ODEs

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1 First order ODEs

1.1 Linear ODEs

An ODE in the form of

$$y'(x) + p(x)y(x) = q(x) \tag{1}$$

is considered **linear**. All linear ODEs can be solved by the following procedure. First we have

$$(y' + py)\mathrm{e}^{\int p\mathrm{d}x} = q\mathrm{e}^{\int p\mathrm{d}x}, \tag{2}$$

and now the LHS is a derivative:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(y\mathrm{e}^{\int p\mathrm{d}x}\right) = q\mathrm{e}^{\int p\mathrm{d}x}, \tag{3}$$

and now we can integrate over x and get

$$y\mathrm{e}^{\int p\mathrm{d}x} = \int q\mathrm{e}^{\int p\mathrm{d}x}\,\mathrm{d}x\,, \tag{4}$$

$$y = \mathrm{e}^{-\int p\mathrm{d}x}\int q\mathrm{e}^{\int p\mathrm{d}x}\,\mathrm{d}x\,. \tag{5}$$

1.2 "Energy-conservation lines" and exact equations

Another way to represent the solution of an ODE is the form $\phi(x, y) = \text{const}$. Note that the RHS contains no variables, and we have

$$0 = \frac{\mathrm{d}\phi}{\mathrm{d}x} = \frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}x}, \tag{6}$$

and thus if

$$y' = f(x, y) \tag{7}$$

is algebraically equivalent to (6), the equation is already solved: We should find M, N such that

$$y' = -\frac{M}{N}, \quad M = \frac{\partial\phi}{\partial x}, \quad N = \frac{\partial\phi}{\partial y}, \tag{8}$$

and then $\phi(x, y)$ solves the equation. In this case we say $y' = -M/N$ is **exact**.

To test for exactness, we only have to test whether

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}, \tag{9}$$

and if so, the existence of ϕ is guaranteed. (Since we work on a topological trivial space, things like cohomology group will not bother us.) We can now use "partial integral" to find ϕ.

Example: suppose in a calculation we find

$$\frac{\partial\phi}{\partial x} = 2y^2 + y\mathrm{e}^{xy}, \quad \frac{\partial\phi}{\partial y} = 4xy + x\mathrm{e}^{xy} + 2y. \tag{10}$$

After partial integration, we find

$$\phi(x, y) = \underbrace{2xy^2 + \mathrm{e}^{xy} + h(y)}_{\int \frac{\partial\phi}{\partial x}\mathrm{d}x} = \underbrace{2xy^2 + \mathrm{e}^{xy} + y^2 + g(x)}_{\int \frac{\partial\phi}{\partial y}\mathrm{d}y}, \tag{11}$$

and we have to choose

$$h(y) = y^2, \quad g(x) = \text{const}, \tag{12}$$

and the solution is

$$\phi(x,y) = 2xy^2 + \mathrm{e}^{xy} + y^2 + \text{const}. \tag{13}$$

Note that even when the decomposition $f = -M/N$ doesn't give an exact equation for us, we can still use the method of exact equations: we can multiply a factor μ to both M and N, and try to guess the form of μ so that

$$\frac{\partial(\mu M)}{\partial y} = \frac{\partial(\mu N)}{\partial x}. \tag{14}$$

An example can be found in solving

$$y' = -\frac{1}{3x - \mathrm{e}^{-2y}}. \tag{15}$$

We have

$$\frac{\partial 1}{\partial y} = 0, \quad \frac{\partial(3x - \mathrm{e}^{-2y})}{\partial x} = 3,$$

so the equation is not exact if we choose $M = 1$ and $N = 3x - \mathrm{e}^{-2y}$. However, (14) can be fulfilled now: it's now

$$\frac{\partial \mu}{\partial y} = 3\mu + \left(3x - \mathrm{e}^{-2y}\right)\frac{\partial \mu}{\partial x},$$

and the most convenient way to solve it (we *don't* need to find all solutions of this equation!) is to let μ contain y only, so the tricky term on the RHS disappears, and thus we choose $\mu = \mathrm{e}^{3y}$, and we get

$$\phi(x,y) = \int \mu M \,\mathrm{d}x = \int \mathrm{e}^{3y}\,\mathrm{d}x = x\mathrm{e}^{3y} + u(y),$$

