Numerical bootstrap

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What's bootstrap

- A quantum theory $=$ expectations of all Hermitian operators; Hamiltonian/Lagrangian $\Leftrightarrow$ "probability distribution"
- Constraint on the system $\Rightarrow$ relation between different $\langle O \rangle$'s ("**data**"); independent $\langle O \rangle$'s $\Leftrightarrow$ parameters in the model
- Inequality constraint (e.g. positivity of $\left\langle O^\dagger O \right\rangle$) $\Rightarrow$ allowed range of $\langle O \rangle$'s
- Solving a class of problems without mentioning explicitly the wave function/path integral: hence the name *bootstrap*

Why we need it

- Because it doesn't fail with strong non-perturbative effects.[1]

[1] arXiv 2108.11416

Example: conformal bootstrap

- The most famous example: **conformal bootstrap**
- Constraints: (spinless) two-point function

$$\langle \mathcal{O}(x)\mathcal{O}(y)\rangle = \frac{1}{|x-y|^{2\Delta_{\mathcal{O}}}}, \tag{1}$$

three-point function

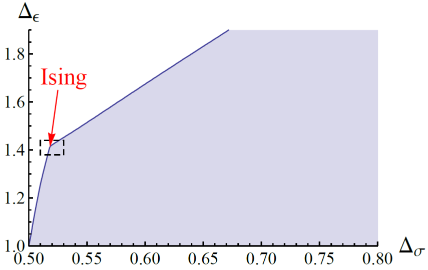
$$\begin{aligned} &\langle \mathcal{A}(x)\mathcal{B}(y)\mathcal{C}(z)\rangle \\ = &\frac{f_{\mathcal{ABC}}}{|x-y|^{\Delta_{\mathcal{A}}+\Delta_{\mathcal{B}}-\Delta_{\mathcal{C}}}|y-z|^{\Delta_{\mathcal{B}}+\Delta_{\mathcal{C}}-\Delta_{\mathcal{A}}}|z-x|^{\Delta_{\mathcal{C}}+\Delta_{\mathcal{A}}-\Delta_{\mathcal{B}}}} \end{aligned} \tag{2}$$

Higher order correlation functions: OPEs.

- Independent parameters: $\{\Delta_{\mathcal{O}}, l_{\mathcal{O}}, f_{\mathcal{ABC}}\}$
- Inequality constraints (self-consistent conditions): determining the range of parameters

Example: conformal bootstrap

- Example of conformal bootstrap: verify whether the critical point of 3D Ising model is a CFT and its position in the allowed region[2]
- Physical picture tells us there are two degrees of freedom: energy density ϵ, spin field σ
- Below is Fig. 3 in the paper: comparing critical exponents of 3D Ising model, and the allowed range from conformal bootstrap



[2] arXiv 1203.6064

How to perform bootstrap for a generic system?

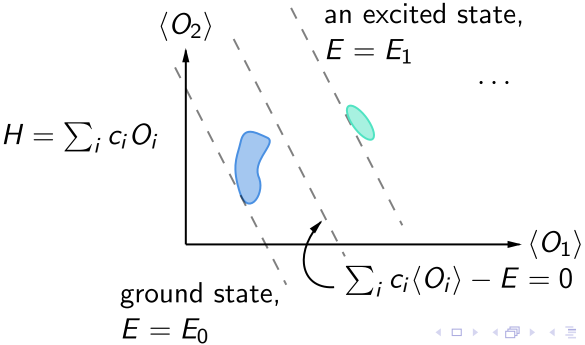
- Correlation functions cannot be determined by countably infinite parameters: no $\{\Delta_{\mathcal{O}}, l_{\mathcal{O}}, f_{\mathcal{ABC}}\}$.
  Solution Store $\langle O_1(x_1) O_2(x_2) \cdots O_n(x_n) \rangle$ separately. Symmetries reduce the size of data: Suppose C is a symmetry,
  $$\langle OC \rangle = \langle CO \rangle \,. \tag{3}$$
  Derivation similar to below.
- Hard to get OPE
  Solution Density matrix is determined solely by H, then
  $$\langle OH \rangle = \mathrm{tr}(\rho(H) OH) = \mathrm{tr}(H\rho(H)O) = \mathrm{tr}(\rho(H)HO) = \langle HO \rangle \tag{4}$$
  for all operators O. For energy eigenstates (i.e with definite E), we have
  $$\langle OH \rangle = E \langle O \rangle \,, \quad E = \langle H \rangle \,. \tag{5}$$

Bootstrap for generic systems

- The equality constraints used together with the positivity constraint

$$\langle O^\dagger O \rangle \geq 0 \tag{6}$$

  for every O defines an allowed region - the feasible domain for the following optimization.
- To get ground state information, minimize $\langle H \rangle$ globally.
- To get information about excited states, find local minima of $\langle H \rangle$.



