ODEs

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1 First order ODEs

1.1 Linear ODEs

An ODE in the form of

$$y'(x) + p(x)y(x) = q(x) \tag{1}$$

is considered **linear**. All linear ODEs can be solved by the following procedure. First we have

$$(y' + py)\mathrm{e}^{\int p\mathrm{d}x} = q\mathrm{e}^{\int p\mathrm{d}x}, \tag{2}$$

and now the LHS is a derivative:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(y\mathrm{e}^{\int p\mathrm{d}x}\right) = q\mathrm{e}^{\int p\mathrm{d}x}, \tag{3}$$

and now we can integrate over x and get

$$y\mathrm{e}^{\int p\mathrm{d}x} = \int q\mathrm{e}^{\int p\mathrm{d}x}\,\mathrm{d}x\,, \tag{4}$$

$$y = \mathrm{e}^{-\int p\mathrm{d}x}\int q\mathrm{e}^{\int p\mathrm{d}x}\,\mathrm{d}x\,. \tag{5}$$

1.2 "Energy-conservation lines" and exact equations

Another way to represent the solution of an ODE is the form $\phi(x, y) = \text{const}$. Note that the RHS contains no variables, and we have

$$0 = \frac{\mathrm{d}\phi}{\mathrm{d}x} = \frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}x}, \tag{6}$$

and thus if

$$y' = f(x, y) \tag{7}$$

is algebraically equivalent to (6), the equation is already solved: We should find M, N such that

$$y' = -\frac{M}{N}, \quad M = \frac{\partial\phi}{\partial x}, \quad N = \frac{\partial\phi}{\partial y}, \tag{8}$$

and then $\phi(x, y)$ solves the equation. In this case we say $y' = -M/N$ is **exact**.

To test for exactness, we only have to test whether

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}, \tag{9}$$

and if so, the existence of ϕ is guaranteed. (Since we work on a topological trivial space, things like cohomology group will not bother us.) We can now use "partial integral" to find ϕ.

Example: suppose in a calculation we find

$$\frac{\partial\phi}{\partial x} = 2y^2 + y\mathrm{e}^{xy}, \quad \frac{\partial\phi}{\partial y} = 4xy + x\mathrm{e}^{xy} + 2y. \tag{10}$$

After partial integration, we find

$$\phi(x, y) = \underbrace{2xy^2 + \mathrm{e}^{xy} + h(y)}_{\int\frac{\partial\phi}{\partial x}\mathrm{d}x} = \underbrace{2xy^2 + \mathrm{e}^{xy} + y^2 + g(x)}_{\int\frac{\partial\phi}{\partial y}\mathrm{d}y}, \tag{11}$$

and we have to choose

$$h(y) = y^2, \quad g(x) = \text{const}, \tag{12}$$

and the solution is

$$\phi(x, y) = 2xy^2 + \mathrm{e}^{xy} + y^2 + \text{const}. \tag{13}$$

Note that even when the decomposition $f = -M/N$ doesn't give an exact equation for us, we can still use the method of exact equations: we can multiply a factor μ to both M and N, and try to guess the form of μ so that

$$\frac{\partial(\mu M)}{\partial y} = \frac{\partial(\mu N)}{\partial x}. \tag{14}$$

An example can be found in solving

$$y' = -\frac{1}{3x - \mathrm{e}^{-2y}}. \tag{15}$$

We have

$$\frac{\partial 1}{\partial y} = 0, \quad \frac{\partial(3x - \mathrm{e}^{-2y})}{\partial x} = 3,$$

so the equation is not exact if we choose $M = 1$ and $N = 3x - \mathrm{e}^{-2y}$. However, (14) can be fulfilled now: it's now

$$\frac{\partial \mu}{\partial y} = 3\mu + \left(3x - \mathrm{e}^{-2y}\right) \frac{\partial \mu}{\partial x},$$

and the most convenient way to solve it (we *don't* need to find all solutions of this equation!) is to let μ contain y only, so the tricky term on the RHS disappears, and thus we choose $\mu = \mathrm{e}^{3y}$, and we get

$$\phi(x, y) = \int \mu M \,\mathrm{d}x = \int \mathrm{e}^{3y} \,\mathrm{d}x = x\mathrm{e}^{3y} + u(y),$$

$$\phi(x, y) = \int \mu N \,\mathrm{d}y = \int (3x\mathrm{e}^{3y} - \mathrm{e}^{y}) \,\mathrm{d}y = x\mathrm{e}^{3y} - \mathrm{e}^{y} + v(x),$$

so

$$\phi(x, y) = x\mathrm{e}^{3y} - \mathrm{e}^{y} + \text{const}. \tag{16}$$