Homework 1

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1 Electric and magnetic field energies of a time-harmonic mode in lossless media

In this problem, we explore the contributions of the electric (U_e) and magnetic energies (U_m) to the total electromagnetic energy (U_{em}) of a mode. There are no free charges in this system, $\mu_r = 1$, and the dielectric distribution, $\varepsilon_r(r)$, can take on an arbitrary spatial distribution; the dielectric is lossless and non-dispersive. We also assume that the electric and magnetic field profiles are squareintegrable. (In other words, the electric and magnetic fields vanish at large distances, $r \to \infty$). For an electromagnetic mode of arbitrary form, show that $U_e = U_m = \frac{1}{2} U_{em}$. Here, we define $U_e = \frac{\varepsilon_o}{4} \int \varepsilon_r(\mathbf{r}) \tilde{\mathbf{E}}^* \cdot \tilde{\mathbf{E}} V$, and $U_m = \frac{\mu_o \mu_r}{4} \int \tilde{\mathbf{H}}^* \cdot \tilde{\mathbf{H}} dV$ as the time-averaged electric and magnetic energy densities.

From $\boldsymbol{\nabla} \times \tilde{\boldsymbol{E}} = \mathrm{i}\omega\mu_0\mu_\mathrm{r}\hat{\boldsymbol{H}}$ we can rewrite the magnetic energy as

$$
\begin{aligned}
U_\mathrm{m} &= \frac{\mu_0\mu_\mathrm{r}}{4} \int \mathrm{d}V \, \tilde{\boldsymbol{H}}^* \cdot \tilde{\boldsymbol{H}} \\
&= \frac{\mu_0\mu_\mathrm{r}}{4} \frac{1}{\omega^2\mu_0^2\mu_\mathrm{r}^2} \int \mathrm{d}V \, \boldsymbol{\nabla} \times \tilde{\boldsymbol{E}}^* \cdot \boldsymbol{\nabla} \times \tilde{\boldsymbol{E}} \\
&= \frac{\mu_0\mu_\mathrm{r}}{4} \frac{1}{\omega^2\mu_0^2\mu_\mathrm{r}^2} \int \mathrm{d}V \, (\tilde{\boldsymbol{E}}^* \cdot \boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \tilde{\boldsymbol{E}}) + \boldsymbol{\nabla} \cdot (\tilde{\boldsymbol{E}}^* \times (\boldsymbol{\nabla} \times \tilde{\boldsymbol{E}}))) \\
&= \frac{\mu_0\mu_\mathrm{r}}{4} \frac{1}{\omega^2\mu_0^2\mu_\mathrm{r}^2} \int \mathrm{d}V \, \tilde{\boldsymbol{E}}^* \cdot \boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \tilde{\boldsymbol{E}}).
\end{aligned}
\tag{1}
$$

From $\boldsymbol{\nabla} \times \tilde{\boldsymbol{H}} = -\mathrm{i}\omega\epsilon_0\epsilon_\mathrm{r}\tilde{\boldsymbol{E}}$, we find

$$
\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \tilde{\boldsymbol{E}}) = \mathrm{i}\omega\mu_0\mu_\mathrm{r}\boldsymbol{\nabla} \times \tilde{\boldsymbol{H}} = \omega^2\epsilon_0\mu_0\epsilon_\mathrm{r}(\boldsymbol{r})\mu_\mathrm{r}\tilde{\boldsymbol{E}},
$$

and hence

$$
U_\mathrm{m} = \frac{\mu_0\mu_\mathrm{r}}{4} \frac{1}{\omega^2\mu_0^2\mu_\mathrm{r}^2} \int \mathrm{d}V \, \omega^2\epsilon_0\mu_0\epsilon_\mathrm{r}(\boldsymbol{r})\mu_\mathrm{r}\tilde{\boldsymbol{E}} \cdot \tilde{\boldsymbol{E}}^* = \frac{\epsilon_0}{4} \int \mathrm{d}V \, \epsilon_\mathrm{r}(\boldsymbol{r})\tilde{\boldsymbol{E}}^* \cdot \tilde{\boldsymbol{E}} = U_\mathrm{e},
\tag{2}
$$

and therefore both U_e and U_e are half of the total energy.

2 Hermitian Operators

(a) When $\boldsymbol{A}$ is a real vector, we have

$$
\int \mathrm{d}^3\boldsymbol{r} \, \boldsymbol{a}^* \cdot (\boldsymbol{A} \times \boldsymbol{b}) = \int \mathrm{d}^3\boldsymbol{r} \, (\boldsymbol{a}^* \times \boldsymbol{A}) \cdot \boldsymbol{b} = \int \mathrm{d}^3\boldsymbol{r} \, (-\boldsymbol{A} \times \boldsymbol{a})^* \cdot \boldsymbol{b}.
\tag{3}
$$

Therefore the Hermitian adjoint of $\boldsymbol{A} \times (\cdots)$ is $-\boldsymbol{A} \times (\cdots)$; the operator is anti-Hermitian.

