Squeezing of quantum noise

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1 Introduction

Noise usually arises from coupling with an unknown external object: when we keep our eyes only on one of the objects (the "system"), we need to average over the state of the other ("the environment"), which corrects the theory with noise and dissipation [3]. When the full quantum treatment is needed, however, another type of noise appears: the state itself does not have a definite value of the observable in question, and any experimental setting – no matter how carefully the system is isolated from the environment – has noisy results. This kind of noise is called **quantum noise**.

In systems that can be well described by the harmonic oscillator picture, in which for each oscillation mode, we have two variables X and P, and $[X, P] = \mathrm{i}$ (here and below we use Planck system of units), and the Hamiltonian is $H \simeq c_1 X^2 + c_2 P^2$.[1] After diagonalization, we get $H \simeq \sum_{\text{modes}} \omega(a^\dagger a + 1/2)$ plus possible interaction terms, and this zero-point energy arises from the non-commutative nature of X and P. Another way to make sense of the $1/2$ term is to notice that at the ground state, though $\langle X \rangle = \langle P \rangle = 0$, and everything seems to be very definite, since we have the zero-point energy, $\langle X^2 \rangle \simeq H/2 \neq 0$. Note that most of the time, the observable actually measured in such systems is $\simeq n = a^\dagger a \sim X^2$, and quantum noise appears in the measurement. People therefore sometimes say that quantum noise comes from the zero point-energy. Of course, in some states, the error on the X degree of freedom – or P, or some combination of them – may be reduced, at the cost of larger error on other degrees of freedom. This is called "squeezing" of the error.

This report focuses on quantum noise in linear quantum optics, which fits in the picture described above. Section 2 discusses more quantitative ways to represent and characterize quantum noise. Section 3 calculates quantum noise in a prevalent phase measurement scheme using interferometer. Section 4 demonstrates how the quantum noise, when the input laser beam is strong enough, can be attributed to the quantum noise in the dark port. Section 5 shows how to squeeze the quantum noise in the output by injecting a squeezed state into the dark port.

2 Quantum optics states and quasi-probability distribution functions

In linear quantum optics, we have

$$H = \frac{1}{2}\epsilon_0 \boldsymbol{E}^2 + \frac{1}{2\mu_0}\boldsymbol{B}^2 = \sum_k \omega_k \left(a_k^\dagger a_k + \frac{1}{2} \right), \tag{1}$$

where k is the label of optical normal modes. It is the momentum and polarization in a large box, and in an spherical optical cavity, it labels spherical harmonics. We have the standard bosonic commutation relations

$$[a_k, a_{k'}] = 0, \quad [a_k, a_{k'}^\dagger] = \delta_{kk'}. \tag{2}$$

As is known in quantum mechanics, the wave function characterizes the state of the system, but cannot be measured directly. One way to visualize a quantum state – and therefore the quantum noise in it – is the so-called quasi-probability distribution functions. For one mode, if a and $a^\dagger$ in an arbitrary operator O is ordered according to a certain convention (a before $a^\dagger$, or $a^\dagger$ before a, or whenever multiplication appears, it is in the form of $(a^\dagger a + a a^\dagger)/2$), we have

$$\langle O(a, a^\dagger) \rangle = \int \mathrm{d}^2\alpha \, f(\alpha, \alpha^*) O(\alpha, \alpha^*), \tag{3}$$

[1] The Planck system of units is used in this report, so we take $\hbar = 1$ if there is no special mention of the units.

we say $f(\alpha, \alpha^*)$ is a quasi-probability distribution function. Note that this definition even extends to the case when we are in a mixed state instead of a pure state, i.e. it is a unified treatment of both quantum noise and thermal noise.

To plot the function as a 2D heatmap, it is often more convenient to change the coordinates into $\operatorname{Re}\alpha$ and $\operatorname{Im}\alpha$. Note that we have

$$[\operatorname{Re} a, \operatorname{Im} a] = \frac{1}{4\mathrm{i}}[a + a^\dagger, a - a^\dagger] = \frac{\mathrm{i}}{2}, \tag{4}$$

and according to the uncertainty principle, we have

$$\Delta(\operatorname{Re} a) \cdot \Delta(\operatorname{Im} a) \geq \frac{1}{4}. \tag{5}$$

Another frequent convention is to define

$$a = \frac{1}{\sqrt{2}}(X + \mathrm{i}P), \quad a^\dagger = \frac{1}{\sqrt{2}}(X - \mathrm{i}P), \tag{6}$$

and we find

$$[X, P] = \frac{\mathrm{i}}{2}[a + a^\dagger, a - a^\dagger] = \mathrm{i}. \tag{7}$$

Here the $\sqrt{2}$ factor is simply there to normalize the commutation relation.

