22F CS-556 B HW 2

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October 11, 2022

1 Problem 1

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 0 & 7 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

A is not RREF because second rows need locate at the bottom of the matrix.
B is RREF.
C is not RREF due to violation above.

2 Problem 2

(a) We can see the matrix is RREF and have 3 pivot, so rk(A1)=3.

(b)

$$A_2 = \begin{bmatrix} 3 & -1 & 7 \\ 1 & 4 & 4 \\ 0 & 0 & 1 \end{bmatrix}$$

$$R2 - R1/3 \begin{bmatrix} 3 & -1 & 7 \\ 0 & \frac{13}{3} & \frac{5}{3} \\ 0 & 0 & 1 \end{bmatrix}$$

There are 3 pivot, so rk(A2)=3.

3 Problem 3

By the theorem, we know that the column and row spaces of an m $\times$ n matrix A both have dimension r, the rank of the matrix. The null space has dimension $n - r$, and the left null space has dimension $m - r$.
We know the matrix of dimensions 7 $\times$ 9 has rank 5.
So Dimension of column space = Dimension of row spaces = 5
Dimension of null space = 9 - 5 = 4
Dimension of left null space = 7 - 5 = 2
Sum of all 4 dimensions = 5 + 5 + 4 + 2 = 16

4 Problem 4

Dimension of column space = rank of matrix = 3
Dimension of left null space = 3 - 3 = 0

5 Problem 5

$$A_2 = \begin{bmatrix} 2 & 2 & 2 & 2 \\ 4 & 3 & 2 & 1 \\ 3 & 3 & 4 & 1 \end{bmatrix} R1/2 \begin{bmatrix} 1 & 1 & 1 & 1 \\ 4 & 3 & 2 & 1 \\ 3 & 3 & 4 & 1 \end{bmatrix} R2 - 4R1 \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & -1 & -2 & -3 \\ 3 & 3 & 4 & 1 \end{bmatrix}$$

$$R3 - 3R1 \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & -1 & -2 & -3 \\ 0 & 0 & 1 & -2 \end{bmatrix} - R2 - 2R3 \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 1 & -2 \end{bmatrix} R1 - R2 - R3 \begin{bmatrix} 1 & 0 & 0 & -4 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 1 & -2 \end{bmatrix}$$

$$A_2x = \begin{bmatrix} 1 & 0 & 0 & -4 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = 0$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 4x_4 \\ -7x_4 \\ 2x_4 \\ x_4 \end{bmatrix}$$

Thus, a basis for the null space is: $\begin{bmatrix} 4 \\ -7 \\ 2 \\ 1 \end{bmatrix}$.

6 Problem 6

$$A = \begin{bmatrix} 1 & -2 & -2 \\ 2 & -5 & -4 \\ 4 & -9 & -8 \end{bmatrix} R2 - 2R1 \begin{bmatrix} 1 & -2 & -2 \\ 0 & -1 & 0 \\ 4 & -9 & -8 \end{bmatrix} R3 - 4R1 \begin{bmatrix} 1 & -2 & -2 \\ 0 & -1 & 0 \\ 0 & -1 & 0 \end{bmatrix}$$

$$R3 - R2 \begin{bmatrix} 1 & -2 & -2 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} R1 - 2R2, -R2 \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

For Ax = b, we can find a solution as just a combination of our pivot columns. Let the free variables all set to 0, then $x_P = \begin{bmatrix} b_1 \\ b_2 \\ 0 \end{bmatrix}$.

For Ax = 0, $\begin{bmatrix} x_1 = 2x_3 \\ x_2 = 0 \\ 1\ x_3 = x_3 \end{bmatrix}, x_N = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$

Hence, we can get

$$x = x_P + x_N = \begin{bmatrix} b_1 + 2 \\ b_2 \\ 1 \end{bmatrix}$$

7 Problem 7

(a)

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \text{ and } b = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$$

Use the formula $p = A\left(A^TA\right)^{-1}A^Tb$

$$A^T = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix}$$

$$\begin{aligned} A^TA &= \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \\ \left(A^TA\right)^{-1} &= \frac{1}{det(A)}\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \\ &= \frac{1}{1 \times 2 - 1 \times 1}\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \end{aligned}$$

$$A^Tb = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$$

Putting everything together,

$$p = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 5 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 2 \\ 5 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix}$$

The magnitude of the vector from b perpendicular to the projection:

$$e = b - p = \begin{bmatrix} 0 \\ 0 \\ 4 \end{bmatrix}$$

$$||b - p|| = \sqrt{16} = 4$$

Hence $p = \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix}$, the magnitude of the vector from b perpendicular to the projection is 4.

