Problem 1 (2.1)

A point charge q is brought to a position a distance d away from an infinite plane conductor held at zero potential. Using the method of images, find:

(a) the surface-charge density induced on the plane, and plot it;

Solution:
For a system with i point charges, the potential is given by

$$\Phi(\mathbf{x}) = \frac{1}{4\pi\epsilon_0}\sum_{i=0}^{n} q_i \frac{1}{|\mathbf{x}-\mathbf{x}_i|} \tag{1}$$

With the method of images, we can change the problem into a different but equivalent system where we add a charge q' a distance $-d$ away and remove the plane conductor. Applying Eq. (1), the potential for this scenario is given by

$$\begin{aligned}\Phi(x,y,z) &= \frac{1}{4\pi\epsilon_0}\left[\frac{q_1}{|\mathbf{x}-\mathbf{x}_1|} + \frac{q_2}{|\mathbf{x}-\mathbf{x}_2|}\right] \\ &= \frac{1}{4\pi\epsilon_0}\left[\frac{q}{\sqrt{(x-0)^2+(y-0)^2+(z-d)^2}} + \frac{q'}{\sqrt{(x-0)^2+(y-0)^2+(z+d)^2}}\right]\end{aligned} \tag{2}$$

Since the conductor is held at zero potential, we can figure out what q' should be to mimic the original problem by evaluating Eq. (2) at the location of the plane conductor:

$$\begin{aligned}0 &= \frac{1}{4\pi\epsilon_0}\left[\frac{q}{\sqrt{x^2+y^2+(0-d)^2}} + \frac{q'}{\sqrt{x^2+y^2+(0+d)^2}}\right] \\ 0 &= \frac{q}{\sqrt{x^2+y^2+d^2}} + \frac{q'}{\sqrt{x^2+y^2+d^2}} \\ 0 &= q+q'\end{aligned} \tag{3}$$

which shows that $q' = q$. Thus, the potential is

$$\Phi(x,y,z) = \frac{1}{4\pi\epsilon_0}\left[\frac{q}{\sqrt{x^2+y^2+(0-d)^2}} - \frac{q}{\sqrt{x^2+y^2+(0+d)^2}}\right] \tag{4}$$

In cylindrical coordinates, note that $x = \rho\cos\phi$ and $y = \rho\sin\phi$. We also have the relation

$$x^2+y^2 = \rho^2\cos^2\phi + \rho^2\sin^2\phi = \rho^2\cancelto{1}{\left(\cos^2\phi+\sin^2\phi\right)} = \rho^2 \tag{5}$$

So, transforming the potential in Eq. (4) from Cartesian to cylindrical coordinates gives

$$\Phi(\rho,\phi,z) = \frac{1}{4\pi\epsilon_0}\left[\frac{q}{\sqrt{\rho^2+(z-d)^2}} - \frac{q}{\sqrt{\rho^2+(z+d)^2}}\right] \tag{6}$$

Now, the induced surface charge density can be obtained from

$$\sigma(\mathbf{x}) = -\epsilon_0\frac{\partial\Phi(\mathbf{x})}{\partial n} \tag{7}$$

Applying this to the problem, we have

$$\begin{aligned}
\sigma(\rho,\phi,z) &= -\epsilon_0 \left.\frac{\partial\Phi(\rho,\phi,z)}{\partial z}\right|_{z=0} \\
&= -\frac{\cancel{\epsilon_0}}{4\pi\cancel{\epsilon_0}}\left[\left(-\frac{1}{\cancel{2}}\right)\frac{q\cdot\cancel{2}(z-d)}{(\rho^2+(z-d)^2)^{3/2}} - \left(-\frac{1}{\cancel{2}}\right)\frac{q\cdot\cancel{2}(z+d)}{(\rho^2+(z+d)^2)^{3/2}}\right]_{z=0} \\
&= -\frac{1}{\cancel{4}^{\,2}\pi}\frac{\cancel{2}qd}{(\rho^2+d^2)^{3/2}} \\
\sigma(\rho,\phi,z) &= -\frac{1}{2\pi}\frac{qd}{(\rho^2+d^2)^{3/2}}
\end{aligned} \tag{8}$$

as the induced surface charge density on the plane. A plot of this density is shown in Fig. 1.

```
q = 1; d = 100;
Plot3D[-1/(2 π) * q d/(x^2 + y^2 + d^2)^(3/2), {x, -1, 1}, {y, -1, 1}, PlotTheme → "Detailed", AxesLabel → Automatic, PlotRange -> All]
```

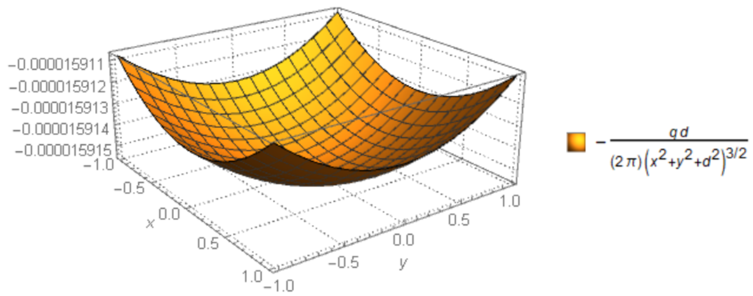

Figure 1: Plotting the surface charge density using Mathematica

(b) the force between the plane and the charge using Coulomb's law for the force between the charge and its image;

Solution:
From Coulomb's law, the force between two point charges is

$$F = \frac{q_1 q_2}{4\pi\epsilon_0}\frac{\mathbf{x}_1-\mathbf{x}_2}{|\mathbf{x}_1-\mathbf{x}_2|^3} \tag{9}$$

Applying this for the image problem, we have

$$\begin{aligned}
\mathbf{F} &= \frac{qq'}{4\pi\epsilon_0}\frac{(x-x)\hat{\boldsymbol{\imath}}+(y-y)\hat{\boldsymbol{\jmath}}+((z-d)-(z+d))\hat{\boldsymbol{k}}}{((x-x)^2+(y-y)^2+((z-d)-(z+d))^2)^{3/2}} \\
&= -\frac{q^2}{4\pi\epsilon_0}\frac{(-2d)\hat{\boldsymbol{k}}}{((-2d)^2)^{3/2}} \\
&= -\frac{1}{4\pi\epsilon_0}\frac{q^2}{4d^2}\hat{\boldsymbol{k}} \\
\mathbf{F} &= -\frac{1}{16\pi\epsilon_0}\frac{q^2}{d^2}\hat{\boldsymbol{k}}
\end{aligned} \tag{10}$$

as the force between the plane and the charge.

