Problem 1 (4.2)

A point dipole with dipole moment $\mathbf{p}$ is located at the point $\mathbf{x}_0$. From the properties of the derivative of a Dirac delta function, show that for calculation of the potential Φ or the energy of a dipole in an external field, the dipole can be described by an effective charge density

$$\rho_{\text{eff}}(\mathbf{x}) = -\mathbf{p} \cdot \boldsymbol{\nabla}\delta\left(\mathbf{x}-\mathbf{x}_0\right) \tag{1}$$

Solution:
In general, the potential is

$$\Phi(\mathbf{x}) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho\left(\mathbf{x}'\right)}{|\mathbf{x}-\mathbf{x}'|}d^3x' \tag{2}$$

Note that the potential for the dipole is of the form

$$\Phi_{\text{dipole}}(\mathbf{x}) = \frac{1}{4\pi\epsilon_0}\frac{\mathbf{p}\cdot\mathbf{x}}{r^3} = \frac{1}{4\pi\epsilon_0}\mathbf{p}\cdot\frac{\mathbf{x}}{|\mathbf{x}|^3} \tag{3}$$

when the dipole is centered at the origin (Eq. 4.10 of Jackson). If it is centered at some point x_0, we have

$$\Phi_{\text{dipole}}(\mathbf{x}) = \frac{1}{4\pi\epsilon_0}\mathbf{p}\cdot\frac{\mathbf{x}-\mathbf{x}_0}{|\mathbf{x}-\mathbf{x}_0|^3} \tag{4}$$

Also, note that

$$\begin{aligned}
\boldsymbol{\nabla}_{\mathbf{x}_0}\left(\frac{1}{|\mathbf{x}-\mathbf{x}_0|}\right) &= \frac{\partial}{\partial x_0}\left(\frac{1}{\sqrt{(x-x_0)^2+(y-y_0)^2}}\right)\hat{\imath} + \frac{\partial}{\partial y_0}\left(\frac{1}{\sqrt{(x-x_0)^2+(y-y_0)^2}}\right)\hat{\jmath} \\
&= -\frac{1}{2}\frac{2(x-x_0)\cdot(-1)}{\left(\sqrt{(x-x_0)^2+(y-y_0)^2}\right)^3}\hat{\imath} - \frac{1}{2}\frac{2(y-y_0)\cdot(-1)}{\left(\sqrt{(x-x_0)^2+(y-y_0)^2}\right)^3}\hat{\jmath} \\
&= \frac{(x-x_0)\hat{\imath}+(y-y_0)\hat{\jmath}}{\left(\sqrt{(x-x_0)^2+(y-y_0)^2}\right)^3} \\
\boldsymbol{\nabla}_{\mathbf{x}_0}\left(\frac{1}{|\mathbf{x}-\mathbf{x}_0|}\right) &= \frac{\mathbf{x}-\mathbf{x}_0}{|\mathbf{x}-\mathbf{x}_0|^3}
\end{aligned} \tag{5}$$

The relation in Eq. (4) is equivalent to

$$\Phi_{\text{dipole}}(\mathbf{x}) = \frac{1}{4\pi\epsilon_0}\int \mathbf{p}\cdot\delta(\mathbf{x}'-\mathbf{x}_0)\frac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|^3}\mathrm{d}^3x' = \frac{1}{4\pi\epsilon_0}\mathbf{p}\cdot\int \delta(\mathbf{x}'-\mathbf{x}_0)\boldsymbol{\nabla}_{\mathbf{x}'}\left(\frac{1}{|\mathbf{x}-\mathbf{x}'|}\right)\mathrm{d}^3x' \tag{6}$$

after applying Eq. (5). Now, we note that the derivative of a Dirac delta function has the property:

$$\int f(x)\delta^{(n)}(x)\mathrm{d}x = -\int \frac{\partial f}{\partial x}\delta^{(n-1)}(x)\mathrm{d}x \tag{7}$$

Thus, we have

$$\int \delta(\mathbf{x}'-\mathbf{x}_0)\boldsymbol{\nabla}_{\mathbf{x}'}\left(\frac{1}{|\mathbf{x}-\mathbf{x}'|}\right)\mathrm{d}^3x' = -\int \frac{1}{|\mathbf{x}-\mathbf{x}'|}\boldsymbol{\nabla}_{\mathbf{x}'}\delta(\mathbf{x}'-\mathbf{x}_0)\mathrm{d}^3x' \tag{8}$$

which we substitute into Eq. (6) to obtain

$$\Phi_{\text{dipole}}(\mathbf{x}) = -\frac{1}{4\pi\epsilon_0}\int \frac{\mathbf{p}\cdot\boldsymbol{\nabla}_{\mathbf{x}'}\delta(\mathbf{x}'-\mathbf{x}_0)}{|\mathbf{x}-\mathbf{x}'|}\mathrm{d}^3x' = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}\mathrm{d}^3x' \tag{9}$$

Notice here that we were able to go back to the general form of the potential given in Eq. (2) where $\rho(\mathbf{x}') = -\mathbf{p}\cdot\boldsymbol{\nabla}_{\mathbf{x}'}\delta(\mathbf{x}'-\mathbf{x}_0)$. Replacing the primed variables with unprimed variables, we get

$$\rho(\mathbf{x}) = -\mathbf{p}\cdot\boldsymbol{\nabla}_{\mathbf{x}}\delta(\mathbf{x}-\mathbf{x}_0) \tag{10}$$

which is just the effective charge density for the dipole given in Eq. (1).
As for the energy of the dipole in an external field, we have

$$W_{\text{dipole}} = -\mathbf{p}\cdot\mathbf{E}(0) \tag{11}$$

when it is centered at the origin (Eq. 4.24 of Jackson). When it is centered at some point $\mathbf{x}_0$, we get

$$W_{\text{dipole}} = -\mathbf{p}\cdot\mathbf{E}(\mathbf{x}_0) \tag{12}$$

Recall that $\mathbf{E} = -\boldsymbol{\nabla}\Phi$. Then, in the same manner as the potential, Eq. (12) is equivalent to

$$W_{\text{dipole}} = \mathbf{p}\cdot\boldsymbol{\nabla}\Phi(\mathbf{x}_0) = \mathbf{p}\cdot\int \delta(\mathbf{x}'-\mathbf{x}_0)\boldsymbol{\nabla}\Phi(\mathbf{x}')\mathrm{d}^3x' \tag{13}$$

Applying again the property of the Dirac delta function's derivative in Eq. (7), we obtain

$$\int \delta(\mathbf{x}'-\mathbf{x}_0)\boldsymbol{\nabla}\Phi(\mathbf{x}')\mathrm{d}^3x' = -\int \Phi(\mathbf{x}')\boldsymbol{\nabla}\delta(\mathbf{x}'-\mathbf{x}_0)\mathrm{d}^3x' \tag{14}$$

Then, Eq. (13) becomes

$$W_{\text{dipole}} = -\mathbf{p}\cdot\int \Phi(\mathbf{x}')\boldsymbol{\nabla}\delta(\mathbf{x}'-\mathbf{x}_0)\mathrm{d}^3x' = \int \Phi(\mathbf{x}')\rho(\mathbf{x}')\mathrm{d}^3x' \tag{15}$$

which is similar to the form of the energy of the system generally given by

$$W = \int \rho(\mathbf{x}')\Phi(\mathbf{x}')\mathrm{d}^3x' \tag{16}$$

only if the charge density is given by Eq. (1). With the potential and the energy of the dipole mimicking their general form if we use the given effective charge density, we have shown that this charge density describes the dipole.

