

# Bosonic modes in Fermi liquid

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# Background

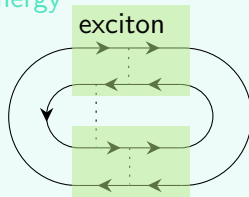
In a Fermi liquid we have ...

- Quasiparticles (electron/hole) with  $\Sigma$ -correction
- Any anything else?

single electron energy



exciton energy



... and more

# Question

## What to do

Finding modes other than the corrected single electron/hole

## Why it's important

Usually not for  $C_V$  but for optical response:  $\epsilon$ ,  $\chi^{(3)}$ , etc.

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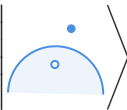
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## Today's topic

Electron-hole bosonic modes in Fermi liquid (with *some* scattering picked up back, i.e. beyond  $\delta E \sim \varepsilon \delta n + f \delta n \delta n$ ), i.e.

$$|\text{single excitation}\rangle = \sum_{\mathbf{k}_1, \mathbf{k}_2} c_{\mathbf{k}_1 \mathbf{k}_2} \left| \begin{array}{c} \bullet \\ \text{---} \\ \text{---} \end{array} \right\rangle \quad (1)$$


No trion, higher order correlation, or even more exotic spinons, etc.  
beyond Fermi liquid

**Serious quantitative prediction** Bethe-Salpeter eq. (BSE)

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**Single electron linear response singularity = bosonic modes**

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**Quantum Boltzmann eq. (QBE)** Easiest kinetic theory for single-electron distribution to external field perturbation.

**Conditions of QBE**

- low (external and inherent)  $\omega$
- long wave length
- well-defined quasiparticles; high order correlation not important

## What to investigate

Stable oscillation modes of QBE ( $\Leftrightarrow$  infinite response to external field  $\Leftrightarrow$  bosonic mode): for  $n_{\mathbf{p}\sigma\sigma'}(\mathbf{r})$ ,  $\varepsilon_{\mathbf{p}\sigma\sigma'} = \varepsilon[\delta n]$ ,

$$\frac{\partial n_{\mathbf{p}}}{\partial t} + \underbrace{\frac{\partial \varepsilon_{\mathbf{p}}}{\partial \mathbf{p}} \cdot \frac{\partial n_{\mathbf{p}}}{\partial \mathbf{r}}}_{\text{diffusion}} - \underbrace{\frac{\partial \varepsilon_{\mathbf{p}}}{\partial \mathbf{r}} \cdot \frac{\partial n_{\mathbf{p}}}{\partial \mathbf{p}}}_{\text{force}} + \underbrace{i[\varepsilon_{\mathbf{p}}, n_{\mathbf{p}}]}_{\text{multi-band}} = \underbrace{I_{\text{Fermi golden rule}}}_{\text{collision}} \quad (2)$$



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## What to expect

Three types of important bosonic modes:

- Zero sound in uncharged Fermi liquid
- Plasmon in charged Fermi liquid = zero sound + long range interaction
- Exciton in charged multi-band Fermi liquid

# Equation governing zero sound

**System** Single-band Fermi liquid with spin ignored

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**Kinetics of uncharged Fermi liquid** *Landau equation* = QBE +

$$\varepsilon_{\mathbf{p}}(\mathbf{r}) = \varepsilon_{\mathbf{p}}^0 + \frac{1}{V} \sum_{\mathbf{p}'} f_{\mathbf{p}\mathbf{p}'} \delta n_{\mathbf{p}}(\mathbf{r}) \quad (3)$$

(assumption:  $\mathbf{q} \rightarrow 0$  in  $c_{\mathbf{p}+\mathbf{q}}^\dagger c_{\mathbf{p}}$ , i.e.  $\delta n_{\mathbf{p}}(\mathbf{r})$  being smooth in  $\mathbf{r}$ )

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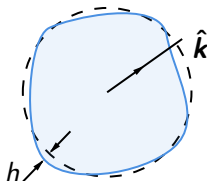
**EOM governing zero sound** Small disturbance, no collision, :

$$\frac{\partial \delta n_{\mathbf{p}}}{\partial t} + \frac{\partial \varepsilon_{\mathbf{p}}^{\text{static}}}{\partial \mathbf{p}} \cdot \frac{\partial \delta n_{\mathbf{p}}}{\partial \mathbf{r}} - \frac{\partial n_{\mathbf{p}}^{\text{static}}}{\partial \mathbf{p}} \cdot \underbrace{\frac{1}{V} \sum_{\mathbf{p}'} f_{\mathbf{p}\mathbf{p}'} \frac{\partial \delta n_{\mathbf{p}}}{\partial \mathbf{r}}}_{\partial \delta \varepsilon_{\mathbf{p}} / \partial \mathbf{r}} = 0 \quad (4)$$

