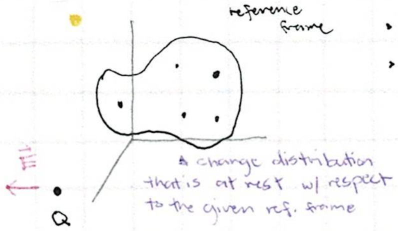
Apr. 6, 2021

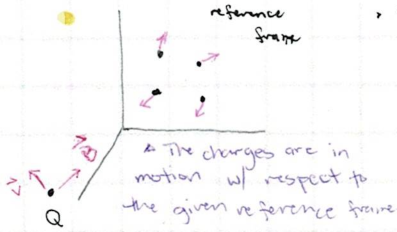
Ch.5: Magnetostatics, Faraday's Law, Quasi-static Fields

Recall: Electrostatics



- The stationary charges set up a force field to interact w/Q
- This force field is embodied by the electric field $\vec{E}$

$$\vec{F}_E = Q\vec{E} \qquad (1)$$



- The moving charge set up a force field that interacts w/ the moving charge. This force field is embodied by the magnetic flux density $\vec{B}$:

$$F_M = Q\vec{v} \times \vec{B} \qquad (2)$$

- The magnetic force arises from the rel. motion of charges

stationary charges $\longrightarrow$ electric force   Moving charges $\longrightarrow$ magnetic force

In general, the force on a given charge is

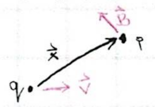
$$F = Q(\vec{E} + \vec{v} \times \vec{B}) \qquad (3)$$

electromagnetic force on charge Q

- Magnetostatics. consider only effects of the magnetic flux density $\vec{B}$

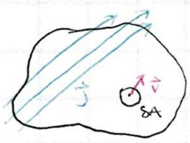
5.1: Introduction

- $\vec{B}$ due to a point charge



$$\vec{B} = \frac{\mu_0}{4\pi} q \frac{\vec{v} \times \vec{x}}{|\vec{x}|^3} \qquad \text{(non-relativistic) } v \ll c \qquad (4)$$

- charges in motion is described by a current density $\vec{J}$

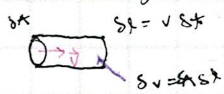


Over some time δt, the amount of charge δQ passes thru δA.

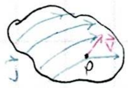
The charge density thru δA is:

$$J = \frac{\delta Q}{\delta A\ \delta t} \qquad (5)$$

Let the charges move w/ velocity $\vec{v}$



$$j = \frac{\delta Q}{\delta A\ \delta l} \frac{\delta l}{\delta t} = \rho v \longrightarrow \vec{J} = \rho\vec{v} \qquad (6)$$



- Charge is conserved. No charge is created or destroyed in any point at any given time

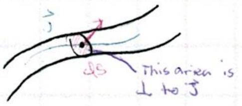
- Charge conservation leads to the continuity equation

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \vec{J} = 0 \qquad (7)$$

In magnetostatics, the charge distribution is constant. Thus

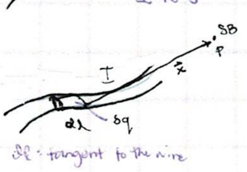
$$\frac{\partial \rho}{\partial t} = 0 \longrightarrow \vec{\nabla} \cdot \vec{J} = 0 \qquad \text{This characterizes magnetostatics} \qquad (8)$$

5.2: Biot - Savart Law



Current charge density confined to wires of small cross sections

The current thru the wire is defined by. $I = \int_S \vec{J} \cdot d\vec{S}$ $\vec{v}$ & $\hat{n}$ are $\parallel$ (9)



$$I\,d\vec{l} = \vec{J} \cdot \vec{A}\,dl = \vec{J} \cdot (\vec{A}\,dl) = \rho\vec{v} \cdot \underbrace{(\vec{A}\,dl)}_{\delta v\,\hat{n}} = \underbrace{(\rho\vec{v} \cdot \delta V)}_{\delta q}\hat{n} = \delta q\,\vec{v} \cdot \hat{n} \qquad (10)$$

$d\vec{l}$: tangent to the wire

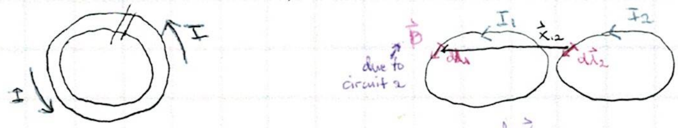
$$\delta\vec{B} = \frac{\mu_0}{4\pi} \frac{\delta q\,\vec{v} \times \vec{x}}{|\vec{x}|^3} = \frac{\mu_0}{4\pi} \frac{I\,d\vec{l} \times \vec{x}}{|\vec{x}|^3} \qquad (11)$$

Integrate to obtain all contributions

$$\vec{B} = \oint \frac{\mu_0}{4\pi} \frac{I\, d\vec{l} \times \vec{x}}{|\vec{x}|^3} \qquad (12)$$

- charge is conserved: an open wire can't sustain flowing charges

- current is sustained only in closed circuits



$$dF = \overbrace{I_1\, d\vec{l}_1}^{dq\,\vec{v}} \times \vec{B} \qquad (13)$$

The total force on circuit 1 is

$$\vec{F}_{12} = \oint I_1\, d\vec{l}_1 \times \vec{B}$$

$$= \oint I_1\, d\vec{l}_1 \times \oint \frac{\mu_0}{4\pi} I_2 \frac{d\vec{l}_2 \times \vec{x}_{12}}{|\vec{x}_{12}|^3}$$

$$= \frac{\mu_0}{4\pi} I_1 I_2 \oint\oint \frac{d\vec{l}_1 \times (d\vec{l}_2 \times \vec{x}_{12})}{|\vec{x}_{12}|^3} \qquad (14)$$

Use the identity: $\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a}\cdot\vec{c})\vec{b} - (\vec{a}\cdot\vec{b})\vec{c}$. Then,

$$d\vec{l}_1 \times (d\vec{l}_2 \times \vec{x}_{12}) = -(d\vec{l}_1 \cdot d\vec{l}_2)\vec{x}_{12} + d\vec{l}_2 (d\vec{l}_1 \cdot \vec{x}_{12}) \qquad (15)$$

Note that $\oint d\vec{l}_2 = 0$. Thus,

$$\vec{F}_{12} = -\frac{\mu_0}{4\pi} I_1 I_2 \oint\oint \frac{(d\vec{l}_1 \cdot d\vec{l}_2)\, \vec{x}_{12}}{|\vec{x}_{12}|^3} \qquad (16)$$

The force on circuit 1 is due to circuit 2

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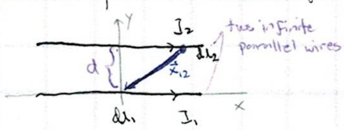
- Example: Obtain the force per unit length on wires



The differential force on dl_1 due to wire -2 is

$$d\vec{F}_1 = -\frac{\mu_0}{4\pi} I_1 I_2 \int_{-\infty}^{\infty} d\vec{l}_1 \cdot d\vec{l}_2 \frac{\vec{x}_{12}}{|\vec{x}_{12}|^3}$$

$$= -\frac{\mu_0}{4\pi} I_1 I_2\, dl_1 \int_{-\infty}^{\infty} \frac{\vec{x}_{12}}{|\vec{x}_{12}|^3}\, dx \qquad (17)$$

The force per unit length on wire -1 is

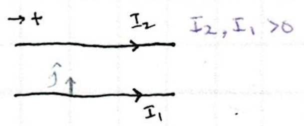
$$\frac{d\vec{F}_1}{dl_1} = -\frac{\mu_0}{4\pi} I_1 I_2 \int_{-\infty}^{\infty} \frac{\vec{x}_{12}}{|\vec{x}_{12}|^3}\, dx, \quad \vec{x}_{12} = -d\hat{j} - x\hat{i}, \quad |\vec{x}_{12}| = \sqrt{d^2 + x^2}$$

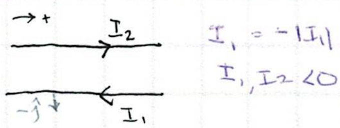
$$= -\frac{\mu_0}{4\pi} I_1 I_2 \int_{-\infty}^{\infty} \frac{(-d\hat{j} - x\hat{i})}{(d^2 + x^2)^{3/2}}\, dx$$

$$= \frac{\mu_0}{4\pi} I_1 I_2 \left[d \int_{-\infty}^{\infty} \frac{dx}{(d^2 + x^2)^{3/2}}\, \hat{j} + \cancelto{0}{\int_{-\infty}^{\infty} \frac{x\, dx}{(d^2 + x^2)^{3/2}}}\, \hat{i} \right]$$

$$= \frac{\mu_0}{4\pi} I_1 I_2 \cancel{d} \cdot \frac{2}{d^2}\, \hat{j}$$

$$\frac{d\vec{F}_1}{dl_1} = \frac{\mu_0 I_1 I_2}{2\pi d}\, \hat{j} \qquad (18)$$





- If I_1 and I_2 are flowing in the same direction, then the force is attractive

If I_1 and I_2 are flowing in opp. direction, the force bet. the wires is repulsive

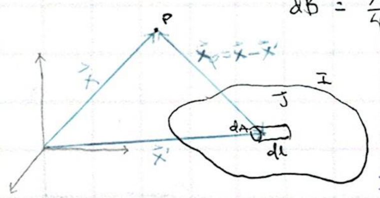
There is a force on the object due to the current density flowing thru it.

$$\vec{F} = \int \vec{J}(\vec{x}) \times \vec{B}(\vec{x})\, d^3x \tag{19}$$

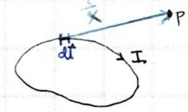
5.3: Differential Equations of Magnetostatics and Ampere's Law

Recall: For a current-carrying wire,

$$d\vec{B} = \frac{\mu_0}{4\pi} \frac{I\, d\vec{l} \times \vec{x}}{|\vec{x}|^3} \tag{20}$$



What is the magnetic field $\vec{B}$ due to $\vec{J}$?



$I\, d\vec{l} = \vec{J}\, dA\, dl$

$$d\vec{B} = \frac{\mu_0}{4\pi} \frac{\vec{J}\, dA\, dl \times \vec{x}_p}{|\vec{x}_p|^3} = \frac{\mu_0}{4\pi} \vec{J} \times \frac{\vec{x} - \vec{x}'}{|\vec{x} - \vec{x}'|^3} \underbrace{dA\, dl}_{d^3x'} \tag{21}$$

Integrating over the entire volume,

$$\vec{B} = \frac{\mu_0}{4\pi} \int \vec{J}(\vec{x}') \times \frac{\vec{x} - \vec{x}'}{|\vec{x} - \vec{x}'|^3}\, d^3x' \tag{22}$$

This is the basic law for the magnetic induction field. Note that

$$\frac{\vec{x} - \vec{x}'}{|\vec{x} - \vec{x}'|^3} = -\vec{\nabla}_{\vec{x}} \left(\frac{1}{|\vec{x} - \vec{x}'|} \right) = \vec{\nabla}_{\vec{x}'} \left(\frac{1}{|\vec{x} - \vec{x}'|} \right) \tag{23}$$

Thus, $$\vec{B} = \frac{\mu_0}{4\pi} \int \vec{J}(\vec{x}') \times \left[-\vec{\nabla}_{\vec{x}} \left(\frac{1}{|\vec{x} - \vec{x}'|} \right) \right] d^3x' = -\frac{\mu_0}{4\pi} \int \vec{J}(\vec{x}') \times \vec{\nabla}_{\vec{x}} \left(\frac{1}{|\vec{x} - \vec{x}'|} \right) d^3x \tag{24}$$

With the identity $\vec{\nabla} \times (\psi \vec{A}) = \psi\, \vec{\nabla} \times \vec{A} + (\nabla \psi) \times \vec{A}$ where we let $\psi = |\vec{x} - \vec{x}'|^{-1}$, $\vec{A} = \vec{J}(\vec{x}')$, we have

$$\vec{\nabla}_{\vec{x}} \times \frac{\vec{J}(\vec{x}')}{|\vec{x} - \vec{x}'|} = \frac{1}{\vec{x} - \vec{x}'} \underbrace{\cancel{\vec{\nabla}_{\vec{x}} \vec{J}(\vec{x}')}}_{0} + \vec{\nabla}_{\vec{x}} \left(\frac{1}{|\vec{x} - \vec{x}'|} \right) \times \vec{J}(\vec{x}') = -\vec{J}(\vec{x}') \times \vec{\nabla}_{\vec{x}} \left(\frac{1}{|\vec{x} - \vec{x}'|} \right) \tag{25}$$

Then, $$\vec{B} = \frac{\mu_0}{4\pi} \int \vec{\nabla}_{\vec{x}} \times \frac{\vec{J}(\vec{x}')}{|\vec{x} - \vec{x}'|}\, d^3x' = \frac{\mu_0}{4\pi} \vec{\nabla} \times \int \frac{\vec{J}(\vec{x}')}{|\vec{x} - \vec{x}'|}\, d^3x' \tag{26}$$

For all vector $\vec{v}$, $\vec{\nabla} \cdot \vec{\nabla} \times \vec{v} = 0$. So,

$$\vec{\nabla} \cdot \vec{B} = \frac{\mu_0}{4\pi} \left(\cancel{\vec{\nabla} \cdot \vec{\nabla} \times \int \frac{\vec{J}(\vec{x}')}{|\vec{x} - \vec{x}'|}\, d^3x'} \right) = 0 \tag{27}$$

Also, $$\vec{\nabla} \times \vec{B} = \frac{\mu_0}{4\pi} \vec{\nabla} \times \vec{\nabla} \times \int \frac{\vec{J}(\vec{x}')}{|\vec{x} - \vec{x}'|}\, d^3x' \tag{28}$$