Problem 2 (4.5)

A localized charge density $\rho(x, y, z)$ is placed in an external electrostatic field described by a potential $\Phi^{(0)}(x, y, z)$. The external potential varies slowly in space over a region where the charge density is different from zero.

(a) From first principles calculate the total force acting on the charge distribution as an expansion in multipole moments times derivatives of the electric field, up to and including the quadrupole moments. Show that the force is

$$\mathbf{F} = q\mathbf{E}^{(0)}(0) + \left\{\nabla\left[\mathbf{p}\cdot\mathbf{E}^{(0)}(\mathbf{x})\right]\right\}_0 + \left\{\nabla\left[\frac{1}{6}\sum_{j,k} Q_{jk}\frac{\partial E_j^{(0)}}{\partial x_k}(\mathbf{x})\right]\right\}_0 + \cdots \tag{1}$$

Compare this to the expansion (4.24) of the energy W. Note that (4.24) is a number – it is not a function of x that can be differentiated! What is its connection to F?

Solution:
First, recall that a function expanded about the origin using Taylor series is written as

$$f(\mathbf{x}) = f(\mathbf{x}')\big|_{\mathbf{x}'=0} + \frac{1}{1!}\mathbf{x}f'(\mathbf{x}')\bigg|_{\mathbf{x}'=0} + \frac{1}{2!}\mathbf{x}^2 f''(\mathbf{x}')\bigg|_{\mathbf{x}'=0} + \dots \tag{2}$$

where the primed variables indicate variables affected by the differentiation. Applying this to the electric field related to the given potential by

$$\mathbf{E}^{(0)} = -\nabla\Phi^{(0)}, \tag{3}$$

the expansion of $\mathbf{E}^{(0)}$ is

$$\begin{aligned}\mathbf{E}^{(0)}(\mathbf{x}) = \sum_i E_i^{(0)}(\mathbf{x}')\Big|_{\mathbf{x}'=0}\hat{\mathbf{x}}_i + \sum_i\sum_j x_j\left(\frac{\partial E_i^{(0)}(\mathbf{x}')}{\partial x_j'}\right)_{\mathbf{x}'=0}\hat{\mathbf{x}}_i \\ + \frac{1}{2}\sum_i\sum_{j,k} x_j x_k\left(\frac{\partial^2 E_i^{(0)}(\mathbf{x}')}{\partial x_j'\,\partial x_k'}\right)_{\mathbf{x}'=0}\hat{\mathbf{x}}_i + \dots\end{aligned} \tag{4}$$

Note that the force acting on the charge distribution is related to this electric field by

$$\mathbf{F} = q\mathbf{E}^{(0)} \tag{5}$$

where $q = \int \rho(x, y, z)\mathrm{d}^3x = \int \rho(\mathbf{x})\mathrm{d}^3x$ is the total charge over the region where the charge

Angeles, Janna May Z.
2015-09948

density is non-zero. Substituting Eq. (4) into Eq. (5), we have

$$
\begin{aligned}
\mathbf{F} &= \left[\int \rho(\mathbf{x})\mathrm{d}^3x\right] \sum_i E_i^{(0)}(\mathbf{x}')\Big|_{\mathbf{x}'=0} \hat{\mathbf{x}}_i \\
&\quad + \left[\int \rho(\mathbf{x})\mathrm{d}^3x\right] \sum_i \sum_j x_j \left(\frac{\partial E_i^{(0)}(\mathbf{x}')}{\partial x'_j}\right)_{\mathbf{x}'=0} \hat{\mathbf{x}}_i \\
&\quad + \frac{1}{2}\left[\int \rho(\mathbf{x})\mathrm{d}^3x\right] \sum_i \sum_{j,k} x_j x_k \left(\frac{\partial^2 E_i^{(0)}(\mathbf{x}')}{\partial x'_j\,\partial x'_k}\right)_{\mathbf{x}'=0} \hat{\mathbf{x}}_i + \ldots \\
\mathbf{F} &= q\,\mathbf{E}^{(0)}(\mathbf{x}')\Big|_{\mathbf{x}'=0} + \sum_i \sum_j \left[\int x_j \rho(\mathbf{x})\mathrm{d}^3x\right] \left(\frac{\partial E_i^{(0)}(\mathbf{x}')}{\partial x'_j}\right)_{\mathbf{x}'=0} \hat{\mathbf{x}}_i \\
&\quad + \frac{1}{2} \sum_i \sum_{j,k} \left[\int x_j x_k \rho(\mathbf{x})\mathrm{d}^3x\right] \left(\frac{\partial^2 E_i^{(0)}(\mathbf{x}')}{\partial x'_j\,\partial x'_k}\right)_{\mathbf{x}'=0} \hat{\mathbf{x}}_i + \ldots
\end{aligned}
\tag{6}
$$

after rearranging some terms. With no magnetic fields involved, the curl of E leads to

$$
\nabla \times \mathbf{E} = 0 \longrightarrow \cancelto{1}{\epsilon_{kij}} \frac{\partial E_j}{\partial x_i} = 0 = \cancelto{-1}{\epsilon_{kji}} \frac{\partial E_i}{\partial x_j} \longrightarrow \frac{\partial E_j}{\partial x_i} = \frac{\partial E_i}{\partial x_j}
\tag{7}
$$

Also, note that $Q_{ij} = \int \left(3x'_i x'_j - r'^2 \delta_{ij}\right) \rho\left(\mathbf{x}'\right) \mathrm{d}^3x'$, the traceless quadrupole moment tensor, can be rearranged to

$$
\int x'_i x'_j \rho\left(\mathbf{x}'\right) \mathrm{d}^3x' = \frac{1}{3}\left(Q_{ij} + \int r'^2 \delta_{ij} \rho\left(\mathbf{x}'\right) \mathrm{d}^3x'\right)
\tag{8}
$$

Applying the relation Eq. (7) to Eq. (6) and substituting in Eq. (8), we have

$$
\begin{aligned}
\mathbf{F} &= q\,\mathbf{E}^{(0)}(\mathbf{x}')\Big|_{\mathbf{x}'=0} + \sum_i \sum_j \left[\int x_j \rho(\mathbf{x})\mathrm{d}^3x\right] \left(\frac{\partial E_j^{(0)}(\mathbf{x}')}{\partial x'_i}\right)_{\mathbf{x}'=0} \hat{\mathbf{x}}_i \\
&\quad + \frac{1}{2} \sum_i \sum_{j,k} \left[\int x_j x_k \rho(\mathbf{x})\mathrm{d}^3x\right] \left(\frac{\partial^2 E_j^{(0)}(\mathbf{x}')}{\partial x'_i\,\partial x'_k}\right)_{\mathbf{x}'=0} \hat{\mathbf{x}}_i + \ldots \\
&= q\,\mathbf{E}^{(0)}(\mathbf{x}')\Big|_{\mathbf{x}'=0} + \sum_i \sum_j \left[\int x_j \rho(\mathbf{x})\mathrm{d}^3x\right] \left(\frac{\partial E_j^{(0)}(\mathbf{x}')}{\partial x'_i}\right)_{\mathbf{x}'=0} \hat{\mathbf{x}}_i \\
&\quad + \frac{1}{2} \sum_i \sum_{j,k} \frac{1}{3}\left[Q_{jk} + \int r'^2 \delta_{jk} \rho\left(\mathbf{x}'\right) \mathrm{d}^3x'\right] \left(\frac{\partial^2 E_j^{(0)}(\mathbf{x}')}{\partial x'_i\,\partial x'_k}\right)_{\mathbf{x}'=0} \hat{\mathbf{x}}_i + \ldots \\
\mathbf{F} &= q\,\mathbf{E}^{(0)}(\mathbf{x}')\Big|_{\mathbf{x}'=0} + \left\{\sum_i \frac{\partial}{\partial x'_i}\hat{\mathbf{x}}_i \sum_j \left[\int x_j \rho(\mathbf{x})\mathrm{d}^3x\right] E_j^{(0)}(\mathbf{x}')\right\}_{\mathbf{x}'=0} \\
&\quad + \frac{1}{6}\left\{\sum_i \frac{\partial}{\partial x'_i}\hat{\mathbf{x}}_i \left[\sum_{j,k} Q_{jk} \frac{\partial E_j^{(0)}(\mathbf{x}')}{\partial x'_k} + \left(\int r^2 \rho\left(\mathbf{x}\right) \mathrm{d}^3x\right) \cancelto{0}{\frac{\partial E_j^{(0)}(\mathbf{x}')}{\partial x'_j}}\right]\right\}_{\mathbf{x}'=0} + \ldots
\end{aligned}
\tag{9}
$$

