Cosmological dynamics in a self-tuning cubic Horndeski theory

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Abstract

We study the cosmological dynamics in a cubic Horndeski theory which admits cosmic inflation as a solution. To this end, we cast the field equations as a dynamical system and show that it admits inflationary era as an attractor. This also supports an existing hypothesis that the most general expansion history should approach the hypersurface of vanishing scalar current.

Keywords: cosmology, dynamical system, scalar-tensor theory

1 Introduction

The success of the Standard Model of Cosmology, often called the ΛCDM model, in explaining the cosmic expansion history and the growth of large scale structure in the observable Universe cements general relativity (GR) as the most accurate theory of gravity for all practically accessible distances. However, its supposition of the existence of dark exotic fluids filling most of the Universe today and the cosmological constant problem, among others, make compelling reasons to consider gravity theories beyond GR [1, 2]. The simplest of such alternative theories is a scalar-tensor theory wherein invisible cosmic ingredients, such as dark energy, are associated with a gravitational degree of freedom contained in a scalar field [3–5]. Scalar-tensor theories are interesting for both theory and practice as they have a simple mathematical structure and yet a very rich phenomenology, even when subjected to constraints from gravitational wave astronomy [6–9].

The cubic sector of Horndeski theory has attracted quite some attention recently [8–12], likely because it evades the strong constraint on gravitational wave propagation. It could also be the last survivor in the scalar-tensor theory family should all future cosmological surveys support GR. In this work, we focus on the cosmological dynamics of a cubic Horndeski theory (Eq. (1)) which includes an inflationary expansion history for cosmological distances while supporting hairy black holes for local distances [13, 14]. This theory belongs to the *self-tuning* class of scalar-tensor theories which is particularly appealing for its power to screen off a large bare cosmological constant to a small effective one. It has been hypothesized that cosmological dynamics in this theory should always end up on the hypersurface of vanishing scalar current [14]. In this work, we investigate this hypothesis from a dynamical systems perspective.

The rest of this work proceeds as follows. First, we introduce the self-tuning cubic Horndeski theory and the relevant field equations for a cosmological background (Section 2). Then, we study the cosmological dynamics of the theory by regarding the field equations as a dynamical system (Section 3). We conclude with a summary of most important results. In the Appendix, we present the fixed points of the dynamical system and their characterization.

For this work, we use geometric units $c = G = 1$ and a mostly plus $(-+++)$ metric signature.

2 Self-tuning cubic Horndeski theory

We consider the self-tuning cubic Horndeski theory given by the action functional [13, 14]

$$S[g_{ab}, \phi] = \int d^4x \sqrt{-g} \left(\kappa R - 2\Lambda - \frac{1}{2} (\partial\phi)^2 + \frac{1}{3H_0} \frac{\Box\phi}{\sqrt{-(\partial\phi)^2}} \right) \tag{1}$$

where g_{ab} is the metric and ϕ is the scalar field. Also, g is the determinant of g_{ab}, R is the Ricci scalar, $(\partial\phi)^2 = (\partial_a\phi)(\partial^a\phi)$, $\Box\phi = \nabla^a\nabla_a\phi$, ∇_a is the covariant derivative, and κ, Λ, and H_0 are constants in which we note that the latter two pertain to the bare cosmological constant and the Hubble parameter's present-day value, respectively. This theory belongs to the subset of Horndeski theory with luminally-propagating gravitational waves and so passes the very tight constraint on the speed of gravitational wave propagation [8, 9]. Cosmological observations and solar system tests should be used to further constrain the theory (1).

In what follows, we consider a spatially-flat cosmological background, a comoving scalar field, and a perfect fluid matter, with energy density ρ and pressure P. In symbols, we have

$$
\begin{aligned}
ds^2 &= -dt^2 + a\left(t\right)^2 d\mathbf{x}^2 && (2)\\
\phi &= \phi\left(t\right) && (3)\\
P\left(t\right) &= w\rho\left(t\right) && (4)
\end{aligned}
$$

where $a\left(t\right)$ is the scale factor and w is the equation of state of the perfect fluid, e.g., $w = 0$ for dust (baryonic fluid) and $w = 1/3$ for a photon gas. From the action (1), the field equations for the metric can be shown to be just the Friedmann equations

