Bosonic modes in Fermi liquid

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1 Introduction

The Fermi liquid theory can be justified by diagrammatic resummation: by summing up a certain family of self-energy diagrams that are believed to be important, we get a correction to the electron band dispersion relation, as well as a finite lifetime. The electron density-dependent part of the self-energy correction is often known as "forward scattering", which has the form of $f_{\boldsymbol{p}\boldsymbol{k}'}\,\delta n\,(\boldsymbol{p})\,\delta n\,(\boldsymbol{p}')$ in the energy functional. But interaction channels beside forward scattering that come from Coulomb interaction do not just disappear; they are still a part of the Hamiltonian and will contribute to the specific heat when the system is heated up. Therefore, it can be expected that a real condensed matter system that is said to be in a Fermi liquid phase contains *more* than electron-like quasiparticles.

Characterization of the full spectrum of a system is generally only possible for exactly solvable systems. This report is constrained on bosonic modes in Fermi liquid,[1] or to be specific, on excitations for which a quantum is essentially a renormalized electron-hole pair. In other words, in this report we are interested in oscillation modes of operators with the shape of $c^\dagger_{\boldsymbol{k}+\boldsymbol{q}/2}c_{\boldsymbol{k}-\boldsymbol{q}/2}$. Three-electron behaviors do exist [1, 2] but are beyond the scope of this report.

2 The formalism

In principle all electro-hole bosonic modes can be found by looking at poles of the four-point Green function, or in other words, by diagonalizing the four-point kernel. This indeed is the usual method in first-principle calculations [3], but is not feasible for semi-quantitative analytical purposes.

One way to proceed is to notice that linear response of two-point Green function to an external field coupled to electrons gives us four-point Green function (Fig. 1). This again is a first-principle approach not feasible for analytical studies [4], but further simplification is possible. For a direct connection between physical observables and the Green function, we work with the so-called *lesser Green function*

$$G^<(\boldsymbol{x}_1,t_1,\boldsymbol{x}_2,t_2)=\mathrm{i}\left\langle\psi^\dagger(2)\psi(1)\right\rangle. \tag{1}$$

Wigner transform of the lesser Green function reads

$$\begin{gathered}G^<(\boldsymbol{X},\boldsymbol{p},T,\omega)=\int\mathrm{d}t\,\mathrm{e}^{\mathrm{i}\omega t}\int\mathrm{d}^3\boldsymbol{r}\,\mathrm{e}^{-\mathrm{i}\boldsymbol{p}\cdot\boldsymbol{x}}G^<(T+t/2,\boldsymbol{X}+\boldsymbol{x}/2,T-t/2,\boldsymbol{X}-\boldsymbol{x}/2),\\ \boldsymbol{x}=\boldsymbol{x}_1-\boldsymbol{x}_2,\quad \boldsymbol{X}=\frac{\boldsymbol{x}_1+\boldsymbol{x}_2}{2},\quad t=t_1-t_2,\quad T=\frac{t_1+t_2}{2}.\end{gathered} \tag{2}$$

The Wigner transform defines the position and momentum variables; note that similar to the single-electron Wigner function, usual positivity conditions expected in the classical case do not hold in general for $G^<(\boldsymbol{X},\boldsymbol{p})$. We then introduce two additional assumptions. The first is the validity of *gradient expansion*: physical quantities involved in the calculation should not have very high order dependence on either $\boldsymbol{X}$ or $\boldsymbol{p}$. The second is that the quasiparticle picture works well in the system so that the peak in the spectral function is sharp enough, and we have

$$G^<(\boldsymbol{X},\boldsymbol{p},T,\omega)=-2\pi\mathrm{i}\delta(\omega-\xi(\boldsymbol{X},\boldsymbol{p}))\cdot f(\boldsymbol{X},\boldsymbol{p},T), \tag{3}$$

[1] There is a terminological confusion here: the term *Fermi liquid* may refer to a system whose Hamiltonian is exactly in the shape of Fermi liquid energy functional, or it may refer to a system in which the behavior of electron Green function follows the Fermi liquid theory, but may contain other excitations. This note uses the latter definition; thus the phrase "a Fermi liquid" is a shorthand for "a real-world condensed matter system demonstrating Fermi liquid behaviors in its single-electron part".

