Homework 1

Jinyuan Wu

February 4, 2024

1 Electric and magnetic field energies of a time-harmonic mode in lossless media

In this problem, we explore the contributions of the electric (U_e) and magnetic energies (U_m) to the total electromagnetic energy (U_{em}) of a mode. There are no free charges in this system, $\mu_r = 1$, and the dielectric distribution, $\varepsilon_r(r)$, can take on an arbitrary spatial distribution; the dielectric is lossless and non-dispersive. We also assume that the electric and magnetic field profiles are squareintegrable. (In other words, the electric and magnetic fields vanish at large distances, $r \to \infty$). For an electromagnetic mode of arbitrary form, show that $U_e = U_m = \frac{1}{2}U_{em}$. Here, we define $U_e = \frac{\varepsilon_o}{4}\int \varepsilon_r(\mathbf{r})\tilde{\mathbf{E}}^* \cdot \tilde{\mathbf{E}}V$, and $U_m = \frac{\mu_o\mu_r}{4}\int \tilde{\mathbf{H}}^* \cdot \tilde{\mathbf{H}}dV$ as the time-averaged electric and magnetic energy densities.

From $\nabla \times \tilde{\boldsymbol{E}} = \mathrm{i}\omega\mu_0\mu_\mathrm{r}\tilde{\boldsymbol{H}}$ we can rewrite the magnetic energy as

$$
\begin{aligned}
U_\mathrm{m} &= \frac{\mu_0\mu_\mathrm{r}}{4}\int \mathrm{d}V\, \tilde{\boldsymbol{H}}^* \cdot \tilde{\boldsymbol{H}} \\
&= \frac{\mu_0\mu_\mathrm{r}}{4}\frac{1}{\omega^2\mu_0^2\mu_\mathrm{r}^2}\int \mathrm{d}V\, \nabla \times \tilde{\boldsymbol{E}}^* \cdot \nabla \times \tilde{\boldsymbol{E}} \\
&= \frac{\mu_0\mu_\mathrm{r}}{4}\frac{1}{\omega^2\mu_0^2\mu_\mathrm{r}^2}\int \mathrm{d}V\, (\tilde{\boldsymbol{E}}^* \cdot \nabla \times (\nabla \times \tilde{\boldsymbol{E}}) + \nabla \cdot (\tilde{\boldsymbol{E}}^* \times (\nabla \times \tilde{\boldsymbol{E}}))) \\
&= \frac{\mu_0\mu_\mathrm{r}}{4}\frac{1}{\omega^2\mu_0^2\mu_\mathrm{r}^2}\int \mathrm{d}V\, \tilde{\boldsymbol{E}}^* \cdot \nabla \times (\nabla \times \tilde{\boldsymbol{E}}).
\end{aligned}
\tag{1}
$$

From $\nabla \times \tilde{\boldsymbol{H}} = -\mathrm{i}\omega\epsilon_0\epsilon_\mathrm{r}\tilde{\boldsymbol{E}}$, we find

$$
\nabla \times (\nabla \times \tilde{\boldsymbol{E}}) = \mathrm{i}\omega\mu_0\mu_\mathrm{r}\nabla \times \tilde{\boldsymbol{H}} = \omega^2\epsilon_0\mu_0\epsilon_\mathrm{r}(\boldsymbol{r})\mu_\mathrm{r}\tilde{\boldsymbol{E}},
$$

and hence

$$
U_\mathrm{m} = \frac{\mu_0\mu_\mathrm{r}}{4}\frac{1}{\omega^2\mu_0^2\mu_\mathrm{r}^2}\int \mathrm{d}V\, \omega^2\epsilon_0\mu_0\epsilon_\mathrm{r}(\boldsymbol{r})\mu_\mathrm{r}\tilde{\boldsymbol{E}} \cdot \tilde{\boldsymbol{E}}^* = \frac{\epsilon_0}{4}\int \mathrm{d}V\, \epsilon_\mathrm{r}(\boldsymbol{r})\tilde{\boldsymbol{E}}^* \cdot \tilde{\boldsymbol{E}} = U_\mathrm{e}, \tag{2}
$$

and therefore both U_e and U_e are half of the total energy.