$$
\begin{aligned}
6\kappa H^2 &= \rho + 2\Lambda + \rho_\phi && (5)\\
4\kappa\dot{H} + 6\kappa H^2 &= -P + 2\Lambda - P_\phi, && (6)
\end{aligned}
$$

where a dot indicates d/dt, $H = \dot{a}/a$ is the Hubble parameter, ρ_ϕ and P_ϕ stand for the scalar field's energy density and pressure, respectively:

$$
\begin{aligned}
\rho_\phi &= \frac{\dot{\phi}^2}{2}\left(1 - 2\frac{H}{H_0}\right) && (7)\\
P_\phi &= \frac{\dot{\phi}^2}{2}\left(1 + \frac{2}{3H_0}\frac{\ddot{\phi}}{\dot{\phi}}\right). && (8)
\end{aligned}
$$

On the other hand, the field equation for the scalar field can be shown to be

$$\dot{J} + 3HJ = 0, \tag{9}$$

where J is the Noether current arising due to the shift symmetry, $\phi \to \phi + \text{constant}$, of the action (1):

$$J = \dot{\phi}\left(1 - \frac{H}{H_0}\right). \tag{10}$$

A cosmological solution $\{a, \rho, \phi\}$ is one which simultaneously satisfies Eqs. (5), (6), and (9).

3 Cosmological solutions: A dynamical systems perspective

In this section, we use a dynamical system analysis to gain further information about the cosmological dynamics of theory (1). This approach is very useful since we are interested in its late-time asymptotics. In the cosmological context, a desirable solution should approach cosmic acceleration at late times [10, 11]. Moreover, explicit general solutions are generally unobtainable in scalar-tensor theories due to the complexity of its field equations. The dynamical systems approach is a reliable tool for obtaining useful insight, in spite of the lack of an explicit general solution.

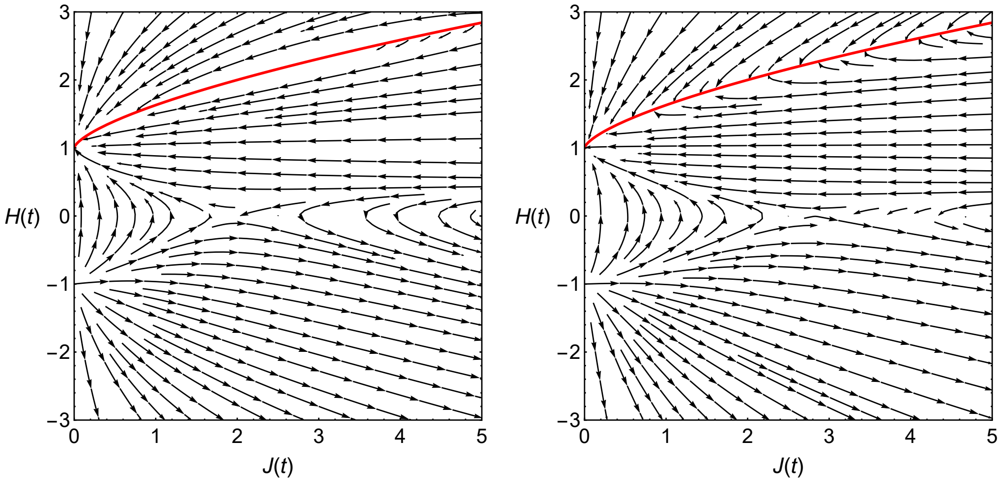
To set up the stage, we take advantage of the fact that ρ can easily be eliminated in Eqs. (5) and (6). By trading also $\dot{\phi}$ for J as a dependent variable, we get to the necessary condition

$$
\begin{aligned}
\dot{H} = \frac{3\left(H_0 - H\right)}{24\kappa(H_0 - H)^3 + 2H_0J^2}\Big[&4H^2(w+1)\left(\Lambda - 3H^2\kappa\right)\\
&- H_0(H_0 - 2H)\left(4(w+1)\left(3H^2\kappa - \Lambda\right) - J^2(w-1)\right)\Big]
\end{aligned} \tag{11}
$$

