# Homework 3

Jinyuan Wu

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**Problem 1**
**Solution**

(a) The conjugate momentum of $\theta$ is

$$p = \frac{\partial L}{\partial \dot{\theta}} = V \left( \frac{\dot{\theta}}{U_0} - \frac{\mu}{U_0} \right), \tag{1}$$

and therefore

$$\dot{\theta} = \frac{U_0}{V} p + \mu. \tag{2}$$

The Hamiltonian is

$$\begin{aligned} H &= p\dot{\theta} - L \\ &= p \left( \frac{U_0}{V} p + \mu \right) - V \left( \frac{1}{2U_0} \left( \frac{U_0}{V} p + \mu \right)^2 - \frac{\mu}{U_0} \left( \frac{U_0}{V} p + \mu \right) \right) \\ &= \frac{1}{2} \frac{U_0}{V} \left( p + \frac{\mu V}{U_0} \right)^2 . \end{aligned} \tag{3}$$

In Heisenberg's picture, the variance of $\theta$ can be evaluated in the follows. We know

$$\begin{aligned} \frac{\mathrm{d}\theta^2}{\mathrm{d}t} &= \frac{1}{\mathrm{i}} [\theta^2, H] \\ &= \frac{U_0}{2\mathrm{i}V} \left[ \theta^2, \left( p + \frac{\mu V}{U_0} \right)^2 \right] \\ &= \frac{U_0}{V} \left( \theta \left( p + \frac{\mu V}{U_0} \right) + \left( p + \frac{\mu V}{U_0} \right) \theta \right), \end{aligned}$$

and therefore

$$\begin{aligned} \frac{\mathrm{d}^2\theta^2}{\mathrm{d}t^2} &= \frac{U_0}{V} \left( \dot{\theta} \left( p + \frac{\mu V}{U_0} \right) + \theta \dot{p} + \dot{p} \theta + \left( p + \frac{\mu V}{U_0} \right) \dot{\theta} \right) \\ &= \frac{2U_0^2}{V^2} \left( p + \frac{\mu V}{U_0} \right)^2 . \end{aligned} \tag{4}$$

Here we use the EOMs

$$\dot{\theta} = \frac{1}{\mathrm{i}} [\theta, H] = \frac{U_0}{V} \left( p + \frac{\mu V}{U_0} \right), \quad \dot{p} = 0. \tag{5}$$

From (4), we have

$$\frac{\mathrm{d}^2 \sigma_\theta^2}{\mathrm{d}t^2} = \frac{\mathrm{d}^2}{\mathrm{d}t^2} \left( \langle \theta^2 \rangle - \langle \theta \rangle^2 \right) = \frac{4U_0}{V} E - \frac{2U_0}{V} \langle \theta \rangle \left( \langle p \rangle + \frac{\mu V}{U_0} \right). \tag{6}$$

Assuming the wave packet doesn't move, we have

$$\frac{\mathrm{d}^2 \sigma_\theta^2}{\mathrm{d}t^2} = \frac{4U_0}{V} E, \tag{7}$$

and therefore

$$\sigma_\theta = \sqrt{\frac{2U_0}{V} E t^2 + \sigma_\theta^2(0)}. \tag{8}$$

The speed sound is

$$v = \sqrt{\frac{\rho_0 U_0}{m}}, \tag{9}$$

so

$$\sigma_\theta = \sqrt{\frac{2U_0}{V} E t^2 + \sigma^2} = \sqrt{\frac{2Emv^2t^2}{\rho_0 V} + \sigma^2}, \tag{10}$$

and the time it takes to have $\sigma_\theta = 2\pi$ is

$$t = \sqrt{\frac{\rho_0 V}{2Emv^2}(4\pi^2 - \sigma^2)}. \tag{11}$$

(b)

**Problem 2**
**Solution**

(a) Repeating the procedure used in ordinary superfluid, we do the decomposition

$$\varphi = \sqrt{\rho} \mathrm{e}^{\mathrm{i}\theta} = \sqrt{\rho_0 + \delta\rho} \mathrm{e}^{\mathrm{i}\theta}, \tag{12}$$

and therefore

$$-\frac{\varphi^* \nabla^2 \varphi}{2m} = \frac{\rho}{2m} (\boldsymbol{\nabla}\theta)^2 + \frac{(\boldsymbol{\nabla}\rho)^2}{8\rho m}, \tag{13}$$

