SPT by Prof. Yang Qi

Jinyuan Wu

February 12, 2022

This article is a lecture note of Prof. Yang Qi's lecture on the topological states of matter seminar on October 8, 2021.

1 One dimensional AKLT chain

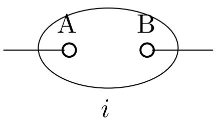
1.1 The AKLT state and the AKLT model

The 1D AKLT chain is first proposed by Affleck, Kennedy, Lieb and Tasaki in [1]. The Haldane conjecture states that an integer spin chain is gapped while a half-integer spin chain is gapless. Since symmetry breaking is impossible in an integer spin chain, it was traditionally regarded as a trivial ferromagnetic system with gapped excitations (magnon). This is definitely true in the bulk - but the possibility that exotic boundary modes exist cannot be ruled out.

Consider a chain, each site of which has a spin-1 degree of freedom on it. The **AKLT state** is a wave function of the spin chain defined on the following way. To make things more understandable we decompose one spin-1 degree of freedom into two spin-1/2 degrees of freedom, i.e. to put two spin-1/2 degrees of freedom on one site, and then use a projector to eliminate redundant Hilbert subspaces. We know that

$$\frac{1}{2} \otimes \frac{1}{2} = 0 \otimes 1,$$

so we need to project the state into *spin triplet subspaces* and throw away the spin singlet states. Suppose we decompose the spin-1 degree of freedom on site i into two spin-1/2 degrees of freedom labeled as A and B, and the projector is visualized as



$$= P = |+\rangle\langle\uparrow\uparrow| + |-\rangle\langle\downarrow\downarrow| + \frac{1}{\sqrt{2}} |0\rangle \left(\langle\uparrow\downarrow| + \langle\downarrow\uparrow| \right). \tag{1}$$

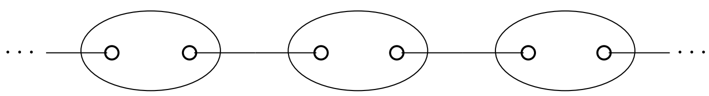
Now we let iB and $i+1$, A get entangled in the following "valence bond"[1]way:

$$i, \mathrm{B} \circ\!\!-\!\!\!-\!\!\!-\!\!\!-\!\!\circ\, i+1, \mathrm{A} = \frac{1}{\sqrt{2}} \left(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle \right), \tag{2}$$

and then apply the projector (1), and now we get an explicit definition of the AKLT state as

$$|\Psi\rangle := \prod_i P_i \, |\Psi_0\rangle \,, \quad |\Psi_0\rangle := \prod_i \frac{1}{\sqrt{2}} \left(|\uparrow\rangle_{i\mathrm{B}} \, |\downarrow\rangle_{i+1,\mathrm{A}} - |\downarrow\rangle_{i\mathrm{B}} \, |\uparrow\rangle_{i+1,\mathrm{A}} \right), \tag{3}$$

or in a visualized way as



$$=: |\Psi\rangle \,. \tag{4}$$

Though (3) is constructed with $2N$ 1/2-spins, after the projection it is already a wave function of a spin-1 model with N sites.

[1] When two spins are in a singlet state, we call it *valence bond state* because the spin part of the two-electron wavefunction in a valence bond is just a singlet state. The two electrons come close in the orbital space, so the spin part of the wave function have to be a singlet to make the whole wave function antisymmetric. If two spins in a singlet states can be directly traced back to two electrons, then it is highly likely that the two electrons are indeed in a valence bond.

Now we try to find a Hamiltonian with (3) as its ground state. [2]Note that the sum of two nearest spin-1 degrees of freedom cannot be 2, because in doing so, we require $S_i^z = S_{i+1}^z = 1$, which, in turn, require that the two 1/2-spins in S_i are all upward and so are the two 1/2-spins in S_{i+1}, which is not possible since the 1/2-spin on i, B and the 1/2-spin on $i+1$, A form a spin singlet. Due to the spin rotational symmetry, suppose

$$\boldsymbol{S} = \boldsymbol{S}_i + \boldsymbol{S}_{i+1},$$

then $S \neq 2$. So we can give a state where $S = 2$ an energy penalty, in an attempt to make (3) a ground state. We may consider a Hamiltonian in the following form:

$$H = \sum_i P_2(\boldsymbol{S}_i + \boldsymbol{S}_{i+1}), \tag{5}$$

where P_2 is an *operator function* which returns 1 when $S = 2$ and returns 0 otherwise. Note that all spins are spin-1 - or otherwise the $(\boldsymbol{S}_i \cdot \boldsymbol{S}_{i+1})^2$ terms in the following discussion make no sense.