The most famous quasi-probability distribution is probably the **Wigner function**, and it is the quasi-probability distribution used in this report. The Wigner function of a density matrix ρ (which is $|\psi\rangle\langle\psi|$ for a pure state) is defined as [2]

$$W(x, p) = \frac{1}{\pi\hbar} \int_{-\infty}^{\infty} \langle x - y|\rho|x + y\rangle e^{-2\mathrm{i}py/\hbar} \mathrm{d}y, \tag{8}$$

and we can use (6) to define $W(\alpha, \alpha^*)$ and hence $W(\operatorname{Re}\alpha, \operatorname{Im}\alpha)$.

Some examples of pure state Wigner functions are given in Fig. 1. Fig. 1(a) is the Wigner function of the vacuum state: there is no photon, and the distribution of probability is focused on $\alpha = 0$. Note that it is not a δ-function, and is somehow "blurred": this is a visualization of (5). Not all two-variable functions can be a Wigner function: they cannot have too strong "spatial resolution".

3 Quantum noise in the Mach-Zehnder interferometer

To have a concrete example of quantum noise, let us move to the **Mach-Zehnder interferometer**, illustrated in Fig. 2. The interferometer contains two beam splitters, two ideal mirrors, and one sample that introduces a 2φ phase shift to the light beam going through it. The two beams created by the first beam splitter gain a phase difference caused by the sample, and are then remixed together by the second beam splitter, so there is interference, and φ can be found by comparing the output signal with the injected laser beam.

We are going to work in the Heisenberg picture, because the whole system is linear and time evolution can be easily described by a linear transformation on the operators. Since this section is just to exemplify the overall idea of quantum noise, for the sake of simplicity we assume the time evolution operator of the beam splitter is

$$S_{\text{beam splitter}} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}. \tag{9}$$

The time evolution operator of the whole system is therefore

$$\begin{aligned} S_{\text{total}}(\varphi) &= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} \mathrm{e}^{\mathrm{i}\varphi} & \\ & \mathrm{e}^{-\mathrm{i}\varphi} \end{pmatrix} \cdot \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \\ &= \begin{pmatrix} \cos\varphi & \mathrm{i}\sin\varphi \\ -\mathrm{i}\sin\varphi & -\cos\varphi \end{pmatrix}, \end{aligned} \tag{10}$$

