Dynamics in an invariant manifold of an interacting dark matter and scalar field system

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Abstract

We construct a cosmological model in a Friedman-Robertson-Walker universe where dark matter and a dynamically evolving scalar field are coupled by a given interaction term. Through a dynamical systems approach, we investigate the behavior of the kinetic and potential components of the scalar field in this model with the absence of dark matter by considering an invariant manifold that pertains to this scenario. We show that the system in this invariant manifold can be analogous to a mechanical system, provided that the Hubble parameter H is constant, and that this analogy breaks down when $H = H(t)$ due to the system becoming non-autonomous in nature.

Keywords: Dynamical systems, invariant manifold, scalar field

1 Introduction

Despite our advancements in modern cosmology, there are still plenty of phenomena in our Universe that lack a firm basis in fundamental theory [1, 2]. Dark energy and dark matter are among these. To understand them, we construct mathematical models that capture their observed behavior, and for dark energy, this often involves a dynamical scalar field ϕ in a potential $V(\phi)$. A dynamical scalar bypasses problems that arise for models that use a cosmological constant Λ for dark energy [3]. A coupling between the said dark components in the manner of an arbitrary interaction term Q is also proposed by some models in the hopes of better understanding the nature of these components and resolving existing cosmological tensions [4, 5].

Understanding these models is often aided by the use of dynamical systems methods in which differential equations that characterize the temporal evolution of the necessary variables are analyzed through linear stability theory and graphically depicted in phase plots [6]. Such a method is advantageous when we want to infer the long-term behaviour of the system without having to solve the equations. Though these models tend to be complicated, the presence of invariant manifolds often lead to simplification and significant insight. These invariant manifolds are subregions of the state space that the flow of the dynamical system maps onto itself. An easy concrete example is when, according the differential equations, the rate of change of some variable becomes zero when the variable itself is zero [7]. In this case, one is able to ignore that variable and focus on the dynamics of the remaining variables.

In this paper, we will first derive the differential equations that will make up our general dynamical system, after which we will focus on an invariant manifold to simplify it. We will then analyze the system's stability for the case where the Hubble parameter is either time dependent or not. Finally, we will discuss the behavior of the components in this invariant manifold for both cases as supplemented by phase plots.

2 Dynamics of a coupled dark matter and scalar field system

Distances in a universe that is homogeneous and isotropic over large scales are characterized by the Robertson-Walker metric

$$\mathrm{d}s^2 = -c^2\mathrm{d}t^2 + a^2(t)\left[\frac{\mathrm{d}r^2}{1-Kr^2} + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2\right)\right] \tag{2.1}$$

where K is the curvature constant and $a(t)$ is a dimensionless scale factor [2]. In supposing that dark matter and the scalar field be the only relevant components in this universe of general curvature, along with the assumption that these components follow the behavior of perfect fluids, this metric would result to the Friedmann equations from Einstein's field equations in the form of

$$H^2 = \frac{\kappa^2}{3}\rho_{\text{tot}} - \frac{K}{a^2} \tag{2.2}$$

$$\frac{\ddot{a}}{a} = -\frac{\kappa^2}{6}\left(\rho_{\text{tot}} + 3P_{\text{tot}}\right) \tag{2.3}$$

where $\kappa^2 = 8\pi$, $H = \dot{a}/a$ is the Hubble parameter, $\rho_{\text{tot}} = \rho_{\text{d}} + \rho_\phi$ and $P_{\text{tot}} = P_{\text{d}} + P_\phi$. Note that ρ_{d} and P_{d} pertain to dark matter's energy density and pressure, respectively, while ρ_ϕ and P_ϕ pertain to that of the scalar field which are given by [3]

$$\rho_\phi = \frac{1}{2}\dot{\phi}^2 + V(\phi), \qquad P_\phi = \frac{1}{2}\dot{\phi}^2 - V(\phi) \tag{2.4}$$