(b) When $\boldsymbol{A}, \boldsymbol{B}$ are both real vectors, we have

$$
\int \mathrm{d}^3\boldsymbol{r} \, \boldsymbol{a}^* \cdot (\boldsymbol{A} \times (\boldsymbol{B} \times \boldsymbol{b})) = \int \mathrm{d}^3\boldsymbol{r} \, ((\boldsymbol{a}^* \times \boldsymbol{A}) \times \boldsymbol{B}) \cdot \boldsymbol{b} = \int \mathrm{d}^3\boldsymbol{r} \, (\boldsymbol{B} \times (\boldsymbol{A} \times \boldsymbol{a}^*)) \cdot \boldsymbol{b}.
\tag{4}
$$

So the Hermitian adjoint of $\boldsymbol{A} \times (\boldsymbol{B} \times \cdots)$ is $\boldsymbol{B} \times (\boldsymbol{A} \times \cdots)$. The operator is neither Hermitian nor anti-Hermitian.

(c) From

$$\boldsymbol{\nabla} \cdot (\boldsymbol{a}^* \times \boldsymbol{b}) = (\boldsymbol{\nabla} \times \boldsymbol{a}^*) \cdot \boldsymbol{b} - \boldsymbol{a}^* \cdot \boldsymbol{\nabla} \times \boldsymbol{b},$$

and the assumption that the integral of LHS over the whole space vanishes, we have

$$\int \mathrm{d}^3\boldsymbol{r}\, \boldsymbol{a}^* \cdot \boldsymbol{\nabla} \times \boldsymbol{b} = \int \mathrm{d}^3\boldsymbol{r}\, (\boldsymbol{\nabla} \times \boldsymbol{a})^* \cdot \boldsymbol{b}, \tag{5}$$

and therefore the Hermitian adjoint of $\boldsymbol{\nabla} \times \cdots$ is itself and the operator is Hermitian.

(d) Since $\boldsymbol{\nabla} \times \cdots$ is Hermitian, so is $\boldsymbol{\nabla} \times \boldsymbol{\nabla} \times \cdots$, and its Hermitian adjoint is again itself.

(e) Using the above facts it's easy to show that

$$\int \mathrm{d}^3\boldsymbol{r}\, \boldsymbol{a}^* \cdot \boldsymbol{\nabla} \times (f\boldsymbol{\nabla} \times \boldsymbol{b}) = \int \mathrm{d}^3\boldsymbol{r}\, (\boldsymbol{\nabla} \times f^*\boldsymbol{\nabla} \times \boldsymbol{a})^* \cdot \boldsymbol{b}. \tag{6}$$

Therefore, the operator $\boldsymbol{\nabla} \times f\boldsymbol{\nabla} \times \cdots$ is Hermitian, if and only if f is a real function.

3 Hermitian eigenvalue problems

3.1 Electromagnetic Modes in Vacuum

In class, we saw that the magnetic vector potential can be used to express the electromagnetic field as a Hermitian eigenvalue problem. We also outlined the steps by which the mode energy can be cast in the form of a simple harmonic oscillator. In this problem, we use operator methods to derive these results in a very slick way. We assume that each mode $(\mathbf{A}_i(r,t) = q_i(t)\mathbf{A}_i^o(r))$ is an eigenfunction of the Hermitian eigenvalue equation $\hat{O}\mathbf{A}_i^o(x) = (\omega_i/c)^2\,\mathbf{A}_i^o(x)$, with a time-dependent amplitude, q_i, that obeys the relation $\ddot{q}_i = -\omega_i^2 q_i$, and $\hat{O}(\ldots) = \nabla \times \nabla \times (\ldots)$.

(a) Let's begin by considering the energy of an individual time-harmonic mode, $\mathbf{A}_i(r,t) = q_i(t)\mathbf{A}_i^o(r)$. From Eqs. 17-18 of Lecture 3 Note N1, notice that the electric field energy can be written as $U_{e,i} = \frac{1}{2}\varepsilon_o\dot{q}_i^2\,(\mathbf{A}_i^o \mid \mathbf{A}_i^o)$, and $U_{m,i} = \frac{1}{2}\mu_o^{-1}q_i^2\,(\nabla \times \mathbf{A}_i^o \mid \nabla \times \mathbf{A}_i^o)$. Using operator properties and the eigenvalue equation above, show that the total energy of the mode can be expressed as $U_{em,i} = \frac{1}{2}\left[\dot{q}_i^2 + \omega_i^2 q_i^2\right]$. How must $\mathbf{A}_i^o(x)$ be normalized to express the energy in this way? [Remember, it is our convention to choose a normalization such that the mass m of our simple harmonic oscillator takes the value $m = 1$.]

What we want is to see

$$\epsilon_0 \langle \boldsymbol{A}_i^0 | \boldsymbol{A}_i^0 \rangle = 1, \quad \frac{1}{\mu_0} \langle \boldsymbol{\nabla} \times \boldsymbol{A}_i^0 | \boldsymbol{\nabla} \times \boldsymbol{A}_i^0 \rangle = \omega_i^2. \tag{7}$$

