22F CS-556 B HW 1

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1. Magnitude of the vector x:

$$\begin{align} ||x|| = \sqrt{x^T x} = \sum_{i=1}^{n} x_i^2 \tag{1} \\ = \sqrt{2^2 + (-3)^2 + 2^2 + 4^2 + (-4)^2} \tag{2} \\ = \sqrt{4 + 9 + 4 + 16 + 16} \tag{3} \\ = 7 \tag{4} \end{align}$$

   So the magnitude of the vector x is 7.

2. By the definition of span, to determine if a vector v is in the span of a set we need to check whether v can be expressed as a linear combination of vectors in S.
   So span of these vector is the set of their linear combinations. The set of linear combinations of $\begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is R^2.
   Hence, the span of vectors is R^2.

3. From question we can get $||a|| = 3, ||b|| = 5$ and $cos(\theta) = \frac{\pi}{2}$.
   By the Cosine Formula for Dot Product:

$$\begin{align} a \cdot b = ||a||\ ||b||\ cos(\theta) \tag{5} \\ = 3 \times 5 \times cos(\frac{\pi}{2}) \tag{6} \\ = 15 \times 0 \tag{7} \\ = 0 \tag{8} \end{align}$$

   So the dot product of a, b is 0.

4. By the function of dot product:

$$\begin{align} u \cdot v = u^T v = \sum_{i=1}^{n} u_i v_i \tag{9} \\ = u_1 v_1 + u_2 v_2 \tag{10} \\ = 1 \times (-2) + 3 \times 7 \tag{11} \\ = 19 \tag{12} \end{align}$$

   Now we have dot product of u and v, to get angle of them we should calculate magnitudes of them first.

$$||u|| = \sqrt{u^T u} = \sum_{i=1}^{n} u_i^2 = \sqrt{1^2 + 3^2} \tag{13}$$
$$= \sqrt{10} \tag{14}$$
$$||v|| = \sqrt{(-2)^2 + 7^2} \tag{15}$$
$$= \sqrt{53} \tag{16}$$

By the Cosine Formula for Dot Product:

$$\theta = cos^{-1}(\frac{u \cdot v}{||u||\ ||v||}) \tag{17}$$
$$= cos^{-1}(\frac{19}{\sqrt{530}}) \tag{18}$$
$$\approx 0.6 \tag{19}$$

Thus we can conclude that the dot product of u, v is 19 and angle between them is 0.6 in radians.

5. By the definition of linear independence, the condition for linear independence is that given a set S of vectors x, y, z and coefficients a, b, c, ax + by + c z = 0 if and only if a = b = c = 0. If $ax + by + cz = 0$, prove $a = b = c = 0$:

$$a \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix} + b \begin{bmatrix} 3 \\ -2 \\ 2 \end{bmatrix} + c \begin{bmatrix} 5 \\ 2 \\ -6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{20}$$
$$\begin{bmatrix} -a + 3b + 5c \\ -2b + 2c \\ 2a + 2b - 6c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{21}$$

So

$$-a + 3b + 5c = 0 \tag{22}$$
$$-2b + 2c = 0 \tag{23}$$
$$2a + 2b - 6c = 0 \tag{24}$$

Let equation (23) +(24), we get

$$a - 2c = 0 \tag{25}$$

By (23) we can get

$$b = c \tag{26}$$
$$\tag{27}$$

Also we know

$$a = 2c \tag{28}$$
$$\tag{29}$$

by (25). But we also get

$$a = 8c = 8b \tag{30}$$
$$\tag{31}$$

from (22) and (23). Hence, we can state that

$$a = b = c = 0 \tag{32}$$

If $a = b = c = 0$, prove $ax + by + cz = 0$:

$$0 * a + 0 * b + 0 * c = 0 \tag{33}$$

Proved.
Thus, we can state vectors x, y, z are linear independent.

6. We want to prove that S is linearly independent iff a linear combination of elements of S with non-zero coefficients does not yield 0.
   Proof 1:
   Suppose that S is linearly independent. We need to prove that a linear combination of elements of S with non-zero coefficients does not yield 0.
   By definition of linearly independent, we can get that: $n_1x_1 + n_2x_2 + n_3x_3... + n_kx_k = 0$ iff $n_1 = n_2 = ... = n_k = 0$.
   That means if n are non-zero coefficients, $n_1x_1 + n_2x_2 + n_3x_3... + n_kx_k \neq 0$, so a linear combination of elements of S with non-zero coefficients does not yield 0.
   Proved.
   Proof 2:
   Suppose if 0 cannot be expressed as a linear combination of elements of S with non-zero coefficients then S is linearly independent.
   By contradiction, suppose that S is not linearly independent. Then there exists an element x in S which is equal to a linear combination of other elements of S:

$$n_1x_1 + n_2x_2 + n_3x_3... + n_kx_k = x \tag{34}$$

$$n_1x_1 + n_2x_2 + n_3x_3... + n_kx_k - x = 0 \tag{35}$$

   Thus we have a linear combination of S which is equal to 0 and not all coefficients of this linear combination are equal to 0. It contradicted with suppose that S is not linearly independent.
   Proved.
   In conclusion, now we can prove that S is linearly independent iff a linear combination of elements of S with non-zero coefficients does not yield 0.