With the identity $\vec{\nabla} \times (\vec{\nabla} \times \vec{A}) = \vec{\nabla}(\vec{\nabla} \cdot \vec{A}) - \vec{\nabla}^2 \vec{A}$, we have

$$\vec{\nabla}_{\vec{x}} \times \vec{\nabla}_{\vec{x}} \times \frac{\vec{J}(\vec{x}')}{|\vec{x} - \vec{x}'|} = \vec{\nabla}_{\vec{x}} \left(\vec{\nabla}_{\vec{x}} \cdot \frac{\vec{J}(\vec{x}')}{|\vec{x} - \vec{x}'|} \right) - \vec{\nabla}^2_{\vec{x}} \frac{\vec{J}(\vec{x}')}{|\vec{x} - \vec{x}'|} \tag{29}$$

* For 2nd term in Eq.(32):

$$-\frac{\mu_0}{4\pi} \int \vec{\nabla}^2_{\vec{x}} \frac{\vec{J}(\vec{x}')}{|\vec{x} - \vec{x}'|}\, d^3x' = -\frac{\mu_0}{4\pi} \int \vec{J}(\vec{x}')\, \vec{\nabla}^2_{\vec{x}} \left(\frac{1}{|\vec{x} - \vec{x}'|} \right) d^3x'$$

$$= -\frac{\mu_0}{4\pi} \int \vec{J}(\vec{x}') \left(-4\pi\, \delta(\vec{x} - \vec{x}') \right) d^3x'$$

$$= \mu_0 \vec{J}(\vec{x})$$

Using $$\vec{\nabla}_{\vec{x}} \left(\frac{1}{|\vec{x} - \vec{x}'|} \right) = -\vec{\nabla}_{\vec{x}'} \left(\frac{1}{|\vec{x} - \vec{x}'|} \right) \tag{30}$$

and $$\vec{\nabla}^2_{\vec{x}} \left(\frac{1}{|\vec{x} - \vec{x}'|} \right) = -4\pi\, \delta(\vec{x} - \vec{x}'), \tag{31}$$

we obtain

$$\vec{\nabla} \times \vec{B} = -\frac{\mu_0}{4\pi} \vec{\nabla} \int \vec{J}(\vec{x}') \cdot \vec{\nabla}_{\vec{x}'} \left(\frac{1}{|\vec{x} - \vec{x}'|} \right) d^3x' + \mu_0 \vec{J}(\vec{x}) = -\frac{\mu_0}{4\pi} \vec{\nabla} \int \frac{\overbrace{\vec{\nabla}_{\vec{x}'} \cdot \vec{J}(\vec{x}')}^{\to 0 \text{ for magnetostatics}}}{|\vec{x} - \vec{x}'|}\, d^3x' + \mu_0 \vec{J}(\vec{x}) \tag{32}$$

Therefore,

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J}(\vec{x}), \qquad \vec{\nabla} \cdot \vec{B} = 0 \tag{33}$$

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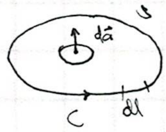
▸ A consequence of the relations in Eq.

$$\int_S (\vec{\nabla}\times\vec{B})\cdot\hat{n}\,da = \mu_0 \int_S (\vec{J}\cdot\hat{n})\,da = \mu_0 I \qquad (34)$$



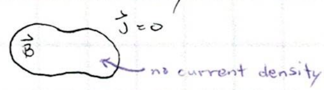
Using Stoke's theorem, we know: $\int_S (\vec{\nabla}\times\vec{B})\cdot\hat{n}\,da = \oint_C \vec{B}\cdot d\vec{l}$ (35)

Thus, we get

$$\oint_C \vec{B}\cdot d\vec{l} = \mu_0 I \qquad (36)$$

▸ Vector Potential (5.4)

Consider solving for $\vec{B}$ in a region free of current density



Recall: $\vec{\nabla}\times\vec{E}=0 \rightarrow \vec{E} = -\vec{\nabla}\Phi_E$ (37)

From $\vec{\nabla}\cdot\vec{B}=0$, we can infer that:

$$\vec{\nabla}\cdot\vec{B} = \vec{\nabla}\cdot(-\nabla\Phi_M) = -\vec{\nabla}^2\Phi_M = 0 \longrightarrow \nabla^2\Phi_M = 0 \text{ for } \vec{J}=0 \qquad (38)$$

where Φ_M is a scalar magnetic potential. Also, from $\vec{\nabla}\times\vec{B}=0$, we get

$$\vec{B} = -\vec{\nabla}\Phi_M \qquad (39)$$

In general, $\vec{\nabla}\cdot\vec{B}=0 \rightarrow \vec{B} = \vec{\nabla}\times\vec{A}$ ← vector potential (40)

Recall: $$\vec{B}(\vec{x}) = \frac{\mu_0}{4\pi}\int \vec{J}(\vec{x}')\times\frac{\vec{x}-\vec{x}'}{|\vec{x}-\vec{x}'|^3}\,d^3x' = \frac{\mu_0}{4\pi}\nabla\times\int\frac{\vec{J}(\vec{x}')}{|\vec{x}-\vec{x}'|}\,d^3x' \qquad (41)$$

This implies: $$\vec{A}(\vec{x}) = \frac{\mu_0}{4\pi}\int\frac{\vec{J}(\vec{x}')}{|\vec{x}-\vec{x}'|}\,d^3x' + \vec{\nabla}\psi \qquad (42)$$ ← this is arbitrary

Note that $\vec{\nabla}\times\vec{\nabla}\psi = 0$, so adding $\vec{\nabla}\psi$ does not change $\vec{B}$.

$\vec{B}(\vec{x})$ is uniquely determined by $\vec{J}(\vec{x})$

$\vec{A}(\vec{x})$ is determined uniquely by $\vec{J}(x)$ up to an arbitrary $\nabla\psi$

If we choose to reformulate magnetostatics in terms of the vector potential $\vec{A}(\vec{x})$, then the transformation on $\vec{A}(\vec{x})$ given by $\vec{A}(\vec{x}) \longrightarrow \vec{A}(\vec{x}) + \nabla\psi$ does not in any way alter the solutions to the field equations ($\nabla\cdot B = 0$ & $\nabla\times B = \mu_0\vec{J}$, then reformulate from $\vec{B}$ to $\vec{A}$). This transformation is known as a gauge transformation

Gauge transformation allows us to choose the most convenient form for $\vec{A}(\vec{x})$. With $\vec{B} = \vec{\nabla}\times\vec{A}$

$$\vec{\nabla}\times\vec{B} = \mu_0\vec{J} \longrightarrow \nabla\times(\vec{\nabla}\times\vec{A}) = \mu_0\vec{J} \qquad (43)$$

$$\vec{\nabla}(\vec{\nabla}\cdot\vec{A}) - \nabla^2\vec{A} = \mu_0\vec{J} \qquad (44)$$

If it happens initially that $\nabla\cdot A \neq 0$, then we can always perform a gauge transformation such that

$$\vec{A} \longrightarrow \vec{A}' = \vec{A} + \vec{\nabla}\psi \qquad \text{where} \quad \vec{\nabla}\cdot\vec{A}' = 0$$

So, $$\vec{\nabla}\cdot\vec{A} = \vec{\nabla}\cdot(\vec{A}+\vec{\nabla}\phi) = \vec{\nabla}\cdot\vec{A} + \nabla^2\phi = 0$$

which leads to: $$\vec{\nabla}^2\Phi = -\vec{\nabla}\cdot\vec{A} \neq 0 \qquad (45)$$

Under the condition $\vec{\nabla}\cdot\vec{A} = 0$,

$$\nabla^2\vec{A} = -\mu_0\vec{J} \begin{cases} \nabla^2 A_x = -\mu_0 J_x \\ \nabla^2 A_y = -\mu_0 J_y \\ \nabla^2 A_z = -\mu_0 J_z \end{cases} \qquad (46)$$

each component of $\vec{A}$ satisfies the Poisson eqn

These equations can be solved by Green's function method

$$\vec{\nabla}^2 G(\vec{x},\vec{x}') = -4\pi\,\delta(\vec{x},\vec{x}') \qquad (47)$$

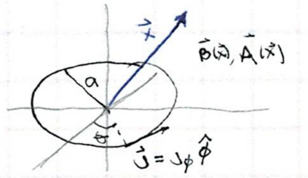
Solution: $$G(\vec{x},\vec{x}') = \frac{1}{|\vec{x}-\vec{x}'|} \qquad (48)$$

This gives the solution

$$A_k(\vec{x}) = \frac{\mu_0}{4\pi}\int \frac{J_k(\vec{x})}{|\vec{x}-\vec{x}'|}\, d^3x' \qquad (49)$$

$$\vec{A}(\vec{x}) = \frac{\mu_0}{4\pi}\int \frac{\vec{J}(\vec{x})}{|\vec{x}-\vec{x}'|}\, d^3x' \qquad (50)$$

Example:



The vector potential is given by Eq. (50). Obtain the current density $\vec{J}$. Jackson writes down

$$J_\phi = I \sin\theta' \,\delta(\cos\theta') \,\frac{\delta(r'-a)}{a} \qquad (51)$$

In general:

$dq\,|\vec{v}| = I\,|d\vec{l}|$

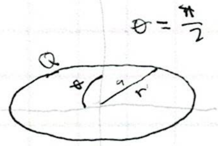
$dq\, v = I dl$

Integrate along the loop:

$$\int dq\, \underbrace{v = \int I}_{\text{constant throughout the loop}}\, dl \qquad (52)$$

where $Qv = I\, 2\pi a \;\rightarrow\; v = (I/Q)\, 2\pi a$

Recall that we are dealing with magnetostatics. The charge distribution ρ is constant in time.



- Constant I, V → the charge along the loop is uniform

$$\rho = Q\, \frac{\delta(r'-a)\,\delta(\cos\theta')}{2\pi r'^2} \qquad (53)$$

$\delta(\cos\theta') \neq 0$ for $\theta' = \pi/2$

Then, the current density is

$$J_\phi = \rho V = \cancel{Q}\,\frac{\delta(r'-a)\,\delta(\cos\theta')}{2\pi a^2}\cdot \frac{I}{\cancel{Q}}\, 2\pi \cancel{a} = \frac{\delta(r'-a)\,\delta(\cos\theta')}{a} \qquad (54)$$

Note: $\sin\theta'\,\delta(\cos\theta') = \sin\frac{\pi}{2}\,\delta(\cos\theta') = \delta(\cos\theta')$. Replace $\delta(\cos\theta') \to \sin\theta'\,\delta(\cos\theta')$

$J_\phi = I\sin\theta'\,\delta(\cos\theta')\left[\delta(r'-a)/a\right] \;\longrightarrow\; \vec{J} = -J_\phi \sin\phi'\,\hat{i} + J_\phi\cos\phi\,\hat{j}$ (55)

To simplify the integration, place field point p on the xz plane since the sys. has ϕ-symmetry

$$\vec{A}(\vec{x}) = \frac{\mu_0}{4\pi}\int\frac{\vec{J}(\vec{x}')}{|\vec{x}-\vec{x}'|}\,d^3x' = \frac{\mu_0}{4\pi}\int\left[-J_\phi \cancelto{0}{\sin\phi'}\,\hat{i} + J_\phi\cos\phi'\,\hat{j}\right]d^3x' = \frac{\mu_0}{4\pi}\int J_\phi\cos\phi'\,\frac{d^3x'}{|\vec{x}-\vec{x}'|}\,\hat{j}$$

$$A_\phi(r,\theta) = \frac{\mu_0 I}{a}\int\frac{\cos\phi'\,\delta(\cos\phi')\,\delta(r'-a)}{|\vec{x}-\vec{x}'|}\sin\theta'\, r'^2 dr'\, d\phi\, d\theta' = \frac{\mu_0 I a}{4\pi}\int_0^{2\pi}\frac{\cos\phi'\, d\phi'}{(a^2+r^2-2ar\sin\theta\cos\phi')^{1/2}}$$

Apr. 15, 2021 ▸ Recall Eq. (50) for the vector potential. Substituting the current density from Eq. (54), we have

$$\vec{A}(\vec{x}) = \frac{\mu_0}{4\pi}\int\left(-J_\phi\cancelto{0}{\sin\phi'}\,\hat{i} + J_\phi\cos\phi\,\hat{j}\right)\frac{d^3x'}{|\vec{x}-\vec{x}'|}$$

$$= \frac{\mu_0}{4\pi}\int J_\phi\cos\phi'\,\frac{d^3x'}{|\vec{x}-\vec{x}'|}\,\hat{j} \qquad (56)$$

Note that $|\vec{x}-\vec{x}'| = \left[r^2 + r'^2 - 2rr'(\cos\theta\cos\theta' + \sin\theta\sin\theta'\cos\phi)\right]^{1/2}$

$$A_\phi(r,\theta) = \frac{\mu_0 I}{4\pi a}\int\frac{\cos\theta'\,\delta(\cos\theta')\,\delta(r'-a)}{|\vec{x}-\vec{x}'|}\sin\theta'\, r'^2\, dr'\, d\phi'\, d\theta' \qquad (57)$$

We also note that $\cos\theta' = 0$ when $\theta' = \pi/2$

$$\int\delta(\cos\theta')\sin\theta'\, d\theta' = \sin\left(\frac{\pi}{2}\right) = 1 \qquad (58)$$

Then,

$$A_\phi(r,\theta) = \frac{\mu_0 I a}{4\pi}\int_0^{2\pi}\frac{\cos\phi'\, d\phi'}{(a^2+r^2-2ar\sin\theta\cos\phi')^{1/2}} \qquad (59)$$

In terms of elliptic integrals:

$$A_\phi(r,\theta) = \frac{\mu_0 I a}{4\pi} \frac{d}{\sqrt{a^2+r^2+2ar\cos\phi' \sin\theta}} \frac{(2-k^2)K(k) - E(\phi)}{k^2} \qquad (60)$$

where $k^2 = (4ar\sin\theta)/(a^2+r^2+2ar\sin\theta)$

Elliptic integrals:

$$K(k) = \int_0^{\pi/2} \frac{d\phi}{\sqrt{1-k^2\sin\phi}}, \qquad E(\phi) = \int_0^{\phi} \sqrt{1-k^2\sin^2\phi}\; d\phi \qquad (61)$$

