Homework 2

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1 Polarizability, absorption cross-section, stimulated emission, and optical amplification

(a) The EOM of a classical atom described by the Drude-Lorentz model is

$$\ddot{x} + \gamma \dot{x} + \omega_0^2 x = \frac{q}{m} E. \tag{1}$$

For a plane wave external electric field whose phasor is

$$\tilde{E} = \tilde{E}_0 \mathrm{e}^{-\mathrm{i}(\omega t - kz)}, \tag{2}$$

the response of x, in the form of phasor, is

$$\tilde{x} = \frac{q}{m} \frac{\tilde{E}}{-\omega^2 + \omega_0^2 - \mathrm{i}\gamma\omega}, \tag{3}$$

and therefore

$$\tilde{\dot{x}} = -\mathrm{i}\omega \frac{q}{m} \frac{\tilde{E}}{-\omega^2 + \omega_0^2 - \mathrm{i}\gamma\omega}, \tag{4}$$

and the time average power of dissipation is

$$\langle P_{\text{abs}} \rangle = \langle F_{\text{dissipation}} \cdot v \rangle = -\gamma \langle v \cdot v \rangle = -\gamma \cdot \frac{1}{2} \operatorname{Re} \tilde{\dot{x}}^* \cdot \tilde{\dot{x}} = -\frac{\gamma q^2}{2m^2} \frac{\omega^2 |\tilde{E}_0|^2}{(\omega^2 - \omega_0^2)^2 + \gamma^2 \omega^2}. \tag{5}$$

(b) The absorption cross section can be calculated by

$$\sigma_{\text{abs}} = \frac{|\langle P_{\text{abs}} \rangle|}{\text{input intensity}} = \frac{|\langle P_{\text{abs}} \rangle|}{\frac{1}{\mu_0} \frac{k}{\omega} \underbrace{E^2}_{\frac{1}{2}|\tilde{E}_0|^2}} = \frac{1}{c\epsilon_0} \frac{\omega^2}{(\omega^2 - \omega_0^2)^2 + \gamma^2 \omega^2} \gamma. \tag{6}$$

The absorption coefficient can then be found by multiplying the number density to σ_{abs}.

(c) Now consider a quantum two-level system described by

$$H = H_0 + H_{\text{dipole}}, \quad H_0 = \hbar \omega_{\mathrm{g}} |\mathrm{g}\rangle\langle \mathrm{g}| + \hbar \omega_{\mathrm{e}} |\mathrm{e}\rangle\langle \mathrm{e}|, \tag{7}$$

where

$$H_{\text{dipole}} = -\boldsymbol{\mu} \cdot \boldsymbol{E} = -\mu_{\mathrm{eg}} |\mathrm{e}\rangle\langle \mathrm{g}| E_{z0} \cos\omega t - \text{h.c.}, \tag{8}$$

and the z direction is set to the direction of $\boldsymbol{\mu}_{\mathrm{eg}}$. We start from the ground state in the $t \to -\infty$ limit, and the first order perturbation is

$$|\psi(t)\rangle = |\mathrm{g}\rangle \mathrm{e}^{-\mathrm{i}\omega_{\mathrm{g}} t} + \gamma_{\mathrm{g}}^{(1)} |\mathrm{g}\rangle \mathrm{e}^{-\mathrm{i}\omega_{\mathrm{g}} t} + \gamma_{\mathrm{e}}^{(1)} |\mathrm{e}\rangle \mathrm{e}^{-\mathrm{i}\omega_{\mathrm{e}} t}, \tag{9}$$

where

$$\frac{\mathrm{d}\gamma_{\mathrm{g}}^{(1)}}{\mathrm{d}t} = \frac{1}{\mathrm{i}\hbar} \gamma_{\mathrm{g}}^{(0)} \cdot \langle \mathrm{g} | H_{\text{dipole}} | \mathrm{g} \rangle = 0 \tag{10}$$

because dipole transition is only allowed between states with different parities, and

$$\begin{aligned} \frac{\mathrm{d}\gamma_{\mathrm{e}}^{(1)}}{\mathrm{d}t} &= \frac{1}{\mathrm{i}\hbar} \gamma_{\mathrm{g}}^{(0)} \mathrm{e}^{\mathrm{i}(\omega_{\mathrm{e}} - \omega_{\mathrm{g}})t} \cdot (-\mu_{\mathrm{eg}} E_{z0} \cos\omega t) \\ \Rightarrow \gamma_{\mathrm{e}}^{(1)}(t) - \underbrace{\Gamma_{\mathrm{e}}^{(1)}(t=0)}_{0} &= \mathrm{i} \frac{\mu_{\mathrm{eg}} E_{z0}}{\hbar} \int_{-\infty}^{t} \mathrm{d}t' \, \mathrm{e}^{\mathrm{i}(\omega_{\mathrm{e}} - \omega_{\mathrm{g}})t} \cos\omega t. \end{aligned} \tag{11}$$

