Homework 1

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1 Maxwell's equations in dielectrics, Lorentz oscillators, and complex notation

1.1 Time-Average Quantities in Complex Notation

It is often important to be able to compute time-averaged quantities, such as the potential energy of a harmonic oscillator $U_{pe} = \frac{k}{2}\left\langle x^2 \right\rangle$ or the electric field energy density $U_{\text{el}} = \frac{\varepsilon_0}{2}\left\langle \mathbf{E}^2 \right\rangle$. Here, the time-average of a function, $f(t)$, is defined as, $\langle f(t) \rangle = (1/T)\int_{t-T/2}^{t+T/2} dt' f(t')$, where T is defined as either the characteristic period of the oscillating system (i.e., $T = 2\pi/\omega$) or infinity. Such time averaging is drastically simplified by using complex notation.

To see this, suppose that we have any two functions $A(t)$ and $B(t)$, both of which take on a time harmonic form. Without loss of generality, we assume that $A(t) = A_0 \cos(\omega t + \phi)$, and $B(t) = B_0 \cos(\omega t + \theta)$, where ϕ and θ are arbitrary phase factors.

1.1.1

We have

$$\begin{aligned}
\langle A(t)B(t) \rangle &= \frac{1}{T}\int_{t-T/2}^{t+T/2} \mathrm{d}t' \, A_0 \cos(\omega t' + \phi) B_0 \cos(\omega t' + \theta) \\
&= A_0 B_0 \frac{1}{T}\int_{t-T/2}^{t+T/2} \mathrm{d}t' \, \frac{1}{2}(\cos(\omega t' + \phi + \omega t' + \theta) + \cos(\omega t' + \phi - \omega t' - \theta)) \\
&= \frac{1}{2} A_0 B_0 \cos(\phi - \theta).
\end{aligned} \tag{1}$$

Here we have used the condition that $T = 2\pi/\omega$ so that the first term vanishes.

1.1.2

We have

$$A(t) = \tilde{A}_0 \mathrm{e}^{-\mathrm{i}\omega t}, \quad B(t) = \tilde{B}_0 \mathrm{e}^{-\mathrm{i}\omega t}, \quad \tilde{A}_0 = A_0 \mathrm{e}^{-\mathrm{i}\phi}, \quad \tilde{B}_0 = B_0 \mathrm{e}^{-\mathrm{i}\theta}, \tag{2}$$

and therefore

$$\operatorname{Re} \tilde{A}_0 B_0 = \operatorname{Re} A_0 \tilde{B}_0 = \operatorname{Re} A_0 B_0 \mathrm{e}^{\mathrm{i}(\phi - \theta)} = A_0 B_0 \cos(\phi - \theta), \tag{3}$$

and hence

$$\langle A(t)B(t) \rangle = \frac{1}{2} \operatorname{Re} \tilde{A}_0 B_0 = \frac{1}{2} \operatorname{Re} A_0 \tilde{B}_0. \tag{4}$$

We can also straightforwardly do the follows. We have

$$\begin{aligned}
\langle A(t)B(t) \rangle &= \left\langle \frac{1}{2}(\tilde{A}(t) + \tilde{A}^*(t)) \cdot \frac{1}{2}(\tilde{B}(t) + \tilde{B}^*(t)) \right\rangle \\
&= \frac{1}{4}\left\langle \tilde{A}_0 \tilde{B}_0 \mathrm{e}^{-2\mathrm{i}\omega t} + \tilde{A}_0 \tilde{B}_0^* + \tilde{A}_0{}^* \tilde{B}_0^* \mathrm{e}^{2\mathrm{i}\omega t} + \tilde{A}_0^* \tilde{B}_0 \right\rangle \\
&= \frac{1}{4}\langle A_0^* B_0 + \text{c.c.} \rangle \\
&= \frac{1}{2} A_0^* B_0 = \frac{1}{2} A_0 B_0^*.
\end{aligned} \tag{5}$$

