Prof. Bambi on General Relativity

Jinyuan Wu

March 11, 2022

This is a note about Prof. Cosimo Bambi's lecture on general relativity on February 25 and March 3, 2022.

This lecture is about the 1-2 chapters in [1]. Nothing quite interesting. [1] itself is very detailed and it seems I don't really need to take much notes.

1 The Christoffel symbols of spherical coordinates

The Christoffel symbol of spherical coordinates, given in (1.83) in [1], can be calculated automatically in this Mathematica notebook.

2 Derivation of special relativity

Section 2.1 and 2.2 seem to be based on Chapter 1 and 2 of Landau's book about field theory. The arguments have been summarized in Section 1.1.2 in this note.

Section 2.3 derives the Lorentz transformations by Wick rotation of Euclidean rotation in $d = 4$ – (2.18) is actually just $\tau = it$ in condensed matter physics. A rotation on xy has the form of (2.20) and we have $C_4^2 = 6$ rotations. Adding 4 translations, we get the total 10 generators of the rotation group in $\mathbb{R}^4$, and hence the Lorentz transformations in the (3+1)-dimensional Minkowski spacetime.

Equations from (2.22) to (2.27) are trying to relate the parameter of $\mathbb{R}^4$ rotations to the relative velocity of the two reference frames.

Note that since the rotation group $SO(4)$ is not Abelian, Lorentz transformations do not commute in general. Problem 2.3

3 Aberration of light and superluminal motion

We should note that *Lorentz contraction* of length (see for example (2.43)) is often *not* the length change we *observe* when an object is moving fast. The point here is that to measure a line we need to detect light signals from its two ends, and generally speaking, two signals arriving at our detector start their journeys at different time points, while on the other hand, from the way we derive Lorentz contraction ((2.41) and (2.42)) we are dealing with events happening at the same time point in the laboratory frame of reference. The conclusion is l in (2.43) is often not the length we *see* of a moving object. (2.48) to (2.57) (2.41) and (2.42)

The fact that the "distance" we see is actually not the authentic space distance between two events with the same time in a given frame of reference means that when calculating the velocity, we are taking the time derivative of two points at different time points, or in other words, taking the derivative of a distance with respect to a time not coherent with the current frame of reference. In this way, some superfluous "superluminal" movement can occur. Consider, for example, the case of Figure 1 on page 2. Suppose at t a beam of light is emitted from the ejected material, and it arrives at the detector at t'. We therefore have

$$\begin{aligned} c(t'-t) = L &= \sqrt{(D - vt\cos\varphi)^2 + v^2t^2\sin^2\varphi} \\ &= D - vt\cos\varphi + \mathcal{O}(v^2t^2/D^2). \end{aligned} \tag{1}$$

(2.51), (2.52)

Therefore, we have

$$t' = \frac{D}{c} + t(1-\beta\cos\varphi). \tag{2}$$

(2.53)

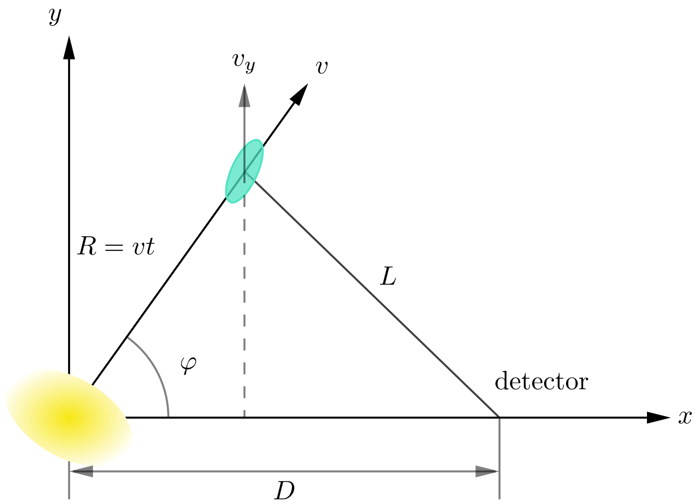

Figure 1: Superluminal motion of a ejected material from a galaxy

Now we try to evaluate the *apparent* velocity on the y direction, which is $\mathrm{d}y/\mathrm{d}t'$. Note that only $\mathrm{d}y/\mathrm{d}t$ is bounded by c, while $\mathrm{d}y/\mathrm{d}t'$ does not have an upper bound. Actually, we have

$$\frac{\mathrm{d}y}{\mathrm{d}t'} = \frac{\mathrm{d}y}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}t'} = v \sin\varphi \times \frac{1}{1-\beta\cos\varphi}. \tag{3}$$ (2.54)

The maximum is shown to be $v\gamma$. Here we can see the apparent velocity is obtained using t' as (2.57)
the time, which is not x^0 in the frame of coordinate attached to the observer.