2 Hermitian Operators

(a) When $\boldsymbol{A}$ is a real vector, we have

$$
\int \mathrm{d}^3\boldsymbol{r}\, \boldsymbol{a}^* \cdot (\boldsymbol{A} \times \boldsymbol{b}) = \int \mathrm{d}^3\boldsymbol{r}\, (\boldsymbol{a}^* \times \boldsymbol{A}) \cdot \boldsymbol{b} = \int \mathrm{d}^3\boldsymbol{r}\, (-\boldsymbol{A} \times \boldsymbol{a})^* \cdot \boldsymbol{b}. \tag{3}
$$

Therefore the Hermitian adjoint of $\boldsymbol{A} \times (\cdots)$ is $-\boldsymbol{A} \times (\cdots)$; the operator is anti-Hermitian.
(b) When $\boldsymbol{A}, \boldsymbol{B}$ are both real vectors, we have

$$
\int \mathrm{d}^3\boldsymbol{r}\, \boldsymbol{a}^* \cdot (\boldsymbol{A} \times (\boldsymbol{B} \times \boldsymbol{b})) = \int \mathrm{d}^3\boldsymbol{r}\, ((\boldsymbol{a}^* \times \boldsymbol{A}) \times \boldsymbol{B}) \cdot \boldsymbol{b} = \int \mathrm{d}^3\boldsymbol{r}\, (\boldsymbol{B} \times (\boldsymbol{A} \times \boldsymbol{a}^*)) \cdot \boldsymbol{b}. \tag{4}
$$

So the Hermitian adjoint of $\boldsymbol{A} \times (\boldsymbol{B} \times \cdots)$ is $\boldsymbol{B} \times (\boldsymbol{A} \times \cdots)$. The operator is neither Hermitian nor anti-Hermitian.

(c) From

$$\boldsymbol{\nabla} \cdot (\boldsymbol{a}^* \times \boldsymbol{b}) = (\boldsymbol{\nabla} \times \boldsymbol{a}^*) \cdot \boldsymbol{b} - \boldsymbol{a}^* \cdot \boldsymbol{\nabla} \times \boldsymbol{b},$$

and the assumption that the integral of LHS over the whole space vanishes, we have

$$\int \mathrm{d}^3\boldsymbol{r}\, \boldsymbol{a}^* \cdot \boldsymbol{\nabla} \times \boldsymbol{b} = \int \mathrm{d}^3\boldsymbol{r}\, (\boldsymbol{\nabla} \times \boldsymbol{a})^* \cdot \boldsymbol{b}, \tag{5}$$

and therefore the Hermitian adjoint of $\boldsymbol{\nabla} \times \cdots$ is itself and the operator is Hermitian.
(d) Since $\boldsymbol{\nabla} \times \cdots$ is Hermitian, so is $\boldsymbol{\nabla} \times \boldsymbol{\nabla} \times \cdots$, and its Hermitian adjoint is again itself.
(e) Using the above facts it's easy to show that

$$\int \mathrm{d}^3\boldsymbol{r}\, \boldsymbol{a}^* \cdot \boldsymbol{\nabla} \times (f \boldsymbol{\nabla} \times \boldsymbol{b}) = \int \mathrm{d}^3\boldsymbol{r}\, (\boldsymbol{\nabla} \times f^* \boldsymbol{\nabla} \times \boldsymbol{a})^* \cdot \boldsymbol{b}. \tag{6}$$

Therefore, the operator $\boldsymbol{\nabla} \times f \boldsymbol{\nabla} \times \cdots$ is Hermitian, if and only if f is a real function.

