PDEs

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1 The linear heat equation

1.1 Homogeneous boundary condition

Consider one-dimensional heat equation

$$\frac{\partial u}{\partial t} = K \frac{\partial^2 u}{\partial x^2}. \tag{1}$$

As an example, we consider the case where the boundaries of the string in question are kept to zero temperature, and the boundary and initial conditions are

$$u(0,t) = u(L,t) = 0, \quad u(x,0) = f(x). \tag{2}$$

By separation of variables

$$u(x,t) = X(x)T(t), \tag{3}$$

we find

$$\frac{T'}{KT} = \frac{X''}{X},$$

which can therefore only be a constant, because otherwise it's impossible for something that only depends on t and something that only depends on x to be equal to each other constantly. So we have

$$X'' = \lambda X, \quad T' = K\lambda T.$$

When $\lambda > 0$, we find

$$X(x) = A\mathrm{e}^{\sqrt{\lambda}x} + B\mathrm{e}^{-\sqrt{\lambda}x},$$

and therefore the boundary conditions mean

$$X(0) = 0 \Rightarrow A + B = 0,$$

$$X(L) = 0 \Rightarrow A(\mathrm{e}^{\sqrt{\lambda}x} - \mathrm{e}^{-\sqrt{\lambda}x}) = 0 \Rightarrow A = B = 0,$$

which gives a trivial solution. Similarly $\lambda = 0$ gives a trivial solution. So we find we should only consider $\lambda < 0$. So now we replace λ by $-\lambda$, and from

$$X'' = -\lambda X$$

we find

$$X(x) = A\cos\left(\sqrt{\lambda}x\right) + B\sin\left(\sqrt{\lambda}x\right).$$

The boundary condition $X(0) = 0$ means $A = 0$, and we should then keep B to be non-zero. Then $X(L) = 0$ means

$$\sqrt{\lambda}L = n\pi, \quad n = 1, 2, 3, \ldots$$

So now λ is completely determined, and the next step is to find T, which is trivial:

$$T(t) \propto \mathrm{e}^{-\lambda t}.$$

So the final solution is

$$u(x,t) = \sum_{n=1}^{\infty} c_n T_n(t) X_n(x), \tag{4}$$

where

$$T_n(t) = \mathrm{e}^{-\frac{n^2\pi^2 K t}{L^2}}, \tag{5}$$

and

$$X_n(x) = \sin\Big(\frac{n\pi x}{L}\Big). \tag{6}$$

The constants $\{c_n\}$ then can be solved from the initial condition:

$$f(x) = \sum_{n=1}^{\infty} c_n \sin\Big(\frac{n\pi x}{L}\Big),$$

which is just a Fourier series, so

$$c_n = \frac{2}{L}\int_0^L f(x)\sin\Big(\frac{n\pi x}{L}\Big)\,\mathrm{d}x\,. \tag{7}$$

1.2 Inhomogeneous boundary condition

We can consider another problem: now the boundaries are still isothermal, but the temperatures there are no longer zero. The conditions are

$$u(0,t) = T_1, \quad u(L,t) = T_2. \tag{8}$$

Note that the temperatures can be different, and when $t \to \infty$, the stable solution may still be non-zero. Linearity guarantees the validity of the following decomposition:

$$u(x,t) = u_0(x,t) + \psi(x), \tag{9}$$

where $\psi(x)$ satisfies

$$\psi''(x) = 0, \quad \psi(0,t) = T_1, \quad \psi(L,t) = T_2, \tag{10}$$

so that $u_0(x,t)$ satisfies the problem (1) plus (2) just solved above – but note that f in (2) should be replaced by $f(x) - \psi(x)$. Now $\psi(x)$ can be found easily: it's just

$$\psi(x) = \frac{T_2 - T_1}{L}x + T_1. \tag{11}$$

1.3 Heat conduction in an infinite medium

Now we consider

$$\frac{\partial u}{\partial t} = K\frac{\partial^2 u}{\partial x^2}, \quad -\infty < x < \infty, \quad u(x,0) = f(x). \tag{12}$$