for a solution of Eqs. (5) and (6). The cosmological dynamics of Eqs. (5), (6), and (9) should then be captured by Eq. (9) and the necessary condition given by Eq. (11). Thus, we could just focus on the dynamical variables (J, H) and regard Eqs. (9) and (11) as the dynamical system for the theory (1). A phase portrait of the vector field $\left(\dot{J}, \dot{H}\right)$ is shown in Figure 1 for baryonic- and radiation-filled universes. For convenience, we fix the theory parameters as $\kappa = \Lambda/3H_0^2$ so that the theory admits de Sitter inflation ($a = a_0 \exp\left(H_0 t\right)$ or $H = H_0$) as a cosmological solution. We also simply focus on $J \geq 0$ since the dynamical system is invariant under the reflection symmetry, $J \to -J$.

Clearly, the dynamical system could be described by its fixed points, defined by $\dot{J} = 0$ and $\dot{H} = 0$. These fixed points turn out to be neatly placed on the axes and are given by

$$(J, H) = \left\{\left(0, \pm H_0\right), \left(\pm 2\sqrt{\Lambda}\sqrt{\frac{1+w}{1-w}}, 0\right)\right\}. \tag{12}$$



(a) baryonic-filled ($w = 0$)   (b) radiation-filled ($w = 1/3$)

Figure 1: Phase portrait of the dynamical system given by Eqs. (11) and (9) for baryonic ($w = 0$)- and radiation ($w = 1/3$)-filled universes with theory parameters $\kappa = \Lambda/3H_0^2$, $\Lambda = 1$, and $H_0 = 1$. The vector field diverges on the solid red line.

Note that $\sqrt{(1+w)/(1-w)}$ is real for $-1 < w < 1$. For the most interesting matter fluids (baryon and radiation), the dynamical system would be described by two fixed points along the J-axis. Furthermore, it could be shown that that the fixed point $(0, H_0)$, describing de Sitter inflation, is stable fixed point. This means that the late-time asymptotic state of a universe described by theory (1) would end up with cosmic inflation. This feature is nicely demonstrated in both cases in Figure 1. On the other hand, the fixed points $(0, -H_0)$ and $\left(2\sqrt{\Lambda}\sqrt{\dfrac{1+w}{1-w}}, 0\right)$ are unstable and saddle points, respectively. The characterization of the fixed points (Eq. (12)) in terms of the eigenvalues of the Jacobian matrix of the dynamical system is shown in the Appendix.

An interesting solution hinted by this dynamical system analysis is a (unstable) contracting phase preceding the (stable) inflationary era. This contracting-to-expanding picture should be the subject of further investigation.

Another notable feature of the phase portraits in Figure 1 is the existence of a curve $H_C(J)$ (solid red line) on which the vector field $\left(\dot{J}, \dot{H}\right)$ blows up. This demands a dynamical system analysis restricted on the curve $H_C(J)$ for completeness. To describe the dynamics on this curve, we simply implement a more appropriate choice of dependent variables. Specifically, by using $\dot{\phi}$ as the dynamical variable instead of J, we could describe this curve by $H_C\left(\dot{\phi}\right) = H_0 + \dot{\phi}^2/(12\kappa H_0)$. The advantage of the coordinate system $\left(\dot{\phi}, H\right)$ is that the derivative $dH/d\dot{\phi}$ remains finite for all points ($|\dot{\phi}| < \infty, |H| < \infty$), including those on $H_C\left(\dot{\phi}\right)$. In particular, on $H_C\left(\dot{\phi}\right)$, we obtain

$$\left.\frac{dH}{d\dot{\phi}}\right|_C = -\frac{H_0}{4\Lambda}\dot{\phi}. \tag{13}$$

This describes the dynamical system on $H_C\left(\dot{\phi}\right)$ with $\dot{\phi}$ acting as the independent variable. This vector field always points to the origin $\dot{\phi} = 0$ on which $H = H_0$. On the curve $H_C\left(\dot{\phi}\right) = 0$, the solution should then always approach the point $\left(\dot{\phi} = 0, H = H_0\right)$. This is expected since cosmic inflation with the Hubble parameter H_0 is a stable fixed point of the dynamical system.