3 Hermitian eigenvalue problems

3.1 Electromagnetic Modes in Vacuum

In class, we saw that the magnetic vector potential can be used to express the electromagnetic field as a Hermitian eigenvalue problem. We also outlined the steps by which the mode energy can be cast in the form of a simple harmonic oscillator. In this problem, we use operator methods to derive these results in a very slick way. We assume that each mode ($\mathbf{A}_i(r,t) = q_i(t)\mathbf{A}_i^o(r)$) is an eigenfunction of the Hermitian eigenvalue equation $\hat{O}\mathbf{A}_i^o(x) = (\omega_i/c)^2\, \mathbf{A}_i^o(x)$, with a time-dependent amplitude, q_i, that obeys the relation $\ddot{q}_i = -\omega_i^2 q_i$, and $\hat{O}(\ldots) = \nabla \times \nabla \times (\ldots)$.
(a) *Let's begin by considering the energy of an individual time-harmonic mode, $\mathbf{A}_i(r,t) = q_i(t)\mathbf{A}_i^o(r)$. From Eqs. 17-18 of Lecture 3 Note N1, notice that the electric field energy can be written as $U_{e,i} = \frac{1}{2}\varepsilon_o \dot{q}_i^2 \left(\mathbf{A}_i^o \mid \mathbf{A}_i^o\right)$, and $U_{m,i} = \frac{1}{2}\mu_o^{-1} q_i^2 \left(\nabla \times \mathbf{A}_i^o \mid \nabla \times \mathbf{A}_i^o\right)$. Using operator properties and the eigenvalue equation above, show that the total energy of the mode can be expressed as $U_{em,i} = \frac{1}{2}\left[\dot{q}_i^2 + \omega_i^2 q_i^2\right]$. How must $\mathbf{A}_i^o(x)$ be normalized to express the energy in this way? [Remember, it is our convention to choose a normalization such that the mass m of our simple harmonic oscillator takes the value $m = 1$.]*

What we want is to see

$$\epsilon_0 \langle \boldsymbol{A}_i^0 | \boldsymbol{A}_i^0 \rangle = 1, \quad \frac{1}{\mu_0} \langle \boldsymbol{\nabla} \times \boldsymbol{A}_i^0 | \boldsymbol{\nabla} \times \boldsymbol{A}_i^0 \rangle = \omega_i^2. \tag{7}$$

The second condition is actually equivalent to the first condition. Since we assume no spatial inhomogeneity of ϵ, the modes follow the Helmholtz equation

$$\left(\nabla^2 + \frac{\omega_i^2}{c^2}\right) \boldsymbol{A}_i^0 = \left(\nabla^2 + \epsilon_0\mu_0\omega_i^2\right) \boldsymbol{A}_i^0 = 0, \tag{8}$$

and therefore

$$\begin{aligned} \frac{1}{\mu_0} \langle \boldsymbol{\nabla} \times \boldsymbol{A}_i^0 | \boldsymbol{\nabla} \times \boldsymbol{A}_i^0 \rangle &= \frac{1}{\mu_0} \int \mathrm{d}^3\boldsymbol{r}\, (\boldsymbol{\nabla} \times \boldsymbol{\nabla} \times \boldsymbol{A}_i^{0*}) \cdot \boldsymbol{A}_i^0 \\ &= \frac{1}{\mu_0} (-\nabla^2 \boldsymbol{A}_i^{0*}) \cdot \boldsymbol{A}_i^0 \\ &= \epsilon_0 \omega_i^2 \int \mathrm{d}^3\boldsymbol{r}\, \boldsymbol{A}_i^{0*} \cdot \boldsymbol{A}_i^0, \end{aligned} \tag{9}$$

where at the second line we use the Hermitian property of $\boldsymbol{\nabla} \times \cdots$, the second line comes from the Coulomb gauge $\boldsymbol{\nabla} \cdot \boldsymbol{A} = 0$, and therefore as long as

$$\epsilon_0 \langle \boldsymbol{A}_i^0 | \boldsymbol{A}_i^0 \rangle = 1, \tag{10}$$

which is just a normalization condition, we can write the energy as

$$H_i = U_{\mathrm{em},i} = \frac{1}{2}(\dot{q}_i^2 + \omega_i^2 q_i^2). \tag{11}$$

(b) *Given an individual mode with the form described in part (a), show that the time average of the electric field energy $\langle U_{e,i}\rangle$ is equal to the time average of the magnetic field energy $\langle U_{m,i}\rangle$. In the case of time-harmonic systems, we define the time average $\langle\ldots\rangle$ as $\frac{1}{T}\int_t^{t+T}(\ldots)dt'$, where $\omega = 2\pi/T$. [Hint: Use integration by parts to make the time averages of the electric and magnetic field energies look identical. You do not need to evaluate the integral.]*