Adding a small imaginary part to ω_e because of spontaneous emission:

$$\omega_\mathrm{e} \to \omega_\mathrm{e} - \mathrm{i}\Gamma_\mathrm{e}, \tag{12}$$

and by completing this integral and not throwing away the high-frequency term, we get

$$\gamma_\mathrm{e}^{(1)} = \mathrm{i}\frac{\mu_\mathrm{eg} E_{z0}}{\hbar} \cdot \frac{1}{2} \left(\frac{\mathrm{e}^{\mathrm{i}(\omega_\mathrm{eg} - \mathrm{i}\Gamma_\mathrm{e} + \omega)t}}{\mathrm{i}(\omega_\mathrm{eg} + \omega - \mathrm{i}\Gamma_\mathrm{e})} + \frac{\mathrm{e}^{\mathrm{i}(\omega_\mathrm{eg} - \mathrm{i}\Gamma_\mathrm{e} - \omega)t}}{\mathrm{i}(\omega_\mathrm{eg} - \omega - \mathrm{i}\Gamma_\mathrm{e})} \right), \quad \omega_\mathrm{eg} = \omega_\mathrm{e} - \omega_\mathrm{g}. \tag{13}$$

Here we have thrown away the terms at $t = -\infty$ because when $t \to -\infty$, $\mathrm{e}^{\Gamma_\mathrm{e} t}$ vanishes. $\Gamma_\mathrm{e}^{(1)}$ now explodes when $t \to \infty$ but this doesn't matter because we still need to multiply $\mathrm{e}^{-\mathrm{i}(\omega_\mathrm{e} - \mathrm{i}\Gamma_\mathrm{e})t}$ to it because ω_e in (9) also needs to be modified by the spontaneous emission rate. The expectation value of the dipole is

$$\begin{aligned} \langle \mu \rangle (t) = \langle \psi(t) | \mu | \psi(t) \rangle &= \Gamma_\mathrm{e}^{(1)} \mathrm{e}^{-\mathrm{i}(\omega_\mathrm{eg} - \mathrm{i}\Gamma_\mathrm{e})t} \mu_\mathrm{ge} + \text{h.c.} \\ &= \frac{|\mu_\mathrm{eg}|^2 E_{0z}}{\hbar} \cdot \frac{1}{2} \left(\frac{\mathrm{e}^{\mathrm{i}\omega t}}{\omega_\mathrm{eg} + \omega - \mathrm{i}\Gamma_\mathrm{e}} + \frac{\mathrm{e}^{-\mathrm{i}\omega t}}{\omega_\mathrm{eg} - \omega - \mathrm{i}\Gamma_\mathrm{e}} + \frac{\mathrm{e}^{-\mathrm{i}\omega t}}{\omega_\mathrm{eg} + \omega + \mathrm{i}\Gamma_\mathrm{e}} + \frac{\mathrm{e}^{\mathrm{i}\omega t}}{\omega_\mathrm{eg} - \omega + \mathrm{i}\Gamma_\mathrm{e}} \right). \end{aligned} \tag{14}$$

Here is a subtlety: in the above calculation, $\langle \psi(t) |$ is calculated using $H_0^\dagger$, because strictly speaking, here we are working with a non-Hermitian system because of the $-\mathrm{i}\Gamma_\mathrm{e}$ correction. Since

$$E = \frac{1}{2} E_{z0} (\mathrm{e}^{\mathrm{i}\omega t} + \mathrm{e}^{-\mathrm{i}\omega t}), \tag{15}$$

by looking at $\mathrm{e}^{-\mathrm{i}\omega t}$ components, we find the frequency-domain response function from E to μ is

$$\begin{aligned} \alpha(\omega) &= \frac{|\mu_\mathrm{eg}|^2}{\hbar} \left(\frac{1}{\omega_\mathrm{eg} - \omega - \mathrm{i}\Gamma_\mathrm{e}} + \frac{1}{\omega_\mathrm{eg} + \omega + \mathrm{i}\Gamma_\mathrm{e}} \right) \\ &\approx \frac{|\mu_\mathrm{eg}|^2}{\hbar} \cdot \frac{2\omega_\mathrm{eg}}{\omega_\mathrm{eg}^2 - \omega^2 - 2\mathrm{i}\Gamma_\mathrm{e}\omega}. \end{aligned} \tag{16}$$