1.1.3

When

$$\boldsymbol{E} = \hat{\boldsymbol{x}}\,\mathrm{Re}\,\tilde{E}_0 \mathrm{e}^{-\mathrm{i}(\omega t - kz)}, \tag{6}$$

from

$$\boldsymbol{\nabla} \times \boldsymbol{E} = -\,\partial \boldsymbol{B}/\partial t \tag{7}$$

we obtain

$$\begin{aligned} \mathrm{i}\boldsymbol{k} \times \boldsymbol{E} &= -(-\mathrm{i}\omega)\boldsymbol{B} \\ \Rightarrow \boldsymbol{B} &= \frac{1}{\omega} k\hat{\boldsymbol{z}} \times \boldsymbol{E} = \frac{k}{\omega}\hat{\boldsymbol{y}}\,\mathrm{Re}\,\tilde{E}_0 \mathrm{e}^{-\mathrm{i}(\omega t - kz)}, \end{aligned} \tag{8}$$

and therefore

$$\langle \boldsymbol{S} \rangle = \frac{1}{\mu_0}\langle \boldsymbol{E} \times \boldsymbol{B} \rangle = \frac{1}{\mu_0} \cdot \frac{1}{2}\,\mathrm{Re}\,\underbrace{\hat{\boldsymbol{x}}\tilde{E}_0 \mathrm{e}^{\mathrm{i}kz}}_{\tilde{\boldsymbol{E}}_0} \times \underbrace{\frac{k}{\omega}\hat{\boldsymbol{y}}\tilde{E}_0^* \mathrm{e}^{-\mathrm{i}kz}}_{\tilde{\boldsymbol{B}}_0} = \frac{k}{2\mu_0\omega}|\tilde{E}_0|^2 \hat{\boldsymbol{z}}, \tag{9}$$

and since the refraction index is n, we eventually get

$$\omega = k \cdot \frac{c}{n} \tag{10}$$

and therefore

$$\langle \boldsymbol{S} \rangle = \frac{n}{2}\sqrt{\frac{\epsilon_0}{\mu_0}}|\tilde{E}_0|^2 \hat{\boldsymbol{z}}. \tag{11}$$

The direction of the energy flow is parallel to the z axis.

1.1.4

The expected value of the electric energy density is

$$\langle u_e \rangle = \frac{1}{2}\epsilon_0\epsilon_\mathrm{r}\langle \boldsymbol{E}^2 \rangle = \frac{1}{2}\epsilon_0 n^2 \cdot \frac{1}{2}|\tilde{E}_0|^2 = \frac{1}{4}\epsilon_0 n^2 |\tilde{E}|^2, \tag{12}$$

and the expected value of the magnetic energy density is

$$\langle u_m \rangle = \frac{1}{2\mu_0}\langle \boldsymbol{B}^2 \rangle = \frac{1}{2\mu_0} \cdot \frac{1}{2}\frac{k^2}{\omega^2}|\tilde{E}_0|^2 = \frac{1}{4}\frac{n^2}{c^2\mu_0}\left|\tilde{E}_0\right|^2 = \frac{1}{4}\epsilon_0 n^2 \left|\tilde{E}_0\right|^2. \tag{13}$$

So we find

$$\frac{\langle u_e \rangle}{\langle u_m \rangle} = 1. \tag{14}$$

2 Lorentz oscillator in an AC field and optical forces

2.1 Optical response of an ensemble of Lorentz oscillators

Consider a dilute ensemble of Lorentz oscillators, uniformly distributed over space with number density N, in an AC electric field given by $\mathbf{E} = \mathrm{Re}\left[\tilde{\mathbf{E}}_0 e^{-i\omega t}\right]$. Each oscillator is driven by the local electric field according to the equation of motion given by

$$\ddot{\mathbf{p}} + \gamma\dot{\mathbf{p}} + \Omega^2\mathbf{p} = \frac{q^2}{m}\mathbf{E}(\mathbf{r}),$$

where $\mathbf{r}, m$, and q are the respective oscillator position, reduced mass, and charge.

2.1.1

The polarization density is

$$\boldsymbol{P} = N\boldsymbol{p}. \tag{15}$$

The EOM for $\boldsymbol{P}$ is

$$\ddot{\boldsymbol{P}} + \gamma\dot{\boldsymbol{P}} + \Omega^2\boldsymbol{P} = \frac{Nq^2}{m}\boldsymbol{E}. \tag{16}$$

We can switch to the Fourier representation. Thus we have

$$((-\mathrm{i}\omega)^2 + \gamma(-\mathrm{i}\omega) + \Omega^2)\tilde{\boldsymbol{P}} = \frac{Nq^2}{m}\tilde{\boldsymbol{E}}, \tag{17}$$

and from

$$\boldsymbol{D} = \epsilon_0\boldsymbol{E} + \boldsymbol{P} \tag{18}$$

we get

$$\tilde{\boldsymbol{D}} = \epsilon_0\epsilon_\mathrm{r}\tilde{\boldsymbol{E}}, \quad \epsilon_\mathrm{r}(\boldsymbol{k}, \omega) = 1 + \frac{\omega_\mathrm{p}^2}{-\omega^2 - \mathrm{i}\gamma\omega + \Omega^2}, \quad \omega_\mathrm{p}^2 = \frac{Nq^2}{m\epsilon_0}. \tag{19}$$

So we already get ϵ_r; it has explicit dependence on ω, but not $\boldsymbol{k}$.

