2 + \frac{1}{2}\int \mathrm{d}\tau \int \mathrm{d}^d\boldsymbol{x}\,\mathrm{d}^d\boldsymbol{y}\,\partial_\tau\theta(\boldsymbol{x},\tau)G(\boldsymbol{x}-\boldsymbol{y})\partial_\tau\theta(\boldsymbol{y}), \end{aligned} \tag{22}$$

where

$$\int \mathrm{d}^d\boldsymbol{y}\,U(\boldsymbol{x}-\boldsymbol{y})G(\boldsymbol{y}-\boldsymbol{z}) - \frac{1}{4\rho_0 m}\nabla_{\boldsymbol{x}}^2 G(\boldsymbol{x}-\boldsymbol{z}) = \delta(\boldsymbol{x}-\boldsymbol{z}). \tag{23}$$

Similar to the procedure in dealing with ordinary superfluid, since we are only interested in the long wave length behaviors of θ, the ∇^2 term can be thrown away, and we have

$$\begin{aligned} \int \frac{\mathrm{d}^d\boldsymbol{p}}{(2\pi)^d}\mathrm{e}^{\mathrm{i}\boldsymbol{p}\cdot(\boldsymbol{x}-\boldsymbol{z})} &= \int \frac{\mathrm{d}^d\boldsymbol{p}}{(2\pi)^d}\int \mathrm{d}^d\boldsymbol{y}\,U(\boldsymbol{x}-\boldsymbol{y})G(\boldsymbol{p})\mathrm{e}^{\mathrm{i}\boldsymbol{p}\cdot(\boldsymbol{y}-\boldsymbol{z})} \\ &= \int \frac{\mathrm{d}^d\boldsymbol{p}}{(2\pi)^d}\int \mathrm{d}^d\boldsymbol{r}\,U(\boldsymbol{r})\mathrm{e}^{-\mathrm{i}\boldsymbol{p}\cdot\boldsymbol{r}}G(\boldsymbol{p})\mathrm{e}^{\mathrm{i}\boldsymbol{p}\cdot(\boldsymbol{x}-\boldsymbol{z})} \quad (\boldsymbol{r}=\boldsymbol{x}-\boldsymbol{y}), \end{aligned}$$

so

$$G(\boldsymbol{r}) = \int \frac{\mathrm{d}^d\boldsymbol{p}}{(2\pi)^d}\mathrm{e}^{\mathrm{i}\boldsymbol{p}\cdot\boldsymbol{r}}G(\boldsymbol{p}), \quad G(\boldsymbol{p}) = \frac{1}{U(\boldsymbol{p})} = \frac{1}{\int \mathrm{d}\boldsymbol{r}\,U(\boldsymbol{r})\mathrm{e}^{-\mathrm{i}\boldsymbol{p}\cdot\boldsymbol{r}}}. \tag{24}$$

To evaluate $G(\boldsymbol{p})$, we need to find

$$U(\boldsymbol{p}) = \int_0^\infty \mathrm{d}r\,\frac{2\pi^{d/2}}{\Gamma(d/2)}r^{d-1}\frac{U_0}{r^{d-\epsilon}} \tag{25}$$

Problem 3
Solution

1. The energy now can be exactly evaluated (N is the number of sites):

$$E = \frac{UN}{2}(M^2 - M) - \mu NM = \frac{N}{2}(UM^2 - (U+2\mu)M). \tag{26}$$

At the ground state, E is minimized. If M were continuous, we would have

$$M = \frac{U+2\mu}{2U} = \frac{1}{2} + \frac{\mu}{U}, \tag{27}$$

but it's not. So we need to find the closest integer to (27). Note that since

$$\frac{1}{2} \leq \frac{1}{2} + \frac{\mu}{U} - \left\lfloor \frac{\mu}{U} \right\rfloor < \frac{3}{2},$$

the following M is always a minimum point:

