Homework 2

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Problem 1

1. The EOM is

$$\frac{\mathrm{d}}{\mathrm{d}t}(\dot{\varphi} - \mathrm{i}\theta/2\pi) = \frac{\mathrm{d}}{\mathrm{d}\varphi} V_0(1-\cos\varphi),$$
$$\ddot{\varphi} = V_0 \sin\varphi. \tag{1}$$

Integrating over φ, we have

$$\frac{1}{2}\dot{\varphi}^2 = -V_0\cos\varphi + C. \tag{2}$$

The range of C is between $\pm V_0$, because it corresponds to the φ when $\dot{\varphi} = 0$. The boundary condition that when $\tau \to \pm\infty$, φ stays zero, so we have $\varphi = 0$ and $\dot{\varphi} = 0$ in the two limits, so $C = V_0$, and therefore

$$\frac{1}{2}\dot{\varphi}^2 = \underbrace{V_0(1-\cos\varphi)}_{V(\varphi)}, \tag{3}$$

$$\pm\sqrt{2V_0}\tau = \int \frac{\mathrm{d}\varphi}{\sqrt{1-\cos\varphi}} = \sqrt{2}\ln\tan\frac{\varphi}{4},$$
$$\varphi = 4\arctan \mathrm{e}^{\pm\sqrt{V_0}(\tau-\tau_0)}.$$

By checking continuity and the $\tau \to \pm\infty$ limits, we find

$$\varphi_+ = 4\arctan\mathrm{e}^{\sqrt{V_0}(\tau-\tau_0)}, \quad \varphi_- = 4\arctan\mathrm{e}^{-\sqrt{V_0}(\tau-\tau_0)} \tag{4}$$

are exactly the two single instanton solutions we need – there is no need "cut and connect" branches of solutions.

We have

$$\begin{aligned} S[\varphi] &= \int \mathrm{d}\tau \left(\frac{1}{2}\dot{\varphi}^2 - \mathrm{i}\frac{\theta}{2\pi}\dot{\varphi} + V(\varphi)\right) \\ &= -\mathrm{i}\frac{\theta}{2\pi}(\varphi(\infty) - \varphi(-\infty)) + 2\int \mathrm{d}\tau\, V(\varphi). \end{aligned}$$

For φ_+, the first term is $-\mathrm{i}\theta$, while for φ_-, the first term is $\mathrm{i}\theta$. For φ_+, the second term is

$$\begin{aligned} &2V_0\int_{-\infty}^{\infty}\mathrm{d}\tau\left(1-\cos\left(4\arctan\mathrm{e}^{\sqrt{V_0}(\tau-\tau_0)}\right)\right) \\ &= 2\sqrt{V_0}\int_{-\infty}^{\infty}\mathrm{d}x\,(1-\cos(4\arctan\mathrm{e}^x)) \\ &= 8\sqrt{V_0}. \end{aligned}$$

The same is true for φ_- because of the time reversal symmetry. So we have

$$S_{0,+}(\theta) = -\mathrm{i}\theta + 8\sqrt{V_0}, \quad S_{0,-}(\theta) = \mathrm{i}\theta + 8\sqrt{V_0}. \tag{5}$$

2. The saddle point approximation, without considering the instantons, gives

$$U(0,T;0,0) = \sqrt{\frac{m\omega}{2\pi\sinh\omega T}}, \tag{6}$$

where the oscillation frequency is just

$$\omega = \sqrt{V_0}. \tag{7}$$

Now we insert instantons into the paths taken into consideration. We make the dilute instanton gas approximation, assuming that the total time T and the distances between instantons are largely enough compared with the temporal size of each instanton ($\sim 1/\sqrt{V_0}$), and in this case, action has additivity, and the contribution to the action of each instanton is approximately the same as the action of the instanton with the $-\infty < \tau < \infty$ time span, which we just evaluated in (5). So for a configuration with n_+ φ_+ instantons and n_- φ_- instantons, the total saddle-point action is

$$K^{n_1+n_2}\mathrm{e}^{-n_+S_{0,+}-n_-S_{0,-}}.$$

The number of the possible orders of the instantons is $\binom{n_1+n_2}{n_1}$, so the path integral is

