ODEs

Jinyuan Wu

January 31, 2023

1 First order ODEs

1.1 Linear ODEs

An ODE in the form of

$$y'(x) + p(x)y(x) = q(x) \tag{1}$$

is considered **linear**. All linear ODEs can be solved by the following procedure. First we have

$$(y' + py)e^{\int p \mathrm{d}x} = q e^{\int p \mathrm{d}x}, \tag{2}$$

and now the LHS is a derivative:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(y e^{\int p \mathrm{d}x}\right) = q e^{\int p \mathrm{d}x}, \tag{3}$$

and now we can integrate over x and get

$$y e^{\int p \mathrm{d}x} = \int q e^{\int p \mathrm{d}x}\, \mathrm{d}x\,, \tag{4}$$

$$y = e^{-\int p \mathrm{d}x}\int q e^{\int p \mathrm{d}x}\, \mathrm{d}x\,. \tag{5}$$

1.2 "Energy-conservation lines" and exact equations

Another way to represent the solution of an ODE is the form $\phi(x, y) = \text{const}$. Note that the RHS contains no variables, and we have

$$0 = \frac{\mathrm{d}\phi}{\mathrm{d}x} = \frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}x}, \tag{6}$$

and thus if

$$y' = f(x, y) \tag{7}$$

is algebraically equivalent to (6), the equation is already solved: We should find M, N such that

$$y' = -\frac{M}{N}, \quad M = \frac{\partial \phi}{\partial x}, \quad N = \frac{\partial \phi}{\partial y}, \tag{8}$$

and then $\phi(x, y)$ solves the equation. In this case we say $y' = -M/N$ is **exact**.

To test for exactness, we only have to test whether

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}, \tag{9}$$

and if so, the existence of ϕ is guaranteed. (Since we work on a topological trivial space, things like cohomology group will not bother us.) We can now use "partial integral" to find ϕ.

Example: suppose in a calculation we find

$$\frac{\partial \phi}{\partial x} = 2y^2 + y e^{xy}, \quad \frac{\partial \phi}{\partial y} = 4xy + x e^{xy} + 2y. \tag{10}$$

After partial integration, we find

$$\phi(x, y) = \underbrace{2xy^2 + e^{xy} + h(y)}_{\int \frac{\partial \phi}{\partial x}\mathrm{d}x} = \underbrace{2xy^2 + e^{xy} + y^2 + g(x)}_{\int \frac{\partial \phi}{\partial y}\mathrm{d}y}, \tag{11}$$

and we have to choose

$$h(y) = y^2, \quad g(x) = \text{const}, \tag{12}$$

and the solution is

$$\phi(x, y) = 2xy^2 + e^{xy} + y^2 + \text{const}. \tag{13}$$

Note that even when the decomposition $f = -M/N$ doesn't give an exact equation for us, we can still use the method of exact equations: we can multiply a factor μ to both M and N, and try to guess the form of μ so that

$$\frac{\partial(\mu M)}{\partial y} = \frac{\partial(\mu N)}{\partial x}. \tag{14}$$

An example can be found in solving

$$y' = -\frac{1}{3x - e^{-2y}}. \tag{15}$$

We have

$$\frac{\partial 1}{\partial y} = 0, \quad \frac{\partial(3x - e^{-2y})}{\partial x} = 3,$$

so the equation is not exact if we choose $M = 1$ and $N = 3x - e^{-2y}$. However, (14) can be fulfilled now: it's now

$$\frac{\partial \mu}{\partial y} = 3\mu + \left(3x - e^{-2y}\right)\frac{\partial \mu}{\partial x},$$

and the most convenient way to solve it (we *don't* need to find all solutions of this equation!) is to let μ contain y only, so the tricky term on the RHS disappears, and thus we choose $\mu = e^{3y}$, and we get

