Homework 2

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1

1.1

Comparison between the approximate and exact susceptibilities. We consider a driven damped simple harmonic oscillator (following L8.N3). We assume that $\dot{p} = \sum_i f_i = -kq - D\dot{q} + f(t)$, where k, D, and $f(t)$ are the spring constant, damping, and driving force, respectively. In what follows, we assume that $f(t) = f_o \cos(\omega t + \phi)$.

(a) *Starting from Newton's equations, find the first order equation of motion in complex form. [Hint: you should find something like $\dot{a} = -\mathrm{i}\omega_o a + (\dots)$, and you're looking to identify all of the (...) terms.]*

When there is no damping the EOMs are

$$\dot{p} = -kq, \quad \dot{q} = \frac{p}{m}, \tag{1}$$

and the following definition of the modes

$$q = \sqrt{\frac{1}{2m\omega_0}}(a^* + a), \quad p = \mathrm{i}\sqrt{\frac{m\omega_0}{2}}(a^* - a), \quad \omega_0^2 = \frac{k}{m} \tag{2}$$

turns the Hamiltonian into $H = \omega_0 a^* a$. Under this definition of a, a^*, the EOM with damping and driving therefore becomes

$$\begin{aligned} \dot{a}^* - \dot{a} &= \mathrm{i}\omega_0(a^* + a) - \frac{D}{m}(a^* - a) - \mathrm{i}\sqrt{\frac{2}{m\omega_0}}f(t), \\ \dot{a}^* + \dot{a} &= \mathrm{i}\omega_0(a^* - a), \end{aligned} \tag{3}$$

from which we find

$$\dot{a} = -\mathrm{i}\omega_0 a + \frac{D}{2m}(a^* - a) + \mathrm{i}\sqrt{\frac{2}{m\omega_0}}f(t) \tag{4}$$

and its complex conjugate

$$\dot{a}^* = \mathrm{i}\omega_0 a^* - \frac{D}{2m}(a^* - a) - \mathrm{i}\sqrt{\frac{2}{m\omega_0}}f(t). \tag{5}$$

(b) *Perform the rotating wave approximation to the solution you obtained in part (a). Identify the terms that are eliminated by this approximation.*

Now we do RWA and let $a = \bar{a}\mathrm{e}^{-\mathrm{i}\omega_0 t}$, and the EOM of a becomes

$$\dot{\bar{a}}\mathrm{e}^{-\mathrm{i}\omega_0 t} + \bar{a}(-\mathrm{i}\omega_0)\mathrm{e}^{-\mathrm{i}\omega_0 t} = -\mathrm{i}\omega_0\bar{a}\mathrm{e}^{-\mathrm{i}\omega_0 t} + \frac{D}{2m}(\bar{a}^*\mathrm{e}^{\mathrm{i}\omega_0 t} - \bar{a}\mathrm{e}^{-\mathrm{i}\omega_0 t}) + \mathrm{i}\sqrt{\frac{2}{m\omega_0}}f(t),$$

or

$$\dot{\bar{a}} = -\frac{D}{2m}(\bar{a} - \bar{a}^*\mathrm{e}^{\mathrm{i}2\omega_0 t}) + \mathrm{i}\sqrt{\frac{2}{m\omega_0}}\mathrm{e}^{\mathrm{i}\omega_0 t}f(t). \tag{6}$$

The fast oscillating terms include the $\frac{D}{2m}\bar{a}^*\mathrm{e}^{\mathrm{i}2\omega_0 t}$ term and the $\mathrm{e}^{\mathrm{i}\omega t}$ part of $f(t)$, assuming that ω is close to ω_0. If these terms are removed, the resulting EOM is

$$\dot{\bar{a}} = -\frac{D}{2m}\bar{a} + \mathrm{i}\sqrt{\frac{1}{2m\omega_0}} \cdot \frac{1}{2}f_0\mathrm{e}^{-\mathrm{i}(\omega-\omega_0)t}. \tag{7}$$

