

# ODEs

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## 1 First order ODEs

### 1.1 Linear ODEs

An ODE in the form of

$$y'(x) + p(x)y(x) = q(x) \quad (1)$$

is considered **linear**. All linear ODEs can be solved by the following procedure. First we have

$$(y' + py)e^{\int p dx} = qe^{\int p dx}, \quad (2)$$

and now the LHS is a derivative:

$$\frac{d}{dx} \left( ye^{\int p dx} \right) = qe^{\int p dx}, \quad (3)$$

and now we can integrate over  $x$  and get

$$ye^{\int p dx} = \int qe^{\int p dx} dx, \quad (4)$$

$$y = e^{-\int p dx} \int qe^{\int p dx} dx. \quad (5)$$

### 1.2 “Energy-conservation lines” and exact equations

Another way to represent the solution of an ODE is the form  $\phi(x, y) = \text{const}$ . Note that the RHS contains no variables, and we have

$$0 = \frac{d\phi}{dx} = \frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y} \frac{dy}{dx}, \quad (6)$$

and thus if

$$y' = f(x, y) \quad (7)$$

is algebraically equivalent to (6), the equation is already solved: We should find  $M, N$  such that

$$y' = -\frac{M}{N}, \quad M = \frac{\partial \phi}{\partial x}, \quad N = \frac{\partial \phi}{\partial y}, \quad (8)$$

and then  $\phi(x, y)$  solves the equation. In this case we say  $y' = -M/N$  is **exact**.

To test for exactness, we only have to test whether

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}, \quad (9)$$

and if so, the existence of  $\phi$  is guaranteed. (Since we work on a topological trivial space, things like cohomology group will not bother us.) We can now use “partial integral” to find  $\phi$ .

Example: suppose in a calculation we find

$$\frac{\partial \phi}{\partial x} = 2y^2 + ye^{xy}, \quad \frac{\partial \phi}{\partial y} = 4xy + xe^{xy} + 2y. \quad (10)$$

After partial integration, we find

$$\phi(x, y) = \underbrace{2xy^2 + e^{xy} + h(y)}_{\int \frac{\partial \phi}{\partial x} dx} = \underbrace{2xy^2 + e^{xy} + y^2 + g(x)}_{\int \frac{\partial \phi}{\partial y} dy}, \quad (11)$$

and we have to choose

$$h(y) = y^2, \quad g(x) = \text{const}, \quad (12)$$

and the solution is

$$\phi(x, y) = 2xy^2 + e^{xy} + y^2 + \text{const}. \quad (13)$$

Note that even when the decomposition  $f = -M/N$  doesn't give an exact equation for us, we can still use the method of exact equations: we can multiply a factor  $\mu$  to both  $M$  and  $N$ , and try to guess the form of  $\mu$  so that

$$\frac{\partial(\mu M)}{\partial y} = \frac{\partial(\mu N)}{\partial x}. \quad (14)$$

An example can be found in solving

$$y' = -\frac{1}{3x - e^{-2y}}. \quad (15)$$

We have

$$\frac{\partial 1}{\partial y} = 0, \quad \frac{\partial(3x - e^{-2y})}{\partial x} = 3,$$

so the equation is not exact if we choose  $M = 1$  and  $N = 3x - e^{-2y}$ . However, (14) can be fulfilled now: it's now

$$\frac{\partial \mu}{\partial y} = 3\mu + (3x - e^{-2y}) \frac{\partial \mu}{\partial x},$$

and the most convenient way to solve it (we *don't* need to find all solutions of this equation!) is to let  $\mu$  contain  $y$  only, so the tricky term on the RHS disappears, and thus we choose  $\mu = e^{3y}$ , and we get

$$\phi(x, y) = \int \mu M \, dx = \int e^{3y} \, dx = xe^{3y} + u(y),$$

$$\phi(x, y) = \int \mu N \, dy = \int (3xe^{3y} - e^y) \, dy = xe^{3y} - e^y + v(x),$$

so

$$\phi(x, y) = xe^{3y} - e^y + \text{const}. \quad (16)$$

### 1.3 Bernoulli equation

Consider the following **Bernoulli equation**

$$y' + P(x)y = R(x)y^\alpha. \quad (17)$$

When  $\alpha = 0, 1$ , the equation can be solved by the standard methods for linear first order ODEs. When this is not the case, we may do the substitution

$$v = y^\beta, \quad (18)$$

and then the equation becomes

$$\begin{aligned} \frac{1}{\beta} v^{1/\beta-1} v' + P(x) v^{1/\beta} &= R(x) v^{\alpha/\beta}, \\ v' + P(x)v &= R(x) v^{1+\frac{\alpha-1}{\beta}}. \end{aligned} \quad (19)$$