The components of magnetic induction are

$$\vec{B} = \vec{\nabla}\times\vec{A} = \frac{1}{r^2\sin\theta}\begin{vmatrix} \hat{r} & r\hat{\theta} & r\sin\theta\hat{\phi} \\ \partial_r & \partial_\theta & \partial_\phi \\ A_r & rA_\theta & r\sin\theta A_\phi \end{vmatrix} \qquad (62)$$

which gives $$\vec{B} = \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}(A_\phi\sin\theta)\,\hat{r} + \frac{1}{r^2\sin\theta}\left(-\frac{\partial}{\partial r}(rA_\phi)\right)\hat{\theta} + 0 \qquad (63)$$

Thus, $$B_r = \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta A_\phi), \quad B_\phi = -\frac{1}{r}\frac{\partial}{\partial r}(A_\phi r), \quad B_\phi = 0 \qquad (64)$$

This is exact, no approx. involved. However, for large r, we can

$$A_\phi(r,\theta) = \frac{\mu_0 I a^2 r\sin\theta}{4(a^2+r^2)k^2}\left[1 + \frac{15a^2r^2\sin^2\theta}{8(a^2+r^2)^2} + \dots\right] \qquad (65)$$

and the components are

$$B_r = \frac{\mu_0 I a^2\cos\theta}{2(a^2+r^2)^{3/2}}\left[1 + \frac{15a^2r^2\sin^2\theta}{4(a^2+r^2)^2} + \dots\right] \qquad (66)$$

$$B_\theta = -\frac{\mu_0 I a^2\sin\theta}{4(a^2+r)^{5/2}}\left[2a^2 - r^2 + \mathcal{O}\left(\frac{1}{r^3}\right)\right] \qquad (67)$$

For the case of $r \gg a$,

$$B_r \approx \frac{\mu_0}{2\pi}(I\pi a^2)\frac{\cos\theta}{r^3} \qquad (68)$$

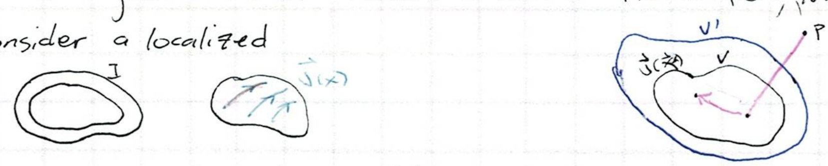
$$B_\theta \approx \frac{\mu_0}{2\pi}(I\pi a^2)\frac{\sin\theta}{r^3} \qquad (69)$$

By analogy w/ electrostatics we define the magnetic dipole moment of the loop to be

$$m = \pi I a^2 \qquad (70)$$

5.6: Magnetic Fields of a Localized Current Distribution, Magnetic Moment

Consider a localized



We know that the vector potential is given by Eq.

We assume that $|\vec{x}| \gg |\vec{x}'|$. Then,

$$|\vec{x}-\vec{x}'| = \sqrt{(\vec{x}-\vec{x}')\cdot(\vec{x}-\vec{x}')} = \sqrt{\vec{x}^2+\vec{x}'^2-2\vec{x}\cdot\vec{x}'} = |\vec{x}|\sqrt{1+\left(\frac{|\vec{x}'|}{|\vec{x}|}\right)^2 - \frac{2\vec{x}\cdot\vec{x}'}{|\vec{x}|^2}} \qquad (71)$$

Note that $|\vec{x}'|^2/|\vec{x}|^2 \ll |2\vec{x}\cdot\vec{x}'|/|\vec{x}^2|$. Thus,

$$|\vec{x}-\vec{x}'| \simeq |\vec{x}|\sqrt{1-\frac{2\vec{x}\cdot\vec{x}'}{|\vec{x}|^2}} = |\vec{x}|\left[1-\frac{1}{2}\frac{2\vec{x}\cdot\vec{x}'}{|\vec{x}|^2}+\dots\right] \qquad (72)$$

Therefore,

$$\frac{1}{|\vec{x}-\vec{x}'|} = \frac{1}{|\vec{x}|}\frac{1}{1-\frac{\vec{x}\cdot\vec{x}'}{|\vec{x}|^2}} \simeq \frac{1}{|\vec{x}|}\left(1+\frac{\vec{x}\cdot\vec{x}'}{|\vec{x}|^2}\right) = \frac{1}{|\vec{x}|}+\frac{\vec{x}\cdot\vec{x}'}{|\vec{x}|^3} \qquad (73)$$

$$\vec{A}(\vec{x}) = \frac{\mu_0}{4\pi}\int\left[\frac{1}{|\vec{x}|} + \frac{\vec{x}\cdot\vec{x}}{|\vec{x}|^3}\right]\vec{J}(\vec{x}')\, d^3x \quad (74)$$

For the component i,

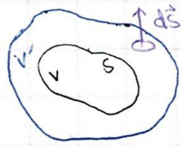
$$A_i(\vec{x}) = \frac{\mu_0}{4\pi}\left[\frac{1}{|\vec{x}|}\cancelto{0}{\int J_i(\vec{x}')d^3x'} + \frac{\vec{x}}{|\vec{x}|^3}\cdot\int J_i(\vec{x})\,\vec{x}'\, d^3x\right] \quad (75)$$

Let $\vec{G} = fg\vec{J}$ where f and g are smooth functions. Assume that $\vec{J}$ is localized.

Then,
$$\int_V \vec{\nabla}\cdot\vec{G}\, dv = \int_{V'}\vec{\nabla}\cdot\vec{G}\, dv' \quad (76)$$

We can convert this to a surface integral using divergence theorem

$\vec{J} = 0$ on the surface S'



$$\int_{V'}\vec{\nabla}\cdot\vec{G}\, dv' = \oint_{S'} G\cdot ds' = 0 \quad (77)$$

Then, we can infer that $\int_V \vec{\nabla}\cdot\vec{G}\, dv = 0$ if $\vec{J}$ is localized

Consider
$$\int\vec{\nabla}\cdot\vec{G}\, dV = \int\left(f\vec{J}\cdot\vec{\nabla}g + g\vec{J}\cdot\vec{\nabla}f + fg\,\cancelto{0 \text{ in magnetostatics}}{\vec{\nabla}\cdot\vec{J}}\right)d^3x$$

$$= \int\left(f\vec{J}\cdot\vec{\nabla}g + g\vec{J}\cdot\vec{\nabla}f\right)d^3x \quad (78)$$

Let $f=1$, $g=x_i$:

(79) $$\int\vec{\nabla}\cdot\vec{G}\, dV = \int\vec{J}\cdot\vec{\nabla}x_i\, d^3x = \int\left(J_x\frac{\partial}{\partial x} + J_y\frac{\partial}{\partial y} + J_z\frac{\partial}{\partial z}\right)x_i\, d^3x = \int J_i(\vec{x})d^3x = 0$$

Therefore, Eq.(75) becomes

$$A_i(\vec{x}) = \frac{\mu_0}{4\pi}\frac{\vec{x}}{|\vec{x}|^3}\int J_i(\vec{x}')\,\vec{x}'\, d^3x' \quad (79)$$

Note that
$$\vec{x}\cdot\int\vec{x}'J_i(\vec{x}')\,d^3x' = \sum_j x_j\int x_j' J_i\, d^3x \quad (80)$$

We make use of Eq.(78) which leads to $\int\left(f\vec{J}\cdot\vec{\nabla}g + g\vec{J}\cdot\vec{\nabla}f\right)d^3x = 0$.
Let $f = x_i'$ and $g = x_j'$. Then, $\int\left(x_i'\vec{J}\cdot\vec{\nabla}x_j' + x_j'\vec{J}\cdot\nabla x_i'\right) = \int\left(x_i'J_j + x_j'J_i\right)d^3x' = 0$

Thus, we get
$$\int x_i' J_j(\vec{x}')\,d^3x' = -\int x_j' J_i(\vec{x}')\,d^3x' \quad (81)$$

Now, Eq.(80) becomes
$$\vec{x}\cdot\int\vec{x}'J_i(\vec{x}')\,d^3x' = \frac{1}{2}\sum\int\left(x_j'J_i(\vec{x}') + x_j'J_j(\vec{x}')\right)d^3x'$$

$$= \frac{1}{2}\sum_j x_j\int\left(x_j'J_i(\vec{x}') - x_i'J_j(\vec{x}')\right)d^3x'$$

$$\vec{x}\cdot\int\vec{x}'J_i(\vec{x}')\,d^3x' = -\frac{1}{2}\left[\vec{x}\times\int(\vec{x}'\times\vec{J})\,d^3x'\right] \quad (82)$$

Define the magnetic moment density / magnetization as

$$\vec{M}(\vec{x}) = \frac{1}{2}\left[\vec{x}\times J(\vec{x})\right] \quad (83)$$

The magnetic moment $\vec{m}$ is given by

$$\vec{m} = \frac{1}{2}\int\vec{x}'\times\vec{J}(\vec{x}')\,d^3x' \quad (84)$$

Then, the leading term in the vector potential is

$$\vec{A}(\vec{x}) = \frac{\mu_0}{4\pi}\frac{\vec{m}\times\vec{x}}{|\vec{x}|^3} \quad (85)$$

This term comes from the magnetic dipole moment. There is no magnetic monopole contribution (i.e. there is no magnetic charge)

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$$\vec{B} = \nabla\times\vec{A} = \nabla\times\left(\frac{\mu_0}{4\pi}\vec{m}\times\frac{\vec{x}}{|\vec{x}|}\right) = \frac{\mu_0}{4\pi}\,\vec{\nabla}\times\vec{m}\times\frac{\vec{x}}{|\vec{x}|^3} \quad (86)$$

$\vec{C}\to\vec{m}$
$\vec{D}\to\frac{\vec{x}}{|\vec{x}|^3}$

Using the identity: $\nabla\times(\vec{C}\times\vec{D}) = C(\nabla\cdot D) - \vec{D}(\nabla\cdot C) + (\vec{D}\cdot\vec{\nabla})\vec{C} - (\vec{C}\cdot\vec{\nabla})\vec{D}$ (86.1)

Applying this + Eq. (86),

$$\vec{B}(\vec{x}) = \frac{\mu_0}{4\pi}\left[\vec{m}\left(\cancel{\vec{\nabla}\cdot\frac{\vec{x}}{|\vec{x}|^3}}\right)^{0} - \frac{\vec{x}}{|\vec{x}|^3}\left(\cancel{\nabla\cdot\vec{m}}\right)^{0} + \left(\cancel{\frac{\vec{x}}{|\vec{x}|^3}\cdot\nabla}\right)^{0}\vec{m} - (\vec{m}\cdot\vec{\nabla})\frac{\vec{x}}{|\vec{x}|^3}\right] \quad (87)$$

Note that by chain rule:

$$\nabla\cdot\left(\frac{\vec{x}}{|\vec{x}|^3}\right) = (\nabla\cdot\vec{x})\frac{1}{|\vec{x}|^3} + \vec{x}\cdot\vec{\nabla}\frac{1}{|\vec{x}|^3} \quad (88)$$

Also, $\vec{\nabla}\cdot\vec{x} = \left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)\cdot(x\hat{i}+y\hat{j}+z\hat{k}) = \frac{\partial x}{\partial x}+\frac{\partial y}{\partial y}+\frac{\partial z}{\partial z} = 3$ (89)

$$\vec{\nabla}\frac{1}{|\vec{x}|^3} = \nabla|\vec{x}|^{-3} = -3|\vec{x}|^{-4}\vec{\nabla}|\vec{x}| = -\frac{3}{|\vec{x}|^4}\left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)(x^2+y^2+z^2)^{1/2}$$

$$= -\frac{3}{|\vec{x}|^4}\frac{\hat{i}x+\hat{j}y+\hat{k}z}{(x^2+y^2+z^2)^{1/2}} = -\frac{3}{|\vec{x}|^4}\frac{\vec{x}}{|\vec{x}|} = -\frac{3}{|\vec{x}|^4}\hat{n}$$

$$\nabla\cdot\frac{\vec{x}}{|\vec{x}|^3} = \frac{3}{|\vec{x}|^3} + \left(-\frac{3}{|\vec{x}|^4}\frac{\vec{x}}{|\vec{x}|}\right)\cdot\vec{x} = 0 \quad (90)$$

Thus, $\vec{B} = \frac{\mu_0}{4\pi}(\vec{m}\cdot\vec{\nabla})\frac{\vec{x}}{|\vec{x}|^3}$ (91)

Note that: $(\vec{m}\cdot\vec{\nabla})\frac{\vec{x}}{|\vec{x}|^3} = \frac{1}{|\vec{x}|^3}(\vec{m}\cdot\vec{\nabla})\vec{x} + \vec{x}(\vec{m}\cdot\vec{\nabla})\frac{1}{|\vec{x}|^3}$ (92)

Also, $(\vec{m}\cdot\vec{\nabla})\vec{x} = \left(m_x\frac{\partial}{\partial x}+m_y\frac{\partial}{\partial y}+m_z\frac{\partial}{\partial z}\right)(x\hat{i}+y\hat{j}+z\hat{k})$

$$= m_x\hat{i}+m_y\hat{j}+m_z\hat{k}$$

$$= \vec{m} \quad (93)$$

$$(\vec{m}\cdot\vec{\nabla})\frac{1}{|\vec{x}|^3} = \left(m_x\frac{\partial}{\partial x}+m_y\frac{\partial}{\partial y}+m_z\frac{\partial}{\partial z}\right)\frac{1}{|\vec{x}|^3}$$