Suppose $q_i = A_i\cos(\omega_i t)$, where by shifting the definition of the $t = 0$ point we eliminate the phase. This means $\dot{q}_i = -\omega_i A_i \sin(\omega_i t)$. The time average of the magnetic energy is

$$\begin{aligned}
\frac{1}{2}\omega_i^2\left\langle q_i^2\right\rangle &= \frac{1}{2}\omega_i^2\cdot\frac{1}{T}\int_0^T \mathrm{d}t\cos^2\omega_i t \\
&= \frac{1}{2}\omega_i^2\cdot\frac{1}{T}\int_0^T\frac{1}{\omega_i}\cos\omega_i t\,\mathrm{d}\sin\omega_i t \\
&= \frac{1}{2}\omega_i^2\cdot\frac{1}{T}\frac{1}{\omega_i}\left(\cos\omega_i t\sin\omega_i t\Big|_0^T - \int_0^T\frac{\mathrm{d}\cos\omega_i t}{\mathrm{d}t}\sin\omega_i t\,\mathrm{d}t\right) \\
&= \frac{1}{2}\omega_i^2\cdot\frac{1}{T}\int_0^T\sin^2\omega_i t\,\mathrm{d}t = \frac{1}{2}\left\langle\dot{q}_i\right\rangle.
\end{aligned} \tag{12}$$

(c) *With the definitions $\mathbf{E}(r,t) = p_i\mathbf{E}_i^o(r) = -p_i\mathbf{A}_i^o(r)$, and $\mathbf{B}(r,t) = q_i\mathbf{B}_i^o(r) = q_i\nabla\times\mathbf{A}_i^o(r)$, use Maxwell's equations find two first order equations for the evolution of p_i and q_i. [Hint: You should find coupled first order equations that are identical to those produced by Hamilton's equations for our simple harmonic oscillator.]*

From $\boldsymbol{\nabla}\times\boldsymbol{E} = -\,\partial\boldsymbol{B}/\partial t$ we get

$$\dot{q}_i = p_i, \tag{13}$$

and from $\boldsymbol{\nabla}\times\boldsymbol{B} = \frac{1}{c^2}\frac{\partial\boldsymbol{E}}{\partial t}$ we get

$$q_i\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\boldsymbol{A}_i^0) = \frac{1}{c^2}(-\dot{p}_i)\boldsymbol{A}_i^0,$$

and from the aforementioned Helmholtz equation we have

$$\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\boldsymbol{A}_i^0) = -\nabla^2\boldsymbol{A}_i^0 = \frac{\omega_i^2}{c^2}\boldsymbol{A}_i^0,$$

and therefore

$$\dot{p}_i = -\omega_i^2 q_i. \tag{14}$$

3.2 Electromagnetic Modes in Dielectric

Next let's consider the modes within a dielectric medium with a real dielectric constant, $\varepsilon_r(r)$, that can vary in space. We assume that each mode ($\mathbf{H}_i(r,t) = q_i(t)\mathbf{H}_i^o(r)$) is an eigenfunction of the Hermitian eigenvalue equation $\hat{O}\mathbf{H}_i^o(r) = (\omega_i/c)^2\,\mathbf{H}_i^o(r)$, with a time-dependent amplitude, q_i, that obeys the relation $\ddot{q}_i = -\omega_i^2 q_i$, and $\hat{O}(\ldots) = \nabla\times\left(\varepsilon_r^{-1}\nabla\times\right)(\ldots)$. Notice we switched from magnetic flux density B to magnetic field strength $\mathbf{H}$. You can directly substitute $\mathbf{B} = \mu_0\mathbf{H}$ where needed, because we usually assume $\mu = \mu_0$ this class.

