Homework 2

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1 Quasiparticle weight in Landau Fermi liquid

1.1 Quasiparticle weight in Na

From Table 2.3 in A&M, in Na, $m^*/m = 1.3$, so $Z = m/m^* = 0.77$. Here m^* is obtained by thermodynamic measurement: the Sommerfield model tells us

$$C_V = \frac{\pi^2}{2} \frac{k_\mathrm{B} T}{E_\mathrm{F}} n k_\mathrm{B}, \quad E_\mathrm{F} = \frac{\hbar^2}{2m} \left(\frac{3\pi^2 N}{V} \right)^{3/2}, \tag{1}$$

and therefore

$$\frac{C_{V,\text{measured}}}{C_{V,\text{free electron}}} = \frac{m^*}{m}. \tag{2}$$

1.2 Direct observation of occupation discontinuity

From [1], a direct measurement of the electron occupation tells us the experimental result for Na: $Z = 0.58$. This is much smaller than the value found in the thermodynamic measurement.

$$\begin{aligned} \frac{1}{\omega - \frac{\boldsymbol{p}^2}{2m} + \mu - \Sigma(\omega, \boldsymbol{p})} &= \frac{1}{\omega - \frac{\boldsymbol{p}^2}{2m} + \mu - \Sigma(\omega = 0, \boldsymbol{p}) - \frac{\partial \Sigma}{\partial \omega} \omega} \\ &= \frac{Z}{\omega - \frac{\boldsymbol{p}^2}{2m^*} + \mu^*}, \end{aligned} \tag{3}$$

where

$$Z := \frac{1}{1 - \frac{\partial \Sigma}{\partial \omega}} < 1, \quad \left(1 - \frac{\partial \Sigma}{\partial \omega} \right) \frac{1}{m^*} = \frac{1}{m} \Rightarrow \frac{m^*}{m} = \frac{1}{Z} > 1. \tag{4}$$

TODO: but $\Sigma(\omega = 0, \boldsymbol{p})$ can also have $\boldsymbol{p}$ dependence???

2 Exotic phenomena in a Landau Fermi liquid

2.1 Heavy fermion systems

Signatures of heavy fermion materials include a low-temperature heat capacity that is, say, 1000 times of the free electron heat capacity, low conductivity, and flat bands that come across the Fermi energy.

2.2 Zero sound

2.2.1 Zero sound in neutral Fermi liquid

Zero sound is a

TODO: zero sound and τ: the spectra of the zero sound and the first sound are of the same magnitude, but then what about the $\omega\tau \ll 1$ or $\gg 1$ conditions?

the interaction between particles has a short characteristic length scale in the real space, which means the corresponding form in the momentum space doesn't show strong dependence on the exchanged momentum $\boldsymbol{q}$. Thus, after renormalization, the interaction between $\delta n_{\boldsymbol{p}}(\boldsymbol{r})$ and $\delta n_{\boldsymbol{p}'}(\boldsymbol{r}')$ – essentially Wigner transforms of $c^\dagger_{\boldsymbol{p}+\boldsymbol{q}} c_{\boldsymbol{p}}$ – has no large differences with $f_{\boldsymbol{p}\boldsymbol{p}'} \delta n_{\boldsymbol{p}} \delta n_{\boldsymbol{p}'}$, because $\boldsymbol{q}$ dependence is not important when $\boldsymbol{q}$ is small (or in other words, when the $\boldsymbol{r}$-dependence of $\delta n_{\boldsymbol{p}}(\boldsymbol{r})$ is smooth enough, which is always required if we want a reasonable Boltzmann equation). This explains why when deriving the kinetic equation of Fermi liquid, we just insert $\delta n_{\boldsymbol{p}}(\boldsymbol{r})$ in place of $\delta n_{\boldsymbol{p}}$: this works only when the interaction is short-range [2]. The resulting kinetic equation system – a fermionic quantum Boltzmann equation plus the relation between $\varepsilon_{\boldsymbol{p}}$, $\varepsilon^0_{\boldsymbol{p}\sigma}$ and $f_{\boldsymbol{p}\boldsymbol{p}'}$ – is known as the **Landau equation**.