$$\begin{aligned}
&\sum_{n_-,n_+}\int_0^T \mathrm{d}\tau_1 \int_{\tau_1}^T \mathrm{d}\tau_2 \cdots \int_{\tau_{n-1}}^T \mathrm{d}\tau \binom{n_1+n_2}{n_1} K^{n_1+n_2}\mathrm{e}^{-n_+S_{0,+}-n_-S_{0,-}}U(0,T;0,0)\\
&= U(0,T;0,0)\sum_{n_-,n_+}\frac{T^{n_++n_-}}{(n_++n_-)!}\frac{(n_++n_-)!}{n_+!n_-!}K^{n_++n_-}\mathrm{e}^{-n_+S_{0,+}-n_-S_{0,-}}\\
&= U(0,T;0,0)\sum_{n_+}\frac{(TK\mathrm{e}^{-S_{0,+}})^{n_+}}{n_+!}\sum_{n-}\frac{(TK\mathrm{e}^{-S_{0,-}})^{n_-}}{n_-!}\\
&= U(0,T;0,0)\mathrm{e}^{TK\mathrm{e}^{-S_{0,+}}}\mathrm{e}^{TK\mathrm{e}^{-S_{0,-}}}.
\end{aligned}$$

So we get

$$\langle 0|\mathrm{e}^{-HT}|0\rangle = U(0,T;0,0)\mathrm{e}^{TK\mathrm{e}^{-S_{0,+}}}\mathrm{e}^{TK\mathrm{e}^{-S_{0,-}}}. \tag{8}$$

3. When $T\to\infty$, we know (in the last homework)

$$U(0,T;0,0)\sim \mathrm{e}^{-\frac{1}{2}\omega T}, \tag{9}$$

Since in the long run

$$\langle 0|\mathrm{e}^{-HT}|0\rangle \sim \mathrm{e}^{-\frac{1}{2}\omega T}\mathrm{e}^{TK(\mathrm{e}^{-S_{0,+}}+\mathrm{e}^{-S_{0,-}})} =: \mathrm{e}^{-ET}, \tag{10}$$

we have

$$\begin{aligned}
E &= \frac{1}{2}\omega - K(\mathrm{e}^{-S_{0,+}}+\mathrm{e}^{-S_{0,-}})\\
&= \frac{1}{2}\omega - K\mathrm{e}^{-8\sqrt{V_0}}(\mathrm{e}^{\mathrm{i}\theta}+\mathrm{e}^{-\mathrm{i}\theta})\\
&= \frac{1}{2}\sqrt{V_0} - 2K\mathrm{e}^{-8\sqrt{V_0}}\cos\theta.
\end{aligned} \tag{11}$$

So the ground state energy oscillates with respect to θ.

Solution

Problem 2

Solution

1. We have

$$\mathcal{T}\,\langle 0|j(t)j(0)|0\rangle = \langle j(t)j(0)\rangle\,,$$

where

$$\langle\cdots\rangle := \frac{\int \mathcal{D}x(\cdots)\mathrm{e}^{\mathrm{i}\int \mathrm{d}tL}}{\int \mathcal{D}x\mathrm{e}^{\mathrm{i}\int \mathrm{d}tL}}.$$

So we can do Fourier expansion to $j(t)$ without fears of details of normal ordering. We have

$$\begin{aligned}
j(t) &= \frac{\mathrm{d}}{\mathrm{d}t}\int \frac{\mathrm{d}\omega}{2\pi}\mathrm{e}^{-\mathrm{i}\omega t}ex(\omega)\\
&= \int \frac{\mathrm{d}\omega}{2\pi}\mathrm{e}^{-\mathrm{i}\omega t}(-\mathrm{i}\omega)ex(\omega),
\end{aligned}$$

so

$$
\begin{aligned}
\int \mathrm{d}t\, \mathrm{e}^{\mathrm{i}\omega t} \langle j(t)j(0)\rangle &= \frac{1}{2\pi\delta(0)} \int \mathrm{d}t\, \mathrm{e}^{\mathrm{i}\omega t} \int \mathrm{d}t_2 \langle j(t+t_2)j(t_2)\rangle \\
&= \frac{1}{2\pi\delta(0)} \int \mathrm{d}t\, \mathrm{e}^{\mathrm{i}\omega t} \int \mathrm{d}t_2 \int \frac{\mathrm{d}\omega_1}{2\pi} (-e\mathrm{i}\omega_1)\mathrm{e}^{-\mathrm{i}\omega_1(t+t_2)} \\
&\quad \cdot \int \frac{\mathrm{d}\omega_2}{2\pi} (-\mathrm{i}e\omega_2)\mathrm{e}^{-\mathrm{i}\omega_2 t_2} \langle j(\omega_1)j(\omega_2)\rangle \\
&= \frac{1}{2\pi\delta(0)} \int \frac{\mathrm{d}\omega_1}{2\pi} \int \frac{\mathrm{d}\omega_2}{2\pi} (-\mathrm{i}e\omega_1)(-\mathrm{i}e\omega_2) 2\pi\delta(\omega-\omega_1) 2\pi\delta(\omega_1+\omega_2) \langle j(\omega_1)j(\omega_2)\rangle \\
&= e^2 \frac{1}{2\pi}\omega^2 \langle j(\omega)j(-\omega)\rangle .
\end{aligned}
$$