4 Outlook

We studied the cosmological dynamics of the self-tuning cubic Horndeski theory (Eq. (1)) using the dynamical system analysis and recovered cosmic inflation with a vanishing scalar current as an attractor. This supports the hypothesis that cosmological solutions outside of the $J = 0$ hypersurface should still approach the $J = 0$ hypersurface provided that the expansion history is unbounded at late times [14].

However, further work must be devoted to study this hypothesis in detail in other scalar-tensor theories. Another potential future work would be to constrain the theory (1) using cosmological surveys and solar system tests.

Appendix: Characterization of the fixed points

Fixed points are classified as stable, unstable, and saddle depending on the signs of the eigenvalues of the Jacobian matrix of the dynamical system. For Eqs. (9) and (11), the fixed points and their corresponding eigenvalues are displayed in Table (1). The characterization is valid for $w \geq 0$.

Table 1: Fixed points and eigenvalues of the Jacobian matrix of the dynamical system given by Eqs. (9) and (11).

fixed point	λ_1	λ_2	type
$(0, H_0)$	$-3H_0$	$-\frac{3H_0(1-w)}{2}$	stable
$(0, -H_0)$	$3H_0$	$3H_0(1+w)$	unstable
$\left(\pm 2\sqrt{\Lambda}\sqrt{\frac{1+w}{1-w}}, 0\right)$	$\frac{3\sqrt{2}H_0}{2}\sqrt{1-w^2}$	$-\frac{3\sqrt{2}H_0}{2}\sqrt{1-w^2}$	saddle

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References

[1] A. G. Riess, A. V. Filippenko, P. Challis, A. Clocchiatti, A. Diercks, P. M. Garnavich, R. L. Gilliland, C. J. Hogan, and S. Jha, Observational Evidence from Supernovae for an Accelerating Universe and a Cosmological Constant, *Astron. J.* **116**, 1009 (1998).

[2] D. N. Spergel, L. Verde, H. V. Peiris, E. Komatsu, M. R. Nolta, C. L. Bennett, M. Halpern, G. Hinshaw, N. Jarosik, A. Kogut, et al., First-year Wilkinson Microwave Anisotropy Probe (WMAP) observations: determination of cosmological parameters, *Astrophys. J., Suppl. Ser.* **148**, 175 (2003).

[3] T. Clifton, P. G. Ferreira, A. Padilla, and C. Skordis, Modified gravity and cosmology, *Phys. Rep.* **513**, 1 (2012).

[4] K. Koyama, Cosmological tests of modified gravity, *Rep. Prog. Phys.* **79**, 046902 (2016).

[5] A. Joyce, B. Jain, J. Khoury, and M. Trodden, Beyond the cosmological standard model, *Phys. Rep.* **568**, 1 (2015), beyond the cosmological standard model.

[6] M. Crisostomi and K. Koyama, Self-accelerating universe in scalar-tensor theories after GW170817, *Phys. Rev. D* **97**, 084004 (2018).

[7] C. Kreisch and E. Komatsu, Cosmological constraints on Horndeski gravity in light of GW170817, *J. Cosmol. Astropart. Phys.* **2018**, 030 (2018).

[8] R. Kase and S. Tsujikawa, Dark energy in Horndeski theories after GW170817: A review, *Int. J. Mod. Phys. D* **28**, 1942005 (2019).

[9] T. Kobayashi, Horndeski theory and beyond: a review, *Rep. Prog. Phys.* **82**, 086901 (2019).

[10] I. S. Albuquerque, N. Frusciante, N. J. Nunes, and S. Tsujikawa, New scaling solutions in cubic Horndeski theories, *Phys. Rev. D* **98**, 064038 (2018).

[11] N. Frusciante, R. Kase, N. J. Nunes, and S. Tsujikawa, Most general cubic-order Horndeski Lagrangian allowing for scaling solutions and the application to dark energy, *Phys. Rev. D* **98**, 123517 (2018).

[12] E. V. Linder, No Run Gravity, *J. Cosmol. Astropart. Phys.* **2019**, 034 (2019).

[13] W. T. Emond, A. Lehébel, and P. M. Saffin, Black holes in self-tuning cubic Horndeski cosmology, arXiv:1912.09199 (2019).

[14] R. C. Bernardo and I. Vega, Tailoring cosmologies in cubic shift-symmetric Horndeski gravity, *J. Cosmol. Astropart. Phys.* **2019**, 058 (2019).