Bosonic modes in Fermi liquid

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1 Introduction

The Fermi liquid theory can be justified by diagrammatic resummation: by summing up a certain family of self-energy diagrams that are believed to be important, we get a correction to the electron band dispersion relation, as well as a finite lifetime. The electron density-dependent part of the self-energy correction is often known as "forward scattering", which has the form of $f_{\boldsymbol{p}\boldsymbol{k}'}\,\delta n\,(\boldsymbol{p})\,\delta n\,(\boldsymbol{p}')$ in the energy functional. But interaction channels beside forward scattering that come from Coulomb interaction do not just disappear; they are still a part of the Hamiltonian and will contribute to the specific heat when the system is heated up. Therefore, it can be expected that a real condensed matter system that is said to be in a Fermi liquid phase contains *more* than electron-like quasiparticles.

Characterization of the full spectrum of a system is generally only possible for exactly solvable systems. This report is constrained on bosonic modes in Fermi liquid,[1] or to be specific, on excitations for which a quantum is essentially a renormalized electron-hole pair. In other words, in this report we are interested in oscillation modes of operators with the shape of $c^\dagger_{\boldsymbol{k}+\boldsymbol{q}/2}c_{\boldsymbol{k}-\boldsymbol{q}/2}$. Three-electron behaviors do exist [1, 2] but are beyond the scope of this report.

2 Landau kinetic equation of neutral Fermi liquid

In principle all electro-hole bosonic modes can be found by looking at poles of the four-point Green function, or in other words, by diagonalizing the four-point kernel; but

Classical Boltzmann equation can be derived using BBGKY hierarchy, if we make the assumption that n-th order correlation functions ($n > 2$) are ignorable; a similar procedure can be applied in the quantum region, giving us the equation of motion of two-point (i.e. single-electron) Green function $G(\boldsymbol{r}_1, t_1; \boldsymbol{r}_2, t_2)$. In the quantum region, and after Wigner transform, we have

$$f \tag{1}$$

[1] There is a terminological confusion here: the term *Fermi liquid* may refer to a system whose Hamiltonian is exactly in the shape of Fermi liquid energy functional, or it may refer to a system in which the behavior of electron Green function follows the Fermi liquid theory, but may contain other excitations. This note uses the latter definition; thus the phrase "a Fermi liquid" is a shorthand for "a real-world condensed matter system demonstrating Fermi liquid behaviors in its single-electron part".

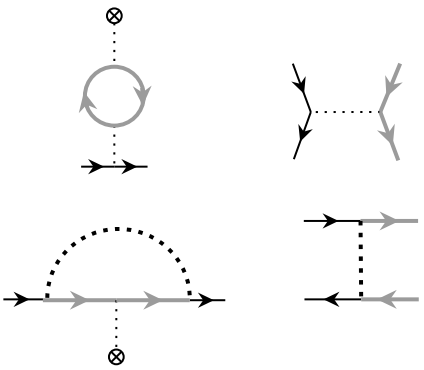

Figure 1: Linear response of two-point Green function gives four-point Green function: the left two Feynman diagrams are

The most generalized derivation involves Keldysh field theory and is beyond the scope of this report; we only emphasize here that the derivation depends on gradience expansion and that the quasiparticle picture works well in the system so that the spectral function is approximately

$$A(\boldsymbol{k},\omega) = \delta(\omega - \operatorname{Re}\Sigma(\boldsymbol{k},\omega)), \tag{2}$$

where the imaginary part of the self-energy is ignored in the spectral function, but appears in the collision integral on the RHS. The second assumption is by definition satisfied with a Fermi liquid; the first assumption however imposes a fundamental constraint on the coverage of the quantum Boltzmann equation to some of well-attested phenomena in Fermi liquid (Section 5).

The gradience expansion condition is at the first glance broken in condensed matter systems, since the crystal potential has a very small characteristic length scale. For a single-band problem, however, we can manually find a "position" operator $\boldsymbol{x}$ as the conjugate variable of the lattice momentum $\boldsymbol{k}$, which represents the center of the wave packet and in the coarse-grained macroscopic limit appears to be the commonly known position; this definition of $\boldsymbol{x}$ however depends on the shape of the band wave function, and when there exist several bands, this procedure is not longer applicable.

The aforementioned procedure does not impose any constraint on $\operatorname{Re}\Sigma$; specifically, it does not dictate that $\operatorname{Re}\Sigma$ cannot have explicit dependence on $G(\boldsymbol{r},\boldsymbol{r}',t,t')$, and hence $f(\boldsymbol{r},\boldsymbol{p},t)$. We can then insert the Fermi liquid effective energy

$$\varepsilon_{\boldsymbol{p}\sigma} = \varepsilon^0_{\boldsymbol{p}\sigma} + \frac{1}{V}\sum_{\boldsymbol{p}',\sigma'} f_{\boldsymbol{p}\boldsymbol{p}'\sigma\sigma'}\,\delta n_{\boldsymbol{p}'\sigma'} \tag{3}$$

into the quantum Boltzmann equation; the resulting equation system is called *Landau equation*. Since now δn has both $\boldsymbol{p}$ and $\boldsymbol{r}$ dependence, $\varepsilon_{\boldsymbol{p}}$ also has $\boldsymbol{r}$ dependence. No spatial dependence is added to $f_{\boldsymbol{p}\boldsymbol{p}'}$, however: the fact that $n_{\boldsymbol{k}\boldsymbol{q}}$ has a non-zero value when $\boldsymbol{q}\neq 0$ is due to external electromagnetic driving, which at the linear level does not change the momentum conservation condition where an interaction line meets with two electron lines.