In the last line we have used the $\Gamma_\mathrm{e} \ll \omega_\mathrm{e} - \omega_\mathrm{g}$ condition.

(d) The prefactor in (16) differs from that in the susceptibility of a harmonic oscillator, and hence the necessity of the f_i factor. But the structure of the denominator is the same: we have

$$\omega_0 = \omega_\mathrm{eg}, \quad \gamma = 2\Gamma_\mathrm{e}. \tag{17}$$

(e) We need to calculate energy loss due to spontaneous emission. Since

$$|\gamma_\mathrm{e}^{(1)}|^2 \propto \mathrm{e}^{-2\mathrm{i}\Gamma_\mathrm{e} t}, \tag{18}$$

we find that in time period $\mathrm{d}t$ the probability of spontaneous emission is $2\Gamma_\mathrm{e}\,\mathrm{d}t$, and therefore power of energy loss is

$$\begin{aligned} P_\mathrm{abs} &= 2\Gamma_\mathrm{e} \cdot \hbar(\omega_\mathrm{e} - \omega_\mathrm{g}) \left\langle \psi_\mathrm{e}^{(1)} \middle| \psi_\mathrm{e}^{(1)} \right\rangle \\ &= 2\Gamma_\mathrm{e} \cdot \hbar\omega_\mathrm{eg} \cdot \frac{|\mu_\mathrm{eg}|^2 E_{z0}^2}{\hbar^2} \cdot \frac{1}{2} \left(\frac{\mathrm{e}^{\mathrm{i}(\omega_\mathrm{eg} - \mathrm{i}\Gamma_\mathrm{e} + \omega)t}}{\omega_\mathrm{eg} + \omega - \mathrm{i}\Gamma_\mathrm{e}} + \frac{\mathrm{e}^{\mathrm{i}(\omega_\mathrm{eg} - \mathrm{i}\Gamma_\mathrm{e} - \omega)t}}{\omega_\mathrm{eg} - \omega - \mathrm{i}\Gamma_\mathrm{e}} \right) \cdot \frac{1}{2} \left(\frac{\mathrm{e}^{-\mathrm{i}(\omega_\mathrm{eg} - \mathrm{i}\Gamma_\mathrm{e} + \omega)t}}{\omega_\mathrm{eg} + \omega + \mathrm{i}\Gamma_\mathrm{e}} + \frac{-\mathrm{e}^{\mathrm{i}(\omega_\mathrm{eg} - \mathrm{i}\Gamma_\mathrm{e} - \omega)t}}{\omega_\mathrm{eg} - \omega + \mathrm{i}\Gamma_\mathrm{e}} \right). \end{aligned} \tag{19}$$

We need to ignore the high oscillation terms, because they are sine waves and have zero expectation; this is the same as what we did for the harmonic oscillator. So

$$\langle P_\mathrm{abs} \rangle = \frac{1}{2} \Gamma_\mathrm{e} \omega_\mathrm{eg} \frac{|\mu_\mathrm{eg}|^2 E_{0z}^2}{\hbar} \left(\frac{1}{(\omega_\mathrm{eg} + \omega)^2 + \Gamma_\mathrm{e}^2} + \frac{1}{(\omega_\mathrm{eg} - \omega)^2 + \Gamma_\mathrm{e}^2} \right). \tag{20}$$

The absorption cross section then is

$$\sigma_\mathrm{abs} = \frac{\langle P_\mathrm{abs} \rangle}{\underbrace{\langle S \rangle}_{\frac{1}{2}\epsilon_0 c E_{0z}^2}} = \frac{|\mu_\mathrm{eg}|^2 \Gamma_\mathrm{e} \omega_\mathrm{eg}}{\hbar \epsilon_0 c} \left(\frac{1}{(\omega_\mathrm{eg} + \omega)^2 + \Gamma_\mathrm{e}^2} + \frac{1}{(\omega_\mathrm{eg} - \omega)^2 + \Gamma_\mathrm{e}^2} \right). \tag{21}$$

