Homework 4

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Exercise 3 in lecture 12

Solution The normalization factor in the time domain is

$$\int_{-\infty}^{\infty} |g(t)|^2 \, \mathrm{d}t = \sigma\sqrt{\pi}. \tag{1}$$

The expectation values are

$$\langle t^2 \rangle = \frac{\int_{-\infty}^{\infty} \mathrm{d}t \, |g(t)|^2 t^2}{\int_{-\infty}^{\infty} \mathrm{d}t \, |g(t)|^2} = t_0^2 + \frac{1}{2}\sigma^2, \tag{2}$$

and

$$\langle t \rangle = \frac{\int_{-\infty}^{\infty} \mathrm{d}t \, |g(t)|^2 t}{\int_{-\infty}^{\infty} \mathrm{d}t \, |g(t)|^2} = t_0. \tag{3}$$

The frequency domain version of g is

$$g[\omega] = \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}\omega t} g(t) = \sigma\sqrt{2\pi} \mathrm{e}^{\mathrm{i}(\omega-\omega_0)t - \frac{1}{2}\sigma^2(\omega-\omega_0)^2}. \tag{4}$$

The normalization factor is

$$\int_{-\infty}^{\infty} |g[\omega]|^2 \, \mathrm{d}\omega = 2\pi\sigma^2 \cdot \sqrt{\frac{\pi}{\sigma^2}} = 2\pi^{3/2}\sigma. \tag{5}$$

The expectation values are

$$\langle \omega^2 \rangle = \frac{\int_{-\infty}^{\infty} \mathrm{d}\omega \, |g[\omega]|^2 \omega^2}{\int_{-\infty}^{\infty} \mathrm{d}\omega \, |g[\omega]|^2} = \frac{1}{2\sigma^2} + \omega_0^2, \tag{6}$$

and

$$\langle \omega \rangle = \omega_0. \tag{7}$$

So

$$\begin{aligned} \sigma_t \sigma_\omega &= \sqrt{\langle t^2 \rangle - \langle t \rangle^2} \sqrt{\langle \omega^2 \rangle - \langle \omega \rangle^2} \\ &= \sqrt{\frac{1}{2}\sigma^2 \cdot \frac{1}{2\sigma^2}} = \frac{1}{2}. \end{aligned} \tag{8}$$

So indeed for $g(t)$, $\sigma_t\sigma_\omega$ reaches its minimum value.

Exercise 1 in lecture 13 We have the following transmission model: the sent signal X has 3 possible values $x = 0, \pm 1$ with $p_0 = p_1 = p_{-1} = 1/3$. The noise Ξ has two possible values $\zeta = \pm 1$. We suppose the channel function F has the following expression $Y = \mathrm{Im}\left[\mathrm{e}^{i2\pi(X+\Xi)/}\right]$. What are all the possible values of Y ? Give the expression of the distributions $w(x), w(y), w(x, y)$ and $w(y \mid x)$.

Solution The possible results are listed in Table 1. The possible values of Y are 0 and $\pm\sqrt{3}/2$. From Table 1 we have

$$w(x) = \begin{cases} \frac{1}{3}, & x = 0, \\ \frac{1}{3}, & x = 1, \\ \frac{1}{3}, & x = -1, \end{cases} \tag{9}$$

$$w(y) = \begin{cases} \frac{1}{3}, & y = 0, \\ \frac{1}{3}, & y = \sqrt{3}/2, \\ \frac{1}{3}, & y = -\sqrt{3}/2, \end{cases} \tag{10}$$

Table 1: Probabilistic distribution of Y; the probability of each row is 1/6.

X	Ξ	$Y = \operatorname{Im} e^{i2\pi(X+\Xi)/3}$
0	1	$\frac{\sqrt{3}}{2}$
	-1	$-\frac{\sqrt{3}}{2}$
1	1	$-\frac{\sqrt{3}}{2}$
	-1	0
-1	1	0
	-1	$\frac{\sqrt{3}}{2}$

and

$$w(x,y) = \begin{cases} \frac{1}{6}, & (x,y) = (0, \sqrt{3}/2), \\ \frac{1}{6}, & (x,y) = (0, -\sqrt{3}/2), \\ \frac{1}{6}, & (x,y) = (1, -\sqrt{3}/2), \\ \frac{1}{6}, & (x,y) = (1, 0), \\ \frac{1}{6}, & (x,y) = (-1, 0), \\ \frac{1}{6}, & (x,y) = (-1, \sqrt{3}/2), \\ 0, & \text{otherwise}. \end{cases} \tag{11}$$

