Homework 2

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1 Polarization of electromagnetic field

1.1 The general form of a pure state

We have (assuming $\hat{\boldsymbol{k}} = \hat{\boldsymbol{z}}$)

$$\boldsymbol{E} = E_x \begin{pmatrix} 1 \\ 0 \end{pmatrix} + E_y \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \sqrt{|E_x|^2 + |E_y|^2} \mathrm{e}^{\mathrm{i}\varphi_x} \begin{pmatrix} \frac{|E_x|}{\sqrt{|E_x|^2+|E_y|^2}} \\ \mathrm{e}^{\mathrm{i}(\varphi_y - \varphi_x)} \frac{|E_y|}{\sqrt{|E_x|^2+|E_y|^2}} \end{pmatrix},$$

and by defining

$$E_0 = \sqrt{|E_x|^2 + |E_y|^2} \mathrm{e}^{\mathrm{i}\varphi_x}, \tag{1}$$

$$\cos\theta = \frac{|E_x|}{\sqrt{|E_x|^2 + |E_y|^2}}, \tag{2}$$

and

$$\phi = \varphi_y - \varphi_x, \tag{3}$$

we find

$$\boldsymbol{E} = E_0 \begin{pmatrix} \cos\theta \\ \mathrm{e}^{\mathrm{i}\phi} \sin\theta \end{pmatrix}. \tag{4}$$

We have

$$|H\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |V\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \tag{5}$$

and therefore after normalization, we have

$$\rho = (\cos\theta\,|H\rangle + \mathrm{e}^{\mathrm{i}\phi}\sin\theta\,|V\rangle)(\cos\theta\,\langle H| + \mathrm{e}^{-\mathrm{i}\phi}\sin\theta\,\langle V|) = \begin{pmatrix} \cos^2\theta & \mathrm{e}^{-\mathrm{i}\phi}\sin\theta\cos\theta \\ \mathrm{e}^{\mathrm{i}\phi}\sin\theta\cos\theta & \sin^2\theta \end{pmatrix}. \tag{6}$$

1.2 The pure state $\rho^2 = \rho$ condition

We can prove the pure state condition $\rho^2 = \rho$ explicitly:

$$\begin{aligned} \rho^2 &= \begin{pmatrix} \cos^4\theta + \sin^2\theta\cos^2\theta & \mathrm{e}^{-\mathrm{i}\phi}\sin\theta\cos^3\theta + \mathrm{e}^{-\mathrm{i}\phi}\sin^3\theta\cos\theta \\ \mathrm{e}^{\mathrm{i}\phi}\sin\theta\cos^3\theta + \mathrm{e}^{\mathrm{i}\phi}\sin^3\theta\cos\theta & \sin^2\theta\cos^2\theta + \sin^4\theta \end{pmatrix} \\ &= \begin{pmatrix} \cos^2\theta & \mathrm{e}^{-\mathrm{i}\phi}\sin\theta\cos\theta \\ \mathrm{e}^{\mathrm{i}\phi}\sin\theta\cos\theta & \sin^2\theta \end{pmatrix} = \rho. \end{aligned} \tag{7}$$

1.3 Mixed state

The condition that ρ is Hermite means it can be written as

$$\rho = R(\sigma^0 + x\sigma^x + y\sigma^y + z\sigma^z),$$

where $R, x, y, z \in \mathbb{R}$, because the σ matrices constitute a basis for all Hermite matrices in $\mathbb{C}^{2\times 2}$. Since $\sigma^{x,y,z}$ are traceless, from the condition $\operatorname{tr}\rho = 1$, we have

$$1 = \operatorname{tr}\rho = R\operatorname{tr}\sigma^0 = 2R \Rightarrow R = \frac{1}{2},$$

so

$$\rho = \frac{1}{2}(\sigma^0 + x\sigma^x + y\sigma^y + z\sigma^z). \tag{8}$$

In the matrix form, we have

$$\rho = \begin{pmatrix} \frac{1+z}{2} & \frac{x-\mathrm{i}y}{2} \\ \frac{x+\mathrm{i}y}{2} & \frac{1-z}{2} \end{pmatrix},$$

