Physics 231

Introduction to Electrostatics (chapter 1)

Sept. 11, 2020

* Historically, electrostatics developed as a science of macroscopic phenomena. ↳ brought about by charges at rest

▸ Coulomb's Law

The force between two small charged bodies separated in air at a distance large compared to their dimensions:

- varies directly as the mag. of each charge
- varies inversely as the square of the distance bet. them
- is directed along the line joining the charges
- is attractive [repulsive] if the bodies are opposite [the same] in charge

▸ Electric Field

↳ force per unit charge acting at a given point

↳ vector fxn of position

We have

$$\boxed{\underbrace{\vec{F}}_{\text{force}} = \underbrace{q}_{\text{charge}}\underbrace{\vec{E}}_{\text{electric field}}}$$

< Assumption: $\vec{F}$ & $\vec{E}$ are evaluated where q is located

* Coulomb's Law

$$\boxed{\vec{F} = k q_1 q_2 \frac{\vec{x}_1 - \vec{x}_2}{|\vec{x}_1 - \vec{x}_2|^3}}$$

< $\vec{F}$ is the force on a pt. charge q_1 located at $\vec{x}_1$ due to another pt. charge q_2 located at $\vec{x}_2$

* $\vec{E}$ field at $\vec{x}$ due to q_1 at $\vec{x}_1$: $\boxed{\vec{E}(\vec{x}) = k q_1 \frac{x - x_1}{|x - x_1|^3}}$

Sept. 12, 2020

▸ Electric field

* $\vec{E}$ field at $\vec{x}$ due to many charges
  - Discrete:

$$\vec{E}(\vec{x}) = \overbrace{\frac{1}{4\pi\epsilon_0}}^{k} \sum_{i=1}^{n} q_i \frac{\vec{x} - \vec{x}_i}{|\vec{x} - \vec{x}_i|^3}$$

  - Continuous:

$$\vec{E}(x) = \frac{1}{4\pi\epsilon_0} \int \underbrace{\rho(\vec{x}')}_{\text{charge density}} \frac{\vec{x} - \vec{x}'}{|\vec{x} - \vec{x}'|^3} \, d^3\vec{x}'$$

< Assumption: The charges are so small & numerous that they can be described by $\rho(\vec{x})$

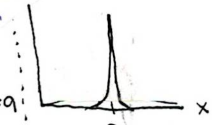
* Dirac Delta function [$\delta(x-a)$ in 1D]

Properties:

① $\delta(x-a) = 0$ for $x \neq a$

② $\int \delta(x-a)\,dx = \begin{cases} 0, & \text{reg. of integration doesn't include } x=a \\ 1, & \text{" " includes } x=a \end{cases}$

plot of $\delta(x-a)$



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③ $\int \underbrace{f(x)}_{\text{just arbitrary fxn}} \delta(x-a)\, dx = f(a)$

④ $\int f(x)\, \delta'(x-a)\, dx = -f'(a)$ ← differentiation wrt x before substituting a

⑤ $\delta(f(x)) = \sum_i \frac{1}{\frac{df}{dx}(x_i)} \delta(x-x_i)$ ◂ Assumption: f(x) have only simple zeroes at $x = x_i$

For N dimensions, we just take N products of $\delta(\ldots)$. In 3D, for example,

⑥ $\delta(\vec{x}-\vec{X}) = \delta(x_1 - X_1)\, \delta(x_2 - X_2)\, \delta(x_3 - X_3)$

⑦ $\int_{\Delta V} \delta(\vec{x} - \vec{X})\, d^3x = \begin{cases} 1 & \text{if } \Delta V \text{ contains } \vec{x} = \vec{X} \\ 0 & \text{" " does not contain } \vec{x} = \vec{X} \end{cases}$

* Using delta fxns, we can describe a discrete set of pt. charges w/ a charge density

$$\rho(\vec{x}) = \sum_{i=1}^{n} q_i\, \delta(\vec{x} - \vec{x}_i)$$

* Coulomb's law: continuous

$$\vec{F} = \frac{1}{4\pi\epsilon_0} \int d^3\vec{x} \int d^3\vec{x}'\, \rho^*(\vec{x})\, \rho(\vec{x}')\, \frac{\vec{x} - \vec{x}'}{|\vec{x} - \vec{x}'|^3}$$

