Homework 1

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1 Problem 1: The Beam Splitter

Since $|t|^2 = |r|^2 = 1/2$, we have

$$\begin{pmatrix} E_c \\ E_d \end{pmatrix} = \underbrace{\begin{pmatrix} \mathrm{e}^{\mathrm{i}\phi_{ta}} & \mathrm{e}^{\mathrm{i}\phi_{rb}} \\ \mathrm{e}^{\mathrm{i}\phi_{ra}} & \mathrm{e}^{\mathrm{i}\phi_{tb}} \end{pmatrix}}_{M} \begin{pmatrix} E_a \\ E_b \end{pmatrix}. \tag{1}$$

The unitary condition means

$$MM^\dagger = I, \tag{2}$$

which in turns means

$$\begin{aligned} I &= \frac{1}{2} \begin{pmatrix} \mathrm{e}^{\mathrm{i}\phi_{ta}} & \mathrm{e}^{\mathrm{i}\phi_{rb}} \\ \mathrm{e}^{\mathrm{i}\phi_{ra}} & \mathrm{e}^{\mathrm{i}\phi_{tb}} \end{pmatrix} \begin{pmatrix} \mathrm{e}^{-\mathrm{i}\phi_{ta}} & \mathrm{e}^{-\mathrm{i}\phi_{ra}} \\ \mathrm{e}^{-\mathrm{i}\phi_{rb}} & \mathrm{e}^{-\mathrm{i}\phi_{tb}} \end{pmatrix} \\ &= \frac{1}{2} \begin{pmatrix} 2 & \mathrm{e}^{\mathrm{i}(\phi_{ta} - \phi_{ra})} + \mathrm{e}^{\mathrm{i}(\phi_{rb} - \phi_{tb})} \\ \mathrm{e}^{-\mathrm{i}(\phi_{ta} - \phi_{ra})} + \mathrm{e}^{-\mathrm{i}(\phi_{rb} - \phi_{tb})} & 2 \end{pmatrix}, \end{aligned}$$

and this is equivalent to

$$\mathrm{e}^{\mathrm{i}(\phi_{ta} - \phi_{ra})} + \mathrm{e}^{\mathrm{i}(\phi_{rb} - \phi_{tb})} = 0,$$

or in other words

$$\phi_{ta} - \phi_{ra} = \phi_{rb} - \phi_{tb} + \pi n, \quad n \text{ odd}. \tag{3}$$

2 Problem 2: Interferometers

Consider a Michelson interferometer, and rotate the beam splitter with an angle of θ, and also rotate one mirror with an angle of 2θ, and we get Figure 1. The change of the optical path of the green ray is

$$\Delta L_{\text{green}} = \frac{l_1 + d}{\cos 2\theta} - (l_1 + d) = (l_1 + d)\left(1 + \frac{1}{2}(2\theta)^2 + \cdots - 1\right) = 2(l_1 + d)\theta^2 + \cdots, \tag{4}$$

and the change of the optical path of the orange ray is

$$\Delta L_{\text{orange}} = l_2 + \frac{d}{\cos 2\theta} - (l_2 + d) = d\left(1 + \frac{1}{2}(2\theta)^2 + \cdots - 1\right) = 2d\theta^2 + \cdots. \tag{5}$$

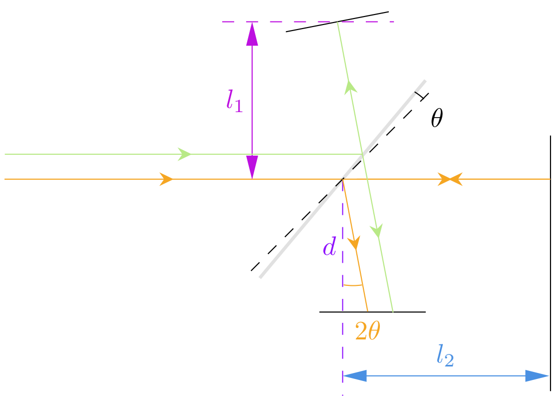

Figure 1: Michelson interferometer with tilted mirrors

Thus the changes of both paths are $\mathcal{O}(\theta^2)$.