and by substitution of variables (this is a three variables to three variables mapping, and therefore is valid)

$$\frac{1}{2}(1-p) + p\cos^2\theta = \frac{1+z}{2}, \quad x = p\cos\phi\sin 2\theta, \quad y = p\sin\phi\sin 2\theta,$$

we get

$$\frac{1-z}{2} = \frac{1}{2}(1-p) + p\sin^2\theta,$$

and therefore

$$\rho = (1-p)\begin{pmatrix} \frac{1}{2} & \\ & \frac{1}{2} \end{pmatrix} + p\begin{pmatrix} \cos^2\theta & \mathrm{e}^{-\mathrm{i}\phi}\sin\theta\cos\theta \\ \mathrm{e}^{\mathrm{i}\phi}\sin\theta\cos\theta & \sin^2\theta \end{pmatrix}. \tag{9}$$

1.4 Jones parameters and Stokes formalism

The definition of Stokes parameters are

$$\begin{aligned} I &= \langle E_x^2 \rangle + \langle E_y^2 \rangle, \\ Q &= \langle E_x^2 \rangle - \langle E_y^2 \rangle, \\ U &= \langle E_a^2 \rangle - \langle E_b^2 \rangle, \\ V &= \langle E_l^2 \rangle - \langle E_r^2 \rangle, \end{aligned} \tag{10}$$

where

$$\hat{\boldsymbol{a}} = \frac{1}{\sqrt{2}}(\hat{\boldsymbol{x}} + \hat{\boldsymbol{y}}), \quad \hat{\boldsymbol{b}} = \frac{1}{\sqrt{2}}(\hat{\boldsymbol{x}} - \hat{\boldsymbol{y}}), \tag{11}$$

and

$$\hat{\boldsymbol{l}} = \frac{1}{\sqrt{2}}(\hat{\boldsymbol{x}} + \mathrm{i}\hat{\boldsymbol{y}}), \quad \hat{\boldsymbol{r}} = \frac{1}{\sqrt{2}}(\hat{\boldsymbol{x}} - \mathrm{i}\hat{\boldsymbol{y}}). \tag{12}$$

Note that $E_{x,y}$ etc. above may be regarded as operators, and we have

$$\begin{aligned} E_x^2 + E_y^2 &= \sigma^0, \\ E_x^2 - E_y^2 &= \sigma^z, \\ E_a^2 - E_b^2 &= \sigma^x, \\ E_l^2 - E_r^2 &= \sigma^y. \end{aligned} \tag{13}$$

From these definitions and the fact that $(\sigma^i)^2 = \sigma^0$ and all other products of σ matrices are traceless, we find

$$I = \langle \sigma^0 \rangle = 1, \tag{14}$$

$$Q = \langle \sigma^z \rangle = \frac{1}{2}z \cdot 2 = 2p\cos^2\theta - p = p\cos 2\theta, \tag{15}$$

$$U = \frac{1}{2}x \cdot 2 = p\cos\phi\sin 2\theta, \tag{16}$$

and

$$V = \frac{1}{2}y \cdot 2 = p\sin\phi\sin 2\theta. \tag{17}$$

Here I is constantly 1, because we are working with the single-photon density matrix.

1.5 Mueller calculus

As is shown above, Mueller calculus actually works on the coefficients in (8), and therefore a Mueller matrix essentially gives the coefficients of $\rho \to U\rho U^\dagger$. It makes sense as long as after its application, the σ^0 component in ρ is still 1/2.

