Prof. Bambi on General Relativity

Jinyuan Wu

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This is a note about Prof. Cosimo Bambi's lecture on general relativity from February 25 to April 8, 2022.

This lecture is about the 1-2 chapters in [1]. Nothing quite interesting. [1] itself is very detailed and it seems I don't really need to take much notes.

1 Relativistic kinetics

1.1 The Christoffel symbols of spherical coordinates

The Christoffel symbol of spherical coordinates, given in (1.83) in [1], can be calculated automatically in this Mathematica notebook.

1.2 Derivation of special relativity

Section 2.1 and 2.2 seem to be based on Chapter 1 and 2 of Landau's book about field theory. The arguments have been summarized in Section 1.1.2 in this note.

Section 2.3 derives the Lorentz transformations by Wick rotation of Euclidean rotation in $d = 4$ – (2.18) is actually just $\tau = it$ in condensed matter physics. A rotation on xy has the form of (2.20) and we have $C_4^2 = 6$ rotations. Adding 4 translations, we get the total 10 generators of the rotation group in $\mathbb{R}^4$, and hence the Lorentz transformations in the (3+1)-dimensional Minkowski spacetime.

Equations from (2.22) to (2.27) are trying to relate the parameter of $\mathbb{R}^4$ rotations to the relative velocity of the two reference frames.

Note that since the rotation group $SO(4)$ is not Abelian, Lorentz transformations do not commute in general. Problem 2.3

The time coordinate in the reference frame attached to a particle (i.e. in the reference frame the particle is at rest) is called the **proper time** of the particle, usually denoted as τ. Since

$$\mathrm{d}s^2 = -\left(c^2 - \left(\frac{\partial x}{\partial t}\right)^2 - \left(\frac{\partial y}{\partial t}\right)^2 - \left(\frac{\partial z}{\partial t}\right)^2\right)\mathrm{d}t^2 = -\left(1 - \frac{v^2}{c^2}\right)c^2\,\mathrm{d}t^2 = -\frac{c^2}{\gamma^2}\,\mathrm{d}t^2 = -c^2\,\mathrm{d}\tau^2\,,$$ (2.36) and (2.37)

we have

$$\mathrm{d}t = \gamma\,\mathrm{d}\tau\,. \tag{1}$$

Since a Lorentz transformation keeps the metrics, we have

$$\mathrm{d}t\,\mathrm{d}V = \mathrm{d}\tau\,\mathrm{d}V_0\,,$$

where $\mathrm{d}V_0$ is the volume metrics in the reference frame attached to the particle. This can also by checked by noting that (here $\boldsymbol{v}$ is along the x axis)

$$\begin{aligned}\mathrm{d}x' \wedge \mathrm{d}t' &= (-\gamma v\,\mathrm{d}t + \gamma\,\mathrm{d}x) \wedge (\gamma\,\mathrm{d}t - \gamma v\,\mathrm{d}x/c^2) \\ &= \gamma^2\,\mathrm{d}x \wedge \mathrm{d}t + \frac{\gamma^2 v^2}{c^2}\,\mathrm{d}t \wedge \mathrm{d}x \\ &= \gamma^2\left(1 - \frac{v^2}{c^2}\right)\mathrm{d}x \wedge \mathrm{d}t = \mathrm{d}x \wedge \mathrm{d}t\,.\end{aligned}$$ from (2.45)

Therefore,

$$\mathrm{d}V = \frac{\mathrm{d}V_0}{\gamma}. \tag{2}$$ (2.44)

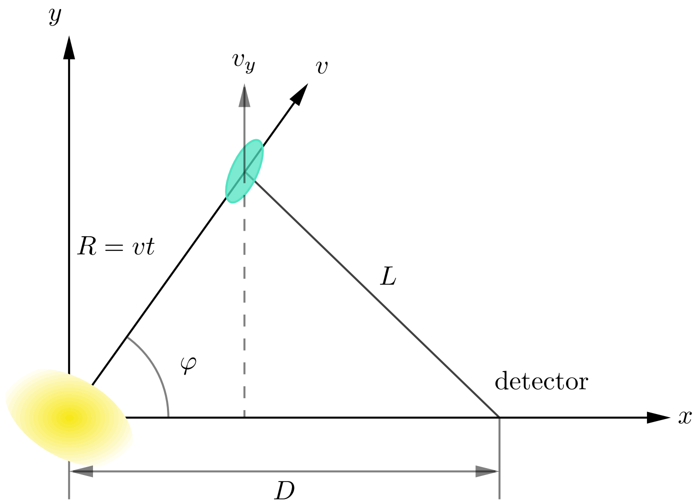

Figure 1: Superluminal motion of a ejected material from a galaxy

1.3 Aberration of light and superluminal motion

We should note that *Lorentz contraction* of length (see for example (2.43)) is often *not* the length change we *observe* when an object is moving fast. The point here is that to measure a line we need to detect light signals from its two ends, and generally speaking, two signals arriving at our detector start their journeys at different time points, while on the other hand, from the way we derive Lorentz contraction ((2.41) and (2.42)) we are dealing with events happening at the same time point in the laboratory frame of reference. The conclusion is l in (2.43) is often not the length we *see* of a moving object. (2.48) to (2.57) (2.41) and (2.42)

