Homework 4

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Problem 1 In this problem we will introduce another way to think about Hall conductance. As discussed in class, the edge of a Chern insulator with Chern number n should have n chiral edge modes. Suppose that the n edge modes move one way near the $y=0$ edge, and the opposite way at the $y=L_y$ edge. 1. First consider the $y=L_y$ edge. Each of the n edge modes near $y=L_y$ can be modeled as a non-interacting free fermion model $H=\sum_k\left(\epsilon_{1k}-\mu_1\right)c_k^{\dagger}c_k$, where ϵ_{1k} is a monotonically increasing function of k, μ_1 corresponds to the chemical potential at the $y=L_y$ edge. We assume that the infinitely many states with $\epsilon_{1k}<\mu_1$ are all filled (in reality, the limit $k\rightarrow-\infty$ corresponds to moving into the bulk, and k is cut off accordingly). Suppose one changes the chemical potential $\mu_1\rightarrow\mu_1+\Delta\mu_1$. What is the corresponding change ΔI_1 in the $y=L_y$ edge current? (Include all n edge modes). 2. Now consider the $y=0$ edge mode, modeled by n chiral modes of the form $H=\sum_k\left(\epsilon_{2k}-\mu_2\right)c_k^{\dagger}c_k$, where ϵ_{2k} is a monotonically decreasing function of k. What is the corresponding change ΔI_2 in the $y=0$ edge current if one changes the chemical potential $\mu_2\rightarrow\mu_2+\Delta\mu_2$? 3. Find the total current $I=I_1+I_2$ induced by a chemical potential difference $\mu_1-\mu_2=\Delta\mu$ between the two edges, assuming that $I=0$ when $\mu_1=\mu_2$. Interpret your result in terms of the Hall conductance σ_H. 4. Now repeat the calculation at finite temperature T. Show that the edge currents and the Hall conductance do not change. (In reality, there is a temperature dependence coming from thermal activation above the bulk gap Δ, but it is exponentially small $\sim e^{-\Delta/T}$).

Solution The problem is illustrated in Figure 1.

(a) The current is

$$\begin{aligned}
I_1&=-e\cdot\text{density of states}\cdot\text{cross section}\cdot v\\
&=-e\frac{1}{L}\sum_{\text{ocupied states}}\frac{1}{\hbar}\frac{\partial\epsilon_{1k}}{\partial k}=-\frac{e}{\hbar}\frac{n}{L}\sum_{\Lambda_1\leq k\leq k_{1\mathrm{F}}}\frac{\partial\epsilon_{1k}}{\partial k}=-n\frac{e}{\hbar}\int_{\Lambda_1}^{k_{1\mathrm{F}}}\frac{\mathrm{d}k}{2\pi}\frac{\partial\epsilon_{1k}}{\partial k}\\
&=-n\frac{e}{h}(\epsilon_{1\mathrm{F}}-\epsilon_{1,k=\Lambda_1})=-n\frac{e}{h}(\mu_1-\epsilon_{1,k=\Lambda_1}).
\end{aligned}\tag{1}$$

So we have

$$\Delta I_1=-n\frac{e}{h}\Delta\mu_1.\tag{2}$$

(b) Similarly, we have

$$\begin{aligned}
I_2&=-e\cdot\text{density of states}\cdot\text{cross section}\cdot v\\
&=-e\frac{1}{L}\sum_{\text{ocupied states}}\frac{1}{\hbar}\frac{\partial\epsilon_{1k}}{\partial k}=-\frac{e}{\hbar}\frac{n}{L}\sum_{k_{2\mathrm{F}}\leq k\leq\Lambda_2}\frac{\partial\epsilon_{2k}}{\partial k}=-n\frac{e}{\hbar}\int_{k_{2\mathrm{F}}}^{\Lambda_2}\frac{\mathrm{d}k}{2\pi}\frac{\partial\epsilon_{2k}}{\partial k}\\
&=-n\frac{e}{h}(\epsilon_{2,k=\Lambda_2}-\epsilon_{2\mathrm{F}})=-n\frac{e}{h}(\epsilon_{2,k=\Lambda_2}-\mu),
\end{aligned}\tag{3}$$

and

$$\Delta I_2=n\frac{e}{h}\Delta\mu_2.\tag{4}$$

(c) We have

$$I=-n\frac{e}{h}(\mu_1-\mu_2+\epsilon_{2,k=\Lambda_2}-\epsilon_{1,k=\Lambda_1}).\tag{5}$$