(c) the total force acting on the plane by integrating $\sigma^2/(2\epsilon_0)$ over the whole plane;

Solution:
Substituting in the surface charge density in Eq. (8), integrating $\sigma^2/(2\epsilon_0)$ over the whole plane gives

$$\begin{aligned} F_{\text{tot}} &= \int \frac{\sigma(\rho,\phi,z)^2}{2\epsilon_0}\text{d}a \\ &= \int_0^{2\pi}\int_0^{\infty}\left(\frac{q^2d^2}{8\pi^2\epsilon_0}\frac{1}{(\rho^2+d^2)^3}\right)\rho\text{d}\rho\text{d}\phi \\ F_{\text{tot}} &= \frac{q^2d^2}{8\pi^2\epsilon_0}\int_0^{2\pi}\text{d}\phi\int_0^{\infty}\frac{\rho}{(\rho^2+d^2)^3}\text{d}\rho \end{aligned} \tag{11}$$

Note that

$$\int_0^{\infty}\frac{\rho}{(\rho^2+d^2)^3}\text{d}\rho = \frac{1}{2}\int_{d^2}^{\infty}\frac{1}{u^3}\text{d}u = \frac{1}{2}\left(-\frac{1}{2}\lim_{a\to\infty}\frac{1}{u^2}\bigg|_{d^2}^{a}\right) = \frac{1}{4}\left(\frac{1}{d^4}-0\right) \tag{12}$$

Thus, we obtain

$$F_{\text{tot}} = \frac{q^2d^2}{8\pi^2\epsilon_0}(2\pi)\left(\frac{1}{4d^4}\right) = \frac{q^2d^2}{16\pi\epsilon_0 d} \tag{13}$$

as the total force.

(d) the work necessary to remove the charge q from its position to infinity;

Solution:
The work done to move a point charge from point a to b is given by

$$W = -\int_a^b \mathbf{F}\cdot\text{d}\mathbf{l} \tag{14}$$

From Eq. (10), we can infer that the force between the plane and the charge as we move the charge to infinity is

$$\mathbf{F}(z) = -\frac{q^2}{16\pi\epsilon_0}\frac{1}{z^2}\hat{\boldsymbol{k}} \tag{15}$$

Thus, we get

$$W = -\int_d^{\infty} -\frac{q^2}{16\pi\epsilon_0}\frac{1}{z^2}\hat{\boldsymbol{k}}\cdot\hat{\boldsymbol{k}}\text{d}z = \frac{q^2}{16\pi\epsilon_0}\left(\frac{1}{d}-0\right) = \frac{q^2}{16\pi\epsilon_0 d} \tag{16}$$

as the work needed in this scenario.

(e) the potential energy between the charge q and its image [compare the answer to part (d) and discuss].

Solution:
In general, the total potential energy of all charges due to the forces acting between them is given by

$$W = \frac{1}{8\pi\epsilon_0}\sum_i\sum_j\frac{q_iq_j}{|\mathbf{x}_i-\mathbf{x}_j|} \tag{17}$$

Applying this to the charge q and its image, we have

$$\begin{aligned} W &= \frac{1}{8\pi\epsilon_0}\left[\frac{q_1 q_2}{|\mathbf{x}_1-\mathbf{x}_2|}+\frac{q_2 q_1}{|\mathbf{x}_2-\mathbf{x}_1|}\right] \\ &= \frac{1}{8\pi\epsilon_0}\left[\frac{qq'}{\sqrt{(x-x)^2+(y-y)^2+((z-d)-(z+d))^2}}\right. \\ &\qquad\qquad \left.+\frac{q'q}{\sqrt{(x-x)^2+(y-y)^2+((z+d)-(z-d))^2}}\right] \\ &= \frac{1}{8\pi\epsilon_0}\left[\frac{-q^2}{\sqrt{(-2d)^2}}+\frac{-q^2}{\sqrt{(2d)^2}}\right] \\ &= \frac{1}{8\pi\epsilon_0}\frac{-2q^2}{2d} \\ W &= -\frac{q^2}{8\pi\epsilon_0 d} \end{aligned} \tag{18}$$

as the potential energy between them. Comparing this to the answer in part (d), we notice that $|W|$ here is twice than that of (d). To know why, let us turn to another form of $|W|$ by evaluating the energy stored in fields:

$$W = \frac{\epsilon_0}{2}\int |\mathbf{E}|^2 d^3x \tag{19}$$

If we consider the charge q and its image as in (e) here, the $z > 0$ and $z < 0$ regions contribute equally due to symmetry to the total E of the system. However, if we consider the original problem as in (d), only the $z > 0$ region contribute to E. This halves $|W|$ for the original problem as compared to the image problem.

(f) Find the answer to part (d) in electron volts for an electron originally one angstrom from the surface.

Solution:
It is given that $q = -1.602 \times 10^{-19}$ C, $d = 1\text{Å} = 10^{-10}$ m, and $\epsilon_0 = 8.85 \times 10^{-12}$ $\mathrm{C^2\,N^{-1}\,m^{-2}}$. Substituting these values to Eq. (16), we calculated that

$$W = 5.77 \times 10^{-19}\,\mathrm{N\,m} = 3.6\,\mathrm{eV} \tag{20}$$

by using the conversion factor $1\mathrm{eV} = 1.602 \times 10^{-19}\mathrm{N\,m}$.

References

Griffiths, D., *Introduction to Electrodynamics* (3rd ed.), Chapter 2

Problem 2 (2.5)

(a) Show that the work done to remove the charge q from a distance $r > a$ to infinity against the force, Eq. (2.6), of a grounded conducting sphere is

$$W = \frac{q^2 a}{8\pi\epsilon_0 \left(r^2 - a^2\right)} \tag{1}$$

Relate this result to the electrostatic potential, Eq. (2.3), and the energy discussion of Section 1.11.