7. Prove $A(B + C) = AB + AC$

$$A(B+C) = \begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 6+1 & 2-2 \\ 3+4 & 2-1 \end{bmatrix} \tag{36}$$

$$= \begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 7 & 0 \\ 7 & 1 \end{bmatrix} \tag{37}$$

$$= \begin{bmatrix} 21+28 & 0+4 \\ 7+14 & 0+2 \end{bmatrix} \tag{38}$$

$$= \begin{bmatrix} 49 & 4 \\ 21 & 2 \end{bmatrix} \tag{39}$$

$$
\begin{aligned}
AB + AC &= \begin{bmatrix} 18+12 & 6+8 \\ 6+6 & 2+4 \end{bmatrix} + \begin{bmatrix} 3+16 & -6-4 \\ 1+8 & -2-2 \end{bmatrix} && (40) \\
&= \begin{bmatrix} 30 & 14 \\ 12 & 6 \end{bmatrix} + \begin{bmatrix} 19 & -10 \\ 9 & -4 \end{bmatrix} && (41) \\
&= \begin{bmatrix} 30+19 & 14+(-10) \\ 12+9 & 6+(-4) \end{bmatrix} && (42) \\
&= \begin{bmatrix} 49 & 4 \\ 21 & 2 \end{bmatrix} = A(B+C) && (43)
\end{aligned}
$$

Hence, for given factors, $A(B+C) = AB + AC$.

8.

$$A = \begin{bmatrix} 3 & 1 & 2 \\ -2 & -4 & 1 \\ 5 & -3 & 2 \end{bmatrix}$$

The minors matrix: a matrix of determinants

$$Minors = \begin{bmatrix} -8+3 & -4-5 & 6+20 \\ 2+6 & 6-10 & -9-5 \\ 1+8 & 3+4 & -12+2 \end{bmatrix} = \begin{bmatrix} -5 & -9 & 26 \\ 8 & -4 & -14 \\ 9 & 7 & -10 \end{bmatrix}$$

The cofactors matrix: the minors matrix element-wise multiplied by a grid of alternating +1 and -1.

$$Cofactors = \begin{bmatrix} -5 & 9 & 26 \\ -8 & -4 & 14 \\ 9 & -7 & -10 \end{bmatrix}$$

The adjugate matrix: the transpose of the cofactors matrix

$$Adjugate = \begin{bmatrix} -5 & -8 & 9 \\ 9 & -4 & -7 \\ 26 & 14 & -10 \end{bmatrix}$$

The inverse matrix: the adjugate matrix divided by the determinant
By the Laplace expansion and definition of determinant

$$\det(\mathbf{A}) = a_{1j}C_{1j} + a_{2j}C_{2j} + a_{3j}C_{3j} + \cdots + a_{nj}C_{nj} = \sum_{i=1}^{n} a_{ij}C_{ij}$$

$$\det(\mathbf{A}) = 3 \times (-5) + 1 \times 9 + 2 \times 26 = 46$$

$$A^{-1} = \frac{Adjugate}{\det(\mathbf{A})} = \begin{bmatrix} \frac{-5}{46} & \frac{-4}{23} & \frac{9}{46} \\ \frac{9}{46} & \frac{-2}{23} & \frac{-7}{46} \\ \frac{13}{23} & \frac{7}{23} & \frac{-5}{23} \end{bmatrix}$$

9. Suppose the three columns in matrix A to be our new basis of interest in R^3.

$$a_1 = \begin{bmatrix} 3 \\ -2 \\ 5 \end{bmatrix}, a_2 = \begin{bmatrix} 1 \\ -4 \\ -3 \end{bmatrix}, a_1 = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$$

From problem we know that

$$x_e = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = A \cdot x_a$$

By A^{-1} of problem 8, so

$$x_a = A^{-1} \cdot x_e \tag{44}$$

$$= \frac{1}{46} \begin{bmatrix} -5 & -8 & 9 \\ 9 & -4 & -7 \\ 26 & 14 & -10 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \tag{45}$$

$$= \frac{1}{46} \begin{bmatrix} -5 - 16 + 27 \\ 9 - 8 - 21 \\ 26 + 28 - 30 \end{bmatrix} \tag{46}$$

$$= \begin{bmatrix} 3/23 \\ -10/23 \\ 12/23 \end{bmatrix} \tag{47}$$

Hence vector x be represented in the basis defined by the matrix A is $\begin{bmatrix} 3/23 \\ -10/23 \\ 12/23 \end{bmatrix}$.