Problem 1 (1.2)

Consider the system in Figure 1 in the case in which all the masses are equal($m_1 = m_2 = m_3 = m$) and the system is released from rest with $x_2 = 0$ and $x_3 = l$.

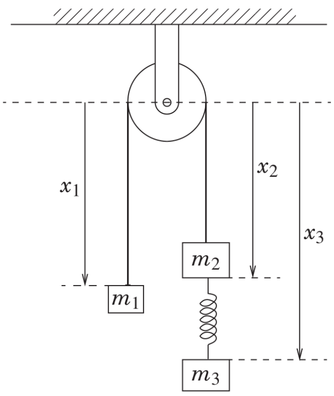

Figure 1: A diagram of the problem (Source: Analytic Mechanics by N. Lemos)

(a) Determine the equations of motion.

Solution:
In this scenario, we will assume that the pulley and the string that connects masses m_1 and m_2 have negligible mass and that the string is inextensible. Also, we assume that the spring constant is k. Looking at Example 1.12 in [NL], we can infer that the Atwood's machine in the system imposes a constraint given by

$$x_1 + x_2 = l_0 \tag{1}$$

where the constant l_0 is dependent on the radius of the pulley and the length of the string. Because of this constraint, this system only has 2 degrees of freedom so we have 2 independent variables which we choose to be x_2 and x_3 (considering that we were given initial conditions for x_2 and x_3, this choice of variables should be better than using x_1 with x_3). Then, we calculate for the kinetic energy of the system given by

$$\begin{aligned} T &= \frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}m_2\dot{x}_2^2 + \frac{1}{2}m_3\dot{x}_3^2 \\ &= \frac{1}{2}m(-\dot{x}_2)^2 + \frac{1}{2}m\dot{x_2}^2 + \frac{1}{2}m\dot{x_3}^2 \\ T &= m\dot{x}_2^2 + \frac{1}{2}m\dot{x}_3^2 \end{aligned} \tag{2}$$

since the masses are equal and $x_1 = l_0 - x_2$ from Eq. (1) which leads to $\dot{x}_1 = -\dot{x}_2$.

As for the potential energy, we set the zero level of the gravitational potential at the center of the pulley. Also, recall that a spring's potential energy is $V_{\text{spring}} = (1/2)kx$ where x is the displacement of the spring. From the fact that $x_2(0) = 0$ and $x_3(0) = l$, we know that $x_3 - x_2 = l$ when the spring is at its initial position. This relation changes once the spring is displaced by x in which we obtain $x_3 - x_2 = l + x$. Thus, the displacement of the spring in this system is given by $x = x_3 - x_2 - l$. Therefore, we have

$$\begin{aligned} V &= V_{\text{grav}} + V_{\text{spring}} \\ &= -m_1 g x_1 - m_2 g x_2 - m_3 g x_3 + \frac{1}{2}k\left(x_3 - x_2 - l\right)^2 \\ &= -mg(l_0 - x_2) - mgx_2 - mgx_3 + \frac{1}{2}k\left(x_3 - x_2 - l\right)^2 \\ V &= -mgl_0 - mgx_3 + \frac{1}{2}k\left(x_3 - x_2 - l\right)^2 \end{aligned} \tag{3}$$

By substituting in Eqs. (2) and (3), the Lagrangian is given by

$$L = T - V = m\dot{x}_2^2 + \frac{1}{2}m\dot{x}_3^2 + mgx_3 - \frac{1}{2}k\left(x_3 - x_2 - l\right)^2 + mgl_0 \tag{4}$$

Now, Lagrange's equations are calculated using the formula

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \frac{\partial L}{\partial q_k} = 0, \quad k = 1, \ldots, n \tag{5}$$

where the generalised coordinates q_k for this system are $q_1 = x_2$ and $q_2 = x_3$. We solve the following partial derivatives:

$$\frac{\partial L}{\partial \dot{x}_2} = 2m\dot{x}_2 \tag{6}$$

$$\frac{\partial L}{\partial x_2} = k\left(x_3 - x_2 - l\right) \tag{7}$$

$$\frac{\partial L}{\partial \dot{x}_3} = m\dot{x}_3 \tag{8}$$

$$\frac{\partial L}{\partial x_3} = mg - k\left(x_3 - x_2 - l\right) \tag{9}$$

and substitute them into Eq. (5) to obtain

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{dt}}\left(\frac{\partial L}{\partial \dot{x}_2}\right) - \frac{\partial L}{\partial x_2} &= \frac{\mathrm{d}}{\mathrm{dt}}\left(2m\dot{x}_2\right) - k\left(x_3 - x_2 - l\right) \\ 0 &= 2m\ddot{x}_2 - k\left(x_3 - x_2 - l\right) \end{aligned} \tag{10}$$

and

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{dt}}\left(\frac{\partial L}{\partial \dot{x}_3}\right) - \frac{\partial L}{\partial x_3} &= \frac{\mathrm{d}}{\mathrm{dt}}\left(m\dot{x}_3\right) - mg + k\left(x_3 - x_2 - l\right) \\ 0 &= m\ddot{x}_3 - mg + k\left(x_3 - x_2 - l\right) \end{aligned} \tag{11}$$

Therefore, the equations of motion for this system are

$$\boxed{\ddot{x}_2 = k\left(x_3 - x_2 - l\right)} \tag{12}$$

$$\boxed{\ddot{x}_3 = g - \frac{k}{m}\left(x_3 - x_2 - l\right)} \tag{13}$$

(b) Solve the equations of motion to show that

$$x_2(t) = \frac{2mg}{9k}(\cos\omega t - 1) + \frac{1}{6}gt^2, \quad \omega = \sqrt{\frac{3k}{2m}} \tag{14}$$

Prove that $\dot{x}_2 > 0$ for all $t > 0$ and conclude that the string always remains taut.