Bootstrap for generic systems

The procedure of numerical bootstrap

Input data:

- Determine N operators $\{O_i\}$ as the basis of all operators. Example: normal ordered, with a length cutoff
- Data of equality constraints: Hamiltonian, symmetry, etc.
- Commutation rules, normal ordering rules, etc. so that $O_i O_j$ can be expanded in terms of $\{O_i\}$
- Hamiltonian $H = \sum_i c_i O_i$

Building the optimization problem:

1. Declare N variables $\{X_i\}$, $X_i = \langle O_i \rangle$ after optimization
2. Impose equality constraints on $\{X_i\}$ according to e.g. $\langle [H, O] \rangle = 0$
3. Imposing semidefinite constraint on $M_{ij} = \langle O_i^\dagger O_j \rangle$, so that after optimization $\langle O^\dagger O \rangle \geq 0$ for every O
4. Optimize $\sum_i c_i X_i$

Some technical aspects of the problem

When building the optimization problem:

- A little symbolic calculation required
- Auto normal ordering of $O_i O_j$ given the operator algebra
- Auto commutator: $[A, B] =$ normal ordered AB - normal ordered BA

For optimization itself

- **linear semidefinite programming (linear SDP)** when using $\langle [H, O] \rangle = 0$ ($\langle HO \rangle$ and $\langle OH \rangle$ being linear combination of $\{O_i\}$)
  - Convex optimization, mature solvers[3]
- **nonlinear semidefinite programming (nonlinear SDP)** when using $\langle HO \rangle = E \langle O \rangle$ because there are optimization variables in E
  - No solver mature enough [4]

[3] See Wikipedia

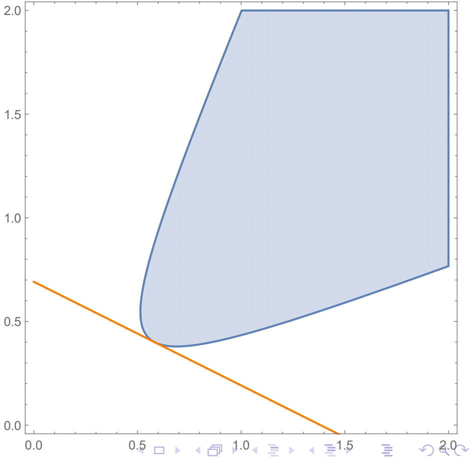
[4] See the discussion before Sec. 1.1 in arXiv 2108.04830. Also, no solver supporting both SDP and nonlinear programming (NLP) is listed in the solver list of JuMP.jl.

Bootstrap for generic systems

- linear SDP (constraints: $\langle [O, H] \rangle = 0$, thermal states allowed) easy
- nonlinear SDP (constraints: $\langle OH \rangle = E \langle O \rangle$, no thermal states) hard

Example of linear SDP (and why it's easy)

$$\max\ M_{11} + 2M_{12},\ \text{s.t.}$$
$$M = M^\top,\ M \geq 0,$$
$$M_{11} + M_{12} + M_{13} = -0.5,$$
$$M_{22} = 2M_{11} + 3M_{12} + 1,$$
$$M_{23} + 4M_{11} = 0,$$
$$M_{33} = 4M_{11} + 5M_{12}.$$



- Convex feasible domain
- Linear objective $\Rightarrow$ minimum at the edge

Example: x^4 nonlinear oscillator

Consider the nonlinear oscillator[5]

$$H = x^2 + p^2 + gx^4. \tag{7}$$

- Famous failure of perturbation theory[6]
- Symmetry: $x \to -x \Rightarrow \langle x^n \rangle = 0$ with odd n
- Building the optimization problem:

[5] The example is provided in arXiv 2004.10212

[6] Carl M. Bender and Tai Tsun Wu, Anharmonic Oscillator. Phys. Rev. 184, 1231.

Example: x^4 nonlinear oscillator

Numerical bootstrap can be quite precise!