$$\varphi^* \partial_\tau \varphi = \underbrace{\frac{1}{2}\partial_\tau \rho}_{\text{time derivative, ignored}} + \mathrm{i}\rho \partial_\tau \theta, \tag{14}$$

$$|\varphi(\boldsymbol{x})| U(\boldsymbol{x} - \boldsymbol{y}) |\varphi(\boldsymbol{y})| = \rho(\boldsymbol{x}) U(\boldsymbol{x} - \boldsymbol{y}) \rho(\boldsymbol{y}), \tag{15}$$

the theory is now

$$S = \int \mathrm{d}\tau \left( \int \mathrm{d}^d \boldsymbol{x} \left( \mathrm{i}\rho \partial_\tau \theta + \frac{\rho}{2m} (\boldsymbol{\nabla}\theta)^2 + \frac{(\boldsymbol{\nabla}\rho)^2}{8\rho m} - \mu\rho \right) + \frac{1}{2} \int \mathrm{d}^d \boldsymbol{x} \int \mathrm{d}^d \boldsymbol{y}\, \rho(\boldsymbol{x}) U(\boldsymbol{x} - \boldsymbol{y}) \rho(\boldsymbol{y}) \right). \tag{16}$$

Around the ground state, we have (note that since we are around a saddle point, the sum of all terms containing $\delta\rho$ only is always zero; the resulting theory has the form of $c_1\, \delta\rho\, \partial_\tau \theta + c_2\, \delta\rho^2$; the chemical potential term is therefore missing in the theory around the saddle point)

$$\mathrm{i}\rho \partial_\tau \theta = \underbrace{\mathrm{i}\rho_0 \partial_\tau \theta}_{\text{time derivative}} + \mathrm{i}\, \delta\rho\, \partial_\tau \theta,$$

and since $\boldsymbol{\nabla}\rho = \boldsymbol{\nabla}\, \delta\rho$, we have

$$\frac{(\boldsymbol{\nabla}\rho)^2}{8\rho m} \approx \frac{(\boldsymbol{\nabla}\, \delta\rho)^2}{8\rho_0 m},$$

ignoring the fluctuation of the $\rho$ in the denominator. Similarly, since we are working on a low energy theory, the fluctuation of $\theta$ shouldn't be large, and we have

$$\frac{\rho}{2m} (\boldsymbol{\nabla}\theta)^2 \approx \frac{\rho_0}{2m} (\boldsymbol{\nabla}\theta)^2.$$

The theory is then

$$S = \int \mathrm{d}^{d+1} x \left( \frac{\rho_0}{2m} (\boldsymbol{\nabla}\theta)^2 + \mathrm{i}\, \delta\rho\, \partial_\tau \theta + \frac{(\boldsymbol{\nabla}\, \delta\rho)^2}{8\rho_0 m} + \frac{1}{2}\, \delta\rho\, (\boldsymbol{x}) \int \mathrm{d}^d \boldsymbol{y}\, U(\boldsymbol{x} - \boldsymbol{y})\, \delta\rho\, (\boldsymbol{y}) \right) + S_{\text{saddle}}. \tag{17}$$

Integrating out $\delta\rho$, we get

$$\begin{aligned} S_{\text{eff}} &= \int \mathrm{d}^{d+1} x\, \frac{\rho_0}{2m} (\boldsymbol{\nabla}\theta)^2 - \frac{1}{2} \int \mathrm{d}\tau \int \mathrm{d}^d \boldsymbol{x}\, \mathrm{d}^d \boldsymbol{y}\, \mathrm{i}\partial_\tau \theta(\boldsymbol{x}, \tau) \frac{1}{\int \mathrm{d}^d \boldsymbol{y}\, U(\boldsymbol{x} - \boldsymbol{y}) - \frac{1}{4\rho_0 m} \nabla^2} \mathrm{i}\partial_\tau \theta(\boldsymbol{y}, \tau) \\ &= \int \mathrm{d}^{d+1} x\, \frac{\rho_0}{2m} (\boldsymbol{\nabla}\theta)^2 + \frac{1}{2} \int \mathrm{d}\tau \int \mathrm{d}^d \boldsymbol{x}\, \mathrm{d}^d \boldsymbol{y}\, \partial_\tau \theta(\boldsymbol{x}, \tau) G(\boldsymbol{x} - \boldsymbol{y}) \partial_\tau \theta(\boldsymbol{y}), \end{aligned} \tag{18}$$

where

$$\int \mathrm{d}^d \boldsymbol{y}\, U(\boldsymbol{x}-\boldsymbol{y})G(\boldsymbol{y}-\boldsymbol{z}) - \frac{1}{4\rho_0 m}\nabla^2_{\boldsymbol{x}} G(\boldsymbol{x}-\boldsymbol{z}) = \delta(\boldsymbol{x}-\boldsymbol{z}). \tag{19}$$

