Fluctuation

Jinyuan Wu

September 16, 2023

1 Motivation

1.1 Brownian motion in a fluid

The experimental motivation to study fluctuation is probably the Brownian motion: in this specific case, large particles in fluid feel the fluctuation of the fluid, and undergo random motion. There are three length scales and three time scales in a typical experimental setting:

- The molecule scales: 1×10^{-10} m, 1×10^{-12} s;
- The particle scales: typically 1×10^{-6} m, 1×10^{-3} s;
- The enclosure (i.e. container of the fluid) scales: at least 1×10^{-2} m, 1×10^{2} s.

The separation between the molecule scales and the particle scales justifies treating the fluid as a continuum, while the separation between the particle scales and the enclosure scales justifies treating the configuration space of the particles as an infinite one. There actually should be another scale: the mean distance between particles; if this is large enough, then particles are independent to each other; but we may want to consider this as the enclosure scales.

When the aforementioned conditions of separation of scales are all true, the behavior of a particle independent to others is described by the **Langevin equation**

$$M \frac{\mathrm{d}^2 X}{\mathrm{d}t^2} + \frac{\partial U}{\partial X} = -c_{\mathrm{F}} \dot{X} + F_{\mathrm{N}}(t), \tag{1}$$

the terms on the LHS being effects about the particle itself, the terms on the RHS being the deterministic damping term and the stochastic Langevin force, accordingly.

Solving (1), therefore, involves two steps: finding reliable descriptions of c_{F} and F_{N}, and solving (1) as a stochastic differential equation. The first problem is essentially about establishing a theory of fluctuation (F_{N}) and response (γ) in the fluid surrounding the particle, and the second problem is about the mathematical treatment of stochastic processes.

The two problems are actually two specific cases of the underlying quantum non-equilibrium theory. [3] contains a neat review of it; the most generic formalism however is not necessary in most of the cases, and this note would rather focus on more concrete cases. Also, phenomena described by the Langevin equation are also described by various master equations, classical or quantum; the two formalisms are also equivalent to each other [4, 2]. The details however are also emitted.

1.2 Quantum circuit

The formalism of Langevin equation is not limited to soft condensed matter systems: since formally Newton's second law is analogous to circuit theory, fluctuation and damping in a circuit immersed in a sea of smaller circuits can also be described in Langevin's formalism. Indeed this is how resistance comes into being In circuit analysis we have another problem: now the Langevin equation should be *quantum*, since although no one would perform a Brownian motion experiment in a low temperature quantum liquid, we indeed can perform quantum circuit experiments at a very low temperature.

Ref.

Commutation relations?

2 Correlation, response, and fluctuation-dissipation theorem

In this section we briefly review the linear response theory. In general, a driven Hamiltonian assumes the following form:

$$H = H_0(x_1, \ldots, x_n, p_1, \ldots, p_n) - A(x_1, \ldots, x_n, p_1, \ldots, p_n) F_A(t), \tag{2}$$

where we have assumed that degrees of freedom in the system follow the canonical commutation relation; this assumption of course is frequently broken (as in, say, a magnetic spin model), but in this note let's focus on the most familiar case. We can readily have two examples: the first is

$$H = \frac{p^2}{2m} + \frac{1}{2} k x^2 - x F(t), \tag{3}$$

of which the EOM is

$$\dot{p} = -kx + F(t), \quad \dot{x} = \frac{p}{m}, \tag{4}$$

as is expected. The second is

$$H = \frac{\Phi^2}{2L} + \frac{Q^2}{2L} - QV, \tag{5}$$

Here we intentionally leave the discussion of damping; it can't be directly treated in the Hamiltonian formalism and will be modeled by an infinite bath of other degrees of freedom. Suppose we have another physical quantity B which is also a polynomial of $x_1, \ldots, x_n$ and $p_1, \ldots, p_n$. When the external driving force $F(t)$ is applied, we get