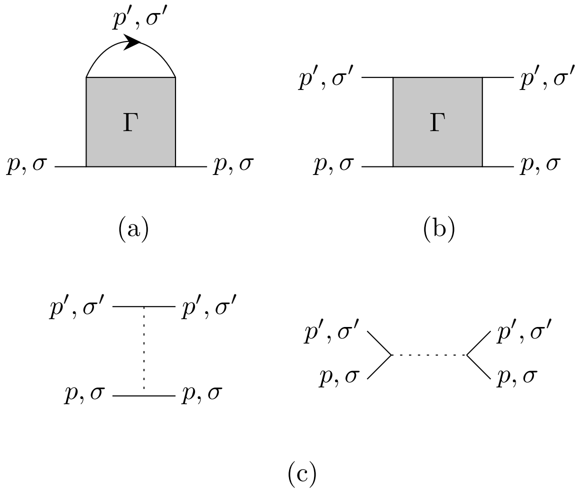

Figure 1: Skeleton diagrams that decide Landau parameters (a) The vertex function decides the single electron self-energy; in other words, f appears in ε. (b) The vertex function appears as the effective interaction channel; in other words, f appears in the energy functional i.e. the effective Hamiltonian. (c) Hartree-Fock approximation in the vertex function. The left one is the Hartree term ($\propto V(\boldsymbol{q}=0)$). The right one is the Fock term ($\propto V(\boldsymbol{q}=\boldsymbol{p}-\boldsymbol{p}')$).

2.2.2 Zero sound (now plasmon) is gapped in metals

It should be noted that kind of zero sound described by the original theory of Landau only exists in charge-neutral systems, such as ^{3}He; in electron systems, the zero sound is essentially the plasmon, which receives a gap ω_{p}. This originates from the long-range property of Coulomb interaction, which means at $\boldsymbol{q}=0$ we have a singularity, and this breaks the aforementioned condition that in the kinetic equation, the quasiparticle-interaction assumes no significant change when $\boldsymbol{q}$ is changed. Essentially, this means the $f_{\boldsymbol{p}\boldsymbol{p}'}$ function is also not well-defined. This can be explicitly checked by naively repeating the procedure to derive $f_{\boldsymbol{p}\boldsymbol{p}'}$ from the microscopic particle interaction potential: if we use the Hartree-Fock approximation to find $f_{\boldsymbol{p}\boldsymbol{p}'}$, we get

$$f(\boldsymbol{p},\boldsymbol{p}') = V(\boldsymbol{q}=0) - \frac{1}{2} V(|\boldsymbol{p}-\boldsymbol{p}'|)(1+\boldsymbol{\sigma}\boldsymbol{\sigma}'), \tag{5}$$

with the first term being the Hartree term and the second term being the Fock term (Fig. 1). This expression gives an infinite result for Coulomb interaction, because $V(\boldsymbol{q}) = 4\pi e^2/\boldsymbol{q}^2$ now diverges at $\boldsymbol{q}=0$ [3].[1]

One way to work around this singularity is to analyze the Hartree term in the real space, while still attributing other corrections in Fig. 1 to $f_{\boldsymbol{p}\boldsymbol{p}'}$. Thus, the kinetic equations for Fermi liquid in a metal now include three equations:

- A quantum Boltzmann equation coupled to a electrostatic field φ created by $\delta n_{\boldsymbol{p}\sigma}(\boldsymbol{r})$.
- The effective single-electron energy equation

$$\varepsilon_{\boldsymbol{p}\sigma}(\boldsymbol{r}) = \varepsilon^0_{\boldsymbol{p}\sigma} + \frac{1}{V}\sum_{\boldsymbol{p}',\sigma'} f_{\boldsymbol{p}\boldsymbol{p}'\sigma\sigma'}\,\delta n_{\boldsymbol{p}'\sigma'}(\boldsymbol{r}). \tag{6}$$

- The Poisson equation

$$\nabla^2\varphi = \frac{e}{\epsilon_0}\sum_{\boldsymbol{p},\sigma}\delta n_{\boldsymbol{p}\sigma}(\boldsymbol{r}). \tag{7}$$

This equation system is usually called **Landau-Silin equation**. The presence of the Poisson equation and the Hartree self-consistent field φ means when $\boldsymbol{q}\to 0$, we see plasma oscillations at frequency ω_{p}. The physical picture on the electron side is still the same: distortion of the

[1]Note that we *can't* correct the Hartree term with screened Coulomb potential! The $\delta n_{\boldsymbol{p}\sigma}(\boldsymbol{r})$ variable used in the Hartree term comes from the renormalized Green function, which already contains, say, the ring diagrams that may appear in the middle of a Coulomb interaction line. If we correct the Coulomb interaction line in the Hartree term, double counting occurs.

Fermi surface propagating around, creating a density mode in the momentum space instead of the real space.

3 Particle number conservation

3.1 Proof of particle number conservation

References

[1] Simo Huotari et al. "Momentum distribution and renormalization factor in sodium and the electron gas". In: *Physical review letters* 105.8 (2010), p. 086403.

[2] David Pines. *Theory of Quantum Liquids: Normal Fermi Liquids*. The "expansion of energy" subsection in Section 1.4 is about the short-range condition. The "charged v.s. neutral system" subsection in Section 3.3 compares plasmon with zero sound. CRC Press, 2018.

[3] VP Silin. "Theory of a degenerate electron liquid". In: *Soviet Physics Jetp-Ussr* 6.2 (1958), pp. 387–391.