2.1.2

The form of a propagating plane wave in the free space is

$$\boldsymbol{E} = \boldsymbol{E}_0\mathrm{e}^{\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{r} - \omega t)} + \text{c.c.}, \quad \boldsymbol{k} = k\hat{\boldsymbol{k}}, \quad k = \frac{\sqrt{\epsilon_\mathrm{r}}\omega}{c} = \frac{n\omega}{c}. \tag{20}$$

Note that it's possible that k has an imaginary part and n is the complex refraction index. The direction of $\operatorname{Re}\boldsymbol{k}$ and $\operatorname{Im}\boldsymbol{k}$ is assumed to be the same. The phase velocity is given by

$$v = \frac{\omega}{\operatorname{Re} k} = \frac{c}{\operatorname{Re} n} = \frac{c}{\operatorname{Re}\sqrt{\epsilon_\mathrm{r}}} = \frac{c}{\operatorname{Re}\sqrt{1 + \frac{\omega_\mathrm{p}^2}{-\omega^2 - \mathrm{i}\gamma\omega + \Omega^2}}}. \tag{21}$$

The group velocity is

$$v_\mathrm{g} = \frac{\mathrm{d}\omega}{\mathrm{d}\operatorname{Re} k} = c\left(\frac{\mathrm{d}\operatorname{Re}\sqrt{1 + \frac{\omega_\mathrm{p}^2}{-\omega^2 - \mathrm{i}\gamma\omega + \Omega^2}}}{\mathrm{d}\omega}\right)^{-1}. \tag{22}$$

2.1.3

Since ϵ_r has frequency dependence, the relation between $\boldsymbol{E}(t)$ and $\boldsymbol{D}(t)$ is not localized in the time domain, and therefore although we still know that the energy would be a quadratic form of $\boldsymbol{E}$ or $\boldsymbol{D}$, since

$$\frac{\partial\boldsymbol{E}}{\partial t}\cdot\boldsymbol{D} \neq \frac{\partial\boldsymbol{D}}{\partial t}\cdot\boldsymbol{E}, \tag{23}$$

the simple relation

$$u_e = \frac{1}{2}\boldsymbol{D}\cdot\boldsymbol{E}$$

no longer holds. Instead, we should start from the most generic theory and utilize

$$\frac{\partial u_e}{\partial t} = \boldsymbol{E}\cdot\frac{\partial\boldsymbol{D}}{\partial t}. \tag{24}$$

To use this equation to get an expression of u_e, we should no longer work with plane waves, or otherwise u_e is a constant and we don't see any change of u_e at all. Below we work with a wave packet centered at $\pm\omega_0$. For the wave packet, the electric field is

$$\boldsymbol{E}(t) = \mathrm{e}^{-\mathrm{i}\omega_0 t}\cdot\underbrace{\int\frac{\mathrm{d}\omega}{2\pi}\mathrm{e}^{-\mathrm{i}(\omega-\omega_0)t}\tilde{\boldsymbol{E}}(\omega)}_{=:\boldsymbol{E}_0(t)}, \tag{25}$$

$$\boldsymbol{D}(t) = \mathrm{e}^{-\mathrm{i}\omega_0 t} \cdot \int \frac{\mathrm{d}\omega}{2\pi} \mathrm{e}^{-\mathrm{i}(\omega-\omega_0)t} \varepsilon(\omega) \tilde{\boldsymbol{E}}(\omega). \tag{26}$$