in which we also applied

$$\nabla \cdot \mathbf{E} = 0 \longrightarrow \frac{\partial E_j}{\partial x_j} = 0 \tag{10}$$

since the source/s of the external electric field is outside the region of interest (*i.e.* where $\rho(x, y, z) = 0$). Now, notice that in index notation

$$\nabla = \sum_i \frac{\partial}{\partial x_i} \hat{\mathbf{x}}_i \tag{11}$$

Also, we have the electric dipole moment given as

$$\mathbf{p} = \int \mathbf{x}' \rho\left(\mathbf{x}'\right) d^3x' \tag{12}$$

Thus, we get

$$\mathbf{F} = q\mathbf{E}^{(0)}(0) + \left\{\nabla\left[\mathbf{p} \cdot \mathbf{E}^{(0)}(\mathbf{x}')\right]\right\}_{\mathbf{x}'=0} + \left\{\nabla\left[\frac{1}{6}\sum_{j,k} Q_{jk} \frac{\partial E_j^{(0)}(\mathbf{x}')}{\partial x'_k}\right]\right\}_{\mathbf{x}'=0} + \ldots \tag{13}$$

which is just Eq. (1).

We can use the relation in Eq. (3) to rewrite Eq. (13) as

$$\begin{aligned}\mathbf{F} &= -q\nabla\Phi^{(0)}(0) + \left\{\nabla\left[\mathbf{p} \cdot \mathbf{E}^{(0)}(\mathbf{x}')\right]\right\}_{\mathbf{x}'=0} + \left\{\nabla\left[\frac{1}{6}\sum_{j,k} Q_{jk} \frac{\partial E_j^{(0)}(\mathbf{x}')}{\partial x'_k}\right]\right\}_{\mathbf{x}'=0} + \ldots \\ &= -\nabla\left[q\Phi^{(0)}(\mathbf{x}') - \mathbf{p} \cdot \mathbf{E}^{(0)}(\mathbf{x}') - \frac{1}{6}\sum_{j,k} Q_{jk} \frac{\partial E_j^{(0)}(\mathbf{x}')}{\partial x'_k} + \ldots\right]_{\mathbf{x}'=0}\end{aligned} \tag{14}$$

If we compare this with the form of the electrostatic energy in Eq. (4.24) of Jackson

$$W = q\Phi(0) - \mathbf{p} \cdot \mathbf{E}(0) - \frac{1}{6}\sum_i \sum_j Q_{tj} \frac{\partial E_j}{\partial x_i}(0) + \ldots, \tag{15}$$

we will notice that we can obtain the following relation:

$$F = -\left.\nabla W\right|_0 \quad \longleftrightarrow \quad W = -\left.\int F \cdot \mathrm{d}\mathbf{x}\right|_0 \tag{16}$$

(b) Repeat the calculation of part a for the total torque. For simplicity, evaluate only one Cartesian component of the torque, say N_1. Show that this component is:

$$N_1 = \left[\mathbf{p} \times \mathbf{E}^{(0)}(0)\right]_1 + \frac{1}{3}\left[\frac{\partial}{\partial x_3}\left(\sum_j Q_{2j} E_j^{(0)}\right) - \frac{\partial}{\partial x_2}\left(\sum_j Q_{3j} E_j^{(0)}\right)\right]_0 + \cdots \tag{17}$$

Solution:

The torque can be calculated using

$$N = \mathbf{x} \times \mathbf{F} = \mathbf{x} \times q\mathbf{E} \tag{18}$$

In index notation, we can write this out as

$$N_i = (\mathbf{x} \times q\mathbf{E})_i = q\epsilon_{ijk} x_j E_k \tag{19}$$

Thus, for N_1, we have

$$N_1 = q\epsilon_{123} x_2 E_3 + q\epsilon_{132} x_3 E_2 = q\left(x_2 E_3 - x_3 E_2\right) = \int \rho(\mathbf{x})\left(x_2 E_3 - x_3 E_2\right) \mathrm{d}^3 x \tag{20}$$

Note that

$$E_2^{(0)}(\mathbf{x}) = E_2^{(0)}(\mathbf{x}')\Big|_{\mathbf{x}'=0} + \sum_j x_j \left(\frac{\partial E_2^{(0)}(\mathbf{x}')}{\partial x'_j}\right)_{\mathbf{x}'=0} + \dots \tag{21}$$

and

$$E_3^{(0)}(\mathbf{x}) = E_3^{(0)}(\mathbf{x}')\Big|_{\mathbf{x}'=0} + \sum_j x_j \left(\frac{\partial E_3^{(0)}(\mathbf{x}')}{\partial x'_j}\right)_{\mathbf{x}'=0} + \dots \tag{22}$$

which we can infer from Eq. (4). Substituting these in Eq. (20),

$$\begin{aligned} N_1 = &\left[\left(\int x_2 \rho(\mathbf{x}) \mathrm{d}^3 x\right) E_3^{(0)}(\mathbf{x}')\Big|_{\mathbf{x}'=0} + \sum_j \left(\int x_2 x_j \rho(\mathbf{x}) \mathrm{d}^3 x\right)\left(\frac{\partial E_3^{(0)}(\mathbf{x}')}{\partial x'_j}\right)_{\mathbf{x}'=0} + \dots\right] \\ &- \left[\left(\int x_3 \rho(\mathbf{x}) \mathrm{d}^3 x\right) E_2^{(0)}(\mathbf{x}')\Big|_{\mathbf{x}'=0} + \sum_j \left(\int x_3 x_j \rho(\mathbf{x}) \mathrm{d}^3 x\right)\left(\frac{\partial E_2^{(0)}(\mathbf{x}')}{\partial x'_j}\right)_{\mathbf{x}'=0} + \dots\right] \end{aligned} \tag{23}$$

