Homework 2

Jinyuan Wu

February 8, 2023

1

Problem

$$y' - 9y = t, \quad y(0) = 5. \tag{1}$$

Solution After Laplace transformation we get

$$sY(s) - 5 - 9Y(s) = \frac{1}{s^2},$$

$$Y(s) = \frac{5}{s-9} + \frac{1}{s^2(s-9)}.$$

The second term can be decomposed (by multiplying x, x^2 or $(x-9)$ and taking the limit $x \to 0$ and $x \to 9$) as

$$\frac{1}{s^2(s-9)} = \frac{1}{81}\frac{1}{s-9} - \frac{1}{81}\frac{1}{s} + \frac{1}{9}\frac{1}{s^2},$$

and

$$Y(s) = \frac{406}{81}\frac{1}{s-9} - \frac{1}{81}\frac{1}{s} - \frac{1}{9}\frac{1}{s^2}.$$

The inverse Laplace transform is

$$y(t) = \frac{406}{81}\mathrm{e}^{9t} - \frac{1}{81} + \frac{1}{9}t. \tag{2}$$

2

Problem

$$y'' - 4y' + 4y = f(t); y(0) = -2, y'(0) = 1 \text{ with}$$
$$f(t) = \begin{cases} t & \text{for } 0 \le t < 3 \\ t+2 & \text{for } t \ge 3 \end{cases}$$

Solution The Laplace transform of the LHS is

$$s^2 Y(s) - sy(0) - y'(0) - 4(sY(s) - y(0)) + 4Y(s) = (s-2)^2 Y(s) + 2s - 9.$$

The RHS is

$$f(t) = t(H(t) - H(t-3)) + (t+2)H(t-3) = tH(t) + 2H(t-3) \xrightarrow{\mathcal{L}} \frac{1}{s^2} + 2 \cdot \mathrm{e}^{-3s} \cdot \frac{1}{s}.$$

So the equation is equivalent to

$$(s-2)^2 Y(s) + 2s - 9 = \frac{1}{s^2} + \frac{2\mathrm{e}^{-3s}}{s},$$

$$Y(s) = -\frac{2}{s-2} + \frac{5}{(s-2)^2} + \frac{1}{s^2(s-2)^2} + \frac{2}{s(s-2)^2}\mathrm{e}^{-3s}. \tag{3}$$

We immediately get

$$\mathcal{L}^{-1}\frac{1}{s-2} = \mathrm{e}^{2t}, \quad \mathcal{L}^{-1}\frac{1}{(s-2)^2} = \mathrm{e}^{2t}t,$$

and from the decomposition

$$\frac{1}{s(s-2)^2} = \frac{1}{4s} - \frac{1}{4(s-2)} + \frac{1}{2}\frac{1}{(s-2)^2},$$