# Homework 5

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## 1

For the surface

$$z - x^2 - y^2 = 0, \tag{1}$$

the direction of normal vectors is given by

$$\boldsymbol{\nabla}\left(z - x^2 - y^2\right) = (-2x, -2y, 1), \tag{2}$$

and a normal vector at $(-1, 1, 2)$ is $(2, -2, 1)$. The corresponding normal line equation is therefore

$$\frac{x+1}{2} = \frac{y-1}{-2} = \frac{z-2}{1}, \tag{3}$$

and the tangent plane is given by

$$2(x+1) - 2(y-1) + (z-2) = 0 \tag{4}$$

or in other words

$$2x - 2y + z = -2. \tag{5}$$

## 2

The equation governing streamlines is

$$\frac{\mathrm{d}x}{\mathrm{d}t} = F_x, \quad \frac{\mathrm{d}y}{\mathrm{d}t} = F_y, \quad \frac{\mathrm{d}z}{\mathrm{d}t} = F_z, \tag{6}$$

and therefore

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \cos y, \quad \frac{\mathrm{d}y}{\mathrm{d}t} = \sin x, \quad \frac{\mathrm{d}z}{\mathrm{d}t} = 0 \Rightarrow z = \text{const}. \tag{7}$$

The equation about $x, y$ is

$$\frac{\mathrm{d}x}{\cos y} = \frac{\mathrm{d}y}{\sin x} \Rightarrow \sin x \, \mathrm{d}x = \cos y \, \mathrm{d}y \Rightarrow -\cos x = \sin y + \text{const}. \tag{8}$$

So the streamline equation is

$$\cos x + \sin y = C_1, \quad z = C_2. \tag{9}$$

## 3

Since $\mathbf{F} = \cos x\mathbf{i} - y\mathbf{j} + xz\mathbf{k}$ and $\mathbf{R} = t\mathbf{i} - t^2\mathbf{j} + \mathbf{k}$, and $0 \leq t \leq 3$, we have

$$\int_C \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{R} = \int_0^3 (\cos x + y \cdot 2t) \, \mathrm{d}t = \int_0^3 (\cos t - 2t^3) \, \mathrm{d}t = \sin 3 - \frac{81}{2}. \tag{10}$$

## 4

$C$ is the circle of radius 4 about $(1, 3)$. $\mathbf{F} = 2y\mathbf{i} - x\mathbf{j}$ so

$$\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} = -3, \tag{11}$$

and

$$\oint_{\partial C} \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{R} = \int_C \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right) \mathrm{d}x \, \mathrm{d}y = -3 \cdot \int_C \mathrm{d}x \, \mathrm{d}y = -3 \cdot \pi \cdot 4^2 = -48\pi. \tag{12}$$

## 5

The equation of the surface is equivalent to

$$z = \frac{1}{10}(25 - 4x - 8y). \tag{13}$$

The integral is therefore

$$\begin{aligned}\iint_\Sigma (x+y)\,\mathrm{d}\sigma &= \iint_{\Sigma'} (x+y)\sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\,\mathrm{d}x\,\mathrm{d}y \\ &= \iint_{\Sigma'} \frac{3}{5}\sqrt{5}(x+y)\,\mathrm{d}x\,\mathrm{d}y\,,\end{aligned} \tag{14}$$

where $\Sigma'$ is the triangle between $(0,0)$, $(1,0)$ and $(1,1)$. Then

$$\iint_{\Sigma'} (x+y)\,\mathrm{d}x\,\mathrm{d}y = \int_0^1 \mathrm{d}x \int_0^x \mathrm{d}y\,(x+y) = \int_0^1 \left(x^2+\frac{x^2}{2}\right)\mathrm{d}x = \frac{1}{2}, \tag{15}$$

so the final result is

$$\iint_\Sigma (x+y)\,\mathrm{d}\sigma = \frac{3}{10}\sqrt{5}. \tag{16}$$

## 6

$\mathbf{F} = x^3\mathbf{i} + y^3\mathbf{j} + z^3\mathbf{k}$ and $\Sigma$ is the sphere of radius 1 about the origin. So

$$\oiint_\Sigma \boldsymbol{F}\cdot\boldsymbol{n}\,\mathrm{d}\sigma = \iiint \boldsymbol{\nabla}\cdot\boldsymbol{F}\,\mathrm{d}V = \iiint 3(x^2+y^2+z^2)\,\mathrm{d}V = 3\cdot 4\pi\int_0^1 r^2\cdot r^2\,\mathrm{d}r = \frac{12}{5}\pi. \tag{17}$$