Commutation relations: cQED?

$$\delta B\,(t) \coloneqq B(t) - B_0 = \int_{-\infty}^{\infty} \mathrm{d}t_1 \, \chi_{BA}(t, t_1) F_A(t_1) + B_{\text{noise}} + \mathcal{O}\,(F_A^2). \tag{6}$$

The first term equals to the linear term in $\langle \delta B\,(t) \rangle$; the noise term contains information about correlation of B with other variables – or maybe itself. It can be statistical noise (noise in classical probability theory) when there is a bath (see below); but even if we are dealing with a pure state theory, $B_{\text{noise}} = B - \langle B \rangle$ will still bring some "quantum noise". The response function is

$$\chi_{BA}(t, t_1) = \chi_{BA}(t - t_1) \tag{7}$$

when H_0 is time-independent.

Now we turn to another aspect: correlation. We define

$$S_{BA}(\omega) = \int \mathrm{e}^{\mathrm{i}\omega t} \langle B(0) A(t) \rangle \, \mathrm{d}t\,. \tag{8}$$

This is known as the greater Green function or the lesser Green function when A and B are field operators. We now state the following **fluctuation-dissipation theorem**, which links the near-equilibrium lesser Green function and the response function (i.e. the retarded function): for a classical system, we have

$$\langle \dot{A}(0) B(t) \rangle \, \theta(t) = k_{\mathrm{B}} T \chi_{BA}(t), \tag{9}$$

and in the frequency domain, if we assume that the response function is even under time reversal operation, we get

$$S_{\dot{A}B}(\omega) = 2 k_{\mathrm{B}} T \operatorname{Re} \chi_{BA}(\omega). \tag{10}$$

and the equation becomes

$$S_{AB}(\omega) = \frac{2 k_{\mathrm{B}} T}{\omega} \operatorname{Im} \chi_{AB}(\omega). \tag{11}$$

In the quantum case we have

Details about time reversal; the point is how to get rid of $\theta(t)$

$$S_{AB}(\omega) = \frac{2\hbar}{1 - \mathrm{e}^{-\hbar\omega / k_{\mathrm{B}} T}} \operatorname{Im} \chi_{AB}(\omega). \tag{12}$$

To save space the proof of the theorem is not shown here; a straightforward derivation can be found in section 9.4 in [1]. Note that in the $T \to 0$ limit, classically S_{AB} vanishes, but in the quantum case we still have a non-vanishing remaining correlation as long as we see a response of A to B which is known as quantum noise: if A responds to the $-Bf(t)$ term in the Hamiltonian, A and B don't commute (or otherwise we don't have non-trivial time evolution), and thus S_{AB} is definitely not zero when $T \to 0$ i.e. when in the pure state theory.

3 Analysis of common noises

Topic: what does noise do to a signal?

4 Langevin equation revisited

4.1 Correlation function of the noise

In the light of fluctuation-dissipation theorem, we see that although the details of $F_{\mathrm{N}}(t)$ and c_{F} are still largely unknown, if $F_{\mathrm{N}}(t)$ and c_{F} "naturally" appear by coupling with the bosonic bath, the properties of $F_{\mathrm{N}}(t)$ and c_{F} in

$$M\ddot{X} + c_{\mathrm{F}}\dot{X} + \frac{\partial U}{\partial X} = F_{\mathrm{N}}(t) \tag{13}$$

have to follow the fluctuation-dissipation theorem. We know the force applied by the fluid to the particle is

$$F_{\mathrm{f}} = -kv + F_{\mathrm{N}}(t), \tag{14}$$

and the coupling Hamiltonian between the particle and the fluid is

$$H_{\mathrm{int}} = -xF_{\mathrm{f}}(t). \tag{15}$$

Thus we choose A and B to be both F_{f}, and fluctuation-dissipation theorem tells us

$$\langle \dot{F}(t')F(t)\rangle\, \theta(t-t') = k_{\mathrm{B}}T\chi_{F_{\mathrm{f}}(t')\to F_{\mathrm{f}}(t)}(t-t'). \tag{16}$$

