General Relativity as an Effective Field Theory

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This article is a reading note of the chapters in Schwartz about how general Relativity (henceforth GR) can be thought as an effective field theory. The idea was first shown by [1, 2].

1 Spin-2 fields

TODO: representations

2 The free spin-2 theory

We first construct a spin-2 field theory with the approach of Schwartz Section 8.7. Consider a symmetric rank 2 tensor $h_{\mu\nu}$. We try to write down a free theory of the theory, which must be quadratic in $h_{\mu\nu}$, and either quadratic or zeroth order in ∂_μ. From $h_{\mu\nu}$ we can construct a list of objects that are first order in $h_{\mu\nu}$ and do not contain $\mathcal{O}(\partial^3)$: Sec. 8.7.2

$$h_{\mu\nu},\ h_{\alpha\alpha},\ \Box h_{\mu\nu},\ \partial_\mu\partial_\nu h_{\mu\nu},\ \partial_\mu\partial_\nu h_{\mu\alpha},\ \Box h_{\alpha\alpha},$$

and we can construct terms that are possible to appear in the free Lagrangian:

$$h_{\mu\nu}^2,\ h_{\mu\nu}\Box h_{\mu\nu},\ h_{\nu\alpha}\partial_\mu\partial_\nu h_{\mu\alpha},\ h_{\alpha\alpha}^2,\ h_{\alpha\alpha}\Box h_{\beta\beta},\ h_{\alpha\alpha}\partial_\mu\partial_\nu h_{\mu\nu},$$

and hence we get (8.126) (8.126)

$$\mathcal{L} = ah_{\mu\nu}\Box h_{\mu\nu} + bh_{\mu\nu}\partial_\mu\partial_\alpha h_{\nu\alpha} + ch\Box h + dh\partial_\mu\partial_\nu h_{\mu\nu} + m^2\left(xh_{\mu\nu}^2 + yh^2\right), \tag{1}$$

where we define $h = h_{\alpha\alpha}$.

Now we consider the "inner structure" of $h_{\mu\nu}$. We first do the decomposition (8.124) to (8.125)

$$h_{\mu\nu} = h^{\mathrm{T}}_{\mu\nu} + \partial_\mu\pi_\nu + \partial_\nu\pi_\mu, \tag{2}$$

where we require

$$h^{\mathrm{T}}_{\mu\nu} = h^{\mathrm{T}}_{\nu\mu}, \quad \partial^\mu h^{\mathrm{T}}_{\mu\nu} = 0. \tag{3}$$

Again we can decompose π_μ into

$$\pi_\mu = \pi^{\mathrm{T}}_\mu + \partial_\mu\pi^{\mathrm{L}}, \quad \partial^\mu\pi^{\mathrm{T}}_\mu = 0. \tag{4}$$

Now we find the decomposition actually imposes strong constraints on (1). First, since

$$xh_{\mu\nu}^2 + yh^2 = 2x(\Box\pi^{\mathrm{L}})^2 + 2y(\partial_\mu\partial_\nu\pi^{\mathrm{L}})^2 \simeq -2x\pi^{\mathrm{L}}\Box^2\pi^{\mathrm{L}} - 2y\partial_\mu\partial_\nu\partial^\mu\partial^\nu\pi^{\mathrm{L}} = -2(x+y)\pi^{\mathrm{L}}\Box^2\pi^{\mathrm{L}},$$

and there should be no $\mathcal{O}(\partial^4)$ terms in (1), we find $x + y = 0$. TODO: derive

$$\mathcal{L} = \frac{1}{2}h_{\mu\nu}\Box h_{\mu\nu} - h_{\mu\nu}\partial_\mu\partial_\alpha h_{\nu\alpha} + h\partial_\mu\partial_\nu h_{\mu\nu} - \frac{1}{2}h\Box h + \frac{1}{2}m^2\left(h_{\mu\nu}^2 - h^2\right) \tag{5}$$ (8.128)

The massless case is

$$\mathcal{L} = \frac{1}{2}h_{\mu\nu}\Box h^{\mu\nu} - h_{\mu\nu}\partial^\mu\partial_\alpha h^{\nu\alpha} + h\partial_\mu\partial_\nu h^{\mu\nu} - \frac{1}{2}h\Box h. \tag{6}$$

From now on we will focus on the massless case, as gravitation seems to have some shared properties with electromagnetism. For an effective theory of gravitation, We require that the basic degrees of freedom is a symmetric spin-2 tensor, and the free part of our theory is (6). We will find that the almost only possibility is general relativity.