So

$$w(y|x) = \frac{w(x,y)}{w(x)} = \begin{cases} \frac{1}{2}, & (x,y) = (0, \sqrt{3}/2), \\ \frac{1}{2}, & (x,y) = (0, -\sqrt{3}/2), \\ \frac{1}{2}, & (x,y) = (1, -\sqrt{3}/2), \\ \frac{1}{2}, & (x,y) = (1, 0), \\ \frac{1}{2}, & (x,y) = (-1, 0), \\ \frac{1}{2}, & (x,y) = (-1, \sqrt{3}/2), \\ 0, & \text{otherwise}. \end{cases} \tag{12}$$

Exercise 2 in lecture 14

Solution Suppose p is the probability of $X = -1$, and we have

$$m = \langle X \rangle = -p + (1-p) = 1 - 2p, \quad -1 \le m \le 1, \tag{13}$$

and therefore

$$p = \frac{1-m}{2}, \tag{14}$$

and

$$\begin{aligned} H_\alpha(X) &= \frac{1}{1-\alpha} \log_2(p^\alpha + (1-p)^\alpha) \\ &= \frac{1}{1-\alpha} \log_2 \left(\left(\frac{1-m}{2} \right)^\alpha + \left(\frac{1+m}{2} \right)^\alpha \right). \end{aligned} \tag{15}$$

The plots can be found in Figure 1.

Exercise 4 in lecture 14

Solution We have

$$\begin{aligned} H(Y|X) &= \sum_x H(Y, X = x) p(x) \\ &= -\sum_x p(x) \sum_y p(y|x) \log_2 p(y|x) \\ &= -\sum_{x,y} p(x,y) \log_2 p(y|x), \end{aligned} \tag{16}$$

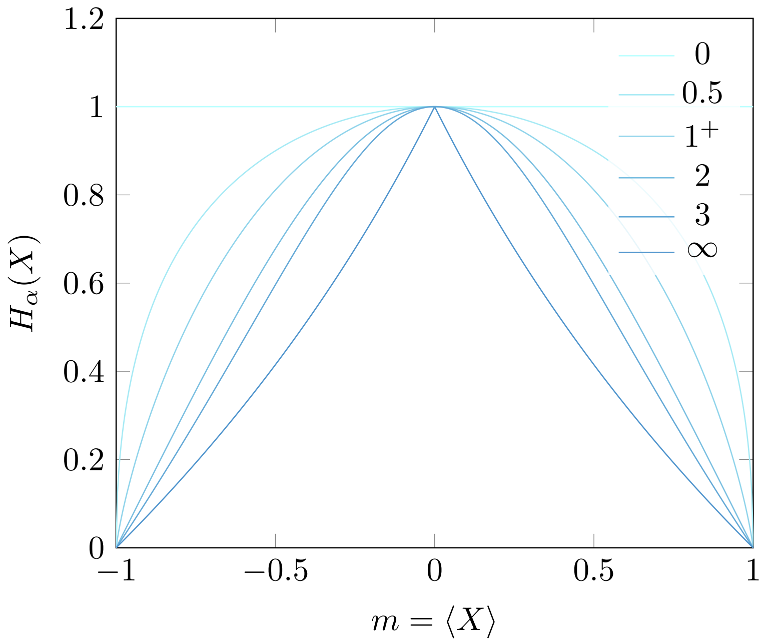

Figure 1: Plots of $H_\alpha(X)$ with different α's

and a direct verification gives

$$
\begin{aligned}
H(X,Y) &= -\sum_{x,y} p(x,y) \log_2 \underbrace{p(x,y)}_{p(x)p(y|x)} \\
&= -\sum_{x,y} p(x,y)(\log_2 p(x) + \log_2 p(y|x)) \\
&= -\sum_{x} p(x) \log_2 p(x) - \sum_{x,y} p(x,y) \log_2 p(y|x) \\
&= H(X) + H(Y|X).
\end{aligned}
\tag{17}
$$

Problem Differences between the classical and quantum entropies
Solution

(a) This can be done by Schmidt decomposition. Suppose $\dim \mathcal{H}_\mathrm{A} = m$, $\dim \mathcal{H}_\mathrm{B} = n$, and without loss of generality $m \geq n$, then we can already find a set of basis, in which

$$
|\psi_\mathrm{AB}\rangle = \sum_{i=1}^{m} c_i \, |u_\mathrm{A}^i\rangle \otimes |v_\mathrm{B}^i\rangle \,. \tag{18}
$$