Solution:
Since the force given in Eq. (2.6) of Jackson is attractive, we can infer that the force between the sphere and the charge as we move the charge is

$$\mathbf{F}(r) = -\frac{1}{4\pi\epsilon_0}\frac{q^2}{a^2}\left(\frac{a}{r}\right)^3\left(1 - \frac{a^2}{r^2}\right)^{-2}\hat{\mathbf{r}} \tag{2}$$

The work done to move a point charge from point a to b is given by

$$W = -\int_a^b \mathbf{F}\cdot \mathrm{d}\mathbf{l} \tag{3}$$

Substituting in Eq. (2) into Eq. (3), we get

$$\begin{aligned}
W &= -\int_r^\infty \mathbf{F}(r')\cdot \mathrm{d}\mathbf{r}' \\
&= \int_r^\infty \frac{1}{4\pi\epsilon_0}\frac{q^2}{a^2}\left(\frac{a}{r'}\right)^3\left(1 - \left(\frac{a}{r'}\right)^2\right)^{-2}\hat{\mathbf{r}}'\cdot\hat{\mathbf{r}}'\mathrm{d}r' \\
W &= \frac{1}{4\pi\epsilon_0}q^2 a\int_r^\infty \left(\frac{1}{r'}\right)^3\left(1 - \left(\frac{a}{r'}\right)^2\right)^{-2}\mathrm{d}r'
\end{aligned} \tag{4}$$

We let $u = 1 - (a/r')^2$ which results to $\mathrm{d}u = 2a^2/r'^3\mathrm{d}r'$. Changing the bounds of integration, the lower bound becomes $u = 1 - (a/r)^2$ and the upper bound becomes $u = 1$. Thus, we have

$$\begin{aligned}
W &= \frac{1}{4\pi\epsilon_0}q^2 a\int_{1-\left(\frac{a}{r}\right)^2}^{1}\frac{1}{u^2}\frac{\mathrm{d}u}{2a^3} \\
&= \frac{1}{8\pi\epsilon_0}\frac{q^2}{a^2}\left[-\frac{1}{u}\right]_{1-\left(\frac{a}{r}\right)^2}^{1} \\
&= -\frac{1}{8\pi\epsilon_0}\left(1 - \frac{1}{1-\left(\frac{a}{r}\right)^2}\right) \\
&= -\frac{1}{8\pi\epsilon_0}\left(-\frac{\left(\frac{a}{r}\right)^2}{1-\left(\frac{a}{r}\right)^2}\right) \\
W &= \frac{1}{8\pi\epsilon_0}\frac{q^2 a}{r^2 - a^2}
\end{aligned} \tag{5}$$

Note that we should also get this result for the work done on the charge through the electrostatic potential in this scenario

$$\Phi(x = a) = \frac{q/4\pi\epsilon_0}{a\left|\mathbf{n} - \frac{y}{a}\mathbf{n}'\right|} - \frac{q/4\pi\epsilon_0}{y'\left|\mathbf{n}' - \frac{a}{y'}\mathbf{n}\right|} \tag{6}$$

with the relation

$$W_i = q_i \Phi(\mathbf{x}_i). \tag{7}$$

(b) Repeat the calculation of the work done to remove the charge q against the force, Eq. (2.9), of an isolated charged conducting sphere. Show that the work done is

$$W = \frac{1}{4\pi\epsilon_0}\left[\frac{q^2 a}{2\left(r^2 - a^2\right)} - \frac{q^2 a}{2r^2} - \frac{qQ}{r}\right] \tag{8}$$

Relate the work to the electrostatic potential, Eq. (2.8), and the energy discussion of Section 1.11.

Solution:
From Eq. (2.9), we can infer that the force between the conducting sphere and the charge as we move the charge is

$$\mathbf{F}(r) = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\left[Q - \frac{qa^3(2r^2 - a^2)}{r(r^2 - a^2)}\right]\hat{\mathbf{r}} \tag{9}$$

Then, in the same manner as the previous part, the work done to remove the charge q against this force is

$$\begin{aligned} W &= -\int_r^\infty \mathbf{F}(r')\cdot \mathrm{d}\mathbf{r}' \\ &= -\int_r^\infty \frac{1}{4\pi\epsilon_0}\frac{q}{r'^2}\left[Q - \frac{qa^3(2r'^2 - a^2)}{r'(r'^2 - a^2)}\right]\hat{\mathbf{r}}'\cdot\hat{\mathbf{r}}'\mathrm{d}r' \\ W &= -\frac{1}{4\pi\epsilon_0}\left[\int_r^\infty \frac{Qq}{r'^2}\mathrm{d}r' - \int_r^\infty \frac{q^2a^3(2r'^2 - a^2)}{r'^3(r'^2 - a^2)}\mathrm{d}r'\right] \end{aligned} \tag{10}$$

Note that the integral in the second term can be expressed as

$$q^2a^3\int_r^\infty \frac{(2r'^2 - a^2)}{r'^3(r'^2 - a^2)}\frac{r'}{r'}\mathrm{d}r' = \frac{q^2a^3}{2}\int_{r^2-a^2}^\infty \frac{2u + \cancelto{a^2}{2a^2 - a^2}}{\left(u + a^2\right)^2 u^2}\mathrm{d}u = \frac{q^2a^3}{2}\int_{r^2(r^2-a^2)}^\infty \frac{1}{v^2}\mathrm{d}v \tag{11}$$

by performing two successive change of variables in which we let $u = r^2 - a^2$ and $v = u(u + a^2)$. Note that we also changed the bounds of integration. Thus, the work done is

$$\begin{aligned} W &= -\frac{1}{4\pi\epsilon_0}\left[Qq\left(-\lim_{a\to\infty}\frac{1}{r'}\bigg|_r^a\right) - \frac{q^2a^3}{2}\left(-\lim_{a\to\infty}\frac{1}{v}\bigg|_{r^2(r^2-a^2)}^a\right)\right] \\ &= -\frac{1}{4\pi\epsilon_0}\left[\frac{Qq}{r} - \frac{q^2a^3}{2r^2(r^2 - a^2)}\right] \end{aligned} \tag{12}$$

