Numerical bootstrap

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Introduction

What's bootstrap

- A quantum theory = expectations of all Hermitian operators; Hamiltonian/Lagrangian $\Leftrightarrow$ "probability distribution"
- Constraint on the system $\Rightarrow$ relation between different $\langle O \rangle$'s ("**data**"); independent $\langle O \rangle$'s $\Leftrightarrow$ parameters in the model
- Inequality constraint (e.g. positivity of $\left\langle O^\dagger O \right\rangle$) $\Rightarrow$ allowed range of $\langle O \rangle$'s
- Solving a class of problems without mentioning explicitly the wave function/path integral: hence the name *bootstrap*

Why we need it

- Because it doesn't fail with strong non-perturbative effects. [1]

[1] arXiv 2108.11416

Example: conformal bootstrap

- The most famous example: **conformal bootstrap**
- Constraints: (spinless) two-point function

$$\langle \mathcal{O}(x)\mathcal{O}(y)\rangle = \frac{1}{|x-y|^{2\Delta_{\mathcal{O}}}}, \tag{1}$$

three-point function

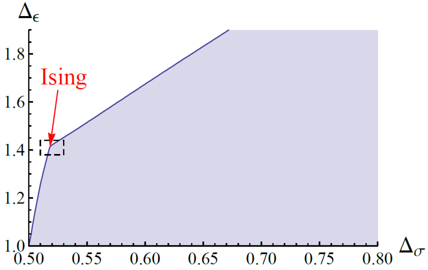
$$\begin{aligned} &\langle \mathcal{A}(x)\mathcal{B}(y)\mathcal{C}(z)\rangle \\ &= \frac{f_{\mathcal{ABC}}}{|x-y|^{\Delta_{\mathcal{A}}+\Delta_{\mathcal{B}}-\Delta_{\mathcal{C}}}|y-z|^{\Delta_{\mathcal{B}}+\Delta_{\mathcal{C}}-\Delta_{\mathcal{A}}}|z-x|^{\Delta_{\mathcal{C}}+\Delta_{\mathcal{A}}-\Delta_{\mathcal{B}}}} \end{aligned} \tag{2}$$

Higher order correlation functions: OPEs.

- Independent parameters: $\{\Delta_{\mathcal{O}}, l_{\mathcal{O}}, f_{\mathcal{ABC}}\}$
- Inequality constraints (self-consistent conditions): determining the range of parameters

Example: conformal bootstrap

- Example of conformal bootstrap: verify whether the critical point of 3D Ising model is a CFT and its position in the allowed region [2]
- Physical picture tells us there are two degrees of freedom: energy density ϵ, spin field σ
- Below is Fig. 3 in the paper: comparing critical exponents of 3D Ising model, and the allowed range from conformal bootstrap



[2] arXiv 1203.6064

How to perform bootstrap for a generic system?

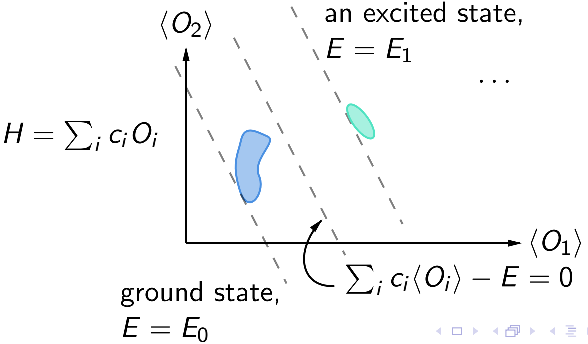
- Correlation functions cannot be determined by countably infinite parameters: no $\{\Delta_{\mathcal{O}}, l_{\mathcal{O}}, f_{\mathcal{ABC}}\}$.
  Solution Store $\langle O_1(x_1) O_2(x_2) \cdots O_n(x_n) \rangle$ separately. Symmetries reduce the size of data: Suppose C is a symmetry,
  $$\langle OC \rangle = \langle CO \rangle . \tag{3}$$
  Derivation similar to below.
- Hard to get OPE
  Solution Suppose we are working on a state whose density matrix is determined solely by H, then we have
  $$\langle OH \rangle = \mathrm{tr}(\rho(H) OH) = \mathrm{tr}(H\rho(H)O) = \mathrm{tr}(\rho(H)HO) = \langle HO \rangle \tag{4}$$
  for all operators O. If we are working on energy eigenstates (i.e with definite E), we have
  $$\langle OH \rangle = E \langle O \rangle , \quad E = \langle H \rangle . \tag{5}$$

Bootstrap for generic systems

- The equality constraints used together with the positivity constraint

$$\langle O^\dagger O \rangle \geq 0 \tag{6}$$

  for every O defines an allowed region - the feasible domain for the following optimization.
- To get ground state information, minimize $\langle H \rangle$ globally.
- To get information about excited states, find local minima of $\langle H \rangle$.



The procedure of numerical bootstrap

- Determine a set of operators

Example: x^4 nonlinear oscillator

$$H = x^2 + p^2 + gx^4. \tag{7}$$

- Symmetry: $x \to -x \Rightarrow \langle x^n \rangle = 0$ with odd n
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