(c) *Use the approximate equation of motion from part (b) to find the complex susceptibility, $\alpha_a[\omega]$, that relates the complex wave amplitude and force as $a[\omega] = \alpha_a[\omega] f[\omega]$.*

Noting that in the stationary solution $\bar{a} \propto \mathrm{e}^{-\mathrm{i}(\omega-\omega_0)t}$, we have

$$a(t) = \mathrm{e}^{-\mathrm{i}\omega_0 t}\bar{a} = \frac{1}{-\mathrm{i}(\omega-\omega_0) + \frac{D}{2m}} \frac{\mathrm{i}}{\sqrt{2m\omega_0}} \cdot \frac{1}{2} f_0 \mathrm{e}^{-\mathrm{i}\omega t}, \tag{8}$$

and therefore the response of a is (we divide $a(t)$ by $f_0\mathrm{e}^{-\mathrm{i}\omega t}/2$)

$$\alpha_a[\omega] = -\frac{1}{\sqrt{2m\omega_0}} \frac{1}{\omega - \omega_0 + \mathrm{i}\frac{D}{2m}}. \tag{9}$$

1.2

(a) *Starting from the second order equation, $m\ddot{q} = \sum_i f_i = -kq - D\dot{q} + f(t)$, find the exact susceptibility $q[\omega] = \alpha_q[\omega] f[\omega]$ of the driven-damped SHO.*

The EOM of q is

$$m\ddot{q} + D\dot{q} + kq = \frac{f_0}{2}(\mathrm{e}^{\mathrm{i}\omega t} + \mathrm{e}^{-\mathrm{i}\omega t}), \tag{10}$$

and since the system is linear, we can only consider the response of q to the $\mathrm{e}^{-\mathrm{i}\omega t}$ component of f, and get

$$\alpha_q[\omega] = \frac{1}{-m\omega^2 - \mathrm{i}D\omega + k} = -\frac{1}{m}\frac{1}{\omega^2 - \omega_0^2 + \mathrm{i}\frac{D}{m}\omega}. \tag{11}$$

(b) *Compare the approximate response you obtained in part (1.1c) with the exact response you found in (1.2a). [Remember, $q(t)$ is contained in $a(t)$.]*

Since $p = m\dot{q}$, we have

$$\alpha_p[\omega] = \frac{\mathrm{i}\omega}{\omega^2 - \omega_0^2 + \mathrm{i}\frac{D}{m}\omega}, \tag{12}$$

and this means

$$\begin{aligned} \alpha_a^{\text{accurate}} &= \frac{1}{2}\left(\sqrt{2m\omega_0}\alpha_q + \mathrm{i}\sqrt{\frac{2}{m\omega_0}}\alpha_p\right) \\ &= -\frac{1}{\sqrt{2m\omega_0}} \frac{\omega + \omega_0}{\omega^2 - \omega_0^2 + \mathrm{i}\frac{D}{m}\omega}. \end{aligned} \tag{13}$$

(c) *Can you find a set of approximations that will make your exact susceptibility converge with the susceptibility you found from the RWA?*

The validity of RWA is equivalent to $\omega \approx \omega_0$, which then means

$$\alpha_a = -\frac{1}{\sqrt{2m\omega_0}} \frac{1}{\frac{\omega^2-\omega_0^2}{\omega+\omega_0} + \mathrm{i}\frac{D}{m}\frac{\omega}{\omega+\omega_0}} \approx -\frac{1}{\sqrt{2m\omega_0}} \frac{1}{\omega - \omega_0 + \mathrm{i}\frac{D}{2m}}. \tag{14}$$

This is expected since when RWA was made, it was already assumed that $\omega \approx \omega_0$.