# Fermi surface vibration

**Ansatz** Disturbance as small as possible ...

$$n_{\mathbf{p}}(\mathbf{r}, t) = e^{i(\mathbf{q} \cdot \mathbf{r} - i\omega t)} \theta(\mu - \varepsilon_{\mathbf{p}}^{\text{stable}} - h(\hat{\mathbf{p}})) \quad (5)$$

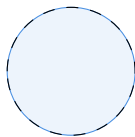


**Eigenvalue problem**

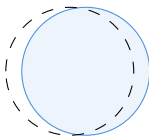
$$(\omega - \mathbf{q} \cdot \mathbf{v})h(\hat{\mathbf{k}}) = \mathbf{q} \cdot \mathbf{v} \int \frac{d\Omega'}{4\pi} F(\vartheta)h(\hat{\mathbf{k}}'). \quad (6)$$

where  $\mathbf{v}$  is single-electron velocity.  $\Rightarrow$  zero sound has linear dispersion.  
Non-trivial zero sound requires  $F \neq 0$  ( $F = 0 \Rightarrow$  bare electron-hole pair)

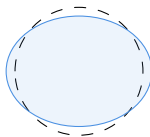
## Shape of Fermi surface when $\mathbf{q} = 0$



$l = 0$

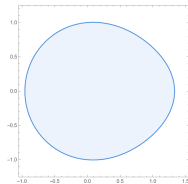


$l = 1$



$l = 2$

**Distortion when  $\mathbf{q} \neq 0$**  More electrons in  $\hat{\mathbf{q}}$ ; less electrons in  $-\hat{\mathbf{q}}$



**Zero sound is not density wave** In zero sound  $V_{\text{Fermi sea}} = \text{const.} \Rightarrow$   
zero sound is not ordinary sound

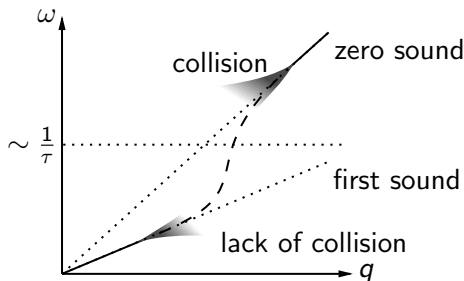
# Comparison with ordinary sound

**Ordinary sound** Fermi liquid theory  $\Rightarrow \partial\rho/\partial P \Rightarrow$  another sound mode (“first sound”, ordinary sound, density mode) from hydrodynamics

## Relation with zero sound

- First sound appears when  $\omega\tau \ll 1$ : ordinary hydrodynamics  $\Leftrightarrow$  local equilibrium  $\Leftrightarrow \tau \ll 1/\omega$
- zero sound appears when  $\omega\tau \gg 1$ : no collision integral  $\Leftrightarrow \tau \gg 1/\omega$

The two are connected: a radical finite- $T$  correction



# What happens with long-range interaction

## The origin of $f_{pp'}$

$$f_{kk'} = \lim_{q \rightarrow 0} \left( \begin{array}{c} k \\ \swarrow \\ \text{---} q \text{---} \\ \searrow \\ k + q \end{array} + \begin{array}{c} k' \\ \swarrow \\ \text{---} k' - k \text{---} \\ \searrow \\ k' + q \end{array} + \begin{array}{c} k \leftarrow \leftarrow k' \\ \text{---} k + q \rightarrow \rightarrow k' + q \\ \text{---} k' - k \text{---} \end{array} \right) \quad (7)$$

Coulomb interaction  $\Rightarrow$  first term divergent in  $\mathbf{k}$  space  $\Rightarrow$  it should be considered in  $\mathbf{r}$  space



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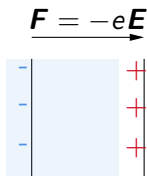
**Landau-Silin eq.**

$$\frac{\partial n_{\mathbf{p}}}{\partial t} + \frac{\partial \varepsilon_{\mathbf{p}}}{\partial \mathbf{p}} \cdot \frac{\partial n_{\mathbf{p}}}{\partial \mathbf{r}} - \frac{\partial (\varepsilon_{\mathbf{p}} - e\varphi(\mathbf{r}))}{\partial \mathbf{r}} \cdot \frac{\partial n_{\mathbf{p}}}{\partial \mathbf{p}} = \underbrace{I_{\text{Fermi golden rule}}}_{\text{collision}} \quad (8)$$

$$\varepsilon_{\mathbf{p}}(\mathbf{r}) = \varepsilon_{\mathbf{p}}^0 + \frac{1}{V} \sum_{\mathbf{p}'} f_{\mathbf{p}\mathbf{p}'} \delta n_{\mathbf{p}'}(\mathbf{r}), \quad \nabla^2 \varphi = e \cdot \frac{1}{V} \sum_{\mathbf{p}} n_{\mathbf{p}}(\mathbf{r}). \quad (9)$$

