22F CS-556 B HW 3

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1 Problem 1

1. (a)

$$\mu = 160, \sigma = 50$$
$$z = \frac{x-\mu}{\sigma} = \frac{95-160}{50} = -1.3$$

By z-score table:

$$P(X < 95) = 0.09680$$

2. (b)

$$P(x >= 95) = 1 - 0.09680 = 0.9032$$
$$P(x >= q3) = 0.9032 * 0.25 = 0.2258$$

2 Problem 2

1. (a)

$$\begin{aligned} E[X] = \mu &= \frac{10}{43} * 0 + \frac{10}{43} * 200 + \frac{10}{43} * 300 + \frac{3}{43} * 500 \\ &= 6500/43 \\ &\approx 151.16 \end{aligned}$$

2. (b)

$$\begin{aligned} Var(X) = \sigma^2 & & = E[X^2] - E[X]^2 \\ &= \frac{10}{43} * 0 + \frac{10}{43} * 200^2 + \frac{10}{43} * 300^2 + \frac{3}{43} * 500^2 - (\frac{6500}{43})^2 \\ &\approx 24824.23 \end{aligned}$$

$$\begin{aligned} \sigma &= \sqrt{Var(X)} = \sqrt{24824.23} \\ &\approx 157.557 \end{aligned}$$

3 Problem 3

$$P(Correct) = x + (1-x)/y$$
$$P(Know|Correct) = \frac{P(Correct|Know)P(Know)}{P(Correct)} = \frac{1 * x}{x + (1-x)/y} = \frac{xy}{xy + 1 - x}$$

4 Problem 4

Let A = email is detected as spam , B = email is spam, C = email is non-spam.
Using Bayes' Theorem:

$$
\begin{aligned}
P(C|A) &= P(A|C)P(C)/P(A) \\
&= P(A|C)P(C)/(P(A|B)P(B) + P(A|C)P(C)) \\
&= \frac{0.05 * 0.5}{0.05 * 0.5 + 0.99 * 0.5} \\
&= 0.025/0.52 \\
&= 0.0481
\end{aligned}
$$

5 Problem 5

Let P = mammogram result is positive, B = tumor is benign, M = tumor is malignant
Bayes' formula:

$$P(M|P) = P(P|M)P(M)/(P(P|M)P(M)+P(P|B)P(B)) = 0.8*0.01/(0.8*0.01+0.1*0.99) \approx 0.075$$

6 Problem 6

(a)

$$
\begin{aligned}
F'(x) = \frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] &= \lim_{h\to 0} \frac{\frac{f(x+h)}{g(x+h)} - \frac{f(x)}{g(x)}}{h} \\
&= \lim_{h\to 0} \left(\frac{1}{h} \cdot \left(\frac{f(x+h)}{g(x+h)} \cdot \frac{g(x)}{g(x)} - \frac{f(x)}{g(x)} \cdot \frac{g(x+h)}{g(x+h)}\right)\right) \\
&= \lim_{h\to 0} \frac{g(x)f(x+h) - f(x)g(x+h)}{h \cdot g(x)g(x+h)} \\
&= \lim_{h\to 0} \frac{g(x)f(x+h) - f(x)g(x) + f(x)g(x) - f(x)g(x+h)}{h \cdot g(x)g(x+h)} \\
&= \lim_{h\to 0} \left(\frac{1}{g(x)g(x+h)} \cdot \frac{g(x)(f(x+h) - f(x)) - f(x)(g(x+h) - g(x))}{h}\right) \\
&= \lim_{h\to 0} \frac{1}{g(x)g(x+h)} \cdot \left(\lim_{h\to 0} g(x)\frac{f(x+h) - f(x)}{h} - \lim_{h\to 0} f(x)\frac{g(x+h) - g(x)}{h}\right) \\
&= \lim_{h\to 0} \frac{1}{g(x)g(x+h)} \cdot \left(\lim_{h\to 0} g(x) \cdot \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} - \lim_{h\to 0} f(x) \cdot \lim_{h\to 0} \frac{g(x+h) - g(x)}{h}\right) \\
&= \frac{1}{[g(x)]^2} \cdot (g(x)f'(x) - f(x)g'(x)) \qquad = g(
\end{aligned}
$$

Proved.

