Squeezing of quantum noise

Jinyuan Wu

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1 Introduction

Noise usually arises from coupling with an unknown external object: when we keep our eyes only on one of the objects (the "system"), we need to average over the state of the other ("the environment"), which corrects the theory with noise and dissipation [1]. When the full quantum treatment is needed, however, another type of noise appears: the state itself does not have a definite value of the observable in question, and any experimental setting – no matter how carefully the system is isolated from the environment – has noisy results. This kind of noise is called **quantum noise**.

In systems that can be well described by the harmonic oscillator picture, in which for each oscillation mode, we have two variables X and P, and $[X, P] = \mathrm{i}$, and the Hamiltonian is $H \simeq c_1 X^2 + c_2 P^2$.[1] After diagonalization, we get $H \simeq \sum_{\text{modes}} \omega(a^\dagger a + 1/2)$ plus possible interaction terms, and this zero-point energy arises from the non-commutative nature of X and P. Another way to make sense of the 1/2 term is to notice that at the ground state, though $\langle X \rangle = \langle P \rangle = 0$, and everything seems to be very definite, since we have the zero-point energy, $\langle X^2 \rangle \simeq H/2 \neq 0$. Note that most of the time, the observable actually measured in such systems is $\simeq n = a^\dagger a \sim X^2$, and quantum noise appears in the measurement. People therefore sometimes say that quantum noise comes from the zero point-energy.

This report focuses on quantum noise in systems mentioned above. This includes linear optics and micromechanics. Section 2 discusses more quantitative ways to represent and characterize quantum noise. Section 3 calculates quantum noise in a prevalent phase measurement scheme using interferometer. Section 4 shows how to squeeze the quantum noise.

2 Overview of representation of states in quantum optics

3 Quantum noise in the Mach-Zehnder interferometer

To have a concrete example of quantum noise, let us move to the Mach-Zehnder interferometer, illustrated in Figure 1. The interferometer contains two beam splitters, two ideal mirrors, and

[1] The Planck system of units is used in this report, so we take $\hbar = 1$ if there is no special mention of the units.

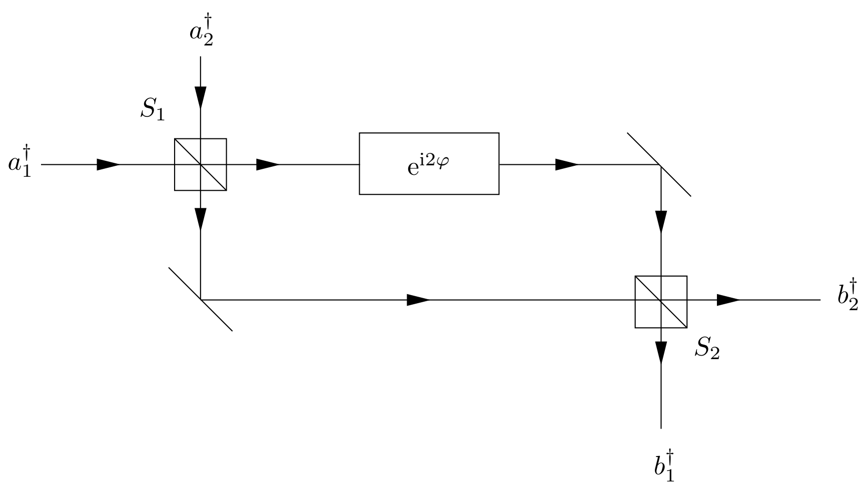

Figure 1: The schematic structure of the Mach-Zehnder interferometer

one sample that introduces a 2φ phase shift to the light beam going through it. The two beams created by the first beam splitter gain a phase difference caused by the sample, and are then remixed together by the second beam splitter, so there is interference, and φ can be found by comparing the output signal with the injected laser beam.