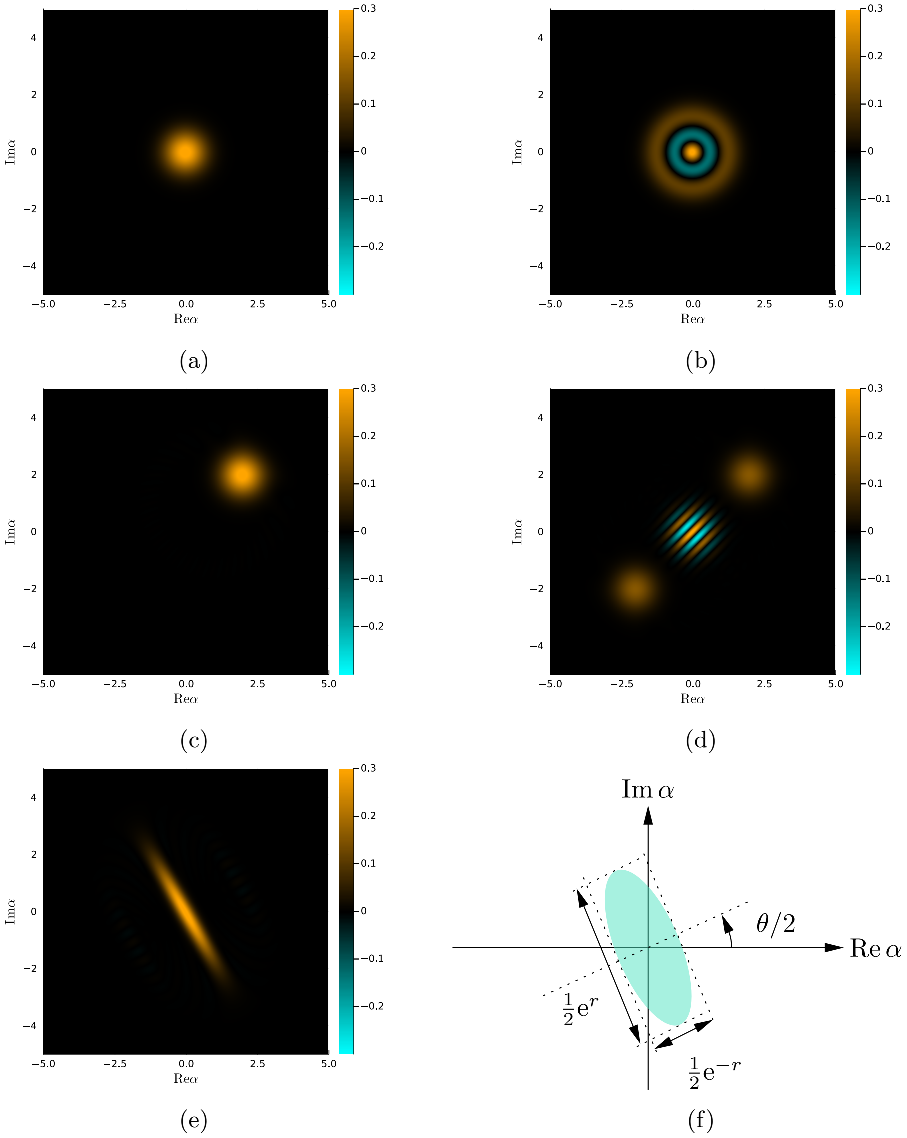

Figure 1: Wigner functions in quantum optics. Figures are plotted using `QuantumOptics.jl` [1]. (a) The Wigner function of the vacuum. Note that this is not a δ-function located at $\alpha = 0$. (b) The Wigner function of a single-photon state. Negative values occur. (c) The Wigner function of $|\alpha = 2 + 2\mathrm{i}\rangle$. (d) The Wigner function of $(|\alpha = 2 + 2\mathrm{i}\rangle + |\alpha = -2 - 2\mathrm{i}\rangle)/\sqrt{2}$. (e) The Wigner function of $S(\mathrm{e}^{\mathrm{i}\frac{\pi}{3}})\,|0\rangle$. (f) Schematic illustration of the Wigner function of a squeezed vacuum state $S(r\mathrm{e}^{\mathrm{i}\theta})\,|0\rangle$. The diagram is adapted from Fig. 2.8 in [2].

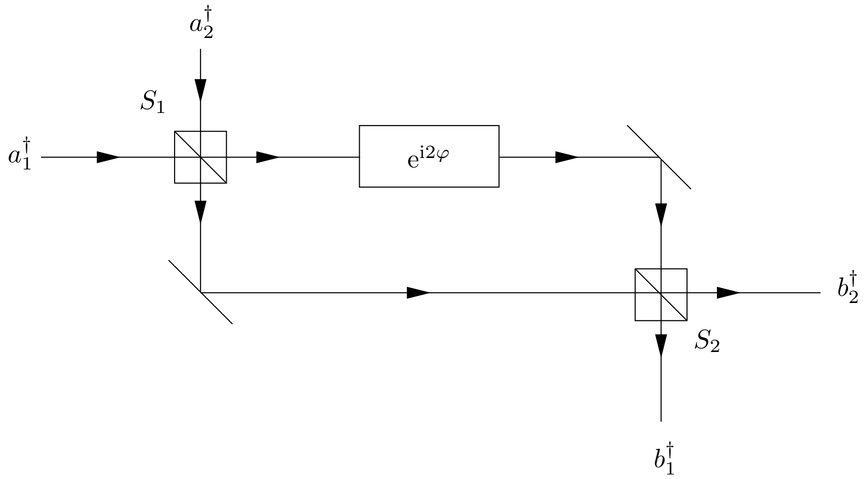

Figure 2: The schematic structure of the Mach-Zehnder interferometer

and therefore[2]

$$\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} \cos\varphi & \mathrm{i}\sin\varphi \\ -\mathrm{i}\sin\varphi & -\cos\varphi \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}, \tag{11}$$

$$\begin{pmatrix} b_1^\dagger \\ b_2^\dagger \end{pmatrix} = \begin{pmatrix} \cos\varphi & -\mathrm{i}\sin\varphi \\ \mathrm{i}\sin\varphi & -\cos\varphi \end{pmatrix} \begin{pmatrix} a_1^\dagger \\ a_2^\dagger \end{pmatrix}, \tag{12}$$

from which we find

$$\begin{pmatrix} a_1^\dagger \\ a_2^\dagger \end{pmatrix} = \begin{pmatrix} \cos\varphi & -\mathrm{i}\sin\varphi \\ \mathrm{i}\sin\varphi & -\cos\varphi \end{pmatrix} \begin{pmatrix} b_1^\dagger \\ b_2^\dagger \end{pmatrix}. \tag{13}$$