Verifying this condition by calculating α_q seems to be not viable, because q is to be calculated from a and a^*, the response function of the latter being $\alpha[-\omega^*]$; but then when RWA around $\omega \approx \omega_0$ works for a, it doesn't work for a^*. RWA for q's response seems to be ill-defined.

2

Time Modulation of Dielectric Constant: Next, we consider the impact of a time-modulated dielectric constant on the modes of a resonator. In this case, we assume that a standing-wave mode of the form $\mathbf{E}(r,t) = (a - a^*)\,\mathbf{E}^o(r)$ with real-valued field distribution $\mathbf{E}^o(r)$ is localized within a structured dielectric, having dielectric distribution $\varepsilon_r(r)$. For example, you could imagine that this is an eigenmode of a dielectric photonic crystal resonator of the type seen as Fig 8 in Chapter 7 of [JJSJ]. In the absence of a dielectric perturbation, the Hamiltonian takes the form $H_a = \omega_o a^* a$, where ω_o is the natural frequency of this time-harmonic mode.

2.1

(a) *As a starting point, find an expression for δH associated with an arbitrary dielectric perturbation $\Delta\varepsilon_r(r)$.*

Using the normalization

$$\boldsymbol{E} = \mathrm{i}(a - a^*)\boldsymbol{E}^0(\boldsymbol{r}), \tag{15}$$

we have

$$\delta H = \int \mathrm{d}^3\boldsymbol{r}\, \frac{1}{2}\epsilon_0 \Delta\epsilon_\mathrm{r} \boldsymbol{E}^2 = \frac{\epsilon_0}{2}(a - a^*)^2 \left\langle \boldsymbol{E}^0 \middle| \Delta\epsilon_\mathrm{r} \middle| \boldsymbol{E}_0 \right\rangle. \tag{16}$$

(b) *Using your result from part (a), find an equations of motion for a and a^*.*

We have

$$\frac{\mathrm{d}a}{\mathrm{d}t} = -\mathrm{i}\frac{\partial H}{\partial a^*} = -\mathrm{i}\omega_0 a + \mathrm{i}\epsilon_0(a - a^*) \left\langle \boldsymbol{E}^0 \middle| \Delta\epsilon_\mathrm{r} \middle| \boldsymbol{E}_0 \right\rangle, \tag{17}$$

and therefore

$$\frac{\mathrm{d}a^*}{\mathrm{d}t} = \mathrm{i}\omega_0 a^* + \mathrm{i}\epsilon_0(a - a^*) \left\langle \boldsymbol{E}^0 \middle| \Delta\epsilon_\mathrm{r} \middle| \boldsymbol{E}_0 \right\rangle. \tag{18}$$

If we formally solve the equation system, we will find the first order correction is

$$\Delta\omega^{(1)} = -\epsilon_0 \left\langle \boldsymbol{E}^0 \middle| \Delta\epsilon_\mathrm{r} \middle| \boldsymbol{E}_0 \right\rangle. \tag{19}$$

Considering the normalization scheme here that leads to $H_0 = \omega a^* a$ is

$$2\epsilon_0 \left\langle \boldsymbol{E}^0 \middle| \epsilon_\mathrm{r} \middle| \boldsymbol{E}^0 \right\rangle = \omega_0 \tag{20}$$

and not

$$\epsilon_0 \left\langle \boldsymbol{E}^0 \middle| \epsilon_\mathrm{r} \middle| \boldsymbol{E}^0 \right\rangle = 1, \tag{21}$$

this result agrees with the first order perturbation in Homework 1.

