Problem 1 (7.3)

Given the Hamiltonian $H(q_1, q_2, p_1, p_2) = q_1p_1 - q_2p_2 - aq_1^2 + bq_2^2$ with a and b constants, prove that

$$F_1 = \frac{p_2 - bq_2}{q_1}, \quad F_2 = q_1q_2, \quad F_3 = q_1e^{-t} \tag{1}$$

are constants of the motion. Identify a fourth constant of the motion independent of these three constants and, using them, obtain the general solution to the equations of motion - that is, $q_1(t)$, $q_2(t)$, $p_1(t)$, $p_2(t)$ involving four arbitrary constants.

Solution:

A function $F(q_i, p_i, t)$ where $i = 1, \ldots, n$ is a constant of the motion if $\mathrm{d}F/\mathrm{d}t = 0$. The time derivative of F can be generally expressed as

$$\frac{\mathrm{d}F}{\mathrm{d}t} = \sum_i \left(\frac{\partial F}{\partial q_i}\frac{\mathrm{d}q_i}{\mathrm{d}t} + \frac{\partial F}{\partial p_i}\frac{\mathrm{d}p_i}{\mathrm{d}t} \right) + \frac{\partial F}{\partial t}\frac{\mathrm{d}t}{\mathrm{d}t} \tag{2}$$

To figure out if F_1, F_2, and F_3 are constants of the motion, it would be good to calculate the Hamilton's equations as follows:

$$\dot{p}_1 = -\frac{\partial H}{\partial q_1} = -p_1 + 2aq_1 \tag{3}$$

$$\dot{p}_2 = -\frac{\partial H}{\partial q_2} = -p_2 - 2bq_2 \tag{4}$$

$$\dot{q}_1 = \frac{\partial H}{\partial p_1} = q_1 \tag{5}$$

$$\dot{q}_2 = \frac{\partial H}{\partial p_2} = -q_2 \tag{6}$$

Then, by substituting Eqs. (3) and (6) into Eq. (2) for $F = F_1$, we get

$$\begin{aligned} \frac{\mathrm{d}F_1}{\mathrm{d}t} &= \frac{\partial F_1}{\partial q_1}\dot{q}_1 + \frac{\partial F_1}{\partial q_2}\dot{q}_2 + \frac{\partial F_1}{\partial p_1}\dot{p}_1 + \frac{\partial F_1}{\partial p_2}\dot{p}_2 + \frac{\partial F_1}{\partial t} \\ &= -\frac{p_2 - bq_2}{q_1^2}\dot{q}_1 - \frac{b}{q_1}\dot{q}_2 + 0\cdot\dot{p}_1 + \frac{1}{q_1}\dot{p}_2 + 0 \\ &= -\frac{p_2}{q_1} + \frac{bq_2}{q_1} + \frac{bq_2}{q_1} + \frac{p_2}{q_1} - \frac{2bq_2}{q_1} \\ \frac{\mathrm{d}F_1}{\mathrm{d}t} &= 0 \end{aligned} \tag{7}$$

which shows that F_1 is a constant of a motion. Applying the same method for F_2 and F_3, we get

$$\begin{aligned} \frac{\mathrm{d}F_2}{\mathrm{d}t} &= \frac{\partial F_2}{\partial q_1}\dot{q}_1 + \frac{\partial F_2}{\partial q_2}\dot{q}_2 + \frac{\partial F_2}{\partial p_1}\dot{p}_1 + \frac{\partial F_2}{\partial p_2}\dot{p}_2 + \frac{\partial F_2}{\partial t} \\ &= q_2\dot{q}_1 + q_1\dot{q}_2 + 0\cdot\dot{p}_1 + 0\cdot\dot{p}_2 + 0 \\ &= q_2q_1 - q_1q_2 \\ \frac{\mathrm{d}F_2}{\mathrm{d}t} &= 0 \end{aligned} \tag{8}$$

$$\begin{aligned}
\frac{\mathrm{d}F_3}{\mathrm{d}t} &= \frac{\partial F_3}{\partial q_1}\dot{q}_1 + \frac{\partial F_3}{\partial q_2}\dot{q}_2 + \frac{\partial F_3}{\partial p_1}\dot{p}_1 + \frac{\partial F_3}{\partial p_2}\dot{p}_2 + \frac{\partial F_3}{\partial t} \\
&= e^{-t}\dot{q}_1 + 0 \cdot \dot{q}_2 + 0 \cdot \dot{p}_1 + 0 \cdot \dot{p}_2 - q_1 e^{-t} \\
&= q_1 e^{-t} - q_1 e^{-t} \\
\frac{\mathrm{d}F_3}{\mathrm{d}t} &= 0
\end{aligned} \tag{9}$$