In actual experiment settings, usually a beam of laser is injected into input port 1, so the state at input port 1 is a coherent state. Then, the wave function is

$$\begin{aligned} |\psi\rangle &= \underbrace{\mathrm{e}^{\alpha a_1 - \alpha^* a_1^\dagger}}_{D_{a_1}(\alpha)} |0\rangle \\ &= \mathrm{e}^{\alpha(\cos\varphi b_1^\dagger - \mathrm{i}\sin\varphi b_2^\dagger) - \alpha^*(\cos\varphi b_1 + \mathrm{i}\sin\varphi b_2)} |0\rangle \\ &= |b_1 = \alpha\cos\varphi, b_2 = -\mathrm{i}\alpha\sin\varphi\rangle\,. \end{aligned} \tag{14}$$

Note that the state in a_2 subspace remains vacuum; it is a convention to call a_2 the **dark port**.

The next step is measurement. Usually, the influence of the sample to the light beam is small, and therefore the signal in the b_2 mode is $\sim \varphi$, while the signal in the b_1 mode is $\sim \varphi^2$ (the signal in the b_1 mode should be defined as $\alpha - \alpha\cos\varphi$, because when $\varphi = 0$, the output state is $|b_1 = \alpha, b_2 = 0\rangle$). So we usually measure the intensity of the b_2 mode. The expectation of photon number is

$$\langle n_{b_2} \rangle = |\alpha|^2 \sin^2\varphi. \tag{15}$$

This output comes with a quantum uncertainty, which is given by

$$\begin{aligned} \Delta n_{b_2} &= \sqrt{\langle n_{b_2}^2 \rangle - \langle n_{b_2} \rangle^2} \\ &= \sqrt{\langle b_2^\dagger (b_2^\dagger b_2 + 1) b_2 \rangle - \langle n_{b_2} \rangle^2} \\ &= \sqrt{|\alpha|^4 \sin^4\varphi + |\alpha|^2 \sin^2\varphi - |\alpha|^4 \sin^4\varphi} \\ &= |\alpha| \sin\varphi. \end{aligned} \tag{16}$$

Here we have used the property of coherent states as eigenstates of the annihilation operator.

So it can be seen that the relative uncertainty of the measurement is

$$\frac{\Delta n_{b_2}}{\langle n_{b_2} \rangle} = \frac{1}{|\alpha|\sin\varphi} = \frac{1}{\sqrt{n_{b_2}}}. \tag{17}$$

[2]Note that the EOM of the annihilation operators in the Heisenberg picture has the same form with the EOM of the wave function in the Schrōdinger picture.

It can be seen from (16) that this standard error comes from the commutation relation

$$[b_2, b_2^\dagger] = 1, \tag{18}$$

which, eventually, comes from the fact that $\boldsymbol{E}$ and $\boldsymbol{A}$ in electromagnetism don't commute. This is therefore a quantum noise.

Of course, (17) can be systematically reduced by using stronger and stronger laser beams, but then another problem occurs: in a real measurement setting, the detecting laser beam of course perturbs the sample, introducing another source of error. (For example, if the sample is an additional optical path, which is the case in gravitational wave detection, then the mirrors used will be heated and begin to have thermal vibration.) In a classical theory, we can always use weaker and weaker laser beams to do the measurement, and do an extrapolation for the measured results to systematically reduce the perturbation of measurement to the system and get results as accurate as we want, but in the quantum theory, this results in stronger quantum noise.

This leads to an astonishing fact: when quantum noise is present, even in principle, we still can't find a way to reduce the error as much as we want. There is a non-zero minimum error, which is met when $|\alpha|$ strikes a balance between the thermal fluctuation caused by large $|\alpha|$ and quantum noise caused by small $|\alpha|$. This actually makes sense, or otherwise we are faced with the problem that an infinitely accurate continuous degree of freedom can store infinite bits of information.