Table 1: The eigenstates and eigenvalues of $(\boldsymbol{S}_i + \boldsymbol{S}_{i+1})^2$ and P_2

s	$(\boldsymbol{S}_i + \boldsymbol{S}_{i+1})^2 =: X$	P_2
0	1	0
1	2	0
2	6	1

We should evaluate the explicit form of P_2. Note that the spectrum of $\boldsymbol{S}_i + \boldsymbol{S}_{i+1}$ carries three irreducible representation of $SU(2)$, namely $s = 0, 1, 2$, and since

$$(\boldsymbol{S}_i + \boldsymbol{S}_{i+1})^2 = s(s+1)$$

we have Table 1 on page 2, and by curve fitting we find

$$P_2 = \frac{1}{24} X(X-2).$$

So the Hamiltonian (5) is

$$H = \sum_i \frac{1}{24} (\boldsymbol{S}_i + \boldsymbol{S}_{i+1})^2 ((\boldsymbol{S}_i + \boldsymbol{S}_{i+1})^2 - 2). \tag{6}$$

Using the formula

$$(\boldsymbol{S}_i + \boldsymbol{S}_{i+1})^2 = 4 + 2\boldsymbol{S}_i \cdot \boldsymbol{S}_{i+1},$$

we have

$$\begin{aligned} H &= \sum_i \frac{1}{24} (\boldsymbol{S}_i + \boldsymbol{S}_{i+1})^2 ((\boldsymbol{S}_i + \boldsymbol{S}_{i+1})^2 - 2) \\ &= \sum_i \frac{1}{24} (4 + 2\boldsymbol{S}_i \cdot \boldsymbol{S}_{i+1})(2 + 2\boldsymbol{S}_i \cdot \boldsymbol{S}_{i+1}) \\ &= \sum_i \left(\frac{1}{2} \boldsymbol{S}_i \cdot \boldsymbol{S}_{i+1} + \frac{1}{6} (\boldsymbol{S}_i \cdot \boldsymbol{S}_{i+1})^2 + \frac{1}{3} \right), \end{aligned}$$

and since (5) is just a penalty function we are free to multiply a coefficient onto it and hence we have a much prettier Hamiltonian

$$H = \sum_i \left(\boldsymbol{S}_i \cdot \boldsymbol{S}_{i+1} + \frac{1}{3} (\boldsymbol{S}_i \cdot \boldsymbol{S}_{i+1})^2 \right), \tag{7}$$

where we ignore the constant term.

We can see that (3) is an eigenstate of (7), because the projector (1) commutes with $(\boldsymbol{S}_i + \boldsymbol{S}_{i+1})^2$,

[2] As is often the case when we investigate topological states of matter, we often write down a wave function *first* and then try to find a model with the wave function as the ground state, or even forget about the model.

1.2 The edge states

The spin-1/2 degrees of freedom can be viewed as fractionalized degrees of freedom in (7), since they do not appear in the basic degrees of freedom in (7) but they do help to understand what is going on in its ground state.

Note that (7) differs with the Heisenberg model with only one $\sum_i (\boldsymbol{S}_i \cdot \boldsymbol{S}_{i+1})^2$ term. It has been numerically demonstrated that the model

$$H = \sum_i (\boldsymbol{S}_i \cdot \boldsymbol{S}_{i+1} + \lambda (\boldsymbol{S}_i \cdot \boldsymbol{S}_{i+1})^2) \tag{8}$$

has no quantum phase transition as λ goes from 1/3 to 0, and therefore the Heisenberg model is somehow in one quantum phase with (7).

1.3 Symmetry of the AKLT model and SPT

The AKLT state is actually an example of **symmetry protected topological phases** - or one may call them symmetry protected *trivial* phases, since there are no anyons in them as is the case in *intrinsic* topological phases. The occurrence of gapless boundary states is quite similar to the Kramers degeneracy, which is protected by some symmetry. Derivation of the Kramers degeneracy invokes no microscopic details of the system. A similar classification of SPTs is required, because SPTs can be formed in many distinct ways. In the AKLT state, the boundary states come from fractionalization of spin-1 degrees of freedom. In free fermion models - or in other words topological band models - the boundary states are actually stationary waves. We need to come up with a definition of SPTs, and try to classify them starting from this abstract definition without any microscopic details.

In the case of the AKLT model, the symmetry involved is $SO(3)$ (not $SU(2)$ - since we are talking about an integer spin system) and the time reversal symmetry $\mathbb{Z}_2^T$.

We consider a subgroup of $SO(3)$:

$$D_2 \simeq \mathbb{Z}_2 \times \mathbb{Z}_2. \tag{9}$$

The group can be generated by two elements satisfying

$$x^2 = 1, \quad y^2 = 1, \quad xy = yx, \tag{10}$$

where x and y correspond to 180° rotations around the x and y axises, respectively. The spin-1/2 spins do not carry an ordinary (or *linear*) representation of D_2, but rather a *projective* one: for a spin-1/2 degree of freedom we have

$$x^2 = y^2 = -1, \quad xy = -yx. \tag{11}$$

2 Classification of SPT states and group cohomology

- All SPT states can be characterized by topological field theories.
- Topological field theories can be classified by group cohomology.
- Group cohomology can also be used to classified projective representations.

We consider the general theory of projective representations.

$$\omega_2(g,h)\omega_2(gh,k) = {}^g\omega_2(h,k)\omega_2(g,hk) \tag{12}$$

$$H^2(\mathbb{Z}_2 \times \mathbb{Z}_2, U(1)) = \mathbb{Z}_2 \tag{13}$$

3 Group cohomology

References

[1] Ian Affleck, Tom Kennedy, Elliott H. Lieb, and Hal Tasaki. Rigorous results on valence-bond ground states in antiferromagnets. *Physical Review Letters*, 59(7):799–802, aug 1987.