# PDEs

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# 1 The linear heat equation

## 1.1 Homogeneous boundary condition

Consider one-dimensional heat equation

$$\frac{\partial u}{\partial t} = K \frac{\partial^2 u}{\partial x^2}. \tag{1}$$

As an example, we consider the case where the boundaries of the string in question are kept to zero temperature, and the boundary and initial conditions are

$$u(0, t) = u(L, t) = 0, \quad u(x, 0) = f(x). \tag{2}$$

By separation of variables

$$u(x, t) = X(x)T(t), \tag{3}$$

we find

$$\frac{T'}{KT} = \frac{X''}{X},$$

which can therefore only be a constant, because otherwise it's impossible for something that only depends on $t$ and something that only depends on $x$ to be equal to each other constantly. So we have

$$X'' = \lambda X, \quad T' = K\lambda T.$$

When $\lambda > 0$, we find

$$X(x) = A\mathrm{e}^{\sqrt{\lambda}x} + B\mathrm{e}^{-\sqrt{\lambda}x},$$

and therefore the boundary conditions mean

$$X(0) = 0 \Rightarrow A + B = 0,$$

$$X(L) = 0 \Rightarrow A(\mathrm{e}^{\sqrt{\lambda}x} - \mathrm{e}^{-\sqrt{\lambda}x}) = 0 \Rightarrow A = B = 0,$$

which gives a trivial solution. Similarly $\lambda = 0$ gives a trivial solution. So we find we should only consider $\lambda < 0$. So now we replace $\lambda$ by $-\lambda$, and from

$$X'' = -\lambda X$$

we find

$$X(x) = A\cos\left(\sqrt{\lambda}x\right) + B\sin\left(\sqrt{\lambda}x\right).$$

The boundary condition $X(0) = 0$ means $A = 0$, and we should then keep $B$ to be non-zero. Then $X(L) = 0$ means

$$\sqrt{\lambda}L = n\pi, \quad n = 1, 2, 3, \ldots$$

So now $\lambda$ is completely determined, and the next step is to find $T$, which is trivial:

$$T(t) \propto \mathrm{e}^{-\lambda t}.$$

So the final solution is

$$u(x, t) = \sum_{n=1}^{\infty} c_n T_n(t) X_n(x), \tag{4}$$

where

$$T_n(t) = \mathrm{e}^{-\frac{n^2\pi^2 K t}{L^2}}, \tag{5}$$

and

$$X_n(x) = \sin\left(\frac{n\pi x}{L}\right). \tag{6}$$

The constants $\{c_n\}$ then can be solved from the initial condition:

$$f(x) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{L}\right),$$

which is just a Fourier series, so

$$c_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)\mathrm{d}x\,. \tag{7}$$

### 1.2 Inhomogeneous boundary condition

We can consider another problem: now the boundaries are still isothermal, but the temperatures there are no longer zero. The conditions are

$$u(0,t) = T_1, \quad u(L,t) = T_2. \tag{8}$$

Note that the temperatures can be different, and when $t \to \infty$, the stable solution may still be non-zero. Linearity guarantees the validity of the following decomposition:

$$u(x,t) = u_0(x,t) + \psi(x), \tag{9}$$

where $\psi(x)$ satisfies

$$\psi''(x) = 0, \quad \psi(0,t) = T_1, \quad \psi(L,t) = T_2, \tag{10}$$

so that $u_0(x,t)$ satisfies the problem (1) plus (2) just solved above – but note that $f$ in (2) should be replaced by $f(x) - \psi(x)$. Now $\psi(x)$ can be found easily: it's just

$$\psi(x) = \frac{T_2 - T_1}{L}x + T_1. \tag{11}$$

### 1.3 Heat conduction in an infinite medium

Now we consider

$$\frac{\partial u}{\partial t} = K\frac{\partial^2 u}{\partial x^2}, \quad -\infty < x < \infty, \quad u(x,0) = f(x). \tag{12}$$