$$\phi(x, y) = \int \mu M \, dx = \int e^{3y} \, dx = xe^{3y} + u(y),$$

$$\phi(x, y) = \int \mu N \, dy = \int (3xe^{3y} - e^y) \, dy = xe^{3y} - e^y + v(x),$$

so

$$\phi(x, y) = xe^{3y} - e^y + \text{const}. \tag{16}$$

1.3 Bernoulli equation

Consider the following **Bernoulli equation**

$$y' + P(x)y = R(x)y^\alpha. \tag{17}$$

When $\alpha = 0, 1$, the equation can be solved by the standard methods for linear first order ODEs. When this is not the case, we may do the substitution

$$v = y^\beta, \tag{18}$$

and then the equation becomes

$$\frac{1}{\beta}v^{1/\beta - 1}v' + P(x)v^{1/\beta} = R(x)v^{\alpha/\beta},$$

$$v' + P(x)v = R(x)v^{1 + \frac{\alpha - 1}{\beta}}. \tag{19}$$

The next step is to choose a good beta so that the equation gets simplified. We may want to make to exponent to be zero, and this means we should choose

$$\beta = 1 - \alpha, \tag{20}$$

and the ODE is now

$$v' + Pu = R, \tag{21}$$

which can then be solved by the method in Section 1.1.

2 Second order ODEs

2.1 Linear 2nd order ODE with initial values

A linear second order ODE has the following form:

$$y'' + p(x)y' + q(x)y = f(x). \tag{22}$$

It usually comes with initial value conditions

$$y(x_0) = A, \quad y'(x_0) = B. \tag{23}$$

This course is about concrete calculations, but knowing what we are doing makes sense is important. Here is an existence and uniqueness theorem: if $p(x), q(x)$, and $f(x)$ are continuous over an interval I, and $x_0 \in I$, then a unique solution exists for (22) with the initial conditions given above.

Usually, we start by looking at the **homogeneous** second order ODE

$$y'' + p(x)y' + q(x)y = 0. \tag{24}$$

The influence of $f(x)$ can be included as the "response" of the LHS. The full solution of (24) takes the form

$$y = c_1 y_1 + c_2 y_2, \tag{25}$$

where c_1, c_2 are constants to be decided by initial conditions, and y_1 and y_2 are linearly independent solutions of (24). The **Wronskian** is defined as

$$W(x) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}. \tag{26}$$

By checking if it is non-zero at most points, we can find whether y_1 and y_2 are truly linearly independent to each other.

There is a method to arrive at y_2 from y_1: we can always take the ansatz

$$y_2 = y_1 u, \tag{27}$$

and therefore we get

$$(u'' y_1 + 2u' y_1' + u y_1'') + p(u' y_1 + u y_1') + q u y_1 = 0,$$

and the condition that y_1 is a solution to (24) means

$$u'' + \underbrace{\frac{2y_1' + py_1}{y_1}}_{g(x)} u' = 0, \tag{28}$$

which is essentially a first order ODE, because we can replace u' by v, and then we find

$$\ln v = -\int g(x)\, \mathrm{d}x,$$

and

$$u(x) = \int e^{-\int g(x)\mathrm{d}x}\, \mathrm{d}x. \tag{29}$$

2.2 Constant coefficients

The equation

$$y'' + Ay + By = 0 \tag{30}$$

can be solved directly by the following construction:

$$y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}, \tag{31}$$

3

where λ_1, λ_2 are solutions of

$$\lambda^2 + A\lambda + B = 0. \tag{32}$$

For example, to solve the equation

$$y'' - 2y' + 10y = 0, \tag{33}$$

we just solve

$$\lambda^2 - 2\lambda + 10 = 0,$$

which gives us

$$\lambda = 1 \pm 3\mathrm{i}, \tag{34}$$

and therefore a general solution is

$$y = \mathrm{e}^x(c_1\mathrm{e}^{3\mathrm{i}x} + c_2\mathrm{e}^{-3\mathrm{i}x}). \tag{35}$$