1.6 Transformation and measurement

We have

$$|45\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$
$$|\mathrm{rcp}\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -\mathrm{i} \end{pmatrix} . \quad (18)$$

The correspond density matrices are

$$\rho_{45} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \quad (19)$$

and

$$\rho_{\mathrm{rcp}} = \frac{1}{2}\begin{pmatrix} 1 & \mathrm{i} \\ -\mathrm{i} & 1 \end{pmatrix}. \quad (20)$$

The operator

$$U_{\mathrm{rcp}} = \begin{pmatrix} 1 & \\ & -\mathrm{i} \end{pmatrix} \quad (21)$$

then turns ρ_{45} to ρ_{rcp}:

$$U_{\mathrm{rcp}}\rho_{45}U_{\mathrm{rcp}}^{\dagger} = \frac{1}{2}\begin{pmatrix} 1 & \\ & -\mathrm{i} \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & \\ & \mathrm{i} \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & \mathrm{i} \\ -\mathrm{i} & 1 \end{pmatrix} = \rho_{\mathrm{rcp}}. \quad (22)$$

The horizontal polarizer operator

$$\mathcal{O} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad (23)$$

is not unitary, because it has non-unitary eigenvalue 0. It is a projection operator: it takes in a beam polarized light and returns its x component. It also represents a measurement: we can use it in a projective measurement setting. In a projective measurement with operator $\mathcal{O}$, $\mathrm{tr}(\rho\mathcal{O})$ is the probability that after measurement, the final state of the system falls into the subspace determined by $\mathcal{O}$. In our case, the subspace determined by $\mathcal{O}$ is the subspace of horizontal polarization, so $\mathrm{tr}(\rho\mathcal{O})$ is the probability that after measurement, we find ρ to be a horizontally polarized state.

Application of $\mathcal{O}$ on (9) is

$$\mathcal{O}\rho\mathcal{O}^{\dagger} = (1-p)\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 0 \end{pmatrix} + p\begin{pmatrix} \cos^2\theta & 0 \\ 0 & 0 \end{pmatrix} = \left(\frac{1}{2} + \frac{p}{2}\cos 2\theta\right)\mathcal{O}. \quad (24)$$

So after the application of $\mathcal{O}$, we get a horizontally polarized state, as is expected. The result is not normalized; the factor before $\mathcal{O}$ is just $\mathrm{tr}(\rho\mathcal{O})$, which is the probability that after measurement, we find ρ to be horizontally polarized. When $p=0$, it's $1/2$, which is expected for the unpolarized state; when $p=1$, it's $\cos^2\theta$, again the correct answer.

2 The $\rho^2 = \rho$ condition for pure states

Suppose

$$\rho = |\psi\rangle\langle\psi|, \quad |\psi\rangle = \sum_m a_m |m\rangle . \quad (25)$$

We have

$$\begin{aligned}
\rho^2 &= \sum_{m,n} a_m^* a_n \left|n\right\rangle \left\langle m\right| \sum_{j,k} a_j^* a_k \left|k\right\rangle \left\langle j\right| \\
&= \sum_{m,n,j,k} a_m^* a_n a_j^* \left|n\right\rangle \left\langle m|k\right\rangle \left\langle j\right| \\
&= \sum_{m,n,j,k} a_m^* a_n a_j^* a_k \left|n\right\rangle\!\left\langle j\right| \delta_{mk} \\
&= \underbrace{\sum_m a_m^* a_m}_{=\langle\psi|\psi\rangle=1} \sum_{n,j} a_n a_j^* \left|n\right\rangle\!\left\langle j\right| \\
&= \sum_{n,j} a_n a_j^* \left|n\right\rangle\!\left\langle j\right| = \rho.
\end{aligned} \tag{26}$$

3 Ammonia molecule

The Hamiltonian of the two low-energy states of ammonia is

$$H = \begin{pmatrix} 0 & \Delta/2 \\ \Delta/2 & 0 \end{pmatrix}, \tag{27}$$

where we set

$$\left|L\right\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \left|R\right\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{28}$$

This Hamiltonian is just a scaled σ^x matrix, and its eigenstates are straightforwardly given by

$$\left|+\right\rangle = \frac{1}{\sqrt{2}}(\left|L\right\rangle + \left|R\right\rangle), \quad \left|-\right\rangle = \frac{1}{\sqrt{2}}(\left|L\right\rangle - \left|R\right\rangle), \tag{29}$$

and the energies are

$$E_+ = \Delta/2, \quad E_- = -\Delta/2. \tag{30}$$

After an electric field is added, in $\left|L\right\rangle$ we have an additional energy contribution, and since the molecular configuration in $\left|R\right\rangle$ is the opposite of the one in $\left|L\right\rangle$, we have

$$H = \begin{pmatrix} dE/2 & \Delta/2 \\ \Delta/2 & -dE/2 \end{pmatrix}. \tag{31}$$