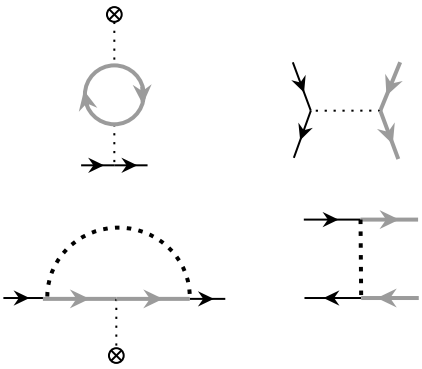

Figure 1: Linear response of two-point Green function gives four-point Green function: the two Feynman diagrams on the left are GW diagrams with the Green function modified by one external field line, representing linear response of the system; the linear susceptibility can be obtained formally by erasing the external field line and we get the two Feynman diagrams on the right, which describes the most frequently considered two terms in the Bethe–Salpeter equation formalism [3].

where $\xi(\boldsymbol{X}, \boldsymbol{p})$ is the single-electron Hamiltonian plus the real part of the self-energy and thus is not necessarily diagonal in the momentum space and has thus undergone Wigner transform. The two assumptions are sufficient to lead to the *quantum Boltzmann equation*

$$\frac{\partial f}{\partial T} + \boldsymbol{\nabla}_{\boldsymbol{p}}\xi \cdot \boldsymbol{\nabla}_{\boldsymbol{R}} f - \boldsymbol{\nabla}_{\boldsymbol{R}}\xi \cdot \boldsymbol{\nabla}_{\boldsymbol{p}} f = \left(\frac{\partial f}{\partial t}\right)_{\mathrm{c}}, \tag{4}$$

where the collision integral on the right-hand side is decided by Fermi golden rule and is

$$\left(\frac{\partial f}{\partial t}\right)_{\mathrm{c}} = \tag{5}$$

Note that the only difference between the quantum Boltzmann equation and the classical Boltzmann equation is the $(1 \pm f)$ factors coming from fermionic/bosonic statistics. The gradient expansion condition is intuitively reflected by the fact that the collision integral depends on $\boldsymbol{R}$ only; also note that the imaginary part of the self-energy is ignored in the spectral function but is picked up back to the collision integral. Below, we replace $\boldsymbol{R}$ by $\boldsymbol{r}$ and T by t for the sake of convenience.

The most generalized derivation of (4) involves Keldysh field theory [5] and is beyond the scope of this report. As a proof of concept, a collision-free linearized quantum Boltzmann equation can be derived using random phase approximation (RPA) from the equation of motion of the electron-hole pair creation operator

$$n^{\dagger}_{\boldsymbol{kq}} = c^{\dagger}_{\boldsymbol{k}+\boldsymbol{q}} c_{\boldsymbol{k}}, \tag{6}$$

where $\boldsymbol{q}$ is the total momentum of the pair and $\boldsymbol{k}$ is the "internal" or relative momentum; recall that $c_{\boldsymbol{k}}$ creates a hole with momentum $-\boldsymbol{k}$. By ignoring $\mathcal{O}(n^3)$ terms in the equation of motion of $n_{\boldsymbol{kq}}$ and switching the equation to the frequency domain we have [6]

$$-\mathrm{i}\dot{n}^{\dagger}_{\boldsymbol{pq}} = (\epsilon_{\boldsymbol{p}+\boldsymbol{q}} - \epsilon_{\boldsymbol{p}}) n^{\dagger}_{\boldsymbol{pq}} + \sum_{\boldsymbol{k}} n_{\boldsymbol{k}} (V_{\boldsymbol{k}-\boldsymbol{p}} - V_{\boldsymbol{k}-\boldsymbol{p}-\boldsymbol{q}}) n^{\dagger}_{\boldsymbol{pq}} - \sum_{\boldsymbol{k}} (V_{\boldsymbol{q}} - V_{\boldsymbol{k}-\boldsymbol{p}})(n_{\boldsymbol{p}+\boldsymbol{q}} - n_{\boldsymbol{p}}) n^{\dagger}_{\boldsymbol{kq}}. \tag{7}$$