It should be noted that c_1, c_2 can be complex, even when we restrict y in $\mathbb{R}$: we can let the imaginary part of y vanish as long as we impose some constraints over c_1, c_2. If we are determined to work in the real space, two alternative linearly independent solutions can be used:

$$y_1(x) = \mathrm{e}^x \cos(3x), \quad y_2(x) = \mathrm{e}^x \sin(3x). \tag{36}$$

Although we can immediately say they are linearly independent, we can use them as a demonstration of the Wronskian method: now we have

$$W(x) = y_1(x)y_2'(x) - y_2(x)y_1'(x) = 3\mathrm{e}^{2x}, \tag{37}$$

which of course isn't constantly zero.

(32) is faced with the problem of having only one solution when $A^2 - 4B = 0$. In this case we need to go back to the standard procedure to get y_2 from y_1. An example is

$$y'' + 6y + 9 = 0, \tag{38}$$

for which (32) only gives

$$y_1 = \mathrm{e}^{-3x}. \tag{39}$$

Suppose $y_2 = u\mathrm{e}^{-3x}$, we have TODO

2.3 Euler equation

A **Euler equation** has the following form:

$$x^2y'' + Axy' + By = 0, \tag{40}$$

where A, B are constants. One solution can be immediate found: it always looks like

$$y = x^a. \tag{41}$$

We then find

$$a(a - 1) + Aa + B = 0. \tag{42}$$

If there are two solutions of the equation, (40) has already been solved. If not, we can use the trick (27).

An example: let's solve

$$x^2y'' + 3xy' + y = 0. \tag{43}$$

The equation about a is now

$$a(a - 1) + 3a + 1 = 0,$$

and it only has one solution $a = -1$. Therefore we have

$$y_1 = \frac{1}{x}.$$

Suppose
$$y_2 = uy_1,$$
we get
$$x^2 \left(\frac{u''}{x^2} - \frac{2u'}{x} + \frac{2u}{x^2} \right) + 3x \left(\frac{u'}{x} - \frac{u}{x^2} \right) + \frac{u}{x} = 0,$$
which is equivalent to
$$v'x + v = 0, \quad v = u'$$
the solution of which is
$$\ln v + \ln x = \text{const},$$
and therefore
$$v = \frac{C'}{x}, \quad u = C' \ln x + C,$$
$$y_2 = \frac{1}{x}(C' \ln x + C).$$

This essentially gives *all* solutions we need: for y_1, we just have $u = 1$, which corresponds to $C = 1$. So now the equation is completely solved.

2.4 Non-homogeneous cases or how to find the linear response

Now we discuss how to solve
$$y'' + p(x)y' + q(x)y = f(x). \tag{44}$$
A general solution is
$$y(x) = y_\text{p}(x) + y_\text{h}(x), \tag{45}$$
where the subscript p means a particular solution, and the subscript h means the general solution of the corresponding homogeneous equation.

We need some common sense to find a particular solution. To solve
$$y'' - y' - 2y = 2x^2 + 5, \tag{46}$$
we don't expect y to be, say, $\cos(2x)$: instead, it's usually the case that y is a polynomial. An ansatz is
$$y = Ax^2 + Bx + C.$$
We don't want a x^3 term because it doesn't appear on RHS. The equation then becomes
$$2A - (2Ax + Bx) - 2(Ax^2 + Bx + C) = 2x^2 + 5,$$
$$-2A = 2, \quad -2A - 2B = 0, \quad 2A - B - 2C = 5,$$
and therefore $A = -1, B = 1, C = -4$. Therefore we get a particular solution:
$$y_\text{p} = -x^2 + x - 4. \tag{47}$$