The next step is to choose a good beta so that the equation gets simplified. We may want to make to exponent to be zero, and this means we should choose

$$\beta = 1 - \alpha, \quad (20)$$

and the ODE is now

$$v' + Pv = R, \quad (21)$$

which can then be solved by the method in Section 1.1.

## 2 Second order ODEs

### 2.1 Linear 2nd order ODE with initial values

A linear second order ODE has the following form:

$$y'' + p(x)y' + q(x)y = f(x). \quad (22)$$

It usually comes with initial value conditions

$$y(x_0) = A, \quad y'(x_0) = B. \quad (23)$$

This course is about concrete calculations, but knowing what we are doing makes sense is important. Here is an existence and uniqueness theorem: if  $p(x)$ ,  $q(x)$ , and  $f(x)$  are continuous over an interval  $I$ , and  $x_0 \in I$ , then a unique solution exists for (22) with the initial conditions given above.

Usually, we start by looking at the **homogeneous** second order ODE

$$y'' + p(x)y' + q(x)y = 0. \quad (24)$$

The influence of  $f(x)$  can be included as the “response” of the LHS. The full solution of (24) takes the form

$$y = c_1 y_1 + c_2 y_2, \quad (25)$$

where  $c_1, c_2$  are constants to be decided by initial conditions, and  $y_1$  and  $y_2$  are linearly independent solutions of (24). The **Wronskian** is defined as

$$W(x) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}. \quad (26)$$

By checking if it is non-zero at most points, we can find whether  $y_1$  and  $y_2$  are truly linearly independent to each other.

There is a method to arrive at  $y_2$  from  $y_1$ : we can always take the ansatz

$$y_2 = y_1 u, \quad (27)$$

and therefore we get

$$(u'' y_1 + 2u' y_1' + u y_1'') + p(u' y_1 + u y_1') + q u y_1 = 0,$$

and the condition that  $y_1$  is a solution to (24) means

$$u'' + \underbrace{\frac{2y_1' + p y_1}{y_1}}_{g(x)} u' = 0, \quad (28)$$

which is essentially a first order ODE, because we can replace  $u'$  by  $v$ , and then we find

$$\ln v = - \int g(x) dx,$$

and

$$u(x) = \int e^{-\int g(x) dx} dx. \quad (29)$$

### 2.2 Constant coefficients

The equation

$$y'' + Ay + By = 0 \quad (30)$$

can be solved directly by the following construction:

$$y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}, \quad (31)$$

where  $\lambda_1, \lambda_2$  are solutions of

$$\lambda^2 + A\lambda + B = 0. \quad (32)$$

For example, to solve the equation

$$y'' - 2y' + 10y = 0, \quad (33)$$

we just solve

$$\lambda^2 - 2\lambda + 10 = 0,$$

which gives us

$$\lambda = 1 \pm 3i, \quad (34)$$

and therefore a general solution is

$$y = e^x(c_1 e^{3ix} + c_2 e^{-3ix}). \quad (35)$$

It should be noted that  $c_1, c_2$  can be complex, even when we restrict  $y$  in  $\mathbb{R}$ : we can let the imaginary part of  $y$  vanish as long as we impose some constraints over  $c_1, c_2$ . If we are determined to work in the real space, two alternative linearly independent solutions can be used:

$$y_1(x) = e^x \cos(3x), \quad y_2(x) = e^x \sin(3x). \quad (36)$$

Although we can immediately say they are linearly independent, we can use them as a demonstration of the Wronskian method: now we have

$$W(x) = y_1(x)y_2'(x) - y_2(x)y_1'(x) = 3e^{2x}, \quad (37)$$

which of course isn't constantly zero.