Solving

$$\det \begin{pmatrix} dE/2 - \lambda & \Delta/2 \\ \Delta/2 & -dE/2 - \lambda \end{pmatrix} = 0,$$

we get

$$E_\pm = \pm\frac{1}{2}\sqrt{\Delta^2 + d^2E^2}, \tag{32}$$

and hence

$$\left|-\right\rangle = \frac{dE - \sqrt{d^2E^2 + \Delta^2}}{\sqrt{\Delta^2 + (\sqrt{d^2E^2 + \Delta^2} - dE)^2}} \left|L\right\rangle + \frac{\Delta}{\sqrt{\Delta^2 + (\sqrt{d^2E^2 + \Delta^2} - dE)^2}} \left|R\right\rangle, \tag{33}$$

and

$$\left|+\right\rangle = \frac{dE + \sqrt{d^2E^2 + \Delta^2}}{\sqrt{\Delta^2 + (\sqrt{d^2E^2 + \Delta^2} + dE)^2}} \left|L\right\rangle + \frac{\Delta}{\sqrt{\Delta^2 + (\sqrt{d^2E^2 + \Delta^2} + dE)^2}} \left|R\right\rangle. \tag{34}$$

When E is large, we have

$$\begin{aligned}
dE - \sqrt{d^2E^2 + \Delta^2} &\approx 0, \\
dE + \sqrt{d^2E^2 + \Delta^2} &\approx 2dE,
\end{aligned}$$

and therefore

$$E_+ = \frac{1}{2}dE, \quad \left|+\right\rangle = \left|L\right\rangle, \tag{35}$$

and

$$E_- = -\frac{1}{2}dE, \quad |-\rangle = |R\rangle\,. \tag{36}$$

This is expected, because when $Ed \gg \Delta$, the non-diagonal terms in the Hamiltonian can be safely ignored.

4 Kaptiza's pendulum

4.1 Integrating out the fast variable

In the $\omega \to \infty$, $F_0 \to 0$ limit, the high-frequency part and the low-frequency part of the solution of

$$mR\ddot{\theta} = (-mg + F_0 \sin\omega t)\sin\theta \tag{37}$$

are not strongly coupled and the high-frequency degree of freedom can be integrated out to get an effective theory of the low-frequency part. We do the decomposition

$$\theta = \theta_f + \theta_s, \tag{38}$$

where θ_f is the fast variable. Observing (37), we find the EOM of θ_f should be

$$mR\ddot{\theta}_f = F_0 \sin\omega t \sin(\theta_f + \theta_s), \tag{39}$$

because the first term on the RHS of (37) has a much lower frequency magnitude compared with ω. We take the first order approximation of (39) and ignore the θ_f dependency on the RHS, and this gives

$$\theta_f = -\frac{F_0}{mR\omega^2}\sin\theta_s \sin\omega t. \tag{40}$$

Putting this back to (37), we get

$$\begin{aligned} mR\left(\frac{F_0}{mR}\sin\theta_s\sin\omega t + \ddot{\theta}_s\right) &= (-mg + F_0\sin\omega t)\sin\left(\theta_s - \frac{F_0}{mR\omega^2}\sin\theta_s\sin\omega t\right) \\ &= (-mg + F_0\sin\omega t)\left(\sin\theta_s - \cos\theta_s\cdot\frac{F_0}{mR\omega^2}\sin\theta_s\sin\omega t\right). \end{aligned}$$

Now we average over all high-frequency time dependencies. The first term on the LHS averages zero, and so do the $-mg\sin\omega t$ term and the $F_0\sin\omega t\sin\theta_s$ term on the RHS. On the other hand, the $\sin^2\omega t$ term on the RHS averages

$$-\frac{F_0^2}{mR\omega^2}\sin\theta_s\cos\theta_s\left\langle\sin^2\omega t\right\rangle = -\frac{1}{2}\frac{F_0^2}{mR\omega^2}\sin\theta_s\cos\theta_s,$$

so the final EOM for θ_s is

$$mR\ddot{\theta}_s = -mg\sin\theta_s - \frac{1}{2}\frac{F_0^2}{mR\omega^2}\sin\theta_s\cos\theta_s. \tag{41}$$