which show that F_2 and F_3 are also constants of the motion. Through trial and error, as well as eyeballing F_1, we get a fourth constant of the motion given by $F_4 = (p_1 - aq_1)/q_2$ as proven in the following calculation:

$$\begin{aligned}
\frac{\mathrm{d}F_4}{\mathrm{d}t} &= \frac{\partial F_4}{\partial q_1}\dot{q}_1 + \frac{\partial F_4}{\partial q_2}\dot{q}_2 + \frac{\partial F_4}{\partial p_1}\dot{p}_1 + \frac{\partial F_4}{\partial p_2}\dot{p}_2 + \frac{\partial F_4}{\partial t} \\
&= -\frac{a}{q_2}\dot{q}_1 - \frac{p_1 - aq_1}{q_2^2}\dot{q}_2 + \frac{1}{q_2}\dot{p}_1 + 0 \cdot \dot{p}_2 + 0 \\
&= -\frac{aq_1}{q_2} + \frac{p_1}{q_2} - \frac{aq_1}{q_2} + -\frac{p_1}{q_2} + \frac{2aq_1}{q_2} \\
\frac{\mathrm{d}F_4}{\mathrm{d}t} &= 0
\end{aligned} \tag{10}$$

Now, from these four constants of the motion, we can obtain

$$q_1(t) = \frac{F_3}{e^{-t}} = F_3 e^t \tag{11}$$

$$q_2(t) = \frac{F_2}{q_1(t)} = \frac{F_2}{F_3} e^{-t} \tag{12}$$

$$p_1(t) = F_4 q_2(t) + aq_1(t) = F_4 \frac{F_2}{F_3} e^{-t} + aF_3 e^t \tag{13}$$

$$p_2(t) = F_1 q_1(t) + bq_2(t) = F_1 F_3 e^t + b\frac{F_2}{F_3} e^{-t} \tag{14}$$

as the general solution to the equations of motion.

**Acknowledgements: I am grateful for the insightful comments of Reinier Ramos when solving this problem.*

References

How to find constant of motion for Hamiltonian system?, `https://physics.stackexchange.com/questions/66717/how-to-find-constant-of-motion-for-hamiltonian-system`

Problem 2 (7.5)

A one-degree-of-freedom mechanical system is described by the following Lagrangian:

$$L(Q, \dot{Q}, t) = \frac{\dot{Q}^2}{2}\cos^2 \omega t - \frac{\omega}{2} Q\dot{Q}\sin 2\omega t - \frac{\omega^2 Q^2}{2}\cos 2\omega t \quad (1)$$

(a) Find the corresponding Hamiltonian.

Solution:

With the given Lagrangian $L(Q, \dot{Q}, t)$,

In[18]:=
```
HoldForm[L] ==
 (L = D[Q[t], t]^2/2 Cos[ω t]^2 - ω/2 Q[t] D[Q[t], t] Sin[2 ω t] - ω^2 Q[t]^2/2 Cos[2 ω t])
```

Out[18]= $L == -\frac{1}{2}\omega^2 \text{Cos}[2\,t\,\omega]\,Q[t]^2 - \frac{1}{2}\omega\,Q[t]\,\text{Sin}[2\,t\,\omega]\,Q'[t] + \frac{1}{2}\text{Cos}[t\,\omega]^2\,Q'[t]^2$

we can calculate the canonical momentum P to Q by
taking the derivative of L with respect to Q as follows :

In[19]:=
```
P[t] == D[L, D[Q[t], t]]
```

Out[19]= $P[t] == -\frac{1}{2}\omega\,Q[t]\,\text{Sin}[2\,t\,\omega] + \text{Cos}[t\,\omega]^2\,Q'[t]$

From this expression, we can isolate $\dot{Q}$ (Q'[t] in this code) to get

In[20]:=
```
Qdot = Solve[%, D[Q[t], t]][[1, 1]]
```

Out[20]= $Q'[t] \to \frac{1}{2}\text{Sec}[t\,\omega]^2\,(2\,P[t] + \omega\,Q[t]\,\text{Sin}[2\,t\,\omega])$

To solve for the Hamiltonian,
we substitute L and $\dot{Q}$ (which we do by applying the rule Qdot) into $H(Q, P, t) =$
$P\dot{Q} - L(Q, \dot{Q}, t)$. Thus, the Hamiltonian $H(Q, P, t)$ of the given system is

In[23]:=
```
HoldForm[H] == (H = D[Q[t], t] P[t] - L /. Qdot // Simplify)
```

Out[23]= $H == \frac{1}{2}\text{Sec}[t\,\omega]^2\,(P[t]^2 + \omega^2\,\text{Cos}[t\,\omega]^4\,Q[t]^2 + \omega\,P[t]\,Q[t]\,\text{Sin}[2\,t\,\omega])$

(b) Is this Hamiltonian a constant of the motion?