$$M = \left\lfloor \frac{\mu}{U} \right\rfloor + 1. \tag{28}$$

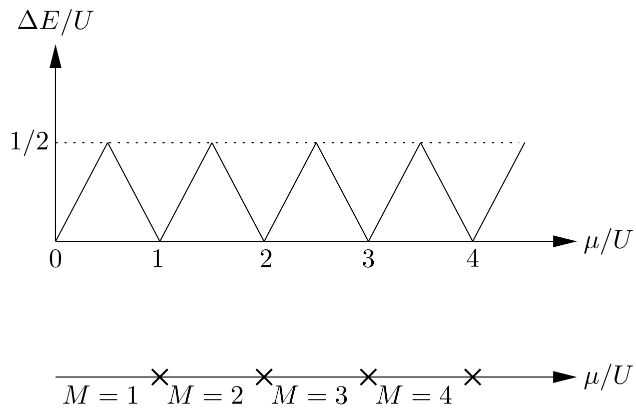

Figure 1: The energy gap plot and the phase diagram when $t = 0$

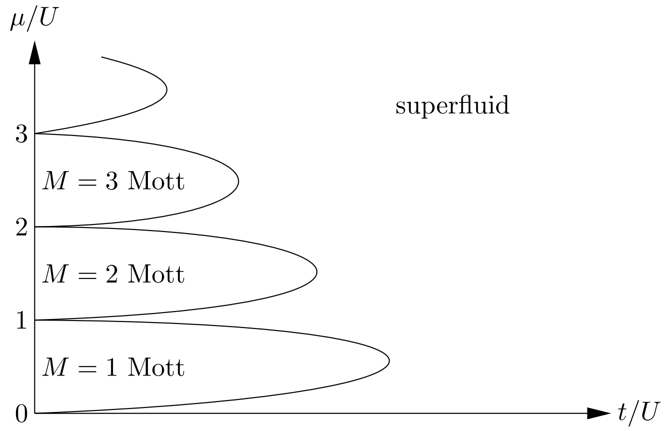

Figure 2: Schematic phase diagram of the boson Hubbard model

When μ/U is an integer, both

$$M = \frac{\mu}{U} \tag{29}$$

and

$$M = \left\lfloor \frac{\mu}{U} \right\rfloor + 1 = \frac{\mu}{U} + 1 \tag{30}$$

can be found in ground states.

The energy gap is

$$\begin{aligned} \Delta E &= \min(E|_{n_i = M+1 \text{ on one site}} - E|_M, E|_{n_i = M-1 \text{ on one site}} - E|_M) \\ &= \min(UM - \mu, U(-M+1) + \mu) \\ &= \begin{cases} 0 \text{ or } U, & \mu/U \text{ integer,} \\ \min\left(U\left\lfloor \frac{\mu}{U} \right\rfloor + U - \mu, \mu - U\left\lfloor \frac{\mu}{U} \right\rfloor\right), & \text{otherwise.} \end{cases} \end{aligned} \tag{31}$$

So

$$\frac{\Delta E}{U} = \begin{cases} \frac{\mu}{U} - \left\lfloor \frac{\mu}{U} \right\rfloor, & \frac{\mu}{U} - \left\lfloor \frac{\mu}{U} \right\rfloor \le \frac{1}{2}, \\ 1 + \left\lfloor \frac{\mu}{U} \right\rfloor - \frac{\mu}{U}, & \frac{\mu}{U} - \left\lfloor \frac{\mu}{U} \right\rfloor \ge \frac{1}{2}. \end{cases} \tag{32}$$

The energy gap and the phase diagram are shown in Figure 1.

2. The gapless points in Figure 1 can only be connected to the superfluid phase, and therefore we get Figure 2.

3. We have

$$\langle n_0 + k' | a | n_0 + k \rangle = \sqrt{n_0 + k} \, \langle n_0 + k' | n_0 + k - 1 \rangle = \sqrt{n_0 + k} \delta_{k', k-1}, \tag{33}$$

and

$$\langle k'|\mathrm{e}^{-\mathrm{i}\theta}|k\rangle = \int_0^{2\pi} \frac{\mathrm{d}\theta}{2\pi} \mathrm{e}^{-\mathrm{i}k'\theta}\mathrm{e}^{-\mathrm{i}\theta}\mathrm{e}^{\mathrm{i}k\theta} = \delta_{k',k-1}. \tag{34}$$

When $k \ll n_0$, we have

$$\langle n_0 + k'|a|n_0 + k\rangle \approx \sqrt{n_0}\, \langle k'|\mathrm{e}^{-\mathrm{i}\theta}|k\rangle \Rightarrow a \approx \sqrt{n_0}\mathrm{e}^{-\mathrm{i}\theta}. \tag{35}$$

And similarly we have

$$\langle n_0 + k'|a^\dagger|n_0 + k\rangle = \sqrt{n_0 + k + 1}\delta_{k',k+1}, \tag{36}$$

and

$$\langle k'|\mathrm{e}^{\mathrm{i}\theta}|k\rangle = \int_0^{2\pi} \frac{\mathrm{d}\theta}{2\pi} \mathrm{e}^{-\mathrm{i}k\theta}\mathrm{e}^{\mathrm{i}\theta}\mathrm{e}^{\mathrm{i}k'\theta} = \delta_{k',k+1}, \tag{37}$$

and in the $k \ll n_0$ limit we have

$$a^\dagger \approx \sqrt{n_0}\mathrm{e}^{\mathrm{i}\theta}. \tag{38}$$

Also,

$$\langle n_0 + k'|n|n_0 + k\rangle = (n_0 + k)\, \langle n_0 + k|n_0 + k'\rangle = (n_0 + k)\delta_{kk'}, \tag{39}$$