We have

$$
\begin{aligned}
\rho_\mathrm{A} &= \sum_{j=1}^{m} \langle v_\mathrm{B}^j|\psi_\mathrm{AB}\rangle \, \langle\psi_\mathrm{AB}|v_\mathrm{B}^j\rangle \\
&= \sum_{j=1}^{m} c_j \, |u_\mathrm{A}^j\rangle \, c_j^* \, \langle u_\mathrm{A}^j| \\
&= \sum_{j=1}^{m} |c_j|^2 \, |u_\mathrm{A}^j\rangle\langle u_\mathrm{A}^j| \,,
\end{aligned}
\tag{19}
$$

and similarly (note that $\{\,|v_\mathrm{B}^j\rangle\}_{j=1}^m$ appearing in the expansion of $|\psi_\mathrm{AB}\rangle$ is over-complete, and the form of ρ_B is exactly the same as ρ_A)

$$
\rho_\mathrm{B} = \sum_{i=1}^{m} |c_i|^2 \, |v_\mathrm{B}^i\rangle\langle v_\mathrm{B}^i| \,. \tag{20}
$$

So we find

$$S(\rho_{\rm A}) = S(\rho_{\rm B}) = -\sum_{j=1}^{m} |c_i|^2 \ln |c_i|^2. \tag{21}$$

Note that here

$$S(\rho_{\rm AB}) = 0, \tag{22}$$

because we are dealing with a pure state. So here the entropy of a system is less than any of its parts, and this is not possible in the classical case.

(b) Let

$$\rho = \rho_{\rm AB}, \quad \sigma = \rho_{\rm A} \otimes \rho_{\rm B},$$

and from

$$\mathrm{tr}(\rho \ln \rho) - \mathrm{tr}(\rho \ln \sigma) \geq 0,$$

we have

$$\begin{aligned} 0 &\leq \mathrm{tr}(\rho_{\rm AB} \ln \rho_{\rm AB}) - \mathrm{tr}(\rho_{\rm AB} \underbrace{\ln \rho_{\rm A} \otimes \rho_{\rm B}}_{\ln \rho_{\rm A} + \ln \rho_{\rm B}}) \\ &= -S(\rho_{\rm AB}) - \mathrm{tr}_{\rm A}\, \mathrm{tr}_{\rm B}\, \rho_{\rm AB} \ln \rho_{\rm A} - \mathrm{tr}_{\rm B}\, \mathrm{tr}_{\rm A}\, \rho_{\rm AB} \ln \rho_{\rm B} \\ &= -S(\rho_{\rm AB}) + S(\rho_{\rm A}) + S(\rho_{\rm B}), \end{aligned}$$

and thus

$$S(\rho_{\rm A}) + S(\rho_{\rm B}) - S(\rho_{\rm AB}) \geq 0. \tag{23}$$

(c) What we need to prove

$$S(\rho_{\rm AB}) \geq S(\rho_{\rm B}) - S(\rho_{\rm A}). \tag{24}$$

Assume the density matrix $\rho_{\rm AB}$ is actually the partial trace of the pure state $|\Psi_{\rm ABC}\rangle$. (23) means

$$S(\rho_{\rm B}) + S(\rho_{\rm C}) \geq S(\rho_{\rm BC}),$$

and from (a), we have

$$S(\rho_{\rm C}) = S(\rho_{\rm AB}), \quad S(\rho_{\rm BC}) = S(\rho_{\rm A}),$$

and therefore

$$\begin{aligned} S(\rho_{\rm B}) + S(\rho_{\rm AB}) &\geq S(\rho_{\rm A}), \\ S(\rho_{\rm AB}) &\geq S(\rho_{\rm A}) - S(\rho_{\rm B}). \end{aligned} \tag{25}$$

Similarly

$$S(\rho_{\rm AB}) \geq S(\rho_{\rm B}) - S(\rho_{\rm A}). \tag{26}$$

(d) We have

$$p_j = \langle j|\rho_{\rm B}|j\rangle\,, \tag{27}$$

and therefore

$$\rho_{{\rm AB}|\Pi_{\rm B}^j} = \frac{1}{p_j} \mathrm{tr}_{\rm B}\, |j\rangle\langle j|\, \langle j|\rho_{\rm AB}|j\rangle = \frac{1}{p_j} \langle j|\rho_{\rm AB}|j\rangle\,. \tag{28}$$

