Photon transferring

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1 Kinetics of photons

1.1 Heuristic derivation of the light transportation equation

In this section we derive the light transportation equation in a way inspired by the quantum Boltzmann equation (QBE). We know in an isotropic medium, the dispersion relation of light is always in the form of

$$\omega = ck, \tag{1}$$

where c is the effective speed of light in the medium, and therefore

$$\boldsymbol{\nabla}_{\boldsymbol{k}} = c\hat{\boldsymbol{k}} =: c\hat{\boldsymbol{s}}, \tag{2}$$

where $\hat{\boldsymbol{s}}$ is the direction of the photon momentum, i.e. the direction of light propagation. Thus, for photons in an isotropic and uniform medium, Suppose $f(\boldsymbol{r}, \boldsymbol{k}, t)$ is the distribution function of photons. In the LHS of QBE, we have

$$\begin{aligned} \text{LHS} &= \frac{\partial f}{\partial t} + \frac{\partial \omega}{\partial \boldsymbol{k}} \cdot \frac{\partial f}{\partial \boldsymbol{r}} - \frac{\partial \omega}{\partial \boldsymbol{r}} \cdot \frac{\partial f}{\partial \boldsymbol{k}} \\ &= \frac{\partial f}{\partial t} + c\hat{\boldsymbol{s}} \cdot \frac{\partial f}{\partial \boldsymbol{r}}. \end{aligned} \tag{3}$$

Ignoring non-linear processes, due to conservation of energy, the magnitude of the $\boldsymbol{k}$ vector is conserved, and therefore we may confine ourselves to a small segment of $f(\boldsymbol{r}, \boldsymbol{k}, t)$ where $|\boldsymbol{k}|$ is a given constant. Since we are only working with single-photon processes, and all scattering events can be thought of as scattering with some kind of "disorders" in the medium, the collision integral in QBE, obtained from Fermi golden rule, is linear with respect to f: it always takes the form of

$$\text{RHS} = -2\pi \sum_{\boldsymbol{k}'} |M(\boldsymbol{k} \to \boldsymbol{k}')|^2 (f(\boldsymbol{r}, \boldsymbol{k}, t) - f(\boldsymbol{r}, \boldsymbol{k}', t)) \delta(\omega_{\boldsymbol{k}} - \omega_{\boldsymbol{k}'}), \tag{4}$$

where the quantum corrections to the incoming and the outgoing terms cancel each other.

Since the QBE with an isotropic and uniform diffusion term and a disorder-induced collision integral outlined above is linear, we can multiply an arbitrary factor to f without modifying the form of the equation. In a macroscopic theory we want to work with quantities that have direct macroscopic meanings; for radiation this means we want to work with energy, i.e. $(n + 1/2)\hbar\omega \approx n\hbar\omega$. Since $|\boldsymbol{k}|$ is conserved, we can decompose the $\boldsymbol{k}$ degree of freedom into the degree of freedom of frequency (i.e. color) and the degree of freedom of wave vector orientation, referred to as $\hat{\boldsymbol{s}}$, the solid angle element of which is $\mathrm{d}\Omega$. We define **radiance** as a function $L(\boldsymbol{r}, \hat{\boldsymbol{s}}, t)$ such that

$$L(\boldsymbol{r}, \hat{\boldsymbol{s}}, t)\hat{\boldsymbol{s}}\, \mathrm{d}\omega\, \mathrm{d}\Omega = \underbrace{f(\boldsymbol{r}, \boldsymbol{k}, t) \cdot k^2\, \mathrm{d}k\, \mathrm{d}\Omega}_{\text{photon number}} \cdot \underbrace{\hbar\omega}_{\text{single photon energy}} \cdot \underbrace{c\hat{\boldsymbol{s}}}_{\text{velocity}}, \quad \omega = ck, \tag{5}$$

and thus the energy flux (also known as the **flux**) on the spectrum range between ω and $\omega + \mathrm{d}\omega$ going through an area element $\mathrm{d}\boldsymbol{A}$ is

$$\mathrm{d}\Phi = L(\boldsymbol{r}, \hat{\boldsymbol{s}}, t)\hat{\boldsymbol{s}} \cdot \mathrm{d}\boldsymbol{A}\, \mathrm{d}\omega\, \mathrm{d}\Omega\,, \tag{6}$$

The unit of L is $\mathrm{W}/(\mathrm{m}^2{\cdot}\mathrm{Hz})$: when $L\,\mathrm{d}\omega$ is understood as an energy current, the unit has clear physical meaning, and we can also understand L's unit as $\mathrm{W}/\mathrm{m}^3{\cdot}\mathrm{m/s}$, which means a part in $L\,\mathrm{d}\omega$ can be seen as energy density times c. Both understandings are correct, as is illustrated in (16).

We also want to parameterize the collision integral of QBE. The RHS of QBE of L is now rewritten as

$$\begin{aligned} \text{RHS} &= c\mu_\text{s} \int \mathrm{d}\Omega' \, (L(\boldsymbol{r}, \hat{\boldsymbol{s}}', t) - L(\boldsymbol{r}, \hat{\boldsymbol{s}}, t)) P(\hat{\boldsymbol{s}}, \hat{\boldsymbol{s}}') \\ &= c\mu_\text{s} \int \mathrm{d}\Omega' \, L(\boldsymbol{r}, \hat{\boldsymbol{s}}', t) P(\hat{\boldsymbol{s}}, \hat{\boldsymbol{s}}') - c\mu_\text{s} L(\boldsymbol{r}, \hat{\boldsymbol{s}}, t), \end{aligned} \tag{7}$$

where we have decomposed the scattering matrix into an overall strength $c\mu_\text{s}$ (the c factor will cancel with the c factor in the second term of LHS), and a function $P(\hat{\boldsymbol{s}}, \hat{\boldsymbol{s}}')$ – for some reason called the **phase function** – measuring how anisotropic the scattering process is, and the normalization condition is

$$\int \mathrm{d}\Omega' \, P(\hat{\boldsymbol{s}}, \hat{\boldsymbol{s}}') = 1, \tag{8}$$

and when the scattering process is also completely isotropic, we have

$$P(\hat{\boldsymbol{s}}, \hat{\boldsymbol{s}}') = \frac{1}{4\pi}. \tag{9}$$

The QBE of photons therefore becomes the following **radiation transfer equation (RTE)**:

$$\frac{1}{c} \frac{\partial L}{\partial t} + \hat{\boldsymbol{s}} \cdot \boldsymbol{\nabla}_{\boldsymbol{r}} L = -\mu_\text{s} L + \mu_\text{s} \int \mathrm{d}\Omega' \, L(\boldsymbol{r}, \hat{\boldsymbol{s}}', t) P(\hat{\boldsymbol{s}}, \hat{\boldsymbol{s}}'). \tag{10}$$

The conditions of the validity of the above equation includes the aforementioned assumptions that <u>the medium is uniform and isotropic, and there is no nonlinear process in the system,</u> and also the fundamental validity condition of QBE: <u>the validity of gradient expansion of the effective density matrix in the Wigner representation</u>. Physically speaking, this means the system should allow the formation of defined wave packets, which has well-defined positions but also looks like a plane wave when we zoom in on it. Specifically, this means diffraction shouldn't be an important factor here; but if a structure with strong diffraction can be modeled as a scattering process, diffraction in this case can still be captured by the collision integral.