(f) Now we consider the same system but the initial state is the excited state. Correspondingly, the imaginary part of the energy now comes to the ground state, which denotes the probability for the atom to be brought to the excited state due to external pumping. Repeating the above procedure but this time replacing ω_g with $\omega_\mathrm{g} - \mathrm{i}\Gamma_\mathrm{g}$, we have

$$\gamma_\mathrm{g}^{(1)}(t) = \mathrm{i}\frac{\mu_\mathrm{ge} E_{z0}}{\hbar} \cdot \frac{1}{2} \left(\frac{\mathrm{e}^{-\mathrm{i}(\omega_\mathrm{eg}+\mathrm{i}\Gamma_\mathrm{g}+\omega)t}}{-\omega_\mathrm{eg} - \omega - \mathrm{i}\Gamma_\mathrm{g}} + \frac{\mathrm{e}^{\mathrm{i}(-\omega_\mathrm{eg}-\mathrm{i}\Gamma_\mathrm{g}+\omega)t}}{-\omega_\mathrm{eg} + \omega - \mathrm{i}\Gamma_\mathrm{g}} \right), \tag{22}$$

and

$$\begin{aligned} \alpha(\omega) &= \frac{|\mu_\mathrm{eg}|^2}{\hbar^2} \left(\frac{1}{-\omega_\mathrm{eg} - \omega - \mathrm{i}\Gamma_\mathrm{g}} + \frac{1}{-\omega_\mathrm{eg} + \omega + \mathrm{i}\Gamma_\mathrm{g}} \right) \\ &\approx \frac{|\mu_\mathrm{eg}|^2}{\hbar^2} \frac{-2\omega_\mathrm{eg}}{\omega_\mathrm{eg}^2 - \omega^2 - 2\mathrm{i}\Gamma_\mathrm{g}\omega}. \end{aligned} \tag{23}$$

This can be obtained by swapping $|\mathrm{e}\rangle$ and $|\mathrm{g}\rangle$ in (16).

(g) Again by swapping the two energy levels, we have

$$\langle P_\mathrm{abs} \rangle = -\frac{1}{2}\Gamma_\mathrm{g}\omega_\mathrm{eg} \frac{|\mu_\mathrm{eg}|^2 E_{0z}^2}{\hbar} \left(\frac{1}{(\omega_\mathrm{eg}+\omega)^2 + \Gamma_\mathrm{g}^2} + \frac{1}{(\omega_\mathrm{eg}-\omega)^2 + \Gamma_\mathrm{g}^2} \right). \tag{24}$$

The absorption power is negative, because the system is gaining energy, not losing it. The stimulated emission cross section then is

$$\sigma_\mathrm{st} = -\frac{|\mu_\mathrm{eg}|^2 \Gamma_\mathrm{g}\omega_\mathrm{eg}}{\hbar\epsilon_0 c} \left(\frac{1}{(\omega_\mathrm{eg}+\omega)^2 + \Gamma_\mathrm{e}^2} + \frac{1}{(\omega_\mathrm{eg}-\omega)^2 + \Gamma_\mathrm{e}^2} \right). \tag{25}$$

(h) Suppose we have a dilute gas containing N atoms per unit volume, and the excited and ground state populations are $N_\mathrm{e} = Np_\mathrm{e}$ and $N_\mathrm{g} = Np_\mathrm{g}$. Since the gas is dilute, we don't need to analyze the interferences of the Γ_g and Γ_e processes, and we have

$$\begin{aligned} \chi(\omega) &= N_\mathrm{g}\alpha_\mathrm{abs}(\omega) + N_\mathrm{e}\alpha_\mathrm{st}(\omega) \\ &= N\frac{|\mu_\mathrm{eg}|^2}{\hbar} \cdot \left(p_\mathrm{g}\frac{2\omega_\mathrm{eg}}{\omega_\mathrm{eg}^2 - \omega^2 - 2\mathrm{i}\Gamma_\mathrm{e}\omega} - p_\mathrm{e}\frac{2\omega_\mathrm{eg}}{\omega_\mathrm{eg}^2 - \omega^2 - 2\mathrm{i}\Gamma_\mathrm{g}\omega} \right). \end{aligned} \tag{26}$$