The second condition is actually equivalent to the first condition. Since we assume no spatial inhomogeneity of ϵ, the modes follow the Helmholtz equation

$$\left(\nabla^2 + \frac{\omega_i^2}{c^2}\right)\boldsymbol{A}_i^0 = \left(\nabla^2 + \epsilon_0\mu_0\omega_i^2\right)\boldsymbol{A}_i^0 = 0, \tag{8}$$

and therefore

$$\begin{aligned}
\frac{1}{\mu_0}\langle \boldsymbol{\nabla} \times \boldsymbol{A}_i^0 | \boldsymbol{\nabla} \times \boldsymbol{A}_i^0 \rangle &= \frac{1}{\mu_0}\int \mathrm{d}^3\boldsymbol{r}\, (\boldsymbol{\nabla} \times \boldsymbol{\nabla} \times \boldsymbol{A}_i^{0*}) \cdot \boldsymbol{A}_i^0 \\
&= \frac{1}{\mu_0}(-\nabla^2\boldsymbol{A}_i^{0*}) \cdot \boldsymbol{A}_i^0 \\
&= \epsilon_0\omega_i^2 \int \mathrm{d}^3\boldsymbol{r}\, \boldsymbol{A}_i^{0*} \cdot \boldsymbol{A}_i^0,
\end{aligned} \tag{9}$$

where at the second line we use the Hermitian property of $\boldsymbol{\nabla} \times \cdots$, the second line comes from the Coulomb gauge $\boldsymbol{\nabla} \cdot \boldsymbol{A} = 0$, and therefore as long as

$$\epsilon_0 \langle \boldsymbol{A}_i^0 | \boldsymbol{A}_i^0 \rangle = 1, \tag{10}$$

which is just a normalization condition, we can write the energy as

$$H_i = U_{\mathrm{em},i} = \frac{1}{2}(\dot{q}_i^2 + \omega_i^2 q_i^2). \tag{11}$$

(b) *Given an individual mode with the form described in part (a), show that the time average of the electric field energy $\langle U_{e,i} \rangle$ is equal to the time average of the magnetic field energy $\langle U_{m,i} \rangle$. In the case of time-harmonic systems, we define the time average $\langle \ldots \rangle$ as $\frac{1}{T} \int_t^{t+T} (\ldots) dt'$, where $\omega = 2\pi/T$. [Hint: Use integration by parts to make the time averages of the electric and magnetic field energies look identical. You do not need to evaluate the integral.]*

Suppose $q_i = A_i \cos(\omega_i t)$, where by shifting the definition of the $t = 0$ point we eliminate the phase. This means $\dot{q}_i = -\omega_i A_i \sin(\omega_i t)$. The time average of the magnetic energy is

$$
\begin{aligned}
\frac{1}{2}\omega_i^2 \left\langle q_i^2 \right\rangle &= \frac{1}{2}\omega_i^2 \cdot \frac{1}{T} \int_0^T dt \cos^2 \omega_i t \\
&= \frac{1}{2}\omega_i^2 \cdot \frac{1}{T} \int_0^T \frac{1}{\omega_i} \cos \omega_i t \, d\sin \omega_i t \\
&= \frac{1}{2}\omega_i^2 \cdot \frac{1}{T} \frac{1}{\omega_i} \left(\cos \omega_i t \sin \omega_i t \Big|_0^T - \int_0^T \frac{d\cos \omega_i t}{dt} \sin \omega_i t \, dt \right) \\
&= \frac{1}{2}\omega_i^2 \cdot \frac{1}{T} \int_0^T \sin^2 \omega_i t \, dt = \frac{1}{2} \left\langle \dot{q}_i \right\rangle .
\end{aligned}
\tag{12}
$$

(c) *With the definitions $\mathbf{E}(r,t) = p_i \mathbf{E}_i^o(r) = -p_i \mathbf{A}_i^o(r)$, and $\mathbf{B}(r,t) = q_i \mathbf{B}_i^o(r) = q_i \nabla \times \mathbf{A}_i^o(r)$, use Maxwell's equations find two first order equations for the evolution of p_i and q_i. [Hint: You should find coupled first order equations that are identical to those produced by Hamilton's equations for our simple harmonic oscillator.]*

From $\nabla \times \boldsymbol{E} = -\partial \boldsymbol{B}/\partial t$ we get

$$
\dot{q}_i = p_i,
\tag{13}
$$

and from $\nabla \times \boldsymbol{B} = \frac{1}{c^2}\frac{\partial \boldsymbol{E}}{\partial t}$ we get

$$
q_i \nabla \times (\nabla \times \boldsymbol{A}_i^0) = \frac{1}{c^2}(-\dot{p}_i)\boldsymbol{A}_i^0,
$$

and from the aforementioned Helmholtz equation we have

$$
\nabla \times (\nabla \times \boldsymbol{A}_i^0) = -\nabla^2 \boldsymbol{A}_i^0 = \frac{\omega_i^2}{c^2} \boldsymbol{A}_i^0,
$$

and therefore

$$
\dot{p}_i = -\omega_i^2 q_i.
\tag{14}
$$

3.2 Electromagnetic Modes in Dielectric

Next let's consider the modes within a dielectric medium with a real dielectric constant, $\varepsilon_r(r)$, that can vary in space. We assume that each mode $(\mathbf{H}_i(r,t) = q_i(t)\mathbf{H}_i^o(r))$ is an eigenfunction of the Hermitian eigenvalue equation $\hat{O}\mathbf{H}_i^o(r) = (\omega_i/c)^2 \mathbf{H}_i^o(r)$, with a time-dependent amplitude, q_i, that obeys the relation $\ddot{q}_i = -\omega_i^2 q_i$, and $\hat{O}(\ldots) = \nabla \times (\varepsilon_r^{-1}\nabla \times) (\ldots)$. Notice we switched from magnetic flux density B to magnetic field strength H. You can directly substitute $\mathbf{B} = \mu_0\mathbf{H}$ where needed, because we usually assume $\mu = \mu_0$ this class.