(b)

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} \text{ and } b = \begin{bmatrix} 4 \\ 4 \\ 6 \end{bmatrix}$$

Use the formula $p = A\left(A^TA\right)^{-1}A^Tb$

$$A^T = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

$$A^TA = \begin{bmatrix} 2 & 2 \\ 2 & 3 \end{bmatrix}$$

$$\left(A^TA\right)^{-1} = \frac{1}{det(A)} \begin{bmatrix} 3 & -2 \\ -2 & 2 \end{bmatrix}$$

$$= \frac{1}{2 \times 3 - 2 \times 2} \begin{bmatrix} 3 & -2 \\ -2 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{3}{2} & -1 \\ -1 & 1 \end{bmatrix}$$

$$A^Tb = \begin{bmatrix} 8 \\ 14 \end{bmatrix}$$

Putting everything together,

$$p = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{3}{2} & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 8 \\ 14 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 0 \\ \frac{1}{2} & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 8 \\ 14 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \\ 6 \end{bmatrix}$$

The magnitude of the vector from b perpendicular to the projection:

$$e = b - p = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$||b - p|| = 0$$

Hence $p = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$, the magnitude of the vector from b perpendicular to the projection is 0.

8 Problem 8

Start with three independent vectors a, b, c

Let $a = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$, $b = \begin{bmatrix} 2 \\ 0 \\ -2 \end{bmatrix}$, $c = \begin{bmatrix} 3 \\ -3 \\ 3 \end{bmatrix}$.

Construct three orthogonal vectors A, B, C, then produce three orthonormal vectors

$A = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$

$B = b - \dfrac{A^T b}{A^T A} A = \begin{bmatrix} 2 \\ 0 \\ -2 \end{bmatrix} - \dfrac{\begin{bmatrix} 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \\ -2 \end{bmatrix}}{\begin{bmatrix} 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ -2 \end{bmatrix} - \dfrac{2}{1+1} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$

$C = c - \dfrac{A^T c}{A^T A} A - \dfrac{B^T c}{B^T B} B = \begin{bmatrix} 3 \\ -3 \\ 3 \end{bmatrix} - \dfrac{6}{2} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} - \dfrac{3-3-6}{1+1+4} \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

$q_1 = \dfrac{A}{||A||} = \dfrac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, q_2 = \dfrac{B}{||B||} = \dfrac{1}{\sqrt{6}} \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}, q_3 = \dfrac{C}{||C||} = \dfrac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

9 Problem 9

We know that $A = V \Lambda V^{-1}$
From the definition of the eigenvector v corresponding to the eigenvalue λ

$$det(A - \lambda I) = det(\begin{bmatrix} 6-\lambda & -1 \\ 2 & 3-\lambda \end{bmatrix}) = 0$$
$$\lambda^2 - 9\lambda + 20 = (\lambda - 4)(\lambda - 5) = 0$$
$$\lambda_1 = 4, \lambda_2 = 5$$

The diagonal matrix (the diagonal entries are the eigenvalues) $\Lambda = \begin{bmatrix} 5 & 0 \\ 0 & 4 \end{bmatrix}$

For $\lambda_1 = 4$,

$$\begin{bmatrix} 2 & -1 \\ 2 & -1 \end{bmatrix} R1/2 \begin{bmatrix} 1 & \frac{-1}{2} \\ 2 & -1 \end{bmatrix} R2 + 2R1 \begin{bmatrix} 1 & \frac{-1}{2} \\ 0 & 0 \end{bmatrix}$$

To find the null space, solve the matrix equation

$$\begin{bmatrix} 1 & \frac{-1}{2} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$
$$x_1 - 0.5x_2 = 0$$
$$X = \begin{bmatrix} \frac{1}{2} * x_2 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}$$

Let $x_2 = 1$, $v_1 = \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}$

For $\lambda_1 = 5$,

$$\begin{bmatrix} 1 & -1 \\ 2 & -2 \end{bmatrix} R2 - 2R1 \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$$