◂ The force that $\rho(\vec{x}')$ exerts $\rho^*(\vec{x})$

- Also, note that the force that ρ^* exerts on ρ is $\vec{F}^* = -\vec{F}$

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* Recall: Maxwell's eqns

$$\nabla \cdot \vec{D} = \rho \qquad \nabla \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0$$
$$\nabla \times \vec{H} - \frac{\partial \vec{D}}{\partial t} = \vec{J} \qquad \nabla \cdot \vec{B} = 0$$

* The foundation of electrodynamics is the ~~presence~~ existence of electric charge

Where for ext. sources in vacuum, $\vec{D} = \epsilon_0 \vec{E}$ and $\vec{B} = \mu_0 \vec{H}$. Then,

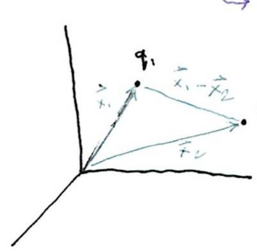
$$\left[\nabla \cdot \vec{D} = \nabla \cdot (\epsilon_0 \vec{E}) = \epsilon_0 (\nabla \cdot \vec{E})\right] = \rho$$
$$\nabla \cdot \vec{E} = \frac{\rho}{\epsilon_0}$$

$$\left[\nabla \times \vec{H} - \frac{\partial \vec{D}}{\partial t} = \nabla \times \frac{\vec{B}}{\mu_0} - \frac{\partial(\epsilon_0 \vec{E})}{\partial t} = \frac{1}{\mu_0}(\nabla \times \vec{B}) - \epsilon_0 \frac{\partial \vec{E}}{\partial t}\right] = \vec{J}$$
$$\nabla \times \vec{B} - \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t} = \mu_0 \vec{J}$$

Note that $\mu_0 \epsilon_0 = \frac{1}{c^2}$.

▸ Lecture notes

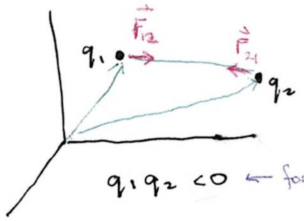
* Charges exert forces on each other
↳ for stationary charges, this is the $\vec{E}$ force



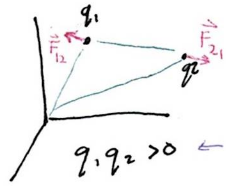
Coulomb's Law

$$\underbrace{\vec{F}_{12}}_{\text{force on } q_1 \text{ due to } q_2} = \frac{1}{4\pi\epsilon_0} \frac{q_1 q_2}{|\vec{x}_1 - \vec{x}_2|^2} \cdot \underbrace{\frac{(\vec{x}_1 - \vec{x}_2)}{|\vec{x}_1 - \vec{x}_2|}}_{\text{unit vector}}$$

* Direction of force is det. by the sign of product $q_1 q_2$

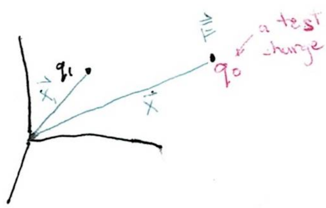


$q_1 q_2 < 0$ ← force is attractive



$q_1 q_2 > 0$ ← force is repulsive

* $\vec{F}_{21}$: force on q_2 due to q_1
* $\vec{F}_{12}$ and $\vec{F}_{21}$: action - reaction forces
  ↳ $|\vec{F}_{12}| = |\vec{F}_{21}|$, $\vec{F}_{12} = -\vec{F}_{21}$
* $\vec{E}$ force bet. 2 stationary charges is dictated by a field called an electric field.
* A charge sets up an electric field in space and this field interacts w/ other charges around.



The charge sets up an electric field $\vec{E}$ at $\vec{x}$

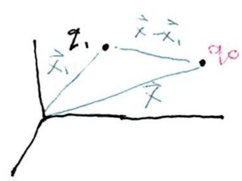
How do we define the $\vec{E}$ field?