When the potential – in optics, the refractive index – is changed, the path of the beam may be changed, but as is outlined above, slight change of the angle of propagation only causes a $\mathcal{O}(\theta^2)$ change on the optical path, so the main contribution of the change of the refractive index is the correction factor to terms like l_1 or d in ΔL_{green} or ΔL_{orange}. If, for example, a sample is placed on l_1, then we have

$$\Delta L_{\text{green}} = n\frac{l_1+d}{\cos 2\theta} - (l_1+d) = (l_1+d)\left(n\left(1+\frac{1}{2}(2\theta)^2+\cdots\right)-1\right) = (l_1+d)(n-1+2n\theta^2+\cdots), \tag{6}$$

and the first order variance of ΔL_{green} comes from the n factor in the $n(l_1+d)/\cos 2\theta$ term.

3 Problem 3: Correlation function and Other Properties of the Blackbody Field

3.1 Energy at ω; Total Energy

3.1.1 Energy of an electromagnetic mode

From

$$\boldsymbol{\nabla}\times\boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t},$$

we have

$$\mathrm{i}\boldsymbol{k}\times\boldsymbol{E}_\omega = \mathrm{i}\omega\boldsymbol{B}_\omega,$$

and therefore

$$|\boldsymbol{B}_\omega| = \frac{k}{\omega}|\boldsymbol{E}_\omega| = \frac{1}{c}|\boldsymbol{E}_\omega|,$$

so

$$\begin{aligned} u_\omega &= \frac{\epsilon_0}{2}|\boldsymbol{E}_\omega^2| + \frac{1}{2\mu_0}|\boldsymbol{B}_\omega|^2 \\ &= \frac{\epsilon_0}{2}|\boldsymbol{E}_\omega^2| + \frac{1}{2\mu_0}\underbrace{\frac{1}{c^2}}_{\mu_0\epsilon_0}|\boldsymbol{E}_\omega|^2 \\ &= \epsilon_0|\boldsymbol{E}_\omega|^2. \end{aligned} \tag{7}$$

Here the notation u_ω may be slightly confusing, because it's just the energy density (spatial density) of *one* photon mode with frequency ω, and the energy density contributed by *all* photon modes with the frequency being between ω and $\omega+\mathrm{d}\omega$ is $n(\omega)\,\mathrm{d}\omega\cdot u_\omega$, where $n(\omega)$ is the density of states.

The expression of $\boldsymbol{E}_\omega$ is

$$\boldsymbol{E}(\boldsymbol{r},t) = \sum_{\boldsymbol{k},\sigma=1,2}\mathrm{i}\sqrt{\frac{\hbar\omega_{\boldsymbol{k}}}{2\epsilon_0 V}}a_{\boldsymbol{k}\sigma}\hat{\boldsymbol{e}}_\sigma\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}-\mathrm{i}\omega_{\boldsymbol{k}}t} + \text{h.c.}, \tag{8}$$

and

3.1.2 Energy density

Now we derive the energy at ω. Between ω and $\omega+\mathrm{d}\omega$, we have

$$\#\text{ of }\boldsymbol{k}\text{ per }\mathrm{d}\omega = \frac{V}{(2\pi)^3}4\pi k^2\,\mathrm{d}k\,,\quad k=\frac{\omega}{c}.$$

Since there are two polarizations for each $\boldsymbol{k}$, the number of states per $\mathrm{d}\omega$ is

$$\#\text{ of state per }\mathrm{d}\omega = 2\cdot\#\text{ of }\boldsymbol{k}\text{ per }\mathrm{d}\omega = \frac{V}{\pi^2c^3}\omega^2\,\mathrm{d}\omega\,. \tag{9}$$

Now since the total energy in the cavity is

$$U = \int \#\text{ of state per }\mathrm{d}\omega\cdot\hbar\omega\cdot\frac{1}{\mathrm{e}^{\hbar\omega/k_{\mathrm{B}}T}-1}, \tag{10}$$

the total energy density – the amount of energy per $\mathrm{d}^3\boldsymbol{r}$ – is

$$u = \int \mathrm{d}\omega \, \frac{\hbar\omega^3}{\pi^2 c^3} \frac{1}{\mathrm{e}^{\hbar\omega/k_\mathrm{B}T} - 1}. \tag{11}$$