(b) We know $sin(a+b) = sinacosb + cosasinb$

$$\begin{aligned} f'(x) = \lim_{h\to 0}\frac{\sin(x+h)-\sin(x)}{h} &= \lim_{h\to 0}\frac{\sin(x)\cos(h)+\cos(x)\sin(h)-\sin(x)}{h} \\ &= \lim_{h\to 0}\frac{\sin(x)(\cos(h)-1)+\cos(x)\sin(h)}{h} \\ &= \lim_{h\to 0}\frac{\sin(x)(\cos(h)-1)}{h} + \cos(x)\lim_{h\to 0}\frac{\sin(h)}{h} \\ &= 0+\cos(x)\times 1 \\ &= \cos(x) \end{aligned}$$

Proved.

7 Problem 7

By quotient rule;

$$f'(x) = \frac{e^x \cdot 2x - x^2 \cdot e^x}{e^{2x}} = \frac{2x-x^2}{e^x}$$

When x=1, then the slope of tangent line is $f'(1) = 1/e, f(x) = 1/e$.

$$y - 1/e = 1/e(x-1)$$

So tangent line is

$$y = x/e$$

8 Problem 8

$$\begin{aligned} f'(x) &= e^x \\ g'(x) = \frac{2x\,(x-1) - 1\cdot\, x^2}{(x-1)^2} &= \frac{x^2-2x}{(x-1)^2} \\ F'(x) = f'(g(x))g'(x) = e^{\frac{x^2}{x-1}} &\frac{x^2-2x}{(x-1)^2} \end{aligned}$$

9 Problem 9

(a)

$$\begin{aligned} F'(x) &= \frac{\left(\frac{1}{2\sqrt{x}}+2\right)\left(7x-4x^2\right) - (7-8x)\left(\sqrt{x}+2x\right)}{\left(7x-4x^2\right)^2} \\ &= \frac{16x^2+16x\sqrt{x}-7\sqrt{x}-4x^{\frac{3}{2}}}{2\left(7x-4x^2\right)^2} \\ &= \frac{16x^2+12x\sqrt{x}-7\sqrt{x}}{98x^2-112x^3+32x^4} \end{aligned}$$

(b)

$$\begin{aligned}
F'(x) &= \frac{3\sqrt{x}}{2}\left(\frac{1}{x^3} - 2\sqrt[3]{x}\right) + \frac{-9x^{\frac{2}{3}} - 2x^4}{3x^{\frac{14}{3}}}\left(1 + \sqrt{x^3}\right) \\
&= \frac{3\left(-2x^3\sqrt[3]{x} + 1\right)}{2x^{\frac{5}{2}}} + \frac{\left(-9x^{\frac{2}{3}} - 2x^4\right)\left(1 + x\sqrt{x}\right)}{3x^{\frac{14}{3}}} \\
&= \frac{3\left(-2x^{\frac{10}{3}} + 1\right)}{2x^{\frac{5}{2}}} + \frac{\left(-9x^{\frac{2}{3}} - 2x^4\right)\left(1 + x^{\frac{3}{2}}\right)}{3x^{\frac{14}{3}}}
\end{aligned}$$

(c)

$$\begin{aligned}
F'(x) &= 10\left(2x+1\right)^4\left(3x-2\right)^7 + 21\left(3x-2\right)^6\left(2x+1\right)^5 \\
&= \left(2x+1\right)^4\left(3x-2\right)^6\left(72x+1\right)
\end{aligned}$$