The problem can be solved by Laplace transform as well as Fourier transform; or we can do Fourier transform in $x$ and Laplace transform in $t$. We have (here we are using $\omega$ to refer to the frequency of $x$, not $t$)

$$\mathcal{F}\left[\frac{\partial u}{\partial t}\right] = \frac{\partial}{\partial t}\hat{u}(\omega,t),$$

and

$$\mathcal{F}\left[\frac{\partial^2 u}{\partial x^2}\right] = (\mathrm{i}\omega)^2\hat{u}(\omega,t).$$

The bulk equation now is

$$\frac{\partial \hat{u}(\omega,t)}{\partial t} = -K\omega^2\hat{u}(\omega,t),$$

and we have

$$\hat{u}(\omega,t) = \mathrm{e}^{-K\omega^2 t}\underbrace{\hat{u}(\omega,0)}_{\hat{f}(\omega)}.$$

So we find

$$u(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\hat{u}(\omega,t)\mathrm{e}^{\mathrm{i}\omega x}\,\mathrm{d}\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty}\hat{f}(\omega)\mathrm{e}^{-K\omega^2 t}\mathrm{e}^{\mathrm{i}\omega x}\,\mathrm{d}\omega\,. \tag{13}$$

A common initial condition is

$$u(x,0) = \delta(x), \tag{14}$$

which means when $t = 0$, all heat is concentrated in a rather small region. Then

$$\hat{f}(\omega) = 1,$$

and (13) tells us

$$u(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{e}^{-K\omega^2 t + \mathrm{i}\omega x}\,\mathrm{d}\omega = \frac{1}{2\sqrt{\pi K t}}\mathrm{e}^{-\frac{x^2}{4Kt}}. \tag{15}$$

Here the integral can be calculated as

$$\begin{aligned}\int_{-\infty}^{\infty} \mathrm{e}^{-K\omega^2 t + \mathrm{i}\omega x}\,\mathrm{d}\omega &= \mathrm{e}^{-\frac{x^2}{4Kt}}\int_{-\infty}^{\infty} \mathrm{e}^{-Kt\left(\omega - \frac{\mathrm{i}x}{2Kt}\right)^2}\,\mathrm{d}\omega \\ &= \sqrt{\frac{\pi}{Kt}}\mathrm{e}^{-\frac{x^2}{4Kt}}.\end{aligned}$$

The solution is always Gaussian, but as time goes by, it becomes wider and wider.

## 1.4 Heat conduction on a semi-infinite domain

Let's then consider the following boundary and initial conditions:

$$u(x,0) = T, \quad u(0,t) = 0. \tag{16}$$

This means we first heat the material and establish a homogeneous temperature field inside it, and then touch it with a colder point. Since this is a half-infinite problem, we can use Laplace transform on the time $t$. We have

$$\mathcal{L}\left[\frac{\partial u}{\partial t}\right] = sU(x,s) - u(x,0) = sU(x,s) - T,$$

and

$$\mathcal{L}\left[\frac{\partial^2 u}{\partial x^2}\right] = \frac{\partial^2 U(x,s)}{\partial x^2}.$$

The bulk equation then becomes

$$sU(x,s) - T = K\frac{\partial^2 U(x,s)}{\partial x^2},$$

$$\frac{\partial^2 U(x,s)}{\partial x^2} - \frac{s}{K}U = -\frac{T}{K}.$$

The homogeneous solution of this equation is just (note that $A$ and $B$ may have $s$ dependence)

$$U = A\mathrm{e}^{\sqrt{s/K}x} + B\mathrm{e}^{-\sqrt{s/K}x}.$$

A specific solution is

$$U = \frac{T}{s}.$$

$A$ has to be zero, because $u(x,t)$ should be finite when $x \to \infty$. So we find

$$U = \frac{T}{s} + B(s)\mathrm{e}^{-\sqrt{s/K}x}.$$

We still need to use the condition $u(0,t) = 0$, which, after Laplace transform, is $U(0,s) = 0$, and we find

$$U = \frac{T}{s}(1 - \mathrm{e}^{-\sqrt{s/K}x}).$$

So

$$u(x,t) = \mathcal{L}^{-1}\left[\frac{T}{s}(1 - \mathrm{e}^{-\sqrt{s/K}x})\right] = T\mathrm{erf}\left(\frac{x}{2\sqrt{Kt}}\right). \tag{17}$$

