Fluctuation

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1 Motivation: Langevin equation

The experimental motivation to study fluctuation is probably the Brownian motion: in this specific case, large particles in fluid feel the fluctuation of the fluid, and undergo random motion. There are three length scales and three time scales in a typical experimental setting:

- The molecule scales: $1 \times 10^{-10}\,\mathrm{m}$, $1 \times 10^{-12}\,\mathrm{s}$;
- The particle scales: typically $1 \times 10^{-6}\,\mathrm{m}$, $1 \times 10^{-3}\,\mathrm{s}$;
- The enclosure (i.e. container of the fluid) scales: at least $1 \times 10^{-2}\,\mathrm{m}$, $1 \times 10^{2}\,\mathrm{s}$.

The separation between the molecule scales and the particle scales justifies treating the fluid as a continuum, while the separation between the particle scales and the enclosure scales justifies treating the configuration space of the particles as an infinite one. There actually should be another scale: the mean distance between particles; if this is large enough, then particles are independent to each other; but we may want to consider this as the enclosure scales.

When the aforementioned conditions of separation of scales are all true, the behavior of a particle independent to others is described by the **Langevin equation**

$$M \frac{\mathrm{d}^2 X}{\mathrm{d}t^2} + \frac{\partial U}{\partial X} = -c_{\mathrm{F}} \dot{X} + F_{\mathrm{N}}(t), \tag{1}$$

the terms on the LHS being effects about the particle itself, the terms on the RHS being the deterministic damping term and the stochastic Langevin force, accordingly.

Solving (1), therefore, involves two steps: finding reliable descriptions of c_{F} and F_{N}, and solving (1) as a stochastic differential equation. The first problem is essentially about establishing a theory of fluctuation (F_{N}) and response (γ) in the fluid surrounding the particle, and the second problem is about the mathematical treatment of stochastic processes.

2 Quantum circuit

The formalism of Langevin equation is not limited to soft condensed matter systems: since formally Newton's second law is analogous to circuit theory, fluctuation and damping in a circuit immersed in a sea of smaller circuits can also be described in Langevin's formalism. Indeed this is how resistance comes into being In circuit analysis we have another problem: now the Langevin equation should be *quantum*, since although no one would perform a Brownian motion experiment in a low temperature quantum liquid, we indeed can perform quantum circuit experiments at a very low temperature.

Ref.

Commutation relations?

3 Correlation, response, and fluctuation-dissipation theorem

In this section we briefly review the linear response theory. In general, a driven Hamiltonian assumes the following form:

$$H = H_0(x_1, \ldots, x_n, p_1, \ldots, p_n) - A(x_1, \ldots, x_n, p_1, \ldots, p_n) F_A(t), \tag{2}$$

where we have assumed that degrees of freedom in the system follow the canonical commutation relation; this assumption of course is frequently broken (as in, say, a magnetic spin model), but in this note let's focus on the most familiar case. We can readily have two examples: the first is

$$H = \frac{p^2}{2m} + \frac{1}{2} k x^2 - x F(t), \tag{3}$$

of which the EOM is

$$\dot{p} = -kx + F(t), \quad \dot{x} = \frac{p}{m}, \tag{4}$$

as is expected. The second is

$$H = \frac{\Phi^2}{2L} + \frac{Q^2}{2L} - QV, \tag{5}$$

Here we intentionally leave the discussion of damping; it can't be directly treated in the Hamiltonian formalism and will be modeled by an infinite bath of other degrees of freedom. Suppose we have another physical quantity B which is also a polynomial of $x_1, \dots, x_n$ and $p_1, \dots, p_n$. When the external driving force $F(t)$ is applied, we get

Commutation relations: cQED?

$$\delta B\,(t) \coloneqq B(t) - B_0 = \int_{-\infty}^{\infty} \mathrm{d}t_1\, \chi_{BA}(t, t_1) F_A(t_1) + B_{\text{noise}} + \mathcal{O}\,(F_A^2). \tag{6}$$

The first term equals to the linear term in $\langle \delta B\,(t) \rangle$; the noise term contains information about correlation of B with other variables – or maybe itself. It can be statistical noise (noise in classical probability theory) when there is a bath (see below); but even if we are dealing with a pure state theory, $B_{\text{noise}} = B - \langle B \rangle$ will still bring some "quantum noise". The response function is

$$\chi_{BA}(t, t_1) = \chi_{BA}(t - t_1) \tag{7}$$

when H_0 is time-independent.