The fact that the "distance" we see is actually not the authentic space distance between two events with the same time in a given frame of reference means that when calculating the velocity, we are taking the time derivative of two points at different time points, or in other words, taking the derivative of a distance with respect to a time not coherent with the current frame of reference. In this way, some superfluous "superluminal" movement can occur. Consider, for example, the case of Figure 1 on page 2. Suppose at t a beam of light is emitted from the ejected material, and it arrives at the detector at t'. We therefore have

$$\begin{aligned} c(t'-t) = L &= \sqrt{(D - vt\cos\varphi)^2 + v^2t^2\sin^2\varphi} \\ &= D - vt\cos\varphi + \mathcal{O}(v^2t^2/D^2). \end{aligned} \tag{3}$$ (2.51), (2.52)

Therefore, we have

$$t' = \frac{D}{c} + t(1 - \beta\cos\varphi). \tag{4}$$ (2.53)

Now we try to evaluate the *apparent* velocity on the y direction, which is $\mathrm{d}y/\mathrm{d}t'$. Note that only $\mathrm{d}y/\mathrm{d}t$ is bounded by c, while $\mathrm{d}y/\mathrm{d}t'$ does not have an upper bound. Actually, we have

$$\frac{\mathrm{d}y}{\mathrm{d}t'} = \frac{\mathrm{d}y}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}t'} = v\sin\varphi \times \frac{1}{1-\beta\cos\varphi}. \tag{5}$$ (2.54)

The maximum is shown to be $v\gamma$. Here we can see the apparent velocity is obtained using t' as the time, which is not x^0 in the frame of coordinate attached to the observer. (2.57)

1.4 Time dilation and cosmic ray muons

The lifetime of an unstable particle is measured according to its proper time, and this causes the difference between the prediction of Newtonian and relativistic theories of the flux of the particle after traveling a certain distance. This is actually a piece of strong evidence of special relativity. Sec. 2.6

2 Relativistic mechanics (single particle)

2.1 The action

We know the Lagrangian of a Newtonian free particle is

$$L = \frac{1}{2} m g_{ij} \dot{x}^i \dot{x}^j . \tag{6}$$ (3.1)

The natural generalization is

$$L = \frac{1}{2} m g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu . \tag{7}$$

Note, however, the meaning of $\dot{x}$ is not clear here: $g_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu$ is already a covariant value, and to make the whole expression covariant, the "time" used in the time derivative $\dot{x}$ should also be a relativistic scalar, which can only be the proper time τ. Similarly, when calculating the action, we need to integrate (7) over τ. So the final action is

$$S = \int \mathrm{d}\tau\, L, \quad L = \frac{1}{2} m g_{\mu\nu} \frac{\mathrm{d}x^\mu}{\mathrm{d}\tau} \frac{\mathrm{d}x^\nu}{\mathrm{d}\tau}. \tag{8}$$

We can repeat the process in (1.71) and find that the EOM is (1.71)

$$\frac{\mathrm{d}^2 x^\mu}{\mathrm{d}\tau^2} + \Gamma^\mu{}_{\nu\rho} \frac{\mathrm{d}x^\nu}{\mathrm{d}\tau} \frac{\mathrm{d}x^\rho}{\mathrm{d}\tau} = 0. \tag{9}$$ (3.6)

Actually Newton's second law can also be written into this form (again see the discussion around (1.71)), but this time we are working with $\mu, \nu, \rho = 0, 1, 1, 3$, not just $1, 2, 3$. Note that (9) is actually the geodesic equation.

(9) is about $\mathrm{d}x^\mu / \mathrm{d}\tau$, which is the tangent vector of the trajectory of the particle and is manifestly a 4-vector. Its components are

$$\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau} = \frac{\mathrm{d}x^\mu}{\mathrm{d}t} \frac{\mathrm{d}t}{\mathrm{d}\tau} = (\gamma c, \gamma \boldsymbol{v}). \tag{10}$$

Note

When we say some expression is **manifestly** covariant, we mean the expression is built up by tensor and Einstein summation notation and can be automatically decided as covariant.

