Bosonic Field Theories in Condensed Matter Physics

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This article is a reading note of Wen's famous textbook [3]. It is mainly a reconstruction of material related to the KT phase transition covered in Chapter 3 .

1 A simplest interacting boson system

Eq. (3.3.1) in Section 3.3.1 provides a simplest interacting bosonic system with a complex scalar field Section 3.3.1

$$S = \int \mathrm{d}^d \boldsymbol{x} \, \mathrm{d}t \, (\mathrm{i}\varphi^* \partial_t \varphi - \frac{1}{2m} \partial_{\boldsymbol{x}} \varphi^* \partial_{\boldsymbol{x}} \varphi + \mu |\varphi|^2 - \frac{V_0}{2} |\varphi|^4). \tag{1}$$

The prefactor of the interaction term makes the corresponding term in the EOM of φ and φ^* not have a numerical factor, but it introduces a numerical factor in the vertex in Feynman diagrams. The sign of the mass term is derived as follows: first we have a $-\varphi^* \nabla^2 \varphi / 2m$ term in the Hamiltonian, and therefore we have a $\varphi^* \nabla^2 \varphi / 2m$ term in the Lagrangian, and by integration by parts we have $\varphi^* \nabla^2 \varphi / 2m \simeq -\partial_{\boldsymbol{x}} \varphi^* \partial_{\boldsymbol{x}} \varphi / 2m$. Eq. (3.3.1)

The semiclassical approximation from Eq. (3.3.1) to Eq. (3.3.2) can be justified when the temperature is high and therefore the most economical path does not have imaginary time evolution at all. It can also be derived using the ideas behind Eq. (3.4.1), where with a finite temperature, we can always integrate out modes with non-zero Matsubara frequencies. This gives a physical picture behind dynamic density functional theories and also explains why "classical" statistical physics is still relevant today. It should be noted that when the temperature is low, parameters in the "classical" theory obtained by integrating out non-zero Matsubara frequencies are different with the truly classical approximation, though they have exactly the same form. The fact can also be seen in the comparison between classical DFT and DFT in the context of hard condensed matter physics.

This, actually, demonstrates the fact that the *interpretation* of what "quantum" actually means is still confusing. In standard textbooks we learned that "a quantum theory uses operators as basic degrees of freedom, while a classical theory do not" and "the Lagrangian of a quantum theory is to be placed into a partition function, while we just need to minimize the Lagrangian in a classical theory". But no one has ever *seen* an operator or the quasi-probability distribution defined in path integral. The idea to integrate out non-zero Matsubara frequencies to modify the parameters in the classical theory gives us a strange sense, that somehow the quantum effect is really just certain additional fluctuation added to a classical theory.

Note that we can only integrate out non-zero Matsubara frequencies when the temperature is not too low. When $T = 0$, we need to integrate out *uncountably* infinite degrees of freedom, which may have some subtlety. This gives rise to some uniquely zero-temperature quantum phenomena.

The following contents from Eq. (3.3.3) to Eq. (3.3.4) are also covered in this note. The discussion between Eq. (3.3.4) to the end of Section 3.3.2 is important, which illustrates the Ginzburg-Landau paradigm and why it is almost always associated with symmetries (or otherwise it is highly unlikely that we have several minima of the energy functional that share the same energy, so that we have a smooth phase transition shown in Fig. 3.5), though the concept of order parameters can also be used in a first-order phase transition (see here, for example). Section 3.3.2

2 Quantum XY model from (1)

Note that there is no BEC when $\mu < 0$. This is often justified by the argument that the Bose-Einstein distribution function is not well-defined when $\epsilon < \mu$, and if $\mu > 0$, there must be a finite number of particles condensed on the ground state. The fact is also shown in the discussion Section 3.3.3

around (86) in this note. The derivation of (3.3.10) is also done in (89) in this note, in the form of an imaginary time field theory.

We call (3.3.10) a quantum XY model. It is easy to see that its classical approximation is exactly the coarse-grained version of a classical XY model, and since θ is a real bosonic field, the simplest way to give it time evolution ("quantum fluctuation") is to add a $(\partial_t \theta)^2$ term. An interesting question may be how can we write down a quantum version of the *lattice* XY model whose continuum field theory is (3.3.10). TODO: is this just [1, 4]?

The method the author used to derive (3.3.13) and (3.3.14) is not really necessary in this case, since we all know what θ actually means, but it is a general method to connect the original degrees of freedom to the degrees of freedom in the effective theory so that we are able to calculate physical quantities defined in the original theory.