Similar to the procedure in dealing with ordinary superfluid, since we are only interested in the long wave length behaviors of $\theta$, the $\nabla^2$ term can be thrown away, and we have

$$\begin{aligned}\int \frac{\mathrm{d}^d \boldsymbol{p}}{(2\pi)^d}\mathrm{e}^{\mathrm{i}\boldsymbol{p}\cdot(\boldsymbol{x}-\boldsymbol{z})} &= \int \frac{\mathrm{d}^d \boldsymbol{p}}{(2\pi)^d}\int \mathrm{d}^d \boldsymbol{y}\, U(\boldsymbol{x}-\boldsymbol{y})G(\boldsymbol{p})\mathrm{e}^{\mathrm{i}\boldsymbol{p}\cdot(\boldsymbol{y}-\boldsymbol{z})} \\ &= \int \frac{\mathrm{d}^d \boldsymbol{p}}{(2\pi)^d}\int \mathrm{d}^d \boldsymbol{r}\, U(\boldsymbol{r})\mathrm{e}^{-\mathrm{i}\boldsymbol{p}\cdot\boldsymbol{r}}G(\boldsymbol{p})\mathrm{e}^{\mathrm{i}\boldsymbol{p}\cdot(\boldsymbol{x}-\boldsymbol{z})} \quad (\boldsymbol{r}=\boldsymbol{x}-\boldsymbol{y}),\end{aligned}$$

so

$$G(\boldsymbol{r}) = \int \frac{\mathrm{d}^d \boldsymbol{p}}{(2\pi)^d}\mathrm{e}^{\mathrm{i}\boldsymbol{p}\cdot\boldsymbol{r}}G(\boldsymbol{p}), \quad G(\boldsymbol{p}) = \frac{1}{U(\boldsymbol{p})} = \frac{1}{\int \mathrm{d}^d\boldsymbol{r}\, U(\boldsymbol{r})\mathrm{e}^{-\mathrm{i}\boldsymbol{p}\cdot\boldsymbol{r}}}. \tag{20}$$

To evaluate $G(\boldsymbol{p})$, we need to find

$$U(\boldsymbol{p}) = \int_0^\infty \mathrm{d}r\, \frac{2\pi^{d/2}}{\Gamma(d/2)} r^{d-1}\frac{U_0}{r^{d-\epsilon}} \tag{21}$$

**Problem 3**
**Solution**

1. The energy now can be exactly evaluated ($N$ is the number of sites):

$$E = \frac{UN}{2}(M^2-M) - \mu NM = \frac{N}{2}(UM^2 - (U+2\mu)M). \tag{22}$$

At the ground state, $E$ is minimized. If $M$ were continuous, we would have

$$M = \frac{U+2\mu}{2U} = \frac{1}{2} + \frac{\mu}{U}, \tag{23}$$

but it's not. So we need to find the closest integer to (23). Note that since

$$\frac{1}{2} \le \frac{1}{2} + \frac{\mu}{U} - \left\lfloor \frac{\mu}{U} \right\rfloor < \frac{3}{2},$$

the following $M$ is always a minimum point:

$$M = \left\lfloor \frac{\mu}{U} \right\rfloor + 1. \tag{24}$$

When $\mu/U$ is an integer, both

$$M = \frac{\mu}{U} \tag{25}$$

and

$$M = \left\lfloor \frac{\mu}{U} \right\rfloor + 1 = \frac{\mu}{U} + 1 \tag{26}$$

can be found in ground states.

The energy gap is

$$\begin{aligned}\Delta E &= \min(E|_{n_i=M+1 \text{ on one site}} - E|_M, E|_{n_i=M-1 \text{ on one site}} - E|_M) \\ &= \min(UM-\mu, U(-M+1)+\mu) \\ &= \begin{cases} 0 \text{ or } U, & \mu/U \text{ integer}, \\ \min\left(U\left\lfloor\frac{\mu}{U}\right\rfloor + U - \mu, \mu - U\left\lfloor\frac{\mu}{U}\right\rfloor\right), & \text{otherwise}.\end{cases}\end{aligned} \tag{27}$$

So

$$\frac{\Delta E}{U} = \begin{cases} \frac{\mu}{U} - \left\lfloor\frac{\mu}{U}\right\rfloor, & \frac{\mu}{U} - \left\lfloor\frac{\mu}{U}\right\rfloor \le \frac{1}{2}, \\ 1 + \left\lfloor\frac{\mu}{U}\right\rfloor - \frac{\mu}{U}, & \frac{\mu}{U} - \left\lfloor\frac{\mu}{U}\right\rfloor \ge \frac{1}{2}.\end{cases} \tag{28}$$

The energy gap and the phase diagram are shown in Figure 1.