The next step is to find the true meaning of μ_s. Assuming a clean background and a stationary configuration, the second term in the RHS of (10) vanishes, we have

$$\frac{\mathrm{d}L}{\mathrm{d}z} = -\mu_\text{s} L,$$

where z is the propagation length. So we find $1/\mu_\text{s}$ is roughly the mean path l_s between two scattering incidents, and (10) can also be alternatively written as

$$\frac{1}{c} \frac{\partial L}{\partial t} + \hat{\boldsymbol{s}} \cdot \boldsymbol{\nabla}_{\boldsymbol{r}} L = \frac{1}{l_\text{s}} \int \mathrm{d}\Omega' \, (L(\boldsymbol{r}, \hat{\boldsymbol{s}}', t) - L(\boldsymbol{r}, \hat{\boldsymbol{s}}, t)) P(\hat{\boldsymbol{s}}, \hat{\boldsymbol{s}}'), \tag{11}$$

from which we have

$$\frac{\mathrm{d}L}{\mathrm{d}z} + \frac{1}{l_\text{s}} L = 0 \tag{12}$$

when the system reaches the stationary state.

Some scattering processes turn the photons into other degrees of freedom, and in the foreseeable future they will not come back; these processes are just absorption processes, and we can just add them to (10) using the relaxation time approximation, and get

$$\begin{aligned} \frac{1}{c} \frac{\partial L}{\partial t} + \hat{\boldsymbol{s}} \cdot \boldsymbol{\nabla}_{\boldsymbol{r}} L &= -\mu_\text{a} L + \underbrace{\frac{1}{l_\text{s}}}_{\mu_\text{s}} \int \mathrm{d}\Omega' \, (L(\boldsymbol{r}, \hat{\boldsymbol{s}}', t) - L(\boldsymbol{r}, \hat{\boldsymbol{s}}, t)) P(\hat{\boldsymbol{s}}, \hat{\boldsymbol{s}}') \\ &= -\underbrace{(\mu_\text{a} + \mu_\text{s})}_{\mu_\text{t}} L + \mu_\text{s} \int \mathrm{d}\Omega' \, L(\boldsymbol{r}, \hat{\boldsymbol{s}}', t) P(\hat{\boldsymbol{s}}, \hat{\boldsymbol{s}}'), \end{aligned} \tag{13}$$

where μ_t is known as the extinction coefficient.

Until now, we have only discussed about "scalar light": L contains no discrete index. This works when all photons in the system are polarized in one direction; this however is not a realistic assumption in actual optical systems. Thus, in (13), we need to replace L and P by

2×2 matrices; $L_{\alpha\beta}$ can be seen as the single-photon reduced density matrix when the spatial indices are fixed: the full single-photon reduced density matrix is $L_{\alpha\boldsymbol{k},\beta\boldsymbol{k}'}$, and $\boldsymbol{k}$ and $\boldsymbol{k}'$ are recast into $\boldsymbol{k}$ and $\boldsymbol{r}$ when we derive the Boltzmann equation. Now (13) is

$$\frac{1}{c}\frac{\partial L_{\alpha\beta}}{\partial t} + \hat{\boldsymbol{s}} \cdot \boldsymbol{\nabla}_{\boldsymbol{r}} L_{\alpha\beta} = -\underbrace{(\mu_{\mathrm{a}} + \mu_{\mathrm{s}})_{\alpha\gamma}}_{\mu_{\mathrm{t}}} L_{\gamma\beta} + \mu_{\mathrm{s}} \int \mathrm{d}\Omega' \, P_{\alpha\gamma}(\hat{\boldsymbol{s}}, \hat{\boldsymbol{s}}') L_{\gamma\beta}(\boldsymbol{r}, \hat{\boldsymbol{s}}', t). \tag{14}$$

Usual detectors that only count photon number detect $\operatorname{tr} L$. It's however possible that $P_{\alpha\gamma}$ randomly turns one polarization to another, and if this polarization relaxation process is fast enough that long before L reaches equilibrium, $L_{\alpha\beta}$ is almost unpolarized, (13) can still be taken literally, since

$$L_{\alpha\beta} = \delta_{\alpha\beta} L/2. \tag{15}$$

A final correction to RTE is the spatial non-uniformity of optical properties, including scattering and change of light speed. This means both $\mu_{\mathrm{a/s}}$ and $P(\hat{\boldsymbol{s}}, \hat{\boldsymbol{s}}')$ may have spatial variance, and so does c; of course, this variance shouldn't break the validity of QBE. In this case even though the correct generalization of RTE contains, say, $\boldsymbol{\nabla}_{\boldsymbol{r}}(cL)$, due to the slow spatial variance of c, we can still rewritten it as $c\boldsymbol{\nabla}_{\boldsymbol{r}} L$, and get back to (13).

1.2 From RTE to the diffusion equation

Under the following assumptions, RTE is reduced to the diffusion equation:

- The radiance is nearly isotropic – and therefore if we do spherical harmonic expansion to it, only the $l = 0, 1$ components need to be kept. This happens when scattering is strong enough so that the direction of light propagation is randomized; to be exact, this means scattering is much stronger than absorption: $1/c\mu_{\mathrm{a}}$ is the time scale of how long radiation lasts in the system before being dampened by absorption, and scattering should randomized the directions of light propagation within this period of time. Note that this also means that if the radiance starts with a highly anisotropic configuration, then there is already a (although short) period of time in the beginning during which the diffusion equation is not a good description of the dynamics of the system.
- The change of current density is much slower than the speed photons pass the mean free path.
- The scattering property has rotational symmetry (although not completely isotropic): thus $P(\hat{\boldsymbol{s}}, \hat{\boldsymbol{s}}') = P(\hat{\boldsymbol{s}} \cdot \hat{\boldsymbol{s}}')$. This can be seen as included in the second approximation, since otherwise it's not likely that the radiance finally converges to an isotropic solution.

In this section we ignore polarization; that's to say, we assume that scattering events relax polarization quickly enough.

If we only keep the first two spherical harmonic components of L (here the first approximation is used), i.e. the $\hat{\boldsymbol{s}}$ dependence of L is either trivial or is proportional to something dot $\hat{\boldsymbol{s}}$, then we have

$$L(\boldsymbol{r}, \hat{\boldsymbol{s}}, t) = \frac{1}{4\pi} \Phi(\boldsymbol{r}, t) + \frac{3}{4\pi} \boldsymbol{J} \cdot \hat{\boldsymbol{s}}, \tag{16}$$

where

$$\Phi(\boldsymbol{r}, t) = \int \mathrm{d}\Omega \, L(\boldsymbol{r}, \hat{\boldsymbol{s}}, t) \tag{17}$$

is c times the energy density at $\boldsymbol{r}$, and

$$\boldsymbol{J}(\boldsymbol{r}, t) = \int \mathrm{d}\Omega \, \hat{\boldsymbol{s}} L(\boldsymbol{r}, \hat{\boldsymbol{s}}, t) \tag{18}$$

is recognized as the current density; this seems intuitive from the microscopic meaning of L, and we are also going to explicitly show that $\boldsymbol{J}$ appears in its expected position in the diffusion equation. We can explicitly verify that the normalization conditions are correct: for example, we have

$$\int \mathrm{d}\Omega \, \frac{3}{4\pi} \boldsymbol{J} \cdot \hat{\boldsymbol{s}} \cdot \underbrace{\cos\theta}_{\hat{s}_z} = J_z, \tag{19}$$

where the $J_{x,y}$ terms, since they contain a $\sin\varphi$ or $\cos\varphi$ factor, vanish under the integral over $\mathrm{d}\Omega$.

