Homework 1

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1 The energy velocity

Since ϵ_r has frequency dependence, the relation between $\boldsymbol{E}(t)$ and $\boldsymbol{D}(t)$ is not localized in the time domain, and therefore although we still know that the energy would be a quadratic form of $\boldsymbol{E}$ or $\boldsymbol{D}$, since

$$\frac{\partial \boldsymbol{E}}{\partial t} \cdot \boldsymbol{D} \neq \frac{\partial \boldsymbol{D}}{\partial t} \cdot \boldsymbol{E}, \tag{1}$$

the simple relation

$$u_e = \frac{1}{2} \boldsymbol{D} \cdot \boldsymbol{E}$$

no longer holds. Instead, we should start from the most generic theory and utilize

$$\frac{\partial u_e}{\partial t} = \boldsymbol{E} \cdot \frac{\partial \boldsymbol{D}}{\partial t}. \tag{2}$$

To use this equation to get an expression of u_e, we should no longer work with plane waves, or otherwise u_e is a constant and we don't see any change of u_e at all. Below we work with a wave packet centered at $\pm\omega_0$. For the wave packet, the electric field is

$$\boldsymbol{E}(t) = \mathrm{e}^{-\mathrm{i}\omega_0 t} \cdot \underbrace{\int \frac{\mathrm{d}\omega}{2\pi} \mathrm{e}^{-\mathrm{i}(\omega-\omega_0)t} \tilde{\boldsymbol{E}}(\omega)}_{=:\boldsymbol{E}_0(t)}, \tag{3}$$

$$\boldsymbol{D}(t) = \mathrm{e}^{-\mathrm{i}\omega_0 t} \cdot \int \frac{\mathrm{d}\omega}{2\pi} \mathrm{e}^{-\mathrm{i}(\omega-\omega_0)t} \varepsilon(\omega) \tilde{\boldsymbol{E}}(\omega). \tag{4}$$

By Taylor expansion of ε we have

$$\begin{aligned}
\partial \boldsymbol{D}/\partial t &= \mathrm{e}^{-\mathrm{i}\omega_0 t} \cdot \int \frac{\mathrm{d}\omega}{2\pi} \mathrm{e}^{-\mathrm{i}(\omega-\omega_0)t} \tilde{\boldsymbol{E}}(\omega)(-\mathrm{i}\omega) \left(\varepsilon(\omega_0) + (\omega-\omega_0) \left.\frac{\mathrm{d}\varepsilon}{\mathrm{d}\omega}\right|_{\omega=\omega_0} + \cdots \right) \\
&\approx \mathrm{e}^{-\mathrm{i}\omega_0 t} \cdot \int \frac{\mathrm{d}\omega}{2\pi} \mathrm{e}^{-\mathrm{i}(\omega-\omega_0)t} \tilde{\boldsymbol{E}}(\omega) \left(-\mathrm{i}\omega_0 \varepsilon(\omega_0) - \mathrm{i}(\omega-\omega_0)\varepsilon(\omega)_0 - \mathrm{i}(\omega-\omega_0)\omega \left.\frac{\mathrm{d}\varepsilon}{\mathrm{d}\omega}\right|_{\omega=\omega_0} \right) \\
&\approx \mathrm{e}^{-\mathrm{i}\omega_0 t} \cdot \int \frac{\mathrm{d}\omega}{2\pi} \mathrm{e}^{-\mathrm{i}(\omega-\omega_0)t} \tilde{\boldsymbol{E}}(\omega) \Bigg(-\mathrm{i}\omega_0 \varepsilon(\omega_0) \underbrace{-\mathrm{i}(\omega-\omega_0)\varepsilon(\omega)_0 - \mathrm{i}(\omega-\omega_0)\omega_0 \left.\frac{\mathrm{d}\varepsilon}{\mathrm{d}\omega}\right|_{\omega=\omega_0}}_{=-\mathrm{i}(\omega-\omega_0) \left.\frac{\mathrm{d}(\omega\varepsilon)}{\mathrm{d}\omega}\right|_{\omega=\omega_0}} \Bigg) \\
&= \mathrm{e}^{-\mathrm{i}\omega_0 t} \underbrace{\left(-\mathrm{i}\omega_0 \varepsilon(\omega_0) \boldsymbol{E}_0(t) + \frac{\mathrm{d}(\omega\varepsilon)}{\mathrm{d}\omega} \frac{\partial \boldsymbol{E}_0}{\partial t} \right)}_{=:\boldsymbol{D}_0(t)}.
\end{aligned} \tag{5}$$