We want $\operatorname{Im}\chi < 0$ to have a negative imaginary part in $\boldsymbol{k}$, and therefore $\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}}$ increases as the light goes forward. Assuming $\Gamma_\mathrm{g} = \Gamma_\mathrm{e}$, we find the condition needed is

$$p_\mathrm{e} > p_\mathrm{g}. \tag{27}$$

2 Spontaneous decay rate computed from Fermi's golden rule and Poynting's Theorem

(a) For a two-level atom, the interaction Hamiltonian between it and the electromagnetic field in vacuum is

$$H_\mathrm{dipole} = -\boldsymbol{\mu}\cdot\boldsymbol{E}. \tag{28}$$

The electric field, as a quantum operator, is

$$\boldsymbol{E}(\boldsymbol{r}) = \sum_{\boldsymbol{k},\sigma} \sqrt{\frac{\hbar\omega_{\boldsymbol{k}}}{2\epsilon_0 V}} \hat{\boldsymbol{k}}\boldsymbol{\epsilon}_\sigma \mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}} a_{\boldsymbol{k}\sigma} + \text{h.c.} \tag{29}$$

The term in H_dipole that are relevant to the excited state-to-ground state transition is

$$\langle \mathrm{g}, n_{\boldsymbol{k}\sigma} = 1 | H_\mathrm{dipole} | \mathrm{e}, 0 \rangle = -\sqrt{\frac{\hbar\omega_{\boldsymbol{k}}}{2\epsilon_0 V}} \mu_\mathrm{ge} \mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}} \cos\theta_{\boldsymbol{k}\sigma}. \tag{30}$$

By Fermi's golden rule, the spontaneous decay rate is

$$\begin{aligned}\Gamma_{\mathrm{e}\to\mathrm{g}} &= \frac{2\pi}{\hbar}\sum_{\boldsymbol{k},\sigma}\left|\langle \mathrm{g}, n_{\boldsymbol{k}\sigma}=1|H_{\text{dipole}}|\mathrm{e},0\rangle\right|^2\delta(\hbar\omega_{\mathrm{e}}-\hbar\omega_{\mathrm{g}}-\hbar\omega_{\boldsymbol{k}\sigma}) \\ &= \frac{2\pi}{\hbar^2}\frac{\hbar}{2\epsilon_0 V}\left|\mu_{\mathrm{eg}}\right|^2\sum_{\boldsymbol{k},\sigma}\omega_{\boldsymbol{k}}\cos^2\theta_{\boldsymbol{k}\sigma}\delta(\omega_{\mathrm{eg}}-\omega_{\boldsymbol{k}\sigma}) \\ &= \frac{2\pi}{\hbar^2}\frac{\hbar}{2\epsilon_0 V}\left|\mu_{\mathrm{eg}}\right|^2\cdot V\int \mathrm{d}\omega\,\omega D(\omega)\int\frac{\mathrm{d}\Omega}{4\pi}\omega\cos^2\theta\delta(\omega_{\mathrm{eg}}-\omega) \\ &= \frac{2\pi}{\hbar^2}\frac{\hbar}{2\epsilon_0 V}\left|\mu_{\mathrm{eg}}\right|^2\cdot V\cdot\omega_{\mathrm{eg}}D(\omega_{\mathrm{eg}})\cdot\frac{1}{3},\end{aligned} \tag{31}$$

where we have decomposed the $\boldsymbol{k}$ dependence of the transition matrix element into dependence on ω and on $\hat{\boldsymbol{k}}$, and by averaging over the latter, the $\cos^2\theta_{\boldsymbol{k}\sigma}$ factor evaluates to $1/3$. The density of state $D(\omega)$ contains a factor of 2 because of polarization degeneracy; for a single polarization the DOS in three-dimensional space is

$$D_{\text{single polarization}} = \frac{\omega^2}{2\pi^2 c^3}, \tag{32}$$

so finally we have

$$\Gamma_{\mathrm{e}\to\mathrm{g}} = \frac{\omega_{\mathrm{eg}}^3\left|\mu_{\mathrm{eg}}\right|^2}{3\pi\hbar\epsilon_0 c^3}. \tag{33}$$