4 Time dilation and cosmic ray muons

The lifetime of an unstable particle is measured according to its proper time, and this causes
the difference between the prediction of Newtonian and relativistic theories of the flux of the Sec. 2.6
particle after traveling a certain distance. This is actually a piece of strong evidence of special relativity.

5 Relativistic mechanics

We know the Lagrangian of a Newtonian free particle is

$$L = \frac{1}{2} m g_{ij} \dot{x}^i \dot{x}^j. \tag{4}$$ (3.1)

The natural generalization is

$$L = \frac{1}{2} m g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu. \tag{5}$$

Note, however, the meaning of $\dot{x}$ is not clear here: $g_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu$ is already a covariant value, and to make the whole expression covariant, the "time" used in the time derivative $\dot{x}$ should also be a relativistic scalar, which can only be the proper time τ. Similarly, when calculating the action, we need to integrate (5) over τ. So the final action is

$$S = \int \mathrm{d}\tau\, L, \quad L = \frac{1}{2} m g_{\mu\nu} \frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau}. \tag{6}$$

We can repeat the process in (1.71) and find that the EOM is (1.71)

$$\frac{\mathrm{d}^2 x^\mu}{\mathrm{d}\tau^2} + \Gamma^\mu{}_{\nu\rho}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau}\frac{\mathrm{d}x^\rho}{\mathrm{d}\tau} = 0. \tag{7}$$ (3.6)

Actually Newton's second law can also be written into this form (again see the discussion around (1.71)), but this time we are working with $\mu, \nu, \rho = 0, 1, 1, 3$, not just $1, 2, 3$. Note that (7) is actually the geodesic equation.

We have a more "geometric" version of (4). Since

$$\mathrm{d}l = \sqrt{g_{ij}\dot{x}^i\dot{x}^j}\,\mathrm{d}t$$

is an increasing function of the integrand of (4), the action corresponding to (4) is equivalent to

$$S = \int \mathrm{d}l\,. \tag{8}$$

Similarly, we may guess the version of (8) corresponding to (6) is

$$S = \int |\mathrm{d}s| = \int \sqrt{-\,\mathrm{d}s^2}. \tag{9}$$

This is indeed the case, since actually by the definition of proper time, we have

$$c^2\,\mathrm{d}\tau^2 = -\,\mathrm{d}s^2 = -g_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu\,, \tag{10}$$ (3.4)

and therefore L in (6) is just a constant, and we have $S \propto \int \mathrm{d}\tau$.

A question is what is the relativistic version of mechanics of *massless* particles. Since m in the action is just a constant, the geodesic equation (7) still works. The Lagrangian therefore can still be written as Sec. 3.3

$$L = -mc\sqrt{-g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu}, \tag{11}$$ (3.38)

but now m should be a constant interpreted as a *coupling constant* with the dimension of mass.

6 Particle collision

Relativistic scattering theory is important in particle physics. Consider a typical reaction:

$$A + B \longrightarrow C + D. \tag{12}$$ (3.41)

Suppose we are working in a frame of reference where A is at rest. Suppose B moves along the x axis. Then we have

$$p_A^\mu = (m_A c, 0, 0, 0), \quad p_B^\mu = (m_B\gamma c, m_B\gamma v, 0, 0). \tag{13}$$ (3.43)

After the reaction, the energies and the momenta of C and D can be quite complicated, so we just try to find some general constraints imposed on them. A reaction is possible if and only if both energy conservation and momentum conservation hold. The momentum conservation condition can be satisfied by working in a reference frame where the total 3-momentum of the system vanishes, and this dictates

$$p_C^\mu = (\sqrt{m_C^2c^2 + \boldsymbol{p}_C^2}, \boldsymbol{p}_C), \quad p_D^\mu = (\sqrt{m_D^2c^2 + \boldsymbol{p}_C^2}, -\boldsymbol{p}_C). \tag{14}$$ (3.44)

Now we just need to impose the energy conservation constraint. Naively doing so is hard because energy itself is not a scalar. However, there *is* a conserved relativistic scalar: we have

$$p_\mu^{\mathrm{i}} p^{\mathrm{i}\mu} = p_\mu^{\mathrm{f}} p^{\mathrm{f}\mu} =: M^2c^2, \tag{15}$$

where M is named the **invariant mass**. Note that we can evaluate the LHS in the reference frame of (13) and the RHS in the reference frame of (14), and this gives

$$(m_A c + m_B\gamma c)^2 - m_B^2\gamma^2v^2 = (m_C c + m_D c)^2 - (2\boldsymbol{p}_C)^2, \tag{16}$$ (3.48)

which can be simplified into

$$2m_A m_B\gamma = m_C^2 + m_D^2 + 2m_C m_D - m_A^2 - m_B^2 - 4\boldsymbol{p}_C^2/c^2. \tag{17}$$ (3.49)

The minimum energy of B is

$$E_B^{\mathrm{th}} = p_B^0 c = m_B\gamma c^2 = \frac{\left(m_C^2 + m_D^2 + 2m_C m_D - m_A^2 - m_B^2\right)c^2}{2m_A}. \tag{18}$$ (3.50)

This is called the **threshold energy**, because if $E_B < E_B^{\mathrm{th}}$, (12) cannot happen.