Solution:

For the Hamiltonian to be a constant of motion,
it must satisfy the relation $\frac{dH}{dt} = 0$. By calculating this,

In[22]:=
```
HoldForm[dH/dt] == (D[H, t] /. Qdot // Expand // FullSimplify)
```

Out[22]= $\frac{dH}{dt} == \frac{1}{2}\text{Sec}[t\,\omega]^2\,(4\,\omega\,P[t]\,(\omega\,Q[t]+P[t]\,\text{Tan}[t\,\omega])+(2\,P[t]+\omega\,Q[t]\,\text{Sin}[2\,t\,\omega])\,P'[t])$

we see that $\frac{dH}{dt} \neq 0$. Thus, H (Q, P, t) is not a constant of motion

(c) Show that the Hamiltonian expressed in terms of the new variable $q = Q\cos\omega t$ and its conjugate momentum does not explicitly depend on time. What physical system does it describe?

Solution:

To express H (Q, P, t) in terms of q,
we can first express L $(Q, \dot{Q}, t)$ in terms of q and then follow the same method in part (a). Using the trigonometric relations sin $(2\,\omega\,t)$ = 2 sin $(\omega\,t)$ cos $(\omega\,t)$ and cos $(2\,\omega\,t)$ = $\cos^2(\omega\,t) - \sin^2(\omega\,t)$, we get

In[35]:=
```
HoldForm[L] == (Lq1 =
   L /. Sin[2 ω t] → 2 Sin[ω t] Cos[ω t] /. Cos[2 ω t] → (Cos[ω t]^2 - Sin[ω t]^2) // Expand)
```

Out[35]= $L == -\frac{1}{2}\omega^2\,\text{Cos}[t\,\omega]^2\,Q[t]^2 + \frac{1}{2}\omega^2\,Q[t]^2\,\text{Sin}[t\,\omega]^2 - \omega\,\text{Cos}[t\,\omega]\,Q[t]\,\text{Sin}[t\,\omega]\,Q'[t] + \frac{1}{2}\text{Cos}[t\,\omega]^2\,Q'[t]^2$

Note that $\dot{q}$ (q'[t] in this code) is given by

In[37]:=
```
HoldForm[dq/dt] == (D[Q[t] Cos[ω t], t])
```

Out[37]= $\frac{dq}{dt} == -\omega\,Q[t]\,\text{Sin}[t\,\omega] + \text{Cos}[t\,\omega]\,Q'[t]$

Using this relation and q, we obtain $\dot{q}$

In[36]:=
```
HoldForm[L] == (Lq2 = Lq1 /. Cos[t ω]^2 Q[t]^2 → q[t]^2 /.
    ω^2 Q[t]^2 Sin[t ω]^2 → (D[Q[t], t] Cos[t ω] - D[q[t], t])^2 // Expand)
```

Out[36]= $L ==$
$-\frac{1}{2}\omega^2\,q[t]^2 + \frac{1}{2}q'[t]^2 - \omega\,\text{Cos}[t\,\omega]\,Q[t]\,\text{Sin}[t\,\omega]\,Q'[t] - \text{Cos}[t\,\omega]\,q'[t]\,Q'[t] + \text{Cos}[t\,\omega]^2\,Q'[t]^2$

Look at the third and fourth terms. Applying $\dot{q}$ again, the two terms simplify to

In[39]:=
```
(-ω Cos[t ω] Q[t] Sin[t ω] Q'[t] - Cos[t ω] q'[t] Q'[t] // Simplify) /.
 ω Q[t] Sin[t ω] → D[Q[t], t] Cos[t ω] - D[q[t], t]
```

Out[39]= $-\text{Cos}[t\,\omega]^2\,Q'[t]^2$

Substituting this back to L, we finally have L $(q, \dot{q}, t)$

In[40]:=
```
HoldForm[L] == (Lq = Lq2 /. -ω Cos[t ω] Q[t] Sin[t ω] Q'[t] - Cos[t ω] q'[t] Q'[t] → %)
```

Out[40]= $L == -\frac{1}{2}\omega^2\,q[t]^2 + \frac{1}{2}q'[t]^2$

Now, in the same manner as part (a),

we can solve for the canonical momentum p to q
by taking the derivative of L $(q, \dot{q}, t)$ with respect to q

In[42]:=
```
p[t] == D[Lq, D[q[t], t]]
```

Out[42]= $p[t] == q'[t]$

and then isolate $\dot{q}$ to get

In[43]:=
```
qdot = Solve[%, D[q[t], t]][[1, 1]]
```

Out[43]= $q'[t] \to p[t]$

Substituting L and $\dot{q}$ (which we do by applying the rule qdot) into H $(q, p, t) =$
$p\,\dot{q} - L\,(q, \dot{q}, t)$, we obtain the Hamiltonian in terms of q

