Elasticity in structural mechanics

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1 Rigid body analysis

2 Elastic medium

Definition The deformation $\boldsymbol{u}(t)$ of the system is completely decided by the external loading at t. Notable counterparts:

- *Fluid.* $\boldsymbol{u} \Leftarrow \boldsymbol{v} \Leftarrow \boldsymbol{F}$: not elastic.
- *Plastic.* $\boldsymbol{u}$ depends on history: not elastic.

Degrees of freedom, with infinitesimal deformation We deal with two sets of variables:

- *Stress* σ_{ij}. $\mathrm{d}F_i = \sigma_{ij}\,\mathrm{d}A_j$. Moment is not needed for bulk equation of equilibrium; but it's needed to capture the spatially fast-varying internal force in low-dimensional systems.
- *Strain* u_{ij}. For small deformation

$$u_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right). \tag{1}$$

- *Constitutive relations.* $\sigma_{ij} = \sigma_{ij}[u_{ij}]$.

Uniform isotropic linear medium Constitutive relation

$$\sigma_{ik} = K u_{ll}\delta_{ik} + 2\mu\left(u_{ik} - \frac{1}{3}\delta_{ik}u_{ll}\right). \tag{2}$$

Temperature expansion The strain induced by temperature change:

$$\frac{\mathrm{d}u}{\mathrm{d}x} = \alpha(T - T_0), \tag{3}$$

where T_0 is the "overall" temperature.

3 Uniform isotropic linear medium, in experiments

Two modes of strain

- *Compression/tension.* Along one direction (for example z):

$$\epsilon = \frac{\delta}{L} = u_{zz}. \tag{4}$$

- *Shear.* On the xy plane:

$$\gamma = \theta_{xx'} + \theta_{yy'} = 2u_{xy}. \tag{5}$$

Young's modulus Relation between tension and force:

$$E = \frac{P}{\epsilon} = \frac{PL}{\delta} \Rightarrow F = PA = \frac{\delta}{L}\cdot EA. \tag{6}$$

Poisson's ratio Relation between transverse strain and axial strain (in Young's modulus experiment):

$$\sigma = \nu = -\frac{\mathrm{d}\epsilon_{\text{transverse}}}{\mathrm{d}\epsilon_{\text{axial}}}. \tag{7}$$

This is how the material becomes thinner when stretched.

Volume modulus Relation between pressure and volume:

$$K = -V\frac{\mathrm{d}P}{\mathrm{d}V}. \tag{8}$$

Here K is that parameter in (2).
Shear modulus Relation between shear stress and shear strain:

$$\mu = G = \frac{\tau}{\gamma}. \tag{9}$$

Here τ is σ_{xy} (or yz or zx); γ is the shear strain.
How many independent parameters? In isothermal process:

$$E = \frac{9K\mu}{3K+\mu}, \quad \sigma = \frac{1}{2}\frac{3K-2\mu}{3K+\mu}. \tag{10}$$

When is the linear elasticity condition broken?

1. Linear region.
2. Proportional limit.
3. Elastic limit.
4. Yield point.
5. Ultimate tensile point.
6. Breaking point.

4 Low dimension system: torsion of cylinder-like rod

Reaction of φ to torque Here T is the torque:

$$\frac{\mathrm{d}\varphi}{\mathrm{d}z} = \frac{T(z)}{JG}, \quad T(z) = \int_0^z \mathrm{d}z' \, \frac{\mathrm{d \; torque}}{\mathrm{d}z'}. \tag{11}$$

Relation between torque and stress

$$\gamma = \gamma_{xz} = \frac{\mathrm{d}\varphi}{\mathrm{d}z} r, \quad \tau = G\gamma, \tag{12}$$

$$\tau_{\max} = G\frac{\mathrm{d}\varphi}{\mathrm{d}z} R = \frac{TR}{J}. \tag{13}$$

Here R may also be written as c.
Note on J It's actually not moment of inertia!

5 Low dimension system: beam, or rod predominantly bended

Important degrees of freedom Suppose the beam is in direction z. Sign convention: for force, deflection: downwards = +; for moment: counterclockwise = +.

- *Deflection.* Referred to as w.
- *Shear force.* The internal force, averaged:

$$\boldsymbol{F}_\perp = \boldsymbol{\sigma} \cdot \boldsymbol{A} = \sigma_{xz} A \hat{\boldsymbol{x}} + \sigma_{yz} A \hat{\boldsymbol{y}}, \tag{14}$$

and usually we only consider one direction (say x), and $\boldsymbol{F}_\perp = V\hat{\boldsymbol{x}}$. Below we change z to x.