$$\phi(x,y) = \int \mu N \,\mathrm{d}y = \int (3x\mathrm{e}^{3y} - \mathrm{e}^{y})\,\mathrm{d}y = x\mathrm{e}^{3y} - \mathrm{e}^{y} + v(x),$$

so

$$\phi(x,y) = x\mathrm{e}^{3y} - \mathrm{e}^{y} + \text{const}. \tag{16}$$

1.3 Bernoulli equation

Consider the following **Bernoulli equation**

$$y' + P(x)y = R(x)y^{\alpha}. \tag{17}$$

When $\alpha = 0, 1$, the equation can be solved by the standard methods for linear first order ODEs. When this is not the case, we may do the substitution

$$v = y^{\beta}, \tag{18}$$

and then the equation becomes

$$\frac{1}{\beta}v^{1/\beta - 1}v' + P(x)v^{1/\beta} = R(x)v^{\alpha/\beta},$$

$$v' + P(x)v = R(x)v^{1 + \frac{\alpha - 1}{\beta}}. \tag{19}$$

The next step is to choose a good beta so that the equation gets simplified. We may want to make to exponent to be zero, and this means we should choose

$$\beta = 1 - \alpha, \tag{20}$$

and the ODE is now

$$v' + Pu = R, \tag{21}$$

which can then be solved by the method in Section 1.1.

2 Second order ODEs

2.1 Linear 2nd order ODE with initial values

A linear second order ODE has the following form:

$$y'' + p(x)y' + q(x)y = f(x). \tag{22}$$

It usually comes with initial value conditions

$$y(x_0) = A, \quad y'(x_0) = B. \tag{23}$$

This course is about concrete calculations, but knowing what we are doing makes sense is important. Here is an existence and uniqueness theorem: if $p(x), q(x)$, and $f(x)$ are continuous over an interval I, and $x_0 \in I$, then a unique solution exists for (22) with the initial conditions given above.

Usually, we start by looking at the **homogeneous** second order ODE

$$y'' + p(x)y' + q(x)y = 0. \tag{24}$$

The influence of $f(x)$ can be included as the "response" of the LHS. The full solution of (24) takes the form

$$y = c_1 y_1 + c_2 y_2, \tag{25}$$

where c_1, c_2 are constants to be decided by initial conditions, and y_1 and y_2 are linearly independent solutions of (24). The **Wronskian** is defined as

$$W(x) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}. \tag{26}$$

By checking if it is non-zero at most points, we can find whether y_1 and y_2 are truly linearly independent to each other.

There is a method to arrive at y_2 from y_1: we can always take the ansatz

$$y_2 = y_1 u, \tag{27}$$

and therefore we get

$$(u'' y_1 + 2u' y_1' + u y_1'') + p(u' y_1 + u y_1') + q u y_1 = 0,$$

and the condition that y_1 is a solution to (24) means

$$u'' + \underbrace{\frac{2y_1' + p y_1}{y_1}}_{g(x)} u' = 0, \tag{28}$$

which is essentially a first order ODE, because we can replace u' by v, and then we find

$$\ln v = -\int g(x) \, \mathrm{d}x \,,$$

and

$$u(x) = \int \mathrm{e}^{-\int g(x) \mathrm{d}x} \, \mathrm{d}x \,. \tag{29}$$

2.2 Constant coefficients

The equation

$$y'' + Ay + By = 0 \tag{30}$$

can be solved directly by the following construction:

$$y = c_1 \mathrm{e}^{\lambda_1 x} + c_2 \mathrm{e}^{\lambda_2 x}, \tag{31}$$

where λ_1, λ_2 are solutions of

$$\lambda^2 + A\lambda + B = 0. \tag{32}$$

For example, to solve the equation

$$y'' - 2y' + 10y = 0, \tag{33}$$

we just solve

$$\lambda^2 - 2\lambda + 10 = 0,$$

which gives us

$$\lambda = 1 \pm 3\mathrm{i}, \tag{34}$$

and therefore a general solution is

$$y = \mathrm{e}^x (c_1 \mathrm{e}^{3\mathrm{i}x} + c_2 \mathrm{e}^{-3\mathrm{i}x}). \tag{35}$$

It should be noted that c_1, c_2 can be complex, even when we restrict y in $\mathbb{R}$: we can let the imaginary part of y vanish as long as we impose some constraints over c_1, c_2. If we are determined to work in the real space, two alternative linearly independent solutions can be used:

$$y_1(x) = \mathrm{e}^x \cos(3x), \quad y_2(x) = \mathrm{e}^x \sin(3x). \tag{36}$$

Although we can immediately say they are linearly independent, we can use them as a demonstration of the Wronskian method: now we have

$$W(x) = y_1(x)y_2'(x) - y_2(x)y_1'(x) = 3\mathrm{e}^{2x}, \tag{37}$$

which of course isn't constantly zero.