We have a more "geometric" version of (6). Since

$$\mathrm{d}l = \sqrt{g_{ij} \dot{x}^i \dot{x}^j}\, \mathrm{d}t$$

is an increasing function of the integrand of (6), the action corresponding to (6) is equivalent to

$$S = \int \mathrm{d}l\, . \tag{11}$$

Similarly, we may guess the version of (11) corresponding to (8) is

$$S = \int |\mathrm{d}s| = \int \sqrt{-\,\mathrm{d}s^2}. \tag{12}$$

This is indeed the case, since actually by the definition of proper time, we have

$$c^2\,\mathrm{d}\tau^2 = -\,\mathrm{d}s^2 = -g_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu\, , \tag{13}$$ (3.4)

and therefore L in (8) is just a constant, and we have $S \propto \int \mathrm{d}\tau$.

A question is what is the relativistic version of mechanics of *massless* particles. Since m in Sec. 3.3 the action is just a constant, the geodesic equation (9) still works. The Lagrangian therefore can still be written as

$$L = -mc\sqrt{-g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu}, \tag{14}$$ (3.38)

but now m should be a constant interpreted as a *coupling constant* with the dimension of mass.

2.2 Particle collision

Relativistic scattering theory is important in particle physics. Consider a typical reaction:

$$A + B \longrightarrow C + D. \tag{15}$$ (3.41)

Suppose we are working in a frame of reference where A is at rest. Suppose B moves along the x axis. Then we have

$$p_A^\mu = (m_A c, 0, 0, 0), \quad p_B^\mu = (m_B \gamma c, m_B \gamma v, 0, 0). \tag{16}$$ (3.43)

After the reaction, the energies and the momenta of C and D can be quite complicated, so we just try to find some general constraints imposed on them. A reaction is possible if and only if both energy conservation and momentum conservation hold. The momentum conservation condition can be satisfied by working in a reference frame where the total 3-momentum of the system vanishes, and this dictates

$$p_C^\mu = (\sqrt{m_C^2 c^2 + \boldsymbol{p}_C^2}, \boldsymbol{p}_C), \quad p_D^\mu = (\sqrt{m_D^2 c^2 + \boldsymbol{p}_C^2}, -\boldsymbol{p}_C). \tag{17}$$ (3.44)

Now we just need to impose the energy conservation constraint. Naively doing so is hard because energy itself is not a scalar. However, there *is* a conserved relativistic scalar: we have

$$p_\mu^{\mathrm{i}} p^{\mathrm{i}\mu} = p_\mu^{\mathrm{f}} p^{\mathrm{f}\mu} =: M^2 c^2, \tag{18}$$

where M is named the **invariant mass**. Note that we can evaluate the LHS in the reference frame of (16) and the RHS in the reference frame of (17), and this gives

$$(m_A c + m_B \gamma c)^2 - m_B^2 \gamma^2 v^2 = (m_C c + m_D c)^2 - (2\boldsymbol{p}_C)^2, \tag{19}$$ (3.48)

which can be simplified into

$$2 m_A m_B \gamma = m_C^2 + m_D^2 + 2 m_C m_D - m_A^2 - m_B^2 - 4\boldsymbol{p}_C^2 / c^2. \tag{20}$$ (3.49)

The minimum energy of B is

$$E_B^{\mathrm{th}} = p_B^0 c = m_B \gamma c^2 = \frac{\left(m_C^2 + m_D^2 + 2 m_C m_D - m_A^2 - m_B^2\right) c^2}{2 m_A}. \tag{21}$$ (3.50)

This is called the **threshold energy**, because if $E_B < E_B^{\mathrm{th}}$, (15) cannot happen.

The procedure can be repeated for different processes and from this we can find another fact about collision that head-on collision is more effective than fixed-target collision.

3 Relativistic perfect fluid

The topic of relativistic idea fluid is discussed in Problem 2.1 and 2.2. First of all, we always have Problem 2.1, 2.2

$$T^{00} = \epsilon = \rho_m c^2,$$

where ϵ is the energy density of the fluid in the rest frame, and ρ_m is the mass density. In a perfect fluid, when we are in the rest-frame, there is no flow, and since momentum is carried by fluid flow, the density of momentum is also zero, i.e.