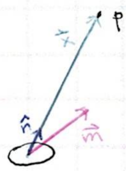
$$\frac{\partial}{\partial x}|\vec{x}|^{-3} = -3|\vec{x}|^{-4}\frac{\partial|\vec{x}|}{\partial x} = -\frac{3}{|\vec{x}|^4}\frac{\partial}{\partial x}(x^2+y^2+z^2)^{1/2}$$

$$= -\frac{3}{|\vec{x}|^4}\frac{x}{(x^2+y^2+z^2)^{1/2}} = -\frac{3}{|\vec{x}|^4}\frac{x}{|\vec{x}|}$$

$$= -\frac{3}{|\vec{x}|^4}\left(\frac{x}{|x|}m_x + \frac{y}{|x|}m_y + \frac{z\,m_z}{|x|}\right)$$

$$= -\frac{3}{|\vec{x}|^4}\frac{\vec{m}\cdot\vec{x}}{|\vec{x}|}, \qquad \hat{n} = \frac{\vec{x}}{|\vec{x}|}$$

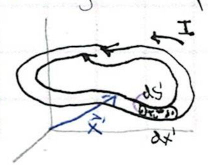
$$= -3\frac{\vec{m}\cdot\hat{n}}{|\vec{x}|^4} \quad (94)$$



Therefore, $\vec{B} = -\frac{\mu_0}{4\pi}\left(\frac{\vec{m}}{|\vec{x}|^3} - \frac{3}{|\vec{x}|^3}(\vec{m}\cdot\hat{n})\,\hat{x}\right) = \frac{\mu_0}{4\pi}\left(\frac{3(\vec{m}\cdot\hat{n})}{|\vec{x}|^3}\frac{\vec{x}}{|\vec{x}|} - \frac{\vec{m}}{|\vec{x}|^3}\right)$

which leads to $\vec{B}(\vec{x}) = \frac{\mu_0}{4\pi}\left(\frac{3\hat{n}(\vec{m}\cdot\hat{n}) - \vec{m}}{|\vec{x}|^3}\right)$ (95)

- Magnetic dipole moment of a current loop



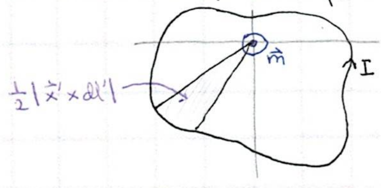
$$m = \frac{1}{2}\int \vec{x}'\times\vec{J}(\vec{x}')\,d^3x' = \frac{1}{2}\int \vec{x}'\times\vec{J}(\vec{x}')\,ds'\,dl'$$

$d^3x' = ds'dx'$

8

which can be written as: $\vec{m} = \frac{1}{2}\int \vec{x}' \times \left(\vec{J}(\vec{x}') \cdot d\vec{s}'\, dl' \right) = \frac{I}{2} \oint \vec{x}' \times d\vec{l}'$ (96)

For a plane loop, we are free to place the origin on the same plane as the loop

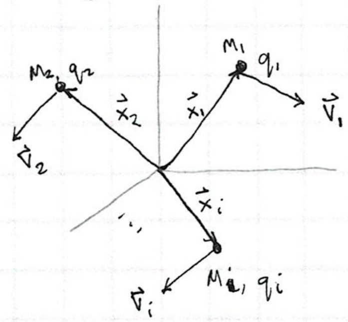


Thus, we get

$$|\vec{m}| = \left| \frac{I}{2} \oint \vec{x}' \times d\vec{l}' \right| = I \underbrace{\oint \frac{1}{2} |\vec{x}' \times d\vec{l}'|}_{\text{total area enclosed by loop}} = I \times (\text{Area}) \quad (97)$$

This holds regardless of the shape of the loop.

- Magnetic dipole moment of moving point charges



The current density is given by

$$\vec{J} = \sum_i q_i \vec{v}_i\, \delta(\vec{x} - \vec{x}_i) \quad (98)$$

The corres. magnetic moment is

$$\vec{m} = \frac{1}{2}\int \vec{x} \times \vec{J}(\vec{x}')\, d^3x'$$

$$= \frac{1}{2}\int \vec{x}' \times \sum_i q_i \vec{v}_i\, \delta(\vec{x} - \vec{x}_i)\, d^3x' \quad (99)$$

which can be written as

$$\vec{m} = \frac{1}{2}\sum_i q_i \int \vec{x}' \times \vec{v}_i\, \delta(\vec{x}' - \vec{x}_i)\, d^3x' = \frac{1}{2}\sum_i q_i (\vec{x}_i \times \overbrace{\vec{v}_i) \frac{M_i}{M_i}}^{p_i}$$

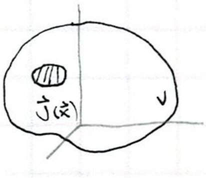
$$= \frac{1}{2}\sum_i \frac{q_i}{M_i} (\vec{x}_i \times \vec{p}_i) = \frac{1}{2}\sum_i \frac{q_i}{M_i} \vec{L}_i \quad (100)$$

If it happens that q_i and M_i are the same for all particles, like electrons

$$\vec{m} = \frac{1}{2}\frac{e}{M}\sum_i \vec{L}_i = \frac{e}{2M} \vec{L} \quad (101)$$

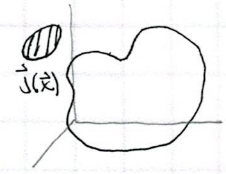
— total angular momentum

- Localized Current Distribution



- The localized current density is inside V

The localized current density is outside V

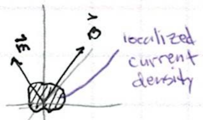


$$\vec{B}(\vec{x}) = \frac{\mu_0}{4\pi}\left[\frac{3\hat{n}(\hat{n}\cdot\vec{m}) - \vec{m}}{|\vec{x}|^3} + \frac{8\pi}{3}\vec{m}\,\delta(\vec{x}) \right] \leftarrow \int_V \vec{B}(\vec{x})\, d^3x = \begin{cases} 4\pi R^3 \vec{B}(0), & \text{inside} \\ \frac{1}{3} 4\pi R^3 \vec{B}(0), & \text{outside} \end{cases} \quad (102)$$

- For all torque and energy of a localized current distribution in an external magnetic field.
- If a localized distribution of current charges is placed in an ext. magnetic field, $\vec{B}(\vec{x})$, it expresses forces and torques acc. to Ampere's law

Assume: The magnetic field varies slowly over the region of the current density.
This allows for the Taylor expansion of $\vec{B}(\vec{x})$ about the origin

$$B_k(\vec{x}) = B_k(0) + \vec{x} \cdot \nabla B_k(0) + \dots \quad (103)$$



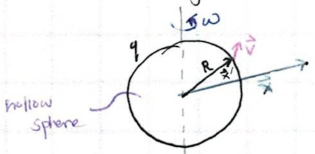
(104) Then, $\vec{F} = \int \vec{J}(\vec{x}') \times \vec{B}(\vec{x})\, d^3x' = \int \vec{J}(\vec{x}') \times \left[\vec{B}(0) + \vec{x}' \cdot \nabla' \vec{B}(0) + \dots \right] d^3x' = \overbrace{\cancel{\int \vec{J}(\vec{x}) \times \vec{B}(0)\, d^3x'}}^{0} + \int \vec{J}(\vec{x}) \times \vec{x}' \cdot \nabla' \vec{B}(0)\, d^3x + \dots$

(105) Thus, $\vec{F} = \int \vec{J}(\vec{x}) \times \vec{x}' \cdot \nabla' \vec{B}(0)\, d^3x' + \dots = \left(\int \vec{J}(\vec{x}') \times \vec{x}'\, d^3x' \right) \cdot \nabla \vec{B}(0) = -\vec{m} \cdot \nabla \vec{B}(0) = \nabla(\vec{m}\cdot\vec{B}) - \vec{m}\overbrace{\cancel{(\nabla \cdot \vec{B})}}^{0}$

$$W = \int \vec{F} \cdot d\vec{l} = \int \nabla(\vec{m}\cdot\vec{B}) \cdot d\vec{l} = \int d(\vec{m} - \vec{B} \quad (106)$$

$W = \vec{m} \cdot \vec{B}$
$\hookrightarrow U = -\vec{m} = \vec{B}$

Apr 22, 2021 ▸ Magnetic field of a charged rotating sphere



- A sphere of radius R w/ a charge q uniformly distributed on its surface is rotating about a diameter at a constant angular velocity
- Calculate the vector potential $\vec{A}(\vec{x})$ and the magnetic induction inside and outside the sphere

Solution: Note Eq. (55) for $\vec{A}(\vec{x})$. In this case, $\vec{J}(\vec{x}') = \rho(\vec{x}')\,\vec{v} = \rho(\vec{x}')\,\vec{\omega}\times\vec{x}'$ where $\rho(\vec{x}') = q\,\delta(r-R)/[4\pi R^2]$. Thus,

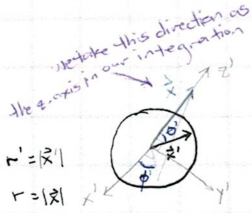
$$\vec{J}(\vec{x}') = \frac{q\,\delta(r-R)}{4\pi R^2}\,\vec{\omega}\times\vec{x}'$$

Then, the vector potential is:

$$\vec{A}(\vec{x}) = \frac{\mu_0}{4\pi}\int \frac{q\,\delta(r-R)}{4\pi R^2}\,\frac{\vec{\omega}\times\vec{x}'}{|\vec{x}-\vec{x}'|}\,d^3x'$$

$$= \frac{\mu_0 q}{(4\pi)^2 R^2}\,\vec{\omega}\times\underbrace{\int \frac{\delta(r-R)\,\vec{x}'}{|\vec{x}-\vec{x}'|}\,d^3x'}$$

$$\vec{A}(\vec{x}) = \frac{\mu_0 q}{(4\pi)^2 R^2}\,\vec{\omega}\times\vec{G} \qquad (107)$$



Note that:

$$\vec{x}' = r'\cos\phi'\sin\theta'\,\hat{\imath} + r'\sin\theta'\sin\phi'\,\hat{\jmath} + r'\cos\theta'\,\hat{k} \qquad (108)$$

Then,

* Note that $\int_0^{2\pi}\sin\phi'\,d\phi' = 0$

$$\vec{G} = \int \frac{\delta(r'-R)\,(r'\cos\phi'\sin\theta'\,\hat{\imath} + r'\sin\theta'\sin\phi'\,\hat{\jmath} + r'\cos\theta'\,\hat{k})}{(r^2+r'^2-2rr'\cos\theta')^{1/2}}\,r'^2\sin\theta'\,dr'\,d\theta'\,d\phi'$$

$$= \int_0^{\pi} d\theta' \int_0^{\infty} \frac{dr'\,\delta(r'-R)\,r'\cos\theta'\,2\pi\sin\theta'\,r'^2}{(r^2+r'^2-2rr'\cos\theta')^{1/2}}\cdot\frac{\vec{x}}{|\vec{x}'|}$$

$$\vec{G} = \int_0^{\pi} d\theta'\,\frac{2\pi R^3\cos\theta'\sin\theta'}{(r^2+R^2-2rR\cos\theta')^{1/2}}\cdot\frac{\vec{x}}{|\vec{x}|} \qquad (109)$$

$\hat{r}$

Apply a change of variables: $\xi = \cos\theta'$

$$\vec{G} = 2\pi R^3\int_{-1}^{1}\frac{\xi\,d\xi}{(r^2+R^2-2rR\xi)^{1/2}}$$

$$= \frac{2\pi R}{3r}\left[(r^2+R^2-rR)(r+R) - (r^2+R^2+rR)(r-R)\right]\hat{r} \qquad (110)$$

Inside the sphere: $r<R$, $|r-R| = R-r$

$$G = \frac{4\pi}{3}Rr \longrightarrow \vec{G} = \frac{4\pi}{3}Rr\cdot\frac{\vec{x}}{\cancel{|\vec{x}|}} = \frac{4\pi}{3}R\vec{x} \qquad (111)$$

Outside the sphere: $r>R$, $|r-R| = r-R$

$$G = \frac{4\pi}{3}\frac{R^4}{r^2} \longrightarrow \vec{G} = \frac{4\pi}{3}\frac{R^4}{r^2}\cdot\frac{\vec{x}}{|\vec{x}|} = \frac{4\pi}{3}\frac{R^4}{|\vec{x}|^3} \qquad (112)$$

Therefore,

$$\vec{G} = \begin{cases} \frac{4\pi}{3}R\vec{x}, & \text{inside} \\ \frac{4\pi}{3}\frac{R^4}{|\vec{x}|^3}\vec{x}, & \text{outside} \end{cases} \qquad (113)$$

Substituting this to Eq.

$$\vec{A}(\vec{x}) = \frac{\mu_0 q}{(4\pi)^2 R^2}\begin{cases} \vec{\omega}\times\frac{4\pi}{3}R\vec{x} \\ \omega\times\frac{4\pi}{3}\frac{R^4}{|\vec{x}|^3}\vec{x} \end{cases} = \begin{cases} \frac{\mu_0 q}{3\cdot 4\pi R}\,\vec{\omega}\times\vec{x}, & \text{inside} \\ \frac{\mu_0 q R^2}{3\cdot 4\pi}\,\omega\times\frac{\vec{x}}{|\vec{x}|^3}, & \text{outside} \end{cases} \qquad (114)$$

The magnetic induction is $\vec{B} = \vec{\nabla}\times\vec{A}$. Recalling the identity in Eq. (86.1):

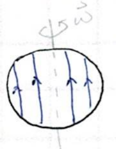
- Inside of the sphere:

$$\vec{B} = \vec{\nabla}\times\frac{\mu_0 q}{3\cdot 4\pi R}\,\vec{\omega}\times\vec{x} = \frac{\mu_0 q}{12\pi R}\left(\vec{\omega}(\vec{\nabla}\cdot\vec{x}) - \vec{\omega}\cdot\nabla\vec{x}\right) \qquad (115)$$