(32) is faced with the problem of having only one solution when  $A^2 - 4B = 0$ . In this case we need to go back to the standard procedure to get  $y_2$  from  $y_1$ . An example is

$$y'' + 6y + 9 = 0, \quad (38)$$

for which (32) only gives

$$y_1 = e^{-3x}. \quad (39)$$

Suppose  $y_2 = ue^{-3x}$ , we have TODO

## 2.3 Euler equation

A **Euler equation** has the following form:

$$x^2 y'' + Axy' + By = 0, \quad (40)$$

where  $A, B$  are constants. One solution can be immediate found: it always looks like

$$y = x^a. \quad (41)$$

We then find

$$a(a-1) + Aa + B = 0. \quad (42)$$

If there are two solutions of the equation, (40) has already been solved. If not, we can use the trick (27).

An example: let's solve

$$x^2 y'' + 3xy' + y = 0. \quad (43)$$

The equation about  $a$  is now

$$a(a-1) + 3a + 1 = 0,$$

and it only has one solution  $a = -1$ . Therefore we have

$$y_1 = \frac{1}{x}.$$

Suppose

$$y_2 = uy_1,$$

we get

$$x^2 \left( \frac{u''}{x^2} - \frac{2u'}{x} + \frac{2u}{x^2} \right) + 3x \left( \frac{u'}{x} - \frac{u}{x^2} \right) + \frac{u}{x} = 0,$$

which is equivalent to

$$v'x + v = 0, \quad v = u'$$

the solution of which is

$$\ln v + \ln x = \text{const},$$

and therefore

$$v = \frac{C'}{x}, \quad u = C' \ln x + C,$$

$$y_2 = \frac{1}{x}(C' \ln x + C).$$

This essentially gives *all* solutions we need: for  $y_1$ , we just have  $u = 1$ , which corresponds to  $C = 1$ . So now the equation is completely solved.

## 2.4 Non-homogeneous cases or how to find the linear response

Now we discuss how to solve

$$y'' + p(x)y' + q(x)y = f(x). \quad (44)$$

A general solution is

$$y(x) = y_p(x) + y_h(x), \quad (45)$$

where the subscript p means a particular solution, and the subscript h means the general solution of the corresponding homogeneous equation.

We need some common sense to find a particular solution. To solve

$$y'' - y' - 2y = 2x^2 + 5, \quad (46)$$

we don't expect  $y$  to be, say,  $\cos(2x)$ : instead, it's usually the case that  $y$  is a polynomial. An ansatz is

$$y = Ax^2 + Bx + C.$$

We don't want a  $x^3$  term because it doesn't appear on RHS. The equation then becomes

$$2A - (2Ax + Bx) - 2(Ax^2 + Bx + C) = 2x^2 + 5,$$

$$-2A = 2, \quad -2A - 2B = 0, \quad 2A - B - 2C = 5,$$

and therefore  $A = -1, B = 1, C = -4$ . Therefore we get a particular solution:

$$y_p = -x^2 + x - 4. \quad (47)$$

A particular solution may also be determined as a "linear combination" of homogeneous solutions, although now the coefficients have temporal variation. That's to say, we take the ansatz

$$y_p(x) = u(x)y_1(x) + v(x)y_2(x). \quad (48)$$

This is quite similar to the procedure introduced in Section 1.1. After substituting  $y$  with (48) in (44), we get

$$u'y'_1 + v'y'_2 = f. \quad (49)$$

Introducing the constraint

$$u'y_1 + v'y_2 = 0, \quad (50)$$

we have

$$\begin{pmatrix} y_1 & y_2 \\ y'_1 & y'_2 \end{pmatrix} \begin{pmatrix} u' \\ v' \end{pmatrix} = \begin{pmatrix} 0 \\ f \end{pmatrix}, \quad (51)$$

from which we find  $u', v'$  and hence  $u, v$ . The Wronskian – the determinant of the matrix on LHS – is non-zero, so the equation always has a solution.

Example: let's solve

$$y'' + y = \tan x. \quad (52)$$

We have

$$y_1 = \cos x, \quad y_2 = \sin x,$$

and therefore

$$W(x) = y_1 y_2' - y_2 y_1' = 1.$$

So

$$\begin{aligned} u' &= - \int \frac{y_2 f}{W(x)} dx = - \int \frac{\sin^2 x}{\cos x} dx = - \int \frac{1 - \cos^2 x}{\cos x} dx \\ &= -\frac{1}{2} \ln \left| \frac{1 + \sin x}{1 - \sin x} \right| + \sin x, \end{aligned}$$

and similarly we have

$$v =$$

## 2.5 Analyticity

The stimulus can be non-analytic.  $f$  is analytic at  $x_0$ , if we can expand it into a power series around  $x_0$ :

$$f(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n \quad (53)$$

in a interval around  $x_0$ . The function  $f(x) = \ln x$ , then, is not analytic at  $x = 0$  – but  $f(x) = \ln(x + 1)$  is analytic at  $x = 0$ , though not at  $x = -1$ .