(a) *Let's begin by considering the energy of an individual time-harmonic mode, with electric and magnetic fields of the form $\mathbf{E}_l(r,t) = \dot{q}_l(t)\mathbf{E}_l^o(r)$ and $\mathbf{H}_l(r,t) = q_l(t)\mathbf{H}_l^o(r)$ respectively. Notice that the electric field energy can be written as $U_{e,i} = \frac{1}{2}\varepsilon_o\dot{q}_i^2\left(\mathbf{E}_i^o\left|\varepsilon_r\right|\mathbf{E}_i^o\right)$, and $U_{m,i} = \frac{1}{2}\mu_o q_i^2\left(\mathbf{H}_i^o\mid\mathbf{H}_i^o\right)$. Using operator properties and the eigenvalue equation above, show that the total energy of the mode can be expressed as $U_{em,i} = \frac{1}{2}\left[\dot{q}_i^2 + \omega_i^2 q_i^2\right]$. How must $\mathbf{E}_i^o(x)$ and $\mathbf{H}_i^o(x)$ be normalized to express the energy in this way? Use Maxwell's equations to show that the normalization conditions for $\mathbf{E}_i^o(x)$ and $\mathbf{H}_i^o(x)$ are equivalent.*

Since $\boldsymbol{H}_i^0 = \tilde{\boldsymbol{H}}/q_i$, the results in Problem 1 can't be immediately used here. From Maxwell's equations we have

$$\boldsymbol{\nabla}\times\boldsymbol{E} = -\,\partial\boldsymbol{B}/\partial t \Rightarrow \boldsymbol{\nabla}\times\boldsymbol{E}_i^0 = -\mu_0\boldsymbol{H}_i^0,$$

and similarly

$$\boldsymbol{\nabla}\times\boldsymbol{H} = \partial\boldsymbol{D}/\partial t \Rightarrow q_i\boldsymbol{\nabla}\times\boldsymbol{H}_i^0 = \epsilon_0\epsilon_{\mathrm{r}}\ddot{q}_i\boldsymbol{E}_i^0 = -\omega_i^2\epsilon_{\mathrm{r}}\epsilon_0 q_i\boldsymbol{E}_i^0,$$

and therefore

$$\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \boldsymbol{E}_i^0) = \epsilon_\mathrm{r} \frac{\omega_i^2}{c^2} \boldsymbol{E}_i^0.$$

Therefore we have

$$\begin{aligned} \mu_0 \left\langle \boldsymbol{H}_i^0 \middle| \boldsymbol{H}_i^0 \right\rangle &= \frac{1}{\mu_0} \int \mathrm{d}^3\boldsymbol{r}\, (\boldsymbol{\nabla} \times \boldsymbol{E}_i^{0*}) \cdot (\boldsymbol{\nabla} \times \boldsymbol{E}_i^0) \\ &= \frac{1}{\mu_0} \int \mathrm{d}^3\boldsymbol{r}\, \boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \boldsymbol{E}_i^{0*}) \cdot \boldsymbol{E}_i^0 \\ &= \frac{\omega_i^2}{\mu_0 c^2} \int \mathrm{d}^3\boldsymbol{r}\, \epsilon_\mathrm{r} \boldsymbol{E}_i^{0*} \cdot \boldsymbol{E}_i^0 \\ &= \omega_i^2 \epsilon_0 \left\langle \boldsymbol{E}_i^0 \middle| \epsilon_\mathrm{r} \middle| \boldsymbol{E}_i^0 \right\rangle. \end{aligned} \tag{15}$$

In order to get a harmonic oscillator form of the total energy, we want to have

$$\epsilon_0 \left\langle \boldsymbol{E}_i^0 \middle| \epsilon_\mathrm{r} \middle| \boldsymbol{E}_i^0 \right\rangle = 1, \quad \mu_0 \left\langle \boldsymbol{H}_i^0 \middle| \boldsymbol{H}_i^0 \right\rangle = \omega_i^2, \tag{16}$$

which, from (15), is equivalent to the normalization condition

$$\epsilon_0 \left\langle \boldsymbol{E}_i^0 \middle| \epsilon_\mathrm{r} \middle| \boldsymbol{E}_i^0 \right\rangle = 1. \tag{17}$$

(b) *Next, we consider superposition of such modes. Use orthogonality to show that the electromagnetic energy stored in a superposition of modes,* $\mathbf{E}(r,t) = \sum_l \dot{q}_l(t) \mathbf{E}_l^o(r), \mathbf{H}(r,t) = \sum_l q_l(t) \mathbf{H}_l^o(r)$*, can be expressed as* $U_{em}^{tot} = \sum_l \frac{1}{2} \left[\dot{q}_l^2 + \omega_l^2 q_l^2 \right]$*.*