$$T^{i0} = 0.$$

The argument used in non-relativistic perfect fluid can be transplanted here for T^{ij}: since an idea fluid is isotropic, and it cannot hold shear force, as long as the time scale we are interested is long enough to hide how the fluid responds to an external shear force, we can assume all shear force components in T^{ij} are zero. Thus we have

$$T^{ij} = \begin{pmatrix} p & & \\ & p & \\ & & p \end{pmatrix}.$$

Putting everything together, we get

$$T^{\mu\nu} = \begin{pmatrix} \epsilon & & & \\ & p & & \\ & & p & \\ & & & p \end{pmatrix}. \tag{22}$$ (2.60)

Now we can get $T^{\mu\nu}$ in any coordinate system with a Lorentz transformation. Applying the Lorentz transformation on x direction:

$$\Lambda^\mu{}_\nu = \begin{pmatrix} \gamma & -\gamma\beta & & \\ -\gamma\beta & \gamma & & \\ & & 1 & \\ & & & 1 \end{pmatrix}, \tag{23}$$ (2.28)

we get (the process can be found in this Mathematica notebook)

$$T'^{\mu\nu} = \begin{pmatrix} \frac{p\beta^2}{1-\beta^2} + \frac{\epsilon}{1-\beta^2} & -\frac{\epsilon\beta}{1-\beta^2} - \frac{p\beta}{1-\beta^2} \\ -\frac{\epsilon\beta}{1-\beta^2} - \frac{p\beta}{1-\beta^2} & \frac{\epsilon\beta^2}{1-\beta^2} + \frac{p}{1-\beta^2} \end{pmatrix}. \tag{24}$$

Suppose the speed of the frame of reference after (23) in the rest frame of the fluid is v, we find (24) is the energy-momentum tensor of a fluid moving with the velocity of $-v\hat{\boldsymbol{e}}_x$. Problem 2.2

(24) is not covariant. We need to generalize it into a covariant version. Solely with information provided in (24), the covariant version cannot be decided, because systems other than a perfect fluid can also has a energy-momentum tensor like (22). Another way to see the point is to note that velocity of the fluid is different on different points, and a global Lorentz transformation cannot turn the fluid into the state of rest. We can do local Lorentz transformation, but this distorts the components of $\eta^{\mu\nu}$, but when deriving (22) we have $\eta = \mathrm{diag}(-1, 1, 1, 1)$. Liang Sec. 6.5

The generic covariant energy-momentum tensor of a perfect fluid is

$$T^{\mu\nu} = \left(\rho_m + \frac{p}{c^2}\right) U^\mu U^\nu + p\eta^{\mu\nu} = \frac{1}{c^2}\left(\epsilon + p\right) U^\mu U^\nu + p\eta^{\mu\nu}. \tag{25}$$

Note that p and ϵ in (25) are defined in a special frame of reference, but this does not eliminate the covariance of (25), because for a fluid there is indeed a special frame of reference, i.e. the rest frame of itself.

Finally we check whether (25) reduces to (24) if the velocity of the fluid is globally $-v\hat{\boldsymbol{e}}$. When in the rest frame of reference, we have

$$U^\mu = \begin{pmatrix} c \\ 0 \\ 0 \\ 0 \end{pmatrix}, \tag{26}$$

and (25) reads

$$T^{\mu\nu} = \frac{1}{c^2}(\epsilon + p) \begin{pmatrix} c^2 & & & \\ & 0 & & \\ & & 0 & \\ & & & 0 \end{pmatrix} + p \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix},$$

which is just (22). After a global Lorentz transformation (23), we have

$$U'^\mu = \Lambda^\mu{}_\nu U^\nu = \begin{pmatrix} \gamma c \\ -\gamma\beta c \\ 0 \\ 0 \end{pmatrix}.$$

Substituting this into (25), we indeed come back to (24). (The process is in this Mathematica notebook).

4 Electromagnetism

4.1 The action

In this section we try to establish a relativistic covariant version of electromagnetism. First, the EOM of particles

$$m\ddot{\boldsymbol{r}} = e\boldsymbol{E} + \frac{e}{c}\dot{\boldsymbol{r}} \times \boldsymbol{B} \tag{27}$$ (4.1)

has to change, because it allows particles to be accelerated without an upper bound. Second, the Maxwell equations must be written into a covariant form. We've already done this in this (4.2) to (4.5) note, but here we need to repeat the procedure with $g_{\mu\nu} = \mathrm{diag}(-1, 1, 1, 1)$ and in the Gaussian unit system.