- *Moment.* The "first-order moment" of internal force:

$$M + \mathrm{d}M = M + V\,\mathrm{d}x \Rightarrow V = \frac{\partial M}{\partial x}. \tag{15}$$

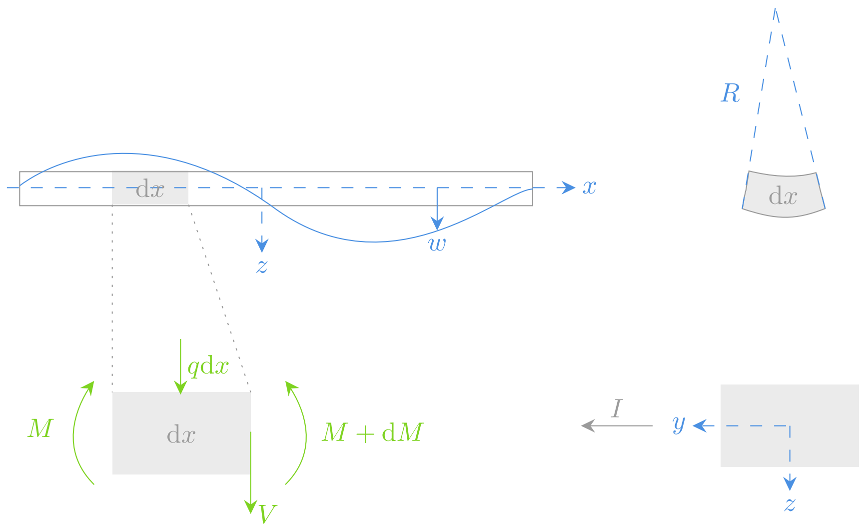

Figure 1: Analysis of a beam

Equation of equilibrium For determining V:

$$\frac{\partial V}{\partial x} + q = 0, \tag{16}$$

where q is force per unit length. The relation between moment and w:

$$M = -EI\frac{\partial^2 w}{\partial x^2}. \tag{17}$$

Here the axis of I is the same as the direction of M.

Details in bending stress Assuming R being large, each beam element can be seen as a beam element feeling stretching only, and thus

$$\frac{\mathrm{d}x'}{\mathrm{d}x} = \frac{R+z}{R}, \quad \sigma := \sigma_{xx} = Eu_{xx} = \frac{z}{R}E, \tag{18}$$

while

$$M = \int \mathrm{d}z\,\mathrm{d}y\,\sigma \cdot z = \frac{E}{R}\underbrace{\int \mathrm{d}z\,\mathrm{d}y\,z^2}_{=:I}. \tag{19}$$

So after M is found from one of the equations above,

$$\sigma = \frac{z}{I}M, \tag{20}$$

and thus at a given point,

$$\sigma_{\max} = \frac{z_{\max}}{I}M, \tag{21}$$

which is needed to determine whether the beam fails.

Boundary condition The boundary condition of concentrated load can be determined if the following rules are followed:

- V is a downward force applied to the right end of a beam element by the beam element following it.
- No V is applied at the left boundary of the first beam element: thus if $F(x=0)$ is upward, then $V(x=0)$ is downward and is positive.
- No V is applied at the right boundary of the last beam element: thus if $F(x=L)$ is upward, then the shear force the $L - \mathrm{d}x$ beam element applies to the beam element at L is downward, and therefore the shear force applied to the beam element at $L - \mathrm{d}x$ is upward, and $V(x=L)$ is negative.

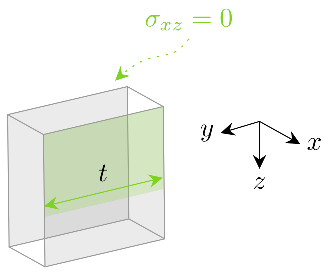

Figure 2: Analysis of shear stress

Details in shear stress Due to V we also have σ_{zx}, and

$$0 = \partial_x \sigma_{xx} + \partial_z \sigma_{zx} \Rightarrow 0 = \frac{1}{I}\frac{\partial M}{\partial x} \int \mathrm{d}z\, \mathrm{d}y\, z + \int \mathrm{d}y\, \sigma_{zx}, \tag{22}$$

and the average shear stress is (t is the width in y coordinate at z)

$$\tau \coloneqq \bar{\sigma}_{zx} = \frac{1}{It}\frac{\partial M}{\partial x} \underbrace{\int_{\text{area above or below } z} z\, \mathrm{d}y\, \mathrm{d}z}_{Q} = \frac{Q}{It} V. \tag{23}$$

The integration range used in calculating Q is the Green region in Fig. 2.
Procedure

1. Finding all reaction forces from the loading.
2. Finding V.
3. Finding M.
4. Finding σ and $\sigma_{\max}$.
5. Finding w if necessary.

6 Problems

- Using deformation to decide forces (that otherwise can't be determined).
- How fast a shaft can rotate: $P = T\omega$. Then $\tau_{\max}$ can be found.
- Beam analysis, and whether it fails because $\sigma_{\max}$ is too large.