4 The quantum noise as the fluctuation of the dark port

One thing to keep in mind is (17) is about the quantum noise of the photon number, not others. This quantity is not the only observable in b_2 mode, but it is the only thing actually measured. Thus it is possible that we reduce the quantum noise of n_{b_2} and in exchange, get a larger quantum noise of other variables, which we do not care. This is called "squeezing" the quantum noise – the term will be visualized in the following discussion. To do so, it is necessary to trace the origin of Δn_{b_2} in terms of quantum noises of a_1 and a_2. For sake of simplicity, here we keep the input state in the a_1 mode $|\alpha\rangle$, without modifying anything, and we also exclude all cross terms between a_1 and a_2 in the wave function, so we have

$$|\psi\rangle = D_{a_1}(\alpha) f(a_2, a_2^\dagger) \left|0\right\rangle, \tag{19}$$

and squeezing Δn_{b_2} therefore reduces to squeezing the quantum error of some operator in mode a_2.

From (12), we have

$$n_{b_2} = b_2^\dagger b_2 = \sin^2\varphi a_1^\dagger a_1 + \cos^2\varphi a_2^\dagger a_2 - \mathrm{i}\sin\varphi\cos\varphi(a_1^\dagger a_2 - a_2^\dagger a_1), \tag{20}$$

and therefore

$$\begin{aligned} n_{b2}^2 = {} & \sin^4\varphi (a_1^\dagger a_1)^2 + \cos^4\varphi (a_2^\dagger a_2)^2 - \sin^2\varphi\cos^2\varphi (a_1^\dagger a_2 - a_2^\dagger a_1)^2 + 2\sin^2\varphi\cos^2\varphi a_1^\dagger a_1 a_2^\dagger a_2 \\ & - \mathrm{i}\sin\varphi\cos^3\varphi \{a_2^\dagger a_2, (a_1^\dagger a_2 - a_2^\dagger a_1)\} - \mathrm{i}\sin^3\varphi\cos\varphi\{a_1^\dagger a_1, (a_1^\dagger a_2 - a_2^\dagger a_1)\}\,. \end{aligned} \tag{21}$$

If we are sure the wave function takes the form of (19), then all a_1 can be replaced by α in the above two equations. Strictly speaking, before that we should first complete normal ordering of a_1 and $a_1^\dagger$, because otherwise the non-trivial commutation relation of a_1 is ignored, and thus the quantum fluctuation of the a_1 mode is ignored. This however involves highly complicated calculation, because we need to do normal ordering for (21).

A huge simplification (and hence a neat explanation of the nature of the quantum noise in the interferometer) however can be made when the following conditions are satisfied. First, we assume φ is small enough, so the fluctuation of the first term in (20) is suppressed by the $\sin^2\varphi$ factor. Then we assume that in (19), in the state of a_2 mode, the expected number of photon is very small compared with $|\alpha|^2$. Then the fluctuation of the second term in (20) is also neglected, because

$$\Delta(a_2^\dagger a_2) = \langle (a_2^\dagger a_2)^2\rangle - \langle a_2^\dagger a_2\rangle^2 = \langle a_2^\dagger a_2^\dagger a_2 a_2\rangle + \langle a_2^\dagger a_2\rangle - \langle a_2^\dagger a_2\rangle^2 \simeq 0, \tag{22}$$

compared with the absolute magnitude of

$$\Delta(a_1^\dagger a_1) = \langle a_1^\dagger a_1^\dagger a_1 a_1\rangle + \langle a_1^\dagger a_1\rangle - \langle a_1^\dagger a_1\rangle^2 = |\alpha|^2. \tag{23}$$

Under the above two conditions, approximately we have

$$\Delta n_{b_2} = \sin\varphi\cos\varphi\Delta(a_1^\dagger a_2 - a_2^\dagger a_1), \tag{24}$$

and again by the argument above that the absolute magnitude of the fluctuation of n_{a_1} is much larger than that of n_{a_2}, we find the final expression of Δn_{b_2}: it is

$$\Delta n_{b_2} = \sin\varphi\cos\varphi\Delta(\alpha^* a_2 - \alpha a_2^\dagger) \approx \varphi\Delta(\alpha^* a_2 - \alpha a_2^\dagger). \tag{25}$$