(32) is faced with the problem of having only one solution when $A^2 - 4B = 0$. In this case we need to go back to the standard procedure to get y_2 from y_1. An example is

$$y'' + 6y + 9 = 0, \tag{38}$$

for which (32) only gives

$$y_1 = \mathrm{e}^{-3x}. \tag{39}$$

Suppose $y_2 = u\mathrm{e}^{-3x}$, we have TODO

2.3 Euler equation

A **Euler equation** has the following form:

$$x^2 y'' + Axy' + By = 0, \tag{40}$$

where A, B are constants. One solution can be immediate found: it always looks like

$$y = x^a. \tag{41}$$

We then find

$$a(a-1) + Aa + B = 0. \tag{42}$$

If there are two solutions of the equation, (40) has already been solved. If not, we can use the trick (27).

An example: let's solve

$$x^2 y'' + 3xy' + y = 0. \tag{43}$$

The equation about a is now

$$a(a-1) + 3a + 1 = 0,$$

and it only has one solution $a = -1$. Therefore we have

$$y_1 = \frac{1}{x}.$$

Suppose

$$y_2 = uy_1,$$

we get

$$x^2\left(\frac{u''}{x^2} - \frac{2u'}{x} + \frac{2u}{x^2}\right) + 3x\left(\frac{u'}{x} - \frac{u}{x^2}\right) + \frac{u}{x} = 0,$$

which is equivalent to

$$v'x + v = 0, \quad v = u'$$

the solution of which is

$$\ln v + \ln x = \text{const},$$

and therefore

$$v = \frac{C'}{x}, \quad u = C' \ln x + C,$$

$$y_2 = \frac{1}{x}(C' \ln x + C).$$

This essentially gives *all* solutions we need: for y_1, we just have $u = 1$, which corresponds to $C = 1$. So now the equation is completely solved.

2.4 Non-homogeneous cases or how to find the linear response

Now we discuss how to solve

$$y'' + p(x)y' + q(x)y = f(x). \tag{44}$$

A general solution is

$$y(x) = y_{\mathrm{p}}(x) + y_{\mathrm{h}}(x), \tag{45}$$

where the subscript p means a particular solution, and the subscript h means the general solution of the corresponding homogeneous equation.

We need some common sense to find a particular solution. To solve

$$y'' - y' - 2y = 2x^2 + 5, \tag{46}$$

we don't expect y to be, say, $\cos(2x)$: instead, it's usually the case that y is a polynomial. An ansatz is

$$y = Ax^2 + Bx + C.$$

We don't want a x^3 term because it doesn't appear on RHS. The equation then becomes

$$2A - (2Ax + Bx) - 2(Ax^2 + Bx + C) = 2x^2 + 5,$$

$$-2A = 2, \quad -2A - 2B = 0, \quad 2A - B - 2C = 5,$$

and therefore $A = -1, B = 1, C = -4$. Therefore we get a particular solution:

$$y_{\mathrm{p}} = -x^2 + x - 4. \tag{47}$$

A particular solution may also be determined as a "linear combination" of homogeneous solutions, although now the coefficients have temporal variation. That's to say, we take the ansatz

$$y_{\mathrm{p}}(x) = u(x)y_1(x) + v(x)y_2(x). \tag{48}$$

This is quite similar to the procedure introduced in Section 1.1. After substituting y with (48) in (44), we get

$$u'y_1' + v'y_2' = f. \tag{49}$$

Introducing the constraint

$$u'y_1 + v'y_2 = 0, \tag{50}$$

we have

$$\begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix} \begin{pmatrix} u' \\ v' \end{pmatrix} = \begin{pmatrix} 0 \\ f \end{pmatrix}, \tag{51}$$

from which we find u', v' and hence u, v. The Wronskian – the determinant of the matrix on LHS – is non-zero, so the equation always has a solution.

