Correlation function and response function

Jinyuan Wu

September 28, 2022

This document gives several examples of the calculation of the correlation function and the response function.

1 Important formulae

For disturbance $H' = \lambda O_1$, the first order response of the expectation of the operator O_1 is

$$D(t_2, t_1) = -\mathrm{i}\theta(t_2 - t_1) \langle [O_2(t_2), O_1(t_1)] \rangle \,. \tag{1}$$

This function, as its definition hints, is hard to calculate. We usually calculate the time-ordered correlation function

$$G(t_2, t_1) = -\mathrm{i} \langle \mathcal{T} O_2(t_2) O_1(t_1) \rangle \,. \tag{2}$$

For calculation of the time-ordered correlation function, we have

$$\langle \mathcal{T} O_2(t_2) O_1(t_1) \rangle = \frac{\int \mathcal{D}x O_2(t_2) O_1(t_1) \mathrm{e}^{\mathrm{i}S[x]}}{\int \mathcal{D}x \mathrm{e}^{\mathrm{i}S[x]}}. \tag{3}$$

Here when calculating the action, we need to replace t with $t(1 - \mathrm{i}\epsilon)$ and then let $\epsilon \to 0$. For free systems, if

$$S = xG^{-1}x, \tag{4}$$

then the correlation function is exactly G, because

$$\frac{\int \mathrm{d}^n\mathbf{x}\, x_k x_l \mathrm{e}^{\frac{\mathrm{i}}{2}\mathbf{x}\mathbf{A}\mathbf{x}}}{\int \mathrm{d}^n\mathbf{x}\, \mathrm{e}^{\frac{\mathrm{i}}{2}\mathbf{x}\mathbf{A}\mathbf{x}}} = \mathrm{i}(\mathbf{A}^{-1})_{kl}. \tag{5}$$

2 The harmonic oscillator

The action is

$$S = \int \mathrm{d}t \left(\frac{1}{2} m\dot{x}^2 - \frac{1}{2} m\omega_0^2 x^2 \right). \tag{6}$$

The kernel is

$$K(t_1, t_2) = (-m\partial_{t_1}^2 - m\omega_0^2)\delta(t_1 - t_2). \tag{7}$$

Here the derivative *doesn't* act on the δ-function. So we need to find the inverse of the kernel. The definition of "summation" in the continuous path integral is $\int \mathrm{d}t$, so we have

$$\int \mathrm{d}t_2\, K(t_1, t_2) G(t_2, t_3) = \delta(t_1 - t_3), \tag{8}$$

and thus we have

$$(-m\partial_{t_1}^2 - m\omega_0^2) G(t_1, t_3) = \delta(t_1, t_3).$$

The equation has time translational symmetry, so we have

$$(-m\partial_t^2 - m\omega_0^2) G(t - t') = \delta(t - t'). \tag{9}$$

By Fourier transformation

$$G(t - t') = \int \frac{\mathrm{d}\omega}{2\pi} \mathrm{e}^{-\mathrm{i}\omega(t-t')} G(\omega), \tag{10}$$

we have

$$\int \frac{\mathrm{d}\omega}{2\pi} \mathrm{e}^{-\mathrm{i}\omega(t-t')} (m\omega^2 - m\omega_0^2) G(\omega) = \int \frac{\mathrm{d}\omega}{2\pi} \mathrm{e}^{-\mathrm{i}\omega(t-t')},$$

so

$$G(\omega) = \frac{1}{m(\omega^2 - \omega_0^2)}, \tag{11}$$

and we get

$$\mathrm{i}G(\omega) = \frac{\mathrm{i}}{m(\omega^2 - \omega_0^2)} \sim \langle xx \rangle \,, \tag{12}$$

which is the propagator in perturbation calculation. We naively do the inverse Fourier transformation

$$G(t - t') = \int \frac{\mathrm{d}\omega}{2\pi} \mathrm{e}^{-\mathrm{i}\omega(t-t')} \frac{1}{m(\omega^2 - \omega_0^2)} \tag{13}$$

and find the integral is not well-defined. The solution is well-known: to add an infinitesimal imaginary part that fits the definition of the time-ordered correlation function, and when we have a ω^2 term in the denominator, it should be

$$G(\omega) = \frac{1}{m(\omega^2 - \omega_0^2 + \mathrm{i}0^+)}. \tag{14}$$

But here I'll show this choice of imaginary part can be directly derived from the infinitesimal imaginary part in the time variable in the path integral. The action, after adding the imaginary part to t, is

$$S = \int \mathrm{d}t \, (1 - \mathrm{i}\epsilon) \left(\frac{m}{2} \frac{1}{(1 - \mathrm{i}\epsilon)^2} \dot{x}^2 - \frac{1}{2} m \omega_0^2 x^2 \right), \tag{15}$$

and the kernel is

$$K(t_1, t_2) = -\frac{m}{2} \left(\frac{1}{1 - \mathrm{i}\epsilon} \partial_{t_1}^2 + (1 - \mathrm{i}\epsilon) \omega_0^2 \right) \delta(t_1 - t_2). \tag{16}$$

So the Green function is now (just replace m by $m/(1 - \mathrm{i}\epsilon)$, and replace ω_0^2 by $(1 - \mathrm{i}\epsilon)^2 \omega_0^2$)

$$G(\omega) = \frac{1 - \mathrm{i}\epsilon}{m((1 - \mathrm{i}\epsilon)^2 \omega^2 - \omega_0^2)}. \tag{17}$$

Now we let $\epsilon \to 0$, and the $-\mathrm{i}\epsilon$ term in the numerator can be thrown away because it doesn't affect the pole structure; in the denominator, the ϵ^2 term is relatively small and can also be ignored. So we get

$$G(\omega) = \frac{1}{m(\omega^2 - 2\mathrm{i}\epsilon\omega^2 - \omega_0^2)}.$$

Since the exact value of ϵ is not important (only its sign is important), we can replace $\epsilon\omega^2$ – which has the same sign with ϵ – by ϵ or 0^+, and thus

$$G(\omega) = \frac{1}{m(\omega^2 - \omega_0^2 + \mathrm{i}\epsilon)}, \tag{18}$$

which is exactly (14).