By Taylor expansion of ε we have

$$\begin{aligned}
\partial \boldsymbol{D}/\partial t &= \mathrm{e}^{-\mathrm{i}\omega_0 t} \cdot \int \frac{\mathrm{d}\omega}{2\pi} \mathrm{e}^{-\mathrm{i}(\omega-\omega_0)t} \tilde{\boldsymbol{E}}(\omega)(-\mathrm{i}\omega) \left(\varepsilon(\omega_0) + (\omega-\omega_0) \left.\frac{\mathrm{d}\varepsilon}{\mathrm{d}\omega}\right|_{\omega=\omega_0} + \cdots \right) \\
&\approx \mathrm{e}^{-\mathrm{i}\omega_0 t} \cdot \int \frac{\mathrm{d}\omega}{2\pi} \mathrm{e}^{-\mathrm{i}(\omega-\omega_0)t} \tilde{\boldsymbol{E}}(\omega) \left(-\mathrm{i}\omega_0 \varepsilon(\omega_0) - \mathrm{i}(\omega-\omega_0)\varepsilon(\omega)_0 - \mathrm{i}(\omega-\omega_0)\omega \left.\frac{\mathrm{d}\varepsilon}{\mathrm{d}\omega}\right|_{\omega=\omega_0} \right) \\
&\approx \mathrm{e}^{-\mathrm{i}\omega_0 t} \cdot \int \frac{\mathrm{d}\omega}{2\pi} \mathrm{e}^{-\mathrm{i}(\omega-\omega_0)t} \tilde{\boldsymbol{E}}(\omega) \Bigg(-\mathrm{i}\omega_0 \varepsilon(\omega_0) \underbrace{-\mathrm{i}(\omega-\omega_0)\varepsilon(\omega)_0 - \mathrm{i}(\omega-\omega_0)\omega_0 \left.\frac{\mathrm{d}\varepsilon}{\mathrm{d}\omega}\right|_{\omega=\omega_0}}_{=-\mathrm{i}(\omega-\omega_0) \left.\frac{\mathrm{d}(\omega\varepsilon)}{\mathrm{d}\omega}\right|_{\omega=\omega_0}} \Bigg) \\
&= \mathrm{e}^{-\mathrm{i}\omega_0 t} \underbrace{\left(-\mathrm{i}\omega_0 \varepsilon(\omega_0) \boldsymbol{E}_0(t) + \frac{\mathrm{d}(\omega\varepsilon)}{\mathrm{d}\omega} \frac{\partial \boldsymbol{E}_0}{\partial t} \right)}_{=:\boldsymbol{D}_0(t)}.
\end{aligned} \tag{27}$$

In the second line we throw away the higher order Taylor terms; in the third line we only keep terms linear to $(\omega - \omega_0)$. These approximations require the wave packet to be focused enough. We use $\langle \cdots \rangle$ to refer to averaging over the fast oscillations; thus, $\boldsymbol{E}_0(t)$ and $\boldsymbol{D}_0(t)$ above can be regarded as constants when applying $\langle \cdots \rangle$, and hence we find

$$\begin{aligned}
\left\langle \frac{\partial u_e}{\partial t} \right\rangle &= \left\langle \boldsymbol{E} \cdot \frac{\partial \boldsymbol{D}}{\partial t} \right\rangle = \frac{1}{2} \cdot \frac{1}{4} \operatorname{Re}(\boldsymbol{D}_0^*(t) \cdot \boldsymbol{E}_0(t) + \boldsymbol{D}_0(t) \cdot \boldsymbol{E}_0^*(t)) \\
&\approx \frac{1}{4} \operatorname{Re} \left(\left(\omega_0 \varepsilon_2(\omega_0) + \frac{\mathrm{d}(\omega\varepsilon_1)}{\mathrm{d}\omega} \right) \frac{\partial \boldsymbol{E}_0^*}{\partial t} \cdot \boldsymbol{E}_0 + \left(\omega_0 \varepsilon_2(\omega_0) + \frac{\mathrm{d}(\omega\varepsilon_1)}{\mathrm{d}\omega} \right) \boldsymbol{E}_0^* \cdot \frac{\partial \boldsymbol{E}_0}{\partial t} \right) \\
&= \frac{1}{4} \left(\omega_0 \varepsilon_2(\omega_0) + \frac{\mathrm{d}(\omega\varepsilon_1)}{\mathrm{d}\omega} \right) \frac{\partial |\boldsymbol{E}_0|^2}{\partial t}.
\end{aligned} \tag{28}$$

In the second line we have considered both the real and imaginary parts of ϵ.[1] Since u_e contains no fast oscillation, we have

$$\frac{\partial \langle u_e \rangle}{\partial t} = \left\langle \frac{\partial u_e}{\partial t} \right\rangle = \frac{1}{4} \left(\omega_0 \varepsilon_2(\omega_0) + \frac{\mathrm{d}(\omega\varepsilon_1)}{\mathrm{d}\omega} \right) \frac{\partial |\boldsymbol{E}_0|^2}{\partial t}. \tag{29}$$