(c) *Next, we assume that perturbation to the dielectric constant takes the form $\Delta\varepsilon_r(t) = \beta\varepsilon_r(r)\cos(2\omega_o t + \phi)$, where β is a small ($\beta \ll 1$) unitless constant. Find the new form of our equations of motion from part (b). [Remember, our field normalization permits us to reduce $\langle \mathbf{E}^o | \varepsilon_o\varepsilon_r(r) | \mathbf{E}^o \rangle$ to a constant already defined above.]*

The EOM of a now is

$$\begin{aligned} \frac{\mathrm{d}a}{\mathrm{d}t} &= -\mathrm{i}\omega_0 a + \mathrm{i}\epsilon_0(a - a^*) \left\langle \boldsymbol{E}^0 \middle| \epsilon_\mathrm{r} \middle| \boldsymbol{E}_0 \right\rangle \beta\cos(2\omega_0 t + \phi) \\ &= -\mathrm{i}\omega_0 a + \frac{\mathrm{i}\omega_0}{2}\beta(a - a^*)\cos(2\omega_0 t + \phi). \end{aligned} \tag{22}$$

(d) *Using a trial solution in the form $a(t) = \bar{a}(t)\mathrm{e}^{-\mathrm{i}\omega_o t}$, apply the rotating wave approximation to the equations of motion from part (c). [Hint: your results should closely resemble the problem involving a child on a swing.]*

Under the ansatz $a(t) = \bar{a}(t)\mathrm{e}^{-\mathrm{i}\omega_0 t}$ we have

$$\begin{aligned} &\mathrm{e}^{-\mathrm{i}\omega_0 t}\frac{\mathrm{d}\bar{a}}{\mathrm{d}t} = \frac{\mathrm{i}\omega_0}{2}\beta(\bar{a}\mathrm{e}^{-\mathrm{i}\omega_0 t} - \bar{a}^*\mathrm{e}^{\mathrm{i}\omega_0 t}) \cdot \frac{1}{2}(\mathrm{e}^{\mathrm{i}(2\omega_0 t + \phi)} + \mathrm{e}^{-\mathrm{i}(2\omega_0 t + \phi)}) \\ \Rightarrow &\frac{\mathrm{d}\bar{a}}{\mathrm{d}t} \approx -\frac{\mathrm{i}\omega_0}{4}\beta\bar{a}^*\mathrm{e}^{-\mathrm{i}\phi}, \end{aligned} \tag{23}$$

and therefore under RWA,

$$\frac{\mathrm{d}\bar{a}^*}{\mathrm{d}t} = \frac{\mathrm{i}\omega_0}{4}\beta\bar{a}\mathrm{e}^{\mathrm{i}\phi}. \tag{24}$$

Therefore, we get

$$\frac{\mathrm{d}^2\bar{a}}{\mathrm{d}t^2} = \frac{\omega_0^2\beta^2}{16}\bar{a}, \tag{25}$$

and the exponential growing solution is

$$a = \bar{a}\mathrm{e}^{-\mathrm{i}\omega_0 t} = \bar{a}(t = 0)\mathrm{e}^{-\mathrm{i}\omega_0 t}\mathrm{e}^{\frac{\omega_0\beta}{4}t}. \tag{26}$$

This expression contains no phase ϕ, but ϕ does appear in the growth of H, which is given by

$$\dot{H} = \omega_0(\dot{a}^* a + a^*\dot{a}) = \omega_0(\dot{\bar{a}}^*\bar{a} + \bar{a}^*\dot{\bar{a}}) = \frac{\omega_0^2}{2}\beta|\bar{a}|^2\sin\phi. \tag{27}$$

(e) *Treating $\Delta\varepsilon_r(r)$ as a perturbation on our original mode, solve for the first order correction to our mode amplitude.*