With the following relation:

$$\frac{q^2 a}{2\left(r^2 - a^2\right)} - \frac{q^2 a}{2r^2} = \frac{q^2r^2a^2 - q^2a(r^2 - a^2)}{2r^2(r^2 - a^2)} = \frac{q^2a^3}{2r^2(r^2 - a^2)} \tag{13}$$

we can write the work done to be

$$W = \frac{1}{4\pi\epsilon_0}\left[-\frac{Qq}{r} + \frac{q^2 a}{2\left(r^2 - a^2\right)} - \frac{q^2 a}{2r^2}\right] \tag{14}$$

In the same manner as the previous part, we should also get this result for the work done on the charge through the electrostatic potential in this scenario

$$\Phi(\mathbf{x}) = \frac{1}{4\pi\epsilon_0}\left[\frac{q}{|\mathbf{x}-\mathbf{y}|} - \frac{aq}{y\left|\mathbf{x}-\frac{a^2}{y^2}\mathbf{y}\right|} + \frac{Q+\frac{a}{y}q}{|\mathbf{x}|}\right] \tag{15}$$

with the relation in Eq. (7).

References

Griffiths, D., *Introduction to Electrodynamics* (3rd ed.), Chapter 2

Problem 3 (2.7)

Consider a potential problem in the half-space defined by $z \geq 0$, with Dirichlet boundary conditions on the plane $z = 0$ (and at infinity).

(a) Write down the appropriate Green function $G(\mathbf{x}, \mathbf{x}')$.

Solution:
To figure out the Green's function appropriate for a plane at $z = 0$ with Dirichlet boundary conditions, we first need to find the potential $\Phi(\mathbf{x})$ such that it is zero at $z = 0$. Let us put a point charge q at $\mathbf{x}' = (x', y', z')$ on the region $z \geq 0$. Using the method of images, we can solve a different but equivalent problem involving an image charge placed at $\mathbf{x}' = (-x', -y', -z')$. Note that for a system with i point charges, the potential is given by

$$\Phi(\mathbf{x}) = \frac{1}{4\pi\epsilon_0} \sum_{i=0}^{n} q_i \frac{1}{|\mathbf{x} - \mathbf{x}_i|} \tag{1}$$

From this, the potential for the image problem is

$$\begin{aligned}
\Phi(\mathbf{x}) &= \frac{1}{4\pi\epsilon_0} \left[\frac{q}{|\mathbf{x} - \mathbf{x}'|} + \frac{q'}{|\mathbf{x} - \mathbf{x}''|} \right] \\
\Phi(x, y, z) &= \frac{1}{4\pi\epsilon_0} \left[\frac{q}{\sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2}} + \frac{q'}{\sqrt{(x-x')^2 + (y-y')^2 + (z+z')^2}} \right]
\end{aligned} \tag{2}$$

Since the potential must be zero at $z = 0$, we have

$$\begin{aligned}
\Phi(x, y, z = 0) &= 0 \\
\frac{1}{4\pi\epsilon_0} \left[\frac{q}{\sqrt{(x-x')^2 + (y-y')^2 + z'^2}} + \frac{q'}{\sqrt{(x-x')^2 + (y-y')^2 + z'^2}} \right] &= 0 \\
q' + q &= 0
\end{aligned} \tag{3}$$

which shows that the image charge mast be $q' = -q$. Then, the Green's function is given by Eq. (2) with $q \to 4\pi\epsilon_0$:

$$G(\mathbf{x}, \mathbf{x}') = \frac{1}{\sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2}} - \frac{1}{\sqrt{(x-x')^2 + (y-y')^2 + (z+z')^2}} \tag{4}$$

For convenience, it would be better to transform $G(\mathbf{x}, \mathbf{x}')$ from Cartesian coordinates to cylindrical coordinates. Note that $x = \rho\cos\phi$ and $y = \rho\sin\phi$. With this, we have

$$(x - x')^2 = x^2 + x'^2 - 2xx' = \rho^2\cos^2\phi + \rho'^2\cos^2\phi' - 2\rho\rho'\cos\phi\cos\phi' \tag{5}$$

We can get a similar relation for $(y - y')^2$. Thus, we get

$$\begin{aligned}
(x-x')^2 + (y-y')^2 &= \rho^2\left(\cos^2\phi + \sin^2\phi\right) + \rho'^2\left(\cos^2\phi' + \sin^2\phi'\right) - 2\rho\rho'\left(\cos\phi\cos\phi' + \sin\phi\sin\phi'\right) \\
&= \rho^2 + \rho'^2 - 2\rho\rho'\cos\left(\phi - \phi'\right)
\end{aligned} \tag{6}$$

Using this, $G(\mathbf{x}, \mathbf{x}')$ is now expressed as

$$G(\mathbf{x}, \mathbf{x}') = \frac{1}{\sqrt{\rho^2 + \rho'^2 - 2\rho\rho' \cos(\phi - \phi') + (z - z')^2}} - \frac{1}{\sqrt{\rho^2 + \rho'^2 - 2\rho\rho' \cos(\phi - \phi') + (z + z')^2}} \tag{7}$$

in cylindrical coordinates.

(b) If the potential on the plane $z = 0$ is specified to be $\Phi = V$ inside a circle of radius a centered on the origin, and $\Phi = 0$ outside that circle, find an integral expression for the potential at the point P specified in terms of cylindrical coordinates (ρ, ϕ, z).