and

$$\langle k'|\pi|k\rangle = \int_0^{2\pi} \mathrm{d}\theta\, \mathrm{e}^{-\mathrm{i}k'\theta}(-\mathrm{i}\partial_\theta)\mathrm{e}^{\mathrm{i}k\theta} = k\delta_{kk'}, \tag{40}$$

so

$$n = n_0 + \pi. \tag{41}$$

4. When the above approximation works,

$$\begin{aligned} H &= -tn_0 \sum_{\langle i,j\rangle} \mathrm{e}^{\mathrm{i}\theta_i}\mathrm{e}^{-\mathrm{i}\theta_j} + \text{h.c.} + \frac{U}{2}\sum_i (n_0 + \pi_i)(n_0 + \pi_i - 1) - \mu \sum_i (n_0 + \pi_i) \\ &= -2tn_0 \sum_{\langle i,j\rangle} \cos(\theta_i - \theta_j) + \frac{U}{2}\sum_i \pi_i^2 + \text{linear terms in } \pi + f(n_0). \end{aligned}$$

The linear terms of π can be safely ignored because we are working around $n = n_0$ that minimize the energy, and $\frac{\partial E}{\partial n} = 0$ at $n = n_0$. Since π is the fluctuation of n, linear terms in π correspond to first order Taylor terms and are also bound to be zero. Indeed, this condition connects n_0 and μ:

$$\frac{U}{2}(2n_0\pi_i - \pi_i) - \mu\pi_0 = 0 \Rightarrow n_0 = \frac{1}{2} + \frac{\mu}{U}. \tag{42}$$

So

$$H = -J \sum_{\langle i,j\rangle} \cos(\theta_i - \theta_j) + u \sum_i \pi_i^2, \quad J = 2n_0 t, \quad u = \frac{U}{2}. \tag{43}$$

5. In the limit of slow spatial varying, we have

$$H = -J \sum_{\langle i,j\rangle} \left(1 - \frac{1}{2}(\theta_i - \theta_j)^2\right) + u \sum_i \pi_i^2. \tag{44}$$

This Hamiltonian has exactly the same form of the lattice phonon Hamiltonian. The operator EOMs are

$$\begin{aligned} \frac{\mathrm{d}\pi_i}{\mathrm{d}t} &= \frac{1}{\mathrm{i}}[\pi_i, H] = -J \sum_{\langle i,j\rangle} (\theta_i - \theta_j), \\ \frac{\mathrm{d}\theta_i}{\mathrm{d}t} &= \frac{1}{\mathrm{i}}[\theta_i, H] = 2u\pi_i, \end{aligned} \tag{45}$$

and therefore

$$\frac{\mathrm{d}^2\theta_i}{\mathrm{d}t^2} = 2Ju \sum_{\langle i,j\rangle} (\theta_j - \theta_i). \tag{46}$$

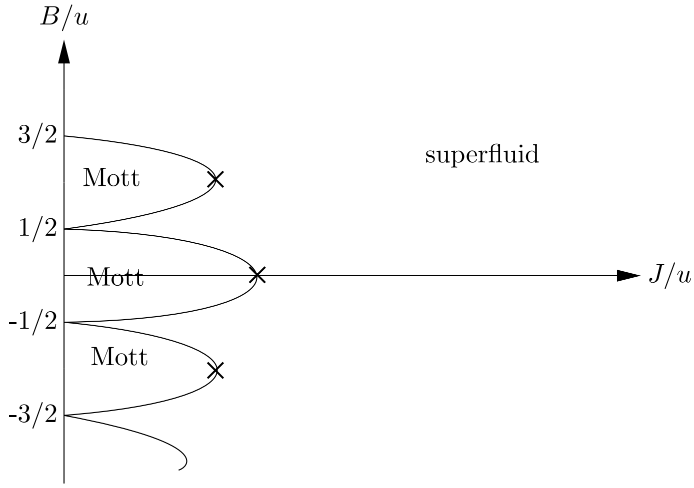

Figure 3: Phase diagram of (52)

Suppose the bond length of the lattice is a. In a normal mode

$$\theta_i \propto \mathrm{e}^{\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{r}-\omega_{\boldsymbol{k}}t)}, \tag{47}$$

we have

$$\begin{aligned}
-\omega_{\boldsymbol{k}}^2 &= 2Ju\sum_{i=1}^{d}(\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot a\hat{\boldsymbol{x}}_i}+\mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot a\hat{\boldsymbol{x}}_i}-2)\\
&= 2Ju\sum_{i=1}^{d}(2\cos(\boldsymbol{k}\cdot a\hat{\boldsymbol{x}}_i)-2)\\
&\approx -2Ju\sum_{i=1}^{d}(\boldsymbol{k}\cdot a\hat{\boldsymbol{x}}_i)^2\\
&= -2Jua^2\boldsymbol{k}^2,
\end{aligned}$$