We will find the action

$$\begin{aligned} S &= S_\mathrm{m} + S_\mathrm{int} + S_\mathrm{em}, \\ S_\mathrm{m} &= -mc\int_\Gamma \sqrt{-\mathrm{d}s^2}, \\ S_\mathrm{int} &= \frac{e}{c}\int_\Gamma A_\mu \mathrm{d}x^\mu, \\ S_\mathrm{em} &= -\frac{1}{16\pi c}\int_\Omega F^{\mu v} F_{\mu v} \mathrm{d}^4\Omega \end{aligned} \tag{28}$$

recovers to the relativistic version of Newton's second law and Maxwell equations. Here $\mathrm{d}^4\Omega =$ (4.15) and $c\,\mathrm{d}t\,\mathrm{d}^3\boldsymbol{r}$. Note that the definition of $F_{\mu\nu}$ here differs with $F_{\mu\nu}$ with $g_{\mu\nu} = (1, -1, -1, -1)$ with a (4.12) global minus sign.

The interaction action S_int can also be written into a "hydrodynamic" form. Before doing (4.19) so, we need to find a relativistic description of flowing. We start from a collective quantity that is invariant between frames of reference, which is total electric charge here. The density is

$$\rho = \frac{\mathrm{d}Q}{\mathrm{d}V} = \gamma \frac{\mathrm{d}Q}{\mathrm{d}V_0} \tag{29}$$

and is not a Lorentz scalar (because $\mathrm{d}Q/\mathrm{d}V_0$ is one). We tentatively define a manifestly 4-vector

$$J^\mu = \frac{\mathrm{d}Q}{\mathrm{d}V_0} U^\mu = \frac{\rho}{\gamma}(\gamma c, \gamma \boldsymbol{v}) = (\rho c, \rho \boldsymbol{v}), \tag{30}$$

where U^μ in the many-body case is the coarse-grained 4-velocity as defined in (10), and in the case where there is only one particle is just (10). It can be immediately found that $\partial_\mu J^\mu = 0$, and therefore J^μ is a good definition of 4-current.

Now in the case with only one particle in the electrodynamic field, we have

$$\rho(\boldsymbol{r}', t) = e\delta(\boldsymbol{r}' - \boldsymbol{r}(t)), \quad \Gamma = \{(t, \boldsymbol{r}(t))\}_t,$$

so

$$\begin{aligned} S_\mathrm{int} &= \frac{1}{c}\int \rho\,\mathrm{d}V \int_\Gamma \mathrm{d}x^\mu\, A_\mu = \frac{1}{c}\int \rho\,\mathrm{d}V \int_{t_1}^{t_2} A_\mu \frac{\mathrm{d}x^\mu}{\mathrm{d}t}\,\mathrm{d}t \\ &= \frac{1}{c^2}\int_\Omega \underbrace{c\,\mathrm{d}t\,\mathrm{d}V}_{\mathrm{d}^4x} \rho \frac{\mathrm{d}x^\mu}{\mathrm{d}t} A_\mu \\ &= \frac{1}{c^2}\int_\Omega \mathrm{d}^4x\, J^\mu_\text{single particle} A_\mu. \end{aligned}$$

Since J^μ in a continuum is just the coarse-grained version of $\sum J^\mu_\text{single particle}$, the many-body version of S_int is

$$S_\mathrm{int} = \frac{1}{c^2}\int_\Omega \mathrm{d}^4x\, J^\mu A_\mu. \tag{31}$$ (4.19)

4.2 Maxwell equations

Maxwell equations don't need to be generalized, because they are already invariant.

5 Riemannian geometry

5.1 Christoffel symbol and covariant derivative

Here we briefly list some important formulae:

$$\nabla_\nu u^\mu = \partial_\nu u^\mu + \Gamma^\mu_{\nu\rho} u^\rho, \qquad (32) \quad (5.15)$$

and from this and $\nabla_\nu (u^\mu w_\mu) = 0$, recursively we have

$$\begin{aligned}\nabla_\lambda T^{\mu_1\mu_2\ldots\mu_r}_{\nu_1\nu_2\ldots\nu_s} = \frac{\partial}{\partial x^\lambda} T^{\mu_1\mu_2\ldots\mu_r}_{\nu_1\nu_2\ldots\nu_s} & \underbrace{+\Gamma^{\mu_1}_{\lambda\sigma} T^{\sigma\mu_2\ldots\mu_r}_{\nu_1\nu_2\ldots\nu_s} + \Gamma^{\mu_2}_{\lambda\sigma} T^{\mu_1\sigma\ldots\mu_r}_{\nu_1\nu_2\ldots\nu_s} + \ldots + \Gamma^{\mu_r}_{\lambda\sigma} T^{\mu_1\mu_2\ldots\sigma}_{\nu_1\nu_2\ldots\nu_s}}_{r\text{ terms}} \\ & \underbrace{-\Gamma^\sigma_{\lambda\nu_1} T^{\mu_1\mu_2\ldots\mu_r}_{\sigma\nu_2\ldots\nu_s} - \Gamma^\sigma_{\lambda\nu_2} T^{\mu_1\mu_2\ldots\mu_r}_{\nu_1\sigma\ldots\nu_s} - \ldots - \Gamma^\sigma_{\lambda\nu_s} T^{\mu_1\mu_2\ldots\mu_r}_{\nu_1\nu_2\ldots\sigma}}_{s\text{ terms}},\end{aligned} \qquad (33) \quad (5.50)$$

and specifically, we have

$$\nabla_\mu W_\nu = \frac{\partial W_\nu}{\partial x^\mu} - \Gamma^\rho_{\mu\nu} W_\rho. \qquad (34) \quad (5.49)$$