On the other hand, we have

$$
\int \frac{\mathrm{d}\omega}{2\pi} \langle x(t)x(0)\rangle = \frac{1}{2\pi} \langle x(\omega)x(-\omega)\rangle ,
$$

and thus

$$
\mathrm{i}G_{jj}(\omega) = \int \mathrm{d}t\, \mathrm{e}^{\mathrm{i}\omega t} \langle j(t)j(0)\rangle = e^2\omega^2 \int \mathrm{d}t\, \mathrm{e}^{\mathrm{i}\omega t} \langle x(t)x(0)\rangle = e^2\omega^2 \frac{\mathrm{i}}{m(\omega^2-\omega_0^2+\mathrm{i}\epsilon)}, \tag{12}
$$

$$
G_{jj}(\omega) = \frac{e^2\omega^2}{m(\omega^2-\omega_0^2+\mathrm{i}\epsilon)}. \tag{13}
$$

2. The EOMs are

$$
\dot{x} = \frac{\partial H}{\partial p} = \frac{p}{m}, \quad \dot{p} = -\frac{\partial H}{\partial x} = -m\omega_0^2 + eE,
$$

$$
m\ddot{x} + m\omega_0^2 x = eE.
$$

After adding a small friction we get

$$
m\ddot{x} + m\epsilon\dot{x} + m\omega_0^2 x = eE. \tag{14}
$$

Again by Fourier transformation

$$
x(t) = \int \frac{\mathrm{d}\omega}{2\pi} \mathrm{e}^{-\mathrm{i}\omega t} x(\omega), \quad E(t) = \int \frac{\mathrm{d}\omega}{2\pi} \mathrm{e}^{-\mathrm{i}\omega t} E(\omega),
$$

we have

$$
(-m\omega^2 + m\omega_0^2 - \mathrm{i}m\epsilon\omega)x(\omega) = eE(\omega),
$$

$$
\sigma(\omega) = \frac{j(\omega)}{E(\omega)} = -\mathrm{i}\omega e \frac{x(\omega)}{E(\omega)} = \mathrm{i}\frac{e^2\omega}{m(\omega^2-\omega_0^2+\mathrm{i}\,\mathrm{sgn}(\omega)\epsilon)}. \tag{15}
$$

3. So when $\omega > 0$, $\mathrm{sgn}(\omega)\epsilon$ is just 0^+, and we get

$$
\sigma(\omega) = C\frac{G_{jj}(\omega)}{\omega}, \quad C = \mathrm{i}. \tag{16}
$$

This is expected: the correlation function corresponding to $\sigma(\omega)$ is $G_{j,ex}$, not G_{jj}. The two all contain a e^2 factor but they differ with a time derivative, which is the origin of the $-\mathrm{i}\omega$ in the denominator.

4. We have

$$
\begin{aligned}
\sigma(\omega) &= \mathrm{i}\frac{e^2}{m}\omega \left(\mathrm{P}\frac{1}{\omega^2-\omega_0^2} - \pi\mathrm{i}\,\mathrm{sgn}(\omega)\delta(\omega^2-\omega_0^2) \right) \\
&= \frac{\pi e^2}{m}\omega\delta(\omega^2-\omega_0^2) + \mathrm{i}\frac{e^2\omega}{m}\mathrm{P}\frac{1}{\omega^2-\omega_0^2}.
\end{aligned} \tag{17}
$$

So the real part is non-zero only when $\omega = \omega_0$.