Now, by applying $\int \mathrm{d}\Omega$ to (13), the LHS becomes

$$\mathrm{LHS} = \frac{1}{c}\frac{\partial \Phi}{\partial t} + \int \mathrm{d}\Omega\, \hat{\boldsymbol{s}} \cdot \boldsymbol{\nabla}_{\boldsymbol{r}} L(\boldsymbol{r}, \hat{\boldsymbol{s}}, t) = \frac{1}{c}\frac{\partial \Phi}{\partial t} + \boldsymbol{\nabla} \cdot \underbrace{\int \mathrm{d}\Omega\, \hat{\boldsymbol{s}} L(\boldsymbol{r}, \hat{\boldsymbol{s}}, t)}_{\boldsymbol{J}}, \tag{20}$$

where we have used the condition

$$\boldsymbol{\nabla} \cdot (\hat{\boldsymbol{s}} L) = \hat{\boldsymbol{s}} \cdot \boldsymbol{\nabla} L \tag{21}$$

since $\hat{\boldsymbol{s}}$ has no spatial dependence, while the RHS becomes

$$\begin{aligned} \mathrm{RHS} &= -\mu_t \Phi + \mu_{\mathrm{s}} \int \mathrm{d}\Omega'\, L(\boldsymbol{r}, \hat{\boldsymbol{s}}', t) \int \mathrm{d}\Omega\, P(\hat{\boldsymbol{s}}, \hat{\boldsymbol{s}}') \\ &= -\mu_t \Phi + \mu_{\mathrm{s}} \int \mathrm{d}\Omega'\, L(\boldsymbol{r}, \hat{\boldsymbol{s}}', t) \\ &= -\mu_t \Phi + \mu_{\mathrm{s}} \Phi = -\mu_{\mathrm{a}} \Phi, \end{aligned} \tag{22}$$

and thus we have

$$\frac{1}{c}\frac{\partial \Phi}{\partial t} + \boldsymbol{\nabla} \cdot \boldsymbol{J} + \mu_{\mathrm{a}} \Phi = 0. \tag{23}$$

Now we see $\boldsymbol{J}$ indeed is the current density.

Then, we apply $\int \mathrm{d}\Omega\, \hat{\boldsymbol{s}}$ to (13). This time the calculation will be slightly more non-trivial. The LHS now is

$$\mathrm{LHS} = \frac{1}{c}\frac{\partial \boldsymbol{J}}{\partial t} + \boldsymbol{\nabla} \cdot \int \mathrm{d}\Omega\, \hat{\boldsymbol{s}}\hat{\boldsymbol{s}} L, \tag{24}$$

where we again use the condition (21). A direct evaluation tells us

$$\frac{1}{4\pi} \int \mathrm{d}\Omega\, \hat{\boldsymbol{s}}\hat{\boldsymbol{s}} \Phi = \frac{1}{3} \Phi, \tag{25}$$

and the contribution from the $\boldsymbol{J}$ part is zero. The (z,z) component of $\int \mathrm{d}\Omega\, \hat{\boldsymbol{s}}\hat{\boldsymbol{s}}$, for example, is

$$\int \mathrm{d}\Omega \cos^2\theta = \int_0^\pi \sin\theta\, \mathrm{d}\theta \cos^2\theta \cdot 2\pi = \frac{2}{3} \cdot 2\pi,$$

and hence the 1/3 factor. On the other hand, the (x,z) or (y,z) components suffer from the existence of a vanishing $\int \mathrm{d}\varphi \sin\varphi$ or $\cos\varphi$ factor, and for the (x,y) component, $\int \mathrm{d}\varphi \sin\varphi \cos\varphi$ is still zero. So now we find the full expression of the LHS:

$$\mathrm{LHS} = \frac{1}{c}\frac{\partial \boldsymbol{J}}{\partial t} + \frac{1}{3} \boldsymbol{\nabla} \Phi. \tag{26}$$

As for the RHS, we have

$$\mathrm{RHS} = -\mu_{\mathrm{t}} \boldsymbol{J} + \mu_{\mathrm{s}} \int \mathrm{d}\Omega'\, L(\boldsymbol{r}, \hat{\boldsymbol{s}}', t) \int \mathrm{d}\Omega\, \hat{\boldsymbol{s}} P(\hat{\boldsymbol{s}} \cdot \hat{\boldsymbol{s}}'). \tag{27}$$

We use $\hat{\boldsymbol{s}}'$ as the z axis, and then $P(\hat{\boldsymbol{s}} \cdot \hat{\boldsymbol{s}}') = P(\cos\theta)$, and therefore $\int \mathrm{d}\Omega\, \hat{\boldsymbol{s}} P(\hat{\boldsymbol{s}} \cdot \hat{\boldsymbol{s}}')$ is always parallel to $\hat{\boldsymbol{s}}'$, since the x and y components contain $\int \mathrm{d}\varphi \sin\varphi$ and $\int \mathrm{d}\varphi \cos\varphi$ and therefore vanish. We therefore write

$$\int \mathrm{d}\Omega\, \hat{\boldsymbol{s}} P(\hat{\boldsymbol{s}} \cdot \hat{\boldsymbol{s}}') = g \hat{\boldsymbol{s}}', \tag{28}$$

where g is an unknown factor. The RHS therefore becomes

$$\mathrm{RHS} = -\mu_{\mathrm{t}} \boldsymbol{J} + \mu_{\mathrm{s}} g \int \mathrm{d}\Omega'\, \hat{\boldsymbol{s}}' L(\boldsymbol{r}, \hat{\boldsymbol{s}}', t) = -\mu_{\mathrm{t}} \boldsymbol{J} + \mu_{\mathrm{s}} g \boldsymbol{J}. \tag{29}$$

So we have

$$\frac{1}{c}\frac{\partial \boldsymbol{J}}{\partial t} + \frac{1}{3} \boldsymbol{\nabla} \Phi = -\mu_{\mathrm{t}} \boldsymbol{J} + \mu_{\mathrm{s}} g \boldsymbol{J} = -(\mu_{\mathrm{a}} + \mu_{\mathrm{s}}') \boldsymbol{J}, \quad \mu_{\mathrm{s}}' = (1-g)\mu_{\mathrm{s}}. \tag{30}$$

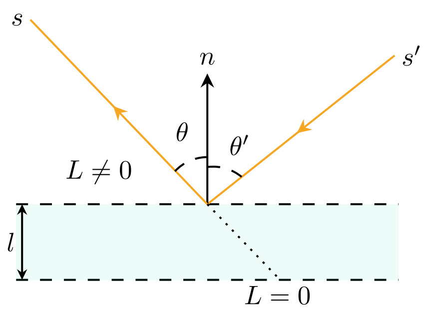

Figure 1: The boundary modeled as a region (the green rectangle) where scattering events are frequent

Now we invoke the condition that the change of the current is slow compared with $c\mu_\mathrm{t}$, and we approximately, we have

$$\frac{1}{3}\boldsymbol{\nabla}\Phi = -\mu_\mathrm{t}\boldsymbol{J} + \mu_\mathrm{s}g\boldsymbol{J} = -(\mu_\mathrm{a} + \mu_\mathrm{s}')\boldsymbol{J}, \quad \mu_\mathrm{s}' = (1-g)\mu_\mathrm{s}. \tag{31}$$

This gives the constitutive relation between the current density and the gradient of radiance.