(d)

$$\begin{aligned}
F'(x) &= \frac{\frac{3x+1}{\sqrt{2x+1}} e^x \sin^3(x) - \left(e^x \sin^3(x) + 3\sin^2(x)\cos(x)\, e^x\right) x\sqrt{2x+1}}{\left(e^x \sin^3(x)\right)^2} \\
&= \frac{-2x^2\sin(x) - 6x^2\cos(x) + 2x\sin(x) - 3x\cos(x) + \sin(x)}{e^x \sin^4(x)\sqrt{2x+1}}
\end{aligned}$$

(e)

$$\begin{aligned}
F'(x) &= \frac{1}{2\sqrt{\frac{(x-1)(x-2)}{(x-3)(x-4)(x-5)}}} \cdot \frac{-x^4 + 6x^3 + 5x^2 - 72x + 86}{\left(x-3\right)^2\left(x-4\right)^2\left(x-5\right)^2} \\
&= \frac{-x^4 + 6x^3 + 5x^2 - 72x + 86}{2\sqrt{\left(x-1\right)\left(x-2\right)}\left(x-3\right)^{\frac{3}{2}}\left(x-4\right)^{\frac{3}{2}}\left(x-5\right)^{\frac{3}{2}}}
\end{aligned}$$

10 Problem 10

(a)

$$\begin{aligned}
\frac{\partial F}{\partial x} &= \frac{\left(3y^2x^2 - 2\right)e^x - e^x\left(x^3y^2 - 2x + 5\right)}{\left(e^x\right)^2} \\
&= \frac{\left(3y^2x^2 - 2\right)e^x - e^x\left(x^3y^2 - 2x + 5\right)}{\left(e^x\right)^2} \\
&= \frac{-x^3y^2 + 3x^2y^2 + 2x - 7}{e^x} \\
\frac{\partial F}{\partial y} &= \frac{1}{e^x}\left(\frac{\partial}{\partial\, y}\left(x^3y^2\right) - \frac{\partial}{\partial\, y}\left(2x\right) + \frac{\partial}{\partial\, y}\left(5\right)\right) \\
&= \frac{1}{e^x}\left(2x^3y - 0 + 0\right) \\
&= \frac{2x^3y}{e^x}
\end{aligned}$$

(b)

$$\begin{aligned}
\frac{\partial F}{\partial x} &= \frac{\partial}{\partial\, x}\left(y^2\ln\left(x+2y\right)\right) - \frac{\partial}{\partial\, x}\left(\ln\left(3z\right)\left(x^3 + y^2 - 4z\right)\right) \\
&= \frac{y^2}{x+2y} - 3x^2\ln\left(3z\right)
\end{aligned}$$

$$
\begin{aligned}
\frac{\partial F}{\partial y} &= \frac{\partial}{\partial\, y}\left(y^2 \ln\left(x+2y\right)\right) - \frac{\partial}{\partial\, y}\left(\ln\left(3z\right)\left(x^3+y^2-4z\right)\right) \\
&= 2y\ln\left(x+2y\right) + \frac{2y^2}{x+2y} - 2y\ln\left(3z\right) \\
\frac{\partial F}{\partial z} &= 0 - \left(\frac{x^3+y^2-4z}{z} - 4\ln\left(3z\right)\right) \\
&= -\frac{y^2+x^3-4z}{z} + 4\ln\left(3z\right)
\end{aligned}
$$

11 Problem 11

(a)

$$
\begin{aligned}
f'(12.5) &= \frac{f(15)-f(10)}{2*2.5} = \frac{240-330}{5} = -18 \\
f'(22.5) &= \frac{f(25)-f(20)}{2*2.5} = \frac{350-300}{5} = 10
\end{aligned}
$$

(b)

$$
f'(15) = \frac{f(20)-f(15)}{5} = \frac{300-240}{5} = 12
$$

(c)

$$
f'(15) = \frac{f(15)-f(10)}{5} = \frac{240-330}{5} = -18
$$