There is a theorem: if  $p, q, f$  are all analytic at  $x_0$ , then (44) together with conditions  $y(x_0) = A$  and  $y'(x_0) = B$  has a unique solution that is analytic at  $x_0$ .

An example:

$$x'' + y' - xy = 0, \quad y(0) = -2, y'(0) = 0. \quad (54)$$

Suppose

$$y(x) = \sum_{n=0}^{\infty} a_n x^n, \quad y'(x) = \sum_{n=0}^{\infty} n a_n x^{n-1}, \quad y''(x) = \sum_{n=0}^{\infty} n(n+1) a_n x^{n-2},$$

we have

$$\begin{aligned} 0 &= \sum_{n=0}^{\infty} n(n-1) a_n x^{n-2} + \sum_{n=0}^{\infty} n a_n x^{n-1} - \sum_{n=0}^{\infty} a_n x^{n+1} \\ &= a_1 + 2a_2 + \sum_{n=1}^{\infty} x^n ((n+2)(n+1)a_{n+2} + (n+1)a_{n+1} - a_{n-1}), \end{aligned}$$

and therefore

$$a_1 + 2a_2 = 0, \quad a_{n+2} = -\frac{a_{n+1}}{n+2} + \frac{a_{n-1}}{(n+1)(n+2)}.$$

The conditions  $y(0) = -2$  and  $y'(0) = 0$  means

$$a_0 = -2, \quad a_1 = 0,$$

and then we can in principle find all  $a_n$ 's – although it's often hard to see a pattern and write down a closed-form expression for  $a_n$ .

Now we consider the equation

$$P(x)y'' + Q(x)y' + R(x)y = F(x). \quad (55)$$

Of course we can divide the equation with  $P(x)$  and go back to (44), but if  $P(x)$  is zero at some points,  $p, q, f$  in (44) are no longer always analytic. Thus the solution isn't guaranteed to be

analytic everywhere. In other words, we no longer have a power series solution – or do we? We can still try a generalized power series, like

$$y(x) = \sum_{n=0}^{\infty} c_n (x - x_0)^{n+r}, \quad (56)$$

where  $r$  can be a fraction. This is guaranteed with

$$(x - x_0)y'' + Q(x)y' + R(x)y = F(x). \quad (57)$$

To demonstrate this, consider

$$y'' + \frac{1}{2x}y' - \frac{1}{4x}y = 0. \quad (58)$$

We plug

$$y(x) = \sum_{n=0}^{\infty} c_n x^{n+r}$$

into

$$4xy'' + 2y' - y = 0,$$

and get

$$\sum_{n=0}^{\infty} (4(n+r)(n+r-1)c_n x^{n+r-1} + 2(n+r)c_n x^{n+r-1} - c_n x^{n+r}) = 0.$$

The coefficient of the  $x^{r-1}$  term is

$$4r(r-1) + 2r = 0,$$

from which we find  $r = 0, 1/2$ . For the rest of the terms, we have

$$4(n+r)(n+r-1)c_n + 2(n+r)c_n - c_{n-1} = 0,$$

$$c_n = \frac{c_{n-1}}{2(n+r)(2n+2r-1)}.$$

When  $r = 0$ , this gives

$$c_n = \frac{c_{n-1}}{2n(2n-1)}, \quad c_n = \frac{c_0}{(2n)!},$$

and we get

$$y(x) = \sum_{n=0}^{\infty} \frac{x^n}{(2n)!}. \quad (59)$$

When  $r = 1/2$ , this gives

$$c_n = \frac{c_{n-1}}{(2n+1) \cdot 2n}, \quad c_n = \frac{c_0}{(2n+1)!},$$

and we get

$$y(x) = \sum_{n=0}^{\infty} \frac{x^{n+1/2}}{(2n+1)!}. \quad (60)$$

So we have already obtained two independent solutions.