Thus, we find with the three – or two – assumptions listed at the start of this section, we have

$$\frac{1}{c}\frac{\partial \Phi}{\partial t} + \boldsymbol{\nabla}\cdot\boldsymbol{J} + \mu_\mathrm{a}\Phi = 0, \quad \boldsymbol{J} = -D\boldsymbol{\nabla}\Phi, \tag{32}$$

where

$$D = \frac{1}{3(\mu_\mathrm{a} + \mu_\mathrm{s}')}, \quad \mu_\mathrm{s}' = (1-g)\mu_\mathrm{s}, \tag{33}$$

and

$$\int \mathrm{d}\Omega\, \hat{\boldsymbol{s}} P(\hat{\boldsymbol{s}}\cdot\hat{\boldsymbol{s}}') = g\hat{\boldsymbol{s}}' \Rightarrow 2\pi\int_0^\pi \sin\theta\,\mathrm{d}\theta\cdot\cos\theta P(\cos\theta) = g. \tag{34}$$

Now we find RTE has been completely reduced to a diffusion equation.

1.3 Volume rendering

In most everyday cases, the time evolution of L can be ignored: the system reaches equilibrium as soon as "the light is turned on", since the speed of light is very fast. It's also frequently assumed that c is <u>one hundred percent uniform in space and not just very slow in its spatial change</u>. In these cases, we get

$$\begin{aligned}\hat{\boldsymbol{s}}\cdot\boldsymbol{\nabla}_{\boldsymbol{r}}L &= -\mu_\mathrm{a}L + \underbrace{\frac{1}{l_\mathrm{s}}}_{\mu_\mathrm{s}}\int \mathrm{d}\Omega'\,(L(\boldsymbol{r},\hat{\boldsymbol{s}}',t) - L(\boldsymbol{r},\hat{\boldsymbol{s}},t))P(\hat{\boldsymbol{s}},\hat{\boldsymbol{s}}') \\ &= -\underbrace{(\mu_\mathrm{a}+\mu_\mathrm{s})}_{\mu_\mathrm{t}}L + \mu_\mathrm{s}\int \mathrm{d}\Omega'\,L(\boldsymbol{r},\hat{\boldsymbol{s}}',t)P(\hat{\boldsymbol{s}},\hat{\boldsymbol{s}}').\end{aligned} \tag{35}$$

The time evolution term is thrown away. This equation is sometimes known as the **scattering equation** or **volume rendering equation**. Again, for polarized light, LP should be replaced by $P_{\alpha\gamma}L_{\gamma\beta}$.

1.4 Surface boundary condition: reflection

If we are not modeling things like clouds, and are interested in, say, the image of a cup or an apple, the bulk part of the scattering equation is easy, since <u>everything is assumed to be isotropic and uniform in the bulk</u>; the main problem then becomes how to treat the boundaries appropriately.

We can think the boundary of an object as a region where scattering is particularly strong, as in Fig. 1. On one side of the region, the radiance takes its value at the edge of the bulk, and on the other side of the region, the radiance decreases to zero. Suppose the thickness of the region is l.

Now we extend the $\hat{\boldsymbol{s}}$ line shown in Fig. 1 opposite its direction, until we reach the $L = 0$ side of the scattering region. We call this path C, and the length of this path is $l/\cos\theta$. We integrate the scattering equation (35) along this path, and get

$$L(\boldsymbol{r}, \hat{\boldsymbol{s}}) = -\mu_{\mathrm{t}} \int_C \mathrm{d}l\, L(\boldsymbol{r}', \hat{\boldsymbol{s}}) + \mu_{\mathrm{s}} \int \mathrm{d}\Omega'\, P(\hat{\boldsymbol{s}}, \hat{\boldsymbol{s}}') \int_C \mathrm{d}l\, L(\boldsymbol{r}', \hat{\boldsymbol{s}}'). \tag{36}$$

$L(\boldsymbol{r}', \hat{\boldsymbol{s}})$ is a linear functional of the radiance distribution around the surface. (It has to be linear, since the scattering equation is a linear equation.) In the simplest case, the reflected light on one point ($L(\boldsymbol{r}, \hat{\boldsymbol{s}})$) mainly comes from incoming light at the same point; thus $L(\boldsymbol{r}', \hat{\boldsymbol{s}})$ and $L(\boldsymbol{r}', \hat{\boldsymbol{s}}')$ on path C can be rewritten as some kind of damping function of $|\boldsymbol{r} - \boldsymbol{r}'|$ times $L(\boldsymbol{r}, \hat{\boldsymbol{s}})$ and $L(\boldsymbol{r}, \hat{\boldsymbol{s}}')$, and we have

$$\mu_{\mathrm{s}} \int_C \mathrm{d}l\, L(\boldsymbol{r}', \hat{\boldsymbol{s}}') = g(\hat{\boldsymbol{s}}, \hat{\boldsymbol{s}}') \frac{\mu_{\mathrm{s}} l}{\cos\theta} L(\boldsymbol{r}, \hat{\boldsymbol{s}}),$$

where $g(\hat{\boldsymbol{s}}, \hat{\boldsymbol{s}}')$ comes from the properties of the scattering layer, and from time reversal symmetry we can assume that it's symmetric, since the phase function is symmetric. This means

$$\begin{aligned}
\mu_{\mathrm{s}} \int \mathrm{d}\Omega'\, P(\hat{\boldsymbol{s}}, \hat{\boldsymbol{s}}') \int_C \mathrm{d}l\, L(\boldsymbol{r}', \hat{\boldsymbol{s}}') &= \mu_{\mathrm{s}} l \int \mathrm{d}\Omega'\, P(\hat{\boldsymbol{s}}, \hat{\boldsymbol{s}}') g(\hat{\boldsymbol{s}}, \hat{\boldsymbol{s}}') \frac{1}{\cos\theta} L(\boldsymbol{r}, \hat{\boldsymbol{s}}) \\
&= \mu_{\mathrm{s}} l \int \cos\theta'\, \mathrm{d}\Omega'\, \underbrace{P(\hat{\boldsymbol{s}}, \hat{\boldsymbol{s}}') g(\hat{\boldsymbol{s}}, \hat{\boldsymbol{s}}') \frac{1}{\cos\theta\cos\theta'}}_{\text{symmetric}} L(\boldsymbol{r}, \hat{\boldsymbol{s}}).
\end{aligned} \tag{37}$$