Now we turn to another aspect: correlation. We define

$$S_{BA}(\omega) = \int \mathrm{e}^{\mathrm{i}\omega t} \langle B(0) A(t) \rangle \, \mathrm{d}t\,. \tag{8}$$

This is known as the greater Green function or the lesser Green function when A and B are field operators. We now state the following **fluctuation-dissipation theorem**: for a classical system, we have

Heisenberg description of the system: from density matrix to??? What is $A(t)$ when we have noise in the system?

$$\langle \dot{A}(0) B(t) \rangle \, \theta(t) = k_{\mathrm{B}} T \chi_{BA}(t), \tag{9}$$

and in the frequency domain,

$$S_{\dot{A}B}(\omega) = 2 k_{\mathrm{B}} T \operatorname{Re} \chi(\omega). \tag{10}$$

Replacing $\dot{A}$ by A, we get a $1/(-\mathrm{i}\omega)$ factor, and the equation becomes

$$S_{AB}(\omega) = \frac{2 k_{\mathrm{B}} T}{\omega} \operatorname{Im} \chi_{AB}(\omega). \tag{11}$$

In the quantum case we have

$$S_{AB}(\omega) = \frac{2\hbar}{1 - \mathrm{e}^{-\hbar\omega / k_{\mathrm{B}} T}} \operatorname{Im} \chi_{AB}(\omega). \tag{12}$$

To save space the proof of the theorem is not shown here; a straightforward derivation can be found in section 9.4 in [1]. Note that in the $T \to 0$ limit, classically S_{AB} vanishes, but in the quantum case we still have a non-vanishing remaining correlation as long as we see a response of A to B which is known as quantum noise: if A responds to the $-Bf(t)$ term in the Hamiltonian, A and B don't commute (or otherwise we don't have non-trivial time evolution), and thus S_{AB} is definitely not zero when $T \to 0$ i.e. when in the pure state theory.

4 Analysis of common noises

5 Langevin equation revisited

Now we can see that although the details of $F_{\mathrm{N}}(t)$ and c_{F} are still largely unknown, if $F_{\mathrm{N}}(t)$ and c_{F} "naturally" appear by coupling with the bosonic bath, the properties of $F_{\mathrm{N}}(t)$ and c_{F} in

Derive F_{N} as correlation function, c_{F} as response

$$M\ddot{X} + c_{\mathrm{F}} \dot{X} + \frac{\partial U}{\partial X} = F_{\mathrm{N}}(t) \tag{13}$$

follow the relation

$$\langle F_\mathrm{N}(0) F_\mathrm{N}(t) \rangle = 2 k_\mathrm{B} T c_\mathrm{F} \delta(t), \tag{14}$$

or, to be accurate, $\delta(t)$ should be replaced by something like

$$\frac{1}{\tau} \mathrm{e}^{-|t|/\tau} \tag{15}$$

if the environment degrees of freedom are “free” and therefore are Gaussian variables. Being more clear here

6 Fokker-Planck equation

The Langevin equation in the last section is a specific case of the following SDE

$$\dot{X} = AW + F, \quad \langle F_i(0) F_j(t) \rangle = B_{ij} \delta(t), \tag{16}$$

the time evolution of the probabilistic distribution $W(X_1, X_2, \ldots, X_n)$ can be rewritten into

$$\frac{\partial W}{\partial t} = \underbrace{-\boldsymbol{\nabla} \cdot AW}_{\text{drift}} + \underbrace{\frac{1}{2} \boldsymbol{\nabla} \cdot \boldsymbol{\nabla} \cdot (BW)}_{\text{diffusion}}. \tag{17}$$

This equation is known as the **Fokker-Planck equation**.

The classical tunneling rate is derivation

$$\Gamma = \frac{\omega_0}{2\pi} \mathrm{e}^{-\Delta U / k_\mathrm{B} T} \tag{18}$$

7 Transmission line and resistance

We can also check the fluctuation-dissipation theorem on the infinite transmission line.

$$Y_m(\omega) = \left(-\mathrm{i}\omega L_m + \frac{1}{-\mathrm{i}\omega C} \right)^{-1} = - \tag{19}$$

References

[1] Piers Coleman. *Introduction to many-body physics*. Cambridge University Press, 2015.