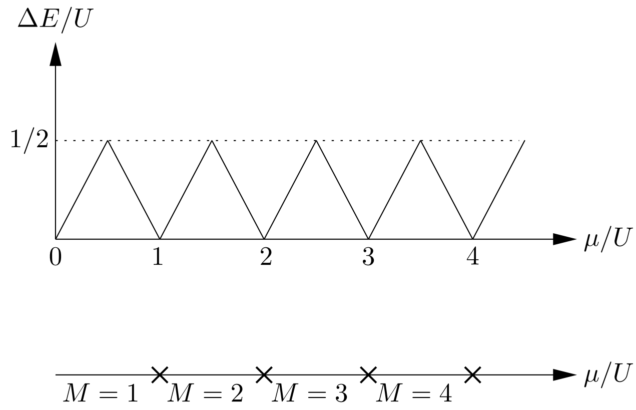


Figure 1: The energy gap plot and the phase diagram when $t = 0$

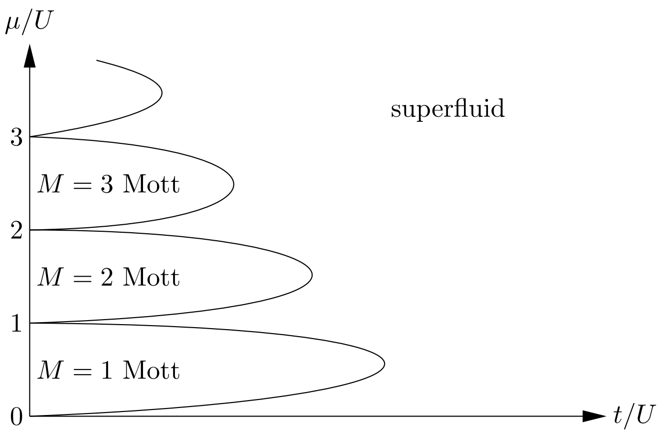


Figure 2: Schematic phase diagram of the boson Hubbard model

2. The gapless points in Figure 1 can only be connected to the superfluid phase, and therefore we get Figure 2.

3. We have

$$\langle n_0+k'|a|n_0+k\rangle = \sqrt{n_0+k}\,\langle n_0+k'|n_0+k-1\rangle = \sqrt{n_0+k}\delta_{k',k-1}, \tag{29}$$

and

$$\langle k'|\mathrm{e}^{-\mathrm{i}\theta}|k\rangle = \int_0^{2\pi}\frac{\mathrm{d}\theta}{2\pi}\mathrm{e}^{-\mathrm{i}k'\theta}\mathrm{e}^{-\mathrm{i}\theta}\mathrm{e}^{\mathrm{i}k\theta} = \delta_{k',k-1}. \tag{30}$$

When $k \ll n_0$, we have

$$\langle n_0+k'|a|n_0+k\rangle \approx \sqrt{n_0}\,\langle k'|\mathrm{e}^{-\mathrm{i}\theta}|k\rangle \Rightarrow a \approx \sqrt{n_0}\mathrm{e}^{-\mathrm{i}\theta}. \tag{31}$$

And similarly we have

$$\langle n_0+k'|a^\dagger|n_0+k\rangle = \sqrt{n_0+k+1}\delta_{k',k+1}, \tag{32}$$

and

$$\langle k'|\mathrm{e}^{\mathrm{i}\theta}|k\rangle = \int_0^{2\pi}\frac{\mathrm{d}\theta}{2\pi}\mathrm{e}^{-\mathrm{i}k\theta}\mathrm{e}^{\mathrm{i}\theta}\mathrm{e}^{\mathrm{i}k'\theta} = \delta_{k',k+1}, \tag{33}$$

and in the $k \ll n_0$ limit we have

$$a^\dagger \approx \sqrt{n_0}\mathrm{e}^{\mathrm{i}\theta}. \tag{34}$$

Also,

$$\langle n_0+k'|n|n_0+k\rangle = (n_0+k)\,\langle n_0+k|n_0+k'\rangle = (n_0+k)\delta_{kk'}, \tag{35}$$

