Homework 2

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1 Polarization of electromagnetic field

1.1 The general form of a pure state

We have (assuming $\hat{\boldsymbol{k}} = \hat{\boldsymbol{z}}$)

$$\boldsymbol{E} = E_x \begin{pmatrix} 1 \\ 0 \end{pmatrix} + E_y \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \sqrt{\left|E_x\right|^2 + \left|E_y\right|^2} \mathrm{e}^{\mathrm{i}\varphi_x} \begin{pmatrix} \frac{|E_x|}{\sqrt{|E_x|^2+|E_y|^2}} \\ \mathrm{e}^{\mathrm{i}(\varphi_y - \varphi_x)} \frac{|E_y|}{\sqrt{|E_x|^2+|E_y|^2}} \end{pmatrix},$$

and by defining

$$E_0 = \sqrt{\left|E_x\right|^2 + \left|E_y\right|^2} \mathrm{e}^{\mathrm{i}\varphi_x}, \tag{1}$$

$$\cos\theta = \frac{|E_x|}{\sqrt{\left|E_x\right|^2 + \left|E_y\right|^2}}, \tag{2}$$

and

$$\phi = \varphi_y - \varphi_x, \tag{3}$$

we find

$$\boldsymbol{E} = E_0 \begin{pmatrix} \cos\theta \\ \mathrm{e}^{\mathrm{i}\phi} \sin\theta \end{pmatrix}. \tag{4}$$

We have

$$|H\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |V\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \tag{5}$$

and therefore after normalization, we have

$$\rho = (\cos\theta \,|H\rangle + \mathrm{e}^{\mathrm{i}\phi} \sin\theta \,|V\rangle)(\cos\theta \,\langle H| + \mathrm{e}^{-\mathrm{i}\phi} \sin\theta \,\langle V|) = \begin{pmatrix} \cos^2\theta & \mathrm{e}^{-\mathrm{i}\phi} \sin\theta\cos\theta \\ \mathrm{e}^{\mathrm{i}\phi} \sin\theta\cos\theta & \sin^2\theta \end{pmatrix}. \tag{6}$$

1.2 The pure state $\rho^2 = \rho$ condition

We can prove the pure state condition $\rho^2 = \rho$ explicitly:

$$\begin{aligned} \rho^2 &= \begin{pmatrix} \cos^4\theta + \sin^2\theta\cos^2\theta & \mathrm{e}^{-\mathrm{i}\phi} \sin\theta\cos^3\theta + \mathrm{e}^{-\mathrm{i}\phi} \sin^3\theta\cos\theta \\ \mathrm{e}^{\mathrm{i}\phi} \sin\theta\cos^3\theta + \mathrm{e}^{\mathrm{i}\phi} \sin^3\theta\cos\theta & \sin^2\theta\cos^2\theta + \sin^4\theta \end{pmatrix} \\ &= \begin{pmatrix} \cos^2\theta & \mathrm{e}^{-\mathrm{i}\phi} \sin\theta\cos\theta \\ \mathrm{e}^{\mathrm{i}\phi} \sin\theta\cos\theta & \sin^2\theta \end{pmatrix} = \rho. \end{aligned} \tag{7}$$

1.3 Mixed state

The condition that ρ is Hermite means it can be written as

$$\rho = R(\sigma^0 + x\sigma^x + y\sigma^y + z\sigma^z),$$

where $R, x, y, z \in \mathbb{R}$, because the σ matrices constitute a basis for all Hermite matrices in $\mathbb{C}^{2\times 2}$. Since $\sigma^{x,y,z}$ are traceless, from the condition $\operatorname{tr}\rho = 1$, we have

$$1 = \operatorname{tr}\rho = R \operatorname{tr}\sigma^0 = 2R \Rightarrow R = \frac{1}{2},$$

so

$$\rho = \frac{1}{2}(\sigma^0 + x\sigma^x + y\sigma^y + z\sigma^z).$$

In the matrix form, we have

$$\rho = \begin{pmatrix} \frac{1+z}{2} & \frac{x-\mathrm{i}y}{2} \\ \frac{x+\mathrm{i}y}{2} & \frac{1-z}{2} \end{pmatrix},$$