Here we have gone back to the double-time formalism since we want to eventually get rid of the time derivative acting on the first F in the correlation function and writing the time of the first F makes the derivation less confusing. Also, note that we are applying the fluctuation-dissipation theorem *to the fluid* (the particle doesn't have linear response anyway) and therefore now x should be regarded as the external field. Given the form of the interaction Hamiltonian, we find

$$\chi_{F_{\mathrm{f}}(t')\to F_{\mathrm{f}}(t)}(t-t') = \frac{\delta F_{\mathrm{f}}(t)}{\delta x(t')}, \tag{17}$$

and therefore we want to rephrase the proportional relation between F_{f} and v as a functional between $F_{\mathrm{f}}(t)$ and $x(t')$. This is done by the follows:

$$F_{\mathrm{f}}(t) = -c_{\mathrm{F}}v = -\int \mathrm{d}t'\, c_{\mathrm{F}}\delta(t-t')\frac{\mathrm{d}x}{\mathrm{d}t'} = \int \mathrm{d}t'\, x(t')c_{\mathrm{F}}\frac{\mathrm{d}}{\mathrm{d}t'}\delta(t-t'). \tag{18}$$

The definition of the $\delta(t-t')$ function however involves a subtlety: in reality, t should always be slightly larger than t' or otherwise causality is broken, and $\delta(t-t')$ therefore should be understood as $\delta(t-t'-0^+)$. The fluctuation-dissipation theorem now reads

$$\langle \dot{F}(t')F(t)\rangle\, \theta(t-t') = k_{\mathrm{B}}T\chi_{F_{\mathrm{f}}(t')\to F_{\mathrm{f}}(t)}(t-t') = k_{\mathrm{B}}T\cdot c_{\mathrm{F}}\frac{\mathrm{d}}{\mathrm{d}t'}\delta(t-t'-0^+). \tag{19}$$

In other words, we have (note that in the equation below, there should be no infinitesimal time shift in δ and stepwise functions)

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t'}(\,\langle F(t')F(t)\rangle\,\theta(t-t')) &= \langle \dot{F}(t')F(t)\rangle\,\theta(t-t') - \langle F(t')F(t)\rangle\,\delta(t-t') \\ &= k_{\mathrm{B}}Tc_{\mathrm{F}}\frac{\mathrm{d}}{\mathrm{d}t'}\delta(t-t') - \langle F(t')F(t)\rangle\,\delta(t-t'),\end{aligned} \tag{20}$$

and we can then integrate over the equation on the both sides over t' and get

$$\langle F(t')F(t)\rangle\,\theta(t-t') = k_{\mathrm{B}}Tc_{\mathrm{F}}\delta(t-t'), \tag{21}$$

where since the δ function in $\langle F(t')F(t)\rangle$ contains an infinitesimal positive shift, when t is *exactly* t' it should be zero. (Or, to be more rigorous, we can always replace $\delta(t-t'-0^+)$ by a smooth function that vanishes when $t<t'$, and we can check that the equations above are all correct.)

The correlation function in fluctuation-dissipation theorem is evaluated when the fluid is at rest, and therefore $F_{\mathrm{f}}(t) = F_{\mathrm{N}}(t)$, and we find

$$\langle F_{\mathrm{N}}(t')F_{\mathrm{N}}(t)\rangle\, \theta(t-t') = k_{\mathrm{B}}Tc_{\mathrm{F}}\delta(t-t'). \tag{22}$$

The $\delta(t-t')$ function in (22) is double-sided: it looks like a Gaussian peak, and therefore

$$\theta(t-t')\delta(t-t') = \frac{1}{2}\delta(t-t'), \tag{23}$$

and eventually, the correlation function of F_{N} reads

$$\langle F_{\mathrm{N}}(0)F_{\mathrm{N}}(t)\rangle = 2k_{\mathrm{B}}Tc_{\mathrm{F}}\delta(t). \tag{24}$$