Using the relation in Eqs. (8) and (12),

$$\begin{aligned} N_1 &= p_2 E_3^{(0)}(0) - p_3 E_2^{(0)}(0) + \sum_j \frac{1}{3}\left[Q_{2j} + \int r^2 \delta_{2j} \rho(\mathbf{x}) \mathrm{d}^3 x\right]\left(\frac{\partial E_3^{(0)}(\mathbf{x}')}{\partial x'_j}\right)_{\mathbf{x}'=0} \\ &\quad - \left[Q_{3j} + \int r^2 \delta_{3j} \rho(\mathbf{x}) \mathrm{d}^3 x\right]\left(\frac{\partial E_2^{(0)}(\mathbf{x}')}{\partial x'_j}\right)_{\mathbf{x}'=0} + \dots \\ &= \left[\mathbf{p} \times \mathbf{E}^{(0)}(0)\right]_1 + \frac{1}{3}\sum_j \left[Q_{2j}\left(\frac{\partial E_3^{(0)}(\mathbf{x}')}{\partial x'_j}\right)_{\mathbf{x}'=0} - Q_{3j}\left(\frac{\partial E_2^{(0)}(\mathbf{x}')}{\partial x'_j}\right)_{\mathbf{x}'=0}\right] \\ &\quad + \frac{1}{3}\sum_j \int r^2 \rho(\mathbf{x}) \cancelto{0}{\left[\left(\frac{\partial E_3^{(0)}(\mathbf{x}')}{\partial x'_2}\right)_{\mathbf{x}'=0} - \left(\frac{\partial E_2^{(0)}(\mathbf{x}')}{\partial x'_3}\right)_{\mathbf{x}'=0}\right]} \mathrm{d}^3 x + \dots \end{aligned} \tag{24}$$

where the last term vanishes due to the relation in Eq. (7). Then, by rearranging the some of the terms, we obtain

$$\begin{aligned} N_1 &= \left[\mathbf{p} \times \mathbf{E}^{(0)}(0)\right]_1 + \frac{1}{3}\sum_j \left[Q_{2j}\left(\frac{\partial E_j^{(0)}(\mathbf{x}')}{\partial x'_3}\right)_{\mathbf{x}'=0} - Q_{3j}\left(\frac{\partial E_j^{(0)}(\mathbf{x}')}{\partial x'_2}\right)_{\mathbf{x}'=0}\right] + \dots \\ &= \left[\mathbf{p} \times \mathbf{E}^{(0)}(0)\right]_1 + \frac{1}{3}\left[\frac{\partial}{\partial x'_3}\left(\sum_j Q_{2j} E_j^{(0)}(\mathbf{x}')\right) - \frac{\partial}{\partial x'_2}\left(\sum_j Q_{3j} E_j^{(0)}(\mathbf{x}')\right)\right]_{\mathbf{x}'=0} + \dots \end{aligned} \tag{25}$$

which is just the given in Eq. (17).

Problem 3 (4.8)

A very long, right circular, cylindrical shell of dielectric constant ϵ/ϵ_0 and inner and outer radii a and b, respectively, is placed in a previously uniform electric field $\mathbf{E}_0$ with its axis perpendicular to the field. The medium inside and outside the cylinder has a dielectric constant of unity.

(a) Determine the potential and electric fields in the three regions, neglecting end effects.

Solution:

Let the uniform electric field be directed towards the positive x-axis (*i.e.* $\mathbf{E}_0 = E_0\hat{\imath}$). With the very long cylindrical shell and neglecting end effects, we can treat this 3D problem as a 2D one. Then, from Eq. 2.71 of Jackson, the general solution of the potential in this scenario is

$$\Phi(\rho,\phi) = \sum_\nu \left(a_\nu \rho^\nu + b_\nu \rho^{-\nu}\right)\left(c_\nu \cos\nu\phi + d_\nu \sin\nu\phi\right) \tag{1}$$

Note that we want to include the origin for the potential in the region of $\rho < a$. Thus, $b_\nu = 0$ in this region. As for the region of $\rho > b$, the potential that gives rise to the external electric field

$$-\int \mathbf{E}_0 \cdot \mathrm{d}x = -E_0 x = -E_0 \rho\cos\phi \tag{2}$$

is the only term for the positive powers of ρ. Thus, $c_\nu = 0$ for $\nu > 1$. With these, the potential for this problem is given by

$$\Phi(\rho,\phi) = \begin{cases} \Phi_I = \sum_\nu \rho^\nu \left(A_\nu \cos\nu\phi + B_\nu \sin\nu\phi\right), & \rho < a \\ \Phi_{II} = \sum_\nu \left[\rho^\nu \left(C_\nu \cos\nu\phi + D_\nu \sin\nu\phi\right) + \rho^{-\nu}\left(F_\nu \cos\nu\phi + G_\nu \sin\nu\phi\right)\right], & a < \rho < b \\ \Phi_{III} = -E_0\rho\cos\phi + \sum_\nu \rho^{-\nu}\left(H_\nu \cos\nu\phi + I_\nu \sin\nu\phi\right), & \rho > b \end{cases} \tag{3}$$

where we let the coefficients of ρ^ν and $\rho^{-\nu}$ be swallowed up by the coefficients of $\cos\nu\phi$ and $\sin\nu\phi$. We need to impose boundary conditions to solve for the coefficients. At $\rho = a$ along the normal direction, we have

$$\epsilon_0 \left.\frac{\mathrm{d}\Phi_I}{\mathrm{d}\rho}\right|_{\rho=a} = \epsilon \left.\frac{\mathrm{d}\Phi_{II}}{\mathrm{d}\rho}\right|_{\rho=a} \tag{4}$$

which leads to

$$\epsilon_0 \sum_\nu \nu a^{\nu-1}\left(A_\nu \cos\nu\phi + B_\nu \sin\nu\phi\right) = \epsilon \sum_\nu \Big[\nu a^{\nu-1}\left(C_\nu \cos\nu\phi + D_\nu \sin\nu\phi\right) - \nu a^{-(\nu+1)}\left(F_\nu \cos\nu\phi + G_\nu \sin\nu\phi\right)\Big] \tag{5}$$

Since the terms of $\cos\nu\phi$ and $\sin\nu\phi$ are independent from each other, we can obtain the following relations:

$$\cos\nu\phi: \quad \epsilon_0 \nu a^{\nu-1} A_\nu = \epsilon\nu(a^{\nu-1}C_\nu - a^{-(\nu+1)}F_\nu) \longrightarrow \frac{\epsilon_0}{\epsilon}A_\nu = C_\nu - a^{-2\nu}F_\nu \tag{6}$$

$$\sin\nu\phi: \quad \epsilon_0 \nu a^{\nu-1} B_\nu = \epsilon\nu(a^{\nu-1}D_\nu - a^{-(\nu+1)}G_\nu) \longrightarrow \frac{\epsilon_0}{\epsilon}B_\nu = D_\nu - a^{-2\nu}G_\nu \tag{7}$$

As for the tangential direction at $\rho = a$, we have

$$\left.\frac{\mathrm{d}\Phi_I}{\mathrm{d}\phi}\right|_{\rho=a} = \left.\frac{\mathrm{d}\Phi_{II}}{\mathrm{d}\rho}\right|_{\phi=a} \tag{8}$$

which leads to

$$\sum_\nu a^\nu\left(-\nu A_\nu \sin\nu\phi + \nu B_\nu \cos\nu\phi\right) = \sum_\nu \left[a^\nu\left(-\nu C_\nu \sin\nu\phi + \nu D_\nu \cos\nu\phi\right) - a^{-\nu}\left(-\nu F_\nu \sin\nu\phi + \nu G_\nu \cos\nu\phi\right)\right] \tag{9}$$

