# Floquet theory

Jinyuan Wu

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## 1 The fundamental formalism

Consider a time-periodic Hamiltonian with period $T = 2\pi/\omega$. Such a Hamiltonian is usually an effective Hamiltonian when the system (hereafter "matter") is coupled with another degree of freedom which doesn't change much in the time evolution; the latter is hereafter called "light", since in condensed matter systems, periodic driving is usually achieved by shedding a beam of light to the matter; it's however possible to use light to stimulate some long-lived degrees of freedom in a solid and let it drive the rest of the system, which sometimes is known as "self-driving". From the Floquet theory of differential equation, we know it's possible to expand an arbitrary state that evolves according to $H$ into a linear combination (the coefficients are constants) of $\{\,|\psi_n(t)\rangle\}$ where

$$|\psi_n(t+T)\rangle = \mathrm{e}^{-\mathrm{i}\varepsilon_n t/\hbar}\,|\Phi_n(t)\rangle\,, \quad |\Phi_n(t+T)\rangle = |\Phi_n(t)\rangle\,. \tag{1}$$

By discrete periodicity of $|\Phi_n(t)\rangle$ we make Fourier expansion

$$|\Phi_n(t)\rangle = \sum_m \mathrm{e}^{-\mathrm{i}m\omega t}\,|\phi_n^{(m)}\rangle\,, \tag{2}$$

where $m$ goes over all integers. Note that here $|\phi_n^{(m)}\rangle$ are *Fourier coefficients* and are not eigenstates of anything; there is no normalization or orthogonality condition. Using $i$ to label the eigenstates of the matter, we have

$$|\Phi_n(t)\rangle = \sum_i \sum_m \mathrm{e}^{-\mathrm{i}m\omega t}\,\langle i|\phi_n(m)\rangle\,|i\rangle\,. \tag{3}$$

*This*, not (2), is the expansion of $|\Phi\rangle$ in a complete, orthogonal basis. The meaning of these vectors can be seen immediately below.

The Schrodinger equation now reads

$$(\varepsilon_n + m\hbar\omega)\,|\phi_n^{(m)}\rangle = \sum_{m'} H^{(m-m')}\,|\phi_n^{(m')}\rangle\,, \tag{4}$$

where

$$H(t) = \sum_m \mathrm{e}^{-\mathrm{i}m\omega t} H^{(m)}. \tag{5}$$

Thus we find that if we use $i$ to label the eigenstates of the matter part, we have

$$\varepsilon_n\,\langle i|\phi_n^{(m)}\rangle = \sum_{m'} (H^{(m-m')} - m\hbar\omega\delta_{mm'})\,\langle i|\phi_n^{(m')}\rangle\,. \tag{6}$$

It doesn't take long to realize that the matrix elements on the RHS are exactly the matrix elements of

$$H^{\text{light}\otimes\text{matter}} = H\otimes 1_{\text{light}} + 1_{\text{matter}}\otimes\hbar\left(b^\dagger b + \frac{1}{2}\right) + \underbrace{bV + b^\dagger V^\dagger}_{H_{\text{light-matter coupling}}} \tag{7}$$

under the basis $|i\rangle\otimes|m\rangle$, where $m$ refers to the photon number. When the dynamics is almost completely decided by the matter part of the system, and we can map the light-matter wave function to the wave function of the matter part only by applying

$$P:\,|\psi\rangle\otimes|\text{whatever the light is}\rangle \mapsto |\psi\rangle\,, \tag{8}$$

which means

$$\langle i, m|\psi_n\rangle \otimes |m\rangle \mapsto \langle i|\phi_n^{(m)}\rangle\,. \tag{9}$$

Now we see the true meaning of $\phi^{(m)_n}$: we are just grouping the components of the complete matter-light wave function $|\phi_n\rangle \otimes |\text{light}\rangle$ with the same photon number $m$ into a vector $\phi_n^{(m)}$.

As an example, when the light field is approximately always in a coherent state $|\alpha \mathrm{e}^{-\mathrm{i}\omega t}\rangle$ ($\alpha$ should be large enough so that the matter part doesn't significantly change the state of the light part), approximately we have

$$H_{\text{light-matter coupling}} \approx \alpha V \mathrm{e}^{-\mathrm{i}\omega t} + \text{h.c.}, \tag{10}$$

and the effective Hamiltonian for the matter part is then

$$H = H_{\text{matter}} + \alpha V \mathrm{e}^{-\mathrm{i}\omega t} + \text{h.c.}. \tag{11}$$

Based on the above perspective, we call the coefficient matrix on the RHS of (6) the effective Hamiltonian in the **extended Hilbert space**, i.e. the space containing both the matter degrees of freedom and the light field. Since the coefficient matrix on the RHS of (6) contains components of the Hamiltonian in the extended space, while we are actually working in the Hilbert space of the matter part, (6)'s solutions are overcomplete. We can actually point out where overcompletion appears: note that if $\varepsilon_n$ satisfies (1), then so does $\varepsilon_n + m\hbar\omega$.

In conclusion, a Floquet system has a set of quasi-eigenstates $\{\,|\psi_n\rangle\}$, the number of which is the same as the dimension of the Hilbert space; but for each quasi-eigenstate, we have countable infinite quasi-energies, the difference between the nearest two being $\hbar\omega$; thus all distinct Floquet quasi-eigenstates can be indexed by quasi-energies that are within one "Floquet-Brillouin zone".

Orthogonal relation between $|\psi_n\rangle$'s?