Similarly, we have

$$\begin{aligned}
\mu_{\mathrm{t}} \int_C \mathrm{d}l\, L(\boldsymbol{r}', \hat{\boldsymbol{s}}) &= \mu_{\mathrm{t}} g(\hat{\boldsymbol{s}}, \hat{\boldsymbol{s}}) \frac{l}{\cos\theta} L(\boldsymbol{r}, \hat{\boldsymbol{s}}) \\
&= \mu_{\mathrm{t}} l \int \cos\theta'\, \mathrm{d}\Omega'\, \underbrace{\delta(\hat{\boldsymbol{s}}, \hat{\boldsymbol{s}}') \frac{1}{\cos\theta\cos\theta'}}_{\text{symmetric}} L(\boldsymbol{r}, \hat{\boldsymbol{s}}'),
\end{aligned} \tag{38}$$

where

$$\delta(\hat{\boldsymbol{s}}, \hat{\boldsymbol{s}}') = \frac{1}{\sin\theta} \delta(\theta - \theta') \delta(\varphi - \varphi'). \tag{39}$$

So finally, from (36) we get

$$L(\boldsymbol{r}, \hat{\boldsymbol{s}}) = \int f(\hat{\boldsymbol{s}}, \hat{\boldsymbol{s}}') L(\boldsymbol{r}, \hat{\boldsymbol{s}}') \cos\theta'\, \mathrm{d}\Omega', \quad f(\hat{\boldsymbol{s}}, \hat{\boldsymbol{s}}') = f(\hat{\boldsymbol{s}}', \hat{\boldsymbol{s}}), \tag{40}$$

where we have packaged all symmetric factors in the equations above into $f(\hat{\boldsymbol{s}}, \hat{\boldsymbol{s}}')$. We find the exact theory (under the aforementioned premises) of light propagation in this case turns out to be (40), which is known as the **rendering equation**. In the rendering equation we have the symmetric function $f(\hat{\boldsymbol{s}}, \hat{\boldsymbol{s}}')$, called the **bidirectional reflectance distribution function (BRDF)**. We can always readjust the BRDF so that it is only non-zero when $\hat{\boldsymbol{s}} \cdot \hat{\boldsymbol{n}}$ and $\hat{\boldsymbol{s}}' \cdot \hat{\boldsymbol{n}}$ have different signs. This can be seen by noticing that an outgoing (i.e. $\hat{\boldsymbol{s}} \cdot \hat{\boldsymbol{n}} > 0$) radiance always comes from an ingoing radiance, and therefore its effect can be alternatively attributed to the latter. This means we can rewrite (40) to

$$L(\boldsymbol{r}, \hat{\boldsymbol{s}}) = \int_{\hat{\boldsymbol{s}}' \cdot \hat{\boldsymbol{n}} < 0} f(\hat{\boldsymbol{s}}, \hat{\boldsymbol{s}}') L(\boldsymbol{r}, \hat{\boldsymbol{s}}') \cos\theta'\, \mathrm{d}\Omega', \quad \hat{\boldsymbol{s}} \cdot \hat{\boldsymbol{n}} > 0, \tag{41}$$

with the BRDF defined appropriately. The photon number conservation condition (essentially the energy conservation condition, since we don't have any frequency changing mechanism) is

$$\int f(\hat{\boldsymbol{s}}, \hat{\boldsymbol{s}}') \cos\theta'\, \mathrm{d}\Omega' \leq 1. \tag{42}$$

This is an inequality, because absorption is also included in the BRDF.

In real materials, the optical properties of the boundary usually observes some spatial variance, but when the variance is usually slow enough so that the above derivation illustrated in (1) still works at a particular spatial point, without any interference or diffraction effects, and in this case we have

$$L(\boldsymbol{r}, \hat{\boldsymbol{s}}) = \int_{\hat{\boldsymbol{s}}' \cdot \hat{\boldsymbol{n}} < 0} f(\boldsymbol{r}, \hat{\boldsymbol{s}}, \hat{\boldsymbol{s}}') L(\boldsymbol{r}, \hat{\boldsymbol{s}}') \cos\theta' \, \mathrm{d}\Omega' \,, \quad \hat{\boldsymbol{s}} \cdot \hat{\boldsymbol{n}} > 0. \tag{43}$$

(43) can be derived in a more straightforward way from energy conservation, where we define

$$\underbrace{\mathrm{d}L(\boldsymbol{r}, \hat{\boldsymbol{s}})}_{\text{outgoing radiance}} = \underbrace{L(\boldsymbol{r}, \hat{\boldsymbol{s}}') \cos\theta' \, \mathrm{d}\Omega'}_{\text{incoming radiance}} \cdot \underbrace{f(\boldsymbol{r}, \hat{\boldsymbol{s}}, \hat{\boldsymbol{s}}')}_{\text{response}}, \tag{44}$$

where the $\cos\theta'$ factor can be obtained intuitively by considering the fact that if a beam of light is incident on the material with an angle θ', the effective cross section of the beam is reduced by a factor of $\cos\theta'$.

1.5 More about surface

One thing that is hard to place this into the framework of scattering of photons in a 3D space is transmission, since now we have two bulk areas with different optical properties, and the Boltzmann transportation approach needs to be corrected to capture this scenario. We still consider TODO: Fabry–Pérot interferometer??? We know a crystal ball also has internal electromagnetic modes, which can only be derived with full wave optics; also multiple reflection/transmission, without polarization relaxation, can also create phenomenon that can't be captured by scalar (13)??

Transmission also brings a notational change. Since at the same spatial position and in the same wave vector direction $\hat{\boldsymbol{s}}$, the radiance can be different on the two sides of the boundary. a good idea is to use subscripts i and o representing "incoming" or "incident" and "outgoing" to refer to the two radiances respectively.

A even more complicated case is when there indeed is a very thing intermediate layer between the two bulk spaces; in this case we need **bidirectional scattering-surface reflectance distribution function (BSSRDF)** and **bidirectional scattering-surface transmittance distribution function (BSSTDF)**, which includes the possibility that light comes into the surface in one position but goes out in another.

There is a major notation difference between the notation above that is more familiar to physicists and the standard notation in computer graphics: in CG, the direction unit vector is usually represented as ω, and $\mathrm{d}\Omega$ is often written as $\mathrm{d}\Omega$; also, in computer graphics, the position is referred to as $\boldsymbol{p}$, which may be confused with the momentum, but since the latter is given no place in computer graphics, this notation change will not cause any confusion. Thus the rendering equation now reads

$$L(\boldsymbol{p}, \omega_{\mathrm{o}}) = \int f(\boldsymbol{p}, \omega_{\mathrm{o}}, \omega_{\mathrm{i}}) L(\boldsymbol{p}, \omega_{\mathrm{i}}) |\cos\theta_{\mathrm{i}}| \, \mathrm{d}\omega_{\mathrm{i}} \,, \tag{45}$$

where $f(\omega_{\mathrm{o}}, \omega_{\mathrm{i}})$ is the sum of BRDF and BTDF; of course polarization may still be important above, and it's also possible that we have BSSTDF and BSSTDF so another integral in the position variable is needed.