so

$$\omega_{\boldsymbol{k}} = \sqrt{2Ju}a|\boldsymbol{k}|. \tag{48}$$

6. When $J/u=0$, we have

$$H = \sum_i (u\pi_i^2 - B\pi_i). \tag{49}$$

Again we apply the same procedure used to derive Figure 2. In the ground state, we have

$$\pi_i = \begin{cases} \left\lfloor \frac{B}{2u}\right\rfloor, & 0 \le \frac{B}{2u}-\left\lfloor\frac{B}{2u}\right\rfloor \le \frac{1}{2},\\ \left\lfloor \frac{B}{2u}\right\rfloor + 1, & \frac{1}{2} \le \frac{B}{2u}-\left\lfloor\frac{B}{2u}\right\rfloor < 1,\end{cases} \tag{50}$$

and when

$$\frac{B}{2u}-\left\lfloor\frac{B}{2u}\right\rfloor = \frac{1}{2}, \tag{51}$$

changing π_i on one site doesn't change the energy, and we get a gapless system. So on the phase diagram, points defined by (51) are connected to the superfluid phase, and the phase diagram of

$$H = -J\sum_{\langle i,j\rangle}\cos(\theta_i-\theta_j) + u\sum_i \pi_i^2 - B\sum_i \pi_i, \tag{52}$$

is given in Figure 3.

7. We have

$$\dot{\theta}_i = \frac{\partial H}{\partial \pi_i} = 2u\pi_i - B, \quad \pi_i = \frac{\dot{\theta}_i + B}{2u}. \tag{53}$$

So

$$\begin{aligned} L &= \sum_i \dot\theta_i \pi_i - H \\ &= \sum_i \dot\theta_i \frac{\dot\theta_i + B}{2u} + J \sum_{\langle i,j\rangle} \cos(\theta_i - \theta_j) - u \sum_i \left(\frac{\dot\theta_i + B}{2u}\right)^2 + B \sum_i \frac{\dot\theta_i + B}{2u} \\ &= \frac{1}{4u} \sum_i (\dot\theta_i + B)^2 + J \sum_{\langle i,j\rangle} \cos(\theta_i - \theta_j). \end{aligned} \tag{54}$$

In the continuous limit, we have

$$\sum_{\langle i,j\rangle} \cos(\theta_i - \theta_j) \to \sum_{\langle i,j\rangle} \left(1 - \frac{1}{2}(\theta_i - \theta_j)^2\right) \simeq - \sum_{\langle i,j\rangle} \frac{1}{2}(\theta_i - \theta_j)^2 = -\frac{1}{2} a^2 (\boldsymbol{\nabla}\theta)^2, \tag{55}$$

and

$$\begin{aligned} L &= \int \frac{\mathrm{d}^d\boldsymbol{x}}{a^d} \frac{1}{4u} (\dot\theta + B)^2 - \int \frac{\mathrm{d}^d\boldsymbol{x}}{a^d} \frac{J}{2} a^2 (\boldsymbol{\nabla}\theta)^2 \\ &= \int \mathrm{d}^d\boldsymbol{x} \left(\frac{1}{4ua^d}(\dot\theta + B)^2 - \frac{J}{2} a^{d-2} (\boldsymbol{\nabla}\theta)^2\right). \end{aligned} \tag{56}$$

8. In the imaginary time path integral,

$$\mathrm{i} \int \mathrm{d}t\, \frac{B}{2u} \dot\theta_i \longrightarrow \mathrm{i} \int_0^\beta \mathrm{d}\tau\, \frac{B}{2u} \partial_\tau \theta = \frac{B}{2u} \mathrm{i}(\theta_i(\beta) - \theta_i(0)). \tag{57}$$

Since in the imaginary time integral, the final state and the initial state have to be the same, $\theta_i(\beta)$ and $\theta_i(0)$ have to be equivalent to each other, and therefore

$$\theta_i(\beta) - \theta_i(0) = 2\pi n, \quad n \in \mathbb{Z}. \tag{58}$$

So when $B/2u$ is an integer, (57) contributes nothing to the partition function. The points are plotted on Figure 3. Comparing the positions of these points on Figure 3 and on the original model Figure 2, we find the condition that the $\partial_\tau \theta$ term in (56) can be ignored is equivalent to

$$\frac{\mu}{U} = \frac{1}{2} + n, \quad n = 0, 1, 2, \ldots,$$

and from (42) we know this is equivalent to the condition that n_0 is an integer.

References

[1] C. T. Lane, Henry A. Fairbank, and William M. Fairbank. Second sound in liquid helium ii. *Phys. Rev.*, 71:600–605, May 1947.