### 3 The Laplace transformation

The **Laplace transformation** is defined for a function  $f(t)$  as

$$\mathcal{L}[f(t)] = F(s) = \int_0^{\infty} f(t)e^{-st} dt, \quad (61)$$

for every  $s$  where the integral converges. It's easy to see that  $\mathcal{L}$  is linear,

### 3.1 Laplace transforms of basic elementary functions

The simplest Laplace transform is

$$\mathcal{L}[1] = \int_0^\infty e^{-st} dt = \frac{1}{s}, \quad s > 0, \quad (62)$$

and by integration by parts, we also have

$$\mathcal{L}[t] = \int_0^\infty te^{-st} dt = \frac{1}{s^2}. \quad (63)$$

By multiple rounds of integration by parts, we have

$$\mathcal{L}[t^2] = \frac{2}{s^3}, \quad (64)$$

and by iteratively using integration by parts we get

$$\mathcal{L}[t^n] = \frac{n!}{s^{n+1}}. \quad (65)$$

Aside from polynomials, we have

$$\mathcal{L}[e^{at}] = \int_0^\infty e^{-(s-a)t} dt = \frac{1}{s-a}, \quad s > a. \quad (66)$$

Since we have

$$\cos x = \frac{e^{ix} + e^{-ix}}{2}, \quad \sin x = \frac{e^{ix} - e^{-ix}}{2i},$$

we have

$$\mathcal{L}[\cos(at)] = \frac{1}{2} \left( \frac{1}{s-ia} + \frac{1}{s+ia} \right) = \frac{s}{s^2 + a^2}. \quad (67)$$

Similarly we have

$$\mathcal{L}[\sin(at)] = \frac{a}{s^2 + a^2}. \quad (68)$$

Consider a pulse signal

$$f(t) = \begin{cases} 0, & t < a \text{ or } t > b, \\ 1, & a \leq t \leq b. \end{cases} \quad (69)$$

This can be easily implemented by the Heaviside function: we have

$$f(t) = H(t-a) - H(t-b). \quad (70)$$

So the Laplace transform is

$$\mathcal{L}[f(t)] = e^{-sa}\mathcal{L}[1] - e^{-sb}\mathcal{L}[1] = \frac{1}{s}(e^{-sa} - e^{-sb}). \quad (71)$$

### 3.2 Laplace transform of differential equations

We have

$$\begin{aligned} \mathcal{L}[f'(t)] &= \int_0^\infty \frac{df}{dt} e^{-st} dt = f e^{-st} \Big|_{s=0}^\infty - \int_0^\infty f \frac{d}{dt} e^{-st} dt \\ &= -f(0) + s \int_0^\infty f(t) e^{-st} dt = s\mathcal{L}[f(t)] - f(0). \end{aligned} \quad (72)$$

Applying this twice, we get

$$\mathcal{L}[f''(t)] = s\mathcal{L}[f'(t)] - f'(0) = s^2\mathcal{L}[f(t)] - sf(0) - f'(0). \quad (73)$$

The general formula is therefore

$$\mathcal{L}[f^{(n)}(t)] = s^n\mathcal{L}[f(t)] - s^{n-1}f(0) - s^{n-2}f'(0) - \dots - f^{(n-1)}(0). \quad (74)$$



This can be used to solve ODEs. Consider, for example,

$$y'' + 4y' + 3y = e^t, \quad y(0) = 0, \quad y'(0) = 2. \quad (75)$$

We have (below we follow the convention to use big letters to refer to functions in the Laplace space)

$$\mathcal{L}[\text{LHS}] = (s^2 Y(s) - sy(0) - y'(0)) + 4(sY(s) - y(0)) + 3Y(s) = (s^2 + 4s + 3)Y(s) - 2,$$

where we have already applied the initial conditions, and

$$\mathcal{L}[\text{RHS}] = \frac{1}{s-1}.$$

So what need to be done is to solve

$$(s^2 + 4s + 3)Y(s) - 2 = \frac{1}{s-1},$$

and we get

$$Y(s) = \frac{2s-1}{(s-1)(s+1)(s+3)}. \quad (76)$$

Thus, once we do the inverse Laplace transformation, we get  $y(t)$ . It's possible to do an inverse integral transformation, but in this case, what's more convenient is to make the decomposition

$$Y(s) = \frac{A}{s-1} + \frac{B}{s+1} + \frac{C}{s+3},$$

and we can find

$$A = \frac{1}{8}, \quad B = \frac{3}{4}, \quad C = -\frac{7}{8}.$$

Then we can read the Laplace transformation table in the inverse direction: from (66), we find

$$y(t) = \frac{1}{8}e^t + \frac{3}{4}e^{-t} - \frac{7}{8}e^{-3t}. \quad (77)$$

So the linear 2nd order ODE is completely solved.

