Fermi liquid

Jinyuan Wu

February 14, 2023

1 Free electron gas

A free electron gas is described by[1]

$$H = \sum_i \frac{\boldsymbol{p}_i^2}{2m},\tag{1}$$

and we get a Fermi sphere in the momentum space containing all occupied states, and a Fermi momentum p_{F}. After we add a periodic potential $V(\boldsymbol{r})$ to the model, we get **bands**: parabolic bands are displaced periodically according to the $\boldsymbol{G}$-grid, and the crossing points of these curves undergo degeneracy breaking, and the spectrum, eventually, becomes several bands confined in the first Brillouin zone. Despite this correction, the number of filled electrons never changes.

For a free electron gas, we have well-known observables like the density of states at E_{F} (here $\omega = \varepsilon - E_{\mathrm{F}}$)

$$N(\omega = 0) = \frac{m}{\pi^2} \frac{p_{\mathrm{F}}}{\hbar^3},\tag{2}$$

the electronic specific heat

$$C_v = \underbrace{\frac{\pi^2 k_{\mathrm{B}}^2}{3} N(\omega = 0)}_{\text{Sommerfield } \gamma} \cdot T,\tag{3}$$

and the magnetic susceptibility

$$\chi = \mu_0 \mu_{\mathrm{B}}^2 \cdot N(\omega = 0).\tag{4}$$

^{3}He atoms are fermions: we have two electrons, two protons, and only one neutron in one ^{3}He atom, and therefore there are odd fermions inside and the composite state is a fermion. In a ^{3}He liquid, since the distance between the atoms is drastically reduced, we can expect the effect of repulsion is much stronger than the case in the gas phase. This is a cold atom model of electrons in solids: we can easily measure the specific heat, etc. of the liquid ^{3}He.

The C_V-T curve measured looks like TODO: fig 1 As is expected, when $T \to 0$, we see a $C_V \propto T$ behavior, qualitatively agreeing with (3). The coefficient however is not what is predicted by (2) and (3). Now if we apply pressure to liquid ^{3}He, C_v/T clearly increases.

For a fermion gas, we can measure the mass according to (3) and (2). Now if we calculate the "mass" of ^{3}Heaccording to the two equations, we get a pressure-dependent (quite surprising because the density of liquids almost stays the same when we press it) "effective mass" that is never the same as the bare mass of ^{3}He.

The next question, then, is whether liquid ^{3}He– a system with known strong interaction – can be equivalently described by something with the effective mass.

2 Assumptions of Fermi liquid theory

2.1 Motivation: scattering changes the distribution of electrons

The general form of Hamiltonian of fermionic systems that is found in condensed matter physics looks like

$$H = \sum_i \frac{\boldsymbol{p}_i^2}{2m} + \sum_{i,j} V(|\boldsymbol{r}_i - \boldsymbol{r}_j|).\tag{5}$$

When we deal with a two-body problem with this dynamics, the collision behavior is completely described by the scattering cross-section. If we extend this picture to a many-body system, we immediately see scattering changes the momentum distribution of particles, because the final states of scattering can obviously be over the Fermi surface, and therefore there is still some distribution above p_{F} now.

---

[1]For some reason, when talking about Fermi liquid, people like to use $\boldsymbol{p}$ instead of $\boldsymbol{k}$.
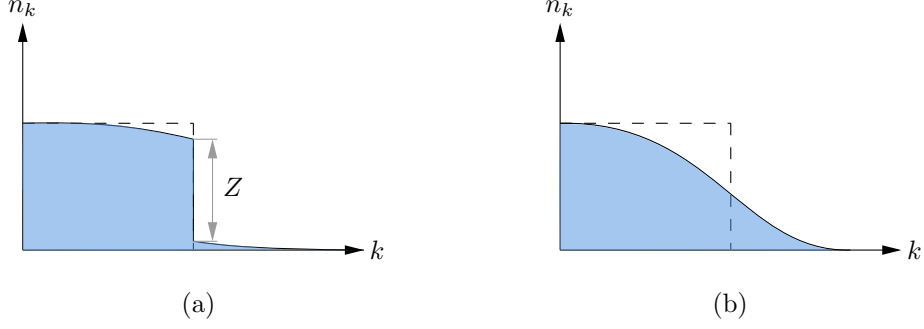
Figure 1: Two types of interaction correction to the electron distribution. (a) The Fermi liquid case: there is still a stepwise change of n with respect to the energy. (b) The non-Fermi liquid case: the change of n is smooth.

