Homework 1

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1 Problem 1: The Beam Splitter

Since $|t|^2 = |r|^2 = 1/2$, we have

$$\begin{pmatrix} E_c \\ E_d \end{pmatrix} = \underbrace{\begin{pmatrix} \mathrm{e}^{\mathrm{i}\phi_{ta}} & \mathrm{e}^{\mathrm{i}\phi_{rb}} \\ \mathrm{e}^{\mathrm{i}\phi_{ra}} & \mathrm{e}^{\mathrm{i}\phi_{tb}} \end{pmatrix}}_{M} \begin{pmatrix} E_a \\ E_b \end{pmatrix}. \tag{1}$$

The unitary condition means

$$MM^\dagger = I, \tag{2}$$

which in turns means

$$\begin{aligned} I &= \frac{1}{2} \begin{pmatrix} \mathrm{e}^{\mathrm{i}\phi_{ta}} & \mathrm{e}^{\mathrm{i}\phi_{rb}} \\ \mathrm{e}^{\mathrm{i}\phi_{ra}} & \mathrm{e}^{\mathrm{i}\phi_{tb}} \end{pmatrix} \begin{pmatrix} \mathrm{e}^{-\mathrm{i}\phi_{ta}} & \mathrm{e}^{-\mathrm{i}\phi_{ra}} \\ \mathrm{e}^{-\mathrm{i}\phi_{rb}} & \mathrm{e}^{-\mathrm{i}\phi_{tb}} \end{pmatrix} \\ &= \frac{1}{2} \begin{pmatrix} 2 & \mathrm{e}^{\mathrm{i}(\phi_{ta}-\phi_{ra})} + \mathrm{e}^{\mathrm{i}(\phi_{rb}-\phi_{tb})} \\ \mathrm{e}^{-\mathrm{i}(\phi_{ta}-\phi_{ra})} + \mathrm{e}^{-\mathrm{i}(\phi_{rb}-\phi_{tb})} & 2 \end{pmatrix}, \end{aligned}$$

and this is equivalent to

$$\mathrm{e}^{\mathrm{i}(\phi_{ta}-\phi_{ra})} + \mathrm{e}^{\mathrm{i}(\phi_{rb}-\phi_{tb})} = 0,$$

or in other words

$$\phi_{ta} - \phi_{ra} = \phi_{rb} - \phi_{tb} + \pi n, \quad n \text{ odd}. \tag{3}$$

2 Problem 2: Interferometers

3 Correlation function and Other Properties of the Black-body Field

3.1 Energy at ω; Total Energy

3.1.1 Energy of an electromagnetic mode

From

$$\boldsymbol{\nabla} \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t},$$

we have

$$\mathrm{i}\boldsymbol{k} \times \boldsymbol{E}_\omega = \mathrm{i}\omega \boldsymbol{B}_\omega,$$

and therefore

$$|\boldsymbol{B}_\omega| = \frac{k}{\omega}|\boldsymbol{E}_\omega| = \frac{1}{c}|\boldsymbol{E}|,$$

so

$$\begin{aligned} u_\omega &= \frac{\epsilon_0}{2}|\boldsymbol{E}_\omega^2| + \frac{1}{2\mu_0}|\boldsymbol{B}_\omega|^2 \\ &= \frac{\epsilon_0}{2}|\boldsymbol{E}_\omega^2| + \frac{1}{2\mu_0}\underbrace{\frac{1}{c^2}}_{\mu_0\epsilon_0}|\boldsymbol{E}|_\omega^2 \\ &= \epsilon_0|\boldsymbol{E}_\omega|^2. \end{aligned} \tag{4}$$

3.1.2 Energy density

Now we derive the energy at ω. Between ω and $\omega + \mathrm{d}\omega$, we have

$$\#\text{ of }\boldsymbol{k}\text{ per }\mathrm{d}\omega = \frac{V}{(2\pi)^3}4\pi k^2\,\mathrm{d}k\,,\quad k = \frac{\omega}{c}.$$

Since there are two polarizations for each $\boldsymbol{k}$, the number of states per $\mathrm{d}\omega$ is

$$\#\text{ of state per }\mathrm{d}\omega = 2\cdot\#\text{ of }\boldsymbol{k}\text{ per }\mathrm{d}\omega = \frac{V}{\pi^2 c^3}\omega^2\,\mathrm{d}\omega\,. \tag{5}$$

Now since the total energy in the cavity is

$$U = \int\#\text{ of state per }\mathrm{d}\omega\cdot\hbar\omega\cdot\frac{1}{\mathrm{e}^{\hbar\omega/k_\mathrm{B}T}-1}, \tag{6}$$

the total energy density – the amount of energy per $\mathrm{d}^3\boldsymbol{r}$ – is

$$u = \int\mathrm{d}\omega\,\frac{\hbar\omega^3}{\pi^2c^3}\frac{1}{\mathrm{e}^{\hbar\omega/k_\mathrm{B}T}-1}. \tag{7}$$

Using

$$\int_0^\infty\frac{x^3\,\mathrm{d}x}{\mathrm{e}^x-1} = \frac{\pi^4}{15},$$

we get

$$u = \frac{\hbar}{\pi^2c^3}\left(\frac{k_\mathrm{B}T}{\hbar}\right)^4\cdot\frac{\pi^4}{15}. \tag{8}$$

The intensity of radiation out of the cavity is

$$I = \sum_{m\text{ outgoing}}A\boldsymbol{n}\cdot\boldsymbol{S}_m,\quad \boldsymbol{S}_m = u_m c\hat{\boldsymbol{k}},$$

where $\boldsymbol{n}$ is the normal vector of the hole between the cavity and the outside word, m is the index of optical modes within the cavity, $\boldsymbol{S}_m$ is the Poynting vector of mode m. We can make use of the spherical symmetry of radiation: suppose $\mathrm{d}\Omega$ is the solid angle element of $\hat{\boldsymbol{k}}$, we have

$$\begin{aligned}J = \frac{I}{A} &= \underbrace{\frac{1}{4\pi}}_{\text{total solid angle}}\int_{\hat{\boldsymbol{k}}\text{ outgoing}}\mathrm{d}\Omega\,\boldsymbol{n}\cdot uc\hat{\boldsymbol{k}}\\ &= uc\cdot\frac{1}{4\pi}\int_{\theta\leq\pi/2}\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\varphi\cos\theta\\ &= uc\cdot\frac{1}{4\pi}\cdot\frac{1}{2}\cdot2\pi = \frac{1}{4}uc,\end{aligned}$$

and finally we get

$$J = \underbrace{\frac{\pi^2k_\mathrm{B}^4}{60\hbar^3c^2}}_{\sigma}T^4. \tag{9}$$

3.2 Correlation Function of the Black Body Field

The experimental definition of the correlation function is

$$R_{xx}(\tau) = \frac{1}{T}\int_{-T/2}^{T/2}\mathrm{d}t\,E_x(t+\tau)E_x(t), \tag{10}$$

and so on. Using the ergodic condition, this is equivalent to

$$R_{xx}(\tau) = \langle E_x(\tau)E_x(0)\rangle\,. \tag{11}$$