The test charge q_0 will experience the force $\vec{F}$ due to q_1

* The $\vec{E}$ field at the loc. of the test charge is given by the limit

$$\boxed{\vec{E} = \lim_{q_0 \to 0} \frac{\vec{F}}{q_0}}$$

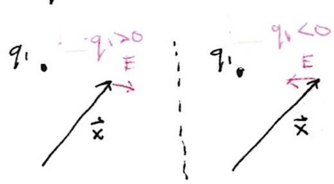
* The $\vec{E}$ field due to a pt. charge:



$$\vec{E} = \lim_{q_0 \to 0} \frac{\vec{F}}{q_0}$$

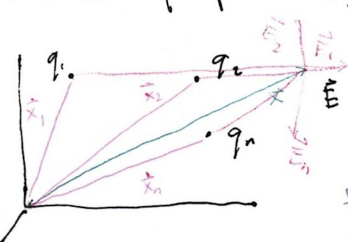
$$= \lim_{q_0 \to 0} \frac{1}{4\pi\epsilon_0} \frac{q_1 \cancel{q_0}}{|\vec{x}-\vec{x}_1|^2} \frac{(\vec{x}-\vec{x}_1)}{|\vec{x}_1-\vec{x}_1|} \cdot \frac{1}{\cancel{q_0}}$$

$$= \frac{1}{4\pi\epsilon_0} \frac{q_1}{|\vec{x}-\vec{x}_1|^2} \frac{\vec{x}_1-\vec{x}_1}{|\vec{x}-\vec{x}_1|}$$



$$\boxed{\vec{E} = \frac{1}{4\pi\epsilon_0} q_1 \frac{(\vec{x}-\vec{x}_1)}{|\vec{x}-\vec{x}_1|^3}}$$

* If there are 2 or more charges, $\vec{E}$ due to all the charges at a given pt. is just the linear superposition of the indiv. $\vec{E}$ due to pt. charge

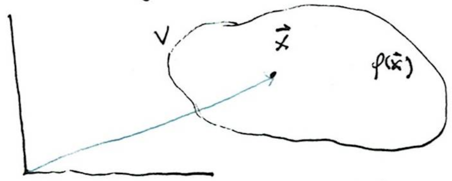


$\vec{E} = \vec{E}_1 + \vec{E}_2 + \dots + \vec{E}_n = \sum_i \vec{E}_i$

(might not be valid for higher energies. But, at least, for our case (our considered energy level), we observed that $\vec{E}$ follows linear superposition

* [Charge density]
  ↳ charge / unit volume

When the charges are numerous, they are described by a charge density



$\int_V \rho(\vec{x})\, d^3x$ = the tot. charge inside V

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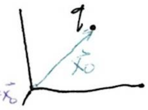
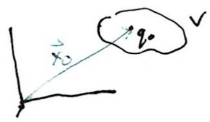
* For a single charge, the charge density is

$$\rho(\vec{x}) = q\,\delta(\vec{x}-\vec{x}_0)$$

A pt. charge q is located at $\vec{x}=\vec{x}_0$

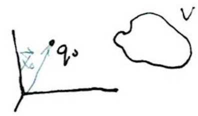


* Recall: Properties of Dirac Delta fxn



q_0 is inside V:

$$\int_V \delta(x-x_0)\,d^3x = 1$$



q_0 is outside V:

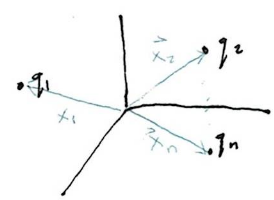
$$\int_V \delta(\vec{x}-\vec{x}_0)\,d^3x = 0$$

* For a single charge q_0 at $\vec{x}_0$,

$$\rho_0(\vec{x}) = q_0\,\delta(\vec{x}-\vec{x}_0)$$

$$\underbrace{\int_V \rho_0(\vec{x})\,d^3x}_{\text{total charge inside V}} = \int_V q_0\,\delta(\vec{x}-\vec{x}_1)\,d^3x = \begin{cases} q_0, & \text{if } q_0 \text{ is inside V} \\ 0, & \text{if } q_0 \text{ is outside V} \end{cases}$$

If there are several pt. charges,



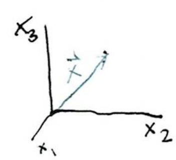
... the charge density for this sys. is

$$\rho(\vec{x}) = \sum_i q_i\,\delta(\vec{x}-\vec{x}_i)$$

How do we describe charge distributions?