4.2 Stable positions of θ_s

We let the LHS of (41) be zero, and the equation becomes

$$\sin\theta_s\left(mg + \frac{1}{2}\frac{F_0^2}{mR\omega^2}\cos\theta_s\right) = 0.$$

Since $F_0 \to 0$, the second factor on the LHS can't be zero, so the equation becomes

$$\sin\theta_s = 0 \Rightarrow \theta_s = 0, \pi. \tag{42}$$

Around $\theta_s = 0$, (41) is approximately

$$mR\ddot{\theta}_s = -mg\theta_s - \frac{1}{2}\frac{F_0^2}{mR\omega^2}\theta_s,$$

and therefore

$$\omega_{\theta=0} = \sqrt{\frac{g}{R} + \frac{F_0^2}{m^2 R^2 \omega^2}}. \tag{43}$$

This is always real, and therefore the $\theta_s = 0$ position is always stable.

Around $\theta_s = \pi$, we rewrite (41) in terms of $\theta_s' = \pi - \theta_s$, and get

$$-mR\ddot{\theta}_s' = mg\theta_s' - \frac{1}{2}\frac{F_0^2}{mR\omega^2}\theta_s' = 0,$$

and

$$\omega_{\theta_s=\pi} = \sqrt{\frac{1}{2}\frac{F_0^2}{m^2 R^2 \omega^2} - \frac{g}{R}}. \tag{44}$$

It can be seen that when $F_0 = 0$, the frequency is imaginary and therefore $\theta_s = \pi$ is not a stable position. However, when

$$\frac{1}{2}\frac{F_0^2}{m^2 R^2 \omega^2} \geq \frac{g}{R}, \tag{45}$$

we do have oscillation behavior around $\theta_s = \pi$.

5 Relaxation of a spin polarization due to an electric field

5.1 The spin-magnetic field coupling

We have

$$H = -\boldsymbol{\mu} \cdot \boldsymbol{B}, \quad \boldsymbol{\mu} = -g\mu_{\mathrm{B}}\boldsymbol{S}, \tag{46}$$

and this means

$$H = \frac{1}{2} g\mu_{\mathrm{B}} \boldsymbol{\sigma} \cdot \boldsymbol{B}. \tag{47}$$

This gives

$$\frac{\mathrm{d}\boldsymbol{S}}{\mathrm{d}t} = \frac{g\mu_{\mathrm{B}}}{\hbar} \boldsymbol{B} \times \boldsymbol{S}. \tag{48}$$

When $\boldsymbol{B}$ is fixed (instead of the motion magnetic field in (55)), we find the oscillation frequency is

$$\omega = \frac{g\mu_{\mathrm{B}}}{\hbar} B, \tag{49}$$

and therefore

$$\gamma = \frac{g\mu_{\mathrm{B}}}{\hbar}. \tag{50}$$

5.2 The $\sigma_\pm$ representation

We define

$$\sigma_\pm = \frac{\sigma_x \pm \mathrm{i}\sigma_y}{2}, \tag{51}$$

and the commutation relations are

$$[\sigma_+, \sigma_-] = \sigma_z, \quad [\sigma^z, \sigma_\pm] = \pm 2\sigma_\pm. \tag{52}$$

The Hamiltonian is then

$$H = b^* \sigma_+ + b\sigma_- + b_z \sigma_z, \tag{53}$$

where

$$b = \frac{1}{2} g\mu_{\mathrm{B}} (B_x + \mathrm{i}B_y), \quad b_z = \frac{1}{2} g\mu_{\mathrm{B}} B_z. \tag{54}$$