In[44]:=
```
HoldForm[H] == (D[q[t], t] p[t] - Lq /. qdot // Simplify // Expand)
```

Out[44]= $H == \frac{p[t]^2}{2} + \frac{1}{2}\omega^2 q[t]^2$

Studying the Lagrangian and the Hamiltonian, we can infer that $T = \frac{1}{2}\dot{q}^2$ and $V = \frac{1}{2}\omega^2 q$. These are the kinetic and potential energies of a mechanical system describing a one - dimensional harmonic oscillator in which $\omega^2 = k / m$ with $m = 1$ as the mass of the oscillator and k as the spring constant.

Problem 3 (7.6)

Consider the n -body problem in the the centre-of-mass frame. The Hamiltonian is given by $H = T + V$ and $T = \sum_i |\mathbf{p}_i|^2 / 2m_i$ and

$$V(\mathbf{r}_1, \ldots, \mathbf{r}_n) = -\frac{1}{2} \sum_{\substack{i,j \\ i \neq j}} \frac{G m_i m_j}{|\mathbf{r}_i - \mathbf{r}_j|} \tag{1}$$

(a) Introduce the quantity $I = \frac{1}{2} \sum_i m_i |\mathbf{r}_i|^2$ and prove that $\ddot{I} = E + T$. Hint: show that V is a homogeneous function and apply Euler's theorem from Appendix C.

Solution:

The quantity $I = \frac{1}{2} \sum_i m_i |\mathbf{r}_i|^2$ is actually the moment of inertia for the said system. Taking its time derivative, we have

$$\dot{I} = \frac{1}{2} \sum_i m_i \frac{\mathrm{d}(|\mathbf{r}_i|^2)}{\mathrm{d}t} = \frac{1}{2} \sum_i m_i \frac{\partial(|\mathbf{r}_i|^2)}{\partial |\mathbf{r}_i|} \frac{\mathrm{d}|\mathbf{r}_i|}{\mathrm{d}t} = \frac{1}{2} \sum_i m_i (2|\mathbf{r}_i|)|\dot{\mathbf{r}}_i| = \sum_i |\mathbf{r}_i||\mathbf{p}_i| \tag{2}$$

where $|\mathbf{p}_i| = m_i |\dot{\mathbf{r}}_i|$ is the linear momentum. Solving for the second time derivative of I, we have

$$\ddot{I} = \sum_i \frac{\mathrm{d}(|\mathbf{r}_i||\mathbf{p}_i|)}{\mathrm{d}t} = \sum_i \left(|\mathbf{r}_i| \frac{\mathrm{d}|\mathbf{p}_i|}{\mathrm{d}t} + |\mathbf{p}_i| \frac{\mathrm{d}|\mathbf{r}_i|}{\mathrm{d}t} \right) = \sum_i \left(|\mathbf{r}_i||\dot{\mathbf{p}}_i| + \frac{|\mathbf{p}_i|^2}{m_i} \right) \tag{3}$$

Now, we note that a function $F(x_1, \ldots, x_n)$ is homogeneous of degree p if

$$F(\lambda x_1, \ldots, \lambda x_n) = \lambda^p F(x_1, \ldots, x_n) \tag{4}$$

for positive λ. If F is differentiable and is shown to be homogeneous, then the following relation is true:

$$\sum_{k=1}^{n} x_k \frac{\partial F}{\partial x_k} = pF \tag{5}$$

By showing that V is a homogeneous function of degree -1

$$V(\lambda \mathbf{r}_1, \ldots, \lambda \mathbf{r}_n) = -\frac{1}{2} \sum_{\substack{i,j \\ i \neq j}} \frac{G m_i m_j}{|\lambda \mathbf{r}_i - \lambda \mathbf{r}_j|} = -\frac{\lambda^{-1}}{2} \sum_{\substack{i,j \\ i \neq j}} \frac{G m_i m_j}{|\mathbf{r}_i - \mathbf{r}_j|} = \lambda^{-1} V(\mathbf{r}_1, \ldots, \mathbf{r}_n) \tag{6}$$

and applying Eq. (5), we can express V as

$$V = -\sum_k |\dot{\mathbf{r}}_k| \frac{\partial V}{\partial |\dot{\mathbf{r}}_k|} = \sum_k |\dot{\mathbf{r}}_k||\dot{\mathbf{p}}_k| \tag{7}$$