In conclusion, if we are sure that the damping force from the fluid is in the form of $F = -kv$, then the EOM of a particle in the fluid always takes the form of

$$M\ddot{X} + c_{\mathrm{F}}\dot{X} + \frac{\partial U}{\partial X} = F_{\mathrm{N}}(t), \quad \langle F_{\mathrm{N}}(0)F_{\mathrm{N}}(t)\rangle = 2k_{\mathrm{B}}Tc_{\mathrm{F}}\delta(t). \tag{25}$$

Thanks to the fluctuation-dissipation theorem, we now find that the statistical properties of the noise is related to the damping coefficient.

4.2 An implementation: fluid as a number of harmonic oscillators

In principle higher order non-trivial correlations are possible; but in most of the systems that are experimentally feasible, the environment can be very effectively modeled as a heat bath containing lots of harmonic oscillators. In this case $F_{\mathrm{N}}(t)$ should be a Gaussian noise. Thus, the smooth, realistic form of the double-sided $\delta(t)$ is something like

$$\frac{1}{\tau}\mathrm{e}^{-|t|/\tau}. \tag{26}$$

In this section we explicitly show how this can be done. If the fluid can be modeled as a pool of harmonic oscillators, the EOMs are

$$M\ddot{X} = -\frac{\partial U}{\partial X} + \sum_i k_i(x_i - X), \quad m_i\ddot{x}_i = -k_i(x_i - X) \Rightarrow \ddot{x}_i + \omega^2 x_i = \omega^2 X, \tag{27}$$

where

$$\omega_i = \sqrt{\frac{k_i}{m_i}}. \tag{28}$$

We can directly verify that

$$x_i(t) = a_i \sin(\omega_i t + \phi_i) + \int_{-\infty}^{t} \mathrm{d}\tau \cos(\omega_i(t-\tau))\dot{X}(\tau) + X(\tau), \tag{29}$$

and therefore the effective EOM for the particle is

$$M\ddot{X} = -\frac{\partial U}{\partial X} + \int_{-\infty}^{t} \underbrace{\sum_i k_i \cos(\omega_i(t-\tau))}_{K(t-\tau)}\dot{X}(\tau) + \underbrace{\sum_i k_i a_i \sin(\omega_i t + \phi_i)}_{F_{\mathrm{N}}(t)}. \tag{30}$$

Now we assume $\{x_i\}$ to be in thermal equilibrium; this is somehow a self-conflicting assumption since we have assumed no interaction between the oscillators and they simply won't thermalize; but we can always assume that the oscillators are coupled with some external reservoir and hence are thermalized, or that there is weak interaction among them, which is weak enough to have no strong correction to the energy but strong enough to lead to thermalization. The energy of each oscillator is

$$E = \frac{p_i^2}{2m_i} + \frac{1}{2}m_i\omega_i^2 x_i^2 = m_i\omega_i^2 a_i^2, \tag{31}$$

and from the equipartition theorem we find

$$\langle E\rangle = k_{\mathrm{B}}T \Rightarrow \langle a_i^2\rangle = \frac{2k_{\mathrm{B}}T}{m_i\omega_i^2}, \tag{32}$$

while

$$\langle a_i a_j\rangle = 0 \quad i \neq j. \tag{33}$$

We thus find

$$\begin{aligned}
\langle F_{\mathrm{N}}(0)F_{\mathrm{N}}(t)\rangle &= \sum_{i,j} \langle a_i a_j\rangle\, k_i k_j \sin\phi_i \sin(\omega_j t + \phi_j) \\
&= \sum_i \frac{2k_{\mathrm{B}}T}{m_i\omega_i^2} k_i^2 \sin\phi_i \sin(\omega_i t + \phi_i) \\
&= k_{\mathrm{B}}T \sum_i 2k_i \sin\phi_i \sin(\omega_i t + \phi_i) \\
&= k_{\mathrm{B}}T \sum_i k_i(\cos(\omega_i t) - \cos(\omega_i t + 2\phi_i)) \\
&\approx k_{\mathrm{B}}T \sum_i k_i \cos(\omega_i t),
\end{aligned} \tag{34}$$