(a) *Let's begin by considering the energy of an individual time-harmonic mode, with electric and magnetic fields of the form $\mathbf{E}_l(r,t) = \dot{q}_l(t)\mathbf{E}_l^o(r)$ and $\mathbf{H}_l(r,t) = q_l(t)\mathbf{H}_l^o(r)$ respectively. Notice that the electric field energy can be written as $U_{e,i} = \frac{1}{2}\varepsilon_o \dot{q}_i^2 (\mathbf{E}_i^o | \varepsilon_r | \mathbf{E}_i^o)$, and $U_{m,i} = \frac{1}{2}\mu_o q_i^2 (\mathbf{H}_i^o | \mathbf{H}_i^o)$. Using operator properties and the eigenvalue equation above, show that the total energy of the mode can be expressed as $U_{em,i} = \frac{1}{2}\left[\dot{q}_i^2 + \omega_i^2 q_i^2\right]$. How must $\mathbf{E}_i^o(x)$ and $\mathbf{H}_i^o(x)$ be normalized to express the energy in this way? Use Maxwell's equations to show that the normalization conditions for $\mathbf{E}_i^o(x)$ and $\mathbf{H}_i^o(x)$ are equivalent.*

Since $\boldsymbol{H}_i^0 = \tilde{\boldsymbol{H}}/q_i$, the results in Problem 1 can't be immediately used here. From Maxwell's equations we have

$$
\nabla \times \boldsymbol{E} = -\partial \boldsymbol{B}/\partial t \Rightarrow \nabla \times \boldsymbol{E}_i^0 = -\mu_0 \boldsymbol{H}_i^0,
$$

and similarly

$$
\nabla \times \boldsymbol{H} = \partial \boldsymbol{D}/\partial t \Rightarrow q_i \nabla \times \boldsymbol{H}_i^0 = \epsilon_0 \epsilon_r \ddot{q}_i \boldsymbol{E}_i^0 = -\omega_i^2 \epsilon_r \epsilon_0 q_i \boldsymbol{E}_i^0,
$$

and therefore

$$\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \boldsymbol{E}_i^0) = \epsilon_\mathrm{r} \frac{\omega_i^2}{c^2} \boldsymbol{E}_i^0.$$

Therefore we have

$$
\begin{aligned}
\mu_0 \left\langle \boldsymbol{H}_i^0 \middle| \boldsymbol{H}_i^0 \right\rangle &= \frac{1}{\mu_0} \int \mathrm{d}^3 \boldsymbol{r} \, (\boldsymbol{\nabla} \times \boldsymbol{E}_i^{0*}) \cdot (\boldsymbol{\nabla} \times \boldsymbol{E}_i^0) \\
&= \frac{1}{\mu_0} \int \mathrm{d}^3 \boldsymbol{r} \, \boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \boldsymbol{E}_i^{0*}) \cdot \boldsymbol{E}_i^0 \\
&= \frac{\omega_i^2}{\mu_0 c^2} \int \mathrm{d}^3 \boldsymbol{r} \, \epsilon_\mathrm{r} \boldsymbol{E}_i^{0*} \cdot \boldsymbol{E}_i^0 \\
&= \omega_i^2 \epsilon_0 \left\langle \boldsymbol{E}_i^0 \middle| \epsilon_\mathrm{r} \middle| \boldsymbol{E}_i^0 \right\rangle .
\end{aligned}
\tag{15}
$$

In order to get a harmonic oscillator form of the total energy, we want to have

$$\epsilon_0 \left\langle \boldsymbol{E}_i^0 \middle| \epsilon_\mathrm{r} \middle| \boldsymbol{E}_i^0 \right\rangle = 1, \quad \mu_0 \left\langle \boldsymbol{H}_i^0 \middle| \boldsymbol{H}_i^0 \right\rangle = \omega_i^2, \tag{16}$$

which, from (15), is equivalent to the normalization condition

$$\epsilon_0 \left\langle \boldsymbol{E}_i^0 \middle| \epsilon_\mathrm{r} \middle| \boldsymbol{E}_i^0 \right\rangle = 1. \tag{17}$$

(b) *Next, we consider superposition of such modes. Use orthogonality to show that the electromagnetic energy stored in a superposition of modes, $\mathbf{E}(r,t) = \sum_l \dot{q}_l(t) \mathbf{E}_l^o(r)$, $\mathbf{H}(r,t) = \sum_l q_l(t) \mathbf{H}_l^o(r)$, can be expressed as $U_{em}^{tot} = \sum_l \frac{1}{2} \left[\dot{q}_l^2 + \omega_l^2 q_l^2 \right]$.*