The Dirac Delta fxn in curvilinear coord's:

- Rectangular (x, y, z)



$x_1 = x$

$x_2 = y$

$x_3 = z$

$$\delta(\vec{x}-\vec{x}_0) = \delta(x-x_0)\,\delta(y-y_0)\,\delta(z-z_0)$$

$$\int_{\text{over all space}} \delta(\vec{x}-\vec{x}_0)\,d^3x = 1$$

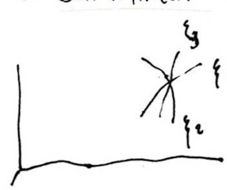
assume that x_0 is inside the range of integration

$$\int_{\text{over all space}} \delta(\vec{x}-\vec{x}_0)\,d^3x = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \delta(x-x_0)\,\delta(y-y_0)\,\delta(z-z_0)\,dx\,dy\,dz$$

$$= \left(\int_{-\infty}^{\infty} \delta(x-x_0)\,dx\right)\left(\int_{-\infty}^{\infty} \delta(y-y_0)\,dy\right)\left(\int_{-\infty}^{\infty} \delta(z-z_0)\,dz\right)$$

$$= 1$$

- Curvilinear (ξ_1, ξ_2, ξ_3) ← assumed to be orthogonal



$$d^3x = (h_1\,d\xi_1)(h_2\,d\xi_2)(h_3\,d\xi_3)$$

$$= \underbrace{h_1 h_2 h_3}_{\text{scale factors}}\,d\xi_1\,d\xi_2\,d\xi_3$$

where

$$h_i = \left[\left(\frac{\partial x}{\partial \xi_i}\right)^2 + \left(\frac{\partial y}{\partial \xi_i}\right)^2 + \left(\frac{\partial z}{\partial \xi_i}\right)^2\right]^{1/2}$$

$$\delta(\vec{x}-\vec{x}_0) = \frac{\delta(q_1 - q_{10})}{h_1} \frac{\delta(q_2 - q_{20})}{h_2} \frac{\delta(q_3 - q_{30})}{h_3}$$

$$= \frac{1}{h_1 h_2 h_3} \delta(q_1 - q_{10})\, \delta(q_2 - q_{20})\, \delta(q_3 - q_{30}) \qquad (1)$$

Note again that $\int_{\text{over all space}} \delta(\vec{x}-\vec{x}_0)\, d^3x = 1$.

$$\int_{\substack{\text{over all}\\ \text{space}}} \delta(\vec{x}-\vec{x}_0)\, d^3x = \int \frac{1}{\cancel{h_1 h_2 h_3}} \delta(q_1 - q_{10})\, \delta(q_2 - q_{20})\, \delta(q_3 - q_{30})\, \cancel{h_1 h_2 h_3}\, dq_1\, dq_2\, dq_3$$

$$= \int \delta(q_1 - q_{10})\, \delta(q_2 - q_{20})\, \delta(q_3 - q_{30})\, dq_1\, dq_2\, dq_3$$

$$= \left(\underbrace{\int \delta(q_1 - q_{10})\, dq_1}_{1}\right)\left(\underbrace{\int \delta(q_2 - q_{20})\, dq_2}_{1}\right)\left(\underbrace{\int \delta(q_3 - q_{30})\, dq_3}_{1}\right)$$

$$= 1$$

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▶ Lecture Notes

Recall: Dirac Delta fxn

In a given curvilinear coordinates (q_1, q_2, q_3):

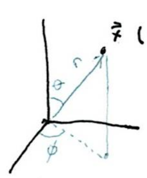
$$d^3x = (h_1 dq_1)(h_2 dq_2)(h_3 dq_3)$$

$$q_1 = q_1(x,y,z) \qquad q_2 = q_2(x,y,z), \qquad q_3 = (x,y,z)$$

$$h_i = \left[\left(\frac{\partial x}{\partial q_i}\right)^2 + \left(\frac{\partial y}{\partial q_i}\right)^2 + \left(\frac{\partial z}{\partial q_i}\right)^3\right]^{1/2} \qquad i = 1,2,3$$

Now, consider $\vec{x}_0 = (q_{01}, q_{02}, q_{03})$. Then, $\delta(\vec{x}-\vec{x}_0)$ is given by Eq. (1) in prev. lecture