Using

$$\int_0^\infty \frac{x^3 \, \mathrm{d}x}{\mathrm{e}^x - 1} = \frac{\pi^4}{15},$$

we get

$$u = \frac{\hbar}{\pi^2 c^3} \left(\frac{k_\mathrm{B} T}{\hbar} \right)^4 \cdot \frac{\pi^4}{15}. \tag{12}$$

The intensity of radiation out of the cavity is

$$I = \sum_{m \text{ outgoing}} A\boldsymbol{n} \cdot \boldsymbol{S}_m, \quad \boldsymbol{S}_m = u_m c \hat{\boldsymbol{k}},$$

where $\boldsymbol{n}$ is the normal vector of the hole between the cavity and the outside word, m is the index of optical modes within the cavity, $\boldsymbol{S}_m$ is the Poynting vector of mode m. We can make use of the spherical symmetry of radiation: suppose $\mathrm{d}\Omega$ is the solid angle element of $\hat{\boldsymbol{k}}$, we have

$$\begin{aligned} J = \frac{I}{A} &= \underbrace{\frac{1}{4\pi}}_{\text{total solid angle}} \int_{\hat{\boldsymbol{k}} \text{ outgoing}} \mathrm{d}\Omega \, \boldsymbol{n} \cdot u c \hat{\boldsymbol{k}} \\ &= uc \cdot \frac{1}{4\pi} \int_{\theta \leq \pi/2} \sin\theta \, \mathrm{d}\theta \, \mathrm{d}\varphi \cos\theta \\ &= uc \cdot \frac{1}{4\pi} \cdot \frac{1}{2} \cdot 2\pi = \frac{1}{4} uc, \end{aligned}$$

and finally we get

$$J = \underbrace{\frac{\pi^2 k_\mathrm{B}^4}{60\hbar^3 c^2}}_{\sigma} T^4. \tag{13}$$

3.2 Correlation Function of the Black Body Field

The experimental definition of the correlation function is

$$R_{xx}(\tau) = \frac{1}{T} \int_{-T/2}^{T/2} \mathrm{d}t \, E_x(t+\tau) E_x(t), \tag{14}$$

and so on. Using the ergodic condition, this is equivalent to

$$R_{xx}(\tau) = \langle E_x(\tau) E_x(0) \rangle . \tag{15}$$

The same applies for R_{xy}, etc.

Now since we are dealing with linear optics, there is no SHG process, etc., and each state in the density matrix $\rho = \sum_n |n\rangle\langle n| \, \mathrm{e}^{-E_n/k_\mathrm{B}t}$ is a photon Fock state. We know E_x contains photon modes for which the polarization vector $\hat{\boldsymbol{e}}$ is in the x direction, while E_y contains photon modes for which the polarization vector $\hat{\boldsymbol{e}}$ is in the y direction. So for each $|n\rangle$ state that is an eigenstate of the density matrix, we have

$$\langle n|E_x E_y|n\rangle = C_1 \langle n|a_{\hat{\boldsymbol{x}}} a_{\hat{\boldsymbol{y}}}|n\rangle + C_2 \langle n|a_{\hat{\boldsymbol{x}}} a^\dagger_{\hat{\boldsymbol{y}}}|n\rangle + C_3 \langle n|a^\dagger_{\hat{\boldsymbol{x}}} a_{\hat{\boldsymbol{y}}}|n\rangle + C_4 \langle n|a^\dagger_{\hat{\boldsymbol{x}}} a^\dagger_{\hat{\boldsymbol{y}}}|n\rangle ,$$

and each term vanishes because after the operators $a_{\hat{\boldsymbol{x}}} a_{\hat{\boldsymbol{y}}}$ etc. are applied to the ket vectors, the photon occupation configurations on the right and the left are different. So for each $|n\rangle$ in ρ, $\langle E_x E_y \rangle = 0$, and therefore $\langle E_x E_y \rangle_\rho$ also vanishes. The same applies for R_{yz} or R_{zx}.

According to Section 3.1.1, we have

$$u = \epsilon_0 |\boldsymbol{E}|^2 = \tag{16}$$

$$R_{xx}(0) = \left\langle E_x(0)^2 \right\rangle = \tag{17}$$