Solution:
In terms of the Green's function, the potential is generally expressed as

$$\Phi(\mathbf{x}) = \frac{1}{4\pi\epsilon_0} \int_V \rho(\mathbf{x}') G(\mathbf{x}, \mathbf{x}') \, \mathrm{d}^3x' + \frac{1}{4\pi} \oint_S \left[G(\mathbf{x}, \mathbf{x}') \frac{\partial \Phi(\mathbf{x}')}{\partial n'} - \Phi(\mathbf{x}') \frac{\partial G(\mathbf{x}, \mathbf{x}')}{\partial n'} \right] \mathrm{d}a' \tag{8}$$

With Dirichlet boundary conditions ($G(\mathbf{x}, \mathbf{x}')$), Eq. (8) simplifies to

$$\Phi(\mathbf{x}) = -\frac{1}{4\pi} \oint_S \Phi(\mathbf{x}') \frac{\partial G(\mathbf{x}, \mathbf{x}')}{\partial n'} \mathrm{d}a' \tag{9}$$

Note that $n' = z'$. Thus, we have

$$\begin{aligned} -\left.\frac{\partial G}{\partial z'}\right|_{z'=0} &= -\left[\frac{-(-2(z - z'))}{2(\rho^2 + \rho'^2 - 2\rho\rho' \cos(\phi - \phi') + (z - z')^2)^{3/2}} \right. \\ &\qquad \left. - \frac{-2(z + z')}{2(\rho^2 + \rho'^2 - 2\rho\rho' \cos(\phi - \phi') + (z + z')^2)^{3/2}} \right] \\ &= -\frac{2z}{(\rho^2 + \rho'^2 - 2\rho\rho' \cos(\phi - \phi') + z^2)^{3/2}} \end{aligned} \tag{10}$$

Substituting this expression to Eq. (9), we get

$$\begin{aligned} \Phi(\rho, \phi, z) &= -\frac{1}{4\pi} \int_0^{2\pi} \int_0^a \Phi(\rho', \phi', 0) \frac{-2z\rho' \mathrm{d}\rho' \mathrm{d}\phi'}{(\rho^2 + \rho'^2 - 2\rho\rho' \cos(\phi - \phi') + z^2)^{3/2}} \\ &= \frac{zV}{2\pi} \int_0^{2\pi} \left(\int_0^a \frac{\rho' \mathrm{d}\rho'}{(\rho^2 + \rho'^2 - 2\rho\rho' \cos(\phi - \phi') + z^2)^{3/2}} \right) \mathrm{d}\phi' \end{aligned} \tag{11}$$

(c) Show that, along the axis of the circle ($\rho = 0$), the potential is given by

$$\Phi = V \left(1 - \frac{z}{\sqrt{a^2 + z^2}} \right) \tag{12}$$

Solution:

With $\rho = 0$, Eq. (11) becomes

$$\begin{aligned}\Phi(\rho=0,\phi,z) &= \frac{zV}{2\pi}\int_0^{2\pi}\left(\int_0^a \frac{\rho' \mathrm{d}\rho'}{(\rho'^2+z^2)^{3/2}}\right)\mathrm{d}\phi' \\ &= \frac{zV}{2\pi}\int_0^{2\pi}\mathrm{d}\phi' \int_{z^2}^{a^2+z^2}\frac{\mathrm{d}u}{2u^{3/2}} \\ &= \frac{zV}{2\pi}\left[\phi'\right]_0^{2\pi}\cdot\frac{1}{2}\left[\frac{u^{-1/2}}{-1/2}\right]_{z^2}^{a^2+z^2} \\ &= \frac{zV}{2\pi}(2\pi)\cdot\frac{1}{2}\left(\frac{1}{\sqrt{z^2}}-\frac{1}{\sqrt{a^2+z^2}}\right) \\ \Phi(\rho=0,\phi,z) &= V\left(1-\frac{z}{\sqrt{a^2+z^2}}\right)\end{aligned} \tag{13}$$

where we let $u = \rho'^2 + z^2$ and changed the bounds of integration according to this variable change.

(d) Show that at large distances ($\rho^2 + z^2 >> a^2$) the potential can be expanded in a power series in $(\rho^2+z^2)^{-1}$, and that the leading terms are

$$\Phi = \frac{Va^2}{2}\frac{z}{(\rho^2+z^2)^{3/2}}\left[1-\frac{3a^2}{4(\rho^2+z^2)}+\frac{5\left(3\rho^2a^2+a^4\right)}{8(\rho^2+z^2)^2}+\cdots\right] \tag{14}$$

Verify that the results of parts (c) and (d) are consistent with each other in their common range of validity.

Solution:
By factoring out $(\rho^2+z^2)^{3/2}$ from the denominator of the integrand, Eq. (11) becomes

$$\begin{aligned}\Phi(\rho,\phi,z) &= \frac{zV}{2\pi}\int_0^{2\pi}\left(\int_0^a \frac{\rho'\,\mathrm{d}\rho'}{(\rho^2+z^2)^{3/2}\left[1+\frac{\rho'^2-2\rho\rho'\cos(\phi-\phi')}{\rho^2+z^2}\right]^{3/2}}\right)\mathrm{d}\phi' \\ &= \frac{zV}{2\pi(\rho^2+z^2)^{3/2}}\int_0^{2\pi}\left(\int_0^a \rho'\left[1+\frac{\rho'^2-2\rho\rho'\cos(\phi-\phi')}{\rho^2+z^2}\right]^{-3/2}\mathrm{d}\rho'\right)\mathrm{d}\phi'\end{aligned} \tag{15}$$

Note that the binomial expansion is given by

$$(1+x)^n = 1+nx+\frac{n(n-1)}{2!}x^2+\frac{n(n-1)(n-2)}{3!}x^3+\ldots \tag{16}$$

Applying this, we obtain

$$\begin{aligned}\left[1+\frac{\rho'^2-2\rho\rho'\cos(\phi-\phi')}{\rho^2+z^2}\right]^{-3/2} = 1 &- \frac{3}{2}\frac{\rho'^2-2\rho\rho'\cos(\phi-\phi')}{\rho^2+z^2} \\ &- \frac{15}{8}\left(\frac{\rho'^2-2\rho\rho'\cos(\phi-\phi')}{\rho^2+z^2}\right)^2+\ldots\end{aligned} \tag{17}$$

where

$$\left(\frac{\rho'^2-2\rho\rho'\cos(\phi-\phi')}{\rho^2+z^2}\right)^2 = \frac{\rho'^4+4\rho^2\rho'^2\cos^2(\phi-\phi')-4\rho\rho'^3\cos(\phi-\phi')}{(\rho^2+z^2)^2} \tag{18}$$