- Spherical coord's



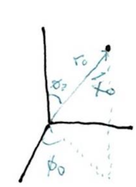
$\vec{x}\,(r,\theta,\phi)$, $0 < r < \infty$, $0 \le \theta \le \pi$, $0 \le \phi < 2\pi$

$$x = r\sin\theta\cos\phi$$
$$y = r\sin\theta\sin\phi$$
$$z = r\cos\theta$$

Let $q_1 = r$, $q_2 = \theta$, $q_3 = \phi$. Then

$$h_1 = 1, \qquad h_2 = r, \qquad h_3 = r\sin\theta$$

★ If we have a particle in $\vec{x}_0$, $\delta(\vec{x}-\vec{x}_0)$ in spherical coords is



$$\delta(\vec{x}-\vec{x}_0) = \frac{\delta(r-r_0)}{1} \cdot \frac{\delta(\theta-\theta_0)}{r} \cdot \frac{\delta(\phi-\phi_0)}{r\sin\theta}$$

$$= \frac{1}{r^2\sin\theta} \delta(r-r_0)\, \delta(\theta-\theta_0)\, \delta(\phi-\phi_0)$$

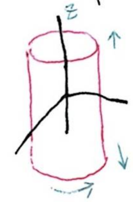
★ When there is a charge at $\vec{x}_0$, the charge density in spherical coord's is

$$\rho(\vec{x}) = \frac{Q}{r^2\sin\theta} \delta(r-r_0)\, \delta(\theta-\theta_0)\, \delta(\phi-\phi_0)$$

★ Each charge gives a $\rho(\vec{x})$ given by this

★ w/o symmetry

* If there is azimuthal symmetry, the phenomena under consideration is indep. of ϕ. For ex., we consider an infinitely long cylinder



- Cylinder oriented at the z-axis
- when rotating about the z-axis, the configuration remains the same

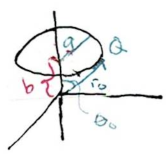
In the presence of this symmetry, ϕ-integral must be projected out. In particular, the denominator

$$\int_0^{2\pi} r^2 \sin\theta \, d\phi = 2\pi r^2 \sin\theta$$

Then, the dirac delta fxn becomes

$$\delta(\vec{x} - \vec{x}_0) = \frac{1}{2\pi r^2 \sin\theta} \delta(r - r_0)\, \delta(\theta - \theta_0)$$

* Consider: A charge Q is uniformly distributed in the ring. What is the charge density?



- has symmetry along ϕ

$$\rho(\vec{x}) = Q\, \delta(\vec{x} - \vec{x}_0)$$
$$= \frac{Q}{2\pi r^2 \sin\theta} \delta(r - r_0)\, \delta(\theta - \theta_0)$$

- where
$r_0 = \sqrt{a^2 + b^2}$
$\theta_0 = \tan^{-1}\left(\frac{a}{b}\right)$

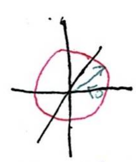
* It is also possible that the phenomena does not depend on θ and ϕ. In this case, we must also project out the θ-integral, along w/ ϕ-integral, in the denominator

$$\int_0^{\pi} d\theta \int_0^{2\pi} d\phi \; r^2 \sin\theta = 4\pi r^2$$

Then, the dirac delta fxn becomes

$$\delta(\vec{x} - \vec{x}_0) = \frac{1}{4\pi r^2} \delta(r - r_0)$$

* Consider: A charge Q uniformly distributed over a sphere of radius r_0. What is the charge density?



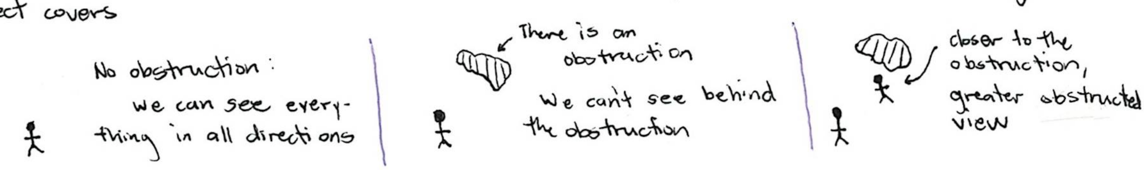
$$\rho(\vec{x}) = \frac{Q}{4\pi r^2} \delta(r - r_0)$$

Shouldn't this be
$$\rho(\vec{x}) = \begin{cases} \frac{Q}{4\pi R^2}, & r \leq R \\ 0, & r > R \end{cases}$$