In the same manner, we obtain the following relations:

$$\cos\phi: \quad \nu a^\nu B_\nu = \nu(a^\nu D_\nu + a^{-\nu} G_\nu) \longrightarrow B_\nu = D_\nu - a^{-2\nu} G_\nu \tag{10}$$

$$\sin\phi: \quad -\nu a^\nu A_\nu = -\nu(a^\nu C_\nu + a^{-\nu} F_\nu) \longrightarrow A_\nu = C_\nu + a^{-2\nu} F_\nu \tag{11}$$

We now consider the normal direction at $r = b$. We have

$$\epsilon \left.\frac{\mathrm{d}\Phi_{II}}{\mathrm{d}\rho}\right|_{\rho=b} = \epsilon_0 \left.\frac{\mathrm{d}\Phi_{III}}{\mathrm{d}\rho}\right|_{\rho=b} \tag{12}$$

which leads to

$$\begin{aligned} &\epsilon \sum_\nu \left[\nu b^{\nu-1}\left(C_\nu \cos\nu\phi + D_\nu \sin\nu\phi\right) - \nu b^{-(\nu+1)}\left(F_\nu \cos\nu\phi + G_\nu \sin\nu\phi\right)\right] \\ &= \epsilon_0 \left[-E_0 \cos\phi - \sum_\nu \nu b^{-(\nu+1)}\left(H_\nu \cos\nu\phi + I_\nu \sin\nu\phi\right)\right] \end{aligned} \tag{13}$$

that gives rise to the following relations:

$$\begin{aligned} \cos\nu\phi: \quad & \epsilon\nu\left(b^{\nu-1} C_\nu - b^{-(\nu+1)} F_\nu\right) = \epsilon_0\left(-E_0 \delta_{1\nu} - \nu b^{-(\nu+1)} H_\nu\right) \\ & \longrightarrow b^{2\nu} C_\nu - F_\nu = \frac{\epsilon_0}{\epsilon}\left(-E_0 b^{\nu+1}\delta_{1\nu} - H_\nu\right) \end{aligned} \tag{14}$$

$$\begin{aligned} \sin\nu\phi: \quad & \epsilon\nu\left(b^{\nu-1} D_\nu - b^{-(\nu+1)} G_\nu\right) = -\epsilon_0 \nu b^{-(\nu+1)} I_\nu \\ & \longrightarrow b^{2\nu} D_\nu - G_\nu = \frac{\epsilon_0}{\epsilon} I_\nu \end{aligned} \tag{15}$$

Finally, at $\rho = b$ along the tangential direction, we have

$$\left.\frac{\mathrm{d}\Phi_{II}}{\mathrm{d}\phi}\right|_{\rho=b} = \left.\frac{\mathrm{d}\Phi_{III}}{\mathrm{d}\rho}\right|_{\phi=b} \tag{16}$$

which leads to

$$\begin{aligned} &\sum_\nu \left[b^\nu\left(-\nu C_\nu \sin\nu\phi + \nu D_\nu \cos\nu\phi\right) + b^{-\nu}\left(-\nu F_\nu \sin\nu\phi + \nu G_\nu \cos\nu\phi\right)\right] \\ &= \nu E_0 b \sin\nu\phi + \sum_\nu \rho^{-\nu}\left(-\nu H_\nu \sin\nu\phi + \nu I_\nu \cos\nu\phi\right) \end{aligned} \tag{17}$$

In the same manner as before, we get

$$\begin{aligned} \cos\nu\phi: \quad & \nu\left(b^\nu D_\nu + b^{-\nu} G_\nu\right) = \nu b^{-\nu} I_\nu \\ & \longrightarrow b^{2\nu} D_\nu + G_\nu = I_\nu \end{aligned} \tag{18}$$

$$\begin{aligned} \sin\nu\phi: \quad & -\nu\left(b^\nu C_\nu + b^{-\nu} F_\nu\right) = \nu\left(E_0 b\, \delta_{1\nu} - b^{-\nu} H_\nu\right) \\ & \longrightarrow b^{2\nu} C_\nu + F_\nu = -E_0 b^{\nu+1} \delta_{1\nu} + H_\nu \end{aligned} \tag{19}$$

Note that $B_\nu = D_\nu = G_\nu = I_\nu = 0$ for all ν to satisfy the system of equations (7), (10), (15), and (18). This is almost also the case for the system of equations (6), (11), (14), and (19) in which $A_\nu = C_\nu = F_\nu = H_\nu = 0$ for $\nu \neq 1$ for the said system to be satisfied. But for $n = 1$, the potential becomes

$$\Phi(\rho,\phi) = \begin{cases} \Phi_I = \rho A_1 \nu \cos\phi, & \rho < a \\ \Phi_{II} = \rho C_1 \cos\phi + \rho^{-1} F_1 \cos\phi, & a < \rho < b \\ \Phi_{III} = -E_0 \rho \cos\phi + \rho^{-1} H_1 \cos\phi, & \rho > b \end{cases} \tag{20}$$

From Eqs. (6) and (11), we can get

$$\frac{\epsilon_0}{\epsilon} A_1 = C_1 - a^{-2}(A_1 - C_1)a^2 \quad \longrightarrow \quad C_1 = \frac{1}{2}\left(1 + \frac{\epsilon_0}{\epsilon}\right) A_1 \tag{21}$$

$$\begin{aligned} \frac{\epsilon_0}{\epsilon}\left(C_1 + a^{-2}F_1\right) =& C_1 - a^{-2}F_1 \\ F_1 =& a^2 \frac{1 - \frac{\epsilon_0}{\epsilon}}{1 + \frac{\epsilon_0}{\epsilon}} C_1 \quad \longrightarrow \quad F_1 = \frac{1}{2} a^2 \left(1 - \frac{\epsilon_0}{\epsilon}\right) A_1 \end{aligned} \tag{22}$$

Also, from Eqs. (14) and (19), we can obtain

$$b^2 C_1 - F_1 = \frac{\epsilon_0}{\epsilon}\left(-E_0 b^2 - H_1\right) \quad \longrightarrow \quad -H_1 = b^2 E_0 + b^2 \frac{\epsilon}{\epsilon_0} C_1 - \frac{\epsilon}{\epsilon_0} F_1 \tag{23}$$

$$b^2 C_1 + F_1 = -E_0 b^2 + H_1 \quad \longrightarrow \quad H_1 = b^2 E_0 + b^2 C_1 + F_1 \tag{24}$$

Equating these relations

$$-b^2 E_0 - b^2 C_1 - F_1 = b^2 E_0 + b^2 \left(\frac{1+\epsilon}{\epsilon_0}\right) C_1 - \frac{\epsilon}{\epsilon_0} F_1 \tag{25}$$

Rearranging this and applying Eqs. (21) and (22), we get

$$\begin{aligned} 0 =& 2b^2 E_0 + \frac{1}{2} b^2 \left(1 + \frac{\epsilon}{\epsilon_0}\right)\left(1 + \frac{\epsilon_0}{\epsilon}\right) A_1 + \frac{1}{2} a^2 \left(1 - \frac{\epsilon}{\epsilon_0}\right)\left(1 - \frac{\epsilon_0}{\epsilon}\right) A_1 \\ 0 =& 2b^2 \epsilon\epsilon_0 E_0 + \frac{1}{2} b^2 (\epsilon + \epsilon_0)^2 A_1 - \frac{1}{2} a^2 (\epsilon - \epsilon_0)^2 A_1 \\ -4b^2 \epsilon\epsilon_0 E_0 =& \left[b^2(\epsilon+\epsilon_0)^2 - a^2(\epsilon-\epsilon_0)^2\right] A_1 \\ \longrightarrow \quad A_1 =& \frac{-4b^2\epsilon\epsilon_0}{b^2(\epsilon+\epsilon_0)^2 - a^2(\epsilon-\epsilon_0)^2} E_0 \end{aligned} \tag{26}$$