Also, an energy density is related to its corresponding pressure by its equation of state (EOS), $P = w\rho$, where w is a dimensionless EOS parameter. Now, we consider a coupling Q between these two components. Their conservation equations are given by

$$\dot{\rho}_{\text{d}} = -3H\rho_{\text{d}}\left(1 + w_{\text{d}}\right) + Q \tag{2.5}$$

$$\ddot{\phi} = -\frac{\mathrm{d}V(\phi)}{\mathrm{d}\phi} - 3H\dot{\phi} - \frac{Q}{\dot{\phi}} \tag{2.6}$$

We also define new dimensionless variables that represent the relative abundances of the universe's components as follows [3]

$$x^2 = \frac{\kappa^2\dot{\phi}^2}{6H^2}, \qquad y^2 = \frac{\kappa^2 V(\phi)}{3H^2}, \qquad s^2 = \frac{\kappa^2\rho_d}{3H^2}, \qquad \Omega_K = -\frac{K}{a^2H^2} \tag{2.7}$$

which, along with Eq. (2.4), leads in Eq. (2.2) to be expressed as $1 = s^2 + x^2 + y^2 + \Omega_K$. Also, the interaction term, which we chose to be $Q = (\kappa^2/3H)\epsilon\rho_\phi\rho_d$ [8] for relating dark matter-scalar field interaction in terms of the abundance of these components and simplifying the derivation of the dynamical systems later on, becomes

$$Q = \frac{3H^3}{\kappa^2}\epsilon\left[x^2 + y^2\right]s^2 \tag{2.8}$$

where ϵ is a positive constant. Notice that the kinetic and potential components of ϕ, in which the latter will be set as $V(\phi) = V_0 e^{-\kappa\lambda\phi}$ [3] in this paper, are related to x^2 and y^2, respectively. In constructing our dynamical system, it will be convenient to obtain the following relation from the time derivative of the Hubble parameter ($\dot{H} = (\dot{a}/a) - H^2$),

$$\frac{\dot{H}}{H^2} = -\left(1 + q\right) \tag{2.9}$$

where the deceleration parameter q is given by

$$q = -\frac{\ddot{a}}{aH^2} = -\frac{\ddot{a}a}{\dot{a}^2} = 2x^2 - y^2 + \frac{1}{2}s^2 \tag{2.10}$$

in terms of the defined variables. To introduce an equivalent but shorter time scale compared to the cosmic time t and easier analysis in the dynamics of the system, it would be useful to identify a new dimensionless time variable $\eta = \ln\left(a/a_0\right)$ where a_0 could be the present-day value of the scale factor. For an arbitrary quantity X, a prime derivative $X' = \mathrm{d}X/\mathrm{d}\eta$ is related to its corresponding dot derivative $\dot{X} = \mathrm{d}X/\mathrm{d}t$ by $X' = \dot{X}/H$. Taking the dot derivatives of the variables in Eq. (2.7) and recasting them into their prime derivatives using this relation, we obtain the following set of equations

$$\begin{cases} x' = \sqrt{\dfrac{3}{2}}\lambda y^2 - 3x + x(1+q) - \dfrac{1}{2x}\epsilon\left(x^2 + y^2\right)s^2 \\ y' = -\sqrt{\dfrac{3}{2}}\lambda yx + y\left(1+q\right) \\ s' = s\left[\dfrac{1}{2}\epsilon\left(x^2+y^2\right) - \dfrac{1}{2} + q\right] \end{cases} \tag{2.11}$$

which becomes the dynamical system of the state variables (x,y,s) that portrays the interaction between dark matter and the scalar field under the given Q (Eq. (2.8)). Now, notice that the plane defined by $s = 0$ (*i.e.* no dark matter is present in the universe) is an invariant manifold as this leads to $s' = 0$ which means that any trajectory with initial conditions lying in this plane will remain in it forever. Considering this manifold, the system in Eq. (2.11) simplifies to

$$\begin{cases} x' = \sqrt{\dfrac{3}{2}}\lambda y^2 - 3x + x(1+q) \\ y' = -\sqrt{\dfrac{3}{2}}\lambda yx + y\left(1+q\right) \end{cases} \tag{2.12}$$

where $q = 2x^2 - y^2$ in this case. As there is no dark matter for the scalar field to interact with, Q becomes zero which results to this system (Eq. (2.12)) being ϵ-independent.