Similarly we have

$$\frac{\partial \langle u_m \rangle}{\partial t} = \left\langle \frac{\partial u_m}{\partial t} \right\rangle = \frac{1}{4} \left(\omega_0 \mu_2(\omega_0) + \frac{\mathrm{d}(\omega\mu_1)}{\mathrm{d}\omega} \right) \frac{\partial |\boldsymbol{H}_0|^2}{\partial t}. \tag{30}$$

When the matter is modeled by harmonic oscillators, μ doesn't undergo any correction, but let's work with a slightly generalized case. Eventually we have

$$\langle u \rangle = \langle u_e + u_m \rangle = \frac{1}{4} \left(\omega_0 \varepsilon_2(\omega_0) + \frac{\mathrm{d}(\omega\varepsilon_1)}{\mathrm{d}\omega} \right) |\boldsymbol{E}_0|^2 + \frac{1}{4} \left(\omega_0 \mu_2(\omega_0) + \frac{\mathrm{d}(\omega\mu_1)}{\mathrm{d}\omega} \right) |\boldsymbol{H}_0|^2. \tag{31}$$

In the case of the Lorentz oscillator, μ is real, and we have

$$\langle u_m \rangle = \frac{1}{4} \mu_0 |\boldsymbol{H}_0|^2 = \frac{1}{4} \mu_0 \cdot \frac{|\varepsilon|}{\mu_0} |\boldsymbol{E}_0|^2 = \frac{1}{4} |\varepsilon| |\boldsymbol{E}_0|^2. \tag{32}$$

The evaluation of the time averaged Poynting vector is more straightforward: since

$$\boldsymbol{\nabla} \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t} \Rightarrow \mathrm{i}\boldsymbol{k} \times \boldsymbol{E} = -(-\mathrm{i}\omega)\boldsymbol{B}, \tag{33}$$

[1] Note that $\epsilon(\omega) = \epsilon(-\omega)^*$ comes from the fact that ϵ is real in the time domain; it says nothing about whether the system is Hermitian; the Hermitian condition is $\epsilon(\omega) = \epsilon(\omega)^*$.

we just have

$$\begin{aligned}\langle \boldsymbol{S}\rangle &= \frac{1}{\mu}\langle \boldsymbol{E}\times\boldsymbol{B}\rangle \\ &= \frac{1}{\mu}\cdot\frac{1}{4}\operatorname{Re}\left(\boldsymbol{E}_0^*\times\boldsymbol{B}_0+\boldsymbol{E}_0\times\boldsymbol{B}_0^*\right) \\ &= \frac{1}{2\mu}\frac{\operatorname{Re}\boldsymbol{k}}{\omega}|\boldsymbol{E}_0|^2 = \frac{1}{2}\operatorname{Re}\sqrt{\frac{\varepsilon}{\mu}}|\boldsymbol{E}_0|^2\hat{\boldsymbol{k}},\end{aligned} \tag{34}$$

where we have used the condition $\boldsymbol{k}\cdot\boldsymbol{E}=0$, and in the third line we have used the condition that the directions of the real and imaginary parts of $\boldsymbol{k}$ are the same, and therefore from $\boldsymbol{k}\cdot\boldsymbol{E}_0=0$ we also have $\boldsymbol{k}^*\cdot\boldsymbol{E}=0$. The energy velocity is therefore

$$v_E = \frac{|\langle \boldsymbol{S}\rangle|}{\langle u_e+u_m\rangle} = \tag{35}$$

2.2 Optical Tweezers

2.2.1

2.3 A simple derivation of electrostrictive pressure

(a) The interaction energy between electromagnetic field and the matter is

$$U_{\text{int}} = -\boldsymbol{P}\cdot\boldsymbol{E} = -\epsilon_0(\epsilon_{\text{r}}-1)\boldsymbol{E}\cdot\boldsymbol{E}, \tag{36}$$

and therefore

$$\langle U_{\text{int}}\rangle = -\epsilon_0 \tag{37}$$

2.4 Fourier-Domain Treatment of the Wave Equation

(a) Separation of variables fails whenever formally carrying out the separation of variable procedure *doesn't* lead to good Sturm-Liouville eigenvalue problems. This happens in several cases. Maybe the structure of the equation is not good, e.g. when the problem is to find how the density change with a given but complicated velocity distribution:

$$\frac{\partial\rho}{\partial t}+\boldsymbol{\nabla}\cdot\rho\boldsymbol{v}=0, \tag{38}$$

from which separation of variable simply can't proceed. Another case is when the boundary condition is very complicated or simply has a weird geometrical shape.
(b) From the 1D wave equation

$$\partial_x^2E(x,t)-\frac{\varepsilon}{c^2}\partial_t^2E(x,t)=0, \tag{39}$$

and the ansatz

$$E(x,t)=\bar{E}(x)f(t),\quad \bar{E}(x)=\int\mathrm{d}k\,\tilde{E}(k)\mathrm{e}^{\mathrm{i}kx},\quad f(t)=\int\mathrm{d}\omega\,\tilde{f}(\omega)\mathrm{e}^{-\mathrm{i}\omega t} \tag{40}$$

we get

$$\int\mathrm{d}\omega\int\mathrm{d}k\left((-\mathrm{i}k)^2-\frac{\varepsilon}{c^2}(-\mathrm{i}\omega)^2\right)\mathrm{e}^{\mathrm{i}(kx-\omega t)}\tilde{f}(\omega)\tilde{E}(k)=0, \tag{41}$$

and therefore the equation in frequency domain is

$$\left(k^2-\frac{\varepsilon\omega^2}{c^2}\right)\tilde{f}(\omega)\tilde{E}(k)=0. \tag{42}$$

(c) The dispersion relation is

$$k=\pm\sqrt{\varepsilon}\frac{\omega}{c}. \tag{43}$$

Since ε is a constant, this means the equation has no dispersion. The solution of the equation therefore can be rewritten as (we here redefine $\tilde{E}$)

$$E(x,t)=\int\frac{\mathrm{d}\omega}{2\pi}\left(\tilde{E}_1(\omega)\mathrm{e}^{\mathrm{i}\left(\sqrt{\varepsilon}\frac{\omega}{c}x-\omega t\right)}+\tilde{E}_2(\omega)\mathrm{e}^{\mathrm{i}\left(-\sqrt{\varepsilon}\frac{\omega}{c}x-\omega t\right)}\right). \tag{44}$$

So there are two solutions corresponding to each ω; by linear recombination of the solutions, the two are left-going and right-going, correspondingly.

(d) We define the complex refraction index as

$$\tilde{n} = \sqrt{\varepsilon} = n + \mathrm{i}\kappa, \tag{45}$$

and

$$k = \frac{\omega}{c}(n + \mathrm{i}\kappa). \tag{46}$$

If we let k be complex, we have

$$\begin{aligned} E(x,t) &= E_0 \mathrm{e}^{\mathrm{i}(kx - \omega t)} + \text{c.c.} \\ &= E_0 \mathrm{e}^{-\kappa \frac{\omega}{c} x} \mathrm{e}^{\mathrm{i}\left(n \frac{\omega}{c} x - \omega t\right)} + \text{c.c.}, \end{aligned} \tag{47}$$

and the decay length is

$$l = \frac{1}{\kappa \frac{\omega}{c}} = \frac{c}{\omega \kappa}. \tag{48}$$

On the other hand, if we let ω be complex, we have

$$\omega = \frac{ck}{n} = \frac{ck}{n^2 + \kappa^2}(n - \mathrm{i}\kappa), \tag{49}$$

and the wave looks like

$$E(x,t) = E_0 \mathrm{e}^{-kc \frac{\kappa}{n^2+\kappa^2} t} \mathrm{e}^{\mathrm{i}\left(kx - kc \frac{n}{n^2+\kappa^2} t\right)} + \text{c.c.} \tag{50}$$

This solution can be achieved by injecting a plane wave into the medium and removing the pumping source, letting the wave "cooling down" within the medium, while the first solution (47), where k is complex, describes the field configuration where at the boundary of the medium the field strength is fixed, probably because of a strong and stable source outside. Now the time scale of damping is

$$\tau = \frac{n^2 + \kappa^2}{\kappa} \frac{1}{kc}, \tag{51}$$

and in this "homogeneously cooling down" case, the speed of the light becomes

$$v = \frac{n}{n^2 + \kappa^2} c, \tag{52}$$

not just the c/n in (47). Now the decay length can be found from τ as

$$l = v\tau = \frac{n}{\kappa k}. \tag{53}$$

If we make identification $c/n = \omega/k$, we find (48) and (53) are the same.

