# Topics in optics and quantum electronics

Jinyuan Wu

January 29, 2024

## 1 Mode description of electromagnetic fields

In this section we mainly focus on the mode description of classical electrodynamics. Modes are seen in resonators, waveguides, photonic crystals, and more. Loss can then be included perturbatively and we get leaky modes.

In the case of electrodynamics in vacuum, in the Coulomb gauge (i.e. $\varphi = 0$), we have

$$\boldsymbol{\nabla} \times \boldsymbol{\nabla} \times \boldsymbol{A} = \left(\frac{\omega}{c}\right)^2 \boldsymbol{A}, \tag{1}$$

and under the inner product definition

$$\langle \boldsymbol{A} | \boldsymbol{B} \rangle \coloneqq \int \mathrm{d}\boldsymbol{r}\, \boldsymbol{A}^* \cdot \boldsymbol{B}, \tag{2}$$

the LHS of the equation on $\boldsymbol{A}$ is Hermitian, guaranteeing that $\omega$ is real.

From the well-known relation $[a, a^\dagger] = 1$, it follows that the classical counterparts of creation and annihilation operators are complex variables which satisfies the relation

$$\{a, a^*\} = -\mathrm{i}. \tag{3}$$

This means if the Poisson brackets are to be defined in terms of $a$ and $a^*$, it should be defined as

$$\{A, B\} = \frac{1}{\mathrm{i}} \left( \frac{\partial A}{\partial a} \frac{\partial B}{\partial a^*} - \frac{\partial A}{\partial a^*} \frac{\partial B}{\partial a} \right), \tag{4}$$

where $a$ and $a^*$ are to be regarded as independent variables. The general relation between $a, a^*$ and $x, p$ is

$$a = \alpha q + \mathrm{i}\beta p, \quad q = \frac{a + a^*}{2\alpha}, \quad p = \frac{a - a^*}{2\mathrm{i}\beta}, \quad 2\alpha\beta = 1. \tag{5}$$

If $q$ and $p$ follow the canonical commutation relation, then $a$ and $a^*$ follow the commutation relation between a creation operator and a annihilation operator, and vice versa.

We can also justify (3) by Hamiltonian's equations. From (5) and the cian rule, we find

$$\begin{aligned} \dot{a} &= \frac{\partial a}{\partial q} \dot{q} + \frac{\partial a}{\partial p} \dot{p} = \frac{\partial a}{\partial q} \frac{\partial H}{\partial p} - \frac{\partial a}{\partial p} \frac{\partial H}{\partial q} \\ &= \frac{\partial a}{\partial q} \left( \frac{\partial H}{\partial a} \frac{\partial a}{\partial p} + \frac{\partial H}{\partial a^*} \frac{\partial a^*}{\partial p} \right) - \frac{\partial a}{\partial p} \left( \frac{\partial H}{\partial a} \frac{\partial a}{\partial q} + \frac{\partial H}{\partial a^*} \frac{\partial a^*}{\partial q} \right) \\ &= -2\mathrm{i}\alpha\beta \frac{\partial H}{\partial a^*}, \end{aligned} \tag{6}$$

and similarly

$$\dot{a}^* = 2\mathrm{i}\alpha\beta \frac{\partial H}{\partial a}. \tag{7}$$

Now that means we have to define the Poisson bracket as (4) to keep the form

$$\frac{\mathrm{d}A}{\mathrm{d}t} = \{A, H\}. \tag{8}$$

The $2\alpha\beta = 1$ constraint can be explained by conservation of area in phase space: roughly speaking the area inside the trajectory of $a(t)$ or $(q(t), p(t))$ is the "strength of oscillation", like the number of photons.

Now going back to the problem of electromagnetic fields. Consider the one dimensional periodic boundary condition problem. We have (in phaser form)

$$\tilde{E}_\pm(x, t) = \tilde{E}^0_\pm \mathrm{e}^{\mathrm{i}kx - \mathrm{i}\omega t}, \tag{9}$$

and we consider $A$ to be the coordinate and hence $-E = \partial A / \partial t$ is the momentum. To decompose $A$ into modes, we write $A$ as static, spatially varying factors weighted by time-dependent real values, and thus $A = q(t) A_0 \mathrm{e}^{\mathrm{i}kz}$,

## 2 Ring resonators

Stationary electrodynamics in terms of $\boldsymbol{E}$ is not a Hermitian problem, but a generalized Hermitian one: the problem is

$$\boldsymbol{\nabla} \times \boldsymbol{\nabla} \times \boldsymbol{E} = \left(\frac{\omega}{c}\right)^2 \epsilon_{\mathrm{r}}(\boldsymbol{r})\boldsymbol{E}(\boldsymbol{r}), \tag{10}$$

and although the operators on both LHS and RHS are Hermitian, the LHS multiplied with the inverse of the RHS is not. The time-independent perturbation theory for this problem is therefore slightly more complicated, and the "energy" being solved is

$$\alpha := \frac{\omega^2}{c^2}, \tag{11}$$

instead of the frequency.

$$\Delta\omega_n^{(1)} = -\frac{\omega_n^{(0)}}{2} TODO \tag{12}$$

The minus sign can be understood using the following line of argumentation: if a piece of dielectric is inserted into an isolated system, the total energy is conserved but some energy is stored within the internal degree of freedom dielectric, so $\omega(a^\dagger a + 1/2)$, the vacuum part of the Hamiltonian has to decrease. One important caveat is that the leading order of energy correction *can't* be evaluated by $\langle \boldsymbol{E}_n | \Delta\epsilon_{\mathrm{r}} | \boldsymbol{E}_n \rangle$: the absolute value of this term is right, but the sign is wrong. TODO: why?