Homework 1

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1

Problem

$$1+e^{y/x}-\frac{y}{x}e^{y/x}+e^{y/x}y'=0,\quad y(1)=-5. \tag{1}$$

Solution The equation is equivalent to

$$\mathrm{e}^{y/x}\,\mathrm{d}y+\left(1+\mathrm{e}^{y/x}-\frac{y}{x}\mathrm{e}^{y/x}\right)\mathrm{d}x=0,$$

and we have

$$\frac{\partial}{\partial x}\mathrm{e}^{y/x}=\frac{\partial}{\partial y}\left(1+\mathrm{e}^{y/x}-\frac{y}{x}\mathrm{e}^{y/x}\right)=-\frac{y}{x^2}\mathrm{e}^{y/x},$$

so the equation is exact. Suppose the solution is $\phi(x,y)=C$, and we have

$$\phi(x,y)=\int\mathrm{d}y\,\mathrm{e}^{y/x}=x\mathrm{e}^{y/x}+f(x),$$

and therefore

$$1+\mathrm{e}^{y/x}-\frac{y}{x}\mathrm{e}^{y/x}=\frac{\partial\phi}{\partial x}=\mathrm{e}^{y/x}+x\cdot\frac{-y}{x^2}\mathrm{e}^{y/x}+\frac{\partial f}{\partial x}\Rightarrow f(x)=x+\text{const}.$$

So the solution is

$$\phi(x,y)=x\mathrm{e}^{y/x}+x=C. \tag{2}$$

When $x=1$, $y=-5$, so we find $C=1+\mathrm{e}^{-5}$, and the final solution is

$$y=x\ln\left(\frac{1+\mathrm{e}^{-5}}{x}-1\right). \tag{3}$$

2

Problem

$$y'=\frac{1}{2x}y^2-\frac{1}{x}y-\frac{4}{x}. \tag{4}$$

Solution We have

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{y^2-2y-8}{2x},$$

and therefore

$$\int\frac{\mathrm{d}x}{2x}=\int\frac{\mathrm{d}y}{y^2-2y-8}=\frac{1}{6}\int\mathrm{d}y\left(\frac{1}{y-4}-\frac{1}{y+2}\right),$$

$$\ln x=\frac{1}{3}(\ln(y-4)-\ln(y+2))+C.$$

Solving out y, we get

$$y=-2+\frac{6}{1-Cx^3}. \tag{5}$$

3

Problem

$$y''(x)-6y'(x)+13y=-e^x,\quad y(0)=-1,\quad y'(0)=1. \tag{6}$$

Solution The solutions of the homogeneous equation is given by

$$y = \mathrm{e}^{\lambda x},$$

$$\lambda^2 - 6\lambda + 13 = 0 \Rightarrow \lambda = 3 \pm 2\mathrm{i},$$

and therefore a linear combination of the solutions corresponding to the two λ's gives

$$y_1 = \mathrm{e}^{3x}\cos(2x), \quad y_2 = \mathrm{e}^{3x}\sin(2x). \tag{7}$$

Now we need to find a particular solution. Using the ansatz $y = A\mathrm{e}^{x}$, we have

$$A - 6A + 13A = -1 \Rightarrow A = -\frac{1}{8}.$$

So the general solution is

$$y = A\mathrm{e}^{3x}\cos(2x) + B\mathrm{e}^{3x}\sin(2x) - \frac{1}{8}\mathrm{e}^{x}. \tag{8}$$

The initial conditions mean

$$A - \frac{1}{8} = -1, \quad 3A + 2B - \frac{1}{8} = 1 \Rightarrow A = -\frac{7}{8}, \quad B = \frac{15}{8},$$

so

$$y = -\frac{7}{8}\mathrm{e}^{3x}\cos(2x) + \frac{15}{8}\mathrm{e}^{3x}\sin(2x) - \frac{1}{8}\mathrm{e}^{x}. \tag{9}$$

4

Problem

$$y'' + 2y' - 3y = 0, \quad y(0) = 6, \quad y'(0) = -2. \tag{10}$$

Solution Following the same recipe above, we try to solve

$$\lambda^2 + 2\lambda - 3 = 0,$$

and we find $\lambda = 1, -3$, so

$$y = A\mathrm{e}^{x} + B\mathrm{e}^{-3x}, \tag{11}$$

and

$$A + B = 6, \quad A - 3B = -2 \Rightarrow A = 4, \quad B = 2.$$

So we get

$$y = 4\mathrm{e}^{x} + 2\mathrm{e}^{-3x}. \tag{12}$$

5

Problem

$$y'' - y' - 6y = 8e^{2x}. \tag{13}$$

Solution Again following the same procedure: the general solutions of the homogeneous equation is

$$y = A\mathrm{e}^{3x} + B\mathrm{e}^{-2x},$$

and taking the ansatz $y = A\mathrm{e}^{2x}$ to find a particular solution, we have

$$A - A - 6A = 8 \Rightarrow A = -\frac{4}{3},$$

so the general solution is

$$y = A\mathrm{e}^{3x} + B\mathrm{e}^{-2x} - \frac{4}{3}\mathrm{e}^{2x}. \tag{14}$$