We are going to work in the Heisenberg picture, because the whole system is linear and time evolution can be easily described by a linear transformation on the operators. Since this section is just to exemplify the overall idea of quantum noise, for the sake of simplicity we assume the time evolution operator of the beam splitter is

$$S_{\text{beam splitter}} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}. \tag{1}$$

The time evolution operator of the whole system is therefore

$$\begin{aligned} S_{\text{total}}(\varphi) &= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \cdot \begin{pmatrix} \mathrm{e}^{\mathrm{i}\varphi} & \\ & \mathrm{e}^{-\mathrm{i}\varphi} \end{pmatrix} \cdot \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \\ &= \begin{pmatrix} \cos\varphi & -\mathrm{i}\sin\varphi \\ \mathrm{i}\sin\varphi & -\cos\varphi \end{pmatrix}, \end{aligned} \tag{2}$$

and therefore

$$\begin{pmatrix} b_1^\dagger \\ b_2^\dagger \end{pmatrix} = \begin{pmatrix} \cos\varphi & -\mathrm{i}\sin\varphi \\ \mathrm{i}\sin\varphi & -\cos\varphi \end{pmatrix} \begin{pmatrix} a_1^\dagger \\ a_2^\dagger \end{pmatrix}, \tag{3}$$

from which we find

$$\begin{pmatrix} a_1^\dagger \\ a_2^\dagger \end{pmatrix} = \begin{pmatrix} \cos\varphi & -\mathrm{i}\sin\varphi \\ \mathrm{i}\sin\varphi & -\cos\varphi \end{pmatrix} \begin{pmatrix} b_1^\dagger \\ b_2^\dagger \end{pmatrix}. \tag{4}$$

In actual experiment settings, usually a beam of laser is injected into input port 1, so the state at input port 1 is a coherent state. Then, the wave function is

$$\begin{aligned} |\psi\rangle &= \underbrace{\mathrm{e}^{\alpha a_1 - \alpha^* a_1^\dagger}}_{D_{a_1}(\alpha)} |0\rangle \\ &= \mathrm{e}^{\alpha(\cos\varphi b_1^\dagger - \mathrm{i}\sin\varphi b_2^\dagger) - \alpha^*(\cos\varphi b_1 + \mathrm{i}\sin\varphi b_2)} |0\rangle \\ &= |b_1 = \alpha\cos\varphi, b_2 = -\mathrm{i}\alpha\sin\varphi\rangle\,. \end{aligned} \tag{5}$$

The next step is measurement. Usually, the influence of the sample to the light beam is small, and therefore the signal in the b_2 mode is $\sim \varphi$, while the signal in the b_1 mode is $\sim \varphi^2$ (the signal in the b_1 mode should be defined as $\alpha - \alpha\cos\varphi$, because when $\varphi = 0$, the output state is $|b_1 = \alpha, b_2 = 0\rangle$). So we usually measure the intensity of the b_2 mode. The expectation of photon number is

$$\langle n_{b_2} \rangle = |\alpha|^2 \sin^2\varphi. \tag{6}$$

This output comes with a quantum uncertainty, which is given by

$$\begin{aligned} \Delta n_{b_2} &= \sqrt{\langle n_{b_2}^2 \rangle - \langle n_{b_2} \rangle^2} \\ &= \sqrt{\langle b_2^\dagger (b_2^\dagger b_2 + 1) b_2 \rangle - \langle n_{b_2} \rangle^2} \\ &= \sqrt{|\alpha|^4 \sin^4\varphi + |\alpha|^2 \sin^2\varphi - |\alpha|^4 \sin^4\varphi} \\ &= |\alpha| \sin\varphi. \end{aligned} \tag{7}$$

Here we have used the property of coherent states as eigenstates of the annihilation operator.

So it can be seen that the relative uncertainty of the measurement is

$$\frac{\Delta n_{b_2}}{\langle n_{b_2} \rangle} = \frac{1}{|\alpha|\sin\varphi} = \frac{1}{\sqrt{n_{b_2}}}. \tag{8}$$

It can be seen from (7) that this standard error comes from the commutation relation

$$[b_2, b_2^\dagger] = 1, \tag{9}$$

which, eventually, comes from the fact that $\boldsymbol{E}$ and $\boldsymbol{A}$ in electromagnetism don't commute. This is therefore a quantum noise.