The eigenvalue problem about $\boldsymbol{H}$ gives us the orthogonality condition

$$\int \mathrm{d}^3 \boldsymbol{r} \, \boldsymbol{H}_i^{0*} \cdot \boldsymbol{H}_j^0 = \frac{\omega_i^2}{\mu_0} \delta_{ij}, \tag{18}$$

and hence the magnetic part of the energy is

$$\frac{1}{2} \mu_0 \int \mathrm{d}^3 \boldsymbol{r} \sum_{i,j} q_i \boldsymbol{H}_i^{0*} q_j \boldsymbol{H}_j^0 = \frac{1}{2} \sum_{i,j} \delta_{ij} q_i q_j \omega_i^2 = \frac{1}{2} \sum_i \omega_i^2 q_i^2. \tag{19}$$

The generalized eigenvalue problem about $\boldsymbol{E}$ is

$$\boldsymbol{\nabla} \times \boldsymbol{\nabla} \times \boldsymbol{E}_i^0 = \frac{\omega_i^2}{c^2} \epsilon_\mathrm{r} \boldsymbol{E}_i^0, \tag{20}$$

and therefore the orthogonality condition is

$$\int \mathrm{d}^3 \boldsymbol{r} \, \boldsymbol{E}_i^{0*} \epsilon_\mathrm{r} \boldsymbol{E}_j^0 = \frac{1}{\epsilon_0} \delta_{ij}, \tag{21}$$

and therefore

$$\frac{1}{2} \epsilon_0 \int \mathrm{d}^3 \boldsymbol{r} \, \epsilon_\mathrm{r} \boldsymbol{E}^2 = \frac{1}{2} \sum_{i,j} \dot{q}_i \dot{q}_j \epsilon_0 \left\langle \boldsymbol{E}_i^0 \middle| \epsilon_\mathrm{r} \middle| \boldsymbol{E}_j^0 \right\rangle = \frac{1}{2} \sum_i \dot{q}_i^2. \tag{22}$$

Therefore

$$U_{\mathrm{em}} = \frac{1}{2} \sum_i (\dot{q}_i^2 + \omega_i^2 q_i^2). \tag{23}$$

Complex Coordinates: Building on the insights from Problem 3.1, one can define real valued fields as $\mathbf{E}(r,t) = p_i \mathbf{E}_i^o(r)$, and $\mathbf{H}(r,t) = q_i \mathbf{H}_i^o(r)$. However, complex coordinates (and complex fields) provide a much more practical description of traveling waves. Using complex coordinates $a_i = \alpha q_i + i\beta p_i$ and $a_i^* = \alpha q_i - i\beta p_i$ to reduce our simple harmonic oscillator to $H_i \left(p_i \left(a_{\tilde{i}}, a_i^* \right), q_i \left(a_i, a_i^* \right) \right) = \omega_i a_i^* a_i$, we can define our complex fields as, $\tilde{\mathbf{E}}(r,t) = i a_i \tilde{\mathbf{E}}_i^o(r)$, and $\tilde{\mathbf{B}}(r,t) = a_i \tilde{\mathbf{B}}_i^o(r)$.

(c) *(c) Following steps similar to Part (a), show that the total electromagnetic energy (or field energy) of an individual mode reduces to $\omega_i a_i^* a_i$. Remember that we must use real-valued fields of the form $\mathbf{E}(r,t) = \left(ia_i\tilde{\mathbf{E}}_i^o - ia_i^*\tilde{\mathbf{E}}_i^{o*}\right)$, and $\mathbf{B}(r,t) = \left(a_i\tilde{\mathbf{B}}_i^o + a_i^*\tilde{\mathbf{B}}_i^{o*}\right)$ to evaluate the energy density. [Hint: do not use time-averaging; most of the terms will cancel!] What field normalization is necessary to express the field energy in this way?*

The electromagnetic energy density of a single mode is

$$
\begin{aligned}
u_{\mathrm{em},i} &= \frac{1}{2}\epsilon_0\epsilon_{\mathrm{r}}\boldsymbol{E}^2 + \frac{1}{2\mu_0}\boldsymbol{B}^2 \\
&= \left(\epsilon_0\epsilon_{\mathrm{r}}\left|\tilde{\boldsymbol{E}}_i^0\right|^2 + \frac{1}{\mu_0}\left|\tilde{\boldsymbol{B}}_i^0\right|^2\right)a_i^*a_i + \frac{1}{2}\left(\frac{1}{\mu_0}(\tilde{\boldsymbol{B}}_i^0)^2 - \epsilon_0\epsilon_{\mathrm{r}}(\tilde{\boldsymbol{E}}_i^0)^2\right)a_i^2 + \mathrm{c.c.},
\end{aligned}
\tag{24}
$$

and the Maxwell's equation dictates the following relation between $\tilde{\boldsymbol{E}}_i^0$ and $\tilde{\boldsymbol{B}}_i^0$:

$$
\boldsymbol{\nabla}\times\boldsymbol{E} = -\,\partial\boldsymbol{B}/\partial t \Rightarrow \boldsymbol{\nabla}\times\boldsymbol{E}_i^0 = \omega_i\boldsymbol{B}_i^0,
\tag{25}
$$

and

$$
\boldsymbol{\nabla}\times\boldsymbol{B} = \mu_0\epsilon_0\epsilon_{\mathrm{r}}\,\partial\boldsymbol{E}/\partial t \Rightarrow \boldsymbol{\nabla}\times\boldsymbol{B}_i^0 = \mu_0\epsilon_0\epsilon_{\mathrm{r}}\omega_i\boldsymbol{E}_i^0.
\tag{26}
$$