2.2 Assumption: correspondence between interactive corrected states and uncorrected states

We assume that we still have "states labeled by $\boldsymbol{p}$, n, etc." in a Fermi liquid, and the ground state and the first several excited states can be totally labeled by $\delta n(\boldsymbol{p})$, which is $n(\boldsymbol{p})$ – some sort of "filling" on these states – minus $n_0(\boldsymbol{p})$ – the Fermi gas filling. These states labeled by $\boldsymbol{p}$, n, etc. are essentially the "single-electron wave function" in the single-electron Green function, and the ground state $n(\boldsymbol{p})$ is essentially the spectral function. We further assume δn is small, so that what we get is Fig. 1(a), instead of Fig. 1(b). This agrees with the observed fact that the $C_v \propto T$ relation is still true, so Fermi liquid should be somehow similar to the Fermi gas. Also, we assume the total number of electrons should change considerably:

$$\delta N = \sum_{\boldsymbol{p}} \delta n(\boldsymbol{p}) \ll \sum_{\boldsymbol{p}} n(\boldsymbol{p}). \tag{6}$$

Note that although we are talking about "partial occupation" or even "quasiparticle decaying" (see the follows), Fermi liquid theory is still a *pure state* theory; as is shown by Green function theory, partial occupation and even decaying can happening in the zero-temperature case, due to quantum fluctuation. A single-electron state can evolve into a multiple-electron state, and therefore the ground state is a mixture of single-electron, double-electron, etc. states, and if we insist on a single-electron theory, certain dissipation channels appear. The Fermi liquid theory, essentially, is assuming that the single-particle picture still works well enough.

2.3 The energy functional

The energy of state $\{\delta n_{\boldsymbol{p}}\}$, E, reads

$$E = E_0 + \delta E, \quad \delta E = \sum_{\boldsymbol{p}} \varepsilon_0(\boldsymbol{p})\,\delta n(\boldsymbol{p}) + \mathcal{O}(\delta n^2). \tag{7}$$

Here E_0 is the ground state energy of the free electron gas, (7) doesn't contain any many-body correction; a more accurate form of the energy of the system is

$$\delta E = \sum_{\boldsymbol{p}} \varepsilon_0(\boldsymbol{p})\,\delta n(\boldsymbol{p}) + \frac{1}{2V} \sum_{\boldsymbol{p},\boldsymbol{p}'} f_{\boldsymbol{p}\boldsymbol{p}'}\,\delta n(\boldsymbol{p})\,\delta n(\boldsymbol{p}') + \mathcal{O}(\delta n^3). \tag{8}$$

This means an electron in a Fermi liquid now receives a (almost trivial) self-energy correction

$$\delta E = \sum_{\boldsymbol{p}} \varepsilon(\boldsymbol{p})\,\delta n(\boldsymbol{p}), \quad \varepsilon(\boldsymbol{p}) = \varepsilon_0(\boldsymbol{p}) + \frac{1}{V} \sum_{\boldsymbol{p}'} f_{\boldsymbol{p}\boldsymbol{p}'}\,\delta n(\boldsymbol{p}'). \tag{9}$$

The chemical potential can be immediately decided:

$$\mu \coloneqq \left. \frac{\delta E}{\delta n(\boldsymbol{p})} \right|_{\text{Fermi surface}} = \varepsilon(p_{\text{F}}). \tag{10}$$

2

For condensed matter systems, we have to make a further modification to the energy functional shown above: we need to introduce spins. So the full theory is

$$\delta E = \sum_{\boldsymbol{p},\sigma} \varepsilon_{0\sigma}(\boldsymbol{p})\,\delta n_\sigma(\boldsymbol{p}) + \frac{1}{2V} \sum_{\boldsymbol{p},\boldsymbol{p}',\sigma,\sigma'} f_{\boldsymbol{p}\boldsymbol{p}'\sigma\sigma'}\,\delta n_\sigma(\boldsymbol{p})\,\delta n_{\sigma'}(\boldsymbol{p}')\,. \tag{11}$$

