Homework 3

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November 5, 2022

Problem 1
Solution

(a) The conjugate momentum of θ is

$$p = \frac{\partial L}{\partial \dot{\theta}} = V \left(\frac{\dot{\theta}}{U_0} - \frac{\mu}{U_0} \right), \tag{1}$$

and therefore

$$\dot{\theta} = \frac{U_0}{V} p + \mu. \tag{2}$$

The Hamiltonian is

$$\begin{aligned} H &= p\dot{\theta} - L \\ &= p \left(\frac{U_0}{V} p + \mu \right) - V \left(\frac{1}{2U_0} \left(\frac{U_0}{V} p + \mu \right)^2 - \frac{\mu}{U_0} \left(\frac{U_0}{V} p + \mu \right) \right) \\ &= \frac{1}{2} \frac{U_0}{V} \left(p + \frac{\mu V}{U_0} \right)^2 . \end{aligned} \tag{3}$$

In Heisenberg's picture, the variance of θ can be evaluated in the follows. We know

$$\begin{aligned} \frac{\mathrm{d}\theta^2}{\mathrm{d}t} &= \frac{1}{\mathrm{i}} [\theta^2, H] \\ &= \frac{U_0}{2\mathrm{i}V} \left[\theta^2, \left(p + \frac{\mu V}{U_0} \right)^2 \right] \\ &= \frac{U_0}{V} \left(\theta \left(p + \frac{\mu V}{U_0} \right) + \left(p + \frac{\mu V}{U_0} \right) \theta \right), \end{aligned}$$

and therefore

$$\begin{aligned} \frac{\mathrm{d}^2\theta^2}{\mathrm{d}t^2} &= \frac{U_0}{V} \left(\dot{\theta} \left(p + \frac{\mu V}{U_0} \right) + \theta \dot{p} + \dot{p} \theta + \left(p + \frac{\mu V}{U_0} \right) \dot{\theta} \right) \\ &= \frac{2U_0^2}{V^2} \left(p + \frac{\mu V}{U_0} \right)^2 . \end{aligned} \tag{4}$$

Here we use the EOMs

$$\dot{\theta} = \frac{1}{\mathrm{i}} [\theta, H] = \frac{U_0}{V} \left(p + \frac{\mu V}{U_0} \right), \quad \dot{p} = 0. \tag{5}$$

From (4), we have

$$\frac{\mathrm{d}^2 \sigma_\theta^2}{\mathrm{d}t^2} = \frac{\mathrm{d}^2}{\mathrm{d}t^2} \left(\langle \theta^2 \rangle - \langle \theta \rangle^2 \right) = \frac{4U_0}{V} E - \frac{2U_0}{V} \langle \theta \rangle \left(\langle p \rangle + \frac{\mu V}{U_0} \right). \tag{6}$$

Assuming the wave packet doesn't move, we have

$$\frac{\mathrm{d}^2 \sigma_\theta^2}{\mathrm{d}t^2} = \frac{4U_0}{V} E, \tag{7}$$

and therefore

$$\sigma_\theta = \sqrt{\frac{2U_0}{V} E t^2 + \sigma_\theta^2(0)}. \tag{8}$$

The speed sound is

$$v = \sqrt{\frac{\rho_0 U_0}{m}}, \tag{9}$$

so

$$\sigma_\theta = \sqrt{\frac{2U_0}{V} E t^2 + \sigma^2} = \sqrt{\frac{2Emv^2t^2}{\rho_0 V} + \sigma^2}, \tag{10}$$

and the time it takes to have $\sigma_\theta = 2\pi$ is

$$t = \sqrt{\frac{\rho_0 V}{2Emv^2}(4\pi^2 - \sigma^2)}. \tag{11}$$

(b)

