Homework 1

Jinyuan Wu

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Exercise 9 in 1.1.7 (**)

Solution Since

$$r^{(n)} = \beta^{(1)}\beta^{(2)}\cdots\beta^{(n)}.\beta^{(n+1)}\beta^{(n+2)}\cdots,$$

we know

$$r^{(n)} = 0.\beta^{(n+1)}\beta^{(n+2)}\cdots. \tag{1}$$

Each digit of $r^{(n)}$ is 0 or 1, and thus the possible range of $r^{(n)}$ is $[0, 1]$.[1] Suppose

$$x = 0.x^{(1)}x^{(2)}\cdots \in [0, 1],$$

we have

$$\begin{aligned} P(r^{(n)} < x) &= P(\beta^{(n+1)} < x^{(1)}) + P(\beta^{(n+1)} = x^{(1)})P(\beta^{(n+2)} < x^{(2)}) + \cdots \\ &= \frac{1}{2}\delta_{x^{(1)},1} + \frac{1}{2}\times\frac{1}{2}\delta_{x^{(2)},1} + \cdots \\ &= 0.x^{(1)}x^{(2)}\cdots = x, \end{aligned}$$

so $r^{(n)}$ has a uniform probabilistic distribution on $[0, 1]$. So the probability of $r^{(n)} < m$ i.e. $\beta^{(n)}_m = 1$ is exactly m, and therefore $\beta^{(n)}_m$ is a realization of B_m, regardless of what n is.

Exercise 3 in 2.2.3.1 (**)

Solution

(a) From (2.15) we have

$$\begin{aligned} \frac{\partial}{\partial t}w(x,t) &= -\frac{1}{2}\sqrt{\frac{1}{4\pi D t^3}}\mathrm{e}^{-\frac{(x-v_d t)^2}{4Dt}} - \sqrt{\frac{1}{4\pi D t}}\mathrm{e}^{-\frac{(x-v_d t)^2}{4Dt}}\frac{1}{4Dt^2}(2v_d(v_d t - x)t - (x - v_d t)^2) \\ &= -\sqrt{\frac{1}{4\pi D t}}\mathrm{e}^{-\frac{(x-v_d t)^2}{4Dt}}\left(\frac{1}{2t} + \frac{(v_d t - x)(v_d t + x)}{4Dt^2}\right), \end{aligned}$$

$$\frac{\partial}{\partial x}w(x,t) = -\sqrt{\frac{1}{4\pi D t}}\mathrm{e}^{-\frac{(x-v_d t)^2}{4Dt}}\frac{x - v_d t}{2Dt},$$

and

$$\frac{\partial^2}{\partial x^2}w(x,t) = -\sqrt{\frac{1}{4\pi D t}}\mathrm{e}^{-\frac{(x-v_d t)^2}{4Dt}}\left(\frac{1}{2Dt} - \left(\frac{x - v_d t}{2Dt}\right)^2\right),$$

The RHS of the Smoluchowski equation is

$$\begin{aligned} D\frac{\partial^2 w}{\partial x^2} - v_d\frac{\partial w}{\partial x} &= -\sqrt{\frac{1}{4\pi D t}}\mathrm{e}^{-\frac{(x-v_d t)^2}{4Dt}}\left(\frac{1}{2t} - \frac{(x - v_d t)^2}{4Dt^2} - v_d\frac{x - v_d t}{2Dt}\right) \\ &= -\sqrt{\frac{1}{4\pi D t}}\mathrm{e}^{-\frac{(x-v_d t)^2}{4Dt}}\left(\frac{1}{2t} - \frac{x^2 - v_d^2 t^2}{4Dt^2}\right), \end{aligned}$$

so we have

$$\frac{\partial}{\partial x}w(x,t) = D\frac{\partial^2 w}{\partial x^2} - v_d\frac{\partial w}{\partial x}.$$

[1] It's actually possible to have $r^{(n)} = 1$, because the binary $0.11111\cdots$ is actually 1, in the same way $0.9999\cdots = 1$ in the decimal case. But the probability to have such a $r^{(n)}$ is $1/2 \times 1/2 \times \cdots = 0$. That is, the event that $r^{(n)} = 1$ is possible but is a null set.

(b) The initial condition is

$$\lim_{t\to 0} w = \delta(x),$$

which can be imposed to (2.26) by adding an "impact":

$$\frac{\partial w}{\partial t} = D\frac{\partial^2 w}{\partial x^2} - v_d\frac{\partial w}{\partial x} + \delta(x)\delta(t). \tag{2}$$

Now by Fourier transformation we have

$$\begin{aligned} w(x,t) &= \int \frac{\mathrm{d}k\,\mathrm{d}\omega}{(2\pi)^2}\mathrm{e}^{-\mathrm{i}(\omega t - kx)}\tilde{w}(k,\omega), \\ -\mathrm{i}\omega\tilde{w} &= D(\mathrm{i}k)^2\tilde{w} - \mathrm{i}kv_d\tilde{w} + 1. \end{aligned}$$

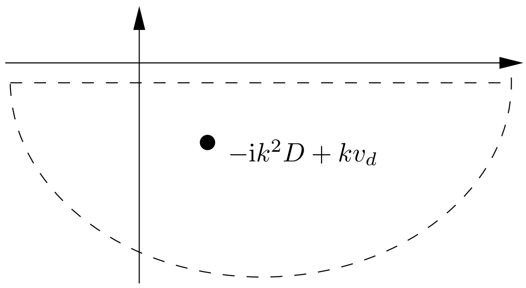
We find

$$\tilde{w} = \frac{1}{-\mathrm{i}\omega + k^2 D + \mathrm{i}kv_d},$$

and thus

$$w(x,t) = \int \frac{\mathrm{d}k\,\mathrm{d}\omega}{(2\pi)^2}\mathrm{e}^{-\mathrm{i}(\omega t - kx)}\frac{1}{-\mathrm{i}\omega + k^2 D + \mathrm{i}kv_d}.$$

We first complete the integral over ω, with the following contour:



$$\int \mathrm{d}\omega\,\mathrm{e}^{-\mathrm{i}(\omega t - kx)}\frac{1}{\omega + \mathrm{i}Dk^2 - kv_d} = -2\pi\mathrm{i}\mathrm{e}^{-\mathrm{i}(-\mathrm{i}k^2Dt + kv_dt - kx)}.$$

Thus

$$\begin{aligned} w(x,t) &= \frac{\mathrm{i}}{(2\pi)^2}\int \mathrm{d}k\,(-2\pi\mathrm{i})\mathrm{e}^{-\mathrm{i}(-\mathrm{i}k^2Dt + kv_dt - kx)} \\ &= \frac{1}{2\pi}\int \mathrm{d}k\,\mathrm{e}^{-k^2Dt - \mathrm{i}k(v_dt - x)} \\ &= \frac{1}{2\pi}\cdot\sqrt{\frac{2\pi}{2Dt}}\mathrm{e}^{\frac{1}{2}\frac{1}{2Dt}(-\mathrm{i}(v_dt - x))^2} \\ &= \sqrt{\frac{1}{4\pi Dt}}\mathrm{e}^{-\frac{(x - v_dt)^2}{4Dt}}. \end{aligned}$$