Homework 4

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1 Casimir-Polder Force

1.1 Interaction potential between two harmonic oscillators

The classical polarizability of a harmonic oscillator is

$$\alpha = \frac{e^2}{\epsilon_0 k}. \tag{1}$$

Since $\omega_0^2 = k/m$, the effective interaction potential

$$V(R) = -\frac{1}{8}\frac{\hbar}{m^2\omega_0^3}\left(\frac{e^2}{2\pi\epsilon_0}\right)^2\frac{1}{R^6} \tag{2}$$

can be rewritten into

$$V(R) = -\frac{1}{32\pi^2}\hbar\omega_0\frac{\alpha^2}{R^6}. \tag{3}$$

1.2 An oscillator and a conducting wall

If we do angle average to the dipole interaction potential

$$\begin{aligned} V(r) &= -\frac{d_1 d_2}{4\pi\epsilon_0 r_{12}^3}\left(\cos\theta_{12} - 3\cos\theta_1\cos\theta_2\right) \\ &= -\frac{d^2}{4\pi\epsilon_0 r_{12}^3}(\cos 2\theta - 3\cos^2\theta) \\ &= -\frac{d^2}{4\pi\epsilon_0 r_{12}^3}\left(-\frac{3}{2} - \frac{1}{2}\cos 2\theta\right), \end{aligned} \tag{4}$$

since the $\cos 2\theta$ term vanishes, we get

$$\langle V(r)\rangle = \frac{3}{2}\frac{\langle d^2\rangle}{4\pi\epsilon_0 r_{12}^3}, \tag{5}$$

where

$$d = ex \tag{6}$$

is the dipole for one oscillator. This means in order to obtain the effective potential between the oscillator and the mirror, we just need to do the following substitution:

$$d_1 d_2 \longrightarrow -\frac{3}{2}e^2\langle x^2\rangle. \tag{7}$$

Ignoring the back action to the internal state of the oscillator, $\langle x^2\rangle$ can be evaluated using the property of a free oscillator:

$$\frac{1}{2}k\langle x^2\rangle = \frac{1}{2}\hbar\omega_0 \Rightarrow \langle d^2\rangle = e^2\frac{\hbar\omega_0}{k}. \tag{8}$$

1.3 Relativistic effect in field propagation

From the boundary condition, in a static field we have (note that the electric field is doubled because of the contribution of the image charge)

$$\frac{\sigma}{\epsilon_0} = \boldsymbol{n} \cdot \boldsymbol{E} \Rightarrow \frac{\sigma(r,\theta)}{\epsilon_0} = -2 \cdot \frac{e}{4\pi\epsilon_0(R^2+r^2)} \frac{R}{\sqrt{R^2+r^2}}, \tag{9}$$

$$\sigma(r) = -\frac{eR}{2\pi(R^2+r^2)^{3/2}}. \tag{10}$$

The total charge is

$$\int_0^\infty 2\pi r \,\mathrm{d}r\, \sigma(r) = -e \int_0^\infty \frac{r}{R}\frac{\mathrm{d}r}{R} \frac{R^3}{(R^2+r^2)^{3/2}} = -e \int_0^\infty \frac{\mathrm{d}x^2}{2} \frac{1}{(x^2+1)^{3/2}} = -e, \tag{11}$$

the expected result.

When R is large enough, it takes time for the electric field to propagate from the oscillator to the mirror and then back to the oscillator. The time cost is

$$T = \frac{2\sqrt{R^2+r^2}}{c}, \tag{12}$$

so at t, the oscillator sees the electric field it spread out at $t-T$; note that at different T values, the oscillator has different phases: the phase factor of $\boldsymbol{d}$ is $\mathrm{e}^{-\mathrm{i}\omega_0(t-T)}$. So finally the phase factor of the contribution of charges at r to the electric field felt by the oscillator at t is $\mathrm{e}^{\mathrm{i}2\omega_0 T}$. We may then take the real part (since we are working with linear electrodynamics) and get

$$\begin{aligned} V_\mathrm{d}(R) &= \int_0^\infty 2\pi r\,\mathrm{d}r\,\sigma(r)\cdot\cos(2\omega_0 T)\cdot e\varphi(r\to R) \\ &= \int_0^\infty 2\pi r\,\mathrm{d}r\,\sigma(r)\cdot\cos\Big(4k_0\sqrt{R^2+r^2}\Big)\frac{e}{4\pi\epsilon_0\sqrt{R^2+r^2}}. \end{aligned} \tag{13}$$

2 p to x matrix element

We know

$$[\boldsymbol{x}, H] = \mathrm{i}\hbar\frac{\boldsymbol{p}}{m}, \tag{14}$$

and therefore

$$\frac{\mathrm{i}\hbar}{m}\langle i|\boldsymbol{p}|j\rangle = \langle i|[\boldsymbol{x},H]|j\rangle = \langle i|\boldsymbol{x}E_j - E_i\boldsymbol{x}|j\rangle = (\hbar\omega_j - \hbar\omega_i)\langle i|\boldsymbol{x}|j\rangle,$$

and thus

$$\langle i|\boldsymbol{p}|j\rangle = \mathrm{i}m\underbrace{(\omega_i-\omega_j)}_{\omega_{ij}}\langle i|\boldsymbol{x}|j\rangle. \tag{15}$$