## 7

We have

$$\log z = \log|z|\,\mathrm{e}^{\mathrm{i}\theta} = \log|z| + \mathrm{i}\arg(z), \tag{18}$$

where $\arg(z)$ is multi-valued. When $z = 1+5\mathrm{i}$, we have

$$\log z = \log\sqrt{26} + \mathrm{i}\arg(z) = \log\sqrt{26} + \mathrm{i}\arccos\frac{1}{\sqrt{26}} + 2\pi n\mathrm{i}, \quad n\in\mathbb{Z}. \tag{19}$$

## 8

No singularities are present and we have

$$\int_\gamma \mathrm{i}z^2\,\mathrm{d}z = \mathrm{i}\left.\frac{z^3}{3}\right|_{1+3\mathrm{i}}^{3+\mathrm{i}} = \mathrm{i}(44+44\mathrm{i}) = -44+44\mathrm{i}. \tag{20}$$

## 9

Suppose $z = r\mathrm{e}^{\mathrm{i}\theta}$. Since $\gamma$ is a circle of radius 5 about the origin, we have

$$\oint_\gamma \frac{1}{\bar{z}}\,\mathrm{d}z = \oint_\gamma \frac{1}{r\mathrm{e}^{-\mathrm{i}\theta}}\,\mathrm{d}(r\mathrm{e}^{\mathrm{i}\theta}) = \oint_\gamma \frac{1}{r\mathrm{e}^{-\mathrm{i}\theta}} r\,\mathrm{d}\mathrm{e}^{\mathrm{i}\theta} = \oint_\gamma \frac{1}{r\mathrm{e}^{-\mathrm{i}\theta}} r\mathrm{e}^{\mathrm{i}\theta}\cdot\mathrm{i}\,\mathrm{d}\theta = \mathrm{i}\int_0^{2\pi} \mathrm{e}^{2\mathrm{i}\theta}\,\mathrm{d}\theta = 0. \tag{21}$$

## 10

The function

$$f(z) = \frac{\cos(z-\mathrm{i})}{(z+2\mathrm{i})^3} \tag{22}$$

has a third-order pole at $z = -2\mathrm{i}$, and therefore the residue is

$$\begin{aligned}\operatorname{Res}_{z=-2\mathrm{i}} f(z) &= \frac{1}{2!} \lim_{z\to -2\mathrm{i}} \frac{\mathrm{d}^2}{\mathrm{d}z^2} (z+2\mathrm{i})^3 f(z) \\ &= \frac{1}{2} \left. \frac{\mathrm{d}^2 \cos(z-\mathrm{i})}{\mathrm{d}z^2} \right|_{z=-2\mathrm{i}} = -\frac{1}{2}\cos(-3\mathrm{i}) = -\frac{1}{4}(\mathrm{e}^3 + \mathrm{e}^{-3}).\end{aligned} \tag{23}$$

The integral is

$$\oint_{\text{near } -2\mathrm{i}} f(z)\,\mathrm{d}z = 2\pi\mathrm{i} \cdot -\frac{1}{4}(\mathrm{e}^3 + \mathrm{e}^{-3}) = -\frac{\mathrm{i}}{2}(\mathrm{e}^3 + \mathrm{e}^{-3}). \tag{24}$$

## 11

We apply the ratio test to $\sum_{n=0}^{\infty} \left(\frac{2i}{5+i}\right)^n (z+3-4i)^n$:

$$\left| \left(\frac{2i}{5+i}\right)(z+3-4i) \right| < 1 \Rightarrow |z+3-4\mathrm{i}| < \frac{|5+\mathrm{i}|}{|2\mathrm{i}|} = \frac{\sqrt{26}}{2}. \tag{25}$$

The convergence radius is $\sqrt{26}/2$; the open disk of convergence is a circle with radius $\sqrt{26}/2$ about $-3+4\mathrm{i}$.

## 12

$\gamma$ is the square of side length 3 and sides parallel to the axes, centered at $-2\mathrm{i}$, and therefore it doesn't contain the $2\mathrm{i}$ pole. Therefore

$$\oint_\gamma \frac{\cos z}{4+z^2}\,\mathrm{d}z = 2\pi\mathrm{i} \operatorname{Res}_{z=-2\mathrm{i}} \frac{\cos z}{4+z^2} = 2\pi\mathrm{i} \lim_{z\to -2\mathrm{i}} (z+2\mathrm{i}) \frac{\cos z}{4+z^2} = 2\pi\mathrm{i} \frac{\cos(-2\mathrm{i})}{-4\mathrm{i}} = -\frac{\pi}{4}(\mathrm{e}^2 + \mathrm{e}^{-2}). \tag{26}$$

## 13

Only the pole 2 is in $\gamma$ the circle of radius 2 about 2. Thus

$$\oint_\gamma \frac{(1-z)^2}{z^3-8}\mathrm{d}z = 2\pi\mathrm{i} \lim_{z\to 2} \frac{(1-z)^2}{z^3-8}(z-2) = \frac{\pi\mathrm{i}}{6}. \tag{27}$$