But I have no idea why we don't make identification $\omega = kcn/(n^2 + \kappa^2)$

2.4.1 Non-instantaneous material response

The most general relation between D and E is

$$D(x,t) = \varepsilon_0 \int_{-\infty}^{\infty} \mathrm{d}t' \int_{-\infty}^{\infty} \mathrm{d}x' \, \epsilon_{\mathrm{r}}(x - x', t - t') E(x', t'). \tag{54}$$

(a) When we have locality and causality, $\epsilon_{\mathrm{r}}(x - x', t - t')$ should be proportional to $\delta(x - x')$ and $\theta(t - t')$, and thus

$$D(x,t) = \varepsilon_0 \int_{-\infty}^{t} \mathrm{d}t' \, \epsilon_{\mathrm{r}}(t - t') E(x, t'), \tag{55}$$

$$\epsilon_{\mathrm{r}}(t - t') = \epsilon_{\mathrm{r}}(t - t')\theta(t - t'). \tag{56}$$

Here $\epsilon_{\mathrm{r}}(x - x', t - t')$ shouldn't have any space dependence other than $\delta(x - x')$ or it's not a function of $x - x'$.

(b) The 1D electro-magnetic wave function then is the consequence of the following equation systems in the 1D case:

$$\boldsymbol{\nabla}\times\boldsymbol{E}=-\frac{\partial\boldsymbol{B}}{\partial t},\quad \boldsymbol{\nabla}\times\boldsymbol{H}=\frac{\partial\boldsymbol{D}}{\partial t},\quad \boldsymbol{B}=\mu_0\boldsymbol{H},\quad \boldsymbol{D}(x,t)=\varepsilon_0\int_{-\infty}^{\infty}\mathrm{d}t'\,\epsilon_{\mathrm{r}}(t-t')\boldsymbol{E}(x,t'). \tag{57}$$

Here we have assumed that ϵ_{r} doesn't map E_x to D_y, etc., or otherwise the problem can't be restricted to the 1D case, and thus from $\boldsymbol{\nabla}\cdot\boldsymbol{D}=0$ we get $\boldsymbol{\nabla}\cdot\boldsymbol{E}=0$. Hence

$$-\nabla^2\boldsymbol{E}=\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\boldsymbol{E})=-\mu_0\frac{\partial}{\partial t}\boldsymbol{\nabla}\times\boldsymbol{H}=-\mu_0\frac{\partial^2}{\partial t^2}\boldsymbol{D}, \tag{58}$$

and therefore the final wave equation is

$$\partial_x^2E(x,t)=\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\int_{-\infty}^{\infty}\mathrm{d}t'\,\epsilon_{\mathrm{r}}(t-t')E(x,t'). \tag{59}$$

(c) We do Fourier transform

$$E(x,t)=\int\frac{\mathrm{d}\omega\,\mathrm{d}k}{(2\pi)^2}\mathrm{e}^{\mathrm{i}(kx-\omega t)}\tilde{E}(k,\omega),\quad \varepsilon(t-t')=\int\frac{\mathrm{d}\omega}{2\pi}\mathrm{e}^{-\mathrm{i}\omega t}\tilde{\varepsilon}_{\mathrm{r}}(\omega), \tag{60}$$

and LHS of (59) becomes

$$\int\frac{\mathrm{d}\omega\,\mathrm{d}k}{(2\pi)^2}(\mathrm{i}k)^2\mathrm{e}^{\mathrm{i}(kx-\omega t)}\tilde{E}(k,\omega), \tag{61}$$

and the RHS becomes

$$\begin{aligned}
&\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\int\mathrm{d}t'\int\frac{\mathrm{d}\omega'}{2\pi}\int\frac{\mathrm{d}\omega\,\mathrm{d}k}{(2\pi)^2}\tilde{\varepsilon}_{\mathrm{r}}(\omega')\mathrm{e}^{-\mathrm{i}\omega'(t-t')}\tilde{E}(k,\omega)\mathrm{e}^{\mathrm{i}(kx-\omega t')}\\
=&\frac{1}{c^2}\int\frac{\mathrm{d}\omega'}{2\pi}\frac{\mathrm{d}\omega}{2\pi}\frac{\mathrm{d}k}{2\pi}(-\mathrm{i}\omega)^2\tilde{\varepsilon}_{\mathrm{r}}(\omega')\tilde{E}(k,\omega)\mathrm{e}^{\mathrm{i}kx}\int\mathrm{d}t'\,\mathrm{e}^{\mathrm{i}(\omega'-\omega)t'}\\
=&\frac{1}{c^2}\int\frac{\mathrm{d}\omega'}{2\pi}\frac{\mathrm{d}\omega}{2\pi}\frac{\mathrm{d}k}{2\pi}(-\mathrm{i}\omega)^2\tilde{\varepsilon}_{\mathrm{r}}(\omega')\tilde{E}(k,\omega)\mathrm{e}^{\mathrm{i}kx-\mathrm{i}\omega' t}\cdot 2\pi\delta(\omega-\omega')\\
=&\frac{1}{c^2}\int\frac{\mathrm{d}\omega\,\mathrm{d}k}{(2\pi)^2}\mathrm{e}^{\mathrm{i}kx-\mathrm{i}\omega t}(-\mathrm{i}\omega)^2\tilde{\varepsilon}_{\mathrm{r}}(\omega)\tilde{E}(k,\omega).
\end{aligned} \tag{62}$$

