22F CS-559 A HW 1

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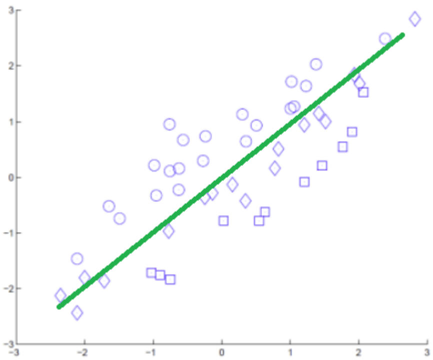
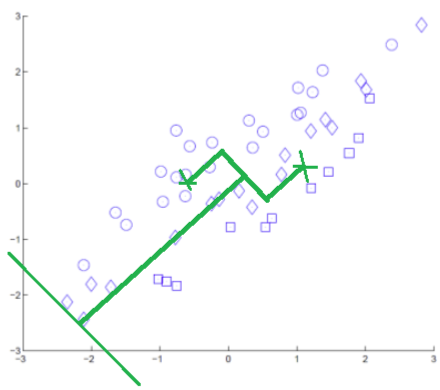
1 Problem 3

1. (a)
   Draw the direction of the first principal component, it's the line that maximizes the variance.

   (b)
   Draw the Fisher's linear discriminant direction, it is to find the projection direction that maximizes the ratio of between-class variance and the within-class variance.

2. (a)

Use the unbiased estimation of the covariance:

$$\mathrm{var}(X) = \frac{1}{N-1}\sum_{n=1}^{N}\left(X_n - \bar{X}\right)^2$$

$$\mathrm{Cov}(X,Y) = \frac{1}{N-1}\sum_{n=1}^{N}\left(X_n - \bar{X}\right)\left(Y_n - \bar{Y}\right)$$

$X_1 = \begin{pmatrix}2\\2\end{pmatrix}$ $X_2 = \begin{pmatrix}0\\0\end{pmatrix}$ $X_3 = \begin{pmatrix}-2\\-2\end{pmatrix}$. So $\bar{X} = \begin{pmatrix}0\\0\end{pmatrix}$.

$$\begin{aligned}
Cov &= \frac{1}{3-1}\sum_{n=1}^{3}\left(X_n - \bar{X}\right)\left(X_n - \bar{X}\right)^T \\
&= \frac{1}{2}(\begin{pmatrix}4 & 4\\4 & 4\end{pmatrix} + \begin{pmatrix}0 & 0\\0 & 0\end{pmatrix} + \begin{pmatrix}4 & 4\\4 & 4\end{pmatrix}) \\
&= \begin{pmatrix}4 & 4\\4 & 4\end{pmatrix}
\end{aligned}$$

From the definition of the eigenvector v corresponding to the eigenvalue λ

$$\begin{aligned}
det(A - \lambda I) = det(\begin{bmatrix}4-\lambda & 4\\4 & 4-\lambda\end{bmatrix}) = 0 \\
\lambda^2 - 8\lambda = (\lambda)(\lambda - 8) = 0 \\
\lambda_1 = 0, \lambda_2 = 8
\end{aligned}$$

We know that: The eigenvector associated with the largest eigenvalue corresponds to the first principal component.

For $\lambda_2 = 8$,

$$\begin{bmatrix}-4 & 4\\4 & -4\end{bmatrix} R1/-4 \begin{bmatrix}1 & -1\\4 & -4\end{bmatrix} R2 - 4R1 \begin{bmatrix}1 & -1\\0 & 0\end{bmatrix}$$

To find the null space, solve the matrix equation

$$\begin{bmatrix}1 & -1\\0 & 0\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix} = 0$$

$$x_1 = x_2$$

$$X = \begin{bmatrix}x_2\\x_2\end{bmatrix} = x_2\begin{bmatrix}1\\1\end{bmatrix}$$

Let $x_2 = 1$, $v_1 = \begin{bmatrix}1\\1\end{bmatrix}$

Hence, the first principal component is $\begin{bmatrix}1\\1\end{bmatrix}$.

(b)

To get the new coordinates, we need find projections of X_i onto the direction $\begin{bmatrix}1\\1\end{bmatrix}$ at first. The projections of them are $\begin{bmatrix} 2 \\ 2 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $\begin{bmatrix} -2 \\ -2 \end{bmatrix}$

$$\begin{bmatrix} 2 \\ 2 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = 0 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} -2 \\ -2 \end{bmatrix} = -2 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

So $X_1 = 2, X_2 = 0, X_3 = -2$
The variance of them is 4.
(c)
Cumulative explained variance of the first principal component $= \frac{\lambda_2}{\lambda_1+\lambda_2} = \frac{8}{8+0} = 1$
So there is not any variance that is not captured by it.