Thus, Eq. (15) can be expressed as

$$\Phi(\rho, \phi, z) = \frac{zV}{2\pi(\rho^2 + z^2)^{3/2}} \left[I_1 + I_2 + I_3 + I_4 + I_5 + I_6 + \ldots\right] \tag{19}$$

in which the following integrals are evaluated as follows:

$$I_1 = \int_0^{2\pi} \mathrm{d}\phi' \int_0^a \rho' \mathrm{d}\rho' = (2\pi)\frac{a^2}{2} = \pi a^2 \tag{20}$$

$$I_2 = -\frac{3}{2}\int_0^{2\pi} \mathrm{d}\phi' \int_0^a \frac{\rho'^3 \mathrm{d}\rho'}{\rho^2 + z^2} = -\frac{3}{2}(2\pi)\frac{a^4}{4(\rho^2 + z^2)} = -\frac{3}{4}\frac{\pi a^4}{\rho^2 + z^2} \tag{21}$$

$$I_3 = -\frac{3}{2}\int_0^{2\pi} \left(\int_0^a \frac{-2\rho\rho' \cos\left(\phi - \phi'\right)}{\rho^2 + z^2}\mathrm{d}\rho'\right) \mathrm{d}\phi' = 0 \tag{22}$$

$$I_4 = -\frac{15}{8}\int_0^{2\pi} \mathrm{d}\phi' \int_0^a \frac{\rho'^5}{\left(\rho^2 + z^2\right)^2}\mathrm{d}\rho' = -\frac{15}{8}(2\pi)\frac{a^6}{6\left(\rho^2 + z^2\right)^2} = -\frac{5}{8}\frac{\pi a^6}{\left(\rho^2 + z^2\right)^2} \tag{23}$$

$$\begin{aligned} I_5 &= -\frac{15}{8}\int_0^{2\pi} \cos^2\left(\phi - \phi'\right) \mathrm{d}\phi' \int_0^a \frac{4\rho^2\rho'^3}{\left(\rho^2 + z^2\right)^2}\mathrm{d}\rho' \\ &= -\frac{15}{8}(-\pi)\frac{4\rho^2 a^4}{4\left(\rho^2 + z^2\right)^2} \\ I_5 &= \frac{15}{8}\frac{\pi a^4 \rho^2}{\left(\rho^2 + z^2\right)^2} \end{aligned} \tag{24}$$

$$I_6 = -\frac{15}{8}\int_0^{2\pi} \left(\int_0^a \frac{-4\rho\rho'^3 \cos\left(\phi - \phi'\right)}{\left(\rho^2 + z^2\right)^2}\mathrm{d}\rho'\right) \mathrm{d}\phi' = 0 \tag{25}$$

where we have used

$$\begin{aligned} \int_0^{2\pi} \cos\left(\phi - \phi'\right) \mathrm{d}\phi' &= -\int_\phi^{\phi - 2\pi} \sin u \mathrm{d}u \\ &= -[\sin(\phi - 2\pi) - \sin\phi] \\ &= -\sin\phi \cancelto{1}{\cos 2\pi} + \cos\phi \cancelto{0}{\sin 2\pi} + \sin\phi \\ \int_0^{2\pi} \cos\left(\phi - \phi'\right) \mathrm{d}\phi' &= 0 \end{aligned} \tag{26}$$

with a variable change by $u = \phi - \phi'$ and

$$\begin{aligned} \int_0^{2\pi} \cos^2\left(\phi - \phi'\right) \mathrm{d}\phi' &= -\int_\phi^{\phi - 2\pi} \frac{\cos 2u - 1}{2}\mathrm{d}u \\ &= -\frac{1}{2}\left(\int_{2\phi}^{2(\phi - 2\pi)} \cos v \mathrm{d}v - \int_\phi^{\phi - 2\pi} \mathrm{d}u\right) \\ &= -\frac{1}{2}\left(\sin(2\phi - 4\pi) - \sin 2\phi + 2\pi\right) \\ &= -\frac{1}{2}\left(\sin 2\phi \cancelto{1}{\cos 4\pi} + \cos 2\phi \cancelto{0}{\sin 4\pi} - \sin 2\phi + 2\pi\right) \\ &= -\frac{1}{2}(2\pi) \\ \int_0^{2\pi} \cos\left(\phi - \phi'\right) \mathrm{d}\phi' &= -\pi \end{aligned} \tag{27}$$

with another variable change by $v = 2u$. Thus, Eq. (19) becomes

$$\begin{aligned}\Phi(\rho,\phi,z) &= \frac{zV}{2\pi(\rho^2+z^2)^{3/2}}\left[\pi a^2 - \frac{3}{4}\frac{\pi a^4}{\rho^2+z^2} - \frac{5}{8}\frac{\pi a^6}{(\rho^2+z^2)^2} + \frac{15}{8}\frac{\pi a^4\rho^2}{(\rho^2+z^2)^2} + \dots\right] \\ &= \frac{za^2V}{2(\rho^2+z^2)^{3/2}}\left[1 - \frac{3}{4}\frac{a^2}{\rho^2+z^2} + \frac{5}{8}\frac{3a^2\rho^2 - a^4}{(\rho^2+z^2)^2} + \dots\right]\end{aligned} \tag{28}$$

where we have shown the given in Eq. (14).

Looking at the scenarios in part (c) and (d), we can see that their common range of validity is at $\rho = 0$ with $z^2 >> a^2$. With $\rho = 0$, Eq. (28) becomes

$$\Phi(\rho=0,\phi,z) = \frac{a^2V}{2z^2}\left[1 - \frac{3}{4}\frac{a^2}{z^2} - \frac{5}{8}\frac{a^4}{z^4} + \dots\right] \tag{29}$$

On the other hand, we can expand Eq. (13) using binomial expansion in Eq. (16) since $z^2 >> a^2$ as follows:

$$\begin{aligned}\Phi(\rho=0,\phi,z) &= V\left(1 - \frac{z}{z\sqrt{1+\frac{a^2}{z^2}}}\right) \\ &= V\left(1 - \left(1+\frac{a^2}{z^2}\right)^{-1/2}\right) \\ &= V\left(1 - \left(1 - \frac{1}{2}\frac{a^2}{z^2} + \frac{3}{8}\frac{a^4}{z^4} - \frac{5}{16}\frac{a^4}{z^4} + \dots\right)\right) \\ \Phi(\rho=0,\phi,z) &= \frac{Va^2}{2z^2}\left(1 - \frac{3}{2}\frac{a^2}{z^2} + \frac{5}{8}\frac{a^2}{z^2} + \dots\right)\end{aligned} \tag{30}$$