To find the null space, solve the matrix equation

$$\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$$x_1 - x_2 = 0$$

$$X = \begin{bmatrix} x_2 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Let $x_2 = 1$, $v_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ The matrix with the eigenvectors as its columns:

$$V = \begin{bmatrix} 1 & \frac{1}{2} \\ 1 & 1 \end{bmatrix}$$

$$V^{-1} = \frac{1}{1 * 1 - 0.5 * 1} \begin{bmatrix} 1 & \frac{-1}{2} \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ -2 & 2 \end{bmatrix}$$

Hence, we can get

$$A = \begin{bmatrix} 1 & \frac{1}{2} \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 5 & 0 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ -2 & 2 \end{bmatrix}$$

10 Problem 10

$$A^T A = \begin{bmatrix} 2 & 0 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 4 & 6 \\ 6 & 13 \end{bmatrix}$$

From the definition of the eigenvector v corresponding to the eigenvalue λ

$$\begin{aligned} det(A - \lambda I) = det(\begin{bmatrix} 4-\lambda & 6 \\ 6 & 13-\lambda \end{bmatrix}) = 0 \\ \lambda^2 - 17\lambda + 16 = (\lambda - 1)(\lambda - 16) = 0 \\ \lambda_1 = 1, \lambda_2 = 16 \end{aligned}$$

For $\lambda_1 = 1$,

$$\begin{bmatrix} 3 & 6 \\ 6 & 12 \end{bmatrix} R1/3 \begin{bmatrix} 1 & 2 \\ 6 & 12 \end{bmatrix} R2 - 6R1 \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$$

To find the null space, solve the matrix equation

$$\begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$$x_1 + 2x_2 = 0$$

$$X = \begin{bmatrix} -2x_2 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

Let $x_2 = 1$, $v_1 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$

For $\lambda_1 = 16$,

$$\begin{bmatrix} -12 & 6 \\ 6 & -3 \end{bmatrix} \begin{bmatrix} 1 & -0.5 \\ 0 & 0 \end{bmatrix}$$

$$X = \begin{bmatrix} 0.5x_2 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} 0.5 \\ 1 \end{bmatrix}$$

Let $x_2 = 1$, $v_2 = \begin{bmatrix} 0.5 \\ 1 \end{bmatrix}$
Orthonormalized eigenvectors:

$$V = v1 + v2 = \begin{bmatrix} \frac{\sqrt{5}}{5} \\ \frac{2\sqrt{5}}{5} \end{bmatrix} + \begin{bmatrix} \frac{-2\sqrt{5}}{5} \\ \frac{\sqrt{5}}{5} \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{5}}{5} & \frac{-2\sqrt{5}}{5} \\ \frac{2\sqrt{5}}{5} & \frac{\sqrt{5}}{5} \end{bmatrix}$$

$$\sigma_1 = \sqrt{16} = 4, \sigma_2 = \sqrt{1} = 1$$

Construct matrix Σ from square roots of the eigenvalues corresponding to eigenvectors: $\Sigma = \begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix}$
By $AV = U\Sigma, U_i = Av_i/\sigma_i$

$$u_1 = \begin{bmatrix} \frac{2\sqrt{5}}{5} \\ \frac{\sqrt{5}}{5} \end{bmatrix} u_2 = \begin{bmatrix} \frac{-\sqrt{5}}{5} \\ \frac{2\sqrt{5}}{5} \end{bmatrix}$$

Hence, we can get that
$A = U\Sigma V^T = \begin{bmatrix} \frac{2\sqrt{5}}{5} & \frac{-\sqrt{5}}{5} \\ \frac{\sqrt{5}}{5} & \frac{2\sqrt{5}}{5} \end{bmatrix} \begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{\sqrt{5}}{5} & \frac{-2\sqrt{5}}{5} \\ \frac{2\sqrt{5}}{5} & \frac{\sqrt{5}}{5} \end{bmatrix}^T$

11 Problem 11

a)
So user 1's 'estimated' rating of the movie Amelie is 0.4256821525055139.
b)
The "strength" of the concept is given by the magnitude of the values in the SVD diagonal matrix. And if you arrange it in decreasing order of the strength, the first element has the highest strength. The strength will be 13.2663997.
c)
So the average rating for Movie 3 across all users in the system is 3.5483758311270606
d)
The movie with the overall highest rating is Harry Potter

12 Problem 12

b)
0.9776852
c)
The strength of each of the two principal components are 4.22824171 and 0.24267075.
d)
So the magnitude of 2 principal components are 1 and 1.