Homework 6

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Lecture 21 Exercise 1

Solution The cumulants of te Poisson distribution can be found by using the characteristic function. It is

$$\begin{aligned}\varphi(t) = \left\langle \mathrm{e}^{\mathrm{i}tn} \right\rangle &= \sum_{n\geq 0} \mathrm{e}^{\mathrm{i}tn} \exp(-\langle n\rangle) \frac{\langle n\rangle^n}{n!} \\ &= \exp(-\langle n\rangle) \sum_{n\geq 0} \frac{1}{n!} (\langle n\rangle \mathrm{e}^{\mathrm{i}t})^n \\ &= \exp(-\langle n\rangle) \exp\left(\mathrm{e}^{\mathrm{i}t} \langle n\rangle\right) = \exp\left((\mathrm{e}^{\mathrm{i}t} - 1)\langle n\rangle\right),\end{aligned} \tag{1}$$

and therefore we have

$$\log \left\langle \mathrm{e}^{\mathrm{i}tn} \right\rangle = (\mathrm{e}^{\mathrm{i}t} - 1)\langle n\rangle = \langle n\rangle \mathrm{i}t + \langle n\rangle \frac{(\mathrm{i}t)^2}{2!} + \langle n\rangle \frac{(\mathrm{i}t)^3}{3!} + \cdots, \tag{2}$$

and the n-th cumulant is the factor before $(\mathrm{i}t)^n/n!$, so we find every cumulant of the Poisson distribution is $\langle n\rangle$. Specifically, we have $C_3 = C_4 = \langle n\rangle$.

Lecture 21 Exercise 4 Isn't it a paradox that shot noise is not Gaussian noise despite the fact the distribution of shots in a long time interval tends towards a Gaussian distribution?

Solution This is not a paradox. What makes the construction of a Poisson distribution is it can be seen as the sum of the number of photon between $t = 0$ and $t = \Delta t$, the number of photon between $t = \Delta t$ and $t = 2\Delta t$, . . . , and each term has a binominal distribution, so it seems we are summing up lots of independent variables and should get a a Gaussian result. The problem here is the distribution of each term involves Δt: as $\Delta t = \text{total time}/N$ goes smaller, the probability that there is a photon occurring between $t = n\Delta t$ and $t = (n + 1)\Delta t$ also goes smaller. This breaks the condition of the central limit theorem, because the central limit theorem is about the sum of "fully encapsulated" random variables which do not vary as we take the $N \to \infty$ limit.

Lecture 22 Exercise 3

Solution For method 2: now by writing down the scattering matrix, we have

$$\begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = \begin{pmatrix} \epsilon & -\mathrm{i} \\ -\mathrm{i} & \epsilon \end{pmatrix} \begin{pmatrix} \mathrm{e}^{\mathrm{i}\phi} & \\ & 1 \end{pmatrix} \begin{pmatrix} \epsilon & \mathrm{i} \\ \mathrm{i} & \epsilon \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix}, \tag{3}$$

and we have

$$A_1 = A + \mathrm{i}\epsilon(\mathrm{e}^{\mathrm{i}\phi} - 1)B, \quad A_2 = -\mathrm{i}\epsilon(\mathrm{e}^{\mathrm{i}\phi} - 1)A + \mathrm{e}^{\mathrm{i}\phi}B. \tag{4}$$

Since B mode is in vacuum, the only operator that carries enough information about ϕ is the amplitude of A_2. So there is no loss of information. The phase of A_2 is mainly determined by the $\mathrm{e}^{\mathrm{i}\phi}B$ part, and since B is in vacuum, measuring it would be impossible. The amplitude and phase of A_1 are mainly determined by the A part, which tell nothing about ϕ.

For method 3: here the wave function of the system is just

$$|\psi\rangle = A^\dagger |0\rangle = \mathrm{i}\mathrm{e}^{\mathrm{i}\phi/2} \left(\sin\frac{\phi}{2} A_1^\dagger + \cos\frac{\phi}{2} A_2^\dagger\right) |0\rangle\,. \tag{5}$$

In the lecture we have already seen that measuring $I_1 + I_2$ tells nothing about ϕ, and since a Fock state is "squeezed in the particle number direction", two phase-related operators in A_1 and A_2 modes also tell nothing about ϕ. So there is no loss of information.

Lecture 23 Exercise 1

Solution The relation between output and input quadratures is

$$
\begin{aligned}
\frac{1}{\sqrt{2}}(a_{\mathrm{out}} + a^\dagger_{\mathrm{out}}) &= \frac{1}{\sqrt{2}}(a_{\mathrm{in}} + a^\dagger_{\mathrm{in}}) \cdot G, \\
\frac{1}{\sqrt{2}\mathrm{i}}(a_{\mathrm{out}} - a^\dagger_{\mathrm{out}}) &= \frac{1}{\sqrt{2}\mathrm{i}}(a_{\mathrm{in}} - a^\dagger_{\mathrm{in}}) \cdot \frac{1}{G},
\end{aligned}
\tag{6}
$$

and therefore

$$
a_{\mathrm{out}} = \frac{1}{2}\left(G + \frac{1}{G}\right) a_{\mathrm{in}} + \frac{1}{2}\left(G - \frac{1}{G}\right) a^\dagger_{\mathrm{in}}, \tag{7}
$$

and therefore

$$
\begin{aligned}
[a_{\mathrm{out}}, a^\dagger_{\mathrm{out}}] &= \frac{1}{4}\left(G + \frac{1}{G}\right)^2 \left[a_{\mathrm{in}}, a^\dagger_{\mathrm{in}}\right] + \frac{1}{4}\left(G - \frac{1}{G}\right)^2 \left[a^\dagger_{\mathrm{in}}, a_{\mathrm{in}}\right] \\
&= \frac{1}{4} \cdot 4 = 1,
\end{aligned}
\tag{8}
$$

and therefore an amplifier that amplifies one quadrature and desamplifies another is at least logically consistent, without the need to introduce another mode – the noise – coupled to a. So in principle such an amplifier can be noiseless.