By exactly the same procedure applied in the previous problems, we have

$$
\frac{1}{\mu_0}\int\mathrm{d}^3r\,\boldsymbol{B}_i^0\cdot\boldsymbol{B}_i^0 = \epsilon_0\int\mathrm{d}^3r\,\boldsymbol{E}_i^0\cdot\epsilon_{\mathrm{r}}\boldsymbol{E}_i^0,
\tag{27}
$$

and

$$
\frac{1}{\mu_0}\int\mathrm{d}^3r\,\boldsymbol{B}_i^{0*}\cdot\boldsymbol{B}_i^0 = \epsilon_0\int\mathrm{d}^3r\,\boldsymbol{E}_i^{0*}\cdot\epsilon_{\mathrm{r}}\boldsymbol{E}_i^0.
\tag{28}
$$

The first equation means the a^2 and $(a^*)^2$ terms in u_{em} vanish after we integrate over the whole space; the second equation means that

$$
U_{\mathrm{em},i} = \int\mathrm{d}^3r\,u_{\mathrm{em},i} = 2a_i^*a_i\int\mathrm{d}^3r\,\epsilon_0\epsilon_{\mathrm{r}}|\tilde{\boldsymbol{E}}_i^0|^2.
\tag{29}
$$

What we want is

$$
U_{\mathrm{em},i} = \omega_i a_i^* a_i,
\tag{30}
$$

and that means the normalization condition has to be

$$
2\int\mathrm{d}^3r\,\epsilon_0\epsilon_{\mathrm{r}}|\tilde{\boldsymbol{E}}_i^0|^2 = \frac{2}{\mu_0}\int\mathrm{d}^3r\,|\tilde{\boldsymbol{B}}_i^0|^2 = \omega_i.
\tag{31}
$$

4 Modal perturbation theory

4.1

We have seen that, through use of separation of variables, classical equations of motion (describing a mode) can often be cast into the form of a Hermitian eigenvalue equation. In particular, many systems that exhibit time-harmonic motion can be cast in the form

$$
\hat{L}f_m(x) = \lambda_m f_m(x),
$$

where $\hat{L}$ is a time-independent Hermitian operator and $\lambda_m\left(\omega_m\right) = \left(\omega_m/v_p\right)^2$ is the eigenvalue. Provided that $\hat{L}$ is Hermitian, the eigenfunctions of such a system form a complete basis. This permits us to formulate generalized methods for modal perturbation. For instance, given an operator $\hat{L}^o$ with corresponding eigen values ω_m^o, we can slightly perturb the system and determine the first order correction, $\Delta\omega_m^1$ to the eigenfrequencies. For instance, if $\hat{L}' = \hat{L}^o + \hat{L}^1$, where $\hat{L}^1$ is a perturbative correction, the new eigenvalue, λ_m' an eigenfrequency $\omega_m' = \omega_m^o + \Delta\omega_m$. Through first order perturbation theory (analagous in form to that of Quantum Mechanics), we find the first order correction to ω_m^o, which we denote as $\Delta\omega_m^1$.

The Setup: To begin with, let's consider a small change in dielectric distribution (i.e., $\varepsilon_r(\mathbf{r}) \to \varepsilon_r(\mathbf{r}) + \Delta\varepsilon_r(\mathbf{r})$). We represent this change in simplified form as a perturbative correction on the operator $\hat{C}^o$, as $\hat{C}' = \hat{C}^0 + \hat{C}^1$. We assume that $f_m^o(x)$ are eigenfunctions of the unperturbed eigen-equation $\hat{L}_x^o f_m^o(x) = \lambda_m^o \hat{C}_x^o f_m^o(x)$, and that they form a complete basis.

(a) *Assuming that $f_m^o(x)$ are eigenfunctions of the unperturbed eigen-equation $\hat{L}_x^o f_m^o(x) = \lambda_m^o \hat{C}_x^o f_m^o(x)$, and that they form a complete basis, find the first order correction to the eigenfrequency, $\Delta\omega_m^1$, produced by $\hat{C}' = \hat{C}^0 + \hat{C}^1$. (Hint: Expand $f_m(x)$ and λ_m and collect terms of similar order.)*

Following the same procedure used in quantum mechanics, the perturbative form of $Lf_m = \lambda_m(C^0 + C^1)f_m$ is

$$L(f_m^0 + f_m^1 + \cdots) = (\lambda_m^0 + \lambda_m^0 + \cdots)(C^0 + C^1)(f_m^0 + f_m^1 + \cdots), \tag{32}$$

and the first order perturbation is given by (note that a term with superscript n is proportional to C^1 to n times)

$$Lf_m^1 = \lambda_m^0 C^1 f_m^0 + \lambda_m^1 C^0 f_m^0 + \lambda_m^0 C^0 f_m^1. \tag{33}$$

From $L^0 f_m^0 = \lambda_m^0 C^0 f_m^0$ we have (below $\langle a|b\rangle$ is defined as $\int \mathrm{d}^n x\, a^* b$, without C_0)

$$\underbrace{\langle f_m^0|L^0|f_m^1\rangle}_{=\lambda_m^0 \langle f_m^0|C^0|f_m^1\rangle} = \lambda_m^0 \langle f_m^0|C^1|f_m^0\rangle + \lambda_m^1 \langle f_m^0|C^0|f_m^0\rangle + \lambda_m^0 \langle f_m^0|C^0|f_m^1\rangle,$$

$$\lambda_m^1 = -\lambda_m^0 \frac{\langle f_m^0|C^1|f_m^0\rangle}{\langle f_m^0|C^0|f_m^0\rangle}. \tag{34}$$

When the wave functions are normalized so that $\langle f_m^0|C^0|f_m^0\rangle = 1$, we only have the numerator.