We can do a sanity check: if the a_2 state is vacuum, then $\langle a_2\rangle = \langle a_2^\dagger\rangle = 0$, and again due to the non-trivial commutation relation, we have

$$\begin{aligned}\Delta(\alpha^* a_2 - \alpha a_2^\dagger) &= \sqrt{|\langle(\alpha^* a_2 - \alpha a_2^\dagger)^2\rangle - \langle(\alpha^* a_2 - \alpha a_2^\dagger)\rangle^2|} \\ &= \sqrt{|\alpha|^2\langle a_2 a_2^\dagger\rangle} \\ &= |\alpha|\sqrt{1 + \langle a_2^\dagger a_2\rangle} = |\alpha|,\end{aligned} \tag{26}$$

so

$$\Delta n_{b_2} = \varphi|\alpha|, \tag{27}$$

which is the small-angle approximation of (16).

Note that (25) can also be written as

$$\begin{aligned}\Delta n_{b_2} &= \varphi|\alpha|\Delta(\mathrm{e}^{-\mathrm{i}\varphi} a_2 - \mathrm{e}^{\mathrm{i}\varphi} a_2^\dagger) \\ &= \varphi|\alpha|\Delta(-2\mathrm{i}\sin\varphi\,\mathrm{Re}\,a_2 + 2\mathrm{i}\cos\varphi\,\mathrm{Im}\,a_2) \\ &= 2\varphi|\alpha|\Delta(-\sin\varphi\,\mathrm{Re}\,a_2 + \cos\varphi\,\mathrm{Im}\,a_2).\end{aligned} \tag{28}$$

The third line has a direct graphic meaning: it is proportion to the projection of the "light spot" in $W(\beta, \beta^*)$ on the axis

$$\cos\varphi\,\mathrm{Re}\,\beta + \sin\varphi\,\mathrm{Im}\,\beta = 0. \tag{29}$$

This leads us to design an appropriate quantum noise squeezing scheme: if we can reduce the variance on this axis, while still keeping the conditions that φ is small and $|\alpha|^2$ is large compared with the photon number expectation in the a_2 mode.

5 Squeezing the quantum noise

If we inject a squeezed state with parameter $r\mathrm{e}^{\mathrm{i}\theta}$ into the dark port, and let it has narrowest variation in the axis defined by (29), then we have

$$\frac{\theta}{2} + \frac{\pi}{2} = \varphi. \tag{30}$$

This means if we increase r, the relative error seen in n_{b_2} changes as

$$\frac{\Delta n_{b_2}}{\langle n_{b_2}\rangle} \approx \frac{1}{|\alpha|\varphi}\mathrm{e}^{-r}. \tag{31}$$

Despite the exponent function factor, injecting a squeezed state into the dark port however does not improve the precision of Mach-Zehnder interferometer endlessly: when ξ is very large, the particle number expectation of the a_2 mode is no longer ignorable, breaking one of the two conditions used to derive (25). This is illustrated in Fig. 3, where $\alpha = 2$, $\varphi = \pi/10$, and (30) is followed. The initial difference between two plots arises because of the error between φ and $\sin\varphi$; it can be seen that as r exceeds 0.15, the shapes of the two curves becomes different, and in reality, increasing r further actually increases the relative error on n_{b_2}.

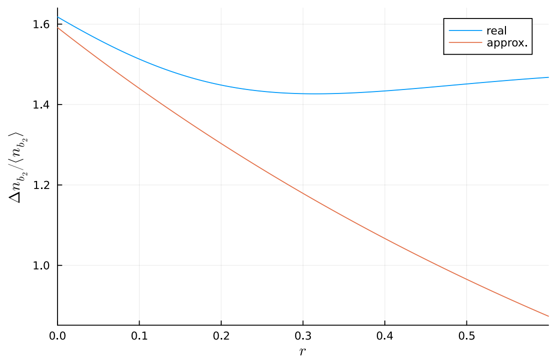

Figure 3: Example of squeezed quantum error: at $r = 0$ we have the standard quantum error. Figure plotted using `QuantumOptics.jl` [1]. In the "real" plot, we calculate Δn_{b_2} strictly by definition, i.e. on the tensor product of the a_1 subspace and the a_2 subspace. The photon number cutoff for both spaces is 30, and α is set to 2 so that $|\alpha|^2$ is much smaller than the cutoff. The "approx." plot is drawn according to (31).

6 Conclusion

This report

References

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[3] Robert Zwanzig. *Nonequilibrium statistical mechanics*. Oxford university press, 2001. The discussion on Mori-Zwanzig formalism or "generalized Langevin equation" can be found on page 149.