Problem 2
Solution

(a) Repeating the procedure used in ordinary superfluid, we do the decomposition

$$\varphi = \sqrt{\rho} \mathrm{e}^{\mathrm{i}\theta} = \sqrt{\rho_0 + \delta\rho} \mathrm{e}^{\mathrm{i}\theta}, \tag{12}$$

and therefore

$$-\frac{\varphi^* \nabla^2 \varphi}{2m} = \frac{\rho}{2m}(\boldsymbol{\nabla}\theta)^2 + \frac{(\boldsymbol{\nabla}\rho)^2}{8\rho m}, \tag{13}$$

$$\varphi^* \partial_\tau \varphi = \underbrace{\frac{1}{2}\partial_\tau \rho}_{\text{time derivative, ignored}} + \mathrm{i}\rho\partial_\tau\theta, \tag{14}$$

$$|\varphi(\boldsymbol{x})| U(\boldsymbol{x}-\boldsymbol{y}) |\varphi(\boldsymbol{y})| = \rho(\boldsymbol{x}) U(\boldsymbol{x}-\boldsymbol{y}) \rho(\boldsymbol{y}), \tag{15}$$

the theory is now

$$S = \int \mathrm{d}\tau \left(\int \mathrm{d}^d\boldsymbol{x} \left(\mathrm{i}\rho\partial_\tau\theta + \frac{\rho}{2m}(\boldsymbol{\nabla}\theta)^2 + \frac{(\boldsymbol{\nabla}\rho)^2}{8\rho m} - \mu\rho \right) + \frac{1}{2}\int \mathrm{d}^d\boldsymbol{x} \int \mathrm{d}^d\boldsymbol{y}\, \rho(\boldsymbol{x}) U(\boldsymbol{x}-\boldsymbol{y}) \rho(\boldsymbol{y}) \right). \tag{16}$$

Around the ground state, we have (note that since we are around a saddle point, the sum of all terms containing $\delta\rho$ only is always zero; the resulting theory has the form of $c_1\, \delta\rho\, \partial_\tau\theta + c_2\, \delta\rho^2$; the chemical potential term is therefore missing in the theory around the saddle point)

$$\mathrm{i}\rho\partial_\tau\theta = \underbrace{\mathrm{i}\rho_0\partial_\tau\theta}_{\text{time derivative}} + \mathrm{i}\, \delta\rho\, \partial_\tau\theta,$$

and since $\boldsymbol{\nabla}\rho = \boldsymbol{\nabla}\, \delta\rho$, we have

$$\frac{(\boldsymbol{\nabla}\rho)^2}{8\rho m} \approx \frac{(\boldsymbol{\nabla}\, \delta\rho)^2}{8\rho_0 m},$$

ignoring the fluctuation of the ρ in the denominator. Similarly, since we are working on a low energy theory, the fluctuation of θ shouldn't be large, and we have

$$\frac{\rho}{2m}(\boldsymbol{\nabla}\theta)^2 \approx \frac{\rho_0}{2m}(\boldsymbol{\nabla}\theta)^2.$$

The theory is then

$$S = \int \mathrm{d}^{d+1}x \left(\frac{\rho_0}{2m}(\boldsymbol{\nabla}\theta)^2 + \mathrm{i}\, \delta\rho\, \partial_\tau\theta + \frac{(\boldsymbol{\nabla}\, \delta\rho)^2}{8\rho_0 m} + \frac{1}{2}\, \delta\rho\, (\boldsymbol{x}) \int \mathrm{d}^d\boldsymbol{y}\, U(\boldsymbol{x}-\boldsymbol{y})\, \delta\rho\, (\boldsymbol{y}) \right) + S_{\text{saddle}}. \tag{17}$$

Integrating out $\delta\rho$, we get

$$\begin{aligned} S_{\text{eff}} &= \int \mathrm{d}^{d+1}x\, \frac{\rho_0}{2m}(\boldsymbol{\nabla}\theta)^2 - \frac{1}{2}\int \mathrm{d}\tau \int \mathrm{d}^d\boldsymbol{x}\, \mathrm{d}^d\boldsymbol{y}\, \mathrm{i}\partial_\tau\theta(\boldsymbol{x},\tau) \frac{1}{\int \mathrm{d}^d\boldsymbol{y}\, U(\boldsymbol{x}-\boldsymbol{y}) - \frac{1}{4\rho_0 m}\nabla^2} \mathrm{i}\partial_\tau\theta(\boldsymbol{y},\tau) \\ &= \int \mathrm{d}^{d+1}x\, \frac{\rho_0}{2m}(\boldsymbol{\nabla}\theta)^2 + \frac{1}{2}\int \mathrm{d}\tau \int \mathrm{d}^d\boldsymbol{x}\, \mathrm{d}^d\boldsymbol{y}\, \partial_\tau\theta(\boldsymbol{x},\tau) G(\boldsymbol{x}-\boldsymbol{y}) \partial_\tau\theta(\boldsymbol{y}), \end{aligned} \tag{18}$$

where

$$\int \mathrm{d}^d \boldsymbol{y}\, U(\boldsymbol{x}-\boldsymbol{y}) G(\boldsymbol{y}-\boldsymbol{z}) - \frac{1}{4\rho_0 m}\nabla_{\boldsymbol{x}}^2 G(\boldsymbol{x}-\boldsymbol{z}) = \delta(\boldsymbol{x}-\boldsymbol{z}). \tag{19}$$