Replacing $\Delta\epsilon_\mathrm{r}$ by $\lambda\Delta\epsilon_\mathrm{r}$, and doing the expansion $a = a^{(0)} + \lambda a^{(1)} + \cdots$, we get

$$\left(\frac{\mathrm{d}}{\mathrm{d}t} + \mathrm{i}\omega_0\right)(a^{(0)} + \lambda a^{(1)} + \cdots) = \mathrm{i}\epsilon_0(a^{(0)} - a^{(0)*} + \lambda(a^{(1)} - a^{(1)*})) \cdot \lambda \left\langle \boldsymbol{E}^0 \middle| \Delta\epsilon_\mathrm{r} \middle| \boldsymbol{E}^0 \right\rangle,$$

and if we only consider the first order perturbation we get

$$\begin{aligned} \dot{a}^{(1)} &= -\mathrm{i}\omega_0 a^{(1)} + \mathrm{i}\epsilon_0(a^{(0)} - a^{(0)*}) \left\langle \boldsymbol{E}^0 \middle| \Delta\epsilon_\mathrm{r} \middle| \boldsymbol{E}^0 \right\rangle \\ &= -\mathrm{i}\omega_0 a^{(1)} - \omega_0\beta \sin\omega_0 t \cos(2\omega_0 t + \phi). \end{aligned} \tag{28}$$

(f) *Under what conditions (i.e., for what values of phase, ϕ) does the mode amplitude (and the energy of the system) grow? Under what conditions does the energy decay?*

When $\sin\phi > 0$, i.e. $0 < \phi < \pi$, the amplitude grows, and otherwise it decays.

(g) *Is this an elastic or inelastic scattering process?*

Since there is energy change between the final state and the initial state, this is an inelastic scattering process.

2.2

Dynamical Bragg Scattering: Next, we consider mode coupling produced by a time-dependent dielectric perturbation. We build on the example presented in Lecture 9-10, which examines the coupling induced between modes of a ring resonator by a dielectric grating. In this case, however, we consider moving grating: $\Delta\varepsilon(z,t) = \Delta\varepsilon^o \cos(\kappa z - \Omega t)$, where $\Delta\varepsilon^o, \kappa,$ and Ω are constants. We consider the impact of this dielectric perturbation on two counter-propagating modes $\tilde{\mathbf{E}}_1(r,t) = a_1\tilde{\mathbf{E}}_1(z) = a_1\tilde{\mathbf{E}}_1^o e^{ik_1 z}$ and $\tilde{\mathbf{E}}_2(r,t) = a_2\tilde{\mathbf{E}}_2(z) = a_2\tilde{\mathbf{E}}_2^o e^{-ik_2 z}$ with distinct natural frequencies ω_1 and ω_2, respectively. In the presence of this perturbation, the Hamiltonian describing this system becomes $H = H_1 + H_2 + \delta H$, where $H_1 = \omega_1 a_1^* a_1, H_2 = \omega_2 a_2^* a_2$, and δH is the interaction term produced by $\Delta\varepsilon(z,t)$. The ring has length L, so the wavevectors are $k_1 = (2\pi/L)n_1, k_2 = (2\pi/L)n_2$ and $\kappa = (2\pi/L)m$, where n_1, n_2, and m are integers.

(a) *Find an expression for δH in terms of the quantities provided above.*

By definition

$$\delta H = \frac{1}{2}\Delta\epsilon^0 \int \mathrm{d}z\, (a_1^* \tilde{\boldsymbol{E}}_1^{0*} \mathrm{e}^{-\mathrm{i}k_1 z} + a_2^* \tilde{\boldsymbol{E}}_2^{0*} \mathrm{e}^{\mathrm{i}k_2 z}) \cdot (a_1 \tilde{\boldsymbol{E}}_1^0 \mathrm{e}^{\mathrm{i}k_1 z} + a_2 \tilde{\boldsymbol{E}}_2^0 \mathrm{e}^{-\mathrm{i}k_2 z}) \cos(\kappa z - \Omega t),$$