So

$$\begin{aligned} \sum_j p_j S\left(\rho_{{\rm AB}|\Pi_{\rm B}^j}\right) &= -\,\mathrm{tr}_{\rm A} \sum_j \langle j|\rho_{\rm AB}|j\rangle \ln \frac{\langle j|\rho_{\rm AB}|j\rangle}{p_j} \\ &= -\,\mathrm{tr}_{\rm A} \sum_j \langle j|\rho_{\rm AB}|j\rangle \ln \langle j|\rho_{\rm AB}|j\rangle + \sum_j \langle j|\rho_{\rm B}|j\rangle \ln \langle j|\rho_{\rm B}|j\rangle \\ &= S(\rho_{\rm AB}^{\rm D}) - S(\rho_{\rm B}^{\rm D}). \end{aligned} \tag{29}$$

The inequality to be proven is equivalent to

$$S(\rho_{\rm A}) - S(\rho_{\rm A}|\rho_{\rm B}) \leq S(\rho_{\rm A}) - H(\rho_{\rm A}|\rho_{\rm B}),$$

which further is equivalent to

$$S(\rho_{\rm A}|\rho_{\rm B}) = \min_{\{\Pi_{\rm B}^j\}} \sum_j p_j S\left(\rho_{{\rm AB}|\Pi_{\rm B}^j}\right) \geq H(\rho_{\rm A}|\rho_{\rm B}). \tag{30}$$

(29) and

$$S(\rho_{\rm AB}^{\rm D}) - S(\rho_{\rm B}^{\rm D}) \geq H(\rho_{\rm A}|\rho_{\rm B}) \tag{31}$$

show that for any $\{\Pi_{\rm B}^j\}$, the inequality is correct, so we have already finished the proof.

(e) The disagreements in the definition of conditional entropy intuitively arises from the entanglement between the two systems, so we expect when

$$\rho_{\mathrm{AB}} = \rho_{\mathrm{A}} \otimes \rho_{\mathrm{B}}, \tag{32}$$

there is no inconsistency in the definition of conditional entropy, which is the classical case. This can be verified directly, because when (32) is correct, we have

$$\rho_{\mathrm{AB}}^{\mathrm{D}} = \rho_{\mathrm{AB}}, \quad \rho_{\mathrm{B}}^{\mathrm{D}} = \rho_{\mathrm{B}},$$

and therefore according to (29), we have

$$S(\rho_{\mathrm{A}}|\rho_{\mathrm{B}}) = S(\rho_{\mathrm{AB}}) - S(\rho_{\mathrm{B}}), \tag{33}$$

and

$$J(\rho_{\mathrm{A}} : \rho_{\mathrm{B}}) = S(\rho_{\mathrm{A}}) + S(\rho_{\mathrm{B}}) - S(\rho_{\mathrm{AB}}) =: I(\rho_{\mathrm{A}} : \rho_{\mathrm{B}}). \tag{34}$$

(f) In a pure state, we have

$$I(\rho_{\mathrm{A}}|\rho_{\mathrm{B}}) = S(\rho_{\mathrm{A}}) + S_{\rho_{\mathrm{B}}} - S_{\rho_{\mathrm{AB}}} = 2S(\rho_{\mathrm{A}}). \tag{35}$$

On the other hand, we have

$$\rho_{\mathrm{AB}|\Pi_{\mathrm{B}}^{j}} = \frac{1}{p_j} \langle j|\psi_{\mathrm{AB}}\rangle \, \langle \psi_{\mathrm{AB}}|j\rangle \,. \tag{36}$$

So now we can use Schmidt decomposition again: We have (18), and therefore (here $|j\rangle$ means a basis vector for B, and $|j\rangle_{\mathrm{A}}$ is used to refer to a basis vector for A)

$$\langle j|\psi_{\mathrm{AB}}\rangle = c_i \, |j\rangle_{\mathrm{A}} \,,$$

so $\rho_{\mathrm{AB}|\Pi_{\mathrm{B}}^{j}}$ is actually a pure state, and therefore we have $S(\rho_{\mathrm{A}}|\rho_{\mathrm{B}}) = 0$, because each term in it is zero. So

$$J(\rho_{\mathrm{A}} : \rho_{\mathrm{B}}) = S(\rho_{\mathrm{A}}), \tag{37}$$

and thus

$$\delta(\rho_{\mathrm{A}} : \rho_{\mathrm{B}}) = 2S(\rho_{\mathrm{A}}) - S(\rho_{\mathrm{A}}) = S(\rho_{\mathrm{A}}) = S(\rho_{\mathrm{B}}). \tag{38}$$