Note that:

$$\vec\nabla\cdot\vec x = \left(\hat\imath \frac{\partial}{\partial x} + \hat\jmath\frac{\partial}{\partial y} + \hat k\frac{\partial}{\partial z}\right)(x\hat\imath + y\hat\jmath + z\hat k) = 1+1+1 = 3 \tag{116}$$

$$(\vec\omega\cdot\vec\nabla)\vec x = \left(\omega_x\frac{\partial}{\partial x} + \omega_y\frac{\partial}{\partial y} + \omega_z\frac{\partial}{\partial z}\right)(x\hat\imath + y\hat\jmath + z\hat k) = \omega_x\hat\imath + \omega_y\hat\jmath + \omega_z\hat k = \vec\omega \tag{117}$$



Then,

$$\vec B_{in} = \frac{\mu_0 q}{12\pi R}(\vec\omega\cdot 3 - \vec\omega) = \underbrace{\frac{\mu_0 q}{6\pi R}\vec\omega}_{\text{indep. of } x} \tag{118}$$

- Outside the sphere:

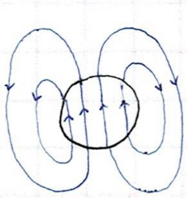
(119) $$\vec B = \nabla\times\frac{\mu_0 q R^2}{3\cdot 4\pi}\vec\omega\times\frac{\vec x}{|\vec x|^3} = \frac{\mu_0 q R^2}{12\pi}\vec\nabla\times\vec\omega\times\frac{\vec x}{|\vec x|^3} = \frac{\mu_0 q R^2}{12\pi}\left(\vec\omega\left(\nabla\cdot\frac{\vec x}{|\vec x|^3}\right) - (\vec\omega\cdot\vec\nabla)\frac{\vec x}{|\vec x|^3}\right)$$

Recalling Eqs.

$$\vec B_{out} = \frac{\mu_0 q R^2}{12\pi}\left[3\left(\vec\omega\cdot\frac{\vec x}{|\vec x|}\right)\frac{\vec x}{|\vec x|} - \omega\right]|\vec x|^{-3} = \frac{3\left[\left(\frac{\mu_0 q R^2}{12\pi}\vec\omega\right)\cdot\hat r\right]\hat r - \left(\frac{\mu_0 q R^2}{12\pi}\vec\omega\right)}{|\vec x|^3} \tag{120}$$

Compare this with Eq. ($\vec B$ due to a magnetic diple). Then,

$$\vec B_{out} = \frac{\mu_0}{4\pi}\left\{3\left[\Big(\underbrace{\frac{qR^2}{3}\vec\omega}_{\vec m}\Big)\cdot\hat r\right]\hat r - \Big(\underbrace{\frac{qR^2}{3}\vec\omega}_{\vec m}\Big)\right\}|\vec x|^{-3} \tag{121}$$



which means that $\vec B$ in this region is due to a magnetic dipole w/ magnetic dipole moment $\vec m = qR^2\vec\omega/3$

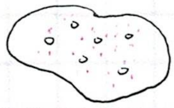
May. 6, 2021

Macroscopic equations, Boundary Conditions on $\vec B$ and $\vec H$ (5.8)

Previous assumption: $\vec J$ is completely known

In macroscopic cases involving bulk materials, it is usually not true that $\vec J$ is known.

The electrons may constitute time varying electric current



In bulk materials, observed magnetic quantities are of ave. values.

The magnetic induction in microscopic sys: $\vec B_{micro}$

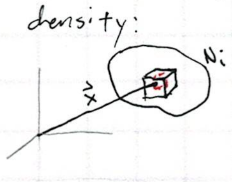
The magnetic intuction in bulk materials: $\vec B_{macro}$ or $\vec B$

We know that: $\vec\nabla\cdot\vec B_{micro} = 0 \longrightarrow \vec\nabla\cdot\vec B = 0$

which implies: $\vec\nabla\cdot\vec B = 0 \longrightarrow \vec B = \vec\nabla\times\vec A$ some vector

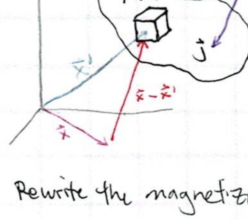
The large number of molecules/atoms per unit volume, each with its molecular magnetic moment ($\vec m_i$) gives rise to an ave. macroscopic magnetization or magnetic moment density:

$$\vec M(\vec x) = \sum_i N_i\langle\vec m_i\rangle \tag{122}$$



ave. no. per unit volume of molecules of type i

ave. molecular moment in a small



macroscopic current density due to free charges in the bulk material

The vector potential is

(123) $$\vec A(\vec x) = \frac{\mu_0}{4\pi}\int\frac{\vec J(\vec x')}{|\vec x - \vec x'|}d^3x' + \frac{\mu_0}{4\pi}\int\frac{\vec M(\vec x')\times(\vec x - \vec x')}{|\vec x - \vec x'|}d^3x'$$

Rewrite the magnetization term:

$$\int\frac{\vec M(\vec x')\times(\vec x - \vec x')}{|\vec x - \vec x'|}d^3x' = \int\vec M(\vec x')\times\vec\nabla'\left(\frac{1}{|\vec x - \vec x'|}\right)d^3x' \tag{123}$$

Note that: $$\nabla\times\left(\frac{\vec M(\vec x')}{|\vec x - \vec x'|}\right) = \frac{1}{|\vec x - \vec x'|}\vec\nabla'\times\vec M(\vec x') + \left(\vec\nabla'\frac{1}{|\vec x - \vec x'|}\right)\times\vec M(\vec x) \tag{124}$$

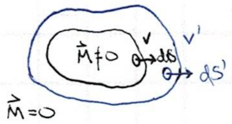
Also: $\vec{M}(\vec{x}') \times \vec{\nabla}'\left(\frac{1}{|\vec{x}-\vec{x}'|}\right) = \frac{1}{|\vec{x}-\vec{x}'|}\vec{\nabla}\times\vec{M}(\vec{x}) - \vec{\nabla}'\times\left(\frac{\vec{M}(\vec{x}')}{|\vec{x}-\vec{x}'|}\right)$ (125)

Thus, $\int \frac{\vec{M}(\vec{x}) \times (\vec{x}-\vec{x}')}{|\vec{x}-\vec{x}'|} d^3x' = \int \left[\frac{1}{|\vec{x}-\vec{x}'|}\vec{\nabla}\times\vec{M}(\vec{x}') - \vec{\nabla}'\times\cancelto{0}{\left(\frac{\vec{M}(\vec{x}')}{|\vec{x}-\vec{x}'|}\right)}\right] d^3x'$ (126)

We consider the ff. term:

$$\int \vec{\nabla}\times\left(\frac{\vec{M}(\vec{x}')}{|\vec{x}-\vec{x}'|}\right) d^3x' = \int \cancelto{0}{\frac{\vec{M}(\vec{x}')}{|\vec{x}-\vec{x}'|}} \times d\vec{S} \quad (127)$$

Assume that $\vec{M}$ is localized. We can expand the volume V to V' without any changes to the expression. Thus,



$$\int \vec{\nabla}\times\left(\frac{M(\vec{x}')}{|\vec{x}-\vec{x}'|}\right) d^3x' = \int \frac{\vec{M}(\vec{x}')}{|\vec{x}-\vec{x}'|} \times dS' = 0 \quad (128)$$

Therefore, we have

$$\int \frac{\vec{M}(\vec{x}') \times (\vec{x}-\vec{x}')}{|\vec{x}-\vec{x}'|} d^3x' = \int \frac{\vec{\nabla}'\times\vec{M}(\vec{x}')}{|\vec{x}-\vec{x}'|} d^3x' \quad (129)$$

The vector potential assumes the form

$$\vec{A}(\vec{x}) = \frac{\mu_0}{4\pi}\int \frac{\vec{J}(\vec{x}') + \vec{\nabla}\times M(\vec{x}')}{|\vec{x}-\vec{x}'|} d^3x' \quad (130)$$

The magnetization introduces the effective current density

$$\vec{J}_M = \vec{\nabla}\times\vec{M} \quad (131)$$

Recall: In the microscopic case, $\vec{\nabla}\times\vec{B}_{micro} = \mu_0 \vec{J}_{micro}$ (132)

For the macroscopic case, we now have

$$\vec{\nabla}\times\vec{B} = \mu_0\left[\vec{J} + \vec{\nabla}\times\vec{M}\right] \quad (133)$$

Combining $\vec{\nabla}\times\vec{M}$ term with $\vec{B}$,

magnetic field

$$\mu_0\vec{J} = \vec{\nabla}\times\vec{B} - \mu_0\vec{\nabla}\times\vec{M} = \mu_0\vec{\nabla}\times\overbrace{\left[\frac{\vec{B}}{\mu_0} - \vec{M}\right]}^{\vec{H}} = \mu_0\nabla\times\vec{H} \quad (134)$$

The magnetic field $\vec{H}$ is a macroscopic quantity as we saw here. Thus,

$$\nabla\times\vec{H} = \vec{J} \quad (135)$$

Eqs. (27) and (135) describe macroscopic magnetostatics.

- For isotropic and diamagnetic and paramagnetic substances, $\vec{B} = \mu\vec{H}$.

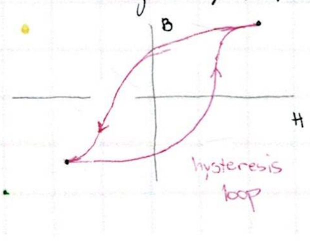
  magnetic permeability

  parameter characteristic of the medium

  $\mu > \mu_0$: paramagnetic substances

  $\mu < \mu_0$: diamagnetic substances

- For ferromagnets, $\vec{B} = \vec{F}(\vec{H})$
- Diamagnetic materials: they are repelled by magnetic fields
  : An applied magnetic field creates an induced magnetic field in in the opp. direction, causing
- Paramagnetic & Ferromagnetic materials: they are attracted by magnetic fields
-



The phenomenon of hysteresis implies that $\vec{B}$ is not a single-valued function of H

The function $\vec{F}(\vec{H})$ depends on the history of preparation of the materials

- The boundary conditions for $\vec{B}$ and $\vec{H}$ at the interface bet. two media are given by

$$\vec{B}_2 - \vec{B}_1 \cdot \hat{n} = 0, \qquad \hat{n} \times (\vec{H}_2 - \vec{H}_1) = \vec{K} \tag{136}$$

2 $\quad \vec{H}_2, \vec{B}_2 \quad \hat{n} \uparrow$ $\vec{K}$: the idealized surface current density

1 $\quad \vec{H}_1, \vec{B}_1$

For media satisfying linear relation of the form $\vec{B} = \mu \vec{H}$ and $\vec{K} = 0$, the b.c. are

$$\vec{B}_2 \cdot \hat{n} = \vec{B}_1 \cdot \hat{n}, \qquad \mu_1 \vec{B}_2 \times \hat{n} = \mu_2 \vec{B}_1 \times \hat{n} \tag{137}$$

or:

$$\vec{H}_2 \cdot \hat{n} = \frac{\mu_1}{\mu_0} \vec{H}_1 \cdot \hat{n}, \qquad \vec{H}_2 \times \hat{n} = \vec{H}_1 \times \hat{n} \tag{138}$$

5.9: Methods of Solving Boundary

The basic equations are $\vec{\nabla} \cdot \vec{B} = 0$ & $\vec{\nabla} \times \vec{M} = \vec{J}$ (Eqs. (27) and (135)) with some constitutive relations bet. $\vec{B}$ and $\vec{H}$

Ⓐ Generally Applicable Method of the Vector Potential

- Since $\vec{\nabla} \cdot \vec{B} = 0 \rightarrow \vec{B} = \vec{\nabla} \times \vec{A}$
- If $\vec{H} = \vec{H}(\vec{B})$, then the second eqn becomes $\nabla \times \vec{H}[\vec{\nabla} \times \vec{A}] = \vec{J}$ which is a complicated eqn to solve in general
- For linear media, $\vec{B} = \mu \vec{H}$. Thus, we have $\vec{\nabla} \times \left(\frac{1}{\mu} \vec{\nabla} \times \vec{A} \right) = \vec{J}$
    - In general: μ is a function of position
    - If μ is indep of position, we get $\vec{\nabla} \times (\vec{\nabla} \times \vec{A}) = \mu \vec{J}$.

      Note that under Coulomb gauge,

      $$\nabla \times (\nabla \times A) = \vec{\nabla}(\cancel{\vec{\nabla} \cdot \vec{A}}^{\,0}) - \vec{\nabla}^2 A = -\vec{\nabla}^2 \vec{A}$$

      Thus, $\vec{\nabla}^2 \vec{A} = -\mu \vec{J}$

      Given that $\vec{\nabla}^2 (1/|\vec{x} - \vec{x}'|) = -4\pi \delta(\vec{x} - \vec{x}')$, then

      $$\vec{A}(\vec{x}) = \frac{\mu}{4\pi} \int \frac{\vec{J}(\vec{x}')}{|\vec{x} - \vec{x}'|} d^3x'$$

May 11, 2021

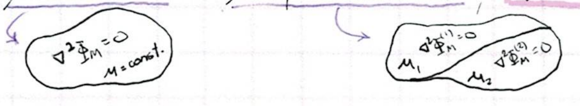
- Recall: The eqns to solve in magnetostatics are $\vec{\nabla} \cdot \vec{B}$ & $\vec{\nabla} \times \vec{M} = \vec{J}$ w/ the required b.c.