and by substitution of variables (this is a three variables to three variables mapping, and therefore is valid)

$$\frac{1}{2}(1-p) + p\cos^2\theta = \frac{1+z}{2}, \quad x = p\cos\phi\sin 2\theta, \quad y = p\sin\phi\sin 2\theta,$$

we get

$$\frac{1-z}{2} = \frac{1}{2}(1-p) + p\sin^2\theta,$$

and therefore

$$\rho = (1-p)\begin{pmatrix} \frac{1}{2} & \\ & \frac{1}{2} \end{pmatrix} + p\begin{pmatrix} \cos^2\theta & \mathrm{e}^{-\mathrm{i}\phi}\sin\theta\cos\theta \\ \mathrm{e}^{\mathrm{i}\phi}\sin\theta\cos\theta & \sin^2\theta \end{pmatrix}. \tag{8}$$

1.4 Jones parameters and Stokes formalism

1.5 Transformation and measurement

We have

$$\begin{aligned} |45\rangle &= \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \\ |\mathrm{rcp}\rangle &= \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -\mathrm{i} \end{pmatrix} \end{aligned}. \tag{9}$$

The correspond density matrices are

$$\rho_{45} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \tag{10}$$

and

$$\rho_{\mathrm{rcp}} = \frac{1}{2}\begin{pmatrix} 1 & \mathrm{i} \\ -\mathrm{i} & 1 \end{pmatrix}. \tag{11}$$

The operator

$$U_{\mathrm{rcp}} = \begin{pmatrix} 1 & \\ & -\mathrm{i} \end{pmatrix} \tag{12}$$

then turns ρ_{45} to ρ_{rcp}:

$$U_{\mathrm{rcp}}\rho_{45}U_{\mathrm{rcp}}^\dagger = \frac{1}{2}\begin{pmatrix} 1 & \\ & -\mathrm{i} \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & \\ & \mathrm{i} \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & \mathrm{i} \\ -\mathrm{i} & 1 \end{pmatrix} = \rho_{\mathrm{rcp}}. \tag{13}$$

The horizontal polarizer operator

$$\mathcal{O} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \tag{14}$$

is not unitary, because it has non-unitary eigenvalue 0. It is a projection operator: it takes in a beam polarized light and returns its x component. It also represents a measurement: we can use it in a projective measurement setting. In a projective measurement with operator $\mathcal{O}$, $\mathrm{tr}(\rho\mathcal{O})$ is the probability that after measurement, the final state of the system falls into the subspace determined by $\mathcal{O}$. In our case, the subspace determined by $\mathcal{O}$ is the subspace of horizontal polarization, so $\mathrm{tr}(\rho\mathcal{O})$ is the probability that after measurement, we find ρ to be a horizontally polarized state.

Application of $\mathcal{O}$ on (8) is

$$\mathcal{O}\rho\mathcal{O}^\dagger = (1-p)\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 0 \end{pmatrix} + p\begin{pmatrix} \cos^2\theta & 0 \\ 0 & 0 \end{pmatrix} = \left(\frac{1}{2} + \frac{p}{2}\cos 2\theta\right)\mathcal{O}. \tag{15}$$

So after the application of $\mathcal{O}$, we get a horizontally polarized state, as is expected. The result is not normalized; the factor before $\mathcal{O}$ is just $\mathrm{tr}(\rho\mathcal{O})$, which is the probability that after measurement, we find ρ to be horizontally polarized. When $p = 0$, it's $1/2$, which is expected for the unpolarized state; when $p = 1$, it's $\cos^2\theta$, again the correct answer.