and

$$\langle k'|\pi|k\rangle = \int_0^{2\pi}\mathrm{d}\theta\,\mathrm{e}^{-\mathrm{i}k'\theta}(-\mathrm{i}\partial_\theta)\mathrm{e}^{\mathrm{i}k\theta} = k\delta_{kk'}, \tag{36}$$

so

$$n = n_0 + \pi. \tag{37}$$

4. When the above approximation works,

$$\begin{aligned} H &= -tn_0\sum_{\langle i,j\rangle}\mathrm{e}^{\mathrm{i}\theta_i}\mathrm{e}^{-\mathrm{i}\theta_j} + \text{h.c.} + \frac{U}{2}\sum_i(n_0+\pi_i)(n_0+\pi_i-1) - \mu\sum_i(n_0+\pi_i) \\ &= -2tn_0\sum_{\langle i,j\rangle}\cos(\theta_i-\theta_j) + \frac{U}{2}\sum_i\pi_i^2 + \text{linear terms in }\pi + f(n_0). \end{aligned}$$

The linear terms of $\pi$ can be safely ignored because we are working around $n = n_0$ that minimize the energy, and $\frac{\partial E}{\partial n} = 0$ at $n = n_0$. Since $\pi$ is the fluctuation of $n$, linear terms in $\pi$ correspond to first order Taylor terms and are also bound to be zero. So

$$H = -J\sum_{\langle i,j\rangle}\cos(\theta_i-\theta_j) + u\sum_i\pi_i^2, \quad J = 2n_0t, \quad u = \frac{U}{2}. \tag{38}$$

5. In the limit of slow spatial varying, we have

$$H = -J\sum_{\langle i,j\rangle}\left(1 - \frac{1}{2}(\theta_i-\theta_j)^2\right) + u\sum_i\pi_i^2. \tag{39}$$

This Hamiltonian has exactly the same form of the lattice phonon Hamiltonian. The operator EOMs are

$$\begin{aligned} \frac{\mathrm{d}\pi_i}{\mathrm{d}t} &= \frac{1}{\mathrm{i}}[\pi_i, H] = -J\sum_{\langle i,j\rangle}(\theta_i-\theta_j), \\ \frac{\mathrm{d}\theta_i}{\mathrm{d}t} &= \frac{1}{\mathrm{i}}[\theta_i, H] = 2u\pi_i, \end{aligned} \tag{40}$$

and therefore

$$\frac{\mathrm{d}^2\theta_i}{\mathrm{d}t^2} = 2Ju\sum_{\langle i,j\rangle}(\theta_j-\theta_i). \tag{41}$$

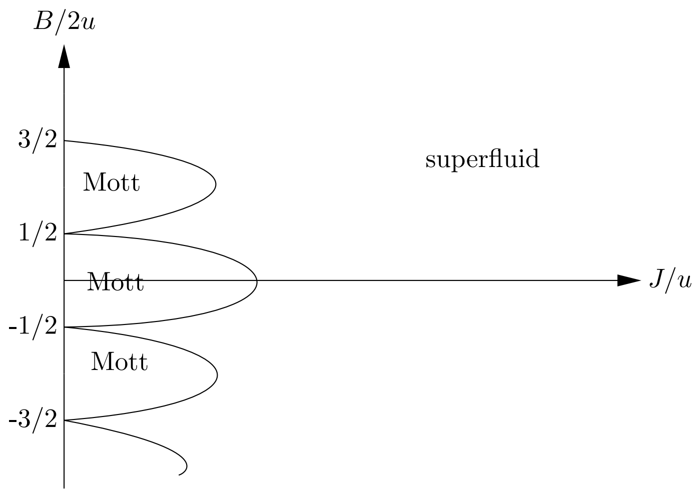


Figure 3: Phase diagram of (47)