Note that

$$\begin{aligned} \int \mathrm{d}^3\boldsymbol{x}\, \mathrm{e}^{-\mathrm{i}\boldsymbol{p}\cdot\boldsymbol{x}} \psi^{\dagger}(\boldsymbol{x}_2)\psi(\boldsymbol{x}_1) &= \int \mathrm{d}^3\boldsymbol{x}\, \mathrm{e}^{-\mathrm{i}\boldsymbol{p}\cdot\boldsymbol{x}} \frac{1}{\sqrt{V}} \sum_{\boldsymbol{k}} \mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot(\boldsymbol{X}-\boldsymbol{x}/2)} c^{\dagger}_{\boldsymbol{k}} \cdot \frac{1}{\sqrt{V}} \sum_{\boldsymbol{k}'} \mathrm{e}^{\mathrm{i}\boldsymbol{k}'\cdot(\boldsymbol{X}+\boldsymbol{x}/2)} c_{\boldsymbol{k}'} \\ &= \sum_{\boldsymbol{k},\boldsymbol{k}'} \mathrm{e}^{\mathrm{i}\boldsymbol{X}\cdot(\boldsymbol{k}'-\boldsymbol{k})} \delta_{\boldsymbol{p},\frac{\boldsymbol{k}+\boldsymbol{k}'}{2}} c^{\dagger}_{\boldsymbol{k}} c_{\boldsymbol{k}'}, \end{aligned} \tag{8}$$

and therefore[2]

$$\int \mathrm{d}^3\boldsymbol{r}\,\mathrm{e}^{-\mathrm{i}\boldsymbol{q}\cdot\boldsymbol{r}} f(\boldsymbol{r},\boldsymbol{p},t) \simeq \langle c^\dagger_{\boldsymbol{k}-\boldsymbol{q}/2} c_{\boldsymbol{k}+\boldsymbol{q}/2}\rangle\,. \tag{9}$$

The gradient expansion condition is equivalent to the condition that the characteristic length scale of f with respect to the $\boldsymbol{r}$ variable is large, which then is equivalent to the condition that $\boldsymbol{q}$ is small. Then, by noticing

$$\xi_{\boldsymbol{p}}(\boldsymbol{r}) = \varepsilon^0_{\boldsymbol{p}} + \sum_{\boldsymbol{k}} (V_0 - V_{\boldsymbol{k}-\boldsymbol{p}}) n_{\boldsymbol{k}}(\boldsymbol{r}) - \mu, \tag{10}$$

taking the $\boldsymbol{q}\to 0$ limit and keep only the $\mathcal{O}(q)$ term in all Taylor expansions with respect to $\boldsymbol{q}$ and noticing that

$$\mathrm{i}\boldsymbol{q} \simeq \frac{\partial}{\partial \boldsymbol{r}},$$

we get a linearized quantum Boltzmann equation with a vanishing collision integral.

Classical Boltzmann equation can be derived using BBGKY hierarchy, if we make the assumption that n-th order correlation functions ($n>2$) are ignorable; here a similar procedure is applied in the quantum region, giving us the equation of motion of two-point (i.e. single-electron) Green function; it is the further two conditions that finally leads to the quantum Boltzmann equation. The second assumption is by definition satisfied for a Fermi liquid. The first assumption – that gradient expansion works – is at the first glance broken in condensed matter systems, since the crystal potential has a very small characteristic length scale. For a single-band problem, however, we can manually find a "position" operator $\boldsymbol{x}$ as the conjugate variable of the lattice momentum $\boldsymbol{k}$, which represents the center of the wave packet and in the coarse-grained macroscopic limit appears to be the commonly known position. The gradient expansion condition therefore is equivalent to the $\boldsymbol{q}\to 0$ limit. When a uniform electric field is applied, it influences $\boldsymbol{x}_1 - \boldsymbol{x}_2$, not $(\boldsymbol{x}_1+\boldsymbol{x}_2)/2$; accordingly, it influences $\boldsymbol{k}$, which is now to be understood as the *relative* momentum between the electron and the hole (absence of an electron with momentum $\boldsymbol{k}$ is equivalent to existence of a hole with momentum $-\boldsymbol{k}$). If, however, the electric field has very strong spatial variance, the electron at $\boldsymbol{r}_1$ feels a different force from that felt at $\boldsymbol{r}_2$, and the electron-hole pair gets driven as a whole, giving a non-zero value to $\boldsymbol{q}$. Thus eventually, the small-$\boldsymbol{q}$ condition is equivalent to the condition that the wave length of the driving electric field should be small compared with the atomic length scale.

