Homework 4

Jinyuan Wu

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1 Casimir-Polder Force

1.1 Interaction potential between two harmonic oscillators

The classical polarizability of a harmonic oscillator is

$$\alpha = \frac{e^2}{\epsilon_0 k}. \tag{1}$$

Since $\omega_0^2 = k/m$, the effective interaction potential

$$V(R) = -\frac{1}{8}\frac{\hbar}{m^2\omega_0^3}\left(\frac{e^2}{2\pi\epsilon_0}\right)^2\frac{1}{R^6} \tag{2}$$

can be rewritten into

$$V(R) = -\frac{1}{32\pi^2}\hbar\omega_0\frac{\alpha^2}{R^6}. \tag{3}$$

1.2 An oscillator and a conducting wall

If we do angle average to the dipole interaction potential

$$\begin{aligned} V(r) &= -\frac{d_1 d_2}{4\pi\epsilon_0 r_{12}^3}\left(\cos\theta_{12} - 3\cos\theta_1\cos\theta_2\right) \\ &= -\frac{d^2}{4\pi\epsilon_0 r_{12}^3}(\cos 2\theta - 3\cos^2\theta) \\ &= -\frac{d^2}{4\pi\epsilon_0 r_{12}^3}\left(-\frac{3}{2} - \frac{1}{2}\cos 2\theta\right), \end{aligned} \tag{4}$$

since the $\cos 2\theta$ term vanishes, we get

$$\langle V(r)\rangle = \frac{3}{2}\frac{\langle d^2\rangle}{4\pi\epsilon_0 r_{12}^3}, \tag{5}$$

where

$$d = ex \tag{6}$$

is the dipole for one oscillator. This means in order to obtain the effective potential between the oscillator and the mirror, we just need to do the following substitution:

$$d_1 d_2 \longrightarrow -\frac{3}{2}e^2\left\langle x^2\right\rangle. \tag{7}$$

Ignoring the back action to the internal state of the oscillator, $\left\langle x^2\right\rangle$ can be evaluated using the property of a free oscillator:

$$\frac{1}{2}k\left\langle x^2\right\rangle = \frac{1}{2}\hbar\omega_0 \Rightarrow \left\langle d^2\right\rangle = e^2\frac{\hbar\omega_0}{k}. \tag{8}$$

1.3 Relativistic effect in field propagation

From the boundary condition, in a static field we have (note that the electric field is doubled because of the contribution of the image charge)

$$\frac{\sigma}{\epsilon_0} = \boldsymbol{n}\cdot\boldsymbol{E} \Rightarrow \frac{\sigma(r,\theta)}{\epsilon_0} = -2\cdot\frac{e}{4\pi\epsilon_0(R^2+r^2)}\frac{R}{\sqrt{R^2+r^2}}, \tag{9}$$

$$\sigma(r) = -\frac{eR}{2\pi(R^2+r^2)^{3/2}}. \tag{10}$$

The total charge is

$$\int_0^\infty 2\pi r\,\mathrm{d}r\,\sigma(r) = -e\int_0^\infty \frac{r}{R}\frac{\mathrm{d}r}{R}\frac{R^3}{(R^2+r^2)^{3/2}} = -e\int_0^\infty \frac{\mathrm{d}x^2}{2}\frac{1}{(x^2+1)^{3/2}} = -e, \tag{11}$$

the expected result.

When R is large enough, it takes time for the electric field to propagate from the oscillator to the mirror and then back to the oscillator. The time cost is

$$T = \frac{2\sqrt{R^2+r^2}}{c}, \tag{12}$$

so at t, the oscillator sees the electric field it spread out at $t-T$; note that at different T values, the oscillator has different phases: the phase factor is $\mathrm{e}^{-\mathrm{i}\omega_0(t-T)}$. So finally the phase factor of the contribution of charges at r to the electric field felt by the oscillator at t is $\mathrm{e}^{\mathrm{i}2\omega_0 T}$. We may then take the real part (since we are working with linear electrodynamics) and the potential felt by a single charge is

$$\begin{aligned} V_\mathrm{d}(R) &= \int_0^\infty 2\pi r\,\mathrm{d}r\,\sigma(r)\cdot\cos(2\omega_0 T)\cdot e\varphi(r\to R) \\ &= \int_0^\infty 2\pi r\,\mathrm{d}r\,\sigma(r)\cdot\cos\Big(4k_0\sqrt{R^2+r^2}\Big)\frac{e}{4\pi\epsilon_0\sqrt{R^2+r^2}} \\ &= -\frac{e^2R}{4\pi\epsilon_0}\int_0^\infty r\,\mathrm{d}r\,\frac{\cos\big(4k_0\sqrt{R^2+r^2}\big)}{(R^2+r^2)^2}. \end{aligned} \tag{13}$$