# Plasmon mode

**Plasmon is gapped** When  $\mathbf{q} \rightarrow 0$  we get to the elementary case



$$m\ddot{\mathbf{x}} = -m\omega^2\mathbf{x} = (-e)\mathbf{E} = -e \cdot \frac{1}{\epsilon_0}en\mathbf{x} \Rightarrow \omega = \sqrt{\frac{ne^2}{\epsilon_0 m}}. \quad (10)$$

**Comparison with zero sound** When  $\mathbf{q} \rightarrow 0$ ,  $\varphi(\mathbf{r}) \Rightarrow$  oscillation:  
long-range interaction  $\Rightarrow$  finite gap

**Comparison with first sound**  $V_{\text{Fermi sea}} = \text{const.}$  in plasmon as well:  
plasmon is not a density mode

## Fermi liquid, uncharged: zero sound

- Linear, gapless
- From  $f_{pp'}$

## Fermi liquid, charged: plasmon

- Divergent Hartree term  $\Rightarrow$  self-energy correction in real space
- When  $\mathbf{q} = 0$ :  $f_{pp'}$  not important; gapped

## Two bands: exciton

# Determining $\mathbf{r}$ and $\mathbf{p}$ at the same time in QBE??

**Why  $n_{\mathbf{p}}(\mathbf{r})$  makes sense** Because it's not a probabilistic distribution function: it's a Wigner function of  $G^<(\mathbf{r}_1, \mathbf{r}_2)$ .

By Fourier transform

$$\int d^3\mathbf{r} n_{\mathbf{p}}(\mathbf{r}) e^{-i\mathbf{q}\cdot\mathbf{r}} \simeq n_{\mathbf{p}\mathbf{q}} \simeq c_{\mathbf{p}}^{\dagger} c_{\mathbf{p}+\mathbf{q}} \quad (11)$$

we get the total momentum of the bosonic mode  $\mathbf{q}$ .

**Why  $\varepsilon_{\mathbf{p}}(\mathbf{r})$  makes sense** Similarly, because of  $\Sigma(\mathbf{r}_1, \mathbf{r}_2)$

**Why  $f_{\mathbf{p}\mathbf{p}'}$  has no spatial dependence**

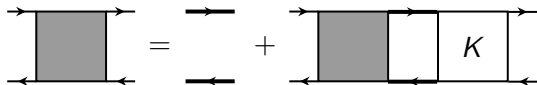
- 1 Interaction channel:  $\delta n_{\mathbf{p}\mathbf{q}} f_{\mathbf{p}\mathbf{p}'} \delta n_{\mathbf{p}'\mathbf{q}'} \delta_{\mathbf{q}\mathbf{q}'} (\sum \mathbf{q} = \text{const.})$
- 2 But we are working with QBE  $\Rightarrow \mathbf{q}$  small
- 3 So we take  $\mathbf{q}, \mathbf{q}' \rightarrow 0$  limit:

$$\sum \delta n_{\mathbf{p}\mathbf{q}} f_{\mathbf{p}\mathbf{p}'} \delta n_{\mathbf{p}'\mathbf{q}'} \delta_{\mathbf{q}\mathbf{q}'} \xrightarrow{\text{Fourier}} \int d^3\mathbf{r} \delta n_{\mathbf{p}}(\mathbf{r}) f_{\mathbf{p}\mathbf{p}'} \delta n_{\mathbf{p}'}(\mathbf{r}) \quad (12)$$

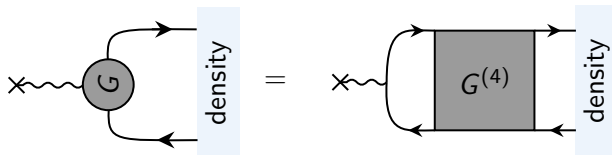
# BSE and single-electron kinetic theory

## Series calculation

Bethe–Salpeter equation (BSE) is for quantitative calculations.