Problem 3

Solution

1. The path integral is

$$\begin{aligned} Z &= \int \mathcal{D}X\mathcal{D}x \mathrm{e}^{\mathrm{i}\int \mathrm{d}t\left(\frac{1}{2}m\dot{x}^2-\frac{1}{2}m\omega_0^2x^2+\frac{1}{2}M\dot{X}^2-\frac{1}{2}M\Omega_0^2X^2-gxX\right)} \\ &= \int \mathcal{D}x \mathrm{e}^{\mathrm{i}\int \mathrm{d}t\left(\frac{1}{2}m\dot{x}^2-\frac{1}{2}m\omega_0^2x^2\right)} \int \mathcal{D}X \mathrm{e}^{\mathrm{i}\int \mathrm{d}t\left(\frac{1}{2}M\dot{X}^2-\frac{1}{2}M\Omega_0^2X^2-gxX\right)}. \end{aligned} \tag{18}$$

We need to integrate out the X variable to obtain an effective theory for x. We have

$$\begin{aligned} &\int \mathcal{D}X \mathrm{e}^{\mathrm{i}\int \mathrm{d}t\left(\frac{1}{2}M\dot{X}^2-\frac{1}{2}M\Omega_0^2X^2-gxX\right)} \\ &= \int \mathcal{D}X \mathrm{e}^{\mathrm{i}\int \mathrm{d}t\left(-\frac{1}{2}MX(\partial_t^2+\Omega_0^2)X-gxX\right)} \\ &= \mathrm{const}\cdot\exp\left(\mathrm{i}\int \mathrm{d}t\, \frac{1}{2}g^2x\frac{1}{M(\partial_t^2+\Omega_0^2)}x\right) \\ &= \mathrm{const}\cdot\exp\left(\mathrm{i}\int \mathrm{d}t\, \frac{1}{2}g^2x\frac{1}{M\Omega_0^2}\left(1-\frac{1}{\Omega_0^2}\frac{\mathrm{d}^2}{\mathrm{d}t^2}+\cdots\right)x\right) \\ &= \mathrm{const}\cdot\exp\left(\mathrm{i}\int \mathrm{d}t\left(\frac{g^2}{2M\Omega_0^2}x^2+\frac{1}{2}\frac{g^2}{M\Omega_0^4}\dot{x}^2+\cdots\right)\right). \end{aligned}$$

Only keeping the first-order correction, we have

$$m^* = m + \frac{g^2}{M\Omega_0^4}, \tag{19}$$

$$m^*(\omega_0^*)^2 = m\omega_0^2 - \frac{g^2}{M\Omega_0^2}. \tag{20}$$

When g is large, (20) becomes negative. This results TODO

2. We just need to replace gx by $gx-E$ in (18). Now after integrating out X, we get

$$\begin{aligned} &\mathrm{const}\cdot\exp\left(\mathrm{i}\int \mathrm{d}t\,\frac{1}{2}(gx-E)\frac{1}{M(\partial_t^2+\Omega_0^2)}(gx-E)\right) \\ &= \mathrm{const}\cdot\exp\left(\mathrm{i}\int \mathrm{d}t\,\frac{1}{2}(gx-E)\frac{1}{M\Omega_0^2}\left(1-\frac{1}{\Omega_0^2}\frac{\mathrm{d}^2}{\mathrm{d}t^2}+\cdots\right)(gx-E)\right) \\ &= \mathrm{const}\cdot\exp\left(\mathrm{i}\int \mathrm{d}t\left(\frac{1}{2M\Omega_0^2}(gx-E)^2+\frac{1}{2}\frac{1}{M\Omega_0^4}\left(\frac{\mathrm{d}(gx-E)}{\mathrm{d}t}\right)^2+\cdots\right)\right) \\ &= \mathrm{const}\cdot\exp\left(\mathrm{i}\int \mathrm{d}t\left(\frac{1}{2M\Omega_0^2}(gx-E)^2+\frac{1}{2}\frac{g^2}{M\Omega_0^4}\dot{x}^2+\cdots\right)\right). \end{aligned}$$

So now the effective theory is

$$L_{\text{eff}} = \frac{1}{2}\left(m+\frac{g^2}{M\Omega_0^4}\right)\dot{x}^2 - \frac{1}{2}m\omega_0^2x^2 + \frac{1}{2M\Omega_0^2}(gx-E)^2. \tag{21}$$

To find the expression of X, we just need to take the derivative of L_{eff} with respect to E, because to find an n-order correlation function of X, we just find the n-th derivative of Z, and if this is done with L_{eff}, then what is averaged over is just $\partial L_{\text{eff}}/\partial E$ to the n. So

$$X = \left.\frac{\partial L_{\text{eff}}}{\partial E}\right|_{E=0} = -\frac{g}{M\Omega_0^2}x, \tag{22}$$

and

$$\dot{X} = -\frac{g}{M\Omega_0^2}\dot{x}. \tag{23}$$

3.