TODO: whether we will encounter things like

$$\mathcal{L}[f] = \frac{1}{s}, \quad s > 100. \quad (78)$$

It seems in most real world problems, we don't really need to worry about the region of allowed  $s$ : we just extend the region of allowed  $s$  as large as we can, solve the algebraic equation in the Laplace space, and then go back.

We can repeat the procedure in the last example for a system of ODEs. Consider for example

$$x' - 2y' = 1, \quad x' - x + y = 0, \quad x(0) = y(0) = 0. \quad (79)$$

For the first equation, Laplace transform gives

$$sX(s) - x(0) - 2(sY(s) - y(0)) = \mathcal{L}[1] = \frac{1}{s},$$

and for the second equation we have

$$sX(s) - x(0) - X(s) + Y(s) = 0.$$

Solving this linear equation system, we get

$$X(s) = \frac{1}{s^2(2s-1)}, \quad Y(s) = -\frac{s-1}{s^2(2s-1)}.$$

Since

$$X(s) = -\frac{2}{s} - \frac{1}{s^2} + 2\frac{1}{s-1/2},$$

we have

$$x(t) = -2 - t + 2e^{t/2}. \quad (80)$$

Similarly,

$$y(t) = -1 + e^{t/2} - t. \quad (81)$$

### 3.3 Shifting of $s$ and $t$

Laplace transform of an integral can also be found. Here we investigate into things like

$$\int_0^t f(t') dt',$$

and we need to pay attention to what variable is inside the integration and what variable is exposed to the Laplace operator. We have

$$\begin{aligned}\mathcal{L}\left[\int_0^t f(t') dt'\right] &= -\frac{1}{s} \int_{t'=0}^{\infty} \int_0^t f(t') dt' de^{-st} \\ &= -\frac{1}{s} \left( e^{-st} \int_0^t f(t') dt' \Big|_0^{\infty} - \int_0^{\infty} f(t) e^{-st} dt \right) \\ &= \frac{1}{s} \mathcal{L}[f(t)].\end{aligned}\tag{82}$$

This is the inverse of the rule of derivatives above, which is expected.

Another theorem is the  **$s$ -shifting theorem**: we have

$$\mathcal{L}[e^{at} f(t)] = F(s-a) = \mathcal{L}[f(t)]_{s \rightarrow s-a}.\tag{83}$$

This is exactly what leads to (66). Correspondingly we have the  $t$ -shifting theorem:

$$\mathcal{L}[f(t-a)H(t-a)] = e^{-sa} \mathcal{L}[f(t)],\tag{84}$$

where  $H(t-a)$  is 1 when  $t \geq 0$  and zero otherwise, and  $a > 0$ .

The theorems can be used to evaluate Laplace transforms of complex functions. We have

$$\mathcal{L}[t^2 e^{-t}] = \mathcal{L}[t^2]_{s \rightarrow s+1} = \frac{2}{(s+1)^3}.\tag{85}$$

### 3.4 Convolution

The **convolution integral** of  $f(t)$  and  $g(t)$  is defined as

$$f \otimes g = \int_0^t f(t') g(t-t') dt'.\tag{86}$$

This can be found very frequently in science: it appears when we deal with interaction: for example, the  $t-t'$  time may come from indirect interaction (where  $t'$  is the time an intermediate step happens).

A way around the reasonable but hard to calculate convolution integral is taking its Laplace transform. We have

$$\begin{aligned}\mathcal{L}[f \otimes g] &= \int_0^{\infty} e^{-st} \int_0^t f(t') g(t-t') dt' dt \\ &= \int_0^{\infty} f(t') \int_{t'}^{\infty} g(t-t') e^{-st} dt dt' \\ &= \int_0^{\infty} f(t') \int_0^{\infty} g(t'') e^{-s(t''+t')} dt'' dt' \\ &= \int_0^{\infty} f(t') e^{-st'} \int_0^{\infty} g(t'') e^{-st''} dt'' \\ &= \mathcal{L}[f] \mathcal{L}[g].\end{aligned}\tag{87}$$

The second line uses another way to see the integration region: by saying  $0 < t < \infty$ ,  $0 < t' < t$ , we also mean  $0 < t' < \infty$ ,  $t' < t < \infty$ . The third line replaces  $t-t'$  with  $t''$ . The final result no longer contains convolution.