Substituting A_1 to Eqs. (21) and (22),

$$C_1 = \frac{-2b^2(\epsilon+\epsilon_0)\epsilon_0}{b^2(\epsilon+\epsilon_0)^2 - a^2(\epsilon-\epsilon_0)^2} E_0 \tag{27}$$

$$F_1 = \frac{-2a^2 b^2(\epsilon-\epsilon_0)\epsilon_0}{b^2(\epsilon+\epsilon_0)^2 - a^2(\epsilon-\epsilon_0)^2} E_0 \tag{28}$$

We can apply these relations to Eq. (24) to obtain

$$\begin{aligned} H_1 =& b^2 E_0 + b^2 \frac{-2b^2(\epsilon+\epsilon_0)\epsilon_0}{b^2(\epsilon+\epsilon_0)^2 - a^2(\epsilon-\epsilon_0)^2} E_0 + \frac{-2a^2 b^2(\epsilon-\epsilon_0)\epsilon_0}{b^2(\epsilon+\epsilon_0)^2 - a^2(\epsilon-\epsilon_0)^2} E_0 \\ =& \frac{b^2\left[b^2(\epsilon+\epsilon_0)^2 - a^2(\epsilon-\epsilon_0)^2 - 2b^2(\epsilon+\epsilon_0)\epsilon_0 - 2a^2 b^2(\epsilon-\epsilon_0)\epsilon_0\right]}{b^2(\epsilon+\epsilon_0)^2 - a^2(\epsilon-\epsilon_0)^2} E_0 \\ H_1 =& \frac{b^2(b^2-a^2)(\epsilon^2-\epsilon_0^2)}{b^2(\epsilon+\epsilon_0)^2 - a^2(\epsilon-\epsilon_0)^2} E_0 \end{aligned} \tag{29}$$

which we simplified using Mathematica as shown in Fig. 1.

```
b² (ϵ + ϵ0)² - a² (ϵ - ϵ0)² - 2 (ϵ + ϵ0) b² ϵ0 - 2 (ϵ - ϵ0) a² ϵ0 // Simplify
- (a² - b²) (ϵ² - ϵ0²)
```

Figure 1: Simplifying the numerator in Eq. using Mathematica

Therefore, our potential has the final form of

$$\Phi(\rho,\phi)=\begin{cases}\Phi_I=\dfrac{-4b^2\epsilon\epsilon_0}{b^2(\epsilon+\epsilon_0)^2-a^2(\epsilon-\epsilon_0)^2}E_0\rho\cos\phi, & \rho<a\\ \Phi_{II}=\dfrac{-2b^2\epsilon_0}{b^2(\epsilon+\epsilon_0)^2-a^2(\epsilon-\epsilon_0)^2}\left[(\epsilon+\epsilon_0)\rho+(\epsilon-\epsilon_0)\dfrac{a^2}{\rho}\right]E_0\cos\phi, & a<\rho<b\\ \Phi_{III}=-E_0\rho\cos\phi+\dfrac{(b^2-a^2)(\epsilon^2-\epsilon_0^2)}{b^2(\epsilon+\epsilon_0)^2-a^2(\epsilon-\epsilon_0)^2}\dfrac{b^2}{\rho}E_0\cos\phi, & \rho>b\end{cases}\tag{30}$$

The electric field associated with this potential can be calculated using

$$\mathbf{E}=-\boldsymbol{\nabla}\Phi(\rho,\phi)=-\left(\frac{\mathrm{d}\Phi}{\mathrm{d}\rho}\hat{\boldsymbol{\rho}}+\frac{1}{\rho}\frac{\mathrm{d}\Phi}{\mathrm{d}\phi}\hat{\boldsymbol{\phi}}\right)\tag{31}$$

Thus, we have

$$\mathbf{E}(\rho,\phi)=\begin{cases}\mathbf{E}_I, & \rho<a\\ \mathbf{E}_{II}, & a<\rho<b\\ \mathbf{E}_{III}, & \rho>b\end{cases}\tag{32}$$

in which

$$\mathbf{E}_I=\frac{4b^2\epsilon\epsilon_0}{b^2(\epsilon+\epsilon_0)^2-a^2(\epsilon-\epsilon_0)^2}E_0\cos\phi\hat{\boldsymbol{\rho}}-\frac{4b^2\epsilon\epsilon_0}{b^2(\epsilon+\epsilon_0)^2-a^2(\epsilon-\epsilon_0)^2}E_0\sin\phi\hat{\boldsymbol{\phi}}\tag{33}$$

$$\begin{aligned}\mathbf{E}_{II}=&\frac{2b^2\epsilon_0}{b^2(\epsilon+\epsilon_0)^2-a^2(\epsilon-\epsilon_0)^2}\left[(\epsilon+\epsilon_0)\rho-(\epsilon-\epsilon_0)\frac{a^2}{\rho^2}\right]E_0\cos\phi\hat{\boldsymbol{\rho}}\\&-\frac{2b^2\epsilon_0}{b^2(\epsilon+\epsilon_0)^2-a^2(\epsilon-\epsilon_0)^2}\left[(\epsilon+\epsilon_0)\rho+(\epsilon-\epsilon_0)\frac{a^2}{\rho^2}\right]E_0\sin\phi\hat{\boldsymbol{\phi}}\end{aligned}\tag{34}$$

$$\begin{aligned}\mathbf{E}_{III}=&\left[E_0\cos\phi+\frac{(b^2-a^2)(\epsilon^2-\epsilon_0^2)}{b^2(\epsilon+\epsilon_0)^2-a^2(\epsilon-\epsilon_0)^2}\frac{b^2}{\rho^2}E_0\cos\phi\right]\hat{\boldsymbol{\rho}}\\&+\left[-E_0\sin\phi+\frac{(b^2-a^2)(\epsilon^2-\epsilon_0^2)}{b^2(\epsilon+\epsilon_0)^2-a^2(\epsilon-\epsilon_0)^2}\frac{b^2}{\rho^2}E_0\sin\phi\right]\hat{\boldsymbol{\phi}}\end{aligned}\tag{35}$$

(b) Sketch the lines of force for a typical case of $b\approx 2a$.

Solution:
Using Mathematica as shown in Fig. 3, we plot the field lines which are shown in Fig. 2.