There is yet one more caveat pertaining to the quantum Boltzmann equation: the naive equation (4) faces a fundamental constraint in multi-band systems (including band splitting caused by an external magnetic field), because, as an example, the counterpart of (7) contains $\varepsilon^{\mathrm{c}}_{\boldsymbol{k}+\boldsymbol{q}} - \varepsilon^{\mathrm{v}}_{\boldsymbol{k}}$. Should the two energies be both from a single band, gradience expansion would possible when $\boldsymbol{q}$ is small, which leads to the $\frac{\partial \varepsilon_{\boldsymbol{k}}}{\partial \boldsymbol{k}}\frac{\partial n}{\partial \boldsymbol{r}}$ term; but for the multi-band case we still need a constant and finite $\varepsilon^{\mathrm{c}}_{\boldsymbol{k}} - \varepsilon^{\mathrm{v}}_{\boldsymbol{k}}$ term even when $\boldsymbol{q}\to 0$. The way to solve this is to introduce $\mathrm{i}[\varepsilon, f]$ in the left-hand side of (4), where $f_{nn'}$ and $\varepsilon_{nn'}$ are seen as matrices (but the commutator does not count $\boldsymbol{r}$ and $\boldsymbol{p}$ as quantum operators, since the corresponding effects are already considered by existing terms in (4)).

3 Landau kinetic theory of neutral Fermi liquid and zero sound

$\mathrm{Re}\,\Sigma$ can have explicit dependence on $G(\boldsymbol{r},\boldsymbol{r}',t,t')$ i.e. $f(\boldsymbol{r},\boldsymbol{p},t)$, as is shown in (10). Below we change the notation once again and use $n_{\boldsymbol{p}}(\boldsymbol{r})$ to refer to the distribution function in place of $f(\boldsymbol{r},\boldsymbol{p})$ to follow the established convention in Fermi liquid theory. For a general Fermi liquid we can insert

$$\varepsilon_{\boldsymbol{p}\sigma}(\boldsymbol{r}) = \varepsilon^0_{\boldsymbol{p}\sigma} + \frac{1}{V}\sum_{\boldsymbol{p}',\sigma'} f_{\boldsymbol{p}\boldsymbol{p}'\sigma\sigma'}\,\delta n_{\boldsymbol{p}'\sigma'}(\boldsymbol{r}) \tag{11}$$

into the quantum Boltzmann equation; the resulting equation system is called *Landau equation*. Since now δn has both $\boldsymbol{p}$ and $\boldsymbol{r}$ dependence, $\varepsilon_{\boldsymbol{p}}$ also has $\boldsymbol{r}$ dependence. No spatial dependence

[2] $\int \mathrm{d}^3\boldsymbol{r}\,\mathrm{e}^{-\mathrm{i}\boldsymbol{q}\cdot\boldsymbol{r}} f(\boldsymbol{r},\boldsymbol{p},t)$ is also proportional to the $|$electron at $\boldsymbol{p}+\boldsymbol{q}/2$, hole at $\boldsymbol{p}-\boldsymbol{q}/2\rangle$, the EOM of the latter being the same as the *annihilation operator* of the electron-hole pair, and therefore there is nothing wrong with the equation below.