and since $\int_0^L \mathrm{d}z \cos(\kappa z - \Omega t) = 0$, we have

$$\begin{aligned} \delta H &= \frac{1}{2}\Delta\epsilon^0 a_1^* a_2 \tilde{\boldsymbol{E}}_1^{0*} \tilde{\boldsymbol{E}}_2^0 \int_0^L \mathrm{d}z\, \mathrm{e}^{-\mathrm{i}(k_1+k_2)z} \cos(\kappa z - \Omega t) + \text{c.c.} \\ &= \frac{L}{4}\Delta\epsilon^0 a_1^* a_2 \tilde{\boldsymbol{E}}_1^{0*} \tilde{\boldsymbol{E}}_2^0 \sum_G (\mathrm{e}^{-\mathrm{i}\Omega t}\delta_{\kappa+G,k_1+k_2} + \mathrm{e}^{\mathrm{i}\Omega t}\delta_{-\kappa+G,k_1+k_2}) + \text{c.c.}, \end{aligned} \tag{29}$$

where

$$G = \frac{2\pi}{L}m, \tag{30}$$

m is an integer.

(b) *Assuming that the two optical modes have distinct k-vector (i.e., $k_1 \neq k_2$), what value of κ is necessary to permit coupling between mode 1 and mode 2 ? This condition is known as phase matching.*

From the Hamiltonian it can be seen that we need

$$\pm\kappa + G = k_1 + k_2 \tag{31}$$

for non-zero coupling between the two modes where G is an arbitrary "reciprocal lattice vector" with the form of $2\pi/L \cdot m$, m being an integer.

(c) *From the total Hamiltonian, H, assuming that phase-matching is satisfied, find the equations of motion that describe coupling between our mode amplitudes a_1 and a_2.*

From δH we find

$$\begin{aligned}\dot{a}_1 &= -\mathrm{i}\omega_1 a_1 - \mathrm{i}\frac{\partial\,\delta H}{\partial a_1^*} \\ &= -\mathrm{i}\omega_1 a_1 - \mathrm{i}\frac{L}{4}\Delta\epsilon_0 \tilde{\boldsymbol{E}}_1^{0*}\tilde{\boldsymbol{E}}_2^{0}\sum_G(\mathrm{e}^{-\mathrm{i}\Omega t}\delta_{\kappa+G,k_1+k_2} + \mathrm{e}^{\mathrm{i}\Omega t}\delta_{-\kappa+G,k_1+k_2})a_2,\end{aligned} \tag{32}$$

and similarly

$$\begin{aligned}\dot{a}_2 &= -\mathrm{i}\omega_2 a_2 - \mathrm{i}\frac{\partial\,\delta H}{\partial a_2^*} \\ &= -\mathrm{i}\omega_2 a_2 - \mathrm{i}\frac{L}{4}\Delta\epsilon_0 \tilde{\boldsymbol{E}}_1^{0}\tilde{\boldsymbol{E}}_2^{0*}\sum_G(\mathrm{e}^{-\mathrm{i}\Omega t}\delta_{\kappa+G,k_1+k_2} + \mathrm{e}^{\mathrm{i}\Omega t}\delta_{-\kappa+G,k_1+k_2})a_1.\end{aligned} \tag{33}$$

(d) *Assuming that our optical modes have a small frequency separation [i.e., $|\omega_2-\omega_1| \ll (\omega_1,\omega_2)$] and that the modulation frequency, Ω, is much smaller than our optical frequencies [i.e., $\Omega \ll (\omega_1,\omega_2)$], apply the rotating wave approximation to the equations of motion from part (c).*

We define $a_{1,2} \coloneqq \bar{a}_{1,2}\mathrm{e}^{-\mathrm{i}\omega_{1,2}t}$. The EOMs of $a_{1,2}$ then becomes

$$\dot{\bar{a}}_1 = -\mathrm{i}\frac{L}{4}\Delta\epsilon_0 \tilde{\boldsymbol{E}}_1^{0*}\tilde{\boldsymbol{E}}_2^{0}\sum_G(\mathrm{e}^{-\mathrm{i}(\Omega+\omega_2-\omega_1)t}\delta_{\kappa+G,k_1+k_2} + \mathrm{e}^{\mathrm{i}(\Omega-\omega_2+\omega_1)t}\delta_{-\kappa+G,k_1+k_2})\bar{a}_2, \tag{34}$$