Problem 4

Solution

1. We have

$$a(\boldsymbol{x},t)=\frac{1}{\sqrt{V}}\sum_{\boldsymbol{k}}\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}-\mathrm{i}\omega_{\boldsymbol{k}}t}a_{\boldsymbol{k}}. \tag{24}$$

So

$$\mathrm{i}G(\boldsymbol{x},t)=\frac{1}{V}\sum_{\boldsymbol{k}}\sum_{\boldsymbol{k}'}\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}\left\langle 0\middle|\mathcal{T}\,\mathrm{e}^{-\mathrm{i}\omega_{\boldsymbol{k}}t}a_{\boldsymbol{k}}a^{\dagger}_{\boldsymbol{k}'}\middle|0\right\rangle.$$

When $t>0$, we have

$$\mathcal{T}\,\mathrm{e}^{-\mathrm{i}\omega_{\boldsymbol{k}}t}a_{\boldsymbol{k}}a^{\dagger}_{\boldsymbol{k}'}=\mathrm{e}^{-\mathrm{i}\omega_{\boldsymbol{k}}t}a_{\boldsymbol{k}}a^{\dagger}_{\boldsymbol{k}'}=\mathrm{e}^{-\mathrm{i}\omega_{\boldsymbol{k}}t}(a^{\dagger}_{\boldsymbol{k}'}a_{\boldsymbol{k}}+\delta_{\boldsymbol{k}\boldsymbol{k}'}),$$

and when $t<0$, we have

$$\mathcal{T}\,\mathrm{e}^{-\mathrm{i}\omega_{\boldsymbol{k}}t}a_{\boldsymbol{k}}a^{\dagger}_{\boldsymbol{k}'}=\mathrm{e}^{-\mathrm{i}\omega_{\boldsymbol{k}}t}a^{\dagger}_{\boldsymbol{k}'}a_{\boldsymbol{k}}.$$

The momentum correlation function is then evaluated as follows:

$$\begin{aligned}\left\langle\Psi_0\middle|\mathcal{T}\,\mathrm{e}^{-\mathrm{i}\omega_{\boldsymbol{k}}t}a_{\boldsymbol{k}}a^{\dagger}_{\boldsymbol{k}'}\middle|\Psi_0\right\rangle&=\theta(t)\delta_{\boldsymbol{k}\boldsymbol{k}'}\mathrm{e}^{-\mathrm{i}\omega_{\boldsymbol{k}}t}+\mathrm{e}^{-\mathrm{i}\omega_{\boldsymbol{k}}t}\left\langle\Psi_0\middle|a^{\dagger}_{\boldsymbol{k}'}a_{\boldsymbol{k}}\middle|\Psi_0\right\rangle\\&=\theta(t)\delta_{\boldsymbol{k}\boldsymbol{k}'}\mathrm{e}^{-\mathrm{i}\omega_{\boldsymbol{k}}t}+\mathrm{e}^{-\mathrm{i}\omega_{\boldsymbol{k}}t}\delta_{\boldsymbol{k},0}\delta_{\boldsymbol{k}',0}\left\langle\Psi_0\middle|a^{\dagger}_0a_0\middle|\Psi_0\right\rangle\\&=\theta(t)\delta_{\boldsymbol{k}\boldsymbol{k}'}\mathrm{e}^{-\mathrm{i}\omega_{\boldsymbol{k}}t}+\mathrm{e}^{-\mathrm{i}\omega_{\boldsymbol{k}}t}\delta_{\boldsymbol{k},0}\delta_{\boldsymbol{k}',0}N.\end{aligned}$$