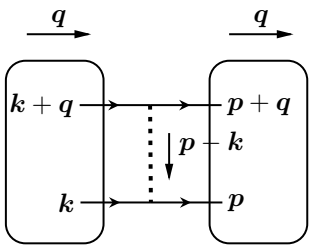

Figure 2: Deriving the quasiparticle interaction function. In the first-order correction the TODO. As long as the corrected vertex is well-defined in the $\boldsymbol{q} \to 0$ limit, we can simply set $\boldsymbol{q} = 0$ when $\boldsymbol{q}$ appears in any of the internal interaction lines; the corrected interaction vertex therefore only depends on $\boldsymbol{p} - \boldsymbol{k}$, as is shown in the figure above. The vertex then can be seen as an interaction channel that keeps the total momentum $\boldsymbol{q}$ of an electron-hole pair but constantly changes its internal momentum $\boldsymbol{p}$.

(i.e. $\boldsymbol{q}$ dependence) is added to $f_{\boldsymbol{p}\boldsymbol{p}'}$, however: the fact that $n_{\boldsymbol{k}\boldsymbol{q}}$ has a non-zero value when $\boldsymbol{q} \neq 0$ is due to external electromagnetic driving, which at the linear level does not change the momentum conservation condition where an interaction line meets with two electron lines. Note that (11) is already beyond the Fermi liquid energy functional

$$\delta E = \sum_{\boldsymbol{p},\sigma} \varepsilon^0_{\boldsymbol{p}}\, \delta n_{\boldsymbol{p}\sigma} + \frac{1}{2V} \sum_{\boldsymbol{p}',\sigma'} f_{\boldsymbol{p}\boldsymbol{p}'\sigma\sigma'}\, \delta n_{\boldsymbol{p}\sigma}\, \delta n_{\boldsymbol{p}'\sigma'}\,, \tag{12}$$

since the energy functional is unable to change the internal relative momentum i.e. $\boldsymbol{p}$ for an electron-hole pair, but that is exactly what happens in (11). (11), then, implicitly assumes that the evolution of $\boldsymbol{p}$ is irrelevant to the value of $\boldsymbol{q}$; since we are working in the small-$\boldsymbol{q}$ region where Boltzmann equation holds, as long as we have a well-defined $f_{\boldsymbol{p}\boldsymbol{p}'}$ function in the first place. Unscreened Coulomb interaction however breaks this condition (Section 5).

The inclusion of the Fermi liquid self-energy correction immediately leads to an important consequence of Fermi liquid: that when the temperature is zero and no collision is possible for quasiparticles, we still have density modes which resemble ordinary sound wave in some aspects. This mode is known as *zero sound*. To show this, we go to Fourier space where

When the temperature is non-zero, $\tau \propto 1/T^2$ is finite and zero sound faces strong damping when its frequency is too slow. In the low frequency domain, where thermal equilibrium is almost always established, we get ordinary sound or "first sound". The first sound can be derived by calculating mechanical properties of the Fermi liquid in question and inserting the compressibility into $v = \sqrt{\partial p/\partial \rho}$ [7]. This approach assumes the usual framework of near-equilibrium hydrodynamics works for Fermi liquid when the frequency is low enough; a direct verification can be found in [8]. The spectrum of first sound is connected to the spectrum of zero sound: the zero sound and the first sound can be derived in a unified way [9], and we may say first sound is merely zero sound with finite temperature correction, but this correction is so severe that the qualitative physical picture is radically changed. First sound can be derived with macroscopic conservation equation usually used in fluid dynamics, while this is no longer possible for zero sound; the two types of sounds also have different dissipation mechanism: zero sound is damped because of the non-zero collision integral, while first sound is damped because the collision is not strong enough, so as a first sound wave propagates, it excites electron-hole pairs and loses energy [10, 8]. The sound spectrum of Fermi liquid is therefore summarized in Fig. 3.