A particular solution may also be determined as a "linear combination" of homogeneous solutions, although now the coefficients have temporal variation. That's to say, we take the ansatz
$$y_\text{p}(x) = u(x)y_1(x) + v(x)y_2(x). \tag{48}$$
This is quite similar to the procedure introduced in Section 1.1. After substituting y with (48) in (44), we get
$$u'y_1' + v'y_2' = f. \tag{49}$$
Introducing the constraint
$$u'y_1 + v'y_2 = 0, \tag{50}$$
we have
$$\begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix} \begin{pmatrix} u' \\ v' \end{pmatrix} = \begin{pmatrix} 0 \\ f \end{pmatrix}, \tag{51}$$

from which we find u', v' and hence u, v. The Wronskian – the determinant of the matrix on LHS – is non-zero, so the equation always has a solution.

Example: let's solve

$$y'' + y = \tan x. \tag{52}$$

We have

$$y_1 = \cos x, \quad y_2 = \sin x,$$

and therefore

$$W(x) = y_1 y_2' - y_2 y_1' = 1.$$

So

$$u' = -\int \frac{y_2 f}{W(x)}\, dx = -\int \frac{\sin^2 x}{\cos x}\, dx = -\int \frac{1 - \cos^2 x}{\cos x}\, dx$$
$$= -\frac{1}{2} \ln \left| \frac{1 + \sin x}{1 - \sin x} \right| + \sin x,$$

and similarly we have

$$v =$$

2.5   Analyticity

The stimulus can be non-analytic. f is analytic at x_0, if we can expand it into a power series around x_0:

$$f(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n \tag{53}$$

in a interval around x_0. The function $f(x) = \ln x$, then, is not analytic at $x = 0$ – but $f(x) = \ln(x + 1)$ is analytic at $x = 0$, though not at $x = -1$.

There is a theorem: if p, q, f are all analytic at x_0, then (44) together with conditions $y(x_0) = A$ and $y'(x_0) = B$ has a unique solution that is analytic at x_0.

An example:

$$x'' + y' - xy = 0, \quad y(0) = -2, y'(0) = 0. \tag{54}$$

Suppose

$$y(x) = \sum_{n=0}^{\infty} a_n x^n, \quad y'(x) = \sum_{n=0}^{\infty} n a_n x^{n-1}, \quad y''(x) = \sum_{n=0}^{\infty} n(n + 1) a_n x^{n-2},$$

we have

$$0 = \sum_{n=0}^{\infty} n(n-1) a_n x^{n-2} + \sum_{n=0}^{\infty} n a_n x^{n-1} - \sum_{n=0}^{\infty} a_n x^{n+1}$$

$$= a_1 + 2a_2 + \sum_{n=1}^{\infty} x^n \left((n+2)(n+1) a_{n+2} + (n+1) a_{n+1} - a_{n-1} \right),$$

and therefore

$$a_1 + 2a_2 = 0, \quad a_{n+2} = -\frac{a_{n+1}}{n+2} + \frac{a_{n-1}}{(n+1)(n+2)}.$$

The conditions $y(0) = -2$ and $y'(0) = 0$ means

$$a_0 = -2, \quad a_1 = 0,$$

and then we can in principle find all a_n's – although it's often hard to see a pattern and write down a closed-form expression for a_n.

Now we consider the equation

$$P(x)y'' + Q(x)y' + R(x)y = F(x). \tag{55}$$

Of course we can divide the equation with $P(x)$ and go back to (44), but if $P(x)$ is zero at some points, p, q, f in (44) are no longer always analytic. Thus the solution isn't guaranteed to be

analytic everywhere. In other words, we no longer have a power series solution – or do we? We can still try a generalized power series, like

$$y(x) = \sum_{n=0}^{\infty} c_n (x - x_0)^{n+r}, \tag{56}$$

where r can be a fraction. This is guaranteed with

$$(x - x_0)y'' + Q(x)y' + R(x)y = F(x). \tag{57}$$

To demonstrate this, consider

$$y'' + \frac{1}{2x}y' - \frac{1}{4x}y = 0. \tag{58}$$

We plug

$$y(x) = \sum_{n=0}^{\infty} c_n x^{n+r}$$

into

$$4xy'' + 2y' - y = 0,$$

and get

$$\sum_{n=0}^{\infty} \left(4(n+r)(n+r-1)c_n x^{n+r-1} + 2(n+r)c_n x^{n+r-1} - c_n x^{n+r} \right) = 0.$$

