$U(1)$ Gauge Theories in Condensed Matter Physics

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This article is a reading note of Wen's famous textbook [1]. It is mainly a reconstruction of material related to $U(1)$ gauge theories covered in Chapter 6.

1 $U(1)$ gauge theory in $2+1$ dimensions

The most general form of a $U(1)$ theory is

$$\mathcal{L}_{U(1)} = -\frac{1}{4g^2} f_{\mu\nu} f^{\mu\nu}, \quad f_{\mu\nu} = \partial_\mu a_\nu - \partial_\nu a_\mu. \tag{1}$$

In the case of a 2+1 dimensional spacetime, we have

$$\begin{aligned} f_{\mu\nu} f^{\mu\nu} &= f^{i0} f_{i0} + f^{0i} f_{0i} + f^{ij} f_{ij} \\ &= e^i e_i + e^i e_i + f^{12} f_{12} + f^{21} f_{21} \\ &= 2e^i e_i + 2f^{21} f_{21}, \end{aligned}$$

where we define

$$e_i = \partial_0 a_i - \partial_i a_0, \quad b = \partial_1 a_2 - \partial_2 a_1, \tag{2}$$

Note that since in the Minkowski space

$$\partial_0 = \partial^0, \quad a_i = -a^i, \quad \partial_{1,2} = -\partial^{1,2}, \quad a_{1,2} = -a^{1,2},$$

we have

$$f_{\mu\nu} f^{\mu\nu} = -2\boldsymbol{e}^2 + 2b^2,$$

so the final Lagrangian is

$$\mathcal{L}_{U(1)} = \frac{1}{2g^2} (\boldsymbol{e}^2 - b^2), \tag{3}$$ Eq. (6.3.2)

where the inner product is now defined in the Euclidean space.

Now we try to quantize the theory. We impose the Coulomb gauge Sec. 6.3.1

$$\nabla \cdot \boldsymbol{a} = 0, \tag{4}$$ Eq. (6.3.3)

and since we are dealing with the free space electromagnetic field, we can actually additionally impose

$$a_0 = 0. \tag{5}$$ Discussion between Eq. (6.3.3) and Eq. (6.3.4)

This **radiation gauge** is well-known when dealing with 3D electromagnetic waves. Here we show explicitly that under (4), a_0 is actually decoupled from other degrees of freedom and can be set to any constant value. Now in the Euclidean space, we have

$$e_i = e^i = -\partial_t a_i - \partial_i a_0, \quad b = \partial_1 a_2 - \partial_2 a_1, \tag{6}$$

and terms involving a_0 Lagrangian all come from the $\boldsymbol{e}^2$ term, and we have

$$\begin{gathered} \partial_i a_0 \partial_i a_0 + \partial_0 a_i \partial_i a_0 + \partial_i a_0 \partial_0 a_i \\ \simeq (\partial_i a_0)^2 - 2a_0 \partial_i \partial_0 a_i = (\partial_i a_0)^2, \end{gathered}$$

so we see that a_0 is decoupled from $\boldsymbol{a}$.

Under the radiation gauge, we can decompose $\boldsymbol{a}$ into different modes.

I feel I'm still unable to understand what he wanted to do ...

Some key points in Section 6.3:

When deriving the equation after Eq. (6.3.4), pay attention to the fact that

$$\int \mathrm{d}^2 \boldsymbol{x} \sim L_1 L_2,$$

and the Fourier transformation of a single $\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}$ can be regarded as zero.

References

[1] Xiao-Gang Wen. *Quantum Field Theory of Many-Body Systems.* Oxford University Press, September 2007.