Notice here that $|\dot{\mathbf{p}}_k| = -\partial V / \partial |\dot{\mathbf{r}}_k|$ which we can infer from

$$\begin{aligned}
\mathbf{F}(\mathbf{r}_1, \ldots, \mathbf{r}_n) &= -\nabla V(\mathbf{r}_1, \ldots, \mathbf{r}_n) \\
\sum_i m_i \ddot{\mathbf{r}}_i &= -\sum_i \frac{\partial V}{\partial |\dot{\mathbf{r}}_i|} \mathbf{r}_i \\
\sum_i \dot{\mathbf{p}}_i &= \\
\sum_i |\dot{\mathbf{p}}_i| \mathbf{r}_i &= -\sum_i \frac{\partial V}{\partial |\dot{\mathbf{r}}_i|} \mathbf{r}_i
\end{aligned} \tag{8}$$

Substituting T and Eq. (7) into Eq. (3), we get

$$\ddot{I} = V + 2T = V + T + T = E + T \tag{9}$$

where $E = T + V$.

(b) Taking into account that T is always positive, perform two successive integrations to prove that $I(t) \geq I(0) + \dot{I}(0)t + Et^2/2$. Conclude, finally, that if the total energy is positive at least one of the bodies will escape to infinity in the limit $t \to \infty$.

Solution:
Since E and T does not depend explicitly on t, let us treat them as constants with respect to time. Integrating Eq. (9), we obtain

$$\begin{aligned} \frac{\mathrm{d}\dot{I}}{\mathrm{d}t} &= E + T \\ \int \mathrm{d}\dot{I} &= (E + T) \int \mathrm{d}t \\ \dot{I}(t) &= (E + T)t + k_1 \end{aligned} \tag{10}$$

At $t = 0$, Eq. (10) becomes $\dot{I}(0) = k_1$. Substituting this into k_1 in Eq. (10) and integrating the said equation we have

$$\begin{aligned} \frac{\mathrm{d}I}{\mathrm{d}t} &= (E + T)t + \dot{I}(0) \\ \int \mathrm{d}I &= \int \left[(E + T)t + \dot{I}(0)\right] \mathrm{d}t \\ I(t) &= (E + T)\frac{t^2}{2} + \dot{I}(0)t + k_2 \end{aligned} \tag{11}$$

At $t = 0$, Eq. (11) becomes $I(0) = k_2$. Substituting this into k_2 in Eq. (11), $I(t)$ is given by

$$I(t) = (E + T)\frac{t^2}{2} + \dot{I}(0)t + I(0) \tag{12}$$

With T as always positive in value, $E + T \geq E$. Thus,

$$I(t) \geq I(0) + \dot{I}(0)t + Et^2/2 \tag{13}$$

So, if the total energy is positive, $I(t)$ becomes infinitely large. The larger moment of inertia an object has, the less easier it is to rotate. Thus, at least one of the bodies can no longer be rotated to keep its orbit and escapes to infinity.

**Acknowledgements: I am grateful for the insightful comments of Reinier Ramos when solving this problem.*

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Problem 4 (7.7)

A planet describes an elliptic orbit about the Sun. Let the spherical coordinates be so chosen that the plane of the motion is given by $\theta = \pi/2$. By Eq. (7.15), in polar coordinates r, ϕ in the orbital plane the Hamiltonian is written

$$H = T + V = \frac{1}{2m}\left(p_r^2 + \frac{p_\phi^2}{r^2}\right) - \frac{k}{r} \tag{1}$$

with $k > 0$. On the face of it, the virial theorem is valid no matter what canonical variables are employed. Show that in terms of the polar coordinates r, ϕ and their conjugate momenta, Eq. (7.32) takes the form

$$2\langle T\rangle = -\langle V\rangle - \left\langle \frac{p_\phi^2}{mr^2}\right\rangle \tag{2}$$

Comparing this result with (7.41), show that $\left\langle p_\phi^2/mr^2\right\rangle = 0$. Using the fact that for elliptic orbits p_ϕ is a non-zero constant of the motion, conclude that $\left\langle 1/r^2\right\rangle = 0$. Argue that this is impossible. Examine carefully the conditions for the validity of the virial theorem in the present case and explain why its use in polar coordinates leads to an absurdity (Chagas, das & Lemos, 1981).