3 Coupling with another field

Now we try to couple (6) to other fields. Note that the π modes never appear in (6) TODO: show this and therefore they cannot be coupled to external degrees of freedom. This gives us a hint that these π modes might be *gauge redundancies*. Therefore, we introduce the minimal coupling between $h_{\mu\nu}$ and another tensor $T_{\mu\nu}$ made up by external fields, and the interaction Lagrangian is (8.115) to (8.116)

$$\mathcal{L}_{\text{int}} \propto h_{\mu\nu} T^{\mu\nu}. \tag{7}$$

Since we have to ensure that

$$h^{\text{T}}_{\mu\nu} T^{\mu\nu} \simeq h_{\mu\nu} T^{\mu\nu} = (h^{\text{T}}_{\mu\nu} + \partial_\mu \pi_\nu + \partial_\nu \pi_\mu) T^{\mu\nu},$$

we must take advantages of integration by parts and assume that

$$(\partial_\mu + \partial_\nu) T^{\mu\nu} = 0.$$

Since $h_{\mu\nu}$ is symmetric, without loss of generality, we assume that $T_{\mu\nu}$ is symmetric, and therefore the above condition is equivalent to $\partial_\mu T^{\mu\nu} = 0$. This in turn means that under a transformation

$$h_{\mu\nu} \longrightarrow h_{\mu\nu} + \partial_\mu \xi_\nu + \partial_\nu \xi_\mu, \tag{8}$$

the theory – both the free part (6) and the coupling part with external fields (7), are invariant, confirming the claim that the π degrees of freedom are indeed gauge degrees of freedom, and (9) is the gauge transformation. From now on, we can rewrite (8) as

$$h_{\mu\nu} \longrightarrow h_{\mu\nu} + \partial_\mu \pi_\nu + \partial_\nu \pi_\mu. \tag{9}$$

Now we already see something familiar in GR here: (9) seems to be how the *metric* transforms, and the gauge group seems to be a local TODO: what group is this?

The fact that $T^{\mu\nu}$ is a conservation current in turn poses an additional constraint. The **Coleman-Mandula theorem** tells us that the *energy-momentum tensor* is the only rank 2 tensor conservation current, so $T^{\mu\nu}$ (up to a constant) must be the energy-momentum tnesor.

4 Self interaction of the gravitation field

We will soon find, however, that (6) plus (7) is still not a self-consistent theory. In this section we will repeat the discussion in Schwartz from (8.132) to (8.146).

Suppose we have a term $\delta^{\mu\nu}\phi/2$ in $T^{\mu\nu}$. This will gives a

$$\mathcal{L}_1 = \frac{1}{2} h \phi \tag{10}$$ (8.131)

term in the Lagrangian. Now if we do the gauge transformation (9), we find this term is not invariant: we have

$$\begin{aligned} \mathcal{L}_1 &\to \frac{1}{2}(h + 2\partial_\mu \pi^\mu)\phi \\ &= \mathcal{L}_1 + \partial_\nu \pi^\nu \phi. \end{aligned} \tag{11}$$ (8.133)

It is possible to eliminate this extra term by letting ϕ transforms as $h_{\mu\nu}$ undergoes (9). If we take the transformation

$$\phi \longrightarrow \phi + \pi_\nu \partial^\nu \phi, \tag{12}$$ (8.135)

and add one term to (10), replacing it with

$$\mathcal{L}_2 = \phi + \frac{1}{2} h \phi \tag{13}$$ (8.134)

we find (8.136)

$$\begin{aligned} \mathcal{L}_2 &\to \phi + \pi_\nu \partial^\nu \phi + \frac{1}{2}(h + 2\partial_\nu \pi^\nu)(\phi + \pi^\nu \partial_\nu \phi) \\ &= \phi + \frac{1}{2} h\phi + \underbrace{\pi_\nu \partial^\nu \phi + \partial_\nu \pi^\nu \phi}_{=\partial^\nu(\pi_\nu \phi)\text{ , surface term}} + \frac{1}{2} h \pi^\nu \partial_\nu \phi + \partial_\mu \pi^\mu \pi^\nu \partial_\nu \phi \\ &\simeq \mathcal{L}_1 + \frac{1}{2} h \pi^\nu \partial_\nu \phi + \partial_\mu \pi^\mu \pi^\nu \partial_\nu \phi. \end{aligned}$$

Now we see that the extra term in (11) disappears, but the cost is introducing more terms. The first term contains h, and the second term is $\sim \mathcal{O}(\pi^2)$. The first term can be further eliminated by introducing more terms in the Lagrangian (in this way introducing more $\mathcal{O}(\pi^n)$ terms), and the second term can be eliminated by introducing more terms in (12).

We now follow the procedures in Schwartz and analyze the first term in

$$\mathcal{L} = \left(1 + \frac{1}{2}h + \frac{1}{8}h^2 + \cdots\right)\phi, \tag{14}$$ (8.142)

It can be seen that this expansion is a taylor expansion of

5 GR comes out

6 The non-renormalizable nature of GR

It has been traditionally claimed that GR is incompatible with quantum mechanics, because GR is not renormalizable. Schwartz says this claim is wrong – non-renormalizable theories are as useful as renormalizable ones. They just tell us themselves when they start to break. The conclusion is that there is, actually, *no* conflict between general relativity and quantum mechanics. Here we

Sec. 22.4

What is really concerning is that GR, in the context of QFT, is not

References

[1] David G Boulware and S Deser. Classical general relativity derived from quantum gravity. *Annals of Physics*, 89(1):193–240, 1975.

[2] S. Deser. Self-interaction and gauge invariance. *General Relativity and Gravitation*, 1(1):9–18, 1970.