The inclusion of the Fermi liquid self-energy correction immediately leads to an important consequence of Fermi liquid: that when the temperature is zero and no collision is possible for quasiparticles, we still have density modes which resemble ordinary sound wave in some aspects. This mode is known as *zero sound*.

When the temperature is non-zero, $\tau \propto 1/T^2$ is finite and zero sound faces strong damping when its frequency is too slow. In the low frequency domain, where thermal equilibrium is almost always established, we get ordinary sound or "first sound". The first sound can be derived by calculating mechanical properties of the Fermi liquid in question and inserting the compressibility into $v = \sqrt{\partial p/\partial\rho}$ [3]. This approach assumes the usual framework of hydrodynamics works for Fermi liquid; a direct verification can be found in [4]. The spectrum of first sound is connected to the spectrum of zero sound: the zero sound and the first sound can be derived in a unified way [5], and we may say first sound is merely zero sound with finite temperature correction, but this correction is so severe that the qualitative physical picture is radically changed. First sound can be derived with macroscopic conservation equation usually used in fluid dynamics, while this is no longer possible for zero sound; the two types of sounds also have different dissipation mechanism: zero sound is damped because of the non-zero collision integral, while first sound is damped because the collision is not strong enough, so as a first sound wave propagates, it excites electron-hole pairs and loses energy [6, 4]. The sound spectrum of Fermi liquid is therefore summarized in Fig. 2.

3 Damping mechanisms

4 Charged Fermi liquid and the plasmon

One more mechanism

5 Microscopic bosonic modes beyond the Landau equation

Not all bosonic modes can be obtained by observing oscillation modes of quantum Boltzmann equation, since the latter only works for excitations with a large characteristic length scale in

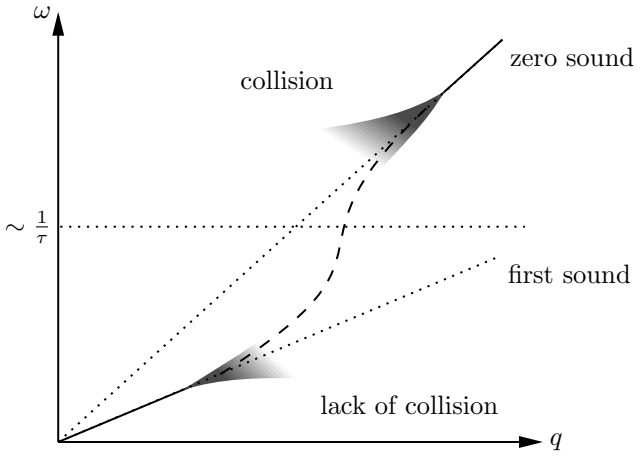

Figure 2: Comparison between zero sound and first sound: zero sound only exists in the $\omega\tau \gg 1$ region, while first sound only exists in the $\omega\tau \ll 1$ region; damping of zero sound is due to collision, while damping of first sound is due to the fact that collision is not strong enough to establish thermal equilibrium. First sound is a finite temperature effect.

the $\boldsymbol{x}_1 - \boldsymbol{x}_2$ variable. This constraint has two consequences.

First, it requires that $\boldsymbol{q}$ is small enough. When a uniform electric field is applied, it influences $\boldsymbol{x}_1 - \boldsymbol{x}_2$, not $(\boldsymbol{x}_1 + \boldsymbol{x}_2)/2$; accordingly, it influences $\boldsymbol{k}$, which is now to be understood as the *relative* momentum between the electron and the hole (absence of an electron with momentum $\boldsymbol{k}$ is equivalent to existence of a hole with momentum $-\boldsymbol{k}$). If, however, the electric field has very strong spatial variance, the electron at $\boldsymbol{r}_1$ feels a different force from that felt at $\boldsymbol{r}_2$, and the electron-hole pair gets driven as a whole, giving a non-zero value to $\boldsymbol{q}$. Thus the small-$\boldsymbol{q}$ condition is equivalent to the condition that the wave length of the driving electric field should be small compared with the atomic length scale.

Second, it rules out the possibility for the naive quantum Boltzmann equation to faithfully represent inter-band phenomena. In the multi-band case the band Hamiltonians H_1 and H_2 (TODO: ref) should be compared directly, both of which have atomic characteristic length scale and gradience expansion fails. Another perspective to see the infeasibility to have a semi-classical picture for a multi-band problem is by directly observing the equation governing exciton energy, which contains $\varepsilon^{\mathrm{c}}_{\boldsymbol{k}+\boldsymbol{q}} - \varepsilon^{\mathrm{v}}_{\boldsymbol{k}}$. Should the two energies be both from a single band, gradience expansion is possible when $\boldsymbol{q}$ is small, which leads to the $\frac{\partial \varepsilon_{\boldsymbol{k}}}{\partial \boldsymbol{k}} \frac{\partial n}{\partial \boldsymbol{r}}$ term; for an exciton however this is not the case. Thus the quantum Boltzmann equation outlined above is unable to describe an exciton mode. This does not mean the quantum Boltzmann equation is useless for excitons: excitons are usually modeled as independent particles themselves *besides* electrons, and exciton formation is modeled as a three-particle vertex [7], while

[8]. We may say zero sound/plasmon is a *hydrodynamic* mode,

6 Conclusion

References

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[8] David Pines. *Theory of Quantum Liquids: Normal Fermi Liquids*. The "expansion of energy" subsection in Section 1.4 is about the short-range condition. The "charged v.s. neutral system" subsection in Section 3.3 compares plasmon with zero sound. The inability of quantum Boltzmann equation to capture microscopic bound states is discussed on p. 56. CRC Press, 2018.