Floquet theory

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1 The fundamental formalism

Consider a time-periodic Hamiltonian with period $T = 2\pi/\omega$. Such a Hamiltonian is usually an effective Hamiltonian when the system (hereafter "matter") is coupled with another degree of freedom which doesn't change much in the time evolution; the latter is hereafter called "light", since in condensed matter systems, periodic driving is usually achieved by shedding a beam of light to the matter; it's however possible to use light to stimulate some long-lived degrees of freedom in a solid and let it drive the rest of the system, which sometimes is known as "self-driving". From the Floquet theory of differential equation, we know it's possible to expand an arbitrary state that evolves according to H into a linear combination (the coefficients are constants) of $\{ \left|\psi_n(t)\right\rangle \}$ where

$$\left|\psi_n(t+T)\right\rangle = \mathrm{e}^{-\mathrm{i}\varepsilon_n t/\hbar} \left|\Phi_n(t)\right\rangle , \quad \left|\Phi_n(t+T)\right\rangle = \left|\Phi_n(t)\right\rangle . \tag{1}$$

By discrete periodicity of $\left|\Phi_n(t)\right\rangle$ we make Fourier expansion

$$\left|\Phi_n(t)\right\rangle = \sum_m \mathrm{e}^{-\mathrm{i}m\omega t} \left|\phi_n^{(m)}\right\rangle , \tag{2}$$

where m goes over all integers. Note that here $\left|\phi_n^{(m)}\right\rangle$ are *Fourier coefficients* and are not eigenstates of anything; there is no normalization or orthogonality condition. Using i to label the eigenstates of the matter, we have

$$\left|\Phi_n(t)\right\rangle = \sum_i \sum_m \mathrm{e}^{-\mathrm{i}m\omega t} \left\langle i|\phi_n(m)\right\rangle \left|i\right\rangle . \tag{3}$$

This, not (2), is the expansion of $\left|\Phi\right\rangle$ in a complete, orthogonal basis. The meaning of these vectors can be seen immediately below.

The Schrodinger equation now reads

$$(\varepsilon_n + m\hbar\omega) \left|\phi_n^{(m)}\right\rangle = \sum_{m'} H^{(m-m')} \left|\phi_n^{(m')}\right\rangle , \tag{4}$$

where

$$H(t) = \sum_m \mathrm{e}^{-\mathrm{i}m\omega t} H^{(m)}. \tag{5}$$

Thus we find that if we use i to label the eigenstates of the matter part, we have

$$\varepsilon_n \left\langle i|\phi_n^{(m)}\right\rangle = \sum_{m'} (H^{(m-m')} - m\hbar\omega\delta_{mm'}) \left\langle i|\phi_n^{(m')}\right\rangle . \tag{6}$$

It doesn't take long to realize that the matrix elements on the RHS are exactly the matrix elements of

$$H^{\text{light}\otimes\text{matter}} = H \otimes 1_{\text{light}} + 1_{\text{matter}} \otimes \hbar \left(b^\dagger b + \frac{1}{2} \right) + \underbrace{bV + b^\dagger V^\dagger}_{H_{\text{light-matter coupling}}} \tag{7}$$

under the basis $\left|i\right\rangle \otimes \left|m\right\rangle$, where m refers to the photon number. When the dynamics is almost completely decided by the matter part of the system, and we can map the light-matter wave function to the wave function of the matter part only by applying

$$P : \left|\psi\right\rangle \otimes \left|\text{whatever the light is}\right\rangle \mapsto \left|\psi\right\rangle , \tag{8}$$

which means

$$\langle i, m | \psi_n \rangle \otimes |m\rangle \mapsto \langle i | \phi_n^{(m)} \rangle . \tag{9}$$

Now we see the true meaning of $\phi^{(m)_n}$: we are just grouping the components of the complete matter-light wave function $|\phi_n\rangle \otimes |\text{light}\rangle$ with the same photon number m into a vector $\phi_n^{(m)}$.

As an example, when the light field is approximately always in a coherent state $|\alpha \mathrm{e}^{-\mathrm{i}\omega t}\rangle$ (α should be large enough so that the matter part doesn't significantly change the state of the light part), approximately we have

$$H_{\text{light-matter coupling}} \approx \alpha V \mathrm{e}^{-\mathrm{i}\omega t} + \text{h.c.}, \tag{10}$$

and the effective Hamiltonian for the matter part is then

$$H = H_{\text{matter}} + \alpha V \mathrm{e}^{-\mathrm{i}\omega t} + \text{h.c.}. \tag{11}$$