(b) *Use this result to find the first order correction to the eigenfrequency, $\Delta\omega_m^1$, produced by $\Delta\varepsilon_r(\mathbf{r})$ on an electromagnetic mode using Eq. 3. (Result should be identical to Eq 28 of Joannopoulos and Johnson.)*

The master equation w.r.t. $\boldsymbol{E}$ is

$$\boldsymbol{\nabla} \times \boldsymbol{\nabla} \times \boldsymbol{E} = \left(\frac{\omega}{c}\right)^2 \epsilon_{\mathrm{r}} \boldsymbol{E}, \tag{35}$$

and therefore $C = \epsilon_{\mathrm{r}}$, and $C^0 = 1$ while $C^1 = \Delta\epsilon_{\mathrm{r}}$, and

$$\lambda_m = \frac{\omega_m^2}{c^2}.$$

The electric field, as is mentioned above, is normalized such that (17) holds, and this condition means

$$\langle f_m^0|C^0|f_m^0\rangle = \frac{1}{\epsilon_0}$$

when there is no perturbation. Therefore under this normalization scheme we have (here we use $\boldsymbol{E}_m$ to refer to $\boldsymbol{E}_m^0$ in (17) so that there aren't too many superscripts, and $\boldsymbol{E}_m^0$ means the zeroth order perturbation)

$$(\omega_m^2)^1 = -(\omega_m^2)^0 \epsilon_0 \left\langle \boldsymbol{E}_m^0 \middle| \Delta\epsilon_{\mathrm{r}} \middle| \boldsymbol{E}_m^0 \right\rangle, \tag{36}$$

and therefore

$$\Delta\omega_m \approx \sqrt{(\omega_m^2)^0 + (\omega_m^2)^1} - \omega_m^0 \approx \omega_m^0 \cdot \frac{(\omega_m^2)^1}{2(\omega_m^2)^0} = -\frac{1}{2}\omega_m^0 \epsilon_0 \left\langle \boldsymbol{E}_m^0 \middle| \Delta\epsilon_{\mathrm{r}} \middle| \boldsymbol{E}_m^0 \right\rangle. \tag{37}$$

Comment and additional guidance: In parts (a) and (b) we found the change in electromagnetic mode frequency, $\Delta\omega_m^1$, produced by a small change in dielectric distribution $\Delta\varepsilon_r(\mathbf{r})$. This expression is APPROXIMATE, and is only valid for small changes in $\Delta\varepsilon_r(\mathbf{r})$.

*** In the next parts of this problem, we change the dielectric distribution in a way that permits us to find an EXACT relationship between the modified eigenfrequencies, ω_m', in terms of the original eigenfrequencies ω_m^o. As we will see, using this exact relationship between ω_m', and ω_m^o, able to find an expression for $\Delta\omega_m^{(n)}$, to all orders of (n) and compare this with our perturbative result.*

(c) *Next, clear your mind of the perturbative approach used in (a) and (b); lets think about the solutions of Eq. 3 with "fresh eyes". Next replace $\varepsilon_r(\mathbf{r})$ in Eq. 3 with scaled-version of the dielctric distribution $\beta \cdot \varepsilon_r(\mathbf{r})$. Here β is a unitles scale-factor. Find the relationship between our new eigenfrequencies of this system, ω'_m, in terms of the original eigenfrequencies ω^o_m. (Your result should be very simple... but it is exact!)*

Now the unperturbed theory is

$$\boldsymbol{\nabla} \times \boldsymbol{\nabla} \times \boldsymbol{E}_m = \epsilon_{\mathrm{r}} \frac{(\omega^0_m)^2}{c^2} \boldsymbol{E}_m, \tag{38}$$

and the perturbed one is

$$\boldsymbol{\nabla} \times \boldsymbol{\nabla} \times \boldsymbol{E}_m = \beta \epsilon_{\mathrm{r}} \frac{\omega^2_m}{c^2} \boldsymbol{E}_m. \tag{39}$$

The relation between the frequencies before and after perturbation is clear:

$$\omega_m = \frac{\omega^0_m}{\sqrt{\beta}}. \tag{40}$$

(d) *Defining $\beta = 1 + \alpha$, where $\alpha \ll 1$, use your results from part (c) to find an expression for the change in frequency to all orders in α.*

Perturbatively, the new frequency is

$$\omega_m = \frac{\omega^0_m}{\sqrt{1+\alpha}} = \omega^0_m \left(1 - \frac{1}{2}\alpha + \frac{3}{8}\alpha^2 + \cdots \right). \tag{41}$$