The eigenvalue problem about $\boldsymbol{H}$ gives us the orthogonality condition

$$\int \mathrm{d}^3\boldsymbol{r}\, \boldsymbol{H}_i^{0*} \cdot \boldsymbol{H}_j^0 = \frac{\omega_i^2}{\mu_0} \delta_{ij}, \tag{18}$$

and hence the magnetic part of the energy is

$$\frac{1}{2}\mu_0 \int \mathrm{d}^3\boldsymbol{r} \sum_{i,j} q_i \boldsymbol{H}_i^{0*} q_j \boldsymbol{H}_j^0 = \frac{1}{2} \sum_{i,j} \delta_{ij} q_i q_j \omega_i^2 = \frac{1}{2} \sum_i \omega_i^2 q_i^2. \tag{19}$$

The generalized eigenvalue problem about $\boldsymbol{E}$ is

$$\boldsymbol{\nabla} \times \boldsymbol{\nabla} \times \boldsymbol{E}_i^0 = \frac{\omega_i^2}{c^2} \epsilon_\mathrm{r} \boldsymbol{E}_i^0, \tag{20}$$

and therefore the orthogonality condition is

$$\int \mathrm{d}^3\boldsymbol{r}\, \boldsymbol{E}_i^{0*} \epsilon_\mathrm{r} \boldsymbol{E}_j^0 = \frac{1}{\epsilon_0} \delta_{ij}, \tag{21}$$

and therefore

$$\frac{1}{2}\epsilon_0 \int \mathrm{d}^3\boldsymbol{r}\, \epsilon_\mathrm{r} \boldsymbol{E}^2 = \frac{1}{2} \sum_{i,j} \dot{q}_i \dot{q}_j \epsilon_0 \left\langle \boldsymbol{E}_i^0 \middle| \epsilon_\mathrm{r} \middle| \boldsymbol{E}_j^0 \right\rangle = \frac{1}{2} \sum_i \dot{q}_i^2. \tag{22}$$

Therefore

$$U_\mathrm{em} = \frac{1}{2} \sum_i (\dot{q}_i^2 + \omega_i^2 q_i^2). \tag{23}$$

Complex Coordinates: Building on the insights from Problem 3.1, one can define real valued fields as $\mathbf{E}(r,t) = p_i \mathbf{E}_i^o(r)$*, and* $\mathbf{H}(r,t) = q_i \mathbf{H}_i^o(r)$*. However, complex coordinates (and complex fields) provide a much more practical description of traveling waves. Using complex coordinates* $a_i = \alpha q_i + i\beta p_i$ *and* $a_i^* = \alpha q_i - i\beta p_i$ *to reduce our simple harmonic oscillator to* $H_i\left(p_i\left(a_{\tilde{i}}, a_i^*\right), q_i\left(a_i, a_i^*\right)\right) = \omega_i a_i^* a_i$*, we can define our complex fields as,* $\tilde{\mathbf{E}}(r,t) = i a_i \tilde{\mathbf{E}}_i^o(r)$*, and* $\tilde{\mathbf{B}}(r,t) = a_i \tilde{\mathbf{B}}_i^o(r)$*.*

(c) *(c) Following steps similar to Part (a), show that the total electromagnetic energy (or field energy) of an individual mode reduces to $\omega_i a_i^* a_i$. Remember that we must use real-valued fields of the form $\mathbf{E}(r,t) = \left(i a_i \tilde{\mathbf{E}}_i^o - i a_i^* \tilde{\mathbf{E}}_i^{o*}\right)$, and $\mathbf{B}(r,t) = \left(a_i \tilde{\mathbf{B}}_i^o + a_i^* \tilde{\mathbf{B}}_i^{o*}\right)$ to evaluate the energy density. [Hint: do not use time-averaging; most of the terms will cancel!] What field normalization is necessary to express the field energy in this way?*