7 Relativistic perfect fluid

The topic of relativistic idea fluid is discussed in Problem 2.1 and 2.2. First of all, we always have Problem 2.1, 2.2

$$T^{00} = \epsilon = \rho_m c^2,$$

where ϵ is the energy density of the fluid in the rest frame, and ρ_m is the mass density. In a perfect fluid, when we are in the rest-frame, there is no flow, and since momentum is carried by fluid flow, the density of momentum is also zero, i.e.

$$T^{i0} = 0.$$

The argument used in non-relativistic perfect fluid can be transplanted here for T^{ij}: since an idea fluid is isotropic, and it cannot hold shear force, as long as the time scale we are interested is long enough to hide how the fluid responds to an external shear force, we can assume all shear force components in T^{ij} are zero. Thus we have

$$T^{ij} = \begin{pmatrix} p & & \\ & p & \\ & & p \end{pmatrix}.$$

Putting everything together, we get

$$T^{\mu\nu} = \begin{pmatrix} \epsilon & & & \\ & p & & \\ & & p & \\ & & & p \end{pmatrix}. \tag{19}$$ (2.60)

Now we can get $T^{\mu\nu}$ in any coordinate system with a Lorentz transformation. Applying the Lorentz transformation on x direction:

$$\Lambda^{\mu}{}_{\nu} = \begin{pmatrix} \gamma & -\gamma\beta & & \\ -\gamma\beta & \gamma & & \\ & & 1 & \\ & & & 1 \end{pmatrix}, \tag{20}$$ (2.28)

we get (the process can be found in this Mathematica notebook)

$$T'^{\mu\nu} = \begin{pmatrix} \frac{p\beta^2}{1-\beta^2} + \frac{\epsilon}{1-\beta^2} & -\frac{\epsilon\beta}{1-\beta^2} - \frac{p\beta}{1-\beta^2} \\ -\frac{\epsilon\beta}{1-\beta^2} - \frac{p\beta}{1-\beta^2} & \frac{\epsilon\beta^2}{1-\beta^2} + \frac{p}{1-\beta^2} \end{pmatrix}. \tag{21}$$

Suppose the speed of the frame of reference after (20) in the rest frame of the fluid is v, we find (21) is the energy-momentum tensor of a fluid moving with the velocity of $-v\hat{\boldsymbol{e}}_x$. Problem 2.2 Liang Sec. 6.5

(21) is not covariant. We need to generalize it into a covariant version. Solely with information provided in (21), the covariant version cannot be decided, because systems other than a perfect fluid can also has a energy-momentum tensor like (19). Another way to see the point is to note that velocity of the fluid is different on different points, and a global Lorentz transformation cannot turn the fluid into the state of rest. We can do local Lorentz transformation, but this distorts the components of $\eta^{\mu\nu}$, but when deriving (19) we have $\eta = \mathrm{diag}(-1, 1, 1, 1)$.

The generic covariant energy-momentum tensor of a perfect fluid is

$$T^{\mu\nu} = \left(\rho_m + \frac{p}{c^2}\right) U^\mu U^\nu + p\eta^{\mu\nu} = \frac{1}{c^2}(\epsilon + p) U^\mu U^\nu + p\eta^{\mu\nu}. \tag{22}$$

Note that p and ϵ in (22) are defined in a special frame of reference, but this does not eliminate the covariance of (22), because for a fluid there is indeed a special frame of reference, i.e. the rest frame of itself. When in the rest frame of reference, we have

$$U^\mu = \begin{pmatrix} \mu \\ 0 \\ 0 \\ 0 \end{pmatrix}, \tag{23}$$

and (22) reads

$$T^{\mu\nu} = \frac{1}{c^2}(\epsilon + p)\begin{pmatrix} c^2 & & & \\ & 0 & & \\ & & 0 & \\ & & & 0 \end{pmatrix} + p\begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix},$$

which is just (19). After a global Lorentz transformation (20), we have

$$U'^{\mu} = \Lambda^{\mu}{}_{\nu} U^{\nu} = \begin{pmatrix} \gamma c \\ -\gamma\beta c \\ 0 \\ 0 \end{pmatrix}.$$

Substituting this into (22), we get (21). (The process is in this Mathematica notebook).

References

[1] Cosimo Bambi. *Introduction to General Relativity: A Course for Undergraduate Students of Physics*. Springer, 2018.