Solution:
Recall that $x_2(0) = 0$, $x_3(0) = l$ and $\dot{x}_2(0) = 0 = \dot{x}_3(0)$ (since the system is released from rest). Using Mathematica to solve Eqs. (12) and (13) with these initial conditions as we see in Figure 2, we show that Eq. (14) results from solving the said equations of motion. Note that $\omega > 0$. ■

Taking the time derivative of $x_2(t)$, we get

$$\dot{x}_2(t) = -\frac{2mg}{9k}\omega \sin\omega t + \frac{1}{3}gt = -\frac{g}{3\omega^{\cancel{2}}}\cancel{\omega}\sin\omega t + \frac{1}{3}gt = \frac{g}{3\omega}(\omega t - \sin\omega t) \tag{15}$$

The only way that $\dot{x}_2(t)$ becomes negative is when $\sin\omega t$ becomes bigger than ωt. However, we see in Figure 3 where we plot $\sin(x)$ and x that this does not happen. Thus, we can say that $\dot{x}_2 > 0$ for all $t > 0$. Because of this, oscillations due to the spring does not affect the direction of $\dot{x}_2$. Thus, the string (assumed to be inextensible) remains taut for all $t > 0$.

```
solns = DSolve[{-k (-l - x2[t] + x3[t]) + 2 m x2''[t] == 0, -g m + k (-l - x2[t] + x3[t]) + m x3''[t] == 0, x2[0] == 0,
   x3[0] == l, x2'[0] == 0, x3'[0] == 0}, {x2[t], x3[t]}, t]
```

$$\left\{\left\{x2[t] \to \frac{e^{-\frac{i\sqrt{\frac{3}{2}}\sqrt{k}\,t}{\sqrt{m}}}\, g\left(2m - 4e^{\frac{i\sqrt{\frac{3}{2}}\sqrt{k}\,t}{\sqrt{m}}}m + 2e^{\frac{i\sqrt{6}\sqrt{k}\,t}{\sqrt{m}}}m + 3e^{\frac{i\sqrt{\frac{3}{2}}\sqrt{k}\,t}{\sqrt{m}}}k\,t^2\right)}{18\,k},\right.\right.$$

$$\left.\left. x3[t] \to -\frac{1}{18\,k}e^{-\frac{i\sqrt{\frac{3}{2}}\sqrt{k}\,t}{\sqrt{m}}}\left(-18\,e^{\frac{i\sqrt{\frac{3}{2}}\sqrt{k}\,t}{\sqrt{m}}}k\,l + 4\,g\,m - 8\,e^{\frac{i\sqrt{\frac{3}{2}}\sqrt{k}\,t}{\sqrt{m}}}g\,m + 4\,e^{\frac{i\sqrt{6}\sqrt{k}\,t}{\sqrt{m}}}g\,m - 3\,e^{\frac{i\sqrt{\frac{3}{2}}\sqrt{k}\,t}{\sqrt{m}}}g\,k\,t^2\right)\right\}\right\}$$

$$x2[t] == (x2[t] \;/.\; solns[[1]] \;//\; ExpToTrig \;//\; Simplify \;//\; Expand) \;/.\; \frac{\sqrt{\frac{3}{2}}\sqrt{k}\,t}{\sqrt{m}} \to \omega\,t$$

$$x2[t] == -\frac{2\,g\,m}{9\,k} + \frac{g\,t^2}{6} + \frac{2\,g\,m\,Cos[t\,\omega]}{9\,k}$$

Figure 2

```
Plot[{x, Sin[x]}, {x, 0, 5}, PlotTheme → "Scientific",
 PlotLegends → Placed[LineLegend[{Style["x", FontSize → 18], Style["sin(x)", FontSize → 18]},
   LegendFunction → (Framed[#, RoundingRadius → 5, FrameStyle → {Gray, Thin}] &), LegendMargins → 5], {After, Top}],
 FrameLabel → {Style["x", FontSize → 20, FontWeight → Bold], None}, FrameTicksStyle → 15]
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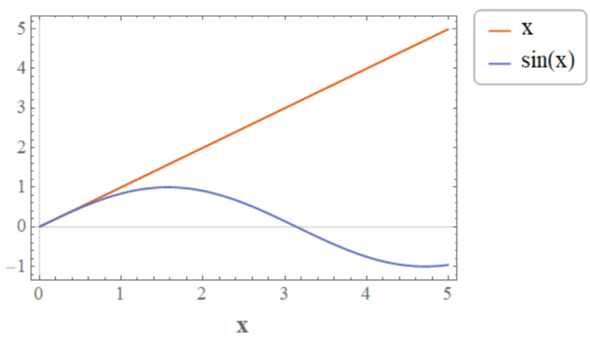

Figure 3: Plot of $\sin x$ and x

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References

Lemos, N., *Analytical Mechanics*, Chapter 1