Similar to the procedure in dealing with ordinary superfluid, since we are only interested in the long wave length behaviors of θ, the ∇^2 term can be thrown away, and we have

$$\begin{aligned}\int \frac{\mathrm{d}^d \boldsymbol{p}}{(2\pi)^d} \mathrm{e}^{\mathrm{i}\boldsymbol{p}\cdot(\boldsymbol{x}-\boldsymbol{z})} &= \int \frac{\mathrm{d}^d \boldsymbol{p}}{(2\pi)^d} \int \mathrm{d}^d \boldsymbol{y}\, U(\boldsymbol{x}-\boldsymbol{y}) G(\boldsymbol{p}) \mathrm{e}^{\mathrm{i}\boldsymbol{p}\cdot(\boldsymbol{y}-\boldsymbol{z})} \\ &= \int \frac{\mathrm{d}^d \boldsymbol{p}}{(2\pi)^d} \int \mathrm{d}^d \boldsymbol{r}\, U(\boldsymbol{r}) \mathrm{e}^{-\mathrm{i}\boldsymbol{p}\cdot\boldsymbol{r}} G(\boldsymbol{p}) \mathrm{e}^{\mathrm{i}\boldsymbol{p}\cdot(\boldsymbol{x}-\boldsymbol{z})} \quad (\boldsymbol{r} = \boldsymbol{x}-\boldsymbol{y}),\end{aligned}$$

so

$$G(\boldsymbol{r}) = \int \frac{\mathrm{d}^d \boldsymbol{p}}{(2\pi)^d} \mathrm{e}^{\mathrm{i}\boldsymbol{p}\cdot\boldsymbol{r}} G(\boldsymbol{p}), \quad G(\boldsymbol{p}) = \frac{1}{U(\boldsymbol{p})} = \frac{1}{\int \mathrm{d}^d \boldsymbol{r}\, U(\boldsymbol{r}) \mathrm{e}^{-\mathrm{i}\boldsymbol{p}\cdot\boldsymbol{r}}}. \tag{20}$$

To evaluate $G(\boldsymbol{p})$, we need to find

$$U(\boldsymbol{p}) = \int_0^\infty \mathrm{d}r\, \frac{2\pi^{d/2}}{\Gamma(d/2)} r^{d-1} \frac{U_0}{r^{d-\epsilon}} \tag{21}$$

Problem 3
Solution

(a) The energy now can be exactly evaluated (N is the number of sites):

$$E = \frac{UN}{2}(M^2 - M) - \mu N M = \frac{N}{2}(UM^2 - (U+2\mu)M). \tag{22}$$

At the ground state, E is minimized. If M were continuous, we would have

$$M = \frac{U+2\mu}{2U} = \frac{1}{2} + \frac{\mu}{U}, \tag{23}$$

but it's not. So we need to find the closest integer to (23). Note that since

$$\frac{1}{2} \le \frac{1}{2} + \frac{\mu}{U} - \left\lfloor \frac{\mu}{U} \right\rfloor < \frac{3}{2},$$

the following M is always a minimum point:

$$M = \left\lfloor \frac{\mu}{U} \right\rfloor + 1. \tag{24}$$

When μ/U is an integer, both

$$M = \frac{\mu}{U} \tag{25}$$

and

$$M = \left\lfloor \frac{\mu}{U} \right\rfloor + 1 = \frac{\mu}{U} + 1 \tag{26}$$

can be found in ground states.