5.3 Eliminating the static magnetic field

The magnetic field felt by an electron wit velocity $\boldsymbol{v}$ when an electric field $\boldsymbol{E}$ is present is

$$\boldsymbol{B}_1 = -\boldsymbol{v} \times \boldsymbol{E}/c^2. \tag{55}$$

We also have a static magnetic field $\boldsymbol{B}_0$ pointing towards $\hat{\boldsymbol{z}}$. Here we assume $\boldsymbol{B}_0$ and the motion magnetic field are orthogonal to each other, and this means the Hamiltonian is

$$H = b^* \sigma_+ + b\sigma_- + \frac{1}{2}\hbar\gamma B_0 = b^*\sigma_+ + b\sigma_- + \underbrace{\frac{1}{2}\hbar\omega_0\sigma_z}_{H_0}. \tag{56}$$

The time evolution corresponding to that ω_0 term is trivial, and using the interaction picture we can get rid of it. We have

$$\mathrm{e}^{\mathrm{i}H_0 t/\hbar} = \begin{pmatrix} \mathrm{e}^{\mathrm{i}\omega_0 t/2} & \\ & \mathrm{e}^{-\mathrm{i}\omega_0 t/2} \end{pmatrix}. \tag{57}$$

The new Hamiltonian is now

$$\begin{aligned} H &= \mathrm{e}^{\mathrm{i}H_0 t/\hbar}(b^*\sigma_+ + b\sigma_-)\mathrm{e}^{-\mathrm{i}H_0 t/\hbar} \\ &= \mathrm{e}^{\mathrm{i}\omega_0 t} b^* \sigma_+ + \mathrm{e}^{-\mathrm{i}\omega_0 t} b\sigma_-. \end{aligned} \tag{58}$$

We will work with this Hamiltonian later.

5.4 Time evolution

Now we use

$$\frac{\partial \rho(t)}{\partial t} = -\frac{1}{\hbar^2}\int_0^t \mathrm{d}t' \,[H(t), [H(t'), \rho(t)]] = -\frac{1}{\hbar^2}\int_0^t \mathrm{d}t' \,[H(t), [H(t-t'), \rho(t)]] \tag{59}$$

to find the time evolution of ρ, considering only second-order correlation in the random variable in H. We again do decomposition

$$\rho = \frac{1}{2}\sigma_0 + \rho_+\sigma_+ + \rho_-\sigma_- + \rho_z\sigma_z, \tag{60}$$

and we have (here b' and b'^* are values taken at $t - t'$)

$$[H(t-t'), \rho(t)] = -\rho_+ \mathrm{e}^{-\mathrm{i}\omega_0(t-t')} b'\sigma_z + \rho_- \mathrm{e}^{\mathrm{i}\omega_0(t-t')} b'^* \sigma_z - 2\rho_z \mathrm{e}^{\mathrm{i}\omega_0(t-t')} b'^*\sigma_+ + 2\rho_z \mathrm{e}^{-\mathrm{i}\omega_0(t-t')} b'\sigma_-,$$

and by component, we get

$$\dot{\rho}_+ = -\frac{1}{\hbar^2}\int_0^t \mathrm{d}t' \left(2\rho_+ \mathrm{e}^{\mathrm{i}\omega_0 t'} b' b^* - 2\rho_- \mathrm{e}^{2\mathrm{i}\omega_0 t - \mathrm{i}\omega_0 t'} b^* b'^*\right), \tag{61}$$

$$\dot{\rho}_- = -\frac{1}{\hbar^2}\int_0^t \mathrm{d}t' \left(2\rho_- \mathrm{e}^{-\mathrm{i}\omega_0 t'} b b'^* - 2\rho_+ \mathrm{e}^{-2\mathrm{i}\omega_0 t + \mathrm{i}\omega_0 t'} b b'\right), \tag{62}$$

and

$$\dot{\rho}_z = -\frac{1}{\hbar^2}\int_0^t \mathrm{d}t' \left(2\rho_z \mathrm{e}^{-\mathrm{i}\omega_0 t'} b b'^* + 2\rho_z \mathrm{e}^{\mathrm{i}\omega_0 t'} b^* b'\right). \tag{63}$$