1.4 Evaluation of the retarded interaction potential

We evaluate (13). We do the substitution

$$u = 4k_0\sqrt{R^2+r^2}, \tag{14}$$

and

$$\mathrm{d}u = 4k_0\frac{r\,\mathrm{d}r}{\sqrt{R^2+r^2}}, \qquad \frac{r\,\mathrm{d}r}{(R^2+r^2)^2} = \frac{1}{4k_0}\frac{\sqrt{R^2+r^2}\,\mathrm{d}u}{(R^2+r^2)} = (4k_0)^2\frac{\mathrm{d}u}{u^3},$$

and we get

$$\begin{aligned} V_\mathrm{d}(R) &= -(4k_0)^2\frac{e^2R}{4\pi\epsilon_0}\int_{4k_0R}^\infty \frac{\mathrm{d}u}{u^3}\cos u \\ &= -(4k_0)^2\frac{e^2R}{4\pi\epsilon_0}\left(-\frac{\sin(4k_0R)}{8k_0R} + \mathrm{Ci}(4k_0R) + \frac{\cos(4k_0R)}{2(4k_0R)^2}\right). \end{aligned} \tag{15}$$

When R is large (but we are still in the near-field region and the quasi-static approximation above still works) we have

$$V_\mathrm{d}(R) = -(4k_0)^2\frac{e^2R}{4\pi\epsilon_0}\cdot -\frac{\sin(4k_0R)}{(4k_0R)^3} = \frac{e^2\sin(4k_0R)}{4\pi\epsilon_0(4k_0)^2R^3}. \tag{16}$$

This is the potential for a single point charge; we can repeat the same procedure for a dipole, and the resulting potential in the $R\to\infty$ limit is $\propto 1/R^4$.

1.5

The $1/R^4$ law agrees with Eq. (3) (the large-cavity case) in Phys. Rev. 73, 360 (1948).

2 p to x matrix element

We know

$$[\boldsymbol{x}, H] = \mathrm{i}\hbar\frac{\boldsymbol{p}}{m}, \tag{17}$$

and therefore

$$\frac{\mathrm{i}\hbar}{m}\langle i|\boldsymbol{p}|j\rangle = \langle i|[\boldsymbol{x}, H]|j\rangle = \langle i|\boldsymbol{x}E_j - E_i\boldsymbol{x}|j\rangle = (\hbar\omega_j - \hbar\omega_i)\langle i|\boldsymbol{x}|j\rangle,$$

and thus

$$\langle i|\boldsymbol{p}|j\rangle = \mathrm{i}m\underbrace{(\omega_i - \omega_j)}_{\omega_{ij}}\langle i|\boldsymbol{x}|j\rangle. \tag{18}$$

The optical transition lifetime is $\sim 1\,\mathrm{ns}$, and the lind width is therefore is $\sim 10 \times 10^9\,\mathrm{Hz}$, which is still very small compared with even the typical frequency of microwave. Therefore, approximately, $\omega_i - \omega_j$ equals to the driving frequency ω.

3 Sum rules

3.1 The x-closure sum rule

We have

$$\sum_k |\langle n|x_i|k\rangle|^2 = \langle n|\, x_i \sum_k |k\rangle\langle k|\, x_i\,|n\rangle = |\langle n|x_i^2|n\rangle|, \tag{19}$$

and therefore

$$\begin{aligned}\sum_k |\langle n|\boldsymbol{x}|k\rangle|^2 &= \sum_k\sum_i |\langle n|x_i|k\rangle|^2 \\ &= \sum_i |\langle n|x_i^2|n\rangle| = |\langle n|\boldsymbol{x}^2|n\rangle|.\end{aligned} \tag{20}$$

3.2 The Thomas-Reiche-Huhn (TRK) sum rule

From

$$\frac{\hbar}{\mathrm{i}} = \langle n|px - xp|n\rangle = \sum_k(\langle n|p|k\rangle\langle k|x|n\rangle - \langle n|x|k\rangle\langle k|p|n\rangle) \tag{21}$$

and

$$[H, x] = \frac{\hbar}{m\mathrm{i}}p, \tag{22}$$

we have

$$\begin{aligned}\frac{\hbar}{\mathrm{i}} &= \sum_k(\frac{m\mathrm{i}}{\hbar}\langle n|[H,x]|k\rangle\langle k|x|n\rangle - \langle n|x|k\rangle\cdot\frac{m\mathrm{i}}{\hbar}\langle k|[H,x]|n\rangle) \\ &= \frac{m\mathrm{i}}{\hbar}\sum_k(\langle n|E_n x - xE_k|k\rangle\langle k|x|n\rangle - \langle n|x|k\rangle\langle k|E_k x - xE_n|n\rangle) \\ &= \frac{m\mathrm{i}}{\hbar}\sum_k(2(E_n - E_k)\langle n|x|k\rangle\langle k|x|n\rangle),\end{aligned}$$

and therefore

$$\sum_k (E_k - E_n)|\langle n|x|k\rangle|^2 = \frac{\hbar^2}{2m}. \tag{23}$$

4 Hydrogen maser

The structure of the device is shown in Foot Fig. 6.4.