Example: let's solve

$$y'' + y = \tan x. \tag{52}$$

We have

$$y_1 = \cos x, \quad y_2 = \sin x,$$

and therefore

$$W(x) = y_1 y_2' - y_2 y_1' = 1.$$

So

$$\begin{aligned} u' &= -\int \frac{y_2 f}{W(x)}\,\mathrm{d}x = -\int \frac{\sin^2 x}{\cos x}\,\mathrm{d}x = -\int \frac{1-\cos^2 x}{\cos x}\,\mathrm{d}x \\ &= -\frac{1}{2}\ln\left|\frac{1+\sin x}{1-\sin x}\right| + \sin x, \end{aligned}$$

and similarly we have

$$v =$$

2.5 Analyticity

The stimulus can be non-analytic. f is analytic at x_0, if we can expand it into a power series around x_0:

$$f(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n \tag{53}$$

in a interval around x_0. The function $f(x) = \ln x$, then, is not analytic at $x = 0$ – but $f(x) = \ln(x+1)$ is analytic at $x = 0$, though not at $x = -1$.

There is a theorem: if p, q, f are all analytic at x_0, then (44) together with conditions $y(x_0) = A$ and $y'(x_0) = B$ has a unique solution that is analytic at x_0.

An example:

$$x'' + y' - xy = 0, \quad y(0) = -2, y'(0) = 0. \tag{54}$$

Suppose

$$y(x) = \sum_{n=0}^{\infty} a_n x^n, \quad y'(x) = \sum_{n=0}^{\infty} n a_n x^{n-1}, \quad y''(x) = \sum_{n=0}^{\infty} n(n+1) a_n x^{n-2},$$

we have

$$\begin{aligned} 0 &= \sum_{n=0}^{\infty} n(n-1) a_n x^{n-2} + \sum_{n=0}^{\infty} n a_n x^{n-1} - \sum_{n=0}^{\infty} a_n x^{n+1} \\ &= a_1 + 2a_2 + \sum_{n=1}^{\infty} x^n((n+2)(n+1)a_{n+2} + (n+1)a_{n+1} - a_{n-1}), \end{aligned}$$

and therefore

$$a_1 + 2a_2 = 0, \quad a_{n+2} = -\frac{a_{n+1}}{n+2} + \frac{a_{n-1}}{(n+1)(n+2)}.$$

The conditions $y(0) = -2$ and $y'(0) = 0$ means

$$a_0 = -2, \quad a_1 = 0,$$

and then we can in principle find all a_n's – although it's often hard to see a pattern and write down a closed-form expression for a_n.

Now we consider the equation

$$P(x)y'' + Q(x)y' + R(x)y = F(x). \tag{55}$$

Of course we can divide the equation with $P(x)$ and go back to (44), but if $P(x)$ is zero at some points, p, q, f in (44) are no longer always analytic. Thus the solution isn't guaranteed to be analytic everywhere. In other words, we no longer have a power series solution – or do we? We can still try a generalized power series, like

$$y(x) = \sum_{n=0}^{\infty} c_n (x - x_0)^{n+r}, \tag{56}$$

where r can be a fraction. This is guaranteed with

$$(x - x_0)y'' + Q(x)y' + R(x)y = F(x). \tag{57}$$

To demonstrate this, consider

$$y'' + \frac{1}{2x}y' - \frac{1}{4x}y = 0. \tag{58}$$

We plug

$$y(x) = \sum_{n=0}^{\infty} c_n x^{n+r}$$

into

$$4xy'' + 2y' - y = 0,$$

and get

$$\sum_{n=0}^{\infty} \left(4(n+r)(n+r-1)c_n x^{n+r-1} + 2(n+r)c_n x^{n+r-1} - c_n x^{n+r}\right) = 0.$$

The coefficient of the x^{r-1} term is

$$4r(r-1) + 2r = 0,$$

from which we find $r = 0, 1/2$. For the rest of the terms, we have

$$4(n+r)(n+r-1)c_n + 2(n+r)c_n - c_{n-1} = 0,$$

$$c_n = \frac{c_{n-1}}{2(n+r)(2n+2r-1)}.$$

When $r = 0$, this gives

$$c_n = \frac{c_{n-1}}{2n(2n-1)}, \quad c_n = \frac{c_0}{(2n)!},$$

and we get

$$y(x) = \sum_{n=0}^{\infty} \frac{x^n}{(2n)!}. \tag{59}$$

When $r = 1/2$, this gives

$$c_n = \frac{c_{n-1}}{(2n+1) \cdot 2n}, \quad c_n = \frac{c_0}{(2n+1)!},$$

and we get

$$y(x) = \sum_{n=0}^{\infty} \frac{x^{n+1/2}}{(2n+1)!}. \tag{60}$$

So we have already obtained two independent solutions.

