Quantum Optics by Prof. Saijun Wu

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1 Master equation

We consider a open quantum system, whose Hamiltonian is H, and at each time step, there is a probability of quantum jump to several given states. Suppose the probability of jumping to state $|i\rangle$ is Γ_i, we define

$$C_i = \sqrt{\Gamma_i}\, |i\rangle\langle i| \,, \tag{1}$$

and the system can be described using a stochastic wave function method shown in previous lectures. If we use a density matrix formalism, we find

$$\rho(t+\Delta t) = \sum_i \Gamma_i \Delta t\, |i\rangle\langle i| + (1 - \sum_i \Gamma_i)\, |\psi_\mathrm{s}(t+\Delta t)\rangle\langle\psi_\mathrm{s}(t+\Delta t)| \,,$$

where $|\psi_\mathrm{s}\rangle$ evolves according to H. We therefore find

$$\dot{\rho} = \frac{1}{\mathrm{i}\hbar}[H_\mathrm{eff}, \rho] + \sum_i C_i \rho C_i^\dagger, \tag{2}$$

where

$$H_\mathrm{eff} = H - \frac{\mathrm{i}\hbar}{2} \sum_i C_i^\dagger C_i. \tag{3}$$

(2) is called **the master equation in Lindblad form**. It should be noted that the "commutator" in the equation is actually the anticommutator for the anti-Hermitian part of H_eff: the two terms in $[H_\mathrm{eff}, \rho]$ correspond to the time evolution of the bra- and ket-part of the density matrix, and since now H_eff is no longer a Hermitian operator, the time evolution of the bra-part is guided by $H_\mathrm{eff}^\dagger$, not H_eff, and therefore actually

$$[H_\mathrm{eff}, \rho] = [H, \rho] - \frac{\mathrm{i}\hbar}{2} \sum_i \{C_i^\dagger C_i, \rho\} \,.$$

The consequence of this construction is the unitarity of (2). After inserting the definition of $[H_\mathrm{eff}, \rho]$ into the equation, we have

$$\operatorname{tr} \dot{\rho} = - \sum_j \operatorname{tr} \left(\frac{1}{2} C_i^\dagger C_i \rho + \frac{1}{2} \rho C_i^\dagger C_i - C_i \rho C_i^\dagger \right) . \tag{4}$$

The last term is called the **recycling term**, which makes the total probability increase, while the first two terms make the total probability decrease. With trace cyclic property, we find the total probability is conserved.

The Lindblad formalism is equivalent to the random wave function approach, which can be immediately seen from (2). The formalism is also useful for a many-body system, because here TODO: is H_eff Σ with imaginary part?

There might also be a question regarding why it's possible to insert C_i's directly into the EOM of density operator besides the terms introduced by the real Hamiltonian H; could there be some sort of interference between the interaction channels we consider dissipative and the interaction channels that merely correct H? In principle, it's possible, actually; but usually we just assume that they don't, which is reasonable most of the case (the spontaneously emitted photons go away very quickly and almost don't participate in the time evolution after their emission, for example). When there are interference between the two kinds of channels, the quantum master equation will be much more complicated, where terms outside $[\cdot, \cdot]$ have memory to the status in the past.

We consider a light-atom interacting system with RWA, where

$$H = \frac{\hbar}{2}\boldsymbol{\Omega}\cdot\boldsymbol{\sigma}, \quad C = \sqrt{\Gamma}\,|g\rangle\langle e|\,, \tag{5}$$

and we have

$$\begin{aligned}
\dot{\rho}_{gg} &= \frac{\mathrm{i}\Omega}{2}\rho_{ge} - \frac{\mathrm{i}\Omega^*}{2}\rho_{eg} + \Gamma\rho_{ee},\\
\dot{\rho}_{ee} &= -\dot{\rho}_{gg},\\
\dot{\rho}_{ge} &= \left(-\frac{\Gamma}{2} + \mathrm{i}\Delta\right)\rho_{ge} - \frac{\mathrm{i}\Omega}{2}(\rho_{ee} - \rho_{gg}),
\end{aligned} \tag{6}$$

where

$$\rho_{ee} + \rho_{gg} = 1, \quad \rho_{eg} = \rho_{ge}^*. \tag{7}$$

We define

$$\langle\sigma^x\rangle = \operatorname{Re}\rho_{eg} =: u, \quad \langle\sigma^y\rangle = \operatorname{Im}\rho_{eg} =: v,\ \langle\sigma^z\rangle = \rho_{ee} - \rho_{gg} =: w, \tag{8}$$

and

$$\boldsymbol{n} = (\,\langle\sigma^x\rangle\,,\,\langle\sigma^y\rangle\,,\,\langle\sigma^z\rangle), \tag{9}$$

and we have

$$\dot{\boldsymbol{n}} = \boldsymbol{\Omega}\times\boldsymbol{n} - \begin{pmatrix}\gamma_{\mathrm{T}}u\\ \gamma_{\mathrm{T}}v\\ \gamma_{\mathrm{L}}(w+1)\end{pmatrix}, \tag{10}$$

where

$$\gamma_{\mathrm{T}} = \frac{\Gamma}{2} \tag{11}$$

is called the **transverse damping rate** and

$$\gamma_{\mathrm{L}} = \Gamma \tag{12}$$

is called the **longitude relaxing rate**. (10) is called the **optical Bloch equation**.