Ⓑ $\vec{J} = 0$; Magnetic Scalar Potential

- Solve by means of magnetic scalar potential Φ_M: $\nabla \times \vec{H} = 0 \longrightarrow \vec{H} = -\nabla \Phi_M$
- This is the same method employed in electrostatics: $\nabla \times E = 0 \longrightarrow E = -\nabla \Phi_E$
- Recall: $\vec{B}$ & $\vec{H}$ are related by $\vec{B} = \vec{B}(\vec{H})$ which leads to $\boxed{\nabla \cdot B[-\nabla \Phi_M] = 0}$
- Assume that the medium is linear: $\vec{B} = \mu \vec{H}$. Then,

$$\nabla \cdot (\mu \vec{H}) = \nabla \cdot [\mu(-\nabla \Phi_M)] = 0 \leftrightarrow \nabla \cdot (\mu \nabla \Phi_M) = 0 \tag{139}$$

For μ = constant or μ = piecewise constant, $\boxed{\mu \nabla^2 \Phi_M = 0}$



Ⓒ Hard Ferromagnets ($\vec{M}$ is given and $\vec{J} = 0$)

A ferromagnet is called "hard" if it has a magnetization ~~of~~ that is essentially indep. of applied fields for moderate field strength. Such mat'ls can be treated as if they have a fixed, specified magnetization $\vec{M}(\vec{x})$

C.1) Scalar potential

Since $J = 0$, the magnetic scalar field Φ_M can be employed

Recall: In the presence of magnetization M

$$\vec{B} = \mu_0 (\vec{H} + \vec{M}) \longrightarrow \vec{\nabla}\cdot\vec{B} = \mu_0 \nabla\cdot(\vec{H}+\vec{M}) = 0 \qquad (140)$$

Which gives $\boxed{\vec{\nabla}\cdot\vec{H} = -\vec{\nabla}\cdot\vec{M}}$. Then, with $\vec{H} = -\nabla\Phi_M$, $\nabla^2\Phi_M = \vec{\nabla}\cdot\vec{H}$.
Comparing this with $\nabla^2\Phi_E = -\rho$ in electrostatics, we infer that

$$\nabla^2\Phi_M = -\rho_M, \qquad \rho_M = -\vec{\nabla}\cdot\vec{M} \qquad (141)$$

Solving this Poisson-like equation

$$\Phi_M(\vec{x}) = -\frac{1}{4\pi}\int \frac{\nabla'\cdot\vec{M}(\vec{x}')}{|\vec{x}-\vec{x}'|}\, d^3x' \qquad (142)$$

Note that:
$$\vec{\nabla}'\left(\frac{1}{|\vec{x}-\vec{x}'|}\right) = -\vec{\nabla}\left(\frac{1}{|\vec{x}-\vec{x}'|}\right) \qquad (143)$$

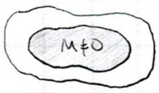
Also, we have:
$$\vec{\nabla}'\cdot\frac{M(\vec{x}')}{|\vec{x}-\vec{x}'|} = \frac{\vec{\nabla}'\cdot M(\vec{x}')}{|\vec{x}-\vec{x}'|} + \vec{M}(\vec{x}')\,\vec{\nabla}'\left(\frac{1}{|\vec{x}-\vec{x}'|}\right) \qquad (144)$$

Applying Eq. (143):
$$\vec{\nabla}'\cdot\frac{\vec{M}(\vec{x}')}{|\vec{x}-\vec{x}'|} = \frac{\vec{\nabla}'\cdot\vec{M}(\vec{x}')}{|\vec{x}-\vec{x}'|} - \vec{M}(\vec{x}')\,\vec{\nabla}\left(\frac{1}{|\vec{x}-\vec{x}'|}\right) \qquad (145)$$

Thus, substituting Eq. after rearranging it,

$$\int \frac{\nabla'\cdot\vec{M}(\vec{x})}{|\vec{x}-\vec{x}'|}\, d^3x' = \int \vec{\nabla}'\cdot\left(\frac{\vec{M}(\vec{x}')}{|\vec{x}-\vec{x}'|}\right) d^3x' + \int M(\vec{x})\,\nabla\left(\frac{1}{|\vec{x}-\vec{x}'|}\right) d^3x' \qquad (146)$$

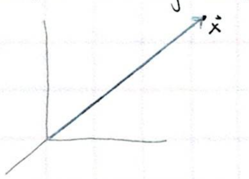
Assuming that M is localized:



$$\vec{M}=0 \Rightarrow \vec{\nabla}'\cdot\frac{\vec{M}(\vec{x}')}{|\vec{x}-\vec{x}'|} = 0$$

Therefore,
$$\Phi_M = -\frac{1}{4\pi}\vec{\nabla}\cdot\int \frac{\vec{M}(\vec{x}')}{|\vec{x}-\vec{x}'|}\, d^3x' \qquad (147)$$

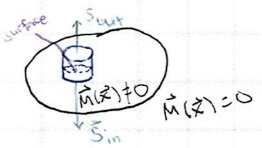
Far from the region of non-vanishing magnetization



$$\Phi_M \simeq -\frac{1}{4\pi}\nabla\left(\frac{1}{r}\right)\cdot\overbrace{\int \vec{M}(\vec{x}')\, d^3x'}^{4\pi\vec{m}}$$

$$= \frac{1}{r^2}\hat{r}\cdot\vec{m} = \frac{\vec{x}}{|\vec{x}|^3}\cdot\vec{m} = \frac{\vec{x}\cdot\vec{m}}{|\vec{x}|^3} \qquad (148)$$

In the case that there is a discontinuity present



Use divergence theorem on the pill-box:
$$\int_V \vec{\nabla}\cdot\vec{F}\, dV = \oint_S \vec{F}\cdot\hat{n}\, d\vec{S} \qquad (149)$$

Then,
$$\int \rho_M(\vec{x}')\, d^3x' = \int -\nabla\cdot\vec{M}\, d^3x' \qquad (150)$$

$$Q_M = -\int \vec{M}\cdot\hat{n}'\, dS'$$

$$= -\int_{S_{in}} \vec{M}\cdot\hat{n}\, d\vec{S} + \int_{side} \vec{M}\cdot\hat{n}'\, dS'$$

$$= \vec{M}\cdot\hat{n}\, S \qquad (151)$$

Thus,
$$\rho_M = \frac{Q_M}{S} = \vec{M}\cdot\hat{n} \qquad (152)$$

The discontinuity of $\vec{M}$ on the surface

Then, the scalar magnetic potential incorporating all possible contributions is given by

contribution from the variation of $\vec{M}$ inside the volume

$$\Phi_M(\vec{x}) = \left(-\frac{1}{4\pi}\int_V \frac{\vec{\nabla}'\cdot\vec{M}(\vec{x}')}{|\vec{x}-\vec{x}'|}\, d^3x'\right) + \underbrace{\frac{1}{4\pi}\oint_S \frac{\hat{n}'\cdot\vec{M}(\vec{x}')}{|\vec{x}-\vec{x}'|}\, da'}_{\text{contribution from the discontinuity of } \vec{M} \text{ across the surface of the medium}} \qquad (153)$$

May 13, 2024 ▶ Recall:

$$\vec{H} = \vec{B}/\mu_0 - \vec{M} \qquad \vec{\nabla}\cdot\vec{B} = 0 \rightarrow \vec{B} = \vec{\nabla}\times\vec{A} \qquad \vec{\nabla}\times\vec{H} = \vec{J} \rightarrow \vec{\nabla}\times\vec{H} = 0$$

(154) Then, $$\nabla \times H = \vec{\nabla}\times\left(\frac{\vec{B}}{\mu_0} - \vec{M}\right) = \frac{1}{\mu_0}\vec{\nabla}\times\vec{B} - \vec{\nabla}\times\vec{M} = \frac{1}{\mu_0}\vec{\nabla}\times\vec{\nabla}\times\vec{A} - \vec{\nabla}\times\vec{M}$$

(155) Then, $$\nabla\times H = \frac{1}{\mu_0}\left[\vec{\nabla}(\vec{\nabla}\cdot\vec{A}) - \vec{\nabla}^2\vec{A}\right] - \vec{\nabla}\times\vec{M} = -\frac{1}{\mu_0}\vec{\nabla}^2\vec{A} - \vec{\nabla}\times\vec{M} = 0$$

Thus, $$\vec{\nabla}^2 A = -\mu_0 \vec{\nabla}\times\vec{M} \qquad (156)$$

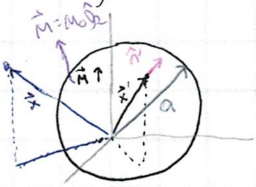
In the absence of surfaces, note that: $$\vec{A}(\vec{x}) = \frac{\mu_0}{4\pi}\int \frac{\vec{\nabla}'\times\vec{M}(\vec{x}')}{|\vec{x}-\vec{x}'|}d^3x' \qquad (157)$$

If the distribution of magnetization is discontinuous, it is necessary to add a surface integral:

$$A(\vec{x}) = \frac{\mu_0}{4\pi}\int_V \frac{\vec{\nabla}'\times\vec{M}(\vec{x}')}{|\vec{x}-\vec{x}'|}d^3x' + \underbrace{\frac{\mu_0}{4\pi}\oint_S \frac{\vec{M}(\vec{x}')\times\hat{n}'}{|\vec{x}-\vec{x}'|}da'}_{\text{contribution from the discontinuity}} \qquad (158)$$

▶ 5.10: Uniformly Magnetized Sphere

Consider a sphere of radius a with a uniform permanent magnetization $\vec{M}$ of magnitude M_0 and parallel to the z-axis embeded in a non



Obtain $\vec{H}$ and $\vec{B}$ inside & outside of the sphere. Note that $\vec{J} = 0$

Using the method

$$\Phi_M(\vec{x}) = -\frac{1}{4\pi}\int_V \cancel{\frac{\overbrace{\vec{\nabla}'\cdot\vec{M}(\vec{x}')}^{\rightarrow 0 \text{ since } M \text{ is constant}}}{|\vec{x}-\vec{x}'|}} d^3x' + \frac{1}{4\pi}\oint_S \frac{\hat{n}'\cdot\vec{M}(\vec{x}')}{|\vec{x}-\vec{x}'|}da' \qquad (159)$$

Thus, $$\Phi_M(\vec{x}) = \frac{M_0 a^2}{4\pi}\int_{\substack{\text{over all}\\ \text{directions}}} d\Omega' \frac{\cos\theta'}{|\vec{x}-\vec{x}'|} \qquad (160)$$

Use the expansion: $$\frac{1}{|\vec{x}-\vec{x}'|} = \sum_{l=0}^{\infty}\frac{r_<^l}{r_>^{l+1}}P_l(\cos\gamma) \qquad (161)$$

where $r_<$ [$r_>$] is the smaller [larger] of $|\vec{x}|$ and $|\vec{x}'|$ and γ is the angle between $\vec{x}$ & $\vec{x}'$.

$$\frac{1}{|\vec{x}-\vec{x}'|} = 4\pi\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{1}{2l+1}\frac{r_<^l}{r_>^{l+1}}Y^*_{lm}(\theta',\phi')Y_{lm}(\theta,\phi) \qquad (162)$$

Recall: $$\int_{\substack{\text{over all}\\ \text{space}}} d\Omega\, Y^*_{l'm'}(\theta,\phi)Y_{lm}(\theta,\phi) = \delta_{l'l}\delta_{m'm} \qquad (163)$$

Also, $$Y_{10} = \sqrt{\frac{3}{4\pi}}\cos\theta' \rightarrow \cos\theta' \propto Y_{10} \qquad (164)$$

Then, we have $$\Phi_M(\vec{x}) = \frac{M_0 a^2}{\cancel{4\pi}}\int d\Omega \times \cancel{4\pi}\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{1}{2l+1}\frac{r_<^l}{r_>^{l+1}}Y^*Y \times \cos\theta' \qquad (165)$$

Only $l=1$ & $m=0$ contributes. Thus,

$$\Phi_M(\vec{x}) = \frac{M_0 a^2}{3}\sqrt{\frac{4\pi}{3}}\frac{r_<}{r_>^2}\underbrace{Y_{10}(\theta,\phi)}_{=\sqrt{\frac{3}{4\pi}}\cos\theta} = \frac{M_0 a^2}{3}\frac{r_<}{r_>^2}\cos\theta \qquad (166)$$

• Inside the sphere: $r_< = |\vec{x}| = r$, $r_> = |\vec{x}'| = a$

$$\Phi_M(\vec{x}) = \frac{1}{3}M_0 a^2\frac{\overbrace{r\cos\theta}^{=z}}{a^2} = \frac{M_0}{3}z \qquad (167)$$

Then, $$\vec{H}_{in} = -\vec{\nabla}\Phi_{M_{in}} = -\vec{\nabla}\left(\frac{M_0 z}{3}\right) = -\frac{1}{3}M_0\hat{z} \qquad (168)$$

$\vec{H}_{in}$ is constant and opposite to the magnetization

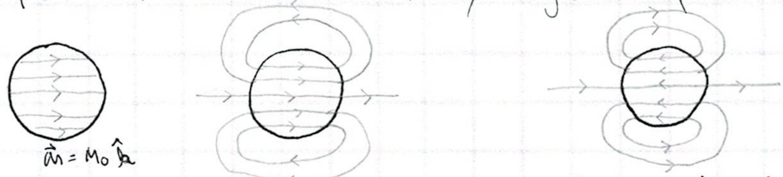
Also, $B_{in} = \mu_0 (H_{in} + M) = \mu_0 \left(-\frac{1}{3} M + M \right) = \frac{2}{3} \mu_0 M$ $\vec{B}_{in}$ is const. & parallel w/ M (169)

Outside the sphere : $r_< = a$, $r_> = r$

$$\Phi_M(\vec{x}) = \frac{1}{3} M_0 \frac{a^3 \cos\theta}{r^2} \qquad (170)$$

This is the potential due to a dipole w/ dipole moment $\vec{m} = 4\pi a^3 \frac{\vec{M}}{3}$

We illustrate the lines of $\vec{B}$ & $\vec{H}$ for a uniformly magnetized sphere

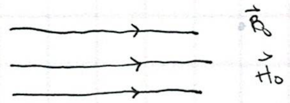


The lines of $\vec{B}$ are closed curves in accordance w/ $\vec{\nabla} \cdot \vec{B} = 0$

The lines of $\vec{B}$ originate on the surface of the sphere where the effective surface magnetic "charge" σ_M resides

May 20, 2021 ▸ 5.11 : Magnetic Shielding

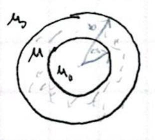
Suppose that a certain magnetic induction $\vec{B}_0 = \mu_0 \vec{H}_0$ exists in a region of empty space. A permeable body is now placed in the region. Let the body be a spherical sphere of inner (outer) radius a (b), made of mat'l of permeability μ



We wish to find the fields $\vec{B}$ and H everywhere in space as fxns of μ.