3 Sum rules

3.1 The x-closure sum rule

We have

$$\sum_k |\langle n|x_i|k\rangle|^2 = \langle n|\,x_i\sum_k|k\rangle\langle k|\,x_i\,|n\rangle = |\langle n|x_i^2|n\rangle|, \tag{16}$$

and therefore

$$\begin{aligned} \sum_k |\langle n|\boldsymbol{x}|k\rangle|^2 &= \sum_k\sum_i |\langle n|x_i|k\rangle|^2 \\ &= \sum_i |\langle n|x_i^2|n\rangle| = |\langle n|\boldsymbol{x}^2|n\rangle|. \end{aligned} \tag{17}$$

3.2 The Thomas-Reiche-Huhn (TRK) sum rule

From

$$\frac{\hbar}{\mathrm{i}} = \langle n|px - xp|n\rangle = \sum_k (\langle n|p|k\rangle \ \langle k|x|n\rangle - \ \langle n|x|k\rangle \ \langle k|p|n\rangle) \tag{18}$$

and

$$[H, x] = \frac{\hbar}{m\mathrm{i}} p, \tag{19}$$

we have

$$\begin{aligned}\frac{\hbar}{\mathrm{i}} &= \sum_k (\frac{m\mathrm{i}}{\hbar} \langle n|[H, x]\,|k\rangle \ \langle k|x|n\rangle - \ \langle n|x|k\rangle \cdot \frac{m\mathrm{i}}{\hbar} \langle k|[H, x]\,|n\rangle) \\ &= \frac{m\mathrm{i}}{\hbar} \sum_k (\langle n|E_n x - x E_k|k\rangle \ \langle k|x|n\rangle - \ \langle n|x|k\rangle \ \langle k|E_k x - x E_n|n\rangle) \\ &= \frac{m\mathrm{i}}{\hbar} \sum_k (2(E_n - E_k) \langle n|x|k\rangle \ \langle k|x|n\rangle),\end{aligned}$$

and therefore

$$\sum_k (E_k - E_n) | \langle n|x|k\rangle |^2 = \frac{\hbar^2}{2m}. \tag{20}$$

4 Hydrogen maser

The structure of the device is shown in Foot Fig. 6.4.

4.1 The initial state

There are four states: $F = 0, M_F = 0$, and $F = 1, M_F = 0, \pm 1$. The initial state is

$$\rho_0 = \frac{1}{Z}(|0,0\rangle\langle 0,0| + \mathrm{e}^{-\beta\Delta E}(|1,-1\rangle\langle 1,-1| + |1,0\rangle\langle 1,0| + |1,1\rangle\langle 1,1|)), \tag{21}$$

where

$$Z = 1 + 3\mathrm{e}^{-\beta\Delta E}, \tag{22}$$

$$\Delta E = \frac{2\mu_{\mathrm{B}} - g_p \mu_{\mathrm{N}}}{2\hbar} B. \tag{23}$$

When the external field is turned off we have

$$\rho_0 = \frac{1}{4}\,\mathrm{diag}(1,1,1,1). \tag{24}$$

The order of basis vectors is the order in the above discussion.

4.2 State selector

As is said in Fig. 6.4, all atoms in $F = 0, M_F = 0$ and $F = 1, M_F = 1$ states are thrown away. Now we add a very strong magnetic field in the x direction, and the interaction Hamiltonian $-\boldsymbol{\mu} \cdot \boldsymbol{B}$ doesn't introduce any split because of M_F, since the latter is along the z direction. Since we only have two states, the Hamiltonian has to take the form

$$H \propto -\sigma_x, \tag{25}$$

and the low energy eigenstate is

$$|\mathrm{ground}\rangle = \frac{1}{\sqrt{2}}(|1,0\rangle + |1,1\rangle), \tag{26}$$

and since the transverse field B is very strong, almost all Hydrogen atoms fall to this ground state, and the density matrix therefore is

$$\rho_{\text{in selector}} = |\mathrm{ground}\rangle\langle\mathrm{ground}| = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \tag{27}$$

or in the full basis,

$$\rho_{\text{in selector}} = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}. \tag{28}$$

4.3 Transition between F states

4.4 The state selector

The state selector

5 Diffracted limited beam

We are dealing with linear optics so the power of the laser beam is irrelevant. The beam radius of a Gaussian beam is

$$w(z) = w_0\sqrt{1+\left(\frac{z}{z_{\mathrm{R}}}\right)^2}, \quad z_{\mathrm{R}} = \frac{\pi w_0^2 n}{\lambda}. \tag{29}$$

When z is very large, we have

$$w(z) = w_0\frac{z}{z_{\mathrm{R}}} = \frac{\lambda}{\pi w_0 n}\cdot z. \tag{30}$$

The wave length of green laser is 532 nm. The distance between earth and moon is 382 500 km, and we can take w_0 to be half of the 1 mm diameter of the laser beam when it leaves the laser, and thus on the moon the radius of the beam is

$$w_{\text{moon}} = \frac{\lambda}{\pi w_0}\cdot R_{\text{earth-moon}} = 129\,\text{km}.$$

This can also be seen as an instance of uncertainty principle: the above equation is equivalent to

$$\underbrace{w_0}_{\Delta x}\cdot\underbrace{\frac{\pi}{\lambda}\cdot\underbrace{\frac{w(z)}{z}}_{\tan\theta}}_{\Delta k} = 1. \tag{31}$$