4.1 The initial state

There are four states: $F = 0, M_F = 0$, and $F = 1, M_F = 0, \pm 1$. The initial state is

$$\rho_0 = \frac{1}{Z}(|0,0\rangle\langle 0,0| + \mathrm{e}^{-\beta\Delta E}(|1,-1\rangle\langle 1,-1| + |1,0\rangle\langle 1,0| + |1,1\rangle\langle 1,1|)), \tag{24}$$

where

$$Z = 1 + 3\mathrm{e}^{-\beta\Delta E}, \tag{25}$$

$$\Delta E = \frac{2\mu_{\mathrm{B}} - g_p\mu_{\mathrm{N}}}{2\hbar}B. \tag{26}$$

When the external field is turned off we have

$$\rho_0 = \frac{1}{4}\,\mathrm{diag}(1,1,1,1). \tag{27}$$

The order of basis vectors is the order in the above discussion.

4.2 State selector

The $F = 1, M_F = 0, 1$ states see energy increase when the magnetic field is applied and are selected. But I have no idea why we get an entangled state ...

4.3 Transition between F states

When the magnetic field inside the cavity is weak, we can consider the magnetic coupling Hamiltonian as a perturbation over the hyperfine splitting. Using formulae from Homework 3.3, we find a magnetic field on z direction doesn't shift the energy of $|F = 1, M_F = 0\rangle$ directly, but there is transition between $|F = 0, M_F = 0\rangle$ and $|F = 1, M_F = 0\rangle$. Using second order perturbation theory, we have

$$\Delta E = \frac{(g_s\mu_{\mathrm{B}} \pm g_p\mu_{\mathrm{N}})^2}{E_{\mathrm{hfs}}}, \tag{28}$$

and the sign is $+$ when the state is $|F = 1, M_F = 1\rangle$ and $-$ when the state is $|F = 1, M_F = 0\rangle$. On the other hand $F = 1, M_F = 1$ receives a first order energy correction. So the frequency between $F = 1, M_F = 0$ and $F = 0, M_F = 0$ is much smaller than the frequency between $F = 1, M_F = 1$ and $F = 0, M_F = 0$.

4.4 Transition between $1, 0$ and $0, 0$

The magnetic moment of electron is much larger than that of proton, so to make things easier we consider the former only. The interaction matrix elements then are

$$\langle F = 0, M_F = 0|g_s\mu_{\mathrm{B}}BS\sigma_{x,y,z}|F = 1, M_F = 0\rangle \approx \mu_{\mathrm{B}}B\,\langle F = 0, M_F = 0|\sigma_{x,y,z}|F = 1, M_F = 0\rangle\,. \tag{29}$$

We already know that since σ_x flips the sign before $|m_s = -1, m_I = 1\rangle$ but leaves $|m_s = 1, m_I = -1\rangle$ unchanged, we have

$$\langle F = 0, M_F = 0|g_s\mu_{\mathrm{B}}BS\sigma_z|F = 1, M_F = 0\rangle = \mu_{\mathrm{B}}B. \tag{30}$$

We have

$$\sigma_x = \sigma_+ + \sigma_-, \quad \sigma_y = -\mathrm{i}\sigma_+ + \mathrm{i}\sigma_-, \tag{31}$$

and therefore we find after we apply $\sigma_{x,y}$ to $|F = 1, M_F = 0\rangle$, we get a mixture of $|m_s = 1/2, m_I = 1/2\rangle$ and $|m_s = -1/2, m_I = -1/2\rangle$, and therefore

$$\langle F = 0, M_F = 0|g_s\mu_{\mathrm{B}}BS\sigma_{x,y}|F = 1, M_F = 0\rangle = 0. \tag{32}$$

4.5 Einstein A and B coefficients

We can first evaluate B and find A from detailed balance. The occupation of the excited state $|F=1, M_F=0\rangle$ is

$$N_2 = N\frac{\Omega^2/4}{(\omega-\omega_0)^2+\gamma^2/4}, \tag{33}$$

where

$$\hbar\Omega = \text{transitional matrix element} = \mu_{\mathrm{B}} B_\omega. \tag{34}$$

Here we use B_ω to refer to the magnetic field. Since the magnetic driving may have a line width we have

$$N_2 = \int \mathrm{d}\omega\, g(\omega) N \frac{\Omega^2/4}{(\omega-\omega_0)^2+\gamma^2/4} \approx N\mu_{\mathrm{B}}^2 B_\omega^2 \int \mathrm{d}\omega\, g(\omega)\frac{1}{(\omega-\omega_0)^2+\gamma^2/4} \approx \frac{2\pi}{\gamma} N\mu_{\mathrm{B}}^2 B_\omega^2. \tag{35}$$