The coefficient of the x^{r-1} term is

$$4r(r-1) + 2r = 0,$$

from which we find $r = 0, 1/2$. For the rest of the terms, we have

$$4(n+r)(n+r-1)c_n + 2(n+r)c_n - c_{n-1} = 0,$$

$$c_n = \frac{c_{n-1}}{2(n+r)(2n+2r-1)}.$$

When $r = 0$, this gives

$$c_n = \frac{c_{n-1}}{2n(2n-1)}, \quad c_n = \frac{c_0}{(2n)!},$$

and we get

$$y(x) = \sum_{n=0}^{\infty} \frac{x^n}{(2n)!}. \tag{59}$$

When $r = 1/2$, this gives

$$c_n = \frac{c_{n-1}}{(2n+1) \cdot 2n}, \quad c_n = \frac{c_0}{(2n+1)!},$$

and we get

$$y(x) = \sum_{n=0}^{\infty} \frac{x^{n+1/2}}{(2n+1)!}. \tag{60}$$

So we have already obtained two independent solutions.

3   The Laplace transformation

The **Laplace transformation** is defined for a function $f(t)$ as

$$\mathcal{L}[f(t)] = F(s) = \int_0^{\infty} f(t)e^{-st}\, dt, \tag{61}$$

for every s where the integral converges. It's easy to see that $\mathcal{L}$ is linear,

3.1 Some examples

The simplest Laplace transform is

$$\mathcal{L}[1] = \int_0^\infty e^{-st}\,\mathrm{d}t = \frac{1}{s}, \quad s > 0, \tag{62}$$

and by integration by parts, we also have

$$\mathcal{L}[t] = \int_0^\infty t e^{-st}\,\mathrm{d}t = \frac{1}{s^2}. \tag{63}$$

By multiple rounds of integration by parts, we have

$$\mathcal{L}[t^2] = \frac{2}{s^3}, \tag{64}$$

and by iteratively using integration by parts we get

$$\mathcal{L}[t^n] = \frac{n!}{s^{n+1}}. \tag{65}$$

Aside from polynomials, we have

$$\mathcal{L}[e^{at}] = \int_0^\infty e^{-(s-a)t}\,\mathrm{d}t = \frac{1}{s-a}, \quad s > a. \tag{66}$$

Since we have

$$\cos x = \frac{e^{ix} + e^{-ix}}{2}, \quad \sin x = \frac{e^{ix} - e^{-ix}}{2i},$$

we have

$$\mathcal{L}[\cos(at)] = \frac{1}{2}\left(\frac{1}{s-ia} + \frac{1}{s+ia}\right) = \frac{s}{s^2+a^2}. \tag{67}$$

Similarly we have

$$\mathcal{L}[\sin(at)] = \frac{a}{s^2+a^2}. \tag{68}$$

3.2 Laplace transform of differential equations

We have

$$\mathcal{L}[f'(t)] = \int_0^\infty \frac{\mathrm{d}f}{\mathrm{d}t}e^{-st}\,\mathrm{d}t = f e^{-st}\Big|_{s=0}^\infty - \int_0^\infty f\frac{\mathrm{d}}{\mathrm{d}t}e^{-st}\,\mathrm{d}t$$
$$= -f(0) + s\int_0^\infty f(t)e^{-st}\,\mathrm{d}t = s\mathcal{L}[f(t)] - f(0). \tag{69}$$

Applying this twice, we get

$$\mathcal{L}[f''(t)] = s\mathcal{L}[f'(t)] - f'(0) = s^2\mathcal{L}[f(t)] - sf(0) - f'(0). \tag{70}$$