The introduction of permeable mat'l will modify magnetic induction density and magnetic field due to the induced magnetization

$\vec{J} = 0$ assuming that there are no free charges



Since $\vec{J} = 0$,

$$\vec{\nabla} \times \vec{H} = 0 \Rightarrow \vec{H} = -\vec{\nabla} \Phi_M \qquad (171)$$

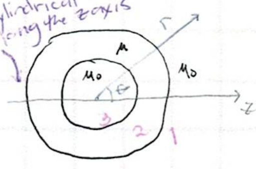
Also, $\vec{\nabla} \cdot \vec{B} = 0 \longrightarrow \vec{\nabla} \cdot [-\mu \nabla \Phi_M] = 0$ (172)

Assuming that the material is linear, this is valid. This means that $\vec{B} = \mu \vec{H}$. Evaluating Eq. in the region where μ is constant, we have

$$\vec{\nabla} \cdot [-\mu \vec{\nabla} \Phi_M] = \vec{\nabla}^2 \Phi_M = 0 \qquad (173)$$

Then, the problem reduces to solving the Laplace equation with the specified boundary conditions

There is cylindrical symmetry along the z axis



The general soln of the Laplace equation for a system with cylindrical symmetry about the z-axis is given by

$$\Phi_M(r,\theta) = \sum_{l=0}^{\infty} \left[A_l r^l + B_l r^{-(l+1)} \right] P_l(\cos\theta) \qquad (174)$$

- Solution in the outer region (Region 1):

Note that $\Phi_M^{(1)} \to -H_0 z$ as $r \to \infty$ or $z = r\cos\theta \to \infty$ so that as $z \to \infty$,

$$H = -\nabla \Phi_M^{(1)} \to -\nabla(-H_0 z) = (\partial_x \hat{i} + \partial_y \hat{j} + \partial_z \hat{k}) H_0 z = H_0 \hat{z} = \vec{H}_0 \qquad (175)$$

Then, $\Phi_M^{(1)} = A_0 P_0(\cos\theta) + \underbrace{A_1 r P_1(\cos\theta)}_{= A_1 r\cos\theta = A_1 z} + A_2 r^2 P_2(\cos\theta) + \dots$

$$+ B_0 r^{-1} P_0(\cos\theta) + B_1 r^{-2} P_1(\cos\theta) + \dots \qquad (176)$$

From this, we infer that $A_0 = A_2 = A_3 = \dots = 0$

Thus, $$\Phi_M^{(1)}(r,\theta) = A_1 r\cos\theta + \sum_{l=0}^{\infty} B_l\, r^{-(l+1)} P_l(\cos\theta) \quad (177)$$

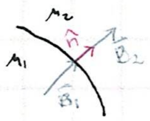
Then, $$\Phi_M^{(1)}(r,\theta) \to H_0 r\cos\theta, \quad r\to\infty$$
$$\Rightarrow A_1 = -H_0 \quad (178)$$

Therefore, $$\Phi_M^{(1)} = -H_0 r\cos\theta + \sum_{l=0}^{\infty} {}_l\, r^{-(l+1)} P_l(\cos\theta) \quad (179)$$

- Solution at the region inside the material (Region 2):
$$\Phi_M^{(2)}(r,\theta) = \sum_{l=0}^{\infty}\left(\beta_l r^l + \gamma_l \frac{1}{r^{l+1}}\right)P_l(\cos\theta) \quad (180)$$

Solution in the hollow region (Region 3):
$$\Phi_M^{(3)}(r,\theta) = \sum_{l=0}^{\infty} \delta_l\, r^l P_l(\cos\theta) \quad (181)$$

- In the absence of current, the boundary conditions are



a) $\vec{B}_2\cdot\hat{n} = \vec{B}_1\cdot\hat{n}$ b) $\mu_2 \vec{H}_2\cdot\hat{n} = \mu_1\vec{H}_1\cdot\hat{n}$ (182)

c) $\mu_1\vec{B}_2\times\hat{n} = \mu_2\vec{B}_1\times\hat{n}$ d) $\vec{H}_2\times\hat{n} = \vec{H}_1\times\hat{n}$

Applying these conditions:

the radial part is continuous

a) $\vec{B}_2\cdot\hat{r} = \vec{B}_1\cdot\hat{r} \to B_{2,r} = B_{1,r}$ (183)

b) $\mu_2\vec{H}_2\cdot\hat{r} = \mu_1\vec{H}_1\cdot\hat{r} \to \mu_2 H_{2,r} = \mu_1 H_{1,r}$

the radial part is NOT continuous

With the assumption that the mat'l is linear:
$$B_{2,r} = \mu_2 H_{2,r} \qquad B_{1,r} = \mu_1 H_{1,r} \quad (184)$$

Recall: $$\vec{H} = -\vec{\nabla}\Phi_M$$
$$= -\left(\frac{\partial}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial}{\partial\theta}\hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\hat{\phi}\right)\Phi_M \quad (185)$$

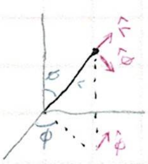
For the radial part: $H_r = -\partial_r\Phi_M$

Then, $$B_{2,r} = \mu_2 H_{2,r} = -\mu_2\frac{\partial}{\partial r}\Phi_{M_2} \quad (186)$$

$$B_{1,r} = \mu_1 H_{1,r} = -\mu_1\frac{\partial}{\partial r}\Phi_{M_1} \quad (187)$$

Using b.c. a), $$\cancel{-}\mu_2\frac{\partial\Phi_{M_2}}{\partial r} = \cancel{-}\mu_1\frac{\partial\Phi_{M_1}}{\partial r} \quad (188)$$

The b.c. b) in Eq. (183) translates to the same boundary condition



For the other conditions:

c) $\mu_1\vec{B}_2\times\hat{r} = \mu_2\vec{B}_1\times\hat{r} \longrightarrow \cancel{-}\mu_1 B_{2,\theta}\cancel{\hat{\phi}} = \cancel{-}\mu_2 B_{1,\theta}\cancel{\hat{\phi}}$

0 due to cylindrical symmetry

Note that $$\vec{B}_2\times\hat{r} = (B_{2,r}\hat{r} + B_{2,\theta}\hat{\theta} + B_{2,\phi}\hat{\phi})\times\hat{r}$$
$$= B_{2,\theta}\hat{\theta}\times\hat{r}$$
$$\vec{B}_2\times\hat{r} = B_{2,\theta}\hat{\phi} \quad (189)$$

With the given assumption,
$$B_{2,\theta} = \mu_2 H_{2,\theta} = -\mu_2 r^{-1}\partial_\theta\Phi_{M_2} \quad (190)$$
$$B_{1,\theta} = \mu_1 H_{1,\theta} = -\mu_1 r^{-1}\partial_\theta\Phi_{M_1}$$

Using b.c. c) in Eq. (183)
$$\mu_1(-\mu_2)\cancel{\frac{1}{r}}\frac{\partial\Phi_{M_2}}{\partial\theta} = \mu_2(-\mu_1)\cancel{\frac{1}{r}}\frac{\partial\Phi_{M_1}}{\partial\theta}$$
$$\partial_\theta\Phi_{M_2} = \partial_\theta\Phi_{M_1} \quad (191)$$

The b.c. d) leads to the same bo

To summarize,

A) $\left.\frac{\partial \Phi_{M_2}}{\partial \theta}\right|_{r=\rho} = \left.\frac{\partial \Phi_{M_1}}{\partial \theta}\right|_{r=\rho}$ B) $\mu_2 \left.\frac{\partial \Phi_{M_2}}{\partial r}\right|_{r=\rho} = \mu_1 \left.\frac{\partial \Phi_{M_1}}{\partial r}\right|_{r=\rho}$ (192)

Applying these boundary conditions.

A) $\left.\frac{\partial \Phi_M^{(1)[(3)]}}{\partial \theta}\right|_{r=b\,[a]} = \left.\frac{\partial \Phi_M^{(2)}}{\partial \theta}\right|_{r=b\,[a]}$ B) $\mu_0 \left.\frac{\partial \Phi_M^{(1)[(3)]}}{\partial r}\right|_{r=b\,[a]} = \mu \left.\frac{\partial \Phi_M^{(2)}}{\partial r}\right|_{r=b\,[a]}$ (193)

May 25, 2021 ▸ All coefficients $(\alpha_\ell, \beta_\ell, \gamma_\ell, \delta_\ell)$ with $l \neq 1$ vanish after imposing b.c. while the $l=1$ coeff.'s satisfy the four simultaneous equations

$$\left.\begin{aligned}
&\alpha_1 - b^3\beta_1 - \gamma_1 = b^3 H_0\\
&2\alpha_1 + \mu' b^3 H_1 - 2\mu'\gamma_1 = -b^3 H_0\\
&a^3\beta_1 + \gamma_1 - a^3\delta_1 = 0\\
&\mu' a^3\beta_1 - 2\mu'\gamma_1 - a^3\delta_1 = 0
\end{aligned}\right\} \quad (194)$$

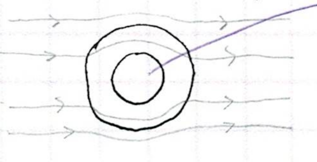
- We find that

$$\left.\begin{aligned}
\alpha_1 &= \frac{(2\mu+1)(\mu'-1)}{(2\mu'+\mu'+2) - \frac{2a^3}{b^3}(\mu'-1)^2}(b^3-a^3)H_0\\
\delta_1 &= -\frac{9\mu'}{(2\mu'+1)(\mu'+\;\;-2\frac{a^3}{b^3}(\mu'-1)^2}H_0
\end{aligned}\right\} \quad (195)$$

For $\mu \gg \mu_0$ or $\mu' = \mu/\mu_0 \gg \Delta$,

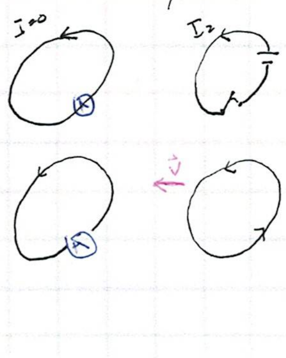
$$\alpha_1 \sim b^3 H_0$$
$$\delta_1 \sim -\frac{1}{2\mu(1-a^3/b^3)} \quad \text{if } \mu_0 \qquad (196)$$

For large μ, δ_1 is significantly smaller than the other non-vanishing coeff.'s, $\delta_1 \ll 1$. This implies that $\Phi_M^{(3)}$ is negligible. $\vec{H}$ and $\vec{B}$ are consequently negligible inside the hollow part of the sphere



magnetic field is negligible inside sphere ▸ This is the concept of magnetic shielding

▸ 5.15: Faraday's Law of Induction



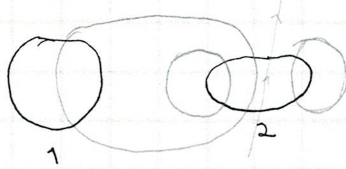
Faraday observed that a transient current is induced in a circuit if:

(a) the steady current in an adjacent circuit is turned on/off

(b) the adj. circuit w/ a steady current flowing is moved relative to the first circuit

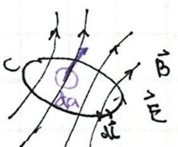
(c) A permanent magnet is thrust into/out of the circuit

- Note that no current flows when either the adj current changes or there is relative motion
- Faraday attributes the transient flow of current to a magnetic flux (changing) linked by the circuit



The changing magnetic flux induces an electric field around the circuit, the line integral of w/c is called the electromotive force, $\mathcal{E}$

This $\mathcal{E}$ causes a current flow acc. to Ohm's Law



The flux through the circuit C is given by

$$F = \int_S \vec{B} \cdot \hat{n} \, da \tag{197}$$

S: the surface bounded by the circuit C

n: normal to the surface S

The flux does not depend on the specifics of the surface bounded by C.

The electromotive force around the circuit is

$$\mathcal{E} = \oint_C \vec{E}' \cdot d\vec{l} \tag{198}$$

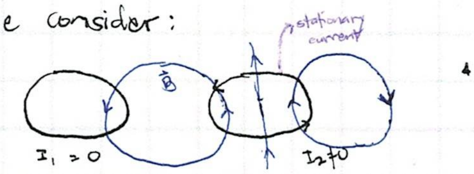
- Faraday's law

This is dictated

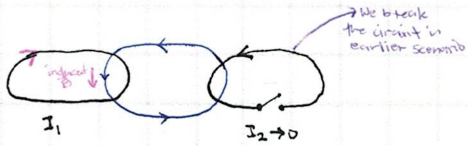
$$\mathcal{E} = -k \frac{dF}{dt} \tag{199}$$

This is Lenz's law, w/c states that the induced current (and accompanying magnetic flux) is in such a direction as to oppose the change of flux thru the circuit

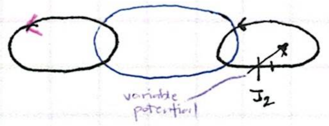
- We consider:



- There would be no induced current in circuit 1



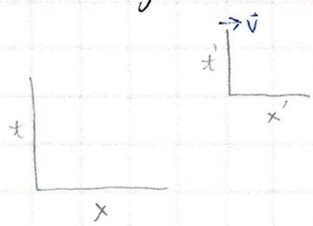
- There is a decrease in the mag. flux through circuit 1



- We can write Faraday's law as

$$\oint_C \vec{E}' \cdot d\vec{l} = -\frac{d}{dt} \int_C \vec{B} \cdot \hat{n} \, da \tag{200}$$

This law was formulated before special theory of relativity was introduced by Einstein. It is thought to be invariant under the Galilean transformation.