In equilibrium, we have

$$B\rho(\omega)N =: N_{\text{pumping per second}} = \gamma N_2, \tag{36}$$

where $\rho = B_\omega^2/\mu_0$ is the electromagnetic energy density, and we get

$$B = 2\pi\mu_0\mu_{\mathrm{B}}^2. \tag{37}$$

We still need to add a 1/3 factor before B since at frequency ω we have 3 directions of $\boldsymbol{k}$ and two polarizations per $\boldsymbol{k}$, and only two modes are oscillating in the $\hat{\boldsymbol{z}}$ direction. So

$$B = \frac{2}{3}\pi\mu_0\mu_{\mathrm{B}}^2. \tag{38}$$

Now from

$$\frac{A}{B} = \frac{\hbar\omega^3}{\pi^2 c^3} \tag{39}$$

we get

$$A = \frac{2\hbar\omega^3\mu_0\mu_{\mathrm{B}}^2}{3\pi c^3}. \tag{40}$$

4.6

4.7 Magnetic field and photon

When the number of photons is large, we have

$$\hbar\omega N = \mu_0 B^2 \cdot V \Rightarrow B = \sqrt{\frac{\hbar\omega N}{\mu_0 V}}. \tag{41}$$

4.8 Equation of motion of particle numbers

The EOMs of particle numbers in the bulb is the follows:

$$\begin{aligned}
\dot{N}_2 &= \Phi - N_2(\gamma_h + \gamma_w) - Bn\rho(\omega) \\
\dot{N}_1 &= -N_1(\gamma_h + \gamma_W) + Bn\rho(\omega) \\
\dot{q} &= -q\gamma_c + Bn\rho(\omega).
\end{aligned} \tag{42}$$

Here $n = N_2 - N_1$ and q is the photon number.

The term $Bn\rho(\omega)$ may be rewritten into

$$Bn\rho(\omega) = B'nq \Rightarrow B' = B\frac{\hbar\omega}{V}. \tag{43}$$

But since q also has frequency dependence we need to do a line width integral similar to (35) and get

$$B' = B\frac{\hbar\omega}{V}\frac{2\pi}{\gamma_c}. \tag{44}$$

4.9 Threshold of maser

From the EOMs, in the equilibrium case, we have

$$q = \frac{\gamma_h + \gamma_w}{2\gamma_c} \left(\frac{\Phi}{\gamma_h + \gamma_w} - \frac{\gamma_c}{B'} \right), \tag{45}$$

and the occupation inversion is constantly

$$n = \frac{\gamma_c}{B'}, \tag{46}$$

independent of the pumping Φ and photon production q. The threshold incoming jet of excited H atoms is

$$\Phi_0 = \frac{\gamma_c(\gamma_h + \gamma_w)}{B'}. \tag{47}$$

We also have

$$N_1 + N_2 = \frac{\Phi}{\gamma_h + \gamma_w}, \tag{48}$$

and therefore we are able to solve N_1 and N_2. Slightly above the threshold, we have

$$N_1 \approx \frac{1}{2} \left(\frac{\Phi_0}{\gamma_h + \gamma_w} - n \right) = 0, \tag{49}$$

and

$$N_2 \approx n = \frac{\gamma_c}{B'}. \tag{50}$$

4.10 Schawlow-Townes limit

5 Diffracted limited beam

We are dealing with linear optics so the power of the laser beam is irrelevant. The beam radius of a Gaussian beam is

$$w(z) = w_0 \sqrt{1 + \left(\frac{z}{z_\mathrm{R}} \right)^2}, \quad z_\mathrm{R} = \frac{\pi w_0^2 n}{\lambda}. \tag{51}$$

When z is very large, we have

$$w(z) = w_0 \frac{z}{z_\mathrm{R}} = \frac{\lambda}{\pi w_0 n} \cdot z. \tag{52}$$

The wave length of green laser is 532 nm. The distance between earth and moon is 382 500 km, and we can take w_0 to be half of the 1 mm diameter of the laser beam when it leaves the laser, and thus on the moon the radius of the beam is

$$w_\mathrm{moon} = \frac{\lambda}{\pi w_0} \cdot R_\text{earth-moon} = 129\,\mathrm{km}.$$

This can also be seen as an instance of uncertainty principle: the above equation is equivalent to

$$\underbrace{w_0}_{\Delta x} \cdot \underbrace{\frac{\pi}{\lambda} \cdot \underbrace{\frac{w(z)}{z}}_{\tan\theta}}_{\Delta k} = 1. \tag{53}$$