When there is no outside spin polarizing factors like a magnetic field, we just have

$$f_{\uparrow\uparrow} = f_{\downarrow\downarrow}, \quad f_{\uparrow\downarrow} = f_{\downarrow\uparrow}, \quad \varepsilon_{0\uparrow} = \varepsilon_{0\downarrow} = \varepsilon_0, \tag{12}$$

and therefore we have

$$\sum_{\boldsymbol{p},\boldsymbol{p}',\sigma,\sigma'} f_{\boldsymbol{p}\boldsymbol{p}'\sigma\sigma'}\,\delta n_\sigma(\boldsymbol{p})\,\delta n_{\sigma'}(\boldsymbol{p}')$$
$$= \sum_{\boldsymbol{p},\boldsymbol{p}'} \frac{1}{2}(f_{\uparrow\uparrow} + f_{\downarrow\uparrow})\,\delta n(\boldsymbol{p})\,\delta n(\boldsymbol{p}) + \sum_{\boldsymbol{p},\boldsymbol{p}'} \frac{1}{2}(f_{\uparrow\uparrow} - f_{\downarrow\uparrow})\,\delta\tilde{n}(\boldsymbol{p})\,\delta\tilde{n}(\boldsymbol{p})\,,$$

where

$$\delta n(\boldsymbol{p}) = \delta n_\uparrow(\boldsymbol{p}) + \delta n_\downarrow(\boldsymbol{p})\,, \quad \delta\tilde{n}(\boldsymbol{p}) = \delta n_\uparrow(\boldsymbol{p}) - \delta n_\downarrow(\boldsymbol{p})\,. \tag{13}$$

We define

$$2f^{\mathrm{S}} = f_{\uparrow\uparrow} + f_{\downarrow\uparrow}, \quad 2f^{\mathrm{A}} = f_{\uparrow\uparrow} - f_{\downarrow\uparrow}, \tag{14}$$

and the interaction part of the energy functional then is

$$\sum_{\boldsymbol{p},\boldsymbol{p}'} \left(f^{\mathrm{S}}_{\boldsymbol{p}\boldsymbol{p}'}\,\delta n(\boldsymbol{p})\,\delta n(\boldsymbol{p}) + f^{\mathrm{A}}_{\boldsymbol{p}\boldsymbol{p}'}\,\delta\tilde{n}(\boldsymbol{p})\,\delta\tilde{n}(\boldsymbol{p}) \right)\,. \tag{15}$$

Now we can make some symmetric arguments. When we calculate physical quantities that only involves δn, like the specific heat capacity, there is no need to deal with $\delta\tilde{n}$; when calculating the spin response the case is the opposite. We can do the angular momentum expansion

$$f^{\mathrm{S/A}}_{\boldsymbol{p}\boldsymbol{p}'} = \sum_{l=0}^{\infty} f^{\mathrm{S/A}}_l\, \mathrm{P}_l(\cos\theta), \tag{16}$$

TODO: only the angle θ between $\boldsymbol{p}$ and $\boldsymbol{p}'$ matters. When we talk about an isotropic excitation mode, only the $l = 0$ term matters, while if we impose a magnetic field along one axis, then usually the $l = 1$ term has the strongest contribution.

The expansion (16) also implies one way Landau Fermi liquid theory breaks: if high l terms have very strong contribution to δE, then even when we still have well-defined $\delta n(\boldsymbol{p})$, Landau Fermi liquid theory will be of limited use. Fortunately this is almost never the case.

We usually define dimensionless Landau parameters

$$F^{\mathrm{S/A}} = N(\omega = 0)f^{\mathrm{S/A}}, \tag{17}$$

where $N(\omega = 0)$ is the density of states at E_{F}.

2.4 Some observables

Following the procedure in electron gas, we define

$$v_{\mathrm{F}} = \boldsymbol{\nabla}_{\boldsymbol{p}}\varepsilon(\boldsymbol{p})|_{p_{\mathrm{F}}}, \tag{18}$$

and

$$m^* = \frac{p_{\mathrm{F}}}{v_{\mathrm{F}}}. \tag{19}$$

The density of states therefore is given by

$$N(\omega = 0) = m^* \frac{p_{\mathrm{F}}}{\pi^2 \hbar^3}. \tag{20}$$

3

This is of course related to the specific heat; let's derive the latter. Suppose we put a Fermi liquid under a finite temperature T. The specific heat capacity can be found by

$$C_V = T \left.\frac{\partial S}{\partial T}\right|_V. \tag{21}$$

The entropy is

$$S = -k_{\mathrm{B}} \ln \Omega = -k_{\mathrm{B}} \frac{1}{V} \sum_{\boldsymbol{p},\sigma} (n_\sigma(\boldsymbol{p}) \ln n_\sigma(\boldsymbol{p}) + (1 - n_\sigma(\boldsymbol{p})) \ln(1 - n_\sigma(\boldsymbol{p}))), \tag{22}$$