As the two expressions coincide in their common range of validity, we have verified the results in both parts

References

Griffiths, D., *Introduction to Electrodynamics* (3rd ed.), Chapter 2

Problem 4 (2.15)

(a) Show that the Green function $G(x,y;x',y')$ appropriate for Dirichlet boundary conditions for a square two-dimensional region, $0 \le x \le 1$, $0 \le y \le 1$, has an expansion

$$G\left(x,y;x',y'\right) = 2\sum_{n=1}^{\infty} g_n\left(y,y'\right)\sin(n\pi x)\sin\left(n\pi x'\right) \tag{1}$$

where $g_n(y,y')$ satisfies

$$\left(\frac{\partial^2}{\partial y'^2} - n^2\pi^2\right) g_n\left(y,y'\right) = -4\pi\delta\left(y'-y\right) \quad \text{and} \quad g_n(y,0) = g_n(y,1) = 0 \tag{2}$$

Solution:
The Green's function satisfies

$$\nabla'^2 G(\mathbf{x},\mathbf{x}') = 4\pi\delta(\mathbf{x}-\mathbf{x}') \tag{3}$$

Applying this, we obtain

$$\begin{aligned}\frac{\partial^2}{\partial x'^2}G(x,y;x',y') + \frac{\partial^2}{\partial y'^2}G(x,y;x',y') &= -4\pi\delta(x-x')\delta(y-y') \\ &= -4\pi\delta(x'-x)\delta(y'-y)\end{aligned} \tag{4}$$

where we used the identity $\delta(x-x') = \delta(x'-x)$. Since Dirichlet boundary conditions imposes that $G(x,y;x',y') = 0$ at $x'=0$, $x'=1$, $y'=0$, and $y'=1$, we can express the Green's function as a Fourier sine series as this series also terminates at the boundaries:

$$G(x,y;x'y') = \sum_{n=1}^{\infty} b_n(x,y;y')\sin\left(\frac{n\pi x'}{L}\right) \tag{5}$$

where $L=1$. Note that the second derivative of $G(x,y;x',y')$ with respect to x' is given by

$$\begin{aligned}\frac{\partial^2}{\partial x'^2}\left(\sum_{n=1}^{\infty} b_n(x,y;y')\sin\left(n\pi x'\right)\right) &= \sum_{n=1}^{\infty} b_n(x,y;y')\frac{\partial^2}{\partial x'^2}\left(\sin\left(n\pi x'\right)\right) \\ &= -n^2\pi^2\sum_{n=1}^{\infty} b_n(x,y;y')\sin\left(n\pi x'\right)\end{aligned} \tag{6}$$

Thus, Eq. (4) becomes

$$\sum_{n=1}^{\infty}\left(-n^2\pi^2 + \frac{\partial^2}{\partial y'^2}\right) b_n(x,y;y')\sin\left(n\pi x'\right) = -4\pi\delta(x'-x)\delta(y'-y) \tag{7}$$

Using the completeness relation

$$\sum_{n=1}^{\infty}\sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{n\pi x'}{L}\right) = \frac{L}{2}\delta(x-x') \tag{8}$$

with $L = 1$ for this problem, we have

$$\sum_{n=1}^{\infty} \sin(n\pi x)\sin\left(n\pi x'\right) = \frac{1}{2}\delta(x'-x) \tag{9}$$

after again applying $\delta(x-x') = \delta(x'-x)$. Substituting in Eq. (9) to Eq. (7), we obtain

$$\sum_{n=1}^{\infty}\left(-n^2\pi^2 + \frac{\partial^2}{\partial y'^2}\right) b_n(x,y;y')\sin\left(n\pi x'\right) = -4\pi\left(2\sum_{n=1}^{\infty}\sin(n\pi x)\sin\left(n\pi x'\right)\right)\delta(y'-y) \tag{10}$$

We define

$$b_n(x,y;y') \equiv g_n(y;y')c_n(x) \tag{11}$$

Substituting this in and cancelling out $\sin(n\pi x')$ in Eq. (10), we have

$$\sum_{n=1}^{\infty}\left(-n^2\pi^2 + \frac{\partial^2}{\partial y'^2}\right) g_n(y;y')c_n(x) = \left[-4\pi\delta(y'-y)\right] 2\sum_{n=1}^{\infty}\sin(n\pi x) \tag{12}$$

where we see that

$$c_n(x) = 2\sin(n\pi x) \tag{13}$$

if $g_n(y;y')$ must satisfy Eq. (2). Thus, with Eqs. (11) and (13), we have

$$\begin{aligned} G(x,y;x'y') &= \sum_{n=1}^{\infty} g_n(y;y')c_n(x)\sin\left(n\pi x'\right) \\ &= 2\sum_{n=1}^{\infty} g_n(y;y')\sin(n\pi x)\sin\left(n\pi x'\right) \end{aligned} \tag{14}$$

in which we have shown Eq. (1)

(b) Taking for $g_n(y,y')$ appropriate linear combinations of $\sinh(n\pi y')$ and $\cosh(n\pi y')$ in the two regions, $y' < y$ and $y' > y$, in accord with the boundary conditions and the discontinuity in slope required by the source delta function, show that the explicit form of G is

$$G\left(x,y;x',y'\right) = 8\sum_{n=1}^{\infty}\frac{1}{n\sinh(n\pi)}\sin(n\pi x)\sin\left(n\pi x'\right)\sinh\left(n\pi y_<\right)\sinh\left[n\pi\left(1-y_>\right)\right] \tag{15}$$

where $y_<(y_>)$ is the smaller (larger) of y and y'.