One application of this fact is shown in the following example. Consider

$$\frac{1}{s^2 - a^2} = \underbrace{\frac{1}{s-a}}_{G(s)} \cdot \underbrace{\frac{1}{s+a}}_{F(s)}.$$

What's its inverse Laplace transform? By shifting theorem, we have

$$f(t) = e^{-at}, \quad g(t) = e^{at},$$

and therefore

$$\mathcal{L}^{-1} \left[ \frac{1}{s^2 - a^2} \right] = \int_0^t e^{-at'} e^{a(t-t')} dt' = e^{at} \int_0^t e^{-2at'} dt' = \frac{1}{a} \sinh(at). \quad (88)$$

We can also use the convolution theorem to give solutions to very generic equations. Consider the following ODE problem:

$$y'' - 5y' + 6y = f(t), \quad f(0) = f'(0) = 0. \quad (89)$$

The point here is we *don't* know what is  $f(t)$ , but still want to give a template of the solution. So we just do Laplace transform:

$$(s^2 - 5s + 6)Y(s) = F(s).$$

To find the Laplace transform of  $1/(s^2 - 5s + 6)$ , we have

$$\mathcal{L}^{-1} \left[ \frac{1}{(s^2 - 5s + 6)} \right] = \mathcal{L}^{-1} \left[ \frac{1}{s-3} - \frac{1}{s-2} \right] = e^{3t} - e^{2t},$$

and

$$\begin{aligned} y(t) &= \mathcal{L}^{-1}[Y(s)] = \mathcal{L}^{-1} \left[ \frac{1}{(s^2 - 5s + 6)} F(s) \right] \\ &= \mathcal{L}^{-1} \left[ \frac{1}{(s^2 - 5s + 6)} \right] \otimes f(t) \\ &= \int_0^t f(t')(e^{3(t-t')} - e^{2(t-t')}) dt'. \end{aligned} \quad (90)$$

## 4 Fourier series

### 4.1 Definition

Consider a real, integrable function  $f(x)$  on  $-L \leq x \leq L$ . The Fourier series of it is

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} \left( a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right) \right). \quad (91)$$

This form has a very intuitive physical picture: any sound can be obtained by mixing a series of simple harmonic oscillations.

The coefficients can be obtained by integrating from  $-L$  to  $L$ . Since

$$\int_{-L}^L \sin\left(\frac{n\pi x}{L}\right) dx = \int_{-L}^L \cos\left(\frac{n\pi x}{L}\right) dx = 0, \quad (92)$$

we have

$$\frac{1}{L} \int_{-L}^L f(x) dx = a_0, \quad (93)$$

and similarly since

$$\int_L^{-L} \cos\left(\frac{n\pi x}{L}\right) \cos\left(\frac{m\pi x}{L}\right) = \int_L^{-L} \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{m\pi x}{L}\right) = \begin{cases} 0, & n \neq m, \\ L, & n = m, \end{cases} \quad (94)$$

and the cross term

$$\int_L^{-L} \sin\left(\frac{n\pi x}{L}\right) \cos\left(\frac{m\pi x}{L}\right) = 0, \quad (95)$$

we find

$$a_m = \frac{1}{L} \int_{-L}^L f(x) \cos\left(\frac{m\pi x}{L}\right) dx, \quad (96)$$

$$b_m = \frac{1}{L} \int_{-L}^L f(x) \sin\left(\frac{m\pi x}{L}\right) dx. \quad (97)$$

Note that the first equation also covers the  $m = 0$  case. The  $1/L$  normalization factor can be found by considering the  $f(x) = 1$  case.

Strictly speaking, the above equations only make sense when we can define the integrals. And another question – whether the Fourier series really *converges* to the original function  $f(x)$  – is a question needing investigation. Another question is whether we can have Fourier series for discontinuous functions, or in other words, how can a stepwise function be simulated by a series of sine waves. Another problem is it seems at  $x = \pm L$ , the Fourier series have to converge to the *same* value, because every term has the same value at  $x = \pm L$ .