The quasiparticles corresponding to the θ field have linear dispersion. They are spin waves when the field $\varphi \sim S^x + \mathrm{i} S^y$. In Section 6.1.1 of this note, we introduced a physical picture which says if there are "phonons" in a bosonic liquid (and in this case φ is the field operator of the bosons forming the fluid), then there may be a superfluid phase. Therefore, sometimes we also call the quasiparticles connected to θ *phonons*.

3 Superfluid as a toy universe

If there are "phonon charges" in a superfluid, then we can always fix a test charge at one particular point, and a static electric field surrounds it. Note, however, that if we regard θ as the potential, then the electric field $-\boldsymbol{\nabla}\theta$ is just the current of the φ bosons. In a static $\boldsymbol{\nabla}\theta$ configuration, we have stale non-zero $\boldsymbol{\nabla}\theta$, meaning that φ bosons continuously flow from the position of the test charge to infinity. This definitely is not allowed in Section 3.3.5

TODO: Still I'm not sure about what a roton is in this picture ...

4 Equivalence to a sine-Gordon model

In this section we show the duality between the 2D XY model and the 2D sine-Gordon model, which can also be thought as an effective model of *clock model* (Wen used this name for the sine-Gordon model, just like using the name "XY model" for the effective field theory of the XY model). Sine-Gordon model is easier to study using RG, and this note discusses the KT phase transition in the *classical* XY model using the sine-Gordon model – which is most frequently seen in introduction to KT phase transition itself. It should be noticed that the phase transition is not restricted to the classical region – we may define quantum XY models, just as is the case in Section 2. TODO: For example, the θ degrees of freedom in this note can be realized with quantum rotors, which are introduced in the first chapter in [2]. Section 3.5.2

The duality is shown by Eq. (3.5.4). Here we show some details in the derivation. From Eq. (3.5.3) we get the first line of Eq. (3.5.4): Eq. (3.5.4)

$$
\begin{aligned}
Z &= \int \mathcal{D}\theta \sum_{k=0}^{\infty} \frac{1}{k!} \left(g \int \mathrm{d}^2 \boldsymbol{x} \, \frac{\mathrm{e}^{\mathrm{i}\theta} + \mathrm{e}^{-\mathrm{i}\theta}}{2} \right)^k \mathrm{e}^{-\int \mathrm{d}^2 \boldsymbol{x} \frac{\kappa}{2} (\boldsymbol{\nabla}\theta)^2} \\
&= \int \mathcal{D}\theta \sum_{k=0}^{\infty} \frac{1}{k!} \left(\frac{g}{2} \right)^k \prod_{i=1}^{k} \int \mathrm{d}^2 \boldsymbol{x}_i \sum_{\{q_i = \pm 1\}} \mathrm{e}^{\sum_{i=1}^k \mathrm{i} q_i \theta(\boldsymbol{x}_i)} \mathrm{e}^{\int \mathrm{d}^2 \boldsymbol{x} \frac{\kappa}{2} \theta \nabla^2 \theta}.
\end{aligned}
$$

Now, note that

$$
\exp\left(\sum_{i=1}^{k} \mathrm{i} q_i \theta(\boldsymbol{x}_i) \right) = \exp\left(\int \mathrm{d}^2 \boldsymbol{x} \, \theta(\boldsymbol{x}) \sum_{i=1}^{k} \mathrm{i} q_i \delta(\boldsymbol{x} - \boldsymbol{x}_i) \right),
$$

we can integrate out θ and get

$$
\begin{aligned}
&\int \mathcal{D}\theta\, \mathrm{e}^{\sum_{i=1}^{k} \mathrm{i} q_i \theta(\boldsymbol{x}_i)} \mathrm{e}^{\int \mathrm{d}^2\boldsymbol{x} \frac{\kappa}{2} \theta \nabla^2 \theta} \\
&= Z_0 \exp\left(-\frac{1}{2} \int \mathrm{d}^2\boldsymbol{x} \int \mathrm{d}^2\boldsymbol{y}\, \frac{1}{2\pi\kappa} \ln|\boldsymbol{x}-\boldsymbol{y}| \sum_{i=1}^{k} \mathrm{i} q_i \delta(\boldsymbol{x}-\boldsymbol{x}_i) \cdot \sum_{j=1}^{k} \mathrm{i} q_j \delta(\boldsymbol{y}-\boldsymbol{x}_j) \right) \\
&= Z_0 \exp\left(\frac{1}{4\pi\kappa} \sum_{i=1}^{k}\sum_{j=1}^{k} q_i q_j \ln r_{ij} \right),
\end{aligned}
$$

