Matrices

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1 Homogeneous linear equation systems

Consider the set of linear equations

$$\begin{aligned} x_1 + x_3 + x_4 &= 0, \\ x_3 + 2x_4 &= 0, \\ x_1 + 2x_3 + 3x_4 &= 0. \end{aligned} \tag{1}$$

The equivalent matrix form is

$$\underbrace{\begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 2 \\ 1 & 0 & 2 & 3 \end{pmatrix}}_{\mathbf{A}} \underbrace{\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}}_{\mathbf{x}} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}. \tag{2}$$

We use the row reduction method to solve the equations. First we keep the first row unchanged, but subtract the first row from the third row, and we get

$$\begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 1 & 2 \end{pmatrix}.$$

Then we can just set the third row to zeros because it duplicates with the second line. Then we can subtract the second line from the first line, and get

$$\begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{3}$$

This is in row echelon form. Now we get the reduced form of $\mathbf{A}$, and from this, we find

$$x_4 = x_1, \quad x_3 = -2x_4, \tag{4}$$

and x_2 can be anything. The general solution is therefore found to be

$$\mathbf{x} = \begin{pmatrix} x_4 \\ x_2 \\ -x_4 \\ x_4 \end{pmatrix} = x_2 \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 1 \\ 0 \\ -2 \\ 1 \end{pmatrix}. \tag{5}$$

So we found there are two independent solutions, which is expected because (3) has only two non-zero rows, so it's rank is 2, and the number of independent variables is the number of columns minus the rank, so we should have $4 - 2 = 2$ independent variables, i.e. 2 independent solutions (one independent variable controls the weight of one independent solution). We define the **row space** of a matrix as the space of spanned by the non-zero row vectors of its reduced matrix. The dimension of the row space is the rank of the matrix.

We can also get (5) by looking at (3). First we need to switch x_2 and x_3 so that the non-zero lines of (3) have the form of $\begin{pmatrix} \mathbf{I} & \mathbf{B} \end{pmatrix}$, and the non-zero columns of $\mathbf{B}$ are $\begin{pmatrix} 0 & 0 \end{pmatrix}^\top$ and $\begin{pmatrix} -1 & 2 \end{pmatrix}^\top$. Now we concatenate the opposites of the two columns with $\begin{pmatrix} 1 & 0 \end{pmatrix}^\top$ and $\begin{pmatrix} 0 & 1 \end{pmatrix}^\top$, respectively, and we get

$$\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ -2 \\ 0 \\ 1 \end{pmatrix},$$

which are solutions for (x_1, x_3, x_2, x_4), and then we switch x_2 and x_3 again and get

$$\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ 0 \\ -2 \\ 1 \end{pmatrix}.$$

2 Determinant

We have **Cramer's rule**: the solution of

$$\mathbf{Ax} = \mathbf{b} \tag{6}$$

can be found by

$$x_k = \frac{\det \mathbf{A}_k}{\det \mathbf{A}}, \tag{7}$$

where

$$\mathbf{A}_k = \begin{pmatrix} \mathbf{A}_1 & \mathbf{A}_2 & \cdots & \mathbf{A}_{k-1} & \mathbf{b} & \mathbf{A}_{k+1} & \cdots & \mathbf{A}_n \end{pmatrix}, \tag{8}$$

and $\mathbf{A}_i$ is the ith column of $\mathbf{A}$. The proof of this rule is simple: we have

$$\mathbf{b} = x_1 \mathbf{A}_1 + x_2 \mathbf{A}_2 + \cdots + x_n \mathbf{A}_n,$$

and therefore

$$\det \mathbf{A}_i = \sum_i x_i \det \begin{pmatrix} \mathbf{A}_1 & \mathbf{A}_2 & \cdots & \mathbf{A}_{k-1} & \mathbf{A}_i & \mathbf{A}_{k+1} & \cdots & \mathbf{A}_n \end{pmatrix},$$

and when $i \neq k$, $\mathbf{A}_i$ must be the same as one of the other columns of the matrix in the RHS and the determinant vanishes, and therefore

$$\det \mathbf{A}_i = x_i \det \mathbf{A},$$

and thus we get the Cramer's rule.

3 Invertible matrix

Suppose $\mathbf{A}$ is an $n \times n$ matrix. The following statements are equivalent:

- Columns of $\mathbf{A}$ are linearly independent;
- $\operatorname{rank} \mathbf{A} = n$;
- The reduced form of $\mathbf{A}$ (Section 1) is the identity matrix;
- $\det \mathbf{A} \neq 0$;
- $\mathbf{Ax} = 0$ only has vanishing solution;
- $\mathbf{Ax} = \mathbf{b}$ has unique solution;
- $\mathbf{A}$ has an inverse.

4 Eigenvalue decomposition

If $n \times n$ matrix $\mathbf{A}$ has n linearly independent eigenvectors, we have

$$\mathbf{A} = \mathbf{PDP}^{-1}, \tag{9}$$

where $\mathbf{D}$ is the diagonal matrix containing all the eigenvalues, while $\mathbf{P}$ is the matrix containing linearly independent eigenvectors as the columns.

If $\mathbf{A}$ has n distinct eigenvalues, then the eigenvectors are linearly independent.