3 The Laplace transformation

The **Laplace transformation** is defined for a function $f(t)$ as

$$\mathcal{L}[f(t)] = F(s) = \int_0^{\infty} f(t)\mathrm{e}^{-st}\,\mathrm{d}t\,, \tag{61}$$

for every s where the integral converges. It's easy to see that $\mathcal{L}$ is linear,

3.1 Laplace transforms of basic elementary functions

The simplest Laplace transform is

$$\mathcal{L}[1] = \int_0^\infty \mathrm{e}^{-st}\,\mathrm{d}t = \frac{1}{s}, \quad s > 0, \tag{62}$$

and by integration by parts, we also have

$$\mathcal{L}[t] = \int_0^\infty t\mathrm{e}^{-st}\,\mathrm{d}t = \frac{1}{s^2}. \tag{63}$$

By multiple rounds of integration by parts, we have

$$\mathcal{L}[t^2] = \frac{2}{s^3}, \tag{64}$$

and by iteratively using integration by parts we get

$$\mathcal{L}[t^n] = \frac{n!}{s^{n+1}}. \tag{65}$$

Aside from polynomials, we have

$$\mathcal{L}[\mathrm{e}^{at}] = \int_0^\infty \mathrm{e}^{-(s-a)t}\,\mathrm{d}t = \frac{1}{s-a}, \quad s > a. \tag{66}$$

Since we have

$$\cos x = \frac{\mathrm{e}^{\mathrm{i}x} + \mathrm{e}^{-\mathrm{i}x}}{2}, \quad \sin x = \frac{\mathrm{e}^{\mathrm{i}x} - \mathrm{e}^{-\mathrm{i}x}}{2\mathrm{i}},$$

we have

$$\mathcal{L}[\cos(at)] = \frac{1}{2}\left(\frac{1}{s-\mathrm{i}a} + \frac{1}{s+\mathrm{i}a}\right) = \frac{s}{s^2+a^2}. \tag{67}$$

Similarly we have

$$\mathcal{L}[\sin(at)] = \frac{a}{s^2+a^2}. \tag{68}$$

Consider a pulse signal

$$f(t) = \begin{cases} 0, & t < a \text{ or } t > b, \\ 1, & a \le t \le b. \end{cases} \tag{69}$$

This can be easily implemented by the Heaviside function: we have

$$f(t) = H(t-a) - H(t-b). \tag{70}$$

So the Laplace transform is

$$\mathcal{L}[f(t)] = \mathrm{e}^{-sa}\mathcal{L}[1] - \mathrm{e}^{-sb}\mathcal{L}[1] = \frac{1}{s}(\mathrm{e}^{-sa} - \mathrm{e}^{-sb}). \tag{71}$$

3.2 Laplace transform of differential equations

We have

$$\begin{aligned} \mathcal{L}[f'(t)] &= \int_0^\infty \frac{\mathrm{d}f}{\mathrm{d}t}\mathrm{e}^{-st}\,\mathrm{d}t = f\mathrm{e}^{-st}\Big|_{s=0}^{\infty} - \int_0^\infty f\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{e}^{-st}\,\mathrm{d}t \\ &= -f(0) + s\int_0^\infty f(t)\mathrm{e}^{-st}\,\mathrm{d}t = s\mathcal{L}[f(t)] - f(0). \end{aligned} \tag{72}$$

Applying this twice, we get

$$\mathcal{L}[f''(t)] = s\mathcal{L}[f'(t)] - f'(0) = s^2\mathcal{L}[f(t)] - sf(0) - f'(0). \tag{73}$$

