Homework 3

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March 19, 2023

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1.1

In order to keep the matrix elements invariant, in the Heisenberg picture we have

$$O_H = \mathrm{e}^{\mathrm{i}Ht/\hbar} O_S \mathrm{e}^{-\mathrm{i}Ht/\hbar}, \tag{1}$$

so that

$${}_H \langle \psi' | O_H | \psi \rangle_H = {}_S \langle \psi' | O_S | \psi \rangle_S . \tag{2}$$

The differential time evolution equation is

$$\frac{\mathrm{d}O_H}{\mathrm{d}t} = \frac{\mathrm{i}}{\hbar} H O_H - O_H \cdot \frac{\mathrm{i}}{\hbar} H + \mathrm{e}^{\mathrm{i}Ht/\hbar} \frac{\partial O_S}{\partial t} \mathrm{e}^{-\mathrm{i}Ht/\hbar}, \tag{3}$$

and therefore

$$\begin{aligned} \frac{\mathrm{d}\langle O \rangle}{\mathrm{d}t} &= \frac{1}{\mathrm{i}\hbar} \langle \psi_H | [O_H, H] | \psi_H \rangle + \langle \psi_H | \mathrm{e}^{\mathrm{i}Ht/\hbar} \frac{\partial O_S}{\partial t} \mathrm{e}^{-\mathrm{i}Ht/\hbar} | \psi_H \rangle \\ &= \frac{1}{\mathrm{i}\hbar} \langle \psi_H | [O_H, H] | \psi_H \rangle + \langle \psi_S | \frac{\partial O_S}{\partial t} | \psi_S \rangle \\ &= \frac{1}{\mathrm{i}\hbar} \langle [O, H] \rangle + \left\langle \frac{\partial O}{\partial t} \right\rangle . \end{aligned} \tag{4}$$

Note: here we assume H has no time dependence, and thus the time evolution operator assumes the simple form $\mathrm{e}^{-\mathrm{i}Ht/\hbar}$. The condition that H has no time dependence also means in the Heisenberg picture, $H_H(t) = H_H(0)$, and on the other hand, $H_H(t)$ can be obtained from $H_H(0)$ by replacing the values of all operators at $t = 0$ to the corresponding values at t, and therefore the commutation relation $[O_H(t), H]$ can be obtained by replacing the occurrences of all operators in $[O_H(t = 0), H] = [O_S, H]$ with their values at t. That's why in the third line, we omit the Heisenberg/Schrodinger picture labels.

1.2

Since the Hamiltonian

$$H = \frac{p^2}{2m} + V \tag{5}$$

is real (and not just Hermitian), we can always obtain real eigenfunctions. Thus

$$\begin{aligned} \int \mathrm{d}x\, \psi^* x (-\mathrm{i}\hbar) \partial_x \psi = -\mathrm{i}\hbar\, \psi^* x \psi \Big|_{x=-\infty}^{\infty} + \mathrm{i}\hbar \int \mathrm{d}x\, \partial_x (\psi^* x) \psi \\ = \mathrm{i}\hbar \int \mathrm{d}x\, \partial_x (x\psi) \psi = \mathrm{i}\hbar \int \mathrm{d}x\, \psi^2 + \mathrm{i}\hbar \int \mathrm{d}x\, x (\partial_x \psi) \psi \\ = \mathrm{i}\hbar \int \mathrm{d}x\, |\psi|^2 - \int \mathrm{d}x\, \psi^* x (-\mathrm{i}\hbar \partial_x) \psi, \end{aligned}$$

and therefore

$$\begin{aligned} 2 \int \mathrm{d}x\, \psi^* x p \psi = \mathrm{i}\hbar, \\ \langle xp \rangle = \mathrm{i}\hbar/2. \end{aligned} \tag{6}$$

Therefore

$$0 = \frac{\mathrm{d}}{\mathrm{d}t} \langle xp \rangle = \frac{1}{\mathrm{i}\hbar} \langle [xp, H] \rangle .$$

The commutation relation can be evaluated as (note the correspondence between Poisson brackets and commutators)

$$\begin{aligned} [xp, H] &= \frac{1}{2m} [xp, p^2] + [xp, V(x)] \\ &= \frac{1}{2m} [x, p^2] \, p + x [p, V(x)] \\ &= \frac{1}{2m} \cdot 2\mathrm{i}\hbar p \cdot p - x \cdot \mathrm{i}\hbar \frac{\partial V(x)}{\partial x}, \end{aligned}$$

so we find

$$0 = \mathrm{i}\hbar \left\langle \frac{p^2}{m} \right\rangle - \mathrm{i}\hbar \left\langle x \frac{\partial V(x)}{\partial x} \right\rangle ,$$

and therefore

$$2 \langle T \rangle = \left\langle x \frac{\partial V(x)}{\partial x} \right\rangle . \tag{7}$$

1.3

1.4

When $V = -1/r$, we have

$$E = \langle V \rangle + \frac{1}{2} \left\langle r \frac{\partial V}{\partial r} \right\rangle = - \left\langle \frac{1}{r} \right\rangle + \frac{1}{2} \left\langle r \cdot \frac{1}{r^2} \right\rangle = - \frac{1}{2} \left\langle \frac{1}{r} \right\rangle < 0, \tag{8}$$

and therefore bound states are possible. When $V = -1/r^2$, however, we have

$$E = \langle V \rangle + \frac{1}{2} \left\langle r \frac{\partial V}{\partial r} \right\rangle = - \left\langle \frac{1}{r^2} \right\rangle + \frac{1}{2} \left\langle r \cdot \frac{2}{r^3} \right\rangle = 0, \tag{9}$$

which causes a contradiction: if a bound state exists, then its energy is not below $E = 0$. Thus the potential $V(r) = -1/r^2$ doesn't assume bound states.

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3

3.1

The dipole interaction matrix is

$$H = -er\mathcal{E} \cos\theta . \tag{10}$$

When the initial state is

$$\psi_0(\boldsymbol{r}) = \frac{1}{\sqrt{\pi}} \mathrm{e}^{-r}, \tag{11}$$

we have

$$E_0^{(1)} = \propto \int \mathrm{d}\Omega \, r^2 \, \mathrm{d}r \, \mathrm{e}^{-2r} r \propto \int_0^\infty r^3 \mathrm{e}^{-2r} = 0 \tag{12}$$

by integration by parts.

3.2