4 Damping mechanisms

5 Charged Fermi liquid and the plasmon

If we try to calculate $f_{\boldsymbol{p}\boldsymbol{p}'}$ by comparing (11) and (10) in a real condensed matter system, we immediately find that the Hartree term V_0 diverges. This radically long-range nature of the Hartree term therefore is better captured by putting the Hartree term to the *spatial* dependence

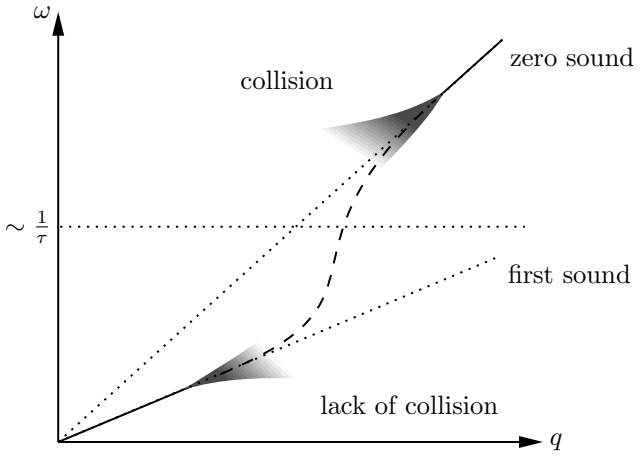

Figure 3: Comparison between zero sound and first sound: they can be seen as one continuous branch on the spectrum, where a large imaginary part is present when $\omega\tau \sim 1$, they have different physical pictures. Zero sound only exists in the $\omega\tau \gg 1$ region, while first sound only exists in the $\omega\tau \ll 1$ region; damping of zero sound is due to collision, while damping of first sound is due to the fact that collision is not strong enough to establish thermal equilibrium. First sound is a finite temperature effect.

of the corrected single-electron energy instead of the momentum dependence. The complete kinetic theory therefore involves three instead of two equations: the Hartree correction

$$\nabla^2\varphi = \frac{e}{\epsilon_0}\sum_{\boldsymbol{p},\sigma}\delta n_{\boldsymbol{p}\sigma}\left(\boldsymbol{r}\right), \tag{13}$$

the single-electron energy corrected by the short-range part of the self-energy

$$\varepsilon_{\boldsymbol{p}\sigma}(\boldsymbol{r}) = \varepsilon^0_{\boldsymbol{p}} + \frac{1}{V}\sum_{\boldsymbol{p}',\sigma'} f_{\boldsymbol{p}\boldsymbol{p}'\sigma\sigma'}\,\delta n_{\boldsymbol{p}'\sigma'}\left(\boldsymbol{r}\right), \tag{14}$$

and the quantum Boltzmann equation

$$\frac{\partial}{\partial t}\delta n_{\boldsymbol{p}\sigma} + \frac{\partial\varepsilon_{\boldsymbol{p}\sigma}}{\partial\boldsymbol{p}}\cdot\frac{\partial\,\delta n_{\boldsymbol{p}\sigma}}{\partial\boldsymbol{r}} - \frac{\partial\,\delta n_{\boldsymbol{p}\sigma}}{\partial\boldsymbol{p}}\cdot\frac{\partial(\varepsilon_{\boldsymbol{p}\sigma} - e\varphi)}{\partial\boldsymbol{r}} = 0. \tag{15}$$

The equation system is known as *Landau-Silin equation*.

6 Microscopic bosonic modes beyond the Landau equation

Not all bosonic modes can be obtained by observing oscillation modes of quantum Boltzmann equation, since the latter only works for excitations with a large characteristic length scale in the $\boldsymbol{x}_1 - \boldsymbol{x}_2$ variable. This constraint has two consequences.

First, it requires that $\boldsymbol{q}$ is small enough.

This does not mean the quantum Boltzmann equation is useless for excitons: excitons are usually modeled as independent particles themselves *besides* electrons, and exciton formation is modeled as a three-particle vertex [11], while

7 Discussion

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[6] David Pines. *Theory of Quantum Liquids: Normal Fermi Liquids*. The "expansion of energy" subsection in Section 1.4 is about the short-range condition. The "charged v.s. neutral system" subsection in Section 3.3 compares plasmon with zero sound. The inability of quantum Boltzmann equation to capture microscopic bound states is discussed on p. 56. The microscopic EOM of electron density operator (and comments regarding its relation with Boltzmann equation) can be found on p. 318. CRC Press, 2018.

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