The general formula is therefore

$$\mathcal{L}[f^{(n)}(t)] = s^n\mathcal{L}[f(t)] - s^{n-1}f(0) - s^{n-2}f'(0) - \cdots - f^{(n-1)}(0). \tag{71}$$

This can be used to solve ODEs. Consider, for example,

$$y'' + 4y' + 3y = e^t, \quad y(0) = 0, \quad y'(0) = 2. \tag{72}$$

We have (below we follow the convention to use big letters to refer to functions in the Laplace space)

$$\mathcal{L}[\text{LHS}] = (s^2Y(s) - sy(0) - y'(0)) + 4(sY(s) - y(0)) + 3Y(s) = (s^2 + 4s + 3)Y(s) - 2,$$

where we have already applied the initial conditions, and

$$\mathcal{L}[\text{RHS}] = \frac{1}{s-1}.$$

So what need to be done is to solve

$$(s^2 + 4s + 3)Y(s) - 2 = \frac{1}{s-1},$$

and we get

$$Y(s) = \frac{2s - 1}{(s-1)(s+1)(s+3)}. \tag{73}$$

Thus, once we do the inverse Laplace transformation, we get $y(t)$. It's possible to do an inverse integral transformation, but in this case, what's more convenient is to make the decomposition

$$Y(s) = \frac{A}{s-1} + \frac{B}{s+1} + \frac{C}{s+3},$$

and we can find

$$A = \frac{1}{8}, \quad B = \frac{3}{4}, \quad C = -\frac{7}{8}.$$

Then we can read the Laplace transformation table in the inverse direction: from (66), we find

$$y(t) = \frac{1}{8}e^t + \frac{3}{4}e^{-t} - \frac{7}{8}e^{-3t}. \tag{74}$$

So the linear 2nd order ODE is completely solved.

TODO: whether we will encounter things like

$$\mathcal{L}[f] = \frac{1}{s}, \quad s > 100. \tag{75}$$

It seems in most real world problems, we don't really need to worry about the region of allowed s: we just extend the region of allowed s as large as we can, solve the algebraic equation in the Laplace space, and then go back.

3.3 More rules

Laplace transform of an integral can also be found. Here we investigate into things like

$$\int_0^t f(t')\,\mathrm{d}t',$$

and we need to pay attention to what variable is inside the integration and what variable is exposed to the Laplace operator. We have

$$\begin{aligned}
\mathcal{L}\left[\int_0^t f(t')\,\mathrm{d}t'\right] &= -\frac{1}{s}\int_{t'=0}^{\infty}\int_0^t f(t')\,\mathrm{d}t'\,\mathrm{d}e^{-st} \\
&= -\frac{1}{s}\left(e^{-st}\int_0^t f(t')\,\mathrm{d}t'\bigg|_0^{\infty} - \int_0^{\infty} f(t)e^{-st}\,\mathrm{d}t\right) \\
&= \frac{1}{s}\mathcal{L}[f(t)].
\end{aligned} \tag{76}$$

This is the inverse of the rule of derivatives above, which is expected.

Another theorem is the s-**shifting theorem**: we have

$$\mathcal{L}[e^{at}f(t)] = F(s-a) = \mathcal{L}[f(t)]_{s\to s-a}. \tag{77}$$

This is exactly what leads to (66). Correspondingly we have the t-shifting theorem:

$$\mathcal{L}[f(t-a)H(t-a)] = e^{-sa}\mathcal{L}[f(t)], \tag{78}$$

where $H(t-a)$ is 1 when $t \geq 0$ and zero otherwise, and $a > 0$.

The theorems can be used to evaluate Laplace transforms of complex functions. We have

$$\mathcal{L}[t^2 e^{-t}] = \mathcal{L}[t^2]|_{s\to s+1} = \frac{2}{(s+1)^3}. \tag{79}$$