(e) *Compare your answer in (d) to the perturbative result that we obtained in (b). Find an expression (in terms of α) that defines the limits in which the first order perturbative calculation be inadequate to capture the true change in frequency? (i.e., when can higher order terms be neglected, and when are they non-negligible?)*

When ϵ_{r} is homogeneous enough,

$$\beta = \epsilon_{\mathrm{r}}, \quad \alpha = \Delta\epsilon_{\mathrm{r}}, \tag{42}$$

and from results in (b) we get

$$\Delta\omega^1_m = -\frac{1}{2}\omega^0_m \Delta\epsilon_{\mathrm{r}} = -\frac{1}{2}\alpha\omega^0_m, \tag{43}$$

which agrees with (41). For the first order perturbation theory to be enough, we need

$$\frac{1}{2}\alpha \gg \frac{3}{8}\alpha^2,$$

and therefore

$$\Delta\epsilon_{\mathrm{r}} \ll \frac{4}{3} \sim 1. \tag{44}$$

5   Hamilton's Equation in Complex Coordinates

As discussed in Lectures 3-4, complex coordinates provide a natural way to describe oscillatory systems. We defined our complex coordinates as $a = \alpha q + i\beta p$ and $a^* = \alpha q - i\beta p$, in close analogy with the quantum mechanical raising/lower operators; here, α and β are real-valued coefficients. Since, we can use the relations $q = (a + a^*)/2\alpha$ and $p = (a - a^*)/2i\beta$ to express any Hamiltonian $H(p,q)$ as $H(p(a,a^*), q(a,a^*))$, our new Hamiltonian can always be expressed as $H(a,a^*)$. Notice that a and a^* are independent coordinates that replace q and p. Our next task is to find a new version of Hamilton's equations in complex coordinates.

(a) *Use the chain rule, in conjunction with relations $\dot{q} = \partial H/\partial p$ and $\dot{p} = -\partial H/\partial q$, to show that $\dot{a} = -i2\alpha\beta\left(\partial H/\partial a^*\right)$ and $\dot{a}^* = +i2\alpha\beta(\partial H/\partial a)$. [Note: We typically choose $2\alpha\beta = 1$ to ensure that our transformation is area preserving. In this case, we obtain $\dot{a} = -i\left(\partial H/\partial a^*\right)$ and $\dot{a}^* = +i(\partial H/\partial a)$.][1]*

We have

$$
\begin{aligned}
\dot{a} = \alpha\dot{q} + i\beta\dot{p} &= \alpha\frac{\partial H}{\partial p} - i\beta\frac{\partial H}{\partial q} \\
&= \alpha\left(\frac{\partial H}{\partial a}\cdot i\beta + \frac{\partial H}{\partial a^*}\cdot(-i\beta)\right) - i\beta\left(\frac{\partial H}{\partial a}\cdot\alpha + \frac{\partial H}{\partial a^*}\alpha\right) \\
&= -2i\alpha\beta\frac{\partial H}{\partial a^*},
\end{aligned}
\tag{45}
$$

and similarly

$$
\begin{aligned}
\dot{a}^* = \alpha\dot{q} - i\beta\dot{p} &= \alpha\frac{\partial H}{\partial p} + i\beta\frac{\partial H}{\partial q} \\
&= \alpha\left(\frac{\partial H}{\partial a}\cdot i\beta + \frac{\partial H}{\partial a^*}\cdot(-i\beta)\right) + i\beta\left(\frac{\partial H}{\partial a}\cdot\alpha + \frac{\partial H}{\partial a^*}\alpha\right) \\
&= 2i\alpha\beta\frac{\partial H}{\partial a}.
\end{aligned}
\tag{46}
$$

[Note: We typically choose $2\alpha\beta = 1$ to ensure that our transformation is area preserving. In this case, we obtain $\dot{a} = -i\left(\partial H/\partial a^*\right)$ and $\dot{a}^* = +i(\partial H/\partial a)$.]

(b) *Using your results from part (a), apply the chain rule to show that*

$$
\frac{dA}{dt} = i\left(\frac{\partial H}{\partial a}\frac{\partial A}{\partial a^*} - \frac{\partial H}{\partial a^*}\frac{\partial A}{\partial a}\right) + \frac{\partial A}{\partial t} = i[H, A]_{a,a^*} + \frac{\partial A}{\partial t}
$$

where A is any function of p, q that we have transformed to complex coordinates as $A(p, q) \to A\left(p\left(a, a^*\right), q\left(a, a^*\right)\right)$. Note that $[C, D]$ represents the Poisson bracket according to the definition $[C, D]_{q,p} = \left(\partial_q C\partial_p D - \partial_p C\partial_q D\right)$.

Now A is written as a function of a, a^*, we have

$$
\begin{aligned}
\frac{\mathrm{d}A}{\mathrm{d}t} &= \frac{\partial A}{\partial a}\dot{a} + \frac{\partial A}{\partial a^*}\dot{a}^* + \frac{\partial A}{\partial t} \\
&= -i\frac{\partial A}{\partial a}\frac{\partial H}{\partial a^*} + i\frac{\partial A}{\partial a^*}\frac{\partial H}{\partial a} + \frac{\partial A}{\partial t}.
\end{aligned}
\tag{47}
$$