2.1 Stability analysis of general case: $H = H(t)$

For this scenario, the dynamical system is exactly given in Eq. (2.12). By setting $x' = 0 = y'$, we can determine its fixed points (x_0, y_0) and analyze their stability from the eigenvalues of their corresponding Jacobian matrix. From these, the characterization of its fixed points are then given in Table 1 where $X_1 = \lambda^{-1}\sqrt{2/3}$, $Y_1 = 2/(\sqrt{3}\lambda)$, $X_2 = \lambda/\sqrt{6}$, $Y_2 = \sqrt{(6-\lambda^2)/6}$, and $\sigma = \lambda^{-2}\sqrt{8\lambda^2 - 3\lambda^4}$.

Table 1: Fixed points' stability analysis of the system in Eq. (2.12)

(x_0, y_0)	Existence	Eigenvalues	Stability
$(-1, 0)$	$\forall\lambda$	$4,\ \frac{1}{2}(6+\sqrt{6}\lambda)$	Repulsor (unstable node) for $\lambda > -\sqrt{6}$ Saddle point for $\lambda < -\sqrt{6}$
$(0, 0)$	$\forall\lambda$	$-2,\ 1$	Saddle point $\forall\lambda$
$(1, 0)$	$\forall\lambda$	$4,\ \frac{1}{2}(6-\sqrt{6}\lambda)$	Repulsor (unstable node) for $\lambda < \sqrt{6}$ Saddle point for $\lambda > \sqrt{6}$
$(X_1, -Y_1)$ (X_1, Y_1)	$\forall\lambda$	$-1-\sigma,\ -1+\sigma$	Attractor (stable spiral) for $\lambda < -2\sqrt{\frac{2}{3}}$ and $\lambda > 2\sqrt{\frac{2}{3}}$ Attractor (stable node) for $-2\sqrt{\frac{2}{3}} \leq \lambda < -\sqrt{2}$ and $\sqrt{2} < \lambda \leq 2\sqrt{\frac{2}{3}}$ Saddle point for $-\sqrt{2} < \lambda < \sqrt{2}$
$(X_2, -Y_2)$ (X_2, Y_2)	$-\sqrt{6} \leq \lambda \leq \sqrt{6}$	$\frac{1}{2}(-6+\lambda^2)$, $-2+\lambda^2$	Attractor (stable node) for $-\sqrt{2} < \lambda < \sqrt{2}$ Saddle point for $-\sqrt{6} < \lambda < -\sqrt{2}$ and $\sqrt{2} < \lambda < \sqrt{6}$

Note that all fixed points except $(0, 0)$ have a corresponding eigenvalue that can become zero depending on whether $\lambda = \pm\sqrt{6}$ or $\lambda = \pm\sqrt{2}$. Linear stability theory would not be enough to completely determine the stability of a fixed point with a zero eigenvalue so we can look into the system's phase plot [9] at these values of λ where we find the points to be a repulsor or attractor depending on the sign of its non-zero eigenvalue.

2.2 Stability analysis of specific case: $H =$ constant

For this scenario, notice that Eq. (2.9) becomes zero which turns the dynamical system of the said invariant manifold in Eq. (2.12) into

$$\begin{cases} x' = \sqrt{\frac{3}{2}}\lambda y^2 - 3x \\ y' = -\sqrt{\frac{3}{2}}\lambda y x \end{cases} \tag{2.13}$$

This system has one fixed point at $(x_0, y_0) = (0, 0)$ with corresponding eigenvalues of -3 and 0. Since an eigenvalue is zero, we must again look into the phase plot to determine this point's stability which we find to be an attracting fixed line along the y-axis when $\lambda = 0$ and a stable node otherwise.