When $\Delta\mu=0$, we have $I=0$, so we find the two cutoff terms cancel each other, and

$$I=-n\frac{e}{h}\Delta\mu.\tag{6}$$

In a Hall effect setting, the difference of the chemical potentials arises from the external electric field: conservation of energy tells us

$$\mu_1-eU=\mu_2,\tag{7}$$

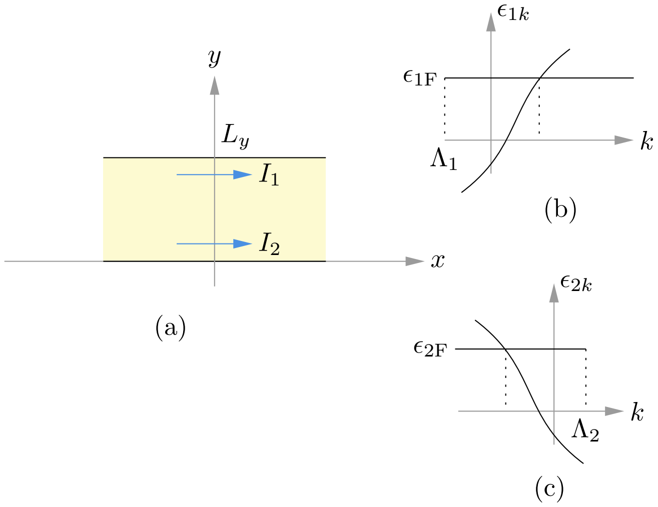

Figure 1: Boundary modes of a 2D Chern insulator (a) The device and sign conventions (b) The spectrum of the boundary modes at $y = L_y$, where Λ_1 is a momentum cutoff (c) The spectrum of the boundary modes at $y = 0$, where Λ_2 is a momentum cutoff

and therefore

$$I = -n\frac{e}{h} \cdot eU = \underbrace{-n\frac{e^2}{h}}_{1/R_{\mathrm{H}}} U. \tag{8}$$

So we get the expected quantized conductance.

(d) Ignoring excited bulk modes, we have

$$\begin{aligned} I_1 &= -n\frac{e}{\hbar}\int_{\Lambda_1}^{\infty} \frac{\mathrm{d}k}{2\pi} \frac{1}{1+\mathrm{e}^{(\epsilon_{1k}-\mu_1)/k_{\mathrm{B}}T}} \frac{\partial \epsilon_{1k}}{\partial k} \\ &= -n\frac{e}{h}\int_{\epsilon_{1,k=\Lambda_1}-\mu_1}^{\infty} \mathrm{d}\xi \, \frac{1}{1+\mathrm{e}^{\xi/k_{\mathrm{B}}T}}, \end{aligned} \tag{9}$$

and similarly

$$\begin{aligned} I_2 &= -n\frac{e}{\hbar}\int_{-\infty}^{\Lambda_2} \frac{\mathrm{d}k}{2\pi} \frac{1}{1+\mathrm{e}^{(\epsilon_{2k}-\mu_2)/k_{\mathrm{B}}T}} \frac{\partial \epsilon_{2k}}{\partial k} \\ &= -n\frac{e}{h}\int_{\infty}^{\epsilon_{2,k=\Lambda_2}-\mu_2} \mathrm{d}\xi \, \frac{1}{1+\mathrm{e}^{\xi/k_{\mathrm{B}}T}} = n\frac{e}{h}\int_{\epsilon_{2,k=\Lambda_2}-\mu_2}^{\infty} \mathrm{d}\xi \, \frac{1}{1+\mathrm{e}^{\xi/k_{\mathrm{B}}T}}, \end{aligned} \tag{10}$$