Galilean relativity: $t = t'$ (time is abs.)

$$x' = x - vt \tag{201}$$

However, for relativistic speeds, we should introduce a correction for this law.

- Let $A = A(x(t), y(t), z(t), t)$. Then,

$$\frac{dA}{dt} = \frac{\partial A}{\partial t}\frac{dt}{dt} + \frac{dx}{dt}\frac{\partial A}{\partial x} + \frac{dy}{dt}\frac{\partial A}{\partial y} + \frac{dz}{dt}\frac{\partial A}{\partial z}$$

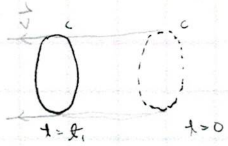
$$= \frac{\partial A}{\partial t} + v_x \frac{\partial A}{\partial x} + v_y \frac{\partial A}{\partial y} + v_z \frac{\partial A}{\partial z}$$

Takes into account the intrinsic time dep. of A

- $$\frac{dA}{dt} = \frac{\partial A}{\partial t} + \vec{v} \cdot \vec{\nabla} A \tag{202}$$

Takes into account the change in A due to motion

convective derivative



Then, we have

$$\frac{d}{dt}\int_S \vec{B}\cdot\hat{u}\,da = \int_S \frac{d}{dt}(\vec{B}\cdot\hat{u})\,da$$

$$= \int_S\left[\frac{\partial(\vec{B}\cdot\hat{u})}{\partial t} + v\cdot\nabla(\vec{B}\cdot\hat{u})\right]da$$

$$= \int_S\left[\frac{\partial\vec{B}}{\partial t} + \vec{v}\cdot\nabla\vec{B}\right]\underbrace{d\vec{a}}_{=\hat{n}\,da} \tag{203}$$

Using the identity:

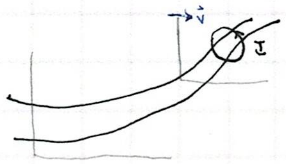
$$(\vec{v}\cdot\vec{\nabla})\vec{B} = \vec{\nabla}\times(\vec{B}\times\vec{v}) + \vec{v}\,\cancel{(\vec{\nabla}\cdot\vec{B})}^{\,=0\text{ for}} \tag{204}$$

Thus, $\displaystyle \frac{d}{dt}\int_S \vec{B}\cdot d\vec{a} = \int_S \frac{\partial\vec{B}}{\partial t}\cdot d\vec{a} + \int_C \vec{\nabla}\times(\vec{B}\times\vec{v})\cdot d\vec{a}$

$$= \int_S \frac{\partial\vec{B}}{\partial t}\cdot d\vec{a} + \oint_C (\vec{B}\times\vec{v})\cdot d\vec{S} \tag{205}$$

Therefore, $$\oint_C \vec{E}'\cdot d\vec{l} = \int_S \frac{\partial\vec{B}}{\partial t}\cdot d\vec{a} + \oint_C(\vec{B}\times\vec{v})\cdot d\vec{S} \tag{206}$$

Arranging this, $$\oint_C\left[\overbrace{\vec{E}'}^{\vec{E}} - \vec{v}\times\vec{B}\right]\cdot d\vec{l} = -\int_S \frac{\partial\vec{B}}{\partial t}\cdot d\vec{S} \tag{207}$$



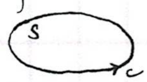
$\vec{E}$: the electric field acc. to the lab frame

$\vec{v}$: velocity of the circuit wrt the lab frame

Note that $\vec{B}(t)$ is wrt the lab frame. In the lab frame, Faraday's law is

$$\oint_C \vec{E}\cdot d\vec{l} = -\int \frac{\partial\vec{B}}{\partial t}\cdot d\vec{a} \tag{208}$$

May 27, 2021 ▸ Using Stoke's Theorem,



$$\oint_C \vec{E}\cdot d\vec{l} = \int_S (\vec{\nabla}\times\vec{E})\cdot d\vec{a} \tag{209}$$

Then, $$\int_S(\vec{\nabla}\times\vec{E})\cdot d\vec{a} = -\int_S \frac{\partial B}{\partial t}\cdot da$$

$$\Rightarrow \int_S\left(\vec{\nabla}\times\vec{E} + \frac{\partial\vec{B}}{\partial t}\right)\cdot d\vec{a} = 0 \tag{210}$$

The circuit C is arbitrary and the focused surface S is likewise arbitrary, the integral must vanish at all points in space:

$$\nabla\times E + \frac{\partial B}{\partial t} = 0 \tag{211}$$

This is the time-dependent generalization of the statement $\vec{\nabla}\times\vec{E}=0$ for electrostatics

▸ 5.18 : Quasi-Static Magnetic Fields

- If the magnetic induction varies in time, an electric field is created, acc. to Faraday's law
  → the situation is nolonger magnetic in character
- If the two is not too rapid, the magnetic fields dominate and the situation is called quasi-static
- "Quasi-static" refers to the regime for w/c the finite speed of light can be neglected and fields treated as it may propagate instantaneously

Here, the relevant equations are

$$\vec{\nabla}\times\vec{H} = \vec{J} \qquad \vec{\nabla}\cdot\vec{B} = 0 \qquad \vec{\nabla}\times\vec{E} + \frac{\partial\vec{B}}{\partial t} = 0 \tag{213}$$

For conducting media, $\vec{J} = \sigma E$ where σ is the conductivity of the conductor.

With $\nabla\cdot B = 0 \rightarrow B = \nabla\times A$, Faraday's Law can be written as

$$\vec{\nabla}\times\vec{E} + \frac{\partial\vec{B}}{\partial t} = \vec{\nabla}\times\vec{E} + \frac{\partial}{\partial t}\vec{\nabla}\times\vec{A}$$

$$= \vec{\nabla}\times\vec{E} + \vec{\nabla}\times\frac{\partial\vec{A}}{\partial t}$$

$$= \vec{\nabla}\times\left(\vec{E} + \frac{\partial\vec{A}}{\partial t} + \vec{\nabla}\Phi\right) = 0 \tag{214}$$

This implies :

$$\underbrace{\vec{E}}_{\text{total }\vec{E}\text{ induced}} = \underbrace{-\frac{\partial\vec{A}}{\partial t}}_{\vec{E}\text{ due to moving }\vec{B}} \underbrace{-\vec{\nabla}\Phi}_{\vec{E}\text{ due to free charges}} \tag{215}$$

Assume that the electric field is due to varying magnetic field $\rightarrow \Phi = 0$

$$\vec{E} = \frac{\partial\vec{A}}{\partial t}, \quad \text{w/o free charges} \tag{216}$$

No free charges implies $\vec{\nabla}\cdot\vec{E} = 0$.

$$\vec{\nabla}\cdot\vec{E} = \vec{\nabla}\cdot -\frac{\partial\vec{A}}{\partial t} = -\frac{\partial}{\partial t}(\vec{\nabla}\cdot\vec{A}) = 0 \tag{217}$$

For media of uniform frequency-indep. permeability μ, Ampere's law can be written as:

$$\nabla\times H = \vec{J} \rightarrow \mu(\nabla\times H) = \nabla\times\mu H = \mu J \tag{218}$$

This implies :

$$\vec{\nabla}\times\vec{B} = \mu\vec{J} \tag{219}$$

For conductors $(\vec{J} = \sigma\vec{E})$, we get :

$$\vec{\nabla}\times\vec{B} = \mu\sigma\vec{E} \tag{220}$$

Eliminating $\vec{B}$ and $\vec{E}$ in favor of $\vec{A}$,

$$\nabla\times B = \nabla\times\nabla\times A = \nabla\cancelto{0}{(\nabla\cdot A)} - \nabla^2 A \tag{221}$$

$$\vec{E} = -\partial_t A \tag{222}$$

Thus,

$$\nabla^2 A = \mu\sigma\frac{\partial A}{\partial t} \tag{223}$$

diffusion eqn

This holds for spatially varying but frequency-indep. σ

From Faraday's law,

$$\vec{\nabla}\times\vec{E} + \frac{\partial\vec{B}}{\partial t} = 0 \rightarrow \vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}$$

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{E}) = -\frac{\partial}{\partial t}(\vec{\nabla}\times\vec{B}) = -\frac{\partial}{\partial t}(\mu\vec{J}) = -\frac{\partial}{\partial t}(\mu\sigma\vec{E}) \tag{224}$$

Then,

$$\vec{\nabla}\cancelto{0}{(\vec{\nabla}\cdot\vec{E})} - \vec{\nabla}^2\vec{E} = -\mu\sigma\frac{\partial\vec{E}}{\partial t} \Rightarrow \nabla^2\vec{E} = \mu\sigma\frac{\partial\vec{E}}{\partial t} \tag{225}$$

The electric field satisfies the diffusion eqn.

How about $\vec{J}$? Again, from the Faraday's law,

$$\nabla\times E = -\frac{\partial B}{\partial t} \longrightarrow \nabla\times\nabla\times E = -\frac{\partial}{\partial t}\nabla\times B$$

$$= -\frac{\partial}{\partial t}\mu\vec{J}$$

$$= -\mu\frac{\partial\vec{J}}{\partial t} \tag{226}$$

Assuming that σ is uniform, we multiply both sides by σ and get

$$\vec{\nabla} \times \vec{\nabla} \times (\sigma \vec{E}) = - \mu \sigma \frac{\partial \vec{J}}{\partial t}$$

$$\vec{\nabla} \times \vec{\nabla} \times \vec{J} =$$

$$\vec{\nabla}(\vec{\nabla} \cdot \vec{J}) - \nabla^2 \vec{J} = - \mu \sigma \frac{\partial \vec{J}}{\partial t} \qquad (227)$$

Note that $\nabla \cdot \vec{J} = \vec{J} = \sigma E = \sigma \nabla \cdot E$. Thus, we get

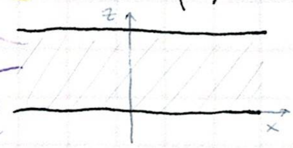
$$\nabla^2 \vec{J} = \mu \sigma \vec{J} \qquad (228)$$

Therefore, $\vec{J}$ satisfies the diffusion eqn

A Skin Depth, Eddy Currents, Induction Heating

semi-infinite conductor

no current on surface



The surface at $z=0$ is subjected to a spatially constant but time varying magnetic field in the x-direction: $H_x(t) = H_0 \cos(\omega t)$

We wish to obtain a steady state soln to $\nabla^2 \vec{B} = \mu \sigma \partial_t \vec{B}$.

Boundary conditions:

→ continuity of the tangential component of $\vec{H}$: $\vec{H}_1 \times \hat{n} = \vec{H}_2 \times \hat{n}$

→ continuity of the normal component of $\vec{B}$: $\vec{B}_1 \cdot \hat{n} = \vec{B}_2 \cdot \hat{n}$

Note that $\vec{\nabla}^2 (\mu \vec{B}) = \mu \sigma \frac{\partial}{\partial t} (\mu \vec{B})$, ⟶ $\vec{\nabla}^2 \vec{H} = \mu \sigma \frac{\partial \vec{H}}{\partial t}$ (229)

we have:

In component form:

$$\nabla^2 H_x^{[y][z]}(x, y, z) = \mu \sigma \frac{\partial H_x^{[y][z]}(xyz)}{\partial t} \qquad (230)$$

The differential equation is linear.

→ H will only have x - component b/c the initial is only along x-axis

→ In the same manner, $H_y = H_z = 0$ for all t

Also, since H_x at $t=0$ is indep. of x and y at $z=0$, the will only depend on z and t → $H_x = H_x(z, t)$

Let $H_x(0, t) = e^{-i\omega t} H_0$ where physical quantities are the real parts.

Assume a relation of the form $H_x(z, t) = h(z) e^{-i\omega t}$

Then, $\nabla^2 H_x = \mu \sigma \frac{\partial H_x}{\partial t}$ → $\frac{d^2 h}{dz^2} e^{-i\omega t} = \mu \sigma h(z)(-i\omega) e^{-i\omega t}$ (231)

and we have:

$$\left(\frac{d^2}{dz^2} + i \mu \sigma \omega\right) h(z) = 0 \qquad (232)$$

A solution is given by $h(z) = e^{ikz}$ [an ansatz]. Then, Eq. becomes

$$-k^2 + i \mu \sigma \omega = 0 \quad \rightarrow \quad k^2 = i \mu \sigma \omega$$

Thus, $k = \pm (1+i) \sqrt{\frac{\delta}{2}}$, w/ $\delta = \sqrt{\frac{2}{\mu \sigma \omega}}$ (233)

The general soln is

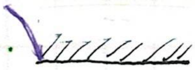
$$H_x(z, t) = \alpha e^{-z/\delta} e^{i(z/\delta - \omega t)} + \beta e^{z/\delta} e^{-i(z/\delta + \omega t)} \qquad (234)$$

For H_x to be finite as $z \to 0$, $\beta = 0$. Thus,

$$H_x(z, t) = \alpha e^{-z/\delta} e^{i(z/\delta - \omega t)} \qquad (235)$$

Hx should be constant across the surface

Note that:



$$H_x(t, t)_{z \to 0} = \alpha e^{-i\omega t} = H_0 e^{-i\omega t} \qquad (236)$$

which implies: $A = H_0$

Then, the solution in the region $z>0$ is

$$H_x(z,t) = H_0 e^{-z/\delta} e^{i(z/\delta - \omega t)} \quad (237)$$

Take the real part

$$H_x(z,t) = H_0 e^{-z/\delta} \cos(z/\delta - \omega t) \quad (238)$$