where we have used the Green function of ∇^2, which is

$$
\frac{1}{\kappa \nabla^2} = \frac{1}{2\pi\kappa} \ln r_{ij},
$$

and Z_0 is the partition function of the action $-\int \mathrm{d}^2\boldsymbol{x}\, (\boldsymbol{\nabla}\theta)^2$. Therefore, we have

$$
Z = Z_0 \sum_{k=0}^{\infty} \frac{1}{k!} \left(\frac{g}{2}\right)^k \prod_{i=1}^{k} \int \mathrm{d}^2\boldsymbol{x}_i \sum_{\{q_i=\pm 1\}} \exp\left(\frac{1}{4\pi\kappa} \sum_{i=1}^{k}\sum_{j=1}^{k} q_i q_j \ln r_{ij} \right). \tag{2}
$$

Note that when $i=j$, $\ln r_{ij}$ diverges, and there are k diverging terms. We can collect these divergence and $(g/2)^k$ together and define

$$
\mathrm{e}^{-kS_{\text{core}}} = \left(\frac{g}{2}\right)^k \exp\left(\frac{1}{4\pi\kappa} k \ln 0 \right), \tag{3}
$$

and now we have

$$
Z = \sum_{k=0}^{\infty} \frac{1}{k!} \mathrm{e}^{-kS_{\mathrm{c}}} \prod_{i=1}^{k} \int \mathrm{d}^2\boldsymbol{x}_i \sum_{\{q_i=\pm 1\}} \exp\left(\frac{1}{4\pi\kappa} 2 \sum_{i<j}^{k} q_i q_j \ln r_{ij} \right). \tag{4}
$$

This is already very close to the second line of Eq. (3.5.4), and we already get the 2D Coulomb interaction. What we need to do to get the final version of Eq. (3.5.4) is to notice that the low energy configurations contain the same number of positive and negative q_i's, so we only have to consider the even k terms. Defining

$$
h = \frac{1}{4\pi\kappa}, \tag{5}
$$

we have

$$
Z \approx \sum_{k=0}^{\infty} \frac{1}{(2k)!} \mathrm{e}^{-2kS_{\mathrm{c}}} \int \prod_{i=1}^{2k} \mathrm{d}^2\boldsymbol{x}_i \sum_{q_i=\pm 1} \mathrm{e}^{\sum_{i<j}^{k} 2 q_i q_j \ln r_{ij}}.
$$

Currently we imposes no restriction on $\{q_i\}$ and we just sum over all possible configurations, but since we can just rearrange $\{\boldsymbol{x}_i\}$, all $\{q_i\}$ configurations with the same k actually give the same weight in the partition function. There are C_{2k}^k possible $\{q_i\}$ configurations with a given k (since we need exactly k positive charges in $2k$ charges), and therefore we have

$$
\begin{aligned}
Z &\approx \sum_{k=0}^{\infty} \frac{1}{(2k)!} \mathrm{e}^{-2kS_{\mathrm{c}}} \int \prod_{i=1}^{2k} \mathrm{d}^2\boldsymbol{x}_i \underbrace{\frac{(2k!)}{k!k!}}_{C_{2k}^k} \mathrm{e}^{\sum_{i<j}^{k} 2 q_i q_j \ln r_{ij}} \\
&= \sum_{k=0}^{\infty} \frac{1}{k!k!} \mathrm{e}^{-2kS_{\mathrm{c}}} \int \prod_{i=1}^{2k} \mathrm{d}^2\boldsymbol{x}_i\, \mathrm{e}^{\sum_{i<j}^{k} 2 q_i q_j \ln r_{ij}}.
\end{aligned}
$$

So now we have completed the derivation of Eq. (3.5.4).

In this note we showed the equivalence between the XY model and the sine-Gordon model using Hubbard-Stratonovich transformation (see Section 4), and here we show the equivalence directly by analyzing the Taylor expansion of the partition function. What we are doing here is actually a path integral version of second quantization, where we have shown that indeed the partition function of a many-body system is the same as the partition function of a field theory.

5 KT phase transition

References

[1] Takeo Matsubara and Hirotsugu Matsuda. A Lattice Model of Liquid Helium, I. *Progress of Theoretical Physics*, 16(6):569–582, 12 1956.

[2] Subir Sachdev. *Quantum Phase Transitions*. Cambridge University Press, 2009.

[3] Xiao-Gang Wen. *Quantum Field Theory of Many-Body Systems*. Oxford University Press, September 2007.

[4] Juan Pablo Álvarez Zúñiga, Gabriel Lemarié, and Nicolas Laflorencie. Spin wave theory for 2d disordered hard-core bosons. In *AIP Conference Proceedings*. AIP Publishing LLC, 2014.