The general formula is therefore

$$\mathcal{L}[f^{(n)}(t)] = s^n\mathcal{L}[f(t)] - s^{n-1}f(0) - s^{n-2}f'(0) - \cdots - f^{(n-1)}(0). \tag{74}$$

This can be used to solve ODEs. Consider, for example,

$$y'' + 4y' + 3y = \mathrm{e}^t, \quad y(0) = 0, \quad y'(0) = 2. \tag{75}$$

We have (below we follow the convention to use big letters to refer to functions in the Laplace space)

$$\mathcal{L}[\text{LHS}] = (s^2 Y(s) - s y(0) - y'(0)) + 4(sY(s) - y(0)) + 3Y(s) = (s^2 + 4s + 3)Y(s) - 2,$$

where we have already applied the initial conditions, and

$$\mathcal{L}[\text{RHS}] = \frac{1}{s-1}.$$

So what need to be done is to solve

$$(s^2 + 4s + 3)Y(s) - 2 = \frac{1}{s-1},$$

and we get

$$Y(s) = \frac{2s - 1}{(s-1)(s+1)(s+3)}. \tag{76}$$

Thus, once we do the inverse Laplace transformation, we get $y(t)$. It's possible to do an inverse integral transformation, but in this case, what's more convenient is to make the decomposition

$$Y(s) = \frac{A}{s-1} + \frac{B}{s+1} + \frac{C}{s+3},$$

and we can find

$$A = \frac{1}{8}, \quad B = \frac{3}{4}, \quad C = -\frac{7}{8}.$$

Then we can read the Laplace transformation table in the inverse direction: from (66), we find

$$y(t) = \frac{1}{8}\mathrm{e}^t + \frac{3}{4}\mathrm{e}^{-t} - \frac{7}{8}\mathrm{e}^{-3t}. \tag{77}$$

So the linear 2nd order ODE is completely solved.

TODO: whether we will encounter things like

$$\mathcal{L}[f] = \frac{1}{s}, \quad s > 100. \tag{78}$$

It seems in most real world problems, we don't really need to worry about the region of allowed s: we just extend the region of allowed s as large as we can, solve the algebraic equation in the Laplace space, and then go back.

We can repeat the procedure in the last example for a system of ODEs. Consider for example

$$x' - 2y' = 1, \quad x' - x + y = 0, \quad x(0) = y(0) = 0. \tag{79}$$

For the first equation, Laplace transform gives

$$sX(s) - x(0) - 2(sY(s) - y(0)) = \mathcal{L}[1] = \frac{1}{s},$$

and for the second equation we have

$$sX(s) - x(0) - X(s) + Y(s) = 0.$$

Solving this linear equation system, we get

$$X(s) = \frac{1}{s^2(2s-1)}, \quad Y(s) = -\frac{s-1}{s^2(2s-1)}.$$

Since

$$X(s) = -\frac{2}{s} - \frac{1}{s^2} + 2\frac{1}{s - 1/2},$$

we have

$$x(t) = -2 - t + 2\mathrm{e}^{t/2}. \tag{80}$$

Similarly,

$$y(t) = -1 + \mathrm{e}^{t/2} - t. \tag{81}$$

3.3 Shifting of s and t

Laplace transform of an integral can also be found. Here we investigate into things like

$$\int_0^t f(t')\,\mathrm{d}t',$$

and we need to pay attention to what variable is inside the integration and what variable is exposed to the Laplace operator. We have

$$\begin{aligned}\mathcal{L}\left[\int_0^t f(t')\,\mathrm{d}t'\right] &= -\frac{1}{s}\int_{t'=0}^{\infty}\int_0^t f(t')\,\mathrm{d}t'\,\mathrm{d}\mathrm{e}^{-st} \\ &= -\frac{1}{s}\left(\mathrm{e}^{-st}\int_0^t f(t')\,\mathrm{d}t'\bigg|_0^\infty - \int_0^\infty f(t)\mathrm{e}^{-st}\,\mathrm{d}t\right) \\ &= \frac{1}{s}\mathcal{L}[f(t)].\end{aligned} \tag{82}$$

This is the inverse of the rule of derivatives above, which is expected.