Of course, (8) can be systematically reduced by using stronger and stronger laser beams, but then another problem occurs: in a real measurement setting, the detecting laser beam of course perturbs the sample, introducing another source of error. (For example, if the sample is an additional optical path, which is the case in gravitational wave detection, then the mirrors used will be heated and begin to have thermal vibration.) In a classical theory, we can always use weaker and weaker laser beams to do the measurement, and do an extrapolation for the measured results to systematically reduce the perturbation of measurement to the system and get results as accurate as we want, but in the quantum theory, this results in stronger quantum noise.

This leads to an astonishing fact: when quantum noise is present, even in principle, we still can't find a way to reduce the error as much as we want. There is a non-zero minimum error, which is met when $|\alpha|$ strikes a balance between the thermal fluctuation caused by large $|\alpha|$ and quantum noise caused by small $|\alpha|$. This actually makes sense, or otherwise we are faced with the problem that an infinitely accurate continuous degree of freedom can store infinite bits of information.

4 Squeezing the quantum noise

One thing to keep in mind is (8) is about the quantum noise of the photon number, not others. This quantity is not the only observable in b_2 mode, but it is the only thing actually measured. Thus it is possible that we reduce the quantum noise of n_{b_2} and in exchange, get a larger quantum noise of other variables, which we do not care. This is called "squeezing" the quantum noise – the term will be visualized in the following discussion. To do so, it is necessary to trace the origin of Δn_{b_2} in terms of quantum noises of a_1 and a_2. For sake of simplicity, here we keep the input state in the a_1 mode $|\alpha\rangle$, without modifying anything, and we also exclude all cross terms between a_1 and a_2 in the wave function, so we have

$$|\psi\rangle = D_{a_1}(\alpha) f(a_2, a_2^\dagger) |0\rangle , \tag{10}$$

and squeezing Δn_{b_2} therefore reduces to squeezing the quantum error of some operator in mode a_2.

From (3), we have

$$n_{b_2} = b_2^\dagger b_2 = \sin^2 \varphi a_1^\dagger a_1 + \cos^2 \varphi a_2^\dagger a_2 - \mathrm{i} \sin\varphi \cos\varphi (a_1^\dagger a_2 - a_2^\dagger a_1), \tag{11}$$

and therefore

$$\begin{aligned} n_{b2}^2 = {} & \sin^4 \varphi (a_1^\dagger a_1)^2 + \cos^4 \varphi (a_2^\dagger a_2)^2 - \sin^2\varphi \cos^2\varphi (a_1^\dagger a_2 - a_2^\dagger a_1)^2 + 2 \sin^2\varphi \cos^2\varphi a_1^\dagger a_1 a_2^\dagger a_2 \\ & - \mathrm{i} \sin\varphi \cos^3\varphi \{a_2^\dagger a_2, (a_1^\dagger a_2 - a_2^\dagger a_1)\} - \mathrm{i} \sin^3\varphi \cos\varphi \{a_1^\dagger a_1, (a_1^\dagger a_2 - a_2^\dagger a_1)\} , \end{aligned} \tag{12}$$

If we are sure the wave function takes the form of (10), then all a_1 can be replaced by α in the above two equations, but before that we should first complete normal ordering of a_1 and $a_1^\dagger$. The expression of n_{b_2} in the subspace of (10) after normal ordering and replacement of a_1 by α is

$$n_{b_2} = |\alpha|^2 \sin^2 \varphi a_2 + \cos^2 \varphi n_{a_2} - \mathrm{i} \sin\varphi \cos\varphi (\alpha^* a_2 - \alpha a_2^\dagger), \tag{13}$$

and the expression of $n_{b_2}^2$ is

$$\begin{aligned} n_{b_2}^2 = {} & \sin^4\varphi (|\alpha|^4 + |\alpha|^2) + \sin^2\varphi \cos^2\varphi (2|\alpha|^2 P(\alpha)^2 + a_2^\dagger a_2) + 2 \sin^2\varphi \cos^2\varphi |\alpha|^2 a_2^\dagger a_2 \\ & - \text{terms that changes } n_{a1} \text{ and } n_{a2}. \end{aligned} \tag{14}$$

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References

[1] Robert Zwanzig. *Nonequilibrium statistical mechanics.* Oxford university press, 2001. The discussion on Mori-Zwanzig formalism or "generalized Langevin equation" can be found on page 149.