The correlation function is therefore

$$\begin{aligned}\mathrm{i}G(\boldsymbol{x},t)&=\frac{1}{V}\sum_{\boldsymbol{k},\boldsymbol{k}'}\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}\left(\theta(t)\delta_{\boldsymbol{k}\boldsymbol{k}'}\mathrm{e}^{-\mathrm{i}\omega_{\boldsymbol{k}}t}+\mathrm{e}^{-\mathrm{i}\omega_{\boldsymbol{k}}t}\delta_{\boldsymbol{k},0}\delta_{\boldsymbol{k}',0}N\right)\\&=\frac{N}{V}+\theta(t)\frac{1}{V}\sum_{\boldsymbol{k}}\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}\mathrm{e}^{-\mathrm{i}\frac{\boldsymbol{k}^2}{2m}t}\\&=\rho_0+\theta(t)\int\frac{\mathrm{d}^3\boldsymbol{k}}{(2\pi)^3}\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}-\mathrm{i}\frac{\boldsymbol{k}^2}{2m}t}.\end{aligned}$$

After completing the integral, we get

$$\mathrm{i}G(\boldsymbol{x},t)=\rho_0+\theta(t)\sqrt{\frac{m^3}{(2\pi\mathrm{i}t)^3}}\mathrm{e}^{\frac{\mathrm{i}}{2}\frac{m\boldsymbol{x}^2}{t}}. \tag{25}$$

When $|\boldsymbol{x}|\to\infty$, the second term oscillates fast, and its average is zero, so we have

$$\mathrm{i}G(\boldsymbol{x},t)\simeq\rho_0, \tag{26}$$

so there is indeed a long-range order in the ground state of BEC.

2. By Wick's theorem and the fact that $\langle 0|\mathcal{T}\,a_1a_2|0\rangle=0$, as well as $\langle 0|a^{\dagger}a|0\rangle$, we have

$$\begin{aligned}&\mathcal{T}\,a^{\dagger}(\boldsymbol{x},t)a(\boldsymbol{x},t)a^{\dagger}(0,0)a(0,0)\\&=:a^{\dagger}(\boldsymbol{x},t)a(\boldsymbol{x},t)a^{\dagger}(0,0)a(0,0):\\&\quad+:a^{\dagger}(\boldsymbol{x},t)a(0,0):\left\langle 0\middle|\mathcal{T}\,a(\boldsymbol{x},t)a^{\dagger}(0,0)\middle|0\right\rangle\\&\quad+:a(\boldsymbol{x},t)a^{\dagger}(0,0):\left\langle 0\middle|\mathcal{T}\,a^{\dagger}(\boldsymbol{x},t)a(0,0)\middle|0\right\rangle\\&\quad+\left\langle 0\middle|\mathcal{T}\,a(\boldsymbol{x},t)a^{\dagger}(0,0)\middle|0\right\rangle\left\langle 0\middle|\mathcal{T}\,a^{\dagger}(\boldsymbol{x},t)a(0,0)\middle|0\right\rangle.\end{aligned} \tag{27}$$

By space and time translational symmetry, we have

$$\begin{aligned}\left\langle 0\middle|\mathcal{T}\,a^{\dagger}(\boldsymbol{x},t)a(0,0)\middle|0\right\rangle&=\left\langle 0\middle|\mathcal{T}\,a(0,0)a^{\dagger}(\boldsymbol{x},t)\middle|0\right\rangle=\left\langle 0\middle|\mathcal{T}\,a(-\boldsymbol{x},-t)a^{\dagger}(0,0)\middle|0\right\rangle\\&=\underbrace{\rho_0}_{=0\text{ for }|0\rangle}+\theta(-t)\int\frac{\mathrm{d}^3\boldsymbol{k}}{(2\pi)^3}\mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}+\mathrm{i}\frac{\boldsymbol{k}^2}{2m}t}.\end{aligned}$$

Thus the last term in (27) vanishes, because it contains both $\theta(t)$ and $\theta(-t)$. So we get

$$\begin{aligned}
&\langle\Psi_0|\mathcal{T}\,a^\dagger(\boldsymbol{x},t)a(\boldsymbol{x},t)a^\dagger(0,0)a(0,0)|\Psi_0\rangle \\
=\;& \langle\Psi_0|\!:a^\dagger(\boldsymbol{x},t)a(\boldsymbol{x},t)a^\dagger(0,0)a(0,0):\!|\Psi_0\rangle \\
&+\langle\Psi_0|\!:a^\dagger(\boldsymbol{x},t)a(0,0):\!|\Psi_0\rangle\,\langle 0|\mathcal{T}\,a(\boldsymbol{x},t)a^\dagger(0,0)|0\rangle \\
&+\langle\Psi_0|\!:a^\dagger(0,0)a(\boldsymbol{x},t):\!|\Psi_0\rangle\,\langle 0|\mathcal{T}\,a(-\boldsymbol{x},-t)a^\dagger(0,0)|0\rangle\,.
\end{aligned}\tag{28}$$