2 The $\rho^2 = \rho$ condition for pure states

Suppose

$$\rho = |\psi\rangle\langle\psi| , \quad |\psi\rangle = \sum_m a_m |m\rangle . \tag{16}$$

We have

$$\begin{aligned}
\rho^2 &= \sum_{m,n} a_m^* a_n |n\rangle \langle m| \sum_{j,k} a_j^* a_k |k\rangle \langle j| \\
&= \sum_{m,n,j,k} a_m^* a_n a_j^* |n\rangle \langle m|k\rangle \langle j| \\
&= \sum_{m,n,j,k} a_m^* a_n a_j^* a_k |n\rangle\langle j| \delta_{mk} \\
&= \underbrace{\sum_m a_m^* a_m}_{=\langle\psi|\psi\rangle=1} \sum_{n,j} a_n a_j^* |n\rangle\langle j| \\
&= \sum_{n,j} a_n a_j^* |n\rangle\langle j| = \rho.
\end{aligned} \tag{17}$$

3 Ammonia molecule

The Hamiltonian of the two low-energy states of ammonia is

$$H = \begin{pmatrix} 0 & \Delta/2 \\ \Delta/2 & 0 \end{pmatrix}, \tag{18}$$

where we set

$$|L\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |R\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{19}$$

This Hamiltonian is just a scaled σ^x matrix, and its eigenstates are straightforwardly given by

$$|+\rangle = \frac{1}{\sqrt{2}}(|L\rangle + |R\rangle), \quad |-\rangle = \frac{1}{\sqrt{2}}(|L\rangle - |R\rangle), \tag{20}$$

and the energies are

$$E_+ = \Delta/2, \quad E_- = -\Delta/2. \tag{21}$$

After an electric field is added, in $|L\rangle$ we have an additional energy contribution, and since the molecular configuration in $|R\rangle$ is the opposite of the one in $|L\rangle$, we have

$$H = \begin{pmatrix} dE/2 & \Delta/2 \\ \Delta/2 & -dE/2 \end{pmatrix}. \tag{22}$$

Solving

$$\det \begin{pmatrix} dE/2 - \lambda & \Delta/2 \\ \Delta/2 & -dE/2 - \lambda \end{pmatrix} = 0,$$

we get

$$E_\pm = \pm\frac{1}{2}\sqrt{\Delta^2 + d^2E^2}, \tag{23}$$

and hence

$$|-\rangle = \frac{dE - \sqrt{d^2E^2 + \Delta^2}}{\sqrt{\Delta^2 + (\sqrt{d^2E^2 + \Delta^2} - dE)^2}} |L\rangle + \frac{\Delta}{\sqrt{\Delta^2 + (\sqrt{d^2E^2 + \Delta^2} - dE)^2}} |R\rangle , \tag{24}$$

and

$$|+\rangle = \frac{dE + \sqrt{d^2E^2 + \Delta^2}}{\sqrt{\Delta^2 + (\sqrt{d^2E^2 + \Delta^2} + dE)^2}} |L\rangle + \frac{\Delta}{\sqrt{\Delta^2 + (\sqrt{d^2E^2 + \Delta^2} + dE)^2}} |R\rangle . \tag{25}$$

When E is large, we have

$$dE - \sqrt{d^2E^2 + \Delta^2} \approx 0,$$

$$dE + \sqrt{d^2E^2 + \Delta^2} \approx 2dE,$$

and therefore

$$E_+ = \frac{1}{2}dE, \quad |+\rangle = |L\rangle, \tag{26}$$

and

$$E_- = -\frac{1}{2}dE, \quad |-\rangle = |R\rangle. \tag{27}$$

This is expected, because when $Ed \gg \Delta$, the non-diagonal terms in the Hamiltonian can be safely ignored.

4 Kaptiza's pendulum

4.1 Integrating out the fast variable

In the $\omega \to \infty$, $F_0 \to 0$ limit, the high-frequency part and the low-frequency part of the solution of

$$mR\ddot{\theta} = (-mg + F_0 \sin \omega t) \sin \theta \tag{28}$$

are not strongly coupled and the high-frequency degree of freedom can be integrated out to get an effective theory of the low-frequency part. We do the decomposition

$$\theta = \theta_f + \theta_s, \tag{29}$$

where θ_f is the fast variable. Observing (28), we find the EOM of θ_f should be

$$mR\ddot{\theta}_f = F_0 \sin \omega t \sin(\theta_f + \theta_s), \tag{30}$$

because the first term on the RHS of (28) has a much lower frequency magnitude compared with ω. We take the first order approximation of (30) and ignore the θ_f dependency on the RHS, and this gives