where in the last line we have assumed that there are so many oscillators and their phases are random, so the second term in the last second line vanishes. Thus we find

$$\langle F_{\mathrm{N}}(0)F_{\mathrm{N}}(t)\rangle = k_{\mathrm{B}}T K(t). \tag{35}$$

Now we get

5 Fokker-Planck equation

The Langevin equation in the last section is a specific case of the following SDE

$$\dot{X} = AW + F, \quad \langle F_i(0)F_j(t)\rangle = B_{ij}\delta(t), \tag{36}$$

the time evolution of the probabilistic distribution $W(X_1, X_2, \ldots, X_n)$ can be rewritten into

$$\frac{\partial W}{\partial t} = \underbrace{-\boldsymbol{\nabla}\cdot AW}_{\text{drift}} + \underbrace{\frac{1}{2}\boldsymbol{\nabla}\cdot\boldsymbol{\nabla}\cdot(BW)}_{\text{diffusion}}. \tag{37}$$

This equation is known as the **Fokker-Planck equation**.

The classical tunneling rate is

derivation

$$\Gamma = \frac{\omega_0}{2\pi}\mathrm{e}^{-\Delta U/k_{\mathrm{B}}T} \tag{38}$$

6 Transmission line and resistance

We can also study the infinite transmission line, the counterpart of the infinite harmonic oscillator array attached to the particle, from the perspective of linear response – and show why it looks like a resistor. This time we proceed in the frequency domain, and we will see it makes many things simpler.

$$Y_m(\omega) = \left(-\mathrm{i}\omega L_m + \frac{1}{-\mathrm{i}\omega C}\right)^{-1} = \mathrm{i}\frac{y_m\omega_m\omega}{\omega^2 - \omega_m^2}. \tag{39}$$

where

$$\omega_m = \frac{1}{\sqrt{L_m C_m}}, \quad y_m = \sqrt{\frac{C_m}{L_m}} \tag{40}$$

This is supposed to be a response function, so let's add a little imaginary part to ω so that all poles are on the lower plane, and we get

$$Y_m(\omega) = \left(-\mathrm{i}\omega L_m + \frac{1}{-\mathrm{i}\omega C}\right)^{-1} = \mathrm{i}y_m\omega_m \frac{\omega}{(\omega + \mathrm{i}0^+)^2 - \omega_m^2} = \chi_{IQ}(\omega). \tag{41}$$

The real part of

On the other hand, the current going into the main circuit is

$$I(t) = \sum_m I_m(t), \tag{42}$$

and the

$$S_{II}(\omega) = 2k_{\mathrm{B}}T \,\mathrm{Re}\, \chi_{IQ}(\omega), \tag{43}$$

where $A = Q$ and $B = I$.

References

[1] Piers Coleman. *Introduction to many-body physics.* Cambridge University Press, 2015.

[2] Michele Coppola, Zoubair Daouma, and Malte Henkel. "From Lindblad master equations to Langevin dynamics and back". In: *arXiv preprint arXiv:2305.06312* (2023).

[3] Lukas M Sieberer, Michael Buchhold, and Sebastian Diehl. "Keldysh field theory for driven open quantum systems". In: *Reports on Progress in Physics* 79.9 (2016), p. 096001.

[4] NG Van Kampen and I Oppenheim. "Langevin and master equation in quantum mechanics". In: *Journal of statistical physics* 87 (1997), pp. 1325–1334.