Elasticity in structural mechanics

Jinyuan Wu

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1 Rigid body analysis

2 Elastic medium

Definition The deformation $\boldsymbol{u}(t)$ of the system is completely decided by the external loading at t. Notable counterparts:

- *Fluid.* $\boldsymbol{u} \Leftarrow \boldsymbol{v} \Leftarrow \boldsymbol{F}$: not elastic.
- *Plastic.* $\boldsymbol{u}$ depends on history: not elastic.

Degrees of freedom, with infinitesimal deformation We deal with two sets of variables:

- *Stress* σ_{ij}. $\mathrm{d}F_i = \sigma_{ij}\,\mathrm{d}A_j$.
- *Strain* u_{ij}. For small deformation

$$u_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right). \tag{1}$$

- *Constitutive relations.* $\sigma_{ij} = \sigma_{ij}[u_{ij}]$.

Uniform isotropic linear medium Constitutive relation

$$\sigma_{ik} = K u_{ll} \delta_{ik} + 2\mu \left(u_{ik} - \frac{1}{3}\delta_{ik} u_{ll}\right). \tag{2}$$

Temperature expansion The strain induced by temperature change:

$$\frac{\mathrm{d}u}{\mathrm{d}x} = \alpha(T - T_0), \tag{3}$$

where T_0 is the "overall" temperature.

3 Uniform isotropic linear medium, in experiments

Two modes of strain

- *Compression/tension.* Along one direction (for example z):

$$\epsilon = \frac{\delta}{L} = u_{zz}. \tag{4}$$

- *Shear.* On the xy plane:

$$\gamma = \theta_{xx'} + \theta_{yy'} = 2u_{xy}. \tag{5}$$

Young's modulus Relation between tension and force:

$$E = \frac{P}{\epsilon} = \frac{PL}{\delta} \Rightarrow F = PA = \frac{\delta}{L} \cdot EA. \tag{6}$$

Poisson's ratio Relation between transverse strain and axial strain (in Young's modulus experiment):

$$\sigma = \nu = -\frac{\mathrm{d}\epsilon_{\text{transverse}}}{\mathrm{d}\epsilon_{\text{axial}}}. \tag{7}$$

This is how the material becomes thinner when stretched.

Volume modulus Relation between pressure and volume:

$$K = -V \frac{\mathrm{d}P}{\mathrm{d}V}. \tag{8}$$

Here K is that parameter in (2).
Shear modulus Relation between shear stress and shear strain:

$$\mu = G = \frac{\tau}{\gamma}. \tag{9}$$

Here τ is σ_{xy} (or yz or zx); γ is the shear strain.
How many independent parameters? In isothermal process:

$$E = \frac{9K\mu}{3K+\mu}, \quad \sigma = \frac{1}{2}\frac{3K-2\mu}{3K+\mu}. \tag{10}$$

When is the linear elasticity condition broken?

1. Linear region.
2. Proportional limit.
3. Elastic limit.
4. Yield point.
5. Ultimate tensile point.
6. Breaking point.

4 Low dimension system: torsion of cylinder-like rod

Reaction of φ to torque

$$\frac{\mathrm{d}\varphi}{\mathrm{d}z} = \frac{T(z)}{JG}, \quad T(z) = \int_0^z \mathrm{d}z' \, \frac{\mathrm{d}\text{ torque}}{\mathrm{d}z'}. \tag{11}$$

Relation between torque and stress

$$\gamma = \frac{\mathrm{d}\varphi}{\mathrm{d}z} r, \quad \tau = G\gamma, \tag{12}$$

$$\tau_{\max} = G\frac{\mathrm{d}\varphi}{\mathrm{d}z}R = \frac{TR}{J}. \tag{13}$$

Here R may also be written as c.

5 Problems

- Using deformation to decide forces (that otherwise can't be determined).
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