2 Describing the boundary

2.1 Example of BDSF: boundary between two dielectrics

In this section we derive the

$$f_{\mathrm{r}} = F_{\mathrm{r}} \frac{\delta(\hat{\boldsymbol{s}}_{\mathrm{i}}, \hat{\boldsymbol{s}}_{\mathrm{r}})}{|\cos\theta_{\mathrm{r}}|} \tag{46}$$

For reflected light, we have

$$
\begin{aligned}
|\boldsymbol{E}_\mathrm{t}|^2 = |t|^2 |\boldsymbol{E}_\mathrm{i}|^2 &\Rightarrow \epsilon_\mathrm{t} |\boldsymbol{E}_\mathrm{t}|^2 = \frac{\epsilon_\mathrm{t}}{\epsilon_\mathrm{i}} |t|^2 \epsilon_\mathrm{i} |\boldsymbol{E}_\mathrm{i}|^2 \\
&\Rightarrow L_\mathrm{t}(\boldsymbol{r}, \hat{\boldsymbol{s}}_\mathrm{t}) = \frac{n_\mathrm{t}^2}{n_\mathrm{i}^2} |t|^2 L_\mathrm{i}(\boldsymbol{r}, \hat{\boldsymbol{s}}_\mathrm{i}).
\end{aligned} \tag{47}
$$

3 Path sampling, ray tracing, and the like

After obtaining enough information about all boundaries and surfaces in the system, and assuming that the bulk properties of the system are completely homogeneous, we can move to boundary rendering.

3.1 Examples of light source: point source

Let's start with the example of the point source. Intuitively, when there is no scattering or absorption in the space, and we have the inverse-square law, and therefore the radiance around a point source assumes the form of

$$
L(\boldsymbol{r}, \hat{\boldsymbol{s}}) = \frac{C}{r^2} \delta(\hat{\boldsymbol{r}}, \hat{\boldsymbol{s}}), \tag{48}
$$

where C is a constant, which is likely to be sometimes times the angular frequency spectral flux, as is implied by dimensional analysis. We can verify this solution explicitly: when there is no scattering and absorption, the volume rendering equation reads

$$
\hat{\boldsymbol{s}} \cdot \boldsymbol{\nabla}_{\boldsymbol{r}} L = 0, \tag{49}
$$

and indeed we have

$$
\begin{aligned}
\hat{\boldsymbol{s}} \cdot \boldsymbol{\nabla}_{\boldsymbol{r}} \left(\frac{1}{r^2} \delta(\hat{\boldsymbol{r}}, \hat{\boldsymbol{s}}) \right) &= \boldsymbol{\nabla}_{\boldsymbol{r}} \cdot \left(\frac{1}{r^2} \delta(\hat{\boldsymbol{r}}, \hat{\boldsymbol{s}}) \hat{\boldsymbol{s}} \right) = \boldsymbol{\nabla}_{\boldsymbol{r}} \cdot \left(\frac{1}{r^2} \delta(\hat{\boldsymbol{r}}, \hat{\boldsymbol{s}}) \hat{\boldsymbol{r}} \right) \\
&= \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \cdot \frac{1}{r^2} \delta(\hat{\boldsymbol{r}}, \hat{\boldsymbol{s}}) \right) = 0.
\end{aligned} \tag{50}
$$

Here the second line makes use of the fact that $\delta(\hat{\boldsymbol{r}}, \hat{\boldsymbol{s}})$ involves no radial dependence.

We can also do (50) in a more straightforward way. We use (θ_0, φ_0) to refer to the spherical coordinates of $\hat{\boldsymbol{s}}$, and use (r, θ, φ) to refer to the spherical coordinates of $\boldsymbol{r}$. Thus

$$
\boldsymbol{\nabla}_{\boldsymbol{r}} = \hat{\boldsymbol{r}} \frac{\partial}{\partial r} + \hat{\boldsymbol{\theta}} \frac{1}{r} \frac{\partial}{\partial \theta} + \hat{\boldsymbol{\varphi}} \frac{1}{r \sin\theta} \frac{\partial}{\partial \varphi}, \tag{51}
$$

$$
\frac{1}{r^2} \delta(\hat{\boldsymbol{r}}, \hat{\boldsymbol{s}}) = \frac{1}{r^2} \frac{1}{\sin\theta} \delta(\theta - \theta_0) \delta(\varphi - \varphi_0) = \frac{1}{r^2} \frac{1}{\sin\theta_0} \delta(\theta - \theta_0) \delta(\varphi - \varphi_0), \tag{52}
$$

and

$$
\begin{aligned}
\hat{\boldsymbol{s}} \cdot \hat{\boldsymbol{\theta}} &= \sin\theta_0 \cos\varphi_0 \cos\theta \cos\varphi + \sin\theta_0 \sin\varphi_0 \cos\theta \sin\varphi - \cos\theta_0 \sin\theta, \\
\hat{\boldsymbol{s}} \cdot \hat{\boldsymbol{\varphi}} &= -\sin\theta_0 \cos\varphi_0 \sin\varphi + \sin\theta_0 \sin\varphi_0 \cos\varphi.
\end{aligned} \tag{53}
$$

Using the fact that

$$
\delta'(x) f(x) = \delta'(x) f(0) - \delta(x) f'(0), \tag{54}
$$

we are able to show that all $\delta'(\theta - \theta_0)$ and $\delta'(\varphi - \varphi_0)$ terms vanish, and indeed we have (50).

Then we proceed to decide the constant C. We have

$$
\begin{aligned}
\text{outgoing power per } \mathrm{d}\omega &= \int_{|\boldsymbol{r}|^2 = \text{const.}} \int \mathrm{d}\Omega \, L(\boldsymbol{r}, \hat{\boldsymbol{s}}) \\
&= \int_{|\boldsymbol{r}|^2 = \text{const.}} \frac{C}{r^2} = 4\pi r^2 \cdot \frac{C}{r^2},
\end{aligned} \tag{55}
$$

and thus we get

$$
L(\boldsymbol{r}, \hat{\boldsymbol{s}}) = \frac{\Phi}{4\pi r^2} \delta(\hat{\boldsymbol{s}}, \hat{\boldsymbol{r}}), \tag{56}
$$

where Φ is the angular frequency spectral flux.

It's instructive to compare this solution with the expression of L obtained from Maxwell equations. TODO: absence of diffraction

When we do have absorption and/or scattering in the system, intuitively we have

$$L(\boldsymbol{r}, \hat{\boldsymbol{s}}) = \Phi \frac{\mathrm{e}^{-\mu_\mathrm{t} r}}{4\pi r^2} \delta(\hat{\boldsymbol{s}}, \hat{\boldsymbol{r}}), \tag{57}$$

which goes back to (56) when $\mu_\mathrm{t} = 0$. Again this can be verified, under the assumption that we can ignore scattered light that comes back: now the volume rendering equation becomes

$$\hat{\boldsymbol{s}} \cdot \boldsymbol{\nabla}_{\boldsymbol{r}} L = -\mu_\mathrm{t} L, \tag{58}$$

and the LHS evaluates into

$$\begin{aligned} \hat{\boldsymbol{s}} \cdot \boldsymbol{\nabla}_{\boldsymbol{r}} \left(\frac{\mathrm{e}^{-\mu_\mathrm{t} r}}{r^2} \right) &= \boldsymbol{\nabla}_{\boldsymbol{r}} \left(\frac{\mathrm{e}^{-\mu_\mathrm{t} r}}{r^2} \delta(\hat{\boldsymbol{r}}, \hat{\boldsymbol{s}}) \hat{\boldsymbol{r}} \right) \\ &= \delta(\hat{\boldsymbol{r}}, \hat{\boldsymbol{s}}) \cdot \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \cdot \frac{\mathrm{e}^{-\mu_\mathrm{t} r}}{r^2} \right) \\ &= \delta(\hat{\boldsymbol{r}}, \hat{\boldsymbol{s}}) \cdot \frac{1}{r^2} (-\mu_\mathrm{r}) \mathrm{e}^{-\mu_\mathrm{t} r} = \mathrm{RHS}. \end{aligned} \tag{59}$$

Interestingly, although the $\mathrm{e}^{-\mu_\mathrm{t} r}$ factor looks diffusional, it's different with the diffusion coefficient given in (32) and (33), and this can be expected: for a point source, the structure of the radiance is rather different with the assumption in (16). The Φ constant in (57) is now the total angular momentum spectral flux in the $r \to 0$ limit.