## 2 The linear wave equation

### 2.1 Homogeneous boundary condition

Now we use the same procedure above the solve

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}. \tag{18}$$

The boundary conditions are

$$y(0,t) = y(L,t) = 0, \tag{19}$$

and the initial condition is

$$y(x,0) = f(x), \quad y'(x,0) = 0. \tag{20}$$

We do a separation of variables

$$y(x,t) = T(t)X(x), \tag{21}$$

and we have

$$XT'' = c^2 TX'' \Rightarrow \frac{T''}{c^2 T} = \frac{X''}{X} = -\lambda. \tag{22}$$

We can check that in order for $X$ to be bounded at $x = 0, L$, $\lambda$ has to be positive. So the solution for the $X$ equation is

$$X(x) = A\cos\sqrt{\lambda}x + B\sin\sqrt{\lambda}x.$$

Then the boundary conditions means

$$A = 0,$$

and

$$\sin\sqrt{\lambda}L = 0 \Rightarrow \sqrt{\lambda}L = n\pi, \quad n = 1, 2, 3, \ldots.$$

So we eventually get

$$X_n(x) = B\sin\sqrt{\lambda_n}x, \quad \lambda_n = \frac{n^2\pi^2}{L^2}. \tag{23}$$

The $T$ equation is

$$T_n'' + \lambda_n c^2 T_n = 0,$$

and we have

$$T_n(t) = A_n \cos\frac{n\pi ct}{L} + B_n \sin\frac{n\pi ct}{L}. \tag{24}$$

Now we can impose the initial condition. From the condition that $y'$ is zero everywhere when $t = 0$, we find

$$0 = \sum_{n=1}^{\infty}\left(-\frac{n\pi c}{L}B_n\right)\sin\frac{n\pi}{L} \Rightarrow B_n = 0,$$

and therefore

$$y(x,t) = \sum_{n=1}^{\infty} c_n \cos\frac{n\pi ct}{L}\sin\frac{n\pi x}{L}.$$

Then we find

$$f(x) = y(x, t=0) = \sum_{n=1}^{\infty} c_n \sin\frac{n\pi x}{L}.$$

This means

$$\int_0^L f(x)\sin\frac{m\pi x}{L}\,\mathrm{d}x = c_m \int_0^L \sin^2\left(\frac{m\pi x}{L}\right)\mathrm{d}x = c_m\frac{L}{2},$$

and we get

$$y(x,t) = \sum_{n=1}^{\infty}\frac{2}{L}\left(\int_0^L f(x')\sin\frac{n\pi x'}{L}\,\mathrm{d}x'\right)\cos\frac{n\pi ct}{L}\sin\frac{n\pi x}{L}. \tag{25}$$

## 2.2 On unbounded interval

Now we want to simulate how a wave packet propagate to infinities. The boundary conditions are now

$$y(x,t), \frac{\partial y}{\partial x} \to \infty \quad \text{as } x \to \pm\infty, \tag{26}$$

and the initial conditions are

$$y(x,0) = f(x), \quad \frac{\partial y}{\partial t} = g(x). \tag{27}$$

The good convergence behaviors of $y$ means we can use Fourier transform to solve for it. We do the transform in $x$ variable:

$$\mathcal{F}\left[\frac{\partial^2 y}{\partial t^2}\right] = \frac{\partial^2 \hat{y}(\omega,t)}{\partial t^2},$$

and

$$\mathcal{F}\left[\frac{\partial^2 y}{\partial x^2}\right] = -\omega^2 \hat{y}(\omega,t).$$