and therefore

$$I = -n\frac{e}{h}\int_{\epsilon_{1,k=\Lambda_1}-\mu_1}^{\epsilon_{2,k=\Lambda_2}-\mu_2} \frac{\mathrm{d}\xi}{1+\mathrm{e}^{\xi/k_{\mathrm{B}}T}}. \tag{11}$$

Since the Fermi-Dirac distribution function is always positive, the condition that when $\Delta\mu = 0$, $I = 0$ implies $\epsilon_{1,k=\Lambda_1} = \epsilon_{2,k=\Lambda_2} =: \epsilon_\Lambda$, so we have

$$I = -n\frac{e}{h}\int_{\epsilon_\Lambda-\mu_1}^{\epsilon_\Lambda-\mu_2} \frac{\mathrm{d}\xi}{1+\mathrm{e}^{\xi/k_{\mathrm{B}}T}}. \tag{12}$$

Since Λ must be large, a very good approximation is

$$I = -n\frac{e}{h}\Delta\mu = -n\frac{e^2}{h}U, \tag{13}$$

so the edge current and the Hall conductance are not affected much by a finite T.

Problem 2

Solution

(a) The only change we need to do is to replace the electric charge $-e$ by the "energy charge" ϵ_k, and following the line of reasoning in Problem 1 (d), we have

$$\begin{aligned} I_{Q1} &= \frac{n}{h}\int_{\epsilon_{1,k=\Lambda_1}}^{\infty} \mathrm{d}\epsilon\, \frac{1}{1+\mathrm{e}^{(\epsilon-\mu_1)/k_\mathrm{B}T_1}}\epsilon \\ &= k_\mathrm{B}T_1\frac{n}{h}\left(k_\mathrm{B}T_1\int_{\frac{\epsilon_{1,k=\Lambda_1}-\mu}{k_\mathrm{B}T_1}}^{\infty}\mathrm{d}x\,\frac{x}{1+\mathrm{e}^x}+\mu\int_{\frac{\epsilon_{1,k=\Lambda_1}-\mu}{k_\mathrm{B}T_1}}^{\infty}\mathrm{d}x\,\frac{1}{1+\mathrm{e}^x}\right), \end{aligned} \tag{14}$$

and similarly

$$I_{Q2} = -k_\mathrm{B}T_2\frac{n}{h}\left(k_\mathrm{B}T_2\int_{\frac{\epsilon_{2,k=\Lambda_2}-\mu}{k_\mathrm{B}T_2}}^{\infty}\mathrm{d}x\,\frac{x}{1+\mathrm{e}^x}+\mu\int_{\frac{\epsilon_{2,k=\Lambda_2}-\mu}{k_\mathrm{B}T_2}}^{\infty}\mathrm{d}x\,\frac{1}{1+\mathrm{e}^x}\right). \tag{15}$$

The sign difference between I_{Q1} and I_{Q2} comes in the same way of (10). Again, when $T_1 = T_2$, we need $I = I_{Q1} + I_{Q2} = 0$, so $\epsilon_{1,k=\Lambda_1} = \epsilon_{2,k=\Lambda_2} =: \epsilon_\Lambda$.

(b) We have

$$\int_t^\infty \mathrm{d}x\,\frac{1}{1+\mathrm{e}^x} = 2\,\mathrm{arctanh}(1+2\mathrm{e}^t), \tag{16}$$

$$\int_t^\infty \mathrm{d}x\,\frac{x}{1+\mathrm{e}^x} = t\ln\big(1+\mathrm{e}^{-t}\big) - \mathrm{Li}_2(-\mathrm{e}^{-t}). \tag{17}$$