These formulae can be remembered using the following key points:

- $\nabla_\nu u^\mu$ is used as a definition of covariant derivative and hence the sign before the Γ term is $+$.
- There is only one upper index μ, and it can't be on u in the Γ term. The lower index ν must be reflected on the Christoffel symbol. So the Γ term has to be the contraction of $\Gamma^\mu_{\nu\cdot}$ and u, and hence $\Gamma^\mu_{\nu\rho} u^\rho$.
- Then from $\nabla_\nu (u^\mu w_\mu) = 0$ we get (34). The $-$ sign comes from the fact that the two Γ terms in (32) and (34) must cancel with each other. There are now two lower indices μ, ν, and neither of them should be on W (or otherwise the upper index of Γ has no lower index to contract with). So both of them should be on Γ, and the Γ term is therefore $-\Gamma^\rho_{\mu\nu} W_\rho$.
- Then (33) can be obtained by remembering that λ is always on the Γ symbol, r terms are like (32), and s terms are like (34).

The Christoffel symbol is given by

$$\Gamma^\mu_{\nu\rho} = \frac{1}{2} g^{\mu\lambda} \left(\frac{\partial g_{\lambda\rho}}{\partial x^\nu} + \frac{\partial g_{\nu\lambda}}{\partial x^\rho} - \frac{\partial g_{\nu\rho}}{\partial x^\lambda} \right). \qquad (35)$$ Conventions

This formula can be remembered by noticing

- it starts with $\frac{1}{2} g^{\mu\lambda}$. The μ index is an upper index, so it must come with another upper index λ, which, then, have to be contracted with a lower index in the brackets.
- The first and the second term can be obtained by placing the lower λ on g. The third term is obtained by placing λ on x.
- There are two positive terms, so the signs have to be $+, +, -$, because after $\nu \leftrightarrow \rho$, the first and the second terms swaps, so they must bear the same sign.

From this we find (5.52). Here we give a step by step derivation of $\nabla_\sigma g^{\mu\nu}$. (5.52)

- Using the tricks described above, we have

  $$\nabla_\sigma g^{\mu\nu} = \partial_\sigma g^{\mu\nu} + \Gamma^\mu_{\sigma\delta} g^{\delta\nu} + \Gamma^\nu_{\sigma\delta} g^{\delta\mu}.$$

  Note that the second and third terms only differ with $\mu \leftrightarrow \nu$.
- Using the tricks described above, we have

  $$\Gamma^\mu_{\sigma\delta} = \frac{1}{2} g^{\mu\lambda} (\partial_\sigma g_{\lambda\delta} + \partial_\delta g_{\lambda\sigma} - \partial_\lambda g_{\sigma\delta}).$$

- Both $\partial_\sigma g^{\mu\nu}$ and $\partial_\sigma g_{\mu\nu}$ appear. Since
$$\partial_\sigma(g_{\lambda\delta}g^{\delta\alpha}) = \partial_\sigma \delta^\alpha_\lambda = 0,$$
we have
$$g_{\lambda\delta}\partial_\sigma g^{\delta\alpha} + g^{\delta\alpha}\partial_\sigma g_{\lambda\delta} = 0.$$
By multiply $g_{\alpha\beta}$ to the equation (i.e. taking the inverse of $g^{\delta\alpha}$), we have
$$\partial_\sigma g_{\lambda\beta} = -g_{\alpha\beta}g_{\lambda\delta}\partial_\sigma g^{\delta\alpha}.$$