The electromagnetic energy density of a single mode is

$$\begin{aligned} u_{\mathrm{em},i} &= \frac{1}{2}\epsilon_0\epsilon_\mathrm{r}\boldsymbol{E}^2 + \frac{1}{2\mu_0}\boldsymbol{B}^2 \\ &= \left(\epsilon_0\epsilon_\mathrm{r}\left|\tilde{\boldsymbol{E}}_i^0\right|^2 + \frac{1}{\mu_0}\left|\tilde{\boldsymbol{B}}_i^0\right|^2\right) a_i^* a_i + \frac{1}{2}\left(\frac{1}{\mu_0}(\tilde{\boldsymbol{B}}_i^0)^2 - \epsilon_0\epsilon_\mathrm{r}(\tilde{\boldsymbol{E}}_i^0)^2\right) a_i^2 + \text{c.c.}, \end{aligned} \tag{24}$$

and the Maxwell's equation dictates the following relation between $\tilde{\boldsymbol{E}}_i^0$ and $\tilde{\boldsymbol{B}}_i^0$:

$$\boldsymbol{\nabla}\times\boldsymbol{E} = -\,\partial\boldsymbol{B}/\partial t \Rightarrow \boldsymbol{\nabla}\times\boldsymbol{E}_i^0 = \omega_i\boldsymbol{B}_i^0, \tag{25}$$

and

$$\boldsymbol{\nabla}\times\boldsymbol{B} = \mu_0\epsilon_0\epsilon_\mathrm{r}\,\partial\boldsymbol{E}/\partial t \Rightarrow \boldsymbol{\nabla}\times\boldsymbol{B}_i^0 = \mu_0\epsilon_0\epsilon_\mathrm{r}\omega_i\boldsymbol{E}_i^0. \tag{26}$$

By exactly the same procedure applied in the previous problems, we have

$$\frac{1}{\mu_0}\int \mathrm{d}^3\boldsymbol{r}\,\boldsymbol{B}_i^0\cdot\boldsymbol{B}_i^0 = \epsilon_0\int \mathrm{d}^3\boldsymbol{r}\,\boldsymbol{E}_i^0\cdot\epsilon_\mathrm{r}\boldsymbol{E}_i^0, \tag{27}$$

and

$$\frac{1}{\mu_0}\int \mathrm{d}^3\boldsymbol{r}\,\boldsymbol{B}_i^{0*}\cdot\boldsymbol{B}_i^0 = \epsilon_0\int \mathrm{d}^3\boldsymbol{r}\,\boldsymbol{E}_i^{0*}\cdot\epsilon_\mathrm{r}\boldsymbol{E}_i^0. \tag{28}$$

The first equation means the a^2 and $(a^*)^2$ terms in u_em vanish after we integrate over the whole space; the second equation means that

$$U_{\mathrm{em},i} = \int \mathrm{d}^3\boldsymbol{r}\,u_{\mathrm{em},i} = 2a_i^* a_i\int \mathrm{d}^3\boldsymbol{r}\,\epsilon_0\epsilon_\mathrm{r}|\tilde{\boldsymbol{E}}_i^0|^2. \tag{29}$$

What we want is

$$U_{\mathrm{em},i} = \omega_i a_i^* a_i, \tag{30}$$

and that means the normalization condition has to be

$$2\int \mathrm{d}^3\boldsymbol{r}\,\epsilon_0\epsilon_\mathrm{r}|\tilde{\boldsymbol{E}}_i^0|^2 = \frac{2}{\mu_0}\int \mathrm{d}^3\boldsymbol{r}\,|\tilde{\boldsymbol{B}}_i^0|^2 = \omega_i. \tag{31}$$

4 Hamilton's Equation in Complex Coordinates

As discussed in Lectures 3-4, complex coordinates provide a natural way to describe oscillatory systems. We defined our complex coordinates as $a = \alpha q + i\beta p$ and $a^* = \alpha q - i\beta p$, in close analogy with the quantum mechanical raising/lower operators; here, α and β are real-valued coefficients. Since, we can use the relations $q = (a + a^*)/2\alpha$ and $p = (a - a^*)/2i\beta$ to express any Hamiltonian $H(p,q)$ as $H(p(a,a^*), q(a,a^*))$, our new Hamiltonian can always be expressed as $H(a,a^*)$. Notice that a and a^* are independent coordinates that replace q and p. Our next task is to find a new version of Hamilton's equations in complex coordinates.