Homework 4

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Exercise 3 in lecture 12
Solution The normalization factor in the time domain is

$$\int_{-\infty}^{\infty} |g(t)|^2 \, \mathrm{d}t = \sigma\sqrt{\pi}. \tag{1}$$

The expectation values are

$$\langle t^2 \rangle = \frac{\int_{-\infty}^{\infty} \mathrm{d}t \, |g(t)|^2 t^2}{\int_{-\infty}^{\infty} \mathrm{d}t \, |g(t)|^2} = t_0^2 + \frac{1}{2}\sigma^2, \tag{2}$$

and

$$\langle t \rangle = \frac{\int_{-\infty}^{\infty} \mathrm{d}t \, |g(t)|^2 t}{\int_{-\infty}^{\infty} \mathrm{d}t \, |g(t)|^2} = t_0. \tag{3}$$

The frequency domain version of g is

$$g[\omega] = \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}\omega t} g(t) = \sigma\sqrt{2\pi}\mathrm{e}^{\mathrm{i}(\omega-\omega_0)t - \frac{1}{2}\sigma^2(\omega-\omega_0)^2}. \tag{4}$$

The normalization factor is

$$\int_{-\infty}^{\infty} |g[\omega]|^2 \, \mathrm{d}\omega = 2\pi\sigma^2 \cdot \sqrt{\frac{\pi}{\sigma^2}} = 2\pi^{3/2}\sigma. \tag{5}$$

The expectation values are

$$\langle \omega^2 \rangle = \frac{\int_{-\infty}^{\infty} \mathrm{d}\omega \, |g[\omega]|^2 \omega^2}{\int_{-\infty}^{\infty} \mathrm{d}\omega \, |g[\omega]|^2} = \frac{1}{2\sigma^2} + \omega_0^2, \tag{6}$$

and

$$\langle \omega \rangle = \omega_0. \tag{7}$$

So

$$\begin{aligned} \sigma_t \sigma_\omega &= \sqrt{\langle t^2 \rangle - \langle t \rangle^2}\sqrt{\langle \omega^2 \rangle - \langle \omega \rangle^2} \\ &= \end{aligned} \tag{8}$$

Exercise 1 in lecture 13 We have the following transmission model: the sent signal X has 3 possible values $x = 0, \pm 1$ with $p_0 = p_1 = p_{-1} = 1/3$. The noise Ξ has two possible values $\zeta = \pm 1$. We suppose the channel function F has the following expression $Y = \mathrm{Im}\left[\mathrm{e}^{i2\pi(X+\Xi)/}\right]$. What are all the possible values of Y ? Give the expression of the distributions $w(x), w(y), w(x, y)$ and $w(y \mid x)$.
Solution The possible results are listed in Table 1. The possible values of Y are 0 and $\pm\sqrt{3}/2$. From Table 1 we have

$$w(x) = \begin{cases} \frac{1}{3}, & x = 0, \\ \frac{1}{3}, & x = 1, \\ \frac{1}{3}, & x = -1, \end{cases} \tag{9}$$

$$w(y) = \begin{cases} \frac{1}{3}, & y = 0, \\ \frac{1}{3}, & y = \sqrt{3}/2, \\ \frac{1}{3}, & y = -\sqrt{3}/2, \end{cases} \tag{10}$$

Table 1: Probabilistic distribution of Y; the probability of each row is 1/6.

X	Ξ	$Y = \operatorname{Im} \mathrm{e}^{\mathrm{i}2\pi(X+\Xi)/3}$
0	1	$\frac{\sqrt{3}}{2}$
	-1	$-\frac{\sqrt{3}}{2}$
1	1	$-\frac{\sqrt{3}}{2}$
	-1	0
-1	1	0
	-1	$\frac{\sqrt{3}}{2}$

and

$$w(x,y) = \begin{cases} \frac{1}{6}, & (x,y) = (0, \sqrt{3}/2), \\ \frac{1}{6}, & (x,y) = (0, -\sqrt{3}/2), \\ \frac{1}{6}, & (x,y) = (1, -\sqrt{3}/2), \\ \frac{1}{6}, & (x,y) = (1, 0), \\ \frac{1}{6}, & (x,y) = (-1, 0), \\ \frac{1}{6}, & (x,y) = (-1, \sqrt{3}/2), \\ 0, & \text{otherwise}. \end{cases} \tag{11}$$

So

$$w(y|x) = \frac{w(x,y)}{w(x)} = \begin{cases} \frac{1}{2}, & (x,y) = (0, \sqrt{3}/2), \\ \frac{1}{2}, & (x,y) = (0, -\sqrt{3}/2), \\ \frac{1}{2}, & (x,y) = (1, -\sqrt{3}/2), \\ \frac{1}{2}, & (x,y) = (1, 0), \\ \frac{1}{2}, & (x,y) = (-1, 0), \\ \frac{1}{2}, & (x,y) = (-1, \sqrt{3}/2), \\ 0, & \text{otherwise}. \end{cases} \tag{12}$$

Exercise 2 in lecture 14

Solution Suppose p is the probability of $X = -1$, and we have

$$m = \langle X \rangle = -p + (1-p) = 1 - 2p, \quad -1 \le m \le 1, \tag{13}$$

and therefore

$$p = \frac{1-m}{2}, \tag{14}$$

and

$$\begin{aligned} H_\alpha(X) &= \frac{1}{1-\alpha} \log_2(p^\alpha + (1-p)^\alpha) \\ &= \frac{1}{1-\alpha} \log_2 \left(\left(\frac{1-m}{2} \right)^\alpha + \left(\frac{1+m}{2} \right)^\alpha \right). \end{aligned} \tag{15}$$

The plots can be found in Figure 1.

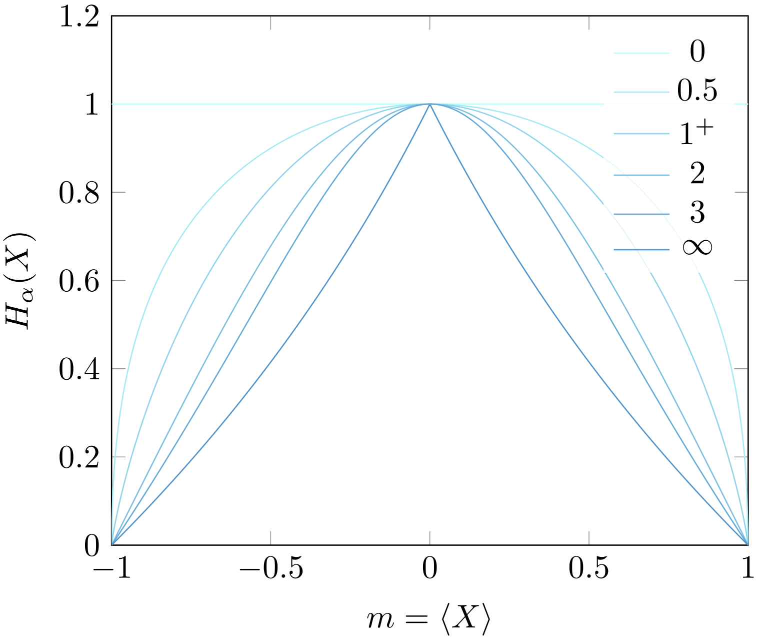

Figure 1: Plots of $H_\alpha(X)$ with different α's