In the second line we throw away the higher order Taylor terms; in the third line we only keep terms linear to $(\omega-\omega_0)$. These approximations require the wave packet to be focused enough. We use $\langle \cdots \rangle$ to refer to averaging over the fast oscillations; thus, $\boldsymbol{E}_0(t)$ and $\boldsymbol{D}_0(t)$ above can be regarded as constants when applying $\langle \cdots \rangle$, and hence we find

$$\begin{aligned}
\left\langle \frac{\partial u_e}{\partial t} \right\rangle &= \left\langle \boldsymbol{E} \cdot \frac{\partial \boldsymbol{D}}{\partial t} \right\rangle = \frac{1}{2} \cdot \frac{1}{4} \operatorname{Re}(\boldsymbol{D}_0^*(t) \cdot \boldsymbol{E}_0(t) + \boldsymbol{D}_0(t) \cdot \boldsymbol{E}_0^*(t)) \\
&\approx \frac{1}{4} \operatorname{Re} \left(\left(\omega_0 \varepsilon_2(\omega_0) + \frac{\mathrm{d}(\omega\varepsilon_1)}{\mathrm{d}\omega} \right) \frac{\partial \boldsymbol{E}_0^*}{\partial t} \cdot \boldsymbol{E}_0 + \left(\omega_0 \varepsilon_2(\omega_0) + \frac{\mathrm{d}(\omega\varepsilon_1)}{\mathrm{d}\omega} \right) \boldsymbol{E}_0^* \cdot \frac{\partial \boldsymbol{E}_0}{\partial t} \right) \\
&= \frac{1}{4} \left(\omega_0 \varepsilon_2(\omega_0) + \frac{\mathrm{d}(\omega\varepsilon_1)}{\mathrm{d}\omega} \right) \frac{\partial |\boldsymbol{E}_0|^2}{\partial t}.
\end{aligned} \tag{6}$$

In the second line we have considered both the real and imaginary parts of ϵ.[1] Since u_e contains no fast oscillation, we have

$$\frac{\partial \langle u_e \rangle}{\partial t} = \left\langle \frac{\partial u_e}{\partial t} \right\rangle = \frac{1}{4} \left(\omega_0 \varepsilon_2(\omega_0) + \frac{\mathrm{d}(\omega \varepsilon_1)}{\mathrm{d}\omega} \right) \frac{\partial |\boldsymbol{E}_0|^2}{\partial t}. \tag{7}$$

Similarly we have

$$\frac{\partial \langle u_m \rangle}{\partial t} = \left\langle \frac{\partial u_m}{\partial t} \right\rangle = \frac{1}{4} \left(\omega_0 \mu_2(\omega_0) + \frac{\mathrm{d}(\omega \mu_1)}{\mathrm{d}\omega} \right) \frac{\partial |\boldsymbol{H}_0|^2}{\partial t}. \tag{8}$$

When the matter is modeled by harmonic oscillators, μ doesn't undergo any correction, but let's work with a slightly generalized case. Eventually we have

$$\langle u \rangle = \langle u_e + u_m \rangle = \frac{1}{4} \left(\omega_0 \varepsilon_2(\omega_0) + \frac{\mathrm{d}(\omega \varepsilon_1)}{\mathrm{d}\omega} \right) |\boldsymbol{E}_0|^2 + \frac{1}{4} \left(\omega_0 \mu_2(\omega_0) + \frac{\mathrm{d}(\omega \mu_1)}{\mathrm{d}\omega} \right) |\boldsymbol{H}_0|^2. \tag{9}$$

In the case of the Lorentz oscillator, μ is real, and we have

$$\langle u_m \rangle = \frac{1}{4} \mu_0 |\boldsymbol{H}_0|^2 = \frac{1}{4} \mu_0 \cdot \frac{|\varepsilon|}{\mu_0} |\boldsymbol{E}_0|^2 = \frac{1}{4} |\varepsilon| |\boldsymbol{E}_0|^2. \tag{10}$$

The evaluation of the time averaged Poynting vector is more straightforward: since

$$\boldsymbol{\nabla} \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t} \Rightarrow \mathrm{i}\boldsymbol{k} \times \boldsymbol{E} = -(-\mathrm{i}\omega)\boldsymbol{B}, \tag{11}$$

we just have

$$\begin{aligned} \langle \boldsymbol{S} \rangle &= \frac{1}{\mu} \langle \boldsymbol{E} \times \boldsymbol{B} \rangle \\ &= \frac{1}{\mu} \cdot \frac{1}{4} \operatorname{Re} (\boldsymbol{E}_0^* \times \boldsymbol{B}_0 + \boldsymbol{E}_0 \times \boldsymbol{B}_0^*) \\ &= \frac{1}{2\mu} \frac{\operatorname{Re} \boldsymbol{k}}{\omega} |\boldsymbol{E}_0|^2 = \frac{1}{2} \operatorname{Re} \sqrt{\frac{\varepsilon}{\mu}} |\boldsymbol{E}_0|^2 \hat{\boldsymbol{k}}, \end{aligned} \tag{12}$$

where we have used the condition $\boldsymbol{k} \cdot \boldsymbol{E} = 0$, and in the third line we have used the condition that the directions of the real and imaginary parts of $\boldsymbol{k}$ are the same, and therefore from $\boldsymbol{k} \cdot \boldsymbol{E}_0 = 0$ we also have $\boldsymbol{k}^* \cdot \boldsymbol{E} = 0$. The energy velocity is therefore

$$v_E = \frac{|\langle \boldsymbol{S} \rangle|}{\langle u_e + u_m \rangle} = \tag{13}$$

[1] Note that $\epsilon(\omega) = \epsilon(-\omega)^*$ comes from the fact that ϵ is real in the time domain; it says nothing about whether the system is Hermitian; the Hermitian condition is $\epsilon(\omega) = \epsilon(\omega)^*$.