Solution:
From the Hamiltonian, we can infer that the kinetic and potential energy of the system are

$$T = \frac{1}{2m}\left(p_r^2 + \frac{p_\phi^2}{r^2}\right), \qquad V = -\frac{k}{r} \tag{3}$$

According to virial theorem, we have the relation

$$\left\langle \sum_i q_i \frac{\partial H}{\partial q_i}\right\rangle = \left\langle \sum_i p_i \frac{\partial H}{\partial p_i}\right\rangle \tag{4}$$

Solving this for the Hamiltonian in Eq. (1), we have

$$\begin{aligned}\left\langle r\frac{\partial H}{\partial r} + \theta\frac{\partial H}{\partial \theta} + \phi\frac{\partial H}{\partial \phi}\right\rangle &= \left\langle p_r\frac{\partial H}{\partial p_r} + p_\theta\frac{\partial H}{\partial p_\theta} + p_\phi\frac{\partial H}{\partial p_\phi}\right\rangle \\ \left\langle r\left(-\frac{1}{m}\frac{p_\phi^2}{r^3} + \frac{k}{r^2}\right) + \theta\cdot 0 + \phi\cdot 0\right\rangle &= \left\langle p_r\frac{p_r}{m} + p_\theta\cdot 0 + p_\phi\frac{p_\phi}{mr^2}\right\rangle \\ \left\langle -\frac{1}{m}\frac{p_\phi^2}{r^2} + \frac{k}{r}\right\rangle &= \left\langle \frac{p_r^2}{m} + \frac{p_\phi^2}{mr^2}\right\rangle\end{aligned} \tag{5}$$

Note that

$$\langle f + g\rangle = \langle f\rangle + \langle g\rangle \tag{6}$$

if f and g exists separately. Thus, we can rearrange Eq. (5) into

$$\begin{aligned}\left\langle -\frac{1}{m}\frac{p_\phi^2}{r^2} - \frac{p_r^2}{m} \right\rangle &= \left\langle -\frac{k}{r} \right\rangle + \left\langle \frac{p_\phi^2}{mr^2} \right\rangle \\ -2\langle T \rangle &= \langle V \rangle + \left\langle \frac{p_\phi^2}{mr^2} \right\rangle \\ 2\langle T \rangle &= -\langle V \rangle - \left\langle \frac{p_\phi^2}{mr^2} \right\rangle\end{aligned} \tag{7}$$

after substituting in T and V from Eq. (3). According to (7.41) of Lemos, the virial theorem establishes that

$$2\langle T \rangle = -\langle V \rangle \tag{8}$$

for a gravitational or electrostatic potential. Comparing Eq. (7) with this relation, we can see that

$$\left\langle \frac{p_\phi^2}{mr^2} \right\rangle = 0 \tag{9}$$

With p_ϕ as a non-zero constant for elliptical orbits, Eq. (9) results to

$$\left\langle \frac{1}{r^2} \right\rangle = 0 \tag{10}$$

However, $\frac{1}{r^2}$ is clearly not zero for a planet orbiting the Sun so Eq. (10) is not true. It turns out that the the virial theorem is not valid in polar coordinates because ϕ is unbounded. We can mathematically show this by solving $\dot{\phi}$

$$\dot{\phi} = -\frac{\partial H}{\partial p_\phi} = \frac{p_\phi}{mr^2} \tag{11}$$

and integrating it to obtain

$$\begin{aligned}\int_{\phi(0)}^{\phi(t)} \mathrm{d}\phi &= \int_0^t \frac{p_\phi}{mr^2}\mathrm{d}t \\ \phi(t) - \phi(0) &= \frac{1}{m}\int_0^t \frac{p_\phi}{r^2}\mathrm{d}t\end{aligned} \tag{12}$$

Note that $r = r(t)$ and $p_\phi = p_\phi(t)$. Now, let R be the maximum value of r (the the radius of the orbit measured at the aphelion). We now that

$$\frac{1}{r^2} \geq \frac{1}{R^2} \tag{13}$$

since R^{-2} goes to zero faster than r^{-2}. Thus, we have

$$\begin{aligned}\frac{1}{m}\int_0^t \frac{p_\phi}{r^2}\mathrm{d}t &\geq \frac{1}{m}\int_0^t \frac{p_\phi}{R^2}\mathrm{d}t \\ \phi(t) - \phi(0) &\geq \frac{1}{m}\int_0^t \frac{p_\phi}{R^2}\mathrm{d}t\end{aligned} \tag{14}$$

after substituting in Eq. (12). Rearranging, we get

$$\phi(t) \geq \phi(0) + \frac{1}{m}\int_0^t \frac{p_\phi}{R^2}\mathrm{d}t \tag{15}$$

which shows that $\phi(t)$ has no maximum value.

References

Um Exeimpla de Como Não Usar Teoremas Matemhticos em Problemas Rsicos, http://sbfisica.org.br/bjp/download/v11/v11a27.pdf

Problem 5 (7.12)

Starting from the Hamiltonian formulation, obtain another formulation of mechanics using as variables p, $\dot{p}$ and a function $Y(p, \dot{p}, t)$. By imitating the example of Eqs. and (7.5), the Legendre transformation that replaces q by $\dot{p}$ must presuppose that the equations $\dot{p}_i = -\partial H/\partial q_i$ can be solved for the q s in terms of p, $\dot{p}$, t. Construct the Legendre transformation that takes from $H(q, p, t)$ to $Y(p, \dot{p}, t)$ and derive the equations of motion in terms of Y. Apply this approach to the particular case in which $H = p^2/2 + \omega^2 q^2/2$ and comment on the results obtained.