$$\dot{\bar{a}}_2 = -\mathrm{i}\frac{L}{4}\Delta\epsilon_0 \tilde{\boldsymbol{E}}_2^{0*}\tilde{\boldsymbol{E}}_1^{0}\sum_G(\mathrm{e}^{-\mathrm{i}(\Omega+\omega_1-\omega_2)t}\delta_{\kappa+G,k_1+k_2} + \mathrm{e}^{\mathrm{i}(\Omega-\omega_1+\omega_2)t}\delta_{-\kappa+G,k_1+k_2})\bar{a}_1. \tag{35}$$

Since the relation between Ω and $|\omega_1-\omega_2|$ is not determined, we can't do RWA yet. RWA works when $\Omega \approx |\omega_1-\omega_2|$, so one of the temporal Fourier components can be ignored. When $\Omega \approx \omega_1-\omega_2$, the RWA EOMs are

$$\dot{\bar{a}}_1 = -\mathrm{i}\frac{L}{4}\Delta\epsilon_0 \tilde{\boldsymbol{E}}_1^{0*}\tilde{\boldsymbol{E}}_2^{0}\sum_G\delta_{\kappa+G,k_1+k_2}\bar{a}_2, \tag{36}$$

$$\dot{\bar{a}}_2 = -\mathrm{i}\frac{L}{4}\Delta\epsilon_0 \tilde{\boldsymbol{E}}_2^{0*}\tilde{\boldsymbol{E}}_1^{0}\sum_G\delta_{-\kappa+G,k_1+k_2}\bar{a}_1, \tag{37}$$

and when $\Omega \approx \omega_2-\omega_1$, the RWA EOMs are

$$\dot{\bar{a}}_1 = -\mathrm{i}\frac{L}{4}\Delta\epsilon_0 \tilde{\boldsymbol{E}}_1^{0*}\tilde{\boldsymbol{E}}_2^{0}\sum_G\delta_{-\kappa+G,k_1+k_2}\bar{a}_2, \tag{38}$$

$$\dot{\bar{a}}_2 = -\mathrm{i}\frac{L}{4}\Delta\epsilon_0 \tilde{\boldsymbol{E}}_2^{0*}\tilde{\boldsymbol{E}}_1^{0}\sum_G\delta_{\kappa+G,k_1+k_2}\bar{a}_1. \tag{39}$$

(e) *What value must Ω take to facilitate resonant coupling between a_1 and a_2 ? [Hint: Your answer should be consistent with phase matching conditions!] Find the reduced form of the coupled mode equations in the case of resonant coupling.*

The condition of resonant coupling is the same as the condition of RWA, and we need $\Omega \approx |\omega_1-\omega_2|$. From the RWA equations above, we also need to find two G's such that

$$\kappa + G_1 = k_1 + k_2, \quad -\kappa + G_2 = k_1 + k_2, \tag{40}$$

which are equivalent to the condition that there exists one $G = 2\pi/Lm$ such that

$$k_1 + k_2 = \frac{G_1+G_2}{2}, \quad \kappa = \frac{G_2-G_1}{2}. \tag{41}$$

When all these conditions are satisfied we get

$$\dot{\bar{a}}_1 = -\mathrm{i}\frac{L}{4}\Delta\epsilon_0 \tilde{\boldsymbol{E}}_1^{0*}\tilde{\boldsymbol{E}}_2^{0}\bar{a}_2, \quad \dot{\bar{a}}_2 = -\mathrm{i}\frac{L}{4}\Delta\epsilon_0 \tilde{\boldsymbol{E}}_2^{0*}\tilde{\boldsymbol{E}}_1^{0}\bar{a}_1, \tag{42}$$

and therefore

$$\ddot{\bar{a}}_{1,2} = \frac{L^2}{16}\Delta\epsilon_0^2|\tilde{\boldsymbol{E}}_1^{0}|^2|\tilde{\boldsymbol{E}}_2^{0}|^2\bar{a}_{1,2}. \tag{43}$$

(f) *Is this an elastic or inelastic scattering process? Is energy conserved? If not, where is the energy coming from?*

The energy is not conserved: it oscillates between $N\omega_1$ and $N\omega_2$, and therefore it's an inelastic scattering process. The energy comes from the field that induces the change of ϵ_{r}.