6

Problem

$$x^2y'' + xy' - 4y = 0. \tag{15}$$

Solution This is a Euler equation. We use the ansatz $y = x^a$ to find a particular solution: we have

$$a(a-1) + a - 4 = 0 \Rightarrow a = \pm 2.$$

So the general solution is

$$y = Ax^2 + \frac{B}{x^2}. \tag{16}$$

7

Problem

$$y'' + xy' + xy = 0. \tag{17}$$

Solution Since the polynomials before y, y', y'' are all analytic, the general form of a solution is

$$y = \sum_{n=0}^{\infty} c_n x^n, \quad xy = \sum_{n=1}^{\infty} c_{n-1} x^n,$$

and therefore

$$y' = \sum_{n=1}^{\infty} c_n n x^{n-1}, \quad xy' = \sum_{n=1}^{\infty} c_n n x^n,$$

and

$$y'' = \sum_{n=2}^{\infty} c_n n(n-1) x^{n-2} = \sum_{n=0}^{\infty} c_{n+2}(n+2)(n+1) x^n.$$

So the equation becomes

$$\begin{aligned} 0 &= \sum_{n=0}^{\infty} c_{n+2}(n+2)(n+1)x^n + \sum_{n=1}^{\infty} c_n n x^n + \sum_{n=1}^{\infty} c_{n-1} x^n \\ &= c_2 + \sum_{n=1}^{\infty} x^n \left(c_{n+2}(n+2)(n+1) + c_n n + c_{n-1} \right) \end{aligned},$$

and we get

$$c_2 = 0, \quad (n+2)(n+1)c_{n+2} + nc_n + c_{n-1} = 0. \tag{18}$$

To pick up two particular solutions, we can set $c_0 = 1, c_1 = 0$, and vice versa. When $c_1 = 0, c_0 = 1$, we have

$$6c_3 + c_1 + c_0 = 0 \Rightarrow c_3 = -\frac{1}{6},$$

and

$$12c_4 + 2c_2 + c_1 = 0 \Rightarrow c_4 = 0.$$

Similarly we get

$$c_5 = \frac{1}{40}, \quad c_6 = \frac{1}{180}, \quad c_7 = -\frac{1}{336}.$$

So we get one solution

$$y_1 = 1 - \frac{1}{6}x^3 + \frac{1}{40}x^5 + \frac{1}{180}x^6 - \frac{1}{336}x^7 + \cdots. \tag{19}$$

When $c_0 = 0, c_1 = 1$, similarly we have

$$c_3 = -\frac{1}{6}, \quad c_4 = -\frac{1}{12}, \quad c_5 = \frac{1}{40}, \quad c_6 = \frac{1}{60},$$

so another solution is found:

$$y_2 = x - \frac{1}{6}x^3 - \frac{1}{12}x^4 + \frac{1}{40}x^5 + \frac{1}{60}x^6 + \cdots. \tag{20}$$

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Problem

$$4xy'' + 2y' + 2y = 0. \tag{21}$$

Solution We take the ansatz

$$y = \sum_{n=0}^{\infty} c_n x^{n+r},$$

and therefore

$$y' = \sum_{n=0}^{\infty} c_n (n+r) x^{n+r-1},$$

$$y'' = \sum_{n=0}^{\infty} c_n (n+r)(n+r-1) x^{n+r-2}.$$

The differential equation therefore becomes

$$\begin{aligned} 0 &= 2\sum_{n=0}^{\infty} c_n (n+r)(n+r-1) x^{n+r-1} + \sum_{n=0}^{\infty} c_n (n+r) x^{n+r-1} + \sum_{n=0}^{\infty} c_n x^{n+r} \\ &= (2r(r-1)+r) c_0 x^{r-1} + \sum_{n=1}^{\infty} \left((2(n+r)(n+r-1) + (n+r)) c_n + c_{n-1} \right) x^{n+r-1}, \end{aligned}$$

and therefore

$$2r(r-1) + r = 0 \Rightarrow r = 0, \frac{1}{2}.$$

Note that here we can't just let $c_0 = 0$, or otherwise because $c_n \propto c_{n-1}$, we just get a trivial solution $y = 0$.

When $r = 0$, we have

$$(2n(n-1) + n) c_n + c_{n-1} = 0.$$

Since our goal is just to find *a* solution, without loss of generality we set $c_0 = 1$, and therefore

$$c_1 = -c_0 = -1, \quad c_2 = -\frac{1}{6} c_1 = \frac{1}{6}, \quad c_3 = -\frac{1}{15} c_2 = -\frac{1}{90}, \quad c_4 = -\frac{1}{28} c_3 = \frac{1}{2520}.$$

So one solutions is found:

$$y_1 = 1 - x + \frac{1}{6} x^2 - \frac{1}{90} x^3 + \frac{1}{2520} x^4 + \cdots. \tag{22}$$

When $r = 1/2$, we have

$$(2n^2 + n) c_n + c_{n-1} = 0,$$

and again we set $c_0 = 1$, and have

$$c_1 = -\frac{1}{3} c_0 = -\frac{1}{3}, \quad c_2 = -\frac{1}{10} c_1 = \frac{1}{30}, \quad c_3 = -\frac{1}{21} c_2 = -\frac{1}{630}, \quad c_4 = -\frac{1}{36} c_3 = \frac{1}{22680}.$$

So we find another solution:

$$y_2 = \sqrt{x} \left(1 - \frac{1}{3} x + \frac{1}{30} x^2 - \frac{1}{630} x^3 + \frac{1}{22680} x^4 + \cdots \right). \tag{23}$$