(b) When the state of the atom is

$$|\psi\rangle = c_{\mathrm{g}}\,|\mathrm{g}\rangle + c_{\mathrm{e}}\,|\mathrm{e}\rangle \tag{34}$$

at $t=0$, the time-dependent wave function is

$$|\psi(t)\rangle = c_{\mathrm{e}}\mathrm{e}^{-\mathrm{i}\omega_{\mathrm{e}}t}\,|\mathrm{e}\rangle + c_{\mathrm{g}}\mathrm{e}^{-\mathrm{i}\omega_{\mathrm{g}}t}\,|\mathrm{g}\rangle\,. \tag{35}$$

Since the dipole operator only has non-zero matrix element between $|\mathrm{g}\rangle$ and $|\mathrm{e}\rangle$, we have

$$\langle\boldsymbol{\mu}\rangle\,(t) = (c_{\mathrm{e}}^* c_{\mathrm{g}}\mu_{\mathrm{eg}}\mathrm{e}^{\mathrm{i}\omega_{\mathrm{eg}}t} + c_{\mathrm{g}}^* c_{\mathrm{e}}\mu_{\mathrm{ge}}\mathrm{e}^{-\mathrm{i}\omega_{\mathrm{eg}}t})\hat{\boldsymbol{z}}. \tag{36}$$

(c) From classical electrodynamics (calculating $\boldsymbol{E}$ and $\boldsymbol{B}$ from the retarded potential radiated by a constantly oscillating dipole), we have

$$P_{\text{rad}} = \frac{\mu_0\omega^4\left|\mu\right|^2}{12\pi c}, \tag{37}$$

where the dipole, in the phasor form, is

$$\tilde{\boldsymbol{\mu}} = \mu\mathrm{e}^{-\mathrm{i}\omega t}\hat{\boldsymbol{z}}. \tag{38}$$

Applying this to (36), whose phasor form is

$$\langle\tilde{\boldsymbol{\mu}}\rangle(t) = 2c_{\mathrm{g}}^* c_{\mathrm{e}}\mu_{\mathrm{ge}}\mathrm{e}^{-\mathrm{i}\omega_{\mathrm{eg}}t}\hat{\boldsymbol{z}}, \tag{39}$$

we find the radiation power of the atom – if we assume its radiation somehow can be captured by classical electrodynamics – is

$$P_{\text{rad}} = \frac{\mu_0\omega_{\mathrm{eg}}^4}{3\pi c}\left|c_{\mathrm{g}}\right|^2\left|c_{\mathrm{e}}\right|^2\left|\mu_{\mathrm{eg}}\right|^2 = \frac{\mu_0\omega_{\mathrm{eg}}^4}{3\pi c}p_{\mathrm{e}}p_{\mathrm{g}}\left|\mu_{\mathrm{eg}}\right|^2 = \frac{\omega_{\mathrm{eg}}^4}{3\pi\epsilon_0 c^3}p_{\mathrm{e}}p_{\mathrm{g}}\left|\mu_{\mathrm{eg}}\right|^2. \tag{40}$$

(d) (40) naturally leads to the following rate equation:

$$\frac{\mathrm{d}E}{\mathrm{d}t} = -P_{\text{rad}}, \tag{41}$$

where

$$E = \hbar\omega_{\mathrm{e}}p_{\mathrm{e}} + \hbar\omega_{\mathrm{eg}}p_{\mathrm{g}}, \quad p_{\mathrm{e}}+p_{\mathrm{g}} = 1. \tag{42}$$

The rate equation concerning p_{e} is therefore

$$\frac{\mathrm{d}p_{\mathrm{e}}}{\mathrm{d}t} = -\frac{\omega_{\mathrm{eg}}^3}{3\pi\epsilon_0 c^3}\left|\mu_{\mathrm{eg}}\right|^2 p_{\mathrm{e}}\underbrace{(1-p_{\mathrm{e}})}_{p_{\mathrm{g}}}. \tag{43}$$

(e) When $p_{\mathrm{e}} \simeq 1$ and $p_{\mathrm{g}} \ll 1$, and we work in a period of time that's much smaller than the decay time (so that the condition that the atom is most likely to be in the excited state is always true), from the rate equation, the decay rate is

$$\Gamma_{\mathrm{e}\to\mathrm{g}} = \frac{\omega_{\mathrm{eg}}^3}{3\pi\epsilon_0 c^3}\left|\mu_{\mathrm{eg}}\right|^2 p_{\mathrm{g}}. \tag{44}$$