Floquet theory

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November 20, 2023

1 The Floquet formalism: quasienergies and quasi-stationary states

In this section we outline the basic formalism of Floquet physics, following the notation in [1]. As is mentioned in the introduction, Floquet effects happen with a time-periodic Hamiltonian; below we let $T = 2\pi/\omega$ be the period. Such a Hamiltonian is usually an effective Hamiltonian when the system (hereafter "matter") is coupled with another degree of freedom which does not change much in the time evolution; the latter is hereafter called "light", since in condensed matter systems, periodic driving is usually achieved by shedding a beam of light to the matter: consider the general form of light-matter interaction Hamiltonian with only one active photon mode

$$H_{\text{full}} = H \otimes 1_{\text{light}} + 1_{\text{matter}} \otimes \hbar \left(b^\dagger b + \frac{1}{2} \right) + \underbrace{bV + b^\dagger V^\dagger}_{H_{\text{light-matter coupling}}}, \tag{1}$$

and we assume that the state of the electromagnetic part is close to a coherent state $|\alpha e^{-i\omega t}\rangle$ with strong intensity that almost has zero time evolution. Under this assumption, we can project out the electromagnetic degree of freedom by

$$P = \sum_i |i\rangle \langle i| \otimes \langle \alpha e^{-i\omega t}|, \tag{2}$$

where i labels eigenstates of the matter degrees of freedom, and this means the Hamiltonian for the matter part is

$$H_{\text{eff}} = P H_{\text{full}} P = H + \underbrace{\hbar|\alpha|^2}_{\text{const.}} + \alpha V e^{-i\omega t} + \alpha^* V^\dagger e^{i\omega t}, \tag{3}$$

which indeed evolves with period $2\pi/\omega$.

From the Floquet theory of differential equation, we know it is possible to expand an arbitrary state that evolves according to H into a linear combination (the coefficients are constants) of $\{ |\psi_n(t)\rangle \}$ where

$$|\psi_n(t+T)\rangle = e^{-i\varepsilon_n t/\hbar} |\Phi_n(t)\rangle, \quad |\Phi_n(t+T)\rangle = |\Phi_n(t)\rangle. \tag{4}$$

By discrete periodicity of $|\Phi_n(t)\rangle$ we make Fourier expansion

$$|\Phi_n(t)\rangle = \sum_m e^{-im\omega t} |\phi_n^{(m)}\rangle, \tag{5}$$

where m goes over all integers. Note that here $|\phi_n^{(m)}\rangle$ are *Fourier coefficients* and are not eigenstates of anything; there is no normalization or orthogonality condition. Using i to label the eigenstates of the matter, we have

$$|\Phi_n(t)\rangle = \sum_i \sum_m e^{-im\omega t} \langle i|\phi_n^{(m)}\rangle \, |i\rangle. \tag{6}$$

The coefficients before $|i\rangle$, not coefficients before $|\phi_n^{(m)}\rangle$ in (5), give the expansion of $|\Phi\rangle$ in a complete, orthogonal basis. The significance of $|\phi\rangle$ vectors can be seen immediately below.

The Schrodinger equation

$$\frac{d}{dt} |\psi_n(t)\rangle = H |\psi_n(t)\rangle \tag{7}$$

now reads

$$(\varepsilon_n + m\hbar\omega) |\phi_n^{(m)}\rangle = \sum_{m'} H^{(m-m')} |\phi_n^{(m')}\rangle , \tag{8}$$

where

$$H(t) = \sum_m \mathrm{e}^{-\mathrm{i}m\omega t} H^{(m)}. \tag{9}$$

Thus we find

$$\varepsilon_n |\phi_n^{(m)}\rangle = \sum_{m'} (H^{(m-m')} - m\hbar\omega\delta_{mm'}) |\phi_n^{(m')}\rangle . \tag{10}$$

Recall that $\langle i|\phi_n^{(m)}\rangle$ is the $m\omega$-frequency component of $|\Phi_n(t)\rangle$ projected on the basis vector $|i\rangle$. The eigenvalues ε_n are known as the *Floquet quasienergy* of the *Floquet quasi-stationary state (or quasi-eigenstate)* $|\Psi_n(t)\rangle$, which can be obtained by diagonalizing

$$H_{\mathrm{Floquet},mm'} = H^{(m-m')} - m\hbar\omega\delta_{mm'}. \tag{11}$$

(11) looks like a light-matter interaction Hamiltonian written in operator form for the matter part and in the Fock basis for the light part (labeled by m and m'). This appearance however is misleading: unlike conventional effective Hamiltonians whose eigenstates can in principle be obtained by applying a projection operator on a subset of eigenstates of the full Hamiltonian, eigenstates of (11) *do not* correspond to any eigenstate of the full Hamiltonian e.g. (1): the light part is in a coherent state, which is far from any eigenstate of the linear electromagnetic Hamiltonian $\hbar\omega(n+\frac{1}{2})$. Instead, Floquet formalisms is to be understood in a more generic framework of non-equilibrium physics: Floquet Green function can be calculated within the Keldysh formalism, and (11) can be understood as the non-equilibrium self-energy [2, 3], with m and m' being labels of external field lines, which is not necessarily an equilibrium effective Hamiltonian.

Moreover, the difference between the Floquet effective Hamiltonian (11) and conventional, "equilibrium" effective Hamiltonians can be observed by the structure of its eigenstates. The dimension of (11), fully expanded into its matrix elements, is the number of the values of m considered times the dimension of the matter Hilbert space, and thus (11)'s eigenstates are overcomplete. We can actually point out where overcompletion appears: note that if ε_n satisfies (4), then so does $\varepsilon_n + m\hbar\omega$. We call (11) the effective Hamiltonian in the **extended Hilbert space**.

In conclusion, a Floquet system has a set of quasi-eigenstates $\{\,|\psi_n\rangle\}$, the number of which is the same as the dimension of the Hilbert space; but for each quasi-eigenstate, we have countable infinite quasi-energies, the difference between the nearest two being $\hbar\omega$; thus all distinct Floquet quasi-eigenstates can be indexed by quasi-energies that are within one "Floquet-Brillouin zone".

Orthogonal relation between $|\psi_n\rangle$'s?

2 Self-driven Floquet effects

it is however possible to use light to stimulate some long-lived degrees of freedom in a solid and let it drive the rest of the system, which sometimes is known as "self-driving".

References

[1] Mark S Rudner and Netanel H Lindner. "The Floquet Engineer's Handbook". In: *arXiv preprint arXiv:2003.08252* (2020).

[2] Andreas Lubatsch and Regine Frank. "Evolution of Floquet topological quantum states in driven semiconductors". In: *The European Physical Journal B* 92 (2019), pp. 1–9.

[3] Hideo Aoki et al. "Nonequilibrium dynamical mean-field theory and its applications". In: *Reviews of Modern Physics* 86.2 (2014), p. 779.