Floquet theory

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1 The Floquet formalism: quasienergies and quasi-stationary states

In this section we outline the basic formalism of Floquet physics, following the notation in [1]. As is mentioned in the introduction, Floquet effects happen with a time-periodic Hamiltonian; below we let $T = 2\pi/\omega$ be the period. Such a Hamiltonian is usually an effective Hamiltonian when the system (hereafter "matter") is coupled with another degree of freedom which does not change much in the time evolution; the latter is hereafter called "light", since in condensed matter systems, periodic driving is usually achieved by shedding a beam of light to the matter.

From the Floquet theory of differential equation, we know it is possible to expand an arbitrary state that evolves according to H into a linear combination (the coefficients are constants) of $\{ |\psi_n(t)\rangle \}$ where

$$|\psi_n(t+T)\rangle = \mathrm{e}^{-\mathrm{i}\varepsilon_n t/\hbar} |\Phi_n(t)\rangle , \quad |\Phi_n(t+T)\rangle = |\Phi_n(t)\rangle . \tag{1}$$

By discrete periodicity of $|\Phi_n(t)\rangle$ we make Fourier expansion

$$|\Phi_n(t)\rangle = \sum_m \mathrm{e}^{-\mathrm{i}m\omega t} |\phi_n^{(m)}\rangle , \tag{2}$$

where m goes over all integers. Note that here $|\phi_n^{(m)}\rangle$ are *Fourier coefficients* and are not eigenstates of anything; there is no normalization or orthogonality condition. Using i to label the eigenstates of the matter, we have

$$|\Phi_n(t)\rangle = \sum_i \sum_m \mathrm{e}^{-im\omega t} \langle i|\phi_n(m)\rangle \ |i\rangle . \tag{3}$$

The coefficients before $|i\rangle$, not coefficients before $|\phi_n^{(m)}\rangle$ in (2), give the expansion of $|\Phi\rangle$ in a complete, orthogonal basis. The significance of $|\phi\rangle$ vectors can be seen immediately below.

The Schrodinger equation

$$\frac{\mathrm{d}}{\mathrm{d}t} |\psi_n(t)\rangle = H |\psi_n(t)\rangle \tag{4}$$

now reads

$$(\varepsilon_n + m\hbar\omega) |\phi_n^{(m)}\rangle = \sum_{m'} H^{(m-m')} |\phi_n^{(m')}\rangle , \tag{5}$$

where

$$H(t) = \sum_m \mathrm{e}^{-\mathrm{i}m\omega t} H^{(m)}. \tag{6}$$

Thus we find that if we use i to label the eigenstates of the matter part, we have

$$\varepsilon_n \langle i|\phi_n^{(m)}\rangle = \sum_{m'} (H^{(m-m')} - m\hbar\omega\delta_{mm'}) \langle i|\phi_n^{(m')}\rangle . \tag{7}$$

Recall that $\langle i|\phi_n^{(m)}\rangle$ is the $m\omega$-frequency component of $|\Phi_n(t)\rangle$ projected on the basis vector $|i\rangle$. ε_n is known as the *Floquet quasienergy* of the *Floquet quasi-stationary state (or quasi-eigenstate)* $|\Psi_n(t)\rangle$.

Floquet formalisms can be understood in a more generic framework of non-equilibrium physics: Floquet Green function can be calculated within the Keldysh formalism, and the RHS of (7) can be understood as the non-equilibrium self-energy [2, 3]. As a simple demonstration in the zero-temperature situation, consider the general form of light-matter interaction Hamiltonian with only one active photon mode

$$H_{\text{full}} = H \otimes 1_{\text{light}} + 1_{\text{matter}} \otimes \hbar \left(b^\dagger b + \frac{1}{2} \right) + \underbrace{bV + b^\dagger V^\dagger}_{H_{\text{light-matter coupling}}}, \tag{8}$$

and we assume that the state of the electromagnetic part is close to a coherent state $|\alpha e^{-i\omega t}\rangle$ with strong intensity that almost has zero time evolution.

We work under the basis $|i\rangle \otimes |m\rangle$, where m refers to the photon number.

Under this assumption, we can project out the electromagnetic degree of freedom by

$$P = \sum_i |i\rangle \langle i| \otimes \langle \alpha e^{-i\omega t}|, \tag{9}$$

where i labels eigenstates of the matter degrees of freedom, and this means the Hamiltonian for the matter part is

$$H_{\text{eff}} = P H_{\text{full}} P = H + \underbrace{\hbar |\alpha|^2}_{\text{const.}} + \alpha V e^{-i\omega t} + \alpha^* V^\dagger e^{i\omega t}, \tag{10}$$

and the relation between the full and projected wave function is

$$\langle i, m| \left(|\psi_n\rangle \otimes |m\rangle \right) \mapsto \langle i|\phi_n^{(m)}\rangle . \tag{11}$$

Now we see the true meaning of $\phi_n^{(m)}$: we are just grouping the components of the complete matter-light wave function $|\phi_n\rangle \otimes |\text{light}\rangle$ with the same photon number m into a vector $\phi_n^{(m)}$.

As an example, when the light field is approximately always in a coherent state $|\alpha e^{-i\omega t}\rangle$ (α should be large enough so that the matter part does not significantly change the state of the light part), approximately we have

$$H_{\text{light-matter coupling}} \approx \alpha V e^{-i\omega t} + \text{h.c.}, \tag{12}$$

and the effective Hamiltonian for the matter part is then

$$H = H_{\text{matter}} + \alpha V e^{-i\omega t} + \text{h.c.}. \tag{13}$$

Based on the above perspective, we call the coefficient matrix on the RHS of (7) the effective Hamiltonian in the **extended Hilbert space**, i.e. the space containing both the matter degrees of freedom and the light field. Since the coefficient matrix on the RHS of (7) contains components of the Hamiltonian in the extended space, while we are actually working in the Hilbert space of the matter part, (7)'s solutions are overcomplete. We can actually point out where overcompletion appears: note that if ε_n satisfies (1), then so does $\varepsilon_n + m\hbar\omega$.

In conclusion, a Floquet system has a set of quasi-eigenstates $\{ |\psi_n\rangle \}$, the number of which is the same as the dimension of the Hilbert space; but for each quasi-eigenstate, we have countable infinite quasi-energies, the difference between the nearest two being $\hbar\omega$; thus all distinct Floquet quasi-eigenstates can be indexed by quasi-energies that are within one "Floquet-Brillouin zone".

Orthogonal relation between $|\psi_n\rangle$'s?

2 Self-driven Floquet effects

it is however possible to use light to stimulate some long-lived degrees of freedom in a solid and let it drive the rest of the system, which sometimes is known as "self-driving".

References

[1] Mark S Rudner and Netanel H Lindner. "The Floquet Engineer's Handbook". In: *arXiv preprint arXiv:2003.08252* (2020).

[2] Andreas Lubatsch and Regine Frank. "Evolution of Floquet topological quantum states in driven semiconductors". In: *The European Physical Journal B* 92 (2019), pp. 1–9.

[3] Hideo Aoki et al. "Nonequilibrium dynamical mean-field theory and its applications". In: *Reviews of Modern Physics* 86.2 (2014), p. 779.