5.5 Statistical average

We now need to average the above three equations over all possible $b(t)$ configurations. Since we have (54), we find

$$\langle b b' \rangle \propto \langle B_x B_x' \rangle - \langle B_y B_y' \rangle + 2\mathrm{i}\,\langle B_x B_y' + B_y B_x' \rangle.$$

The imaginary part of this vanishes, because there is no correlation between orthogonal directions, and the real part also vanishes because of spatial rotational symmetry. On the other hand, we have

$$\begin{aligned}\langle bb'^{*}\rangle &= \left(\frac{1}{2}g\mu_{\mathrm{B}}\right)^2 \langle (B_x + \mathrm{i}B_y)(B'_x - \mathrm{i}B'_y)\rangle \\ &= \frac{1}{4}\hbar^2\gamma^2(\langle B_x B'_x\rangle + \langle B_y B'_y\rangle) \\ &= \frac{1}{2}\hbar^2\gamma^2 R(\tau),\end{aligned} \tag{64}$$

and similarly

$$\langle b^* b'\rangle = \frac{1}{2}\hbar^2\gamma^2(\tau), \tag{65}$$

where we define

$$R(\tau) = \langle B_x(\tau)B_x(0)\rangle = \langle B_y(\tau)B_y(0)\rangle\,. \tag{66}$$

So after averaging over possible $\boldsymbol{B}(t)$ configurations, the three time evolution equations become

$$\dot{\rho}_+ = -\gamma^2\rho_+ \int_0^t \mathrm{e}^{\mathrm{i}\omega_0 t} R(t')\,\mathrm{d}t'\,, \tag{67}$$

$$\dot{\rho}_- = -\gamma^2\rho_- \int_0^t \mathrm{e}^{-\mathrm{i}\omega_0 t} R(t')\,\mathrm{d}t'\,, \tag{68}$$

and

$$\dot{\rho}_z = -\gamma^2\rho_z \int_0^t (\mathrm{e}^{-\mathrm{i}\omega_0 t'} R(t') + \mathrm{e}^{\mathrm{i}\omega_0 t'} R(t')). \tag{69}$$

We assume the time scale interested is much larger than τ_{c}, and the integrals above are just Fourier transforms. Using the condition that $R(t)$ is real and $R(t) = R(-t)$, we have $R(\omega) = R(-\omega)$, and therefore

$$\dot{\rho}_+ + \frac{1}{T_2}\rho_+ = 0, \quad \dot{\rho}_- + \frac{1}{T_2}\rho_- = 0, \quad \frac{1}{T_2} = \gamma^2 R(\omega_0), \tag{70}$$

and

$$\dot{\rho}_z + \frac{1}{T_1}\rho_z = 0, \quad \frac{1}{T_1} = 2\gamma^2 R(\omega_0). \tag{71}$$

5.6 Evaluation of the correlation function

From (55) we have

$$R(t) = \frac{E^2}{c^4}\langle v_x(t)v_x(0)\rangle\,. \tag{72}$$

A common form of the velocity correlation function is

$$\langle v_x(t)v_x(0)\rangle = \langle v_x^2\rangle\,\mathrm{e}^{-t/\tau_{\mathrm{c}}}, \tag{73}$$

where by the equipartition theorem, we have

$$\frac{1}{2}m\langle v_x^2\rangle = \frac{1}{2}k_{\mathrm{B}}T, \tag{74}$$

and therefore

$$R(\omega) = \frac{E^2}{c^4}\int_{-\infty}^{\infty}\mathrm{d}t\,\mathrm{e}^{\mathrm{i}\omega t}\langle v_x^2\rangle\,\mathrm{e}^{-t/\tau_{\mathrm{c}}} = \frac{E^2}{c^4}\frac{k_{\mathrm{B}}T}{m}\frac{2\tau_{\mathrm{c}}}{1+\omega^2\tau_{\mathrm{c}}^2}. \tag{75}$$

So

$$T_1 = \frac{1}{2\gamma^2}\frac{c^4}{E^2}\frac{m}{k_{\mathrm{B}}T}\frac{1+\omega^2\tau_{\mathrm{c}}^2}{2\tau_{\mathrm{c}}}, \quad T_2 = 2T_1. \tag{76}$$