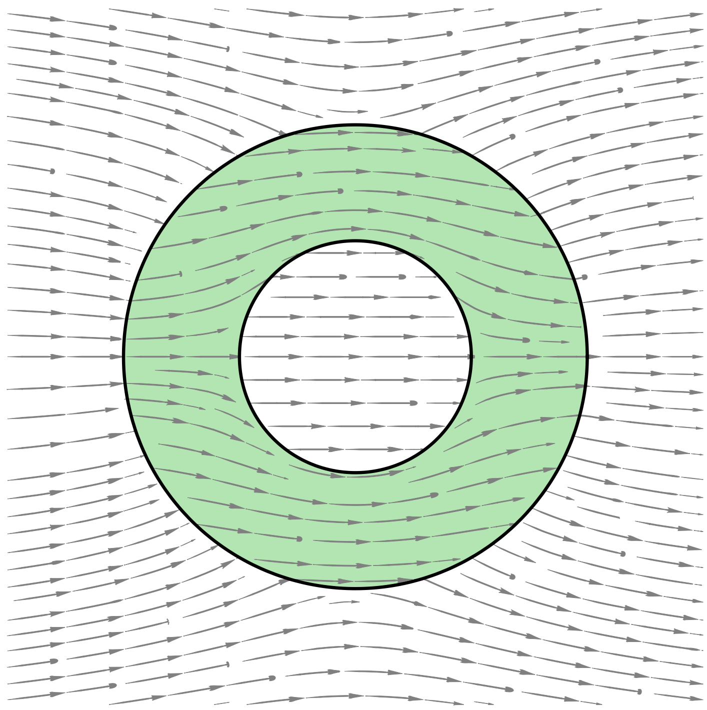

Figure 2: A plot of the field lines where $b = 2a$ and $\epsilon_0/\epsilon = 1/4$

We first need to assume values for a, b, ϵ_0, ϵ, and E_0 :

```
a = 3/4; b = 2 a; ϵ0 = 1; ϵ = 4 ϵ0; E0 = 1;
```

We also plot the top view of the cylinder and shade the dielectric for clarity,

```
InnerRadius = PolarPlot[a, {θ, 0, 2 Pi}, PlotTheme → "Minimal", PlotStyle → Black, AxesStyle → LightGray, Axes → None, PolarAxesOrigin → {0, 1}];
OuterRadius = PolarPlot[b, {θ, 0, 2 Pi}, PlotTheme → "Minimal", PlotStyle → Black, AxesStyle → LightGray, Axes → None, PolarAxesOrigin → {0, 1}];
Dielectric = Graphics[{Opacity[0.3], Darker[Green], Disk[{0, 0}, b], Opacity[1], White, Disk[{0, 0}, a]}];
```

Since the calculated electric field is in polar coordinates, we need to convert the expression into Cartesian coordinates to be able to plot field lines :

```
FI[x_, y_] = TransformedField["Polar" → "Cartesian", {(4 ϵ ϵ0 b^2)/(b^2 (ϵ + ϵ0)^2 - a^2 (ϵ - ϵ0)^2) E0 Cos[ϕ], -(4 ϵ ϵ0 b^2)/(b^2 (ϵ + ϵ0)^2 - a^2 (ϵ - ϵ0)^2) E0 Sin[ϕ]}, {ρ, ϕ} → {x, y}];

FII[x_, y_] = TransformedField["Polar" → "Cartesian", {(2 ϵ0 b^2)/(b^2 (ϵ + ϵ0)^2 - a^2 (ϵ - ϵ0)^2) ((ϵ + ϵ0) - (ϵ - ϵ0) a^2/ρ^2) E0 Cos[ϕ], -(2 ϵ0 b^2)/(b^2 (ϵ + ϵ0)^2 - a^2 (ϵ - ϵ0)^2) ((ϵ + ϵ0) + (ϵ - ϵ0) a^2/ρ^2) E0 Sin[ϕ]},
    {ρ, ϕ} → {x, y}];

FIII[x_, y_] = TransformedField["Polar" → "Cartesian", {E0 Cos[ϕ] + ((b^2 - a^2) (ϵ^2 - ϵ0^2))/(b^2 (ϵ + ϵ0)^2 - a^2 (ϵ - ϵ0)^2) (b/ρ)^2 E0 Cos[ϕ], -E0 Sin[ϕ] + ((b^2 - a^2) (ϵ^2 - ϵ0^2))/(b^2 (ϵ + ϵ0)^2 - a^2 (ϵ - ϵ0)^2) (b/ρ)^2 E0 Sin[ϕ]}, {ρ, ϕ} → {x, y}];
```

Here, we plot the field lines :

```
EF[x_, y_] := Piecewise[{{FI[x, y], x^2 + y^2 < a^2}, {FII[x, y], a^2 <= x^2 + y^2 <= b^2}, {FIII[x, y], x^2 + y^2 > b^2}}];
EFplot = StreamPlot[PiecewiseExpand[EF[x, y]], {x, -3/2 b, 3/2 b}, {y, -3/2 b, 3/2 b}, PlotRange → 3/2 b, StreamPoints → Fine, StreamStyle → {Gray, "PinDart"}, Frame → False];
```

Combining these plots, we get

```
Show[{Dielectric, EFplot, InnerRadius, OuterRadius}, ImageSize → Medium]
```

Figure 3: Mathematica code for the plot

(c) Discuss the limiting forms of your solution appropriate for a solid dielectric cylinder in a uniform field, and a cylindrical cavity in a uniform dielectric.

Solution:
For the case of the solid dielectric cylinder in a uniform field, we consider that $a \to 0$ so

region I vanishes and the potential becomes

$$\Phi(\rho,\phi) \approx \begin{cases} \dfrac{-2\epsilon_0}{(\epsilon+\epsilon_0)} E_0 \rho\cos\phi, & \rho < b \\ -E_0\rho\cos\phi + \dfrac{\epsilon-\epsilon_0}{\epsilon+\epsilon_0}\dfrac{b^2}{\rho} E_0\cos\phi, & \rho > b \end{cases} \tag{36}$$

which gives rise to the electric field

$$\mathbf{E}(\rho,\phi) \approx \begin{cases} \dfrac{2\epsilon_0}{(\epsilon+\epsilon_0)} E_0 \left(\cos\phi\hat{\boldsymbol{\rho}} - \sin\phi\hat{\boldsymbol{\phi}}\right), & \rho < b \\ E_0\left(\cos\phi\hat{\boldsymbol{\rho}} - \sin\phi\hat{\boldsymbol{\phi}}\right) + \dfrac{\epsilon-\epsilon_0}{\epsilon+\epsilon_0}\dfrac{b^2}{\rho^2} E_0 \left(\cos\phi\hat{\boldsymbol{\rho}} + \sin\phi\hat{\boldsymbol{\phi}}\right), & \rho > b \end{cases} \tag{37}$$

On the other hand, the case of a cylindrical cavity in a uniform dielectric requires that $b \to \infty$ so region III vanishes instead. The potential then becomes

$$\Phi(\rho,\phi) \approx \begin{cases} \dfrac{-4\epsilon\epsilon_0}{(\epsilon+\epsilon_0)^2} E_0 \rho\cos\phi, & \rho < a \\ \dfrac{-2\epsilon_0}{(\epsilon+\epsilon_0)^2}\left[(\epsilon+\epsilon_0)\rho + (\epsilon-\epsilon_0)\dfrac{a^2}{\rho}\right] E_0\cos\phi, & \rho > a \end{cases} \tag{38}$$

which gives rise to

$$\mathbf{E}(\rho,\phi) \approx \begin{cases} \dfrac{4\epsilon\epsilon_0}{(\epsilon+\epsilon_0)^2} E_0 \left(\cos\phi\hat{\boldsymbol{\rho}} - \sin\phi\hat{\boldsymbol{\phi}}\right), & \rho < a \\ \dfrac{2\epsilon_0}{(\epsilon+\epsilon_0)} E_0\left(\cos\phi\hat{\boldsymbol{\rho}} - \sin\phi\hat{\boldsymbol{\phi}}\right) - \dfrac{2\epsilon_0(\epsilon-\epsilon_0)}{(\epsilon+\epsilon_0)^2}\dfrac{a^2}{\rho^2} E_0 \left(\cos\phi\hat{\boldsymbol{\rho}} + \sin\phi\hat{\boldsymbol{\phi}}\right), & \rho > a \end{cases} \tag{39}$$

Problem 4 (4.13)

Two long, coaxial, cylindrical conducting surfaces of radii a and b are lowered vertically into a liquid dielectric. If the liquid rises an average height h between the electrodes when a potential difference V is established between them, show that the susceptibility of the liquid is

$$\chi_e = \frac{\left(b^2 - a^2\right) \rho g h \ln(b/a)}{\epsilon_0 V^2} \tag{1}$$

where ρ is the density of the liquid, g is the acceleration due to gravity, and the susceptibility of air is neglected.