where now due to thermal fluctuation, we have TODO: ε has δn dependence, so what will happen in the Fermi-Dirac distribution? A tentative solution: by summing over $n_\sigma(\boldsymbol{p}')$ ($\boldsymbol{p} \neq \boldsymbol{p}'$) first, we have

$$n_\sigma(\boldsymbol{p}) = \frac{\left\langle \mathrm{e}^{-(\varepsilon_\sigma(\boldsymbol{p}) - \mu)/k_{\mathrm{B}}T} \right\rangle}{1 + \left\langle \mathrm{e}^{-(\varepsilon_\sigma(\boldsymbol{p}) - \mu)/k_{\mathrm{B}}T} \right\rangle},$$

where the expectation brackets mean to average over non-$\boldsymbol{p}$ degrees of freedom, and since for fermions $n^2 = n$, we can move the expectation brackets into the exponent, and thus we have the self-consistent Fermi-Dirac distribution

$$n_\sigma(\boldsymbol{p}) = \frac{1}{1 + \mathrm{e}^{(\varepsilon_\sigma(\boldsymbol{p}) - \mu)/k_{\mathrm{B}}T}}, \tag{23}$$

where $\varepsilon_\sigma(\boldsymbol{p})$ is decided by (9), where the $\delta n_\sigma(\boldsymbol{p})$ in (9) is just in the Fermi-Dirac distribution. This can also be confirmed by thinking about the Hartree term in imaginary time field theory: $n_\sigma(\boldsymbol{p})$ can be obtained from the single-electron Green function in the same way in the free electron gas, and the loop electron line in the Hartree diagram gives exactly (the temperature-dependent) $n_\sigma(\boldsymbol{p})$.

$$\delta S = -\frac{1}{TV} \sum_{\boldsymbol{p},\sigma} (\varepsilon_\sigma(\boldsymbol{p}) - \mu)\, \delta n_\sigma(\boldsymbol{p}) = -\frac{1}{TV} \sum_{\boldsymbol{p},\sigma} (\varepsilon_\sigma(\boldsymbol{p}) - \mu) \left(\frac{\partial n_\sigma(\boldsymbol{p})}{\partial \varepsilon_\sigma(\boldsymbol{p})} \delta \epsilon_\sigma(\boldsymbol{p}) + \frac{\partial n_\sigma(\boldsymbol{p})}{\partial T} \delta T \right). \tag{24}$$

TODO: eventually we find

$$\delta S = -\frac{1}{TV} \int \mathrm{d}\varepsilon\, N(\omega)(\varepsilon - \mu) \frac{\varepsilon - \mu}{T}\, \mathrm{d}T, \tag{25}$$

and

$$C_V = \frac{\pi^3}{3} N(\omega = 0) k_{\mathrm{B}}^2 T. \tag{26}$$

> **Box .1: What is really $n_\sigma(p)$?**
>
> When I first learned Fermi liquid theory, what puzzled me the most is what exactly is $n_\sigma(\boldsymbol{p})$: is it a quantum operator (and thus a label of the many-body wave function), or is it the single-electron Fermi-Dirac distribution (which is stored in the single-electron Matsubara Green function)? From the above discussion, we can find *both* views are correct. This is a trivial instance of the fact that Feynman diagram components, like the single-electron self-energy or the double-electron kernel, do have counterparts in the wave-function-plus-operator language (often referred to as second quantization, although technically second quantization is used everywhere, including Feynman diagram techniques), to which we can do everything we do for ordinary quantum operators.

The next question is what's m^*. There is actually a constraint imposed by the translational symmetry. Suppose we add a very small velocity $\boldsymbol{v}$ globally. This means

$$\boldsymbol{p} \to \boldsymbol{p}' = \boldsymbol{p} - m\boldsymbol{v}, \quad E \to E' = E - \boldsymbol{p} \cdot \boldsymbol{v} + \frac{1}{2} m v^2. \tag{27}$$

Taylor expansion tells us

$$\varepsilon(\boldsymbol{p} - m\boldsymbol{v}) = \varepsilon(\boldsymbol{p}) - \left(\frac{m}{m^*} - 1\right) \boldsymbol{p} \cdot \boldsymbol{v}, \tag{28}$$

and this has to agree with the TODO:

$$\frac{m^*}{m} = 1 + \frac{F_1^{\mathrm{S}}}{3}. \tag{29}$$

This means the effective mass only carries information about the symmetric part of $f_{\boldsymbol{p}\boldsymbol{p}'\sigma\sigma'}$.

$$\frac{m^*}{m} \qquad 5 \quad + \frac{F_1^{\mathrm{S}}}{3}. \tag{29}$$