Suppose the bond length of the lattice is $a$. In a normal mode

$$\theta_i \propto \mathrm{e}^{\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{r}-\omega_{\boldsymbol{k}}t)}, \tag{42}$$

we have

$$\begin{aligned}
-\omega_{\boldsymbol{k}}^2 &= 2Ju\sum_{i=1}^{d}(\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot a\hat{\boldsymbol{x}}_i}+\mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot a\hat{\boldsymbol{x}}_i}-2)\\
&= 2Ju\sum_{i=1}^{d}(2\cos(\boldsymbol{k}\cdot a\hat{\boldsymbol{x}}_i)-2)\\
&\approx -2Ju\sum_{i=1}^{d}(\boldsymbol{k}\cdot a\hat{\boldsymbol{x}}_i)^2\\
&= -2Jua^2\boldsymbol{k}^2,
\end{aligned}$$

so

$$\omega_{\boldsymbol{k}} = \sqrt{2Ju}a|\boldsymbol{k}|. \tag{43}$$

6. When $J/u=0$, we have

$$H = \sum_i (u\pi_i^2 - B\pi_i). \tag{44}$$

Again we apply the same procedure used to derive Figure 2. In the ground state, we have

$$\pi_i = \begin{cases} \left\lfloor \frac{B}{2u}\right\rfloor, & 0 \le \frac{B}{2u} - \left\lfloor \frac{B}{2u}\right\rfloor \le \frac{1}{2},\\ \left\lfloor \frac{B}{2u}\right\rfloor + 1, & \frac{1}{2} \le \frac{B}{2u} - \left\lfloor \frac{B}{2u}\right\rfloor < 1, \end{cases} \tag{45}$$

and when

$$\frac{B}{2u} - \left\lfloor \frac{B}{2u}\right\rfloor = \frac{1}{2}, \tag{46}$$

changing $\pi_i$ on one site doesn't change the energy, and we get a gapless system. So on the phase diagram, points defined by (46) are connected to the superfluid phase, and the phase diagram of

$$H = -J\sum_{\langle i,j\rangle}\cos(\theta_i-\theta_j) + u\sum_i \pi_i^2 - B\sum_i \pi_i, \tag{47}$$

is given in Figure 3.

7. We have

$$\dot{\theta}_i = \frac{\partial H}{\partial \pi_i} = 2u\pi_i - B, \quad \pi_i = \frac{\dot{\theta}_i + B}{2u}. \tag{48}$$

So

$$
\begin{aligned}
L &= \sum_i \dot{\theta}_i \pi_i - H \\
&= \sum_i \dot{\theta}_i \frac{\dot{\theta}_i + B}{2u} + J \sum_{\langle i,j \rangle} \cos(\theta_i - \theta_j) - u \sum_i \left( \frac{\dot{\theta}_i + B}{2u} \right)^2 + B \sum_i \frac{\dot{\theta}_i + B}{2u} \\
&= \frac{1}{4u} \sum_i (\dot{\theta}_i + B)^2 + J \sum_{\langle i,j \rangle} \cos(\theta_i - \theta_j).
\end{aligned}
\tag{49}
$$

In the continuous limit, we have

$$
\sum_{\langle i,j \rangle} \cos(\theta_i - \theta_j) \to \sum_{\langle i,j \rangle} \left( 1 - \frac{1}{2}(\theta_i - \theta_j)^2 \right) \simeq - \sum_{\langle i,j \rangle} \frac{1}{2}(\theta_i - \theta_j)^2 = -\frac{1}{2} a^2 (\boldsymbol{\nabla}\theta)^2,
\tag{50}
$$

and

$$
\begin{aligned}
L &= \int \frac{\mathrm{d}^d \boldsymbol{x}}{a^d} \frac{1}{4u} (\dot{\theta} + B)^2 - \int \frac{\mathrm{d}^d \boldsymbol{x}}{a^d} \frac{J}{2} a^2 (\boldsymbol{\nabla}\theta)^2 \\
&= \int \mathrm{d}^d \boldsymbol{x} \left( \frac{1}{4ua^d} (\dot{\theta} + B)^2 - \frac{J}{2} a^{d-2} (\boldsymbol{\nabla}\theta)^2 \right).
\end{aligned}
\tag{51}
$$

8. In the imaginary time path integral,

$$
\mathrm{i} \int \mathrm{d}t \, \frac{B}{2u} \dot{\theta}_i \longrightarrow \mathrm{i} \int_0^\beta \mathrm{d}\tau \, \frac{B}{2u} \partial_\tau \theta = \frac{B}{2u} \mathrm{i} (\theta_i(\beta) - \theta_i(0)).
\tag{52}
$$

Since in the imaginary time integral, the final state and the initial state have to be the same, $\theta_i(\beta)$ and $\theta_i(0)$ have to be equivalent to each other, and therefore

$$
\theta_i(\beta) - \theta_i(0) = 2\pi n, \quad n \in \mathbb{Z}.
\tag{53}
$$

So when $B/2u$ is an integer, (52) contributes nothing to the partition function.