The energy gap is

$$\begin{aligned}\Delta E &= E|_{n_i = M+1 \text{ on one site}} - E|_M \\ &= \frac{U}{2}((M+1)^2 - (M+1)) - \mu(M+1) - \frac{U}{2}(M^2 - M) + \mu M \\ &= UM - \mu = \begin{cases} 0 \text{ or } U, & \mu/U \text{ integer}, \\ U, & \text{otherwise}. \end{cases}\end{aligned} \tag{27}$$

Since when μ changes we observe discontinuous change of M, we may say when μ/U is an integer, a phrase transition happens, so we pick $\Delta E = 0$ when μ/U is an integer and get

$$\frac{\Delta E}{U} = M - \frac{\mu}{U} = \begin{cases} 0, & \mu/U \text{ integer}, \\ 1, & \text{otherwise}. \end{cases} \tag{28}$$

The energy gap and the phase diagram are shown in Figure 1.

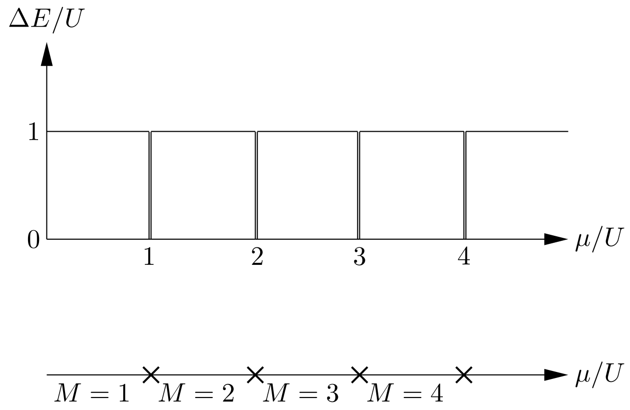

Figure 1: Phase diagram when $t = 0$

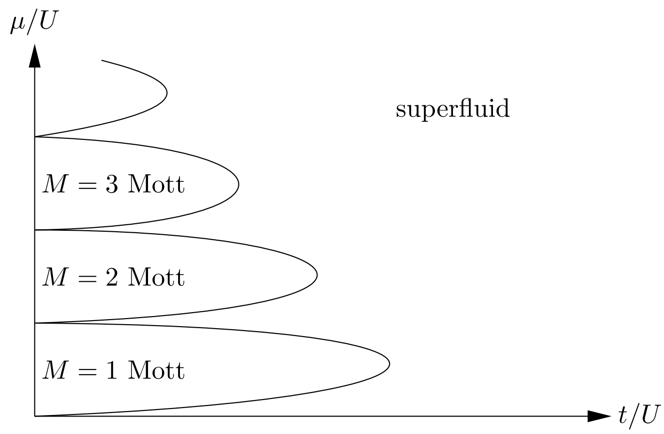

Figure 2: Schematic phase diagram of the boson Hubbard model

(b) The gapless points in Figure 1 can only be connected to the superfluid phase, and therefore we get Figure 2.

(c) We have

$$\langle n_0 + k'|a|n_0 + k\rangle = \sqrt{n_0 + k}\,\langle n_0 + k'|n_0 + k - 1\rangle = \sqrt{n_0 + k}\delta_{k',k-1}, \tag{29}$$

and

$$\langle k'|\mathrm{e}^{-\mathrm{i}\theta}|k\rangle = \int_0^{2\pi} \frac{\mathrm{d}\theta}{2\pi}\mathrm{e}^{-\mathrm{i}k'\theta}\mathrm{e}^{-\mathrm{i}\theta}\mathrm{e}^{\mathrm{i}k\theta} = \delta_{k',k-1}. \tag{30}$$

When $k \ll n_0$, we have

$$\langle n_0 + k'|a|n_0 + k\rangle \approx \sqrt{n_0}\,\langle k'|\mathrm{e}^{-\mathrm{i}\theta}|k\rangle \Rightarrow a \approx \sqrt{n_0}\mathrm{e}^{-\mathrm{i}\theta}. \tag{31}$$

And similarly we have

$$\langle n_0 + k'|a^\dagger|n_0 + k\rangle = \sqrt{n_0 + k + 1}\delta_{k',k+1}, \tag{32}$$

and

$$\langle k'|\mathrm{e}^{\mathrm{i}\theta}|k\rangle = \int_0^{2\pi} \frac{\mathrm{d}\theta}{2\pi}\mathrm{e}^{-\mathrm{i}k\theta}\mathrm{e}^{\mathrm{i}\theta}\mathrm{e}^{\mathrm{i}k'\theta} = \delta_{k',k+1}, \tag{33}$$

and in the $k \ll n_0$ limit we have