Note that the second step has already used the boundary conditions, because to show $\mathcal{F}y' = -\mathrm{i}\omega\hat{y}$, we have used the condition

$$\int_{-\infty}^{\infty} \mathrm{d}\left(\frac{\partial y}{\partial x}\mathrm{e}^{-\mathrm{i}\omega x}\right) = 0,$$

which means $y(x=\infty,t) = 0$ and $y(x=-\infty,t) = 0$. The wave equation after Fourier transform is

$$\frac{\partial^2 \hat{y}}{\partial t^2} = -c^2\omega^2\hat{y},$$

and we have

$$\hat{y}(\omega,t) = A(\omega)\cos\omega ct + B(\omega)\sin\omega ct.$$

We can do Fourier transform to $f$ and $g$ as well, and this immediately gives

$$\hat{y}(\omega, t=0) = A(\omega) = \hat{f}(\omega),$$

and

$$\frac{\partial \hat{y}}{\partial t} = \omega c B(\omega) = \hat{g}(\omega).$$

So we find

$$\hat{y}(\omega,t) = \hat{f}(\omega)\cos\omega ct + \frac{\hat{g}}{\omega c}\sin\omega ct.$$

The inverse Fourier transform of this equation tells us

$$y(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}\omega x}\left(\hat{f}(\omega)\cos\omega ct + \frac{\hat{g}(\omega)}{\omega c}\sin\omega ct\right)\mathrm{d}\omega\,. \tag{28}$$

## 2.3 The d'Alembert's solution

We define

$$\xi = x - ct, \quad \eta = x + ct. \tag{29}$$

The two variables are paths in which a wave may move. So we have

$$\frac{\partial}{\partial t} = \frac{\partial \xi}{\partial t}\frac{\partial}{\partial \xi} + \frac{\partial \eta}{\partial t}\frac{\partial}{\partial \eta} = -c\frac{\partial}{\partial \xi} + c\frac{\partial}{\partial \eta},$$

and

$$\frac{\partial}{\partial x} = \frac{\partial \xi}{\partial x}\frac{\partial}{\partial \xi} + \frac{\partial \eta}{\partial t}\frac{\partial}{\partial \eta} = \frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta}.$$

So we find

$$\frac{\partial^2}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} = 4\frac{\partial}{\partial \xi}\frac{\partial}{\partial \eta},$$

So the wave equation now reads

$$\frac{\partial}{\partial \xi}\frac{\partial}{\partial \eta} y = 0,$$

and therefore

$$y(\xi, \eta) = F(\xi) + G(\eta) = F(x - ct) + G(x + ct). \tag{30}$$

So now from the initial conditions, we get

$$y(x, 0) = f(x) = F + G, \quad \left.\frac{\partial y}{\partial t}\right|_{x,0} = g(x) = -cF' + cG'.$$

Note that the $F'$ symbol in the second equation can be understood as putting $x - ct$ into $\partial F/\partial \xi$, as well as $\partial F(x - ct)/\partial x$. This means we have

$$\int_0^x g(x')\,\mathrm{d}x' = -cF + cG,$$

and note that we are currently at $t = 0$, we get

$$F(x) = \frac{1}{2} f(x) - \frac{1}{2c}\int_0^x g(x')\,\mathrm{d}x'\,,$$

and

$$G(x) = \frac{1}{2} f(x) + \frac{1}{2c}\int_0^x g(x')\,\mathrm{d}x'\,,$$

and eventually

$$y(x, t) = \frac{1}{2} f(x - ct) + \frac{1}{2} f(x + ct) + \frac{1}{2c}\int_{x-ct}^{x+ct} g(x')\,\mathrm{d}x'\,. \tag{31}$$