Now we try to find a stable solution of (10). It is

$$\rho_{ee}^{\text{stable}} = \frac{(\Omega/\Gamma)^2}{1 + 2(\Omega/\Gamma)^2 + 4(\Delta/\Gamma)^2}, \tag{13}$$

and

$$\rho_{ge}^{\text{stable}} = -\frac{\Omega/2}{\Delta - \mathrm{i}\Gamma/2}(2\rho_{ee} - 1). \tag{14}$$

We define

$$S = 2\left(\frac{\Omega}{\Gamma}\right)^2. \tag{15}$$

We can also evaluate the response of the electric dipole. We have

$$\langle d\rangle = \rho_{eg}d_{ge} + \text{h.c.} = \alpha E + \text{c.c.}, \quad \alpha = \frac{1}{1 + S + 4(\Delta/\Gamma)^2}\left(\frac{2\Delta}{\Gamma} + \mathrm{i}\right)\frac{3\lambda^3}{4\pi^2}\epsilon_0 =: \alpha(I). \tag{16}$$

saturated absorption, Saturated absorption spectroscopy **Lamb dip**

2 Rate equation

We choose a *adiabatic* basis, which are dressed states of H_{eff}. In this basis, assuming that the non-diagonal elements of the density matrix damp quickly enough, we have

$$\dot{\rho}_{nn} = -\gamma_n\rho_{nn} + \sum_{m\neq n}\gamma_{nm}\rho_{mm}, \quad \rho_{mn}|_{m\neq n} = 0, \tag{17}$$

which is called the **rate equation**.

Again for a two-level system where RWA works, we have

$$\gamma_{\tilde{g}} = \Gamma \sin^2\theta, \quad \gamma_{\tilde{e}} = \Gamma\cos^2\theta, \gamma_{\tilde{g}\tilde{e}} = \Gamma\sin^4\theta, \gamma_{\tilde{e}\tilde{g}} = \Gamma\cos^4\theta, \tag{18}$$

and the rate equation is

$$\dot{\rho}_{\tilde{g}\tilde{g}} = (-\sin^4\theta\rho_{\tilde{g}\tilde{g}} + \cos^4\theta\rho_{\tilde{e}\tilde{e}})\Gamma. \tag{19}$$

The stable solution is

$$\rho_{\tilde{g}\tilde{g}}^{\text{stable}} = \frac{\cos^4\theta}{\cos^4\theta + \sin^4\theta}, \quad \rho_{\tilde{e}\tilde{e}}^{\text{stable}} = \frac{\sin^4\theta}{\cos^4\theta+\sin^4\theta}. \tag{20}$$

3 How atoms move in the space

All previous discussions were based on the assumption that atoms are somehow "fixed" or "trapped" at a given point. This is of course possible (using laser trap or something), but a more interesting case is when atoms are not that constrained. In this case, we need to take the spacial motion of atoms into account.

Consider a stationary mode in a cavity:

$$E = E_0\cos(kx)\mathrm{e}^{\mathrm{i}\omega t} + \text{c.c.}, \tag{21}$$

and at each point the energies of the ground state and the excited state of a two-level atom are different. Since

$$H = \frac{\hbar}{2}\boldsymbol{\Omega}\cdot\boldsymbol{\sigma} \propto E, \tag{22}$$

we have

$$m\left\langle \ddot{\boldsymbol{r}}\right\rangle = -\boldsymbol{\nabla}\left\langle H\right\rangle \propto -\boldsymbol{\nabla}E. \tag{23}$$

Sisyphus cooling

Note: there are some subtleties here.

$$\langle F\rangle = \operatorname{Re}\frac{\boldsymbol{\nabla}\Omega}{\Omega}\alpha_r|E|^2 + \operatorname{Im}\frac{\boldsymbol{\nabla}\Omega}{\Omega}\alpha_i|E|^2 + \tag{24}$$

We find that the conservative force causes cooling, while the scattering force causes heating.

Doppler cooling is another cooling approach We will find

$$\Delta \to \Delta - kv, \tag{25}$$

$$F = -\beta v \tag{26}$$

What Doppler cooling fails to take into account is the Sisyphus cooling or heating mechanism above.

4 A four-level atom model, and when polarization is important

$$H = \hbar\Delta(|e+\rangle\langle e+| + |e-\rangle\langle e-|) + \frac{\hbar\Omega_{\mathrm{T}}}{2}(\,|e+\rangle\langle g+| + \,|e-\rangle\langle g-|) + \text{h.c.} + \frac{\hbar\Omega_{\sigma-}}{2}(\,|e-\rangle\langle g+| + \text{h.c.}), \tag{27}$$

and the dressed states are

$$|\tilde{g}+\rangle = |g+\rangle + \frac{\Omega_{\sigma-}/2}{\Delta}|e-\rangle, \quad |\tilde{g}-\rangle = |g-\rangle. \tag{28}$$

We find $|g-\rangle$ actually has no coupling with the optical field and therefore is not dressed at all. States like this are often called **dark states**, because if a atom falls on such a state, no radiation will be seen. It does not mean these states are completely irrelevant, because quantum hopping may drive an atom to such a state.

In such a system we can realize **sub-Doppler cooling**, again by Sisyphus cooling. This was first found by Steven Chu, who found his cooling device mysteriously worked better than the estimation of Doppler cooling.