(56) and (57) work as well for finite, spherical sources, of which the wave vector of radiation at the boundary is parallel to the normal vector. We want to extend the $1/r^2$ law to a more generalized boundary.

3.2 Tracing the radiance along rays

The last piece needed to render a system with trivial bulk properties where all interesting things happen at the boundaries is how to link $L(\boldsymbol{r}, \hat{\boldsymbol{s}})$ at the boundaries with $L(\boldsymbol{r}, \hat{\boldsymbol{s}})$ in the space.

The integral form of (49) is

$$L(\boldsymbol{r}, \hat{\boldsymbol{s}}) = L(\boldsymbol{r} + \hat{\boldsymbol{s}} t, \hat{\boldsymbol{s}}), \tag{60}$$

where t is a parameter; the equation

$$\boldsymbol{x} = \boldsymbol{r} + \hat{\boldsymbol{s}} t \tag{61}$$

has a clear physical meaning: it's the equation of the **ray** in the direction $\hat{\boldsymbol{s}}$ that passes $\boldsymbol{r}$. Thus, we find that along a ray, the radiance in the direction of the ray doesn't change.

It should be noted that the coverage of (60) is narrower than the original differential volume rendering equation; specifically, it can be easily seen that it conflicts with inverse square law (56) when $\hat{\boldsymbol{s}}$ is set to $\hat{\boldsymbol{r}}$. The root cause is we can't always freely change the order of the partial derivative $\hat{\boldsymbol{s}} \cdot \boldsymbol{\nabla}_{\boldsymbol{r}}$ and the substitution of the variables with concrete values when the function following the derivative operator contains δ functions. When deriving (60), what we do is choose a small t and argue that

$$(60) \Leftrightarrow L(\boldsymbol{r} + \hat{\boldsymbol{s}} t) - L(\boldsymbol{r}, \hat{\boldsymbol{s}}) = t \underbrace{(\hat{\boldsymbol{s}} \cdot \boldsymbol{\nabla}_{\boldsymbol{r}}) L}_{=0}, \tag{62}$$

where $\hat{\boldsymbol{s}} \cdot \boldsymbol{\nabla}_{\boldsymbol{r}}$ is seen as a one-dimensional gradient operator. This, of course, is wrong for (56): if we set $\hat{\boldsymbol{s}} = \hat{\boldsymbol{r}}$, we get

$$(\hat{\boldsymbol{s}} \cdot \boldsymbol{\nabla}_{\boldsymbol{r}}) L = \hat{\boldsymbol{r}} \cdot \left(\hat{\boldsymbol{r}} \frac{\partial}{\partial r} + \hat{\boldsymbol{\theta}} \frac{1}{r} \frac{\partial}{\partial \theta} + \hat{\boldsymbol{\varphi}} \frac{1}{r \sin\theta} \frac{\partial}{\partial \varphi} \right) L = \frac{\partial}{\partial r} L = -\frac{\Phi}{2\pi r^3} \delta(\hat{\boldsymbol{s}}, \hat{\boldsymbol{r}}) \neq 0. \tag{63}$$

On the other hand, if we apply $\boldsymbol{\nabla}_{\boldsymbol{r}}$ to L first, we get $\delta'(\theta - \theta_0)$ and $\delta'(\varphi - \varphi_0)$ factors, and since now $\hat{\boldsymbol{s}} \cdot \hat{\boldsymbol{\theta}}$ and $\hat{\boldsymbol{s}} \cdot \hat{\boldsymbol{\varphi}}$ haven't been evaluated under the assumption that $\hat{\boldsymbol{s}} = \hat{\boldsymbol{r}}$, they still have θ, φ dependence, and then (54) gives us a $\partial_\theta (\hat{\boldsymbol{s}} \cdot \hat{\boldsymbol{\theta}})$ term and a $\partial_\varphi (\hat{\boldsymbol{s}} \cdot \hat{\boldsymbol{\varphi}})$ term, which, even when $\hat{\boldsymbol{r}} = \hat{\boldsymbol{s}}$, are still non-zero. Usually, when dealing with PDEs, we don't really need to consider

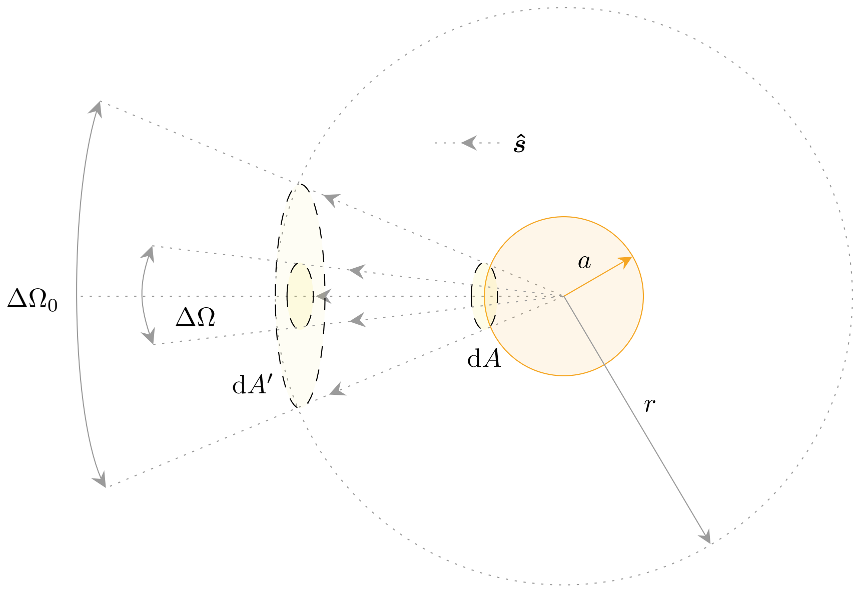

Figure 2: An approximate point source. The orange circle is the surface of the light source; $\Delta\Omega$ is the width of the angular distribution of L. The grey lines are rays. When the uncertainty of $\hat{\boldsymbol{s}}$ distribution at the light source $\boldsymbol{r}_0$ is given by $\Delta\Omega_0$, it's given by $\Delta\Omega$ at $\boldsymbol{r}_0 + r\hat{\boldsymbol{s}}$; this means the approximation of $\delta(\hat{\boldsymbol{r}}, \hat{\boldsymbol{s}})$ actually has spatial dependence.

such corner cases, since δ functions usually appear as a "source" and the field to be solved has reasonable behaviors even with such a δ function source. (56) is indeed a very rare case in physics in which we need to seriously consider the unusual properties of δ function. The corner case of (56) however doesn't need any specific treatment in real world problems: we can always approximate $\delta(\hat{\boldsymbol{r}}, \hat{\boldsymbol{s}})$ by a, say, gaussian peak, or a stepwise function like

$$\delta(\hat{\boldsymbol{r}}, \hat{\boldsymbol{s}}) \approx \frac{1}{\Delta\theta\Delta\varphi\sin\theta}\theta(\Delta\theta/2 - |\theta - \theta_0|)\theta(\Delta\varphi/2 - |\varphi - \varphi_0|). \tag{64}$$