$$\theta_f = -\frac{F_0}{mR\omega^2} \sin \theta_s \sin \omega t. \tag{31}$$

Putting this back to (28), we get

$$\begin{aligned} mR\left(\frac{F_0}{mR} \sin \theta_s \sin \omega t + \ddot{\theta}_s\right) &= (-mg + F_0 \sin \omega t) \sin\left(\theta_s - \frac{F_0}{mR\omega^2} \sin \theta_s \sin \omega t\right) \\ &= (-mg + F_0 \sin \omega t)\left(\sin \theta_s - \cos \theta_s \cdot \frac{F_0}{mR\omega^2} \sin \theta_s \sin \omega t\right). \end{aligned}$$

Now we average over all high-frequency time dependencies. The first term on the LHS averages zero, and so do the $-mg \sin \omega t$ term and the $F_0 \sin \omega t \sin \theta_s$ term on the RHS. On the other hand, the $\sin^2 \omega t$ term on the RHS averages

$$-\frac{F_0^2}{mR\omega^2} \sin \theta_s \cos \theta_s \left\langle \sin^2 \omega t \right\rangle = -\frac{1}{2}\frac{F_0^2}{mR\omega^2} \sin \theta_s \cos \theta_s,$$

so the final EOM for θ_s is

$$mR\ddot{\theta}_s = -mg \sin \theta_s - \frac{1}{2}\frac{F_0^2}{mR\omega^2} \sin \theta_s \cos \theta_s. \tag{32}$$

4.2 Stable positions of θ_s

We let the LHS of (32) be zero, and the equation becomes

$$\sin\theta_s\left(mg + \frac{1}{2}\frac{F_0^2}{mR\omega^2}\cos\theta_s\right) = 0.$$

Since $F_0 \to 0$, the second factor on the LHS can't be zero, so the equation becomes

$$\sin\theta_s = 0 \Rightarrow \theta_s = 0, \pi. \tag{33}$$

Around $\theta_s = 0$, (32) is approximately

$$mR\ddot{\theta}_s = -mg\theta_s - \frac{1}{2}\frac{F_0^2}{mR\omega^2}\theta_s,$$

and therefore

$$\omega_{\theta=0} = \sqrt{\frac{g}{R} + \frac{F_0^2}{m^2R^2\omega^2}}. \tag{34}$$

This is always real, and therefore the $\theta_s = 0$ position is always stable.

Around $\theta_s = \pi$, we rewrite (32) in terms of $\theta_s' = \pi - \theta_s$, and get

$$-mR\ddot{\theta}_s' = mg\theta_s' - \frac{1}{2}\frac{F_0^2}{mR\omega^2}\theta_s' = 0,$$

and

$$\omega_{\theta_s=\pi} = \sqrt{\frac{1}{2}\frac{F_0^2}{m^2R^2\omega^2} - \frac{g}{R}}. \tag{35}$$

It can be seen that when $F_0 = 0$, the frequency is imaginary and therefore $\theta_s = \pi$ is not a stable position. However, when

$$\frac{1}{2}\frac{F_0^2}{m^2R^2\omega^2} \geq \frac{g}{R}, \tag{36}$$

we do have oscillation behavior around $\theta_s = \pi$.

5 Relaxation of a spin polarization due to an electric field

The magnetic field felt by an electron wit velocity $\boldsymbol{v}$ when an electric field $\boldsymbol{E}$ is present is

$$\boldsymbol{B} = -\boldsymbol{v} \times \boldsymbol{E}/c^2. \tag{37}$$

From

$$H = -\boldsymbol{\mu}\cdot\boldsymbol{B}, \quad \boldsymbol{\mu} = -g\mu_{\mathrm{B}}\boldsymbol{S}, \tag{38}$$

we have

$$\frac{\mathrm{d}\boldsymbol{S}}{\mathrm{d}t} = \frac{g\mu_{\mathrm{B}}}{\hbar}\boldsymbol{B}\times\boldsymbol{S} = \frac{g\mu_{\mathrm{B}}}{\hbar}\boldsymbol{S}\times\frac{\boldsymbol{v}\times\boldsymbol{E}}{c^2}. \tag{39}$$