- Therefore
$$\begin{aligned}\Gamma^\mu_{\sigma\delta}g^{\delta\nu} &= \frac{1}{2}g^{\mu\lambda}g^{\delta\nu}(-g_{\alpha\delta}g_{\lambda\beta}\partial_\sigma g^{\beta\alpha} - g_{\alpha\sigma}g_{\lambda\gamma}\partial_\delta g^{\gamma\alpha} + g_{\alpha\delta}g_{\sigma\gamma}\partial_\lambda g^{\lambda\alpha}) \\ &= -\frac{1}{2}\partial_\sigma g^{\mu\nu} - \frac{1}{2}g^{\delta\nu}g_{\alpha\sigma}\partial_\delta g^{\mu\alpha} + \frac{1}{2}g^{\mu\lambda}g_{\sigma\gamma}\partial_\lambda g^{\gamma\nu}.\end{aligned}$$
Swapping μ and ν, we have
$$\Gamma^\nu_{\sigma\delta}g^{\delta\mu} = -\frac{1}{2}\partial_\sigma g^{\mu\nu} - \frac{1}{2}g^{\delta\mu}g_{\alpha\sigma}\partial_\delta g^{\nu\alpha} + \frac{1}{2}g^{\nu\lambda}g_{\sigma\gamma}\partial_\lambda g^{\gamma\mu}.$$
So we find
$$\Gamma^\mu_{\sigma\delta}g^{\delta\nu} + \Gamma^\nu_{\sigma\delta}g^{\delta\mu} = -\partial_\sigma g^{\mu\nu},$$
and therefore we complete the proof.

5.2 Riemann tensor

Now we go on to Riemann tensor. First we explain the equivalence between (5.67) and (5.83) (if we take the latter as a definition of the Riemann tensor and ignore its explicit expression in the second equation). The transport of u^μ along an infinitesimal vector p^μ, (5.80), is derived from (5.80) to (5.83)

$$\begin{aligned}0 = p^\mu\nabla_\mu u^\nu &= p^\mu(\partial_\mu u^\nu + \Gamma^\nu_{\mu\sigma}u^\sigma) \\ &= u^\nu_{A\to B} - u^\nu + p^\mu\Gamma^\nu_{\mu\sigma}u^\sigma\end{aligned}$$

and therefore
$$u^\nu_{A\to B} = u^\nu - p^\mu\Gamma^\nu_{\mu\sigma}u^\sigma. \tag{36}$$ (5.80)

Now we have
$$u^\mu_{A\to B\to D} = (1 + q^\nu\nabla_\nu)u^\mu_{A\to B} = (1 + q^\nu\nabla_\nu)(1 + p^\rho\nabla_\rho)u^\mu,$$
and similarly
$$u^\mu_{A\to C\to D} = (1 + p^\rho\nabla_\rho)(1 + q^\nu\nabla_\nu)u^\mu.$$
So
$$\begin{aligned}u^\mu_{A\to B\to D} - u^\mu_{A\to C\to D} &= q^\nu p^\rho(\nabla_\nu\nabla_\rho - \nabla_\rho\nabla_\nu)u^\mu \\ &= q^\nu p^\rho(\nabla_\nu\nabla_\rho - \nabla_\rho\nabla_\nu)u_\sigma g^{\mu\sigma} \\ &= q^\nu p^\rho g^{\mu\sigma}R^\lambda{}_{\sigma\rho\nu}u_\lambda.\end{aligned}$$
The equivalence between (5.67) and (5.83) is therefore reduced to the following equation:
$$R^\mu{}_{\tau\nu\rho}u^\tau q^\nu p^\rho = q^\nu p^\rho g^{\mu\sigma}R^\lambda{}_{\sigma\rho\nu}u_\lambda. \tag{37}$$
We have
$$\begin{aligned}\text{LHS} &= g^{\mu\sigma}R_{\sigma\tau\nu\rho}u^\tau q^\nu p^\rho = g^{\mu\sigma}R_{\tau\sigma\rho\nu}u^\tau q^\nu p^\rho, \\ \text{RHS} &= q^\nu p^\rho g^{\mu\sigma}g^{\tau\lambda}R_{\tau\sigma\rho\nu}u_\lambda = q^\nu p^\rho g^{\mu\sigma}R_{\tau\sigma\rho\nu}u^\tau = \text{LHS},\end{aligned}$$ (5.77)
so indeed (5.67) and (5.83) are equivalent as definitions of the Riemann tensor.

6 Physics with a background gravitational field

Note that [1] uses the term *general relativity* to denote any metric theory of gravity.