The problem can be solved by Laplace transform as well as Fourier transform; or we can do Fourier transform in x and Laplace transform in t. We have (here we are using ω to refer to the frequency of x, not t)

$$\mathcal{F}\left[\frac{\partial u}{\partial t}\right] = \frac{\partial}{\partial t}\hat{u}(\omega,t),$$

and

$$\mathcal{F}\left[\frac{\partial^2 u}{\partial x^2}\right] = (\mathrm{i}\omega)^2\hat{u}(\omega,t).$$

The bulk equation now is

$$\frac{\partial \hat{u}(\omega,t)}{\partial t} = -K\omega^2\hat{u}(\omega,t),$$

and we have

$$\hat{u}(\omega,t) = \mathrm{e}^{-K\omega^2 t}\underbrace{\hat{u}(\omega,0)}_{\hat{f}(\omega)}.$$

So we find

$$u(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\hat{u}(\omega,t)\mathrm{e}^{\mathrm{i}\omega x}\,\mathrm{d}\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty}\hat{f}(\omega)\mathrm{e}^{-K\omega^2 t}\mathrm{e}^{\mathrm{i}\omega x}\,\mathrm{d}\omega\,. \tag{13}$$

A common initial condition is

$$u(x,0) = \delta(x), \tag{14}$$

which means when $t = 0$, all heat is concentrated in a rather small region. Then

$$\hat{f}(\omega) = 1,$$

and (13) tells us

$$u(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{e}^{-K\omega^2 t + \mathrm{i}\omega x}\,\mathrm{d}\omega = \frac{1}{2\sqrt{\pi K t}}\mathrm{e}^{-\frac{x^2}{4Kt}}. \tag{15}$$

Here the integral can be calculated as

$$\begin{aligned}\int_{-\infty}^{\infty} \mathrm{e}^{-K\omega^2 t + \mathrm{i}\omega x}\,\mathrm{d}\omega &= \mathrm{e}^{-\frac{x^2}{4Kt}}\int_{-\infty}^{\infty} \mathrm{e}^{-Kt\left(\omega - \frac{\mathrm{i}x}{2Kt}\right)^2}\,\mathrm{d}\omega \\ &= \sqrt{\frac{\pi}{Kt}}\mathrm{e}^{-\frac{x^2}{4Kt}}.\end{aligned}$$

The solution is always Gaussian, but as time goes by, it becomes wider and wider.

1.4 Heat conduction on a semi-infinite domain

Let's then consider the following boundary and initial conditions:

$$u(x,0) = T, \quad u(0,t) = 0. \tag{16}$$

This means we first heat the material and establish a homogeneous temperature field inside it, and then touch it with a colder point. Since this is a half-infinite problem, we can use Laplace transform on the time t. We have

$$\mathcal{L}\left[\frac{\partial u}{\partial t}\right] = sU(x,s) - u(x,0) = sU(x,s) - T,$$

and

$$\mathcal{L}\left[\frac{\partial^2 u}{\partial x^2}\right] = \frac{\partial^2 U(x,s)}{\partial x^2}.$$

The bulk equation then becomes

$$sU(x,s) - T = K\frac{\partial^2 U(x,s)}{\partial x^2},$$

$$\frac{\partial^2 U(x,s)}{\partial x^2} - \frac{s}{K}U = -\frac{T}{K}.$$

The homogeneous solution of this equation is just (note that A and B may have s dependence)

$$U = A\mathrm{e}^{\sqrt{s/K}x} + B\mathrm{e}^{-\sqrt{s/K}x}.$$

A specific solution is

$$U = \frac{T}{s}.$$

A has to be zero, because $u(x,t)$ should be finite when $x \to \infty$. So we find

$$U = \frac{T}{s} + B(s)\mathrm{e}^{-\sqrt{s/K}x}.$$

We still need to use the condition $u(0,t) = 0$, which, after Laplace transform, is $U(0,s) = 0$, and we find

$$U = \frac{T}{s}(1 - \mathrm{e}^{-\sqrt{s/K}x}).$$

So

$$u(x,t) = \mathcal{L}^{-1}\left[\frac{T}{s}(1 - \mathrm{e}^{-\sqrt{s/K}x})\right] = T\mathrm{erf}\left(\frac{x}{2\sqrt{Kt}}\right). \tag{17}$$