Solution:
Since $\dot{p}_i = -\partial H/\partial q_i$, the Legendre transformation that takes from $H(q, p, t)$ to $Y(p, \dot{p}, t)$ is a little different from what we usually do. To construct the transform for this case, let us suppose that we have a function $F(x, y)$ in which

$$\frac{\partial F(x,y)}{\partial y} = -z \quad \longrightarrow \quad -\mathrm{d}F = z\mathrm{d}y \tag{1}$$

which we want to transform into $G(x, z)$ in which

$$\frac{\partial G(x,z)}{\partial z} = y \quad \longrightarrow \quad \mathrm{d}G = y\mathrm{d}z \tag{2}$$

Combining Eqs. (1) and (2), we obtain

$$\begin{aligned} -\mathrm{d}F + \mathrm{d}G &= z\mathrm{d}y + y\mathrm{d}z \\ \int \mathrm{d}(G - F) &= \int \mathrm{d}(yz) \\ G - F &= yz \end{aligned} \tag{3}$$

Rearranging Eq. (3), the Legendre transformation that takes from $F(x, y)$ to $G(x, z)$ is

$$G(x, z) = yz + F(x, y) \tag{4}$$

Applying this relation, we have

$$Y(p, \dot{p}, t) = \sum_i q_i \dot{p}_i + H(q, p, t) \tag{5}$$

as the Legendre transformation that takes from $H(q, p, t)$ to $Y(p, \dot{p}, t)$. Using Hamilton's equations

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial q_i}, \tag{6}$$

in solving $\mathrm{d}Y$ from Eq. (5), we get

$$\begin{aligned} \mathrm{d}Y &= \sum_i (q_i \mathrm{d}\dot{p}_i + \dot{p}_i \mathrm{d}q_i) + \mathrm{d}H \\ &= \sum_i (q_i \mathrm{d}\dot{p}_i + \dot{p}_i \mathrm{d}q_i) + \sum_i \left(\frac{\partial H}{\partial q_i} \mathrm{d}q_i + \frac{\partial H}{\partial p_i} \mathrm{d}p_i \right) + \frac{\partial H}{\partial t} \mathrm{d}t \\ &= \sum_i (q_i \mathrm{d}\dot{p}_i + \dot{p}_i \mathrm{d}q_i) + \sum_i (-\dot{p}_i \mathrm{d}q_i + \dot{q}_i \mathrm{d}p_i) + \frac{\partial H}{\partial t} \mathrm{d}t \\ \mathrm{d}Y &= \sum_i (q_i \mathrm{d}\dot{p}_i + \dot{q}_i \mathrm{d}p_i) + \frac{\partial H}{\partial t} \mathrm{d}t \end{aligned} \tag{7}$$

Comparing this with the total differential of Y

$$\mathrm{d}Y = \sum_i \left(\frac{\partial Y}{\partial p_i} \mathrm{d}p_i + \frac{\partial H}{\partial \dot{p}_i} \mathrm{d}\dot{p}_i \right) + \frac{\partial Y}{\partial t} \mathrm{d}t, \tag{8}$$

we obtain the following equations of motion in terms of Y.

$$q_i = \frac{\partial Y}{\partial p_i}, \quad \dot{q}_i = \frac{\partial H}{\partial \dot{p}_i} \tag{9}$$

For the harmonic oscillator where $H = p^2/2 + \omega^2 q^2/2$ (as mentioned in problem 2), we can transform this into the $Y(p, \dot{p}, t)$ formulation by calculating

$$\dot{p} = -\frac{\partial H}{\partial q} = -\omega^2 q, \quad \rightarrow \quad q = -\frac{\dot{p}}{\omega^2} \tag{10}$$

and substituting this in the equation obtained using Eq. (5):

$$\begin{aligned} Y &= -q\dot{p} + \frac{p^2}{2} + \frac{\omega^2 q^2}{2} \\ &= -\frac{\dot{p}^2}{\omega^2} + \frac{p^2}{2} + \frac{1}{2}\frac{\dot{p}^2}{\omega^2} \\ Y &= \frac{p^2}{2} - \frac{1}{2}\frac{\dot{p}^2}{\omega^2} \end{aligned} \tag{11}$$