The final equation therefore becomes

$$\left(k^2-\tilde{\varepsilon}_{\mathrm{r}}(\omega)\frac{\omega^2}{c^2}\right)\tilde{E}(k,\omega)=0. \tag{63}$$

(d) The dispersion relation now is

$$\omega=\frac{c|k|}{\sqrt{\tilde{\varepsilon}_{\mathrm{r}}(\omega)}}. \tag{64}$$

2.4.2 Energy absorption

(a) In the specific case of

$$E(x,t)=\bar{E}(x)\mathrm{e}^{-\mathrm{i}\omega t}+\bar{E}^*(x)\mathrm{e}^{\mathrm{i}\omega t}, \tag{65}$$

we have

$$\begin{aligned}
D(x,t)&=\int\mathrm{d}t'\,\varepsilon(t-t')(\bar{E}(x)\mathrm{e}^{-\mathrm{i}\omega t'}+\bar{E}^*(x)\mathrm{e}^{\mathrm{i}\omega t'})\\
&=\bar{E}(x)\mathrm{e}^{-\mathrm{i}\omega t}\int\mathrm{d}t'\,\mathrm{e}^{\mathrm{i}\omega(t-t')}\varepsilon(t-t')+\bar{E}^*(x)\mathrm{e}^{\mathrm{i}\omega t}\int\mathrm{d}t'\,\mathrm{e}^{-\mathrm{i}\omega(t-t')}\varepsilon(t-t')\\
&=\bar{E}(x)\mathrm{e}^{-\mathrm{i}\omega t}\tilde{\varepsilon}(\omega)+\bar{E}^*(x)\mathrm{e}^{\mathrm{i}\omega t}\tilde{\varepsilon}(-\omega).
\end{aligned} \tag{66}$$

Since $\varepsilon(t)$ is real, we can rewrite

$$\tilde{\varepsilon}(-\omega)=\tilde{\varepsilon}(\omega)^*. \tag{67}$$

(b) Following the practice in (24), we have

$$\begin{aligned}\frac{\partial u_e}{\partial t} &= E \cdot \frac{\partial D}{\partial t} \\ &= (\bar{E}(x)\mathrm{e}^{-\mathrm{i}\omega t} + \bar{E}^*(x)\mathrm{e}^{\mathrm{i}\omega t}) \cdot ((-\mathrm{i}\omega)\bar{E}(x)\mathrm{e}^{-\mathrm{i}\omega t}\tilde{\varepsilon}(\omega) + (\mathrm{i}\omega)\bar{E}^*(x)\mathrm{e}^{\mathrm{i}\omega t}\tilde{\varepsilon}(-\omega)) \\ &= -2\omega\left|\bar{E}(\omega)\right|^2\varepsilon_2(\omega) + \bar{E}(x)^2(-\mathrm{i}\omega)\mathrm{e}^{-2\mathrm{i}\omega t}\tilde{\varepsilon}(\omega) + \text{c.c.}\end{aligned} \tag{68}$$

Here ε_1 and ε_2 are real and imaginary parts of ε, respectively. The sum of the second and third terms is a sine function; it oscillates and has zero average value. The first term gives the damping. Thus, the energy absorption rate is

$$\partial_t \langle u_e \rangle = -2\omega\left|\bar{E}(\omega)\right|^2\varepsilon_2(\omega). \tag{69}$$

(c) The energy absorbed is proportional to the imaginary part of the Fourier transform of the dielectric constant, which is expected since it's related to the imaginary part of the energy spectrum.