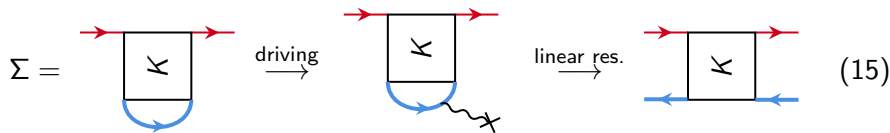

$$\text{Diagram} = \text{Diagram} + \text{Diagram} \times K \quad (13)$$

**What we need** Linear response of single-electron under external field = BSE (simplest single-electron theory: QBE)


$$\text{Diagram} = \text{Diagram} \quad (14)$$

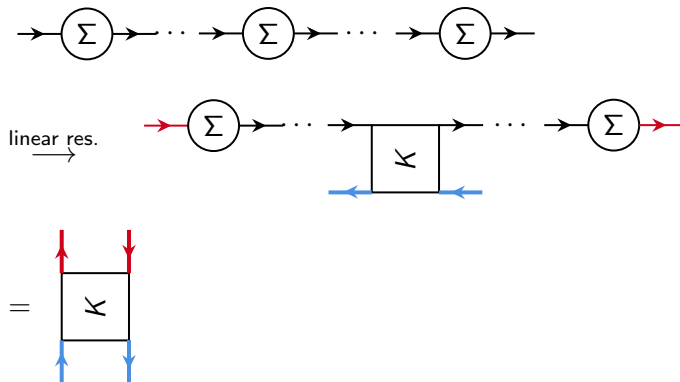
**Next step: relation between  $K$  and  $\Sigma$**

## Linear response of a single self-energy diagram



# Linking $\Sigma$ with $K$

## Whole picture



(16)

## Example: linear response from time-dependent $GW = \text{BSE}$

$$\Sigma = \text{[Diagram 1]} + \text{[Diagram 2]}, \quad (17)$$

The diagram for  $\Sigma$  consists of two terms. The first term is a blue circle with a clockwise arrow, connected to a red horizontal line above it by a vertical dotted line. The second term is a blue horizontal line with a rightward arrow, connected to a red horizontal line above it by a semi-circular dotted line.

$$K = \text{[Diagram 3]} + \text{[Diagram 4]}$$

The diagram for  $K$  consists of two terms. The first term is a red bracket pointing right and a blue bracket pointing left, connected by a horizontal dotted line. The second term is a vertical dotted line connecting a red horizontal line at the top to a blue horizontal line at the bottom.

- First term = Electron Hartree term = Electron direct term = Exciton exchange term;  $+1$  prefactor;
- Second term = Electron Fock term = Electron exchange term = Exciton direct term;  $(-1)$  prefactor.

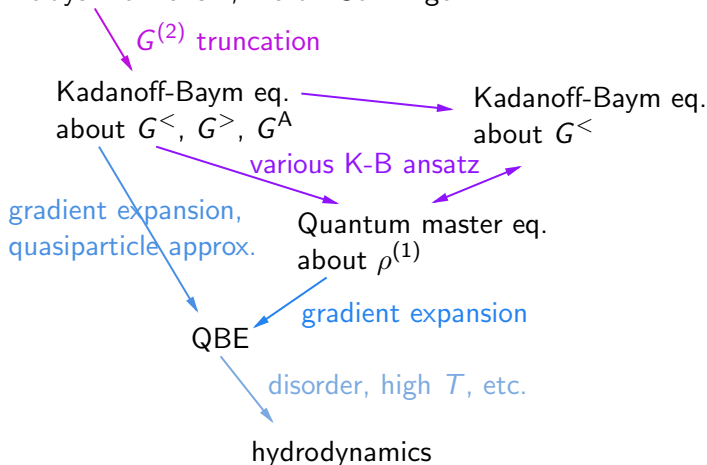


# Justifying quantum Boltzmann equation

## Is QBE reliable?

Yes! When we intuitively expect it to work –

Keldysh formalism, Martin-Schwinger



# Fermi liquid is really liquid

**In small  $T$  limit** Infinite degrees of freedom; QBE still works but the system is not hydrodynamic in the normal sense; bosonization (note that  $n_p(\mathbf{r})$  may also be seen as the field operator or single-electron hole pair wave function)

**In large  $T$  limit** Only five hydrodynamic equations; described by Navier-Stokes equation.

# Subtlety about plasmon

**Classical derivation of plasmon involves a positive background;  
where does this background go in Landau-Silin equation?**

Note that only when  $\delta n \neq 0$  do we have non-zero  $\varphi$ : we have already assumed a positive jellium background.