Solution:
Since we are considering values where $y' \neq y$, Eq. (2) becomes

$$\left(\frac{\partial^2}{\partial y'^2} - n^2\pi^2\right) g_n\left(y,y'\right) = 0 \tag{16}$$

which is a homogeneous second order differential equation. Solving its characteristic equation, we get solutions of $e^{\pm n\pi y'}$. In terms of hyperbolic functions,

$$g_n(y,y') = C_1\cosh n\pi y' + C_2\sinh n\pi y' \tag{17}$$

which we can express as

$$g_n(y,y') = \begin{cases} g_{n>} & \equiv C_{1>}\cosh(n\pi y') + C_{2>}\sinh(n\pi y'), \qquad y' > y \\ g_{n<} & \equiv C_{1<}\cosh(n\pi y') + C_{2<}\sinh(n\pi y'), \qquad y' < y \end{cases} \tag{18}$$

Recall the boundary conditions: $g_n(y,0) = g_n(y,1) = 0$. Also, recall that $0 \leq y \leq 1$. At $y' = 0$, we use $g_{n<}$:

$$\begin{aligned} g_{n<}(y,y'=0) &= C_{1<}\cancelto{1}{\cosh(n\pi\cdot 0)} + C_{2<}\cancelto{0}{\sinh(n\pi\cdot 0)} \\ 0 &= C_{1<} \end{aligned} \tag{19}$$

and $g_{n>}$ at $y' = 1$:

$$\begin{aligned} g_{n>}(y,y'=1) &= C_{1>}\cosh(n\pi) + C_{2>}\sinh(n\pi) \\ 0 &= C_{1>}\cosh(n\pi) + C_{2>}\sinh(n\pi) \end{aligned} \tag{20}$$

where we see that

$$C_{1>} = -C_{2>}\frac{\sinh(n\pi)}{\cosh(n\pi)} = -C_{2>}\tanh(n\pi) \tag{21}$$

Thus, Eq. (18) becomes

$$g_n(y,y') = \begin{cases} g_{n>} & = C_{2>}\left[-\tanh(n\pi)\cosh(n\pi y') + \sinh(n\pi y')\right], \qquad y' > y \\ g_{n<} & = C_{2<}\sinh(n\pi y'), \qquad y' < y \end{cases} \tag{22}$$

From the continuity condition, we have $g_{n>} = g_{n<}$ at $y' = y$. This results to the relation

$$C_{2>}\left[-\tanh(n\pi)\cosh(n\pi y) + \sinh(n\pi y)\right] - C_{2<}\sinh(n\pi y) = 0 \tag{23}$$

We also have the condition from the discontinuity of the slope at $y' = y$ given by $\partial_{y'}g_{n>} - \partial_{y'}g_{n<} = -4\pi$. This results to the relation

$$\begin{aligned} C_{2>}\left[-n\pi\tanh(n\pi)\sinh(n\pi y) + n\pi\cosh(n\pi y)\right] - n\pi C_{2<}\cosh(n\pi y) &= -4\pi \\ C_{2>}\left[-\tanh(n\pi)\sinh(n\pi y) + \cosh(n\pi y)\right] - C_{2<}\cosh(n\pi y) &= -\frac{4}{n} \end{aligned} \tag{24}$$

We can express Eqs. (23) and (24) in matrix form:

$$\begin{bmatrix} -\tanh(n\pi)\cosh(n\pi y) + \sinh(n\pi y) & -\sinh(n\pi y) \\ -\tanh(n\pi)\sinh(n\pi y) + \cosh(n\pi y) & -\cosh(n\pi y) \end{bmatrix} \begin{bmatrix} C_{2>} \\ C_{2<} \end{bmatrix} = \begin{bmatrix} 0 \\ -4/n \end{bmatrix} \tag{25}$$

From this, we can solve for the constants $C_{2>}$ and $C_{2<}$ by taking the inverse of the 2×2 matrix. Note that the inverse of a 2×2 matrix is given by

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \tag{26}$$

Thus, we have

$$\begin{bmatrix} C_{2>} \\ C_{2<} \end{bmatrix} = \frac{1}{\tanh(n\pi)}\begin{bmatrix} -\cosh(n\pi y) & \sinh(n\pi y) \\ \tanh(n\pi)\sinh(n\pi y) - \cosh(n\pi y) & -\tanh(n\pi)\cosh(n\pi y) + \sinh(n\pi y) \end{bmatrix}\begin{bmatrix} 0 \\ -\frac{4}{n} \end{bmatrix} \tag{27}$$

which results to

$$\begin{bmatrix} C_{2>} \\ C_{2<} \end{bmatrix} = \frac{1}{\tanh(n\pi)} \begin{bmatrix} -\frac{4}{n}\sinh(n\pi y) \\ -\frac{4}{n}\left(-\tanh(n\pi)\cosh(n\pi y) + \sinh(n\pi y)\right) \end{bmatrix} \\ = \frac{-4}{n\sinh(n\pi)} \begin{bmatrix} \cosh(n\pi)\sinh(n\pi y) \\ -\sinh(n\pi)\cosh(n\pi y) + \cosh(n\pi)\sinh(n\pi y) \end{bmatrix} \tag{28}$$

With the constants determined, $g_n(y, y')$ becomes

$$g_n(y, y') = \frac{-4}{n\sinh(n\pi)} \begin{cases} \sinh(n\pi y)\left[-\sinh(n\pi)\cosh(n\pi y') + \cosh(n\pi)\sinh(n\pi y')\right], & y' > y \\ \sinh(n\pi y')\left[-\sinh(n\pi)\cosh(n\pi y) + \cosh(n\pi)\sinh(n\pi y)\right], & y' < y \end{cases} \tag{29}$$

Using the identity $\sinh[a(1-\theta)] = \sinh a \cosh a\theta - \cosh a \sinh a\theta$, we can express $g_n(y, y')$

$$g_n(y, y') = \frac{-4}{n\sinh(n\pi)} \begin{cases} -\sinh(n\pi y)\sinh[n\pi(1-y')], & y' > y \\ -\sinh(n\pi y')\sinh[n\pi(1-y)], & y' < y \end{cases} \tag{30}$$

Let $y_<$ and $y_>$ be the smaller and larger of y and y'. We can then express $g_n(y, y')$ into a compact form of

$$g_n(y, y') = \frac{4}{n\sinh(n\pi)}\sinh(n\pi y_<)\sinh[n\pi(1-y_>)] \tag{31}$$

which shows the given in Eq. (**??**) by substituting the resulting into Eq. (**??**)

References

Griffiths, D., *Introduction to Electrodynamics* (3^{rd} ed.), Chapter 2