Suppose that the light source is actually a small sphere with radius a placed at $\boldsymbol{r}_0$, and that at the light source, the spot size of the radiance in the $\hat{\boldsymbol{s}}$-space at $\boldsymbol{r}_0$ is $\Delta\Omega_0$ (see Fig. 2); due to momentum conservation, photons that arrive at an imaginary sphere with radius r around $\boldsymbol{r}_0$ are still distributed in a solid angle of $\Delta\Omega_0$, but not all of them comes to $\boldsymbol{r}_0 + r\hat{\boldsymbol{s}}$: if we only look at the flux at $\boldsymbol{r}_0 + r\hat{\boldsymbol{s}}$ though $\mathrm{d}A$, the spot size in the $\hat{\boldsymbol{s}}$-space becomes

$$\Delta\Omega = \frac{a^2}{r^2}\Delta\Omega_0. \tag{65}$$

When we give an approximation of $\delta(\hat{\boldsymbol{r}}, \hat{\boldsymbol{s}})$ at $\boldsymbol{r}_0 + r\hat{\boldsymbol{s}}$, we use $\Delta\Omega$, not $\Delta\Omega_0$, and therefore there is a r^2 factor in the approximation of $\delta(\hat{\boldsymbol{r}}, \hat{\boldsymbol{s}})$; that's to say, if at the light source, the δ function is only an approximate one, then at each space position, the δ function should also be an approximation one, and the approximation depends on the space point to get us a solution of (49). thus the radiance becomes a constant along a ray, and we have

$$\begin{aligned} L(\boldsymbol{r}, \hat{\boldsymbol{s}}) &= \frac{\Phi}{4\pi r^2} \cdot \frac{1}{\sin\theta\Delta\Omega}\theta(\Delta\theta/2 - |\theta - \theta_0|)\theta(\Delta\varphi/2 - |\varphi - \varphi_0|) \\ &= \frac{\Phi}{4\pi a^2}\frac{1}{\Delta\Omega_0\sin\theta}\theta(\Delta\theta/2 - |\theta - \theta_0|)\theta(\Delta\varphi/2 - |\varphi - \varphi_0|). \end{aligned} \tag{66}$$

Indeed this is a constant along a ray: if we go along a ray, θ and φ don't change, and hence $L(\boldsymbol{r}, \hat{\boldsymbol{s}})$ is a constant; the inverse square behavior can still be obtained if we integrate over $\hat{\boldsymbol{s}}$, while $L(\boldsymbol{r}, \hat{\boldsymbol{s}})$ satisfies (60) well. Of course, when we come far away from the light source, the spot size will finally be reduced to a rather small value, and (56) eventually is reached.

3.3 Rendering as a path summation

Now we have (60), and after we get the emission spectrum of all light sources in the system, we are able to decide the radiance in the whole space. The problem is shown in Fig. 3: for a spatial

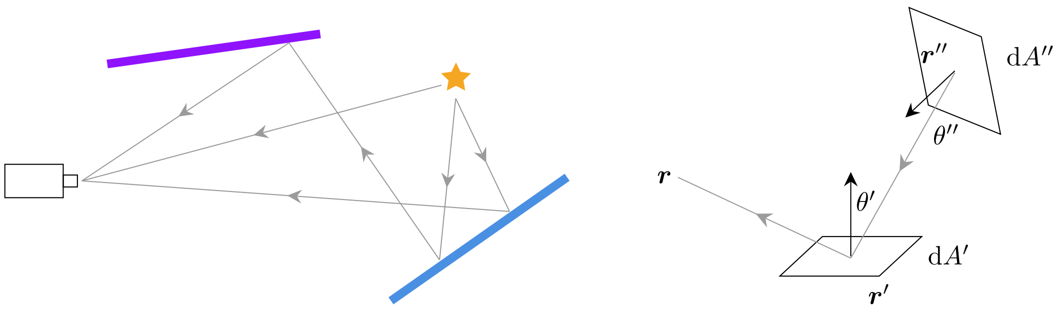

Figure 3: An illustration of a surface rendering problem.

point $\boldsymbol{r}$ (represented by the camera), we enumerate over all possible paths from light sources to it; a path contains several rays, connected by reflection and/or refraction. Note that we don't impose Snell's law or law of reflection at the boundaries, since it might be the case that we have diffuse reflection or refraction and hence the BSDF doesn't dictate the angle of the outgoing ray, but rather gives a distribution of radiance over the outgoing ray angle. We use

$$L(\boldsymbol{x} \to \boldsymbol{x}') \coloneqq L(\boldsymbol{x}, \widehat{\boldsymbol{x}' - \boldsymbol{x}}) = L(\boldsymbol{x}', \widehat{\boldsymbol{x}' - \boldsymbol{x}}) \tag{67}$$

to refer to the radiance of the ray going from $\boldsymbol{x}$ to $\boldsymbol{x}'$, i.e. the radiance at $\boldsymbol{x}$ (or $\boldsymbol{x}'$ – they are the same, since we have (60)) in the direction $\widehat{\boldsymbol{x}' - \boldsymbol{x}}$. To connect the radiances of two rays connected at a boundary, we use the rendering equation: the equation now reads

$$L(\boldsymbol{r}' \to \boldsymbol{r}) = L_{\text{emit}} + L_{\text{scatter}}, \quad L_{\text{scatter}}(\boldsymbol{r}' \to \boldsymbol{r}) = \int \mathrm{d}\Omega' \, f(\boldsymbol{r}'' \to \boldsymbol{r}' \to \boldsymbol{r}) L(\boldsymbol{r}'' \to \boldsymbol{r}') \left|\cos\theta'\right|, \tag{68}$$

where L_{emit} refers to the radiance emitted exactly by something at $\boldsymbol{r}'$ in the direction $\boldsymbol{r}' \to \boldsymbol{r}$, $\boldsymbol{r}''$ is a point on the ray determined by $\boldsymbol{r}'$ and the incoming direction $\hat{\boldsymbol{s}}'$, and

$$f(\boldsymbol{r}'' \to \boldsymbol{r}' \to \boldsymbol{r}) \coloneqq f(\boldsymbol{r}', \boldsymbol{s} = \widehat{\boldsymbol{r}' - \boldsymbol{r}''}, \hat{\boldsymbol{s}}' = \widehat{\boldsymbol{r} - \boldsymbol{r}'}). \tag{69}$$

Since $\boldsymbol{r}''$ is arbitrary, we can push it away until we reach the other side of the $\boldsymbol{r}'' \to \boldsymbol{r}'$ ray, and now $\boldsymbol{r}''$ is on the boundary which the $\boldsymbol{r}'' \to \boldsymbol{r}'$ ray comes from, and we find

$$\mathrm{d}\Omega' = \frac{\cos\theta'' \, \mathrm{d}A''}{\left|\boldsymbol{r}'' - \boldsymbol{r}'\right|^2}. \tag{70}$$

Putting the equations above together, we have already explicitly constructed a systematic way to decide $L(\boldsymbol{r}, \hat{\boldsymbol{s}})$.