Solution:
We assume that the cylindrical surfaces are long enough that we can treat the 3D orientation as a 2D one in calculating for the potential. Then, from Eq. 2.71 of Jackson, the general solution of the potential in this scenario is

$$\Phi(r,\phi) = A_0 + B_0 \ln r + \sum_{n=1}^{\infty} A_n r^n \sin(n\phi + \alpha_n) + \sum_{n=1}^{\infty} B_n r^{-n} \sin(n\phi + \beta_n) \tag{2}$$

Since there is azimuthal symmetry, this general solution simplifies to

$$\Phi(r,\phi) = A_0 + B_0 \ln r \tag{3}$$

With the potential difference of V, let us set $\Phi(r,\phi) = 0$ at $r = a$. Applying this boundary condition, we have

$$0 = A_0 + B_0 \ln a \quad \longrightarrow \quad A_0 = -B_0 \ln a \tag{4}$$

Updating the potential, we have

$$\Phi(r,\phi) = -B_0 \ln a + B_0 \ln r = B_0 \ln \frac{r}{a} \tag{5}$$

On the other hand, we also set $\Phi(r,\phi) = V$ at $r = b$. Then,

$$V = B_0 \ln \frac{b}{a} \quad \longrightarrow \quad B_0 = \frac{V}{\ln \frac{b}{a}} \tag{6}$$

Thus, the potential is given by

$$\Phi(r,\phi) = V \frac{\ln \frac{r}{a}}{\ln \frac{b}{a}} \tag{7}$$

The electric field associated with this potential can be calculated using

$$\mathbf{E} = -\boldsymbol{\nabla}\Phi(r,\phi) = -\left(\frac{\mathrm{d}\Phi}{\mathrm{d}r}\hat{\boldsymbol{r}} + \frac{1}{r}\frac{\mathrm{d}\Phi}{\mathrm{d}\phi}\hat{\boldsymbol{\phi}}\right) \tag{8}$$

Thus, we obtain

$$\mathbf{E}(r,\phi) = -\frac{V}{\ln \frac{b}{a}} \frac{\hat{\boldsymbol{r}}}{r} \tag{9}$$

By lowering the cylindrical surfaces to the liquid dielectric and establishing a potential difference, the electrostatic energy between the two surfaces changes. This change is calculated to be

$$\begin{aligned}
\mathrm{d}W_e &= W_{e,f} - W_{e,i} \\
&= \frac{1}{2}\int \mathbf{E}_f \cdot \mathbf{D}_f \mathrm{d}^3x + \frac{1}{2}\int \mathbf{E}_i \cdot \mathbf{D}_i \mathrm{d}^3x \\
&= \frac{1}{2}\int_h^L \int_0^{2\pi}\int_a^b \mathbf{E}_{\mathrm{air}} \cdot \mathbf{D}_{\mathrm{air}} r\mathrm{d}r\mathrm{d}\phi\mathrm{d}z + \frac{1}{2}\int_0^h \int_0^{2\pi}\int_a^b \mathbf{E}_{\mathrm{ld}} \cdot \mathbf{D}_{\mathrm{ld}} r\mathrm{d}r\mathrm{d}\phi\mathrm{d}z \\
&\qquad - \frac{1}{2}\int_0^L \int_0^{2\pi}\int_a^b \mathbf{E}_{\mathrm{air}} \cdot \mathbf{D}_{\mathrm{air}} r\mathrm{d}r\mathrm{d}\phi\mathrm{d}z \\
&= \frac{1}{2}2\pi\left[(L-h)\epsilon_0\int_a^b E_{\mathrm{air}}^2 r\mathrm{d}r + h\epsilon\int_a^b E_{\mathrm{ld}}^2 r\mathrm{d}r - L\epsilon_0\int_a^b E_{\mathrm{air}}^2 r\mathrm{d}r\right] \\
\mathrm{d}W_e &= \pi h\left[\epsilon\int_a^b E_{\mathrm{ld}}^2 r\mathrm{d}r - \epsilon_0\int_a^b E_{\mathrm{air}}^2 r\mathrm{d}r\right]
\end{aligned} \tag{10}$$

where $D_{\mathrm{air}} = \epsilon_0\mathbf{E}_{\mathrm{air}}$ and $D_{\mathrm{ld}} = \epsilon\mathbf{E}_{\mathrm{ld}}$ in which $\epsilon/\epsilon_0 = 1+\chi_e$. The electric fields passing through the air and the dielectric are actually the same (*i.e.* $\mathbf{E}_{\mathrm{air}} = \mathbf{E}_{\mathrm{ld}} = \mathbf{E}$). Thus,

$$\begin{aligned}
\mathrm{d}W_e &= \pi h\left[(1+\chi_e)\epsilon_0\int_a^b E^2 r\mathrm{d}r - \epsilon_0\int_a^b E^2 r\mathrm{d}r\right] \\
&= \pi h\chi_e\epsilon_0\int_a^b E^2 r\mathrm{d}r \\
&= \pi h\chi_e\epsilon_0\int_a^b \frac{V^2}{\ln^2\frac{b}{a}}\frac{1}{r^2} r\mathrm{d}r \\
&= \pi h\chi_e\epsilon_0\frac{V^2}{\ln^2\frac{b}{a}}(\ln b - \ln a) \\
\mathrm{d}W_e &= \frac{\pi h\chi_e\epsilon_0 V^2}{\ln\frac{b}{a}}
\end{aligned} \tag{11}$$

This change of electrostatic energy must be balanced out by the change in gravitational potential energy brought about by the rising of the liquid dielectric to an average height h. This change in gravitational potential energy is given by

$$\mathrm{d}W_g = W_{g,f} - W_{g,i} = mg(h-0) = mgh \tag{12}$$

Now, our region of interest is space between the two surfaces. The volume $\mathcal{V}$ of the liquid in this space is given by

$$\mathcal{V} = \pi b^2 h - \pi a^2 h = \pi(b^2-a^2)h \tag{13}$$

The density of the liquid between the two surfaces can give rise to the following relation:

$$\rho = \frac{m}{\mathcal{V}} \quad\longrightarrow\quad m = \rho\mathcal{V} \tag{14}$$

Thus, we obtain

$$\mathrm{d}W_g = \rho\pi(b^2-a^2)gh^2 \tag{15}$$

Equating Eqs. (11) and (15),

$$\begin{aligned} \mathrm{d}W_e =& \mathrm{d}W_g \\ \frac{\pi h \chi_e \epsilon_0 V^2}{\ln \frac{b}{a}} =& \rho\pi(b^2 - a^2) g h^2 \end{aligned} \tag{16}$$

which can be rearranged to

$$\chi_e = \frac{\rho(b^2 - a^2) g h}{\epsilon_0 V^2} \ln \frac{b}{a}. \tag{17}$$