Homework 3

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1.1

For the free electron gas, we have

$$\mu = \frac{1}{2m}\left(\frac{3\pi^2 N}{V}\right)^{2/3}, \tag{1}$$

where we have considered the fact that electrons are spin-1/2 particles. The free electron gas Green function is

$$G_\sigma(\boldsymbol{k}, \omega) = \frac{1}{\omega - \frac{\boldsymbol{k}^2}{2m} + \mu + \mathrm{i}\,\mathrm{sgn}(\omega) 0^+}. \tag{2}$$

The imaginary part is therefore

$$\begin{aligned} A_\sigma(\boldsymbol{k}, \omega) &= -\frac{1}{\pi} \operatorname{Im} G_\sigma(\boldsymbol{k}, \omega) = -\frac{1}{\pi} \cdot \operatorname{Im}(-\pi \mathrm{i}) \operatorname{sgn}(\omega) \delta\left(\omega - \frac{\boldsymbol{k}^2}{2m} + \mu\right) \\ &= \operatorname{sgn}\left(\frac{\boldsymbol{k}^2}{2m} - \mu\right) \delta\left(\omega - \frac{\boldsymbol{k}^2}{2m} + \mu\right). \end{aligned} \tag{3}$$

This function is not really analytical, since its peak has infinitesimal width and infinite height; but adding a small imaginary part to the energy of a single electron, the spectral function should be analytical.

1.2

When

$$\Sigma = \lambda\omega^2 + \mathrm{i}(\alpha\omega^2 + \beta T^2), \tag{4}$$

we have

$$\begin{aligned} G_\sigma(\boldsymbol{k}, \omega) &= \frac{1}{\omega - \frac{\boldsymbol{k}^2}{2m} - \Sigma + \mathrm{i}\,\mathrm{sgn}(\omega) 0^+} \\ &= \frac{1}{(1-\lambda)\omega - \frac{\boldsymbol{k}^2}{2m} + \mu - \mathrm{i}(\alpha\omega^2 + \beta T^2)} \\ &= \frac{(1-\lambda)\omega - \frac{\boldsymbol{k}^2}{2m} + \mu + \mathrm{i}(\alpha\omega^2 + \beta T^2)}{\left((1-\lambda)\omega - \frac{\boldsymbol{k}^2}{2m} + \mu\right)^2 + (\alpha\omega^2 + \beta T^2)^2}, \end{aligned} \tag{5}$$

where the infinitesimal imaginary part can be ignored since we already have a finite imaginary part. The imaginary part gives

$$A_\sigma(\boldsymbol{k}, \omega) = -\frac{1}{\pi} \frac{\alpha\omega^2 + \beta T^2}{\left((1-\lambda)\omega - \frac{\boldsymbol{k}^2}{2m} + \mu\right)^2 + (\alpha\omega^2 + \beta T^2)^2}. \tag{6}$$

When ω is already given and is a real number, $\boldsymbol{k}$ should satisfy

$$(1-\lambda)\omega - \frac{\boldsymbol{k}^2}{2m} + \mu = 0 \Rightarrow \omega = \frac{\boldsymbol{k}^2}{2m^*} - \mu^*, \quad m^* = (1-\lambda)m \tag{7}$$

so that $A_\sigma(\boldsymbol{k}, \omega)$ is maximized. This is the effective dispersion relation measured by ARPES; the effective mass is $(1-\lambda)m$. If ω is complex, the maximizing procedure will be much more tedious, but this isn't physical, since in ARPES the frequency of the input beam can't be complex.