3

3.1

Third order nonlinear polarization: In this problem, we consider the effect of third order nonlinearity on a single time-harmonic mode. A single optical mode, $\mathbf{E}(\mathbf{r}, t)$, interacts with a polarizable medium. We assume that the medium produces a polarization field $\mathbf{P} = \chi^{(3)}|E|^2\mathbf{E}$, where $\chi^{(3)}$ is a constant. In the absence of the polarizable medium (i.e., $\chi^{(3)} = 0$), the Hamiltonian for the system is $H(a, a^*) = \omega a^* a$ where a and a^* are the complex mode amplitudes. As we have done in lecture, we assume that the mode amplitude is related to the field by $\mathbf{E}(\mathbf{r}, t) = (a - a^*)\, \mathbf{E}^o(\mathbf{r})$.

(a) *Find an expression for the interaction Hamiltonian, δH, in terms of the electric fields.*

The nonlinear interaction Hamiltonian is

$$\begin{aligned} \delta H = \int \boldsymbol{E} \cdot \delta \boldsymbol{D} &= \int \boldsymbol{E} \cdot \chi^{(3)}(|\boldsymbol{E}|^2 \delta \boldsymbol{E} + 2\boldsymbol{E}\boldsymbol{E} \cdot \delta \boldsymbol{E}) \\ &= \chi^{(3)} \int 3|\boldsymbol{E}|^2 \boldsymbol{E} \cdot \delta \boldsymbol{E} \\ &= \frac{3}{4}\chi^{(3)}|\boldsymbol{E}|^4. \end{aligned} \tag{44}$$

(b) *Express your interaction Hamiltonian from part (a) as products of a, a^*; be sure to define all of the coefficients in this expression. [The coefficients will include integrals over space.]*

$$\delta H = \frac{3}{4}\chi^{(3)}(a - a^*)^4 \int \mathrm{d}^3\boldsymbol{r}\, |\boldsymbol{E}^0|^4. \tag{45}$$

(c) *Find $\dot{a}$ using Poisson brackets.*

$$\begin{aligned} \frac{\mathrm{d}a}{\mathrm{d}t} &= -\mathrm{i}\omega_0 a - \mathrm{i}\frac{\partial\, \delta H}{\partial a^*} \\ &= -\mathrm{i}\omega_0 a + \mathrm{i}3\chi^{(3)}(a - a^*)^3 \int \mathrm{d}^3\boldsymbol{r}\, |\boldsymbol{E}^0|^4. \end{aligned} \tag{46}$$

(d) *From part (c), show that the mode acquires intensity dependent frequency shift. Explain the significance of this term.*

Applying RWA to (46), we have

$$\dot{\bar{a}} = -\mathrm{i}9\chi^{(3)}|a|^2\bar{a} \int \mathrm{d}^3\boldsymbol{r}\, |\boldsymbol{E}^0|^4, \tag{47}$$

and therefore the RWA EOM of a is

$$\dot{a} = -\mathrm{i}\left(\omega_0 + 9\chi^{(3)}|a|^2 \int \mathrm{d}^3\boldsymbol{r}\, |\boldsymbol{E}^0|^4\right) a, \tag{48}$$

and it can be seen that the frequency of a is now modified by $|a|^2$ i.e. the intensity of the mode. This effect leads to self-phase modulation and other Kerr nonlinearity phenomena.