Another theorem is the s**-shifting theorem**: we have

$$\mathcal{L}[\mathrm{e}^{at}f(t)] = F(s-a) = \mathcal{L}[f(t)]_{s\to s-a}. \tag{83}$$

This is exactly what leads to (66). Correspondingly we have the t-shifting theorem:

$$\mathcal{L}[f(t-a)H(t-a)] = \mathrm{e}^{-sa}\mathcal{L}[f(t)], \tag{84}$$

where $H(t-a)$ is 1 when $t \geq 0$ and zero otherwise, and $a > 0$.

The theorems can be used to evaluate Laplace transforms of complex functions. We have

$$\mathcal{L}[t^2\mathrm{e}^{-t}] = \mathcal{L}[t^2]|_{s\to s+1} = \frac{2}{(s+1)^3}. \tag{85}$$

3.4 Convolution

The **convolution integral** of $f(t)$ and $g(t)$ is defined as

$$f\otimes g = \int_0^t f(t')g(t-t')\,\mathrm{d}t'. \tag{86}$$

This can be found very frequently in science: it appears when we deal with interaction: for example, the $t-t'$ time may come from indirect interaction (where t' is the time an intermediate step happens).

A way around the reasonable but hard to calculate convolution integral is taking its Laplace transform. We have

$$\begin{aligned}\mathcal{L}[f\otimes g] &= \int_0^\infty \mathrm{e}^{-st}\int_0^t f(t')g(t-t')\,\mathrm{d}t'\,\mathrm{d}t \\ &= \int_0^\infty f(t')\int_{t'}^\infty g(t-t')\mathrm{e}^{-st}\,\mathrm{d}t\,\mathrm{d}t' \\ &= \int_0^\infty f(t')\int_0^\infty g(t'')\mathrm{e}^{-s(t''+t')}\,\mathrm{d}t''\,\mathrm{d}t' \\ &= \int_0^\infty f(t')\mathrm{e}^{-st'}\int_0^\infty g(t'')\mathrm{e}^{-st''}\,\mathrm{d}t'' \\ &= \mathcal{L}[f]\mathcal{L}[g].\end{aligned} \tag{87}$$

The second line uses another way to see the integration region: by saying $0 < t < \infty$, $0 < t' < t$, we also mean $0 < t' < \infty$, $t' < t < \infty$. The third line replaces $t - t'$ with t''. The final result no longer contains convolution.

One application of this fact is shown in the following example. Consider

$$\frac{1}{s^2-a^2} = \underbrace{\frac{1}{s-a}}_{G(s)}\cdot\underbrace{\frac{1}{s+a}}_{F(s)}.$$

What's its inverse Laplace transform? By shifting theorem, we have

$$f(t) = \mathrm{e}^{-at}, \quad g(t) = \mathrm{e}^{at},$$

and therefore

$$\mathcal{L}^{-1}\left[\frac{1}{s^2 - a^2}\right] = \int_0^t \mathrm{e}^{-at'}\mathrm{e}^{a(t-t')}\,\mathrm{d}t' = \mathrm{e}^{at}\int_0^t \mathrm{e}^{-2at'}\,\mathrm{d}t' = \frac{1}{a}\sinh(at). \tag{88}$$

We can also use the convolution theorem to give solutions to very generic equations. Consider the following ODE problem:

$$y'' - 5y' + 6y = f(t), \quad f(0) = f'(0) = 0. \tag{89}$$

The point here is we *don't* know what is $f(t)$, but still want to give a template of the solution. So we just do Laplace transform:

$$(s^2 - 5s + 6)Y(s) = F(s).$$

To find the Laplace transform of $1/(s^2 - 5s + 6)$, we have

$$\mathcal{L}^{-1}\left[\frac{1}{(s^2 - 5s + 6)}\right] = \mathcal{L}^{-1}\left[\frac{1}{s-3} - \frac{1}{s-2}\right] = \mathrm{e}^{3t} - \mathrm{e}^{2t},$$

and

$$\begin{aligned} y(t) = \mathcal{L}^{-1}[Y(s)] &= \mathcal{L}^{-1}\left[\frac{1}{(s^2 - 5s + 6)}F(s)\right] \\ &= \mathcal{L}^{-1}\left[\frac{1}{(s^2 - 5s + 6)}\right] \otimes f(t) \\ &= \int_0^t f(t)(\mathrm{e}^{3(t-t')} - \mathrm{e}^{2(t-t')})\,\mathrm{d}t'\,. \end{aligned} \tag{90}$$