Complex analysis

Jinyuan Wu

April 27, 2023

The order of a zero point z_0 of a complex function $f(z)$ is the order of the first non-zero term in the Taylor series around z_0. If we have a set of complex functions $\{h_i(z)\}$, each of which has zero of order m_i about z_0, then $\prod_i h_i(z)$ has zero of order $\sum_i m_i$ at z_0. The proof is straightforward.

Let $f(z) = h(z)/g(z)$ with h and g differentiable on $0 < |z - z_0| < R$. If $h(z)$ has order k at z_0, and $g(z)$ has order m at z_0, and $m > k$, then $f(z)$ has a pole of order $m - k$ at z_0. The proof is also straightforward. Note that $f(z)$ doesn't have $(z - z_0)^{k-m-1}$ or $(z - z_0)^{k-m-2}$ or components with an order smaller than $k - m$, although formally, since $g(z)$ may contain $m + 1$ or $m + 2$ components, we might think $(z - z_0)^{k-m-1}$ or $(z - z_0)^{k-m-2}$ terms exist.

We have the so-called residue theorem:

$$\oint_\Gamma f(z)\,\mathrm{d}z = 2\pi\mathrm{i}\,\mathrm{Res}_{z_0} f, \tag{1}$$

where $\mathrm{Res}_{z_0} f$ is the residue of f at $z = z_0$, which is c_{-1} of its Laurent series at $z = z_0$. Other terms in the Laurent series contribute nothing to the contour integral; but they do have contribution in, say,

$$\oint_\Gamma (z - z_0)^n f(z)\,\mathrm{d}z\,. \tag{2}$$

Indeed we have

$$c_n = \frac{1}{2\pi\mathrm{i}} \oint_\Gamma \frac{f(z)}{(z - z_0)^{n+1}}. \tag{3}$$

The proof can be straightforwardly obtained from multiplying $(z - z_0)$ to the Laurent series and then applying the residue theorem.