

Homework 4

Jinyuan Wu

May 6, 2023

1 Casimir-Polder Force

1.1 Interaction potential between two harmonic oscillators

The classical polarizability of a harmonic oscillator is

$$\alpha = \frac{e^2}{\epsilon_0 k}. \quad (1)$$

Since $\omega_0^2 = k/m$, the effective interaction potential

$$V(R) = -\frac{1}{8} \frac{\hbar}{m^2 \omega_0^3} \left(\frac{e^2}{2\pi\epsilon_0} \right)^2 \frac{1}{R^6} \quad (2)$$

can be rewritten into

$$V(R) = -\frac{1}{32\pi^2} \hbar \omega_0 \frac{\alpha^2}{R^6}. \quad (3)$$

1.2 An oscillator and a conducting wall

2 p to x matrix element

We know

$$[\mathbf{x}, H] = i\hbar \frac{\mathbf{p}}{m}, \quad (4)$$

and therefore

$$\frac{i\hbar}{m} \langle i|\mathbf{p}|j\rangle = \langle i|[\mathbf{x}, H]|j\rangle = \langle i|\mathbf{x}E_j - E_i\mathbf{x}|j\rangle = (\hbar\omega_j - \hbar\omega_i) \langle i|\mathbf{x}|j\rangle,$$

and thus

$$\langle i|\mathbf{p}|j\rangle = im \underbrace{(\omega_i - \omega_j)}_{\omega_{ij}} \langle i|\mathbf{x}|j\rangle. \quad (5)$$

3 Sum rules

3.1 The x -closure sum rule

We have

$$\sum_k |\langle n|x_i|k\rangle|^2 = \langle n|x_i \sum_k |k\rangle\langle k|x_i|n\rangle = |\langle n|x_i^2|n\rangle|, \quad (6)$$

and therefore

$$\begin{aligned} \sum_k |\langle n|\mathbf{x}|k\rangle|^2 &= \sum_k \sum_i |\langle n|x_i|k\rangle|^2 \\ &= \sum_i |\langle n|x_i^2|n\rangle| = |\langle n|\mathbf{x}^2|n\rangle|. \end{aligned} \quad (7)$$

3.2 The Thomas-Reiche-Huhn (TRK) sum rule

From

$$\frac{\hbar}{i} = \langle n | px - xp | n \rangle = \sum_k (\langle n | p | k \rangle \langle k | x | n \rangle - \langle n | x | k \rangle \langle k | p | n \rangle) \quad (8)$$

and

$$[H, x] = \frac{\hbar}{mi} p, \quad (9)$$

we have

$$\begin{aligned} \frac{\hbar}{i} &= \sum_k \left(\frac{mi}{\hbar} \langle n | [H, x] | k \rangle \langle k | x | n \rangle - \langle n | x | k \rangle \cdot \frac{mi}{\hbar} \langle k | [H, x] | n \rangle \right) \\ &= \frac{mi}{\hbar} \sum_k (\langle n | E_n x - x E_k | k \rangle \langle k | x | n \rangle - \langle n | x | k \rangle \langle k | E_k x - x E_n | n \rangle) \\ &= \frac{mi}{\hbar} \sum_k (2(E_n - E_k) \langle n | x | k \rangle \langle k | x | n \rangle), \end{aligned}$$

and therefore

$$\sum_k (E_k - E_n) |\langle n | x | k \rangle|^2 = \frac{\hbar^2}{2m}. \quad (10)$$

4 Hydrogen maser

The structure of the device is shown in Foot Fig. 6.4.

4.1 The initial state

There are four states: $F = 0, M_F = 0$, and $F = 1, M_F = 0, \pm 1$. The initial state is

$$\rho_0 = \frac{1}{Z} (|0, 0\rangle\langle 0, 0| + e^{-\beta \Delta E} (|1, -1\rangle\langle 1, -1| + |1, 0\rangle\langle 1, 0| + |1, 1\rangle\langle 1, 1|)), \quad (11)$$

where

$$Z = 1 + 3e^{-\beta \Delta E}, \quad (12)$$

$$\Delta E = \frac{2\mu_B - g_p \mu_N}{2\hbar} B. \quad (13)$$

When the external field is turned off we have

$$\rho_0 = \frac{1}{4} \text{diag}(1, 1, 1, 1). \quad (14)$$

The order of basis vectors is the order in the above discussion.

4.2 State selector

As is said in Fig. 6.4, all atoms in $F = 0, M_F = 0$ and $F = 1, M_F = 1$ states are thrown away. Now we add a very strong magnetic field in the x direction, and the interaction Hamiltonian $-\boldsymbol{\mu} \cdot \mathbf{B}$ doesn't introduce any split because of M_F , since the latter is along the z direction. Since we only have two states, the Hamiltonian has to take the form

$$H \propto -\sigma_x, \quad (15)$$

and the low energy eigenstate is

$$|\text{ground}\rangle = \frac{1}{\sqrt{2}} (|1, 0\rangle + |1, 1\rangle), \quad (16)$$

and since the transverse field B is very strong, almost all Hydrogen atoms fall to this ground state, and the density matrix therefore is

$$\rho_{\text{in selector}} = |\text{ground}\rangle\langle \text{ground}| = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \quad (17)$$

or in the full basis,

$$\rho_{\text{in selector}} = \frac{1}{2} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}. \quad (18)$$

4.3 Transition between F states

4.4 The state selector

The state selector