This seems somewhat similar to the harmonic oscillator's Lagrangian where

$$L(q, \dot{q}, t) = \frac{\dot{q}^2}{2} - \frac{\omega^2 q^2}{2} \tag{12}$$

Indeed, there is a special relation between L and Y for a harmonic oscillator which we see in Eq. (11) after substituting back in $\dot{p}$ from Eq. (10):

$$Y = \frac{p^2}{2} - \frac{1}{2}\frac{(-\omega^2 q)^2}{\omega^2} = \frac{\dot{q}^2}{2} - \frac{\omega^2 q^2}{2} = L \tag{13}$$

where we note that $p = \dot{q}$. We say special because Y is not necessarily equal to L. We can use the Legendre transform for L

$$L(q, \dot{q}, t) = \sum_i p_i \dot{q}_i - H(q, p, t) \tag{14}$$

and Y in Eq. (5) to cancel H and show that

$$Y(p, \dot{p}, t) = \sum_i (q_i \dot{p}_i + p_i \dot{q}_i) - L(q, \dot{q}, t) = \left[\sum_i (q_i \dot{p}_i + p_i \dot{q}_i) - 2L(q, \dot{q}, t) \right] + L(q, \dot{q}, t) \tag{15}$$

We notice here that $Y = L$ only when

$$L = \sum_i \frac{1}{2} (q_i \dot{p}_i + p_i \dot{q}_i) \tag{16}$$

which must have been satisfied by the harmonic oscillator's Lagrangian as we have shown that $Y = L$ for this case.

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References

Morin, D., *Chapter 15: The Hamiltonian method*, `https://scholar.harvard.edu/files/david-morin/files/cmchap15.pdf`

Problem 6 (7.21)

A one-degree-of-freedom mechanical system obeys the following equations of motion: $\dot{q} = pf(q), \dot{p} = g(q,p)$. What restrictions must be imposed on the functions f and g in order that this system be Hamiltonian? If the system is Hamiltonian, what is the most general form of Hamilton's function H?

Solution:
Hamilton's equations are given by

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial q_i} \tag{1}$$

From the given and these relations, we know that

$$\frac{\partial H}{\partial p} = pf(q) \quad \rightarrow \quad f(q) = \frac{1}{p}\frac{\partial H}{\partial p} \tag{2}$$

and

$$-\frac{\partial H}{\partial q} = g(q,p) \tag{3}$$

We can infer from this that $f(q)$ and $g(q,p)$ must be smooth, differentiable and must satisfy Eqs. (2) and (3) for the said system to be Hamiltonian. By integrating Eq. (2) wih respect to p,

$$\begin{aligned} \int \mathrm{d}H &= \int pf(q)\mathrm{d}p \\ H &= f(q)\frac{p^2}{2} + f(q)C(q) \end{aligned} \tag{4}$$

we obtain the most general form of H where $C(q)$ is some function in q that satisfy the relation taken from integrating Eq. (3):

$$\begin{aligned} -\frac{\partial H}{\partial q} &= -\frac{\partial}{\partial q}\left(\frac{1}{2}f(q)p^2 + f(q)C(q)\right) \\ g(q,p) &= -\frac{1}{2}\frac{\partial f(q)}{\partial q}p^2 - C(q)\frac{\partial f(q)}{\partial q} - f(q)\frac{\partial C(q)}{\partial q} \end{aligned} \tag{5}$$

Let us have an example to see if this general form is true. We consider the one-dimensional harmonic oscillator's Hamiltonian $H = p^2/2 + \omega^2 q^2/2$. We have

$$\dot{q} = p, \qquad \dot{p} = -\omega^2 q \tag{6}$$

by solving Hamilton's equations. Comparing $\dot{q}$ and $\dot{p}$ in Eq. (6) with $\dot{q} = pf(q)$ and $\dot{p} = g(q,p)$, we can infer that $f(q) = 1$ and $g(q,p) = -\omega^2 q$. Substituting this into Eq. (5),

$$-\omega^2 q = -\frac{1}{2}\cancelto{0}{\frac{\partial f(q)}{\partial q}} p^2 - C(q)\cancelto{0}{\frac{\partial f(q)}{\partial q}} - 1 \cdot \frac{\partial C(q)}{\partial q} \tag{7}$$

Integrating this with respect to q,

$$\begin{aligned} \int C(q)\mathrm{d}q &= \int \omega^2 q\mathrm{d}q \\ C(q) &= \frac{\omega^2 q^2}{2} + k \end{aligned} \tag{8}$$

where k is some constant. Substituting $f(q) = 1$ and Eq. (8) into Eq. (4), we obtain

$$H = \frac{p^2}{2} + \frac{\omega^2 q^2}{2} + k \tag{9}$$

which shows that, up to a constant, the one-dimensional harmonic oscillator's Hamiltonian is a specific case in the general form of H given in Eq. (4) for a one-degree-of-freedom mechanical system with $\dot{q} = pf(q)$ and $\dot{p} = g(q, p)$.