The normal ordered operator factor in the second term is

$$\begin{aligned}
\langle\Psi_0|\!:a^\dagger(\boldsymbol{x},t)a(0,0):\!|\Psi_0\rangle &= \frac{1}{V}\sum_{\boldsymbol{k},\boldsymbol{k}'}\mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}\mathrm{e}^{\mathrm{i}\omega_{\boldsymbol{k}}t}\,\langle\Psi_0|a^\dagger_{\boldsymbol{k}}a_{\boldsymbol{k}'}|\Psi_0\rangle \\
&= \frac{1}{V}\sum_{\boldsymbol{k},\boldsymbol{k}'}\mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}\mathrm{e}^{\mathrm{i}\omega_{\boldsymbol{k}}t}N\delta_{\boldsymbol{k},0}\delta_{\boldsymbol{k}',0} = \frac{N}{V},
\end{aligned}$$

and similarly the normal ordered operator factor in the third term is N/V. The first term is

$$\begin{aligned}
&\langle\Psi_0|\!:a^\dagger(\boldsymbol{x},t)a(\boldsymbol{x},t)a^\dagger(0,0)a(0,0):\!|\Psi_0\rangle \\
&= \frac{1}{V^2}\sum_{\boldsymbol{k}_1,\boldsymbol{k}_2,\boldsymbol{k}_3,\boldsymbol{k}_4}\mathrm{e}^{-\mathrm{i}\boldsymbol{k}_1\cdot\boldsymbol{x}+\mathrm{i}\omega_{\boldsymbol{k}_1}t}\mathrm{e}^{\mathrm{i}\boldsymbol{k}_2\cdot\boldsymbol{x}-\mathrm{i}\omega_{\boldsymbol{k}_2}t}\,\langle\Psi_0|a^\dagger_{\boldsymbol{k}_1}a^\dagger_{\boldsymbol{k}_3}a_{\boldsymbol{k}_2}a_{\boldsymbol{k}_4}|\Psi_0\rangle \\
&= \frac{1}{V^2}\sum_{\boldsymbol{k}_1,\boldsymbol{k}_2,\boldsymbol{k}_3,\boldsymbol{k}_4}\mathrm{e}^{-\mathrm{i}\boldsymbol{k}_1\cdot\boldsymbol{x}+\mathrm{i}\omega_{\boldsymbol{k}_1}t}\mathrm{e}^{\mathrm{i}\boldsymbol{k}_2\cdot\boldsymbol{x}-\mathrm{i}\omega_{\boldsymbol{k}_2}t}\delta_{\boldsymbol{k}_1,0}\delta_{\boldsymbol{k}_2,0}\delta_{\boldsymbol{k}_3,0}\delta_{\boldsymbol{k}_4,0}N(N-1) \\
&= \frac{N(N-1)}{V^2}.
\end{aligned}$$

So the final result is

$$\begin{aligned}
&\langle\Psi_0|\mathcal{T}\,\rho(\boldsymbol{x},t)a(\boldsymbol{x},t)\rho(0,0)|\Psi_0\rangle \\
&= \frac{N(N-1)}{V^2} + \frac{N}{V}\theta(t)\int\frac{\mathrm{d}^3\boldsymbol{k}}{(2\pi)^3}\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}-\mathrm{i}\frac{\boldsymbol{k}^2}{2m}t} + \frac{N}{V}\theta(-t)\int\frac{\mathrm{d}^3\boldsymbol{k}}{(2\pi)^3}\mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}+\mathrm{i}\frac{\boldsymbol{k}^2}{2m}t}.
\end{aligned}\tag{29}$$

3. We have
$$\langle 0|\mathcal{T}\,\rho(\boldsymbol{x},t)\rho(0,0)|0\rangle = \rho_0^2\,\langle 0|\mathcal{T}\,\mathrm{e}^{\mathrm{i}\theta(\boldsymbol{x},t)}|0\rangle \tag{30}$$

4. The second and third terms of (29) are all rapidly oscillating in the same way we see in (25). So we have
$$\langle\Psi_0|\mathcal{T}\,\rho(\boldsymbol{x},t)a(\boldsymbol{x},t)\rho(0,0)|\Psi_0\rangle \overset{|\boldsymbol{x}|\to\infty}{\longrightarrow} \rho_0^2. \tag{31}$$