6.1 Absorbing inertial forces into the metric

6.2 Absorbing Newtonian gravity into the metric

In the Newtonian limit, suppose the gravitational potential is Φ, the Lagrangian is

$$L = -mc^2 + \frac{1}{2}m\boldsymbol{v}^2 - m\Phi. \tag{38}$$ (5.3)

Note that $mc^2 \gg m\boldsymbol{v}^2/2, m\Phi$. This Lagrangian has a natural high-speed completion: (5.4)

$$L = -mc\sqrt{c^2 - \boldsymbol{v}^2 + 2\Phi} = -mc^2\left(1 + \frac{-\boldsymbol{v}^2 + 2\Phi}{c^2} + \mathcal{O}(\boldsymbol{v}^4/c^4, \Phi^2/c^4)\right). \tag{39}$$

Note that this is the Lagrangian of a free-falling particle, and if it's a metric theory, it can be rephrased into

$$S = -mc\int\sqrt{-g_{\mu\nu}\dot{x^\mu}\dot{x^\nu}}\,\mathrm{d}t\,, \quad \dot{x}^\mu := \frac{\mathrm{d}x^\mu}{\mathrm{d}t}. \tag{40}$$

Now obviously, we have

$$g_{\mu\nu} = \begin{pmatrix} -\left(1+\frac{2\Phi}{c^2}\right) & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}. \tag{41}$$ (5.7)

So we find Newtonian gravity is a low-speed, low gravitational force limit of a metric theory with metrics (41).

Note that it's not necessary that in the low speed limit of the particle and weak gravitational field limit (actually, since gravitational force can be used to accelerate the particle, a low particle speed already implies a weak gravitational field) the only kind of gravitational potential is the Newtonian one. For example, see the PPN formalism. This is because it's possible that $g_{\mu\nu}$ has components deviate from the Lorentz metrics other than g_{00}. However, even with the presence of such components, Newtonian gravity is *still* the low-speed weak-field theory of gravity, because a particle moving slow enough is *unable to feel* these components. This is what is actually shown in Section 6.3. We see that the derivation ends in (6.14), which is the g_{00} component of (41) and (6.4) to
is only about g_{tt} but not other components – but other components are not *relevant*, anyway. (6.14)

6.3 Local inertial reference frame and moving frame

In this section we show the existence of a local inertial reference frame, i.e. to show that a metric theory always satisfy Einstein's equivalence principle. First, we can always diagonalize the metrics at a certain point by the following coordinate transformation:

$$\mathrm{d}x^\mu \to \mathrm{d}x'^\mu = e^{\mu'}_\nu\,\mathrm{d}x^\nu\,, \tag{42}$$

where $e^{\mu'}_\nu$'s are obtained by the following diagonalization (the eigenvalues are absorbed into $e^{\mu'}_\nu$)

$$g_{\mu\nu} = e^{\alpha'}_\mu e^{\beta'}_\nu \eta_{\alpha'\beta'}. \tag{43}$$

So without loss of generality, we assume

$$g_{\mu\nu}(0) = \eta_{\mu\nu}. \tag{44}$$

The metric tensor can be expanded as

$$g_{\mu v}(x) = g_{\mu v}(0) + \left.\frac{\partial g_{\mu v}}{\partial x^\rho}\right|_0 x^\rho + \frac{1}{2}\left.\frac{\partial^2 g_{\mu v}}{\partial x^\rho \partial x^\sigma}\right|_0 x^\rho x^\sigma + \cdots\,. \tag{45}$$ (6.21)

Now suppose we do the coordinate transformation

$$x^\mu \to x'^\mu = x^\mu + \frac{1}{2}\Gamma^\mu_{\rho\sigma}(0)x^\rho x^\sigma + \cdots\,. \tag{46}$$ (6.22)

The inverse is

$$x^\mu = x'^\mu - \frac{1}{2}\Gamma^\mu_{\rho\sigma}(0)x'^\rho x'^\sigma + \cdots\,, \tag{47}$$

so we have

$$\frac{\partial x^\alpha}{\partial x'^\mu} = \delta^\alpha_\mu - \Gamma^\alpha_{\mu\nu}(0)x'^\nu + \cdots \,. \tag{48}$$

Under these transformations we find (6.29), so (6.26) evaluates to zero. By the definition of $\Gamma^\sigma_{\mu\nu}$, (6.26) to
after the transformation, we see all Christoffel symbols become zero, so locally, we get $\nabla_\mu \to g_\mu$, (6.29)
$g_{\mu\nu} \to \eta_{\mu\nu}$, and hence $\{x'^\mu\}$ is a local inertial reference frame.

7 Time slowing down in gravitational field

As long as gravity is created by the metrics, clocks placed in a gravitational field is slower than a clock far from any gravity source. For example, with (41), we have

$$\mathrm{d}\tau^2 = \left(1 + \frac{2\Phi}{c^2}\right)\mathrm{d}t^2 \,. \tag{49}$$

When the metrics is not that simple, but the clock moves very slowly, we still have (49).

8 Einstein's gravity

References

[1] Cosimo Bambi. *Introduction to General Relativity: A Course for Undergraduate Students of Physics.* Springer, 2018.