To lay a rigorous foundation for Fourier series, we state the following theorem: if  $f(x)$  is piecewise smooth on  $[-L, L]$ , then the Fourier series of  $f(x)$  converges to  $(f(x^+) + f(x^-))/2$  at each  $x \in (-L, L)$ , and to  $(f(-L) + f(L))/2$  at  $x = \pm L$ . Here the term **piecewise smooth** means that the function may have jumps in  $[-L, L]$ , but there may be one or more point at which  $f(x^+)$  and  $f(x^-)$  are different. At such points of confusion, we just take an average of the two options.

As an example, let's try Fourier series on

$$f(x) = \begin{cases} 1, & 0 \leq x \leq L \\ -1, & -L \leq x < 0. \end{cases} \quad (98)$$

So

$$a_0 = \frac{1}{L}(-L + L) = 0,$$

and indeed  $a_m = 0$  for all  $m$ 's: the function in question is odd, and all the  $a$  terms are even, so automatically they all vanish. For the odd terms, we have

$$\begin{aligned} b_n &= \frac{1}{L} \int_{-L}^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx \\ &= \frac{1}{L} \left( - \int_{-L}^0 \sin\left(\frac{n\pi x}{L}\right) dx + \int_0^L \sin\left(\frac{n\pi x}{L}\right) dx \right) \\ &= \frac{1}{n\pi} (2 - 2 \cos(n\pi)), \end{aligned}$$

and therefore

$$b_n = \begin{cases} \frac{4}{n\pi}, & n = 1, 3, 5, \dots, \\ 0, & \text{otherwise,} \end{cases} \quad (99)$$

and

$$f(x) = \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{1}{2n-1} \sin\left(\frac{(2n-1)\pi x}{L}\right). \quad (100)$$

## 4.2 Parseval's theorem

We have

$$\int_{-L}^L f(x)^2 dx = \frac{1}{2} a_0^2 \int_{-L}^L dx + \sum_{n=1}^{\infty} \left( a_n \int_{-L}^L \cos\left(\frac{n\pi x}{L}\right) f(x) dx + b_n \int_{-L}^L \sin\left(\frac{n\pi x}{L}\right) f(x) dx \right),$$

and the integral in each term again gives a Fourier coefficient, and in the end we get

$$\frac{1}{L} \int_{-L}^L f(x)^2 dx = \frac{1}{2} a_0^2 + \sum_{n=1}^{\infty} (a_n^2 + b_n^2). \quad (101)$$

This is called **Parseval's theorem**.

### 4.3 Complex Fourier series

We can also change the way we show the Fourier series into the follows:

$$f(x) = \sum_{n=-\infty}^{\infty} d_n e^{in\pi x/L}. \quad (102)$$

This is merely a linear combination of the original definition in terms of sin and cos. Repeating the integral over  $[-L, L]$ , we get

$$\begin{aligned} \int_{-L}^L f(x) e^{-im\pi x/L} dx &= \sum_{n=-\infty}^{\infty} d_n \int_{-L}^L e^{i(n-m)\pi x/L} dx \\ &= \sum_{n=-\infty}^{\infty} d_n \frac{2L}{(n-m)\pi} \sin(n-m)\pi. \end{aligned}$$

When  $n \neq m$ , the RHS is zero; when  $n = m$ , we need to take the  $n - m \rightarrow 0$  limit and get

$$\frac{2L}{(n-m)\pi} \sin(n-m)\pi = 2L.$$

So the conclusion is

$$d_m = \frac{1}{2L} \int_{-L}^L f(x) e^{-im\pi x/L} dx. \quad (103)$$

This version of Fourier series immediately leads to Fourier transform when  $L \rightarrow \infty$ : We define

$$\omega_n = \frac{n\pi}{L}, \quad (104)$$

and we get

$$\Delta\omega = \frac{\pi}{L}. \quad (105)$$

Therefore

$$\begin{aligned} f(x) &= \sum_{n=-\infty}^{\infty} e^{in\pi x/L} \cdot \frac{1}{2L} \int_{-L}^L f(x') e^{-in\pi x'/L} dx' \\ &= \sum_{n=-\infty}^{\infty} \frac{\Delta\omega}{2\pi} e^{i\omega_n x} \cdot \int_{-L}^L f(x') e^{-i\omega_n x'} dx' \\ &= \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{i\omega x} \cdot \int_{-\infty}^{\infty} f(x') e^{-i\omega x'} dx'. \end{aligned}$$