3 Discussion

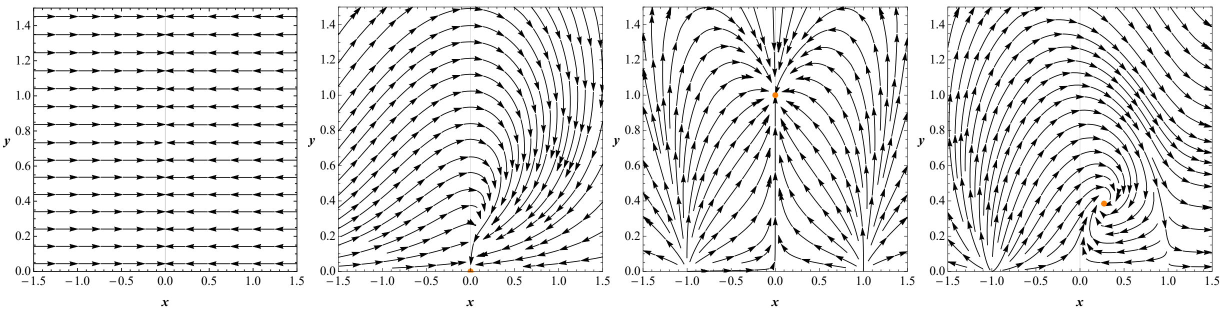
Recall Eq. (2.6) which is also the temporal equation of evolution for the scalar field. In the absence of dark matter, Q vanishes which turns it into an equation that is analogous to the motion of a particle in 1D with mass m and displacement x that is affected by a frictional or damping term and by the force due to the potential $V(x)$. This is expressed as

$$\ddot{x} = -\frac{1}{m}\frac{\mathrm{d}V(x)}{\mathrm{d}x} - \frac{\alpha}{m}\dot{x} \tag{3.1}$$

which is applicable when this force is conservative (*i.e.* the total work done to move it between two points is path-independent) in which $F(x) = -\mathrm{d}V(x)/\mathrm{d}x$. With α as a constant, Eq. (3.1) physically translates to a situation where any motion of the particle with potential $V(x)$ is counteracted by friction due to

$-(\alpha/m)\dot{x}$. Thus, we can say that $-3H\dot{\phi}$ is kind of a friction term on the scalar field. Considering the similarities to this mechanical system, the state of kinetic and potential energies of such a particle that we usually picture as a ball rolling around the given potential should be reflected by the behavior of the system in the considered invariant manifold which we can observe using phase plots.

Let us consider sample values of λ. For the constant H case as shown in Figures 1a and 1b, any initial kinetic component eventually disappears when $\lambda = 0$ as the potential stays the same while an initial negative kinetic component with some potential when $\lambda = 3$ would first have an increase in its kinetic and potential values before vanishing. Noting that we have an exponential potential and applying the rolling ball analogy, this is the expected behaviour as the potential becomes flat that is set at a value determined by V_0 when $\lambda = 0$ with the friction term decreasing the speed (*i.e.* the kinetic component) of the ball until it stops. On the other hand, negative kinetic values when $\lambda = 3$ mean that the ball would roll up the potential before rolling down and stopping due to friction. However, this is not what we see in the $H(t)$ case as shown in Figures 1c and 1d. This is due to the fact that the mechanical system analogy is only viable when the Hubble parameter H is constant as Eq. (2.6) becomes non-autonomous (*i.e.* no longer explicitly independent of the independent variable which in this case is time) with a time-varying $H(t)$.



(a) $\lambda = 0,\ H = constant$ (b) $\lambda = 3,\ H = constant$ (c) $\lambda = 0,\ H = H(t)$ (d) $\lambda = 3,\ H = H(t)$

Figure 1: Phase plots of the systems in Eqs. (2.13) and (2.12) (Note that negative values of y and λ does not give new information on the behavior of these plots). Orange points represent stable points.

4 Conclusion and recommendations

We have analyzed the dynamics of an invariant manifold in interacting dark matter and scalar field. We showed that the behavior of the dynamical system in this invariant manifold can be analogous to a mechanical system, provided that the Hubble parameter H is constant. We also showed that this analogy breaks down when $H = H(t)$ due to the system becoming non-autonomous in nature. As an extension of this study, one can try to find a non-autonomous mechanical system that would be analogous to the time-varying $H(t)$ case.

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