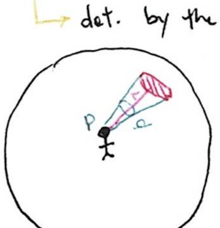
- there is symmetry along θ and ϕ; this means that the charges are only distributed on the surface of the sphere

Solid Angles

↳ a measure of the field of view from some particular point that a given object covers



↳ det. by the area of the obstruction and the distance of the eyes to the obstruction



A will subtend a solid angle wrt the point p or

Ω: the solid angle subtended by A

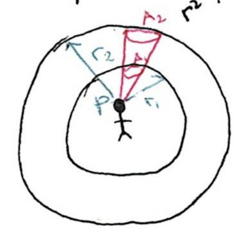
$$\Omega = \frac{A}{r^2}$$

The solid angle subtended by the entire sphere is $\Omega_s = \frac{A_s}{r^2} = \frac{4\pi r^2}{r^2} = 4\pi$

A_s: surface area of the sphere

$\Omega_s = 4\pi$: No obj obstructs our view in all directions

* Why $\Omega = \frac{A}{r^2}$? Consider : Two concentric spheres w/ our eyes at the center



A_1 : on the surface of sphere 1

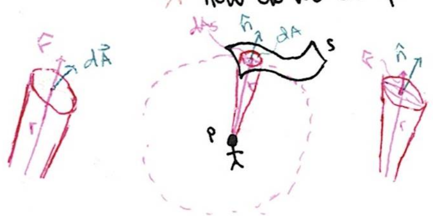
A_2 : on the surface of sphere 2

Note that the view that A_1 obstructs is exactly the same view that A_2 obstructs. Thus, A_1 & A_2 substends the solid angle about the apex P :

$$\Omega_1 = \Omega_2$$

This is only true if the solid angle is defined by $\Omega = \frac{A}{r^2}$

* Two objects may substend the same solid angle
* How do we compute for the solid angle for the general case ?



Since dA and dA_s are intercepts of same cone of view, they substend the equal solid angle about P

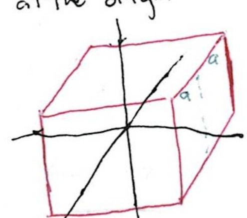
The differential solid angle substended by dA is

$$d\Omega = \frac{dA}{r^2} = \frac{d\vec{A}\cdot\hat{n}}{r^2}$$

The solid angle subitended by the area about P

$$\Omega = \int_S \frac{\hat{r}\cdot d\vec{A}}{r^2}$$

* Example : Obtain the solid angle subtended by one face of a cube of side length 2a centered at the origin



The expected solid angle is

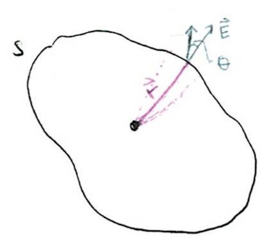
$$\frac{4\pi}{6} = \frac{2\pi}{6}.$$

total solid angle in all directions

total no. of equal side surfaces

Sept. 21, 2020

Gauss' Law



$$\vec{E}\cdot\vec{n}\, da = \frac{q}{4\pi\epsilon_0}\frac{\cos\theta}{r^2}da$$

Recall prev. lecture on solid angles

$$\vec{E}\cdot\vec{n}\, da = \frac{q}{4\pi\epsilon_0}\, d\Omega$$

* Integrating the normal component of $\vec{E}$ over the whole surface S

$$\oint_S \vec{E}\cdot\vec{n}\, da = \frac{q}{4\pi\epsilon_0}\overbrace{\oint_S d\Omega}^{4\pi\,?} = \frac{q}{\epsilon_0}$$

if q is inside S. While $\oint_S \vec{E}\cdot\hat{n}\, da = 0$ if its outside.

* For a discrete set of charges,

$$\oint_S \vec{E}\cdot\hat{n}\, da = \frac{1}{\epsilon_0}\sum_i q_i$$

For a cont. charge density,

$$\oint_S \vec{E}\cdot\hat{n}\, da = \frac{1}{\epsilon_0}\int \rho(\vec{x})\, d^3x$$

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▶ Differential form of Gauss' Law

* Divergence theorem

$$\oint_S \vec{A}\cdot\vec{n}\,da = \int_V \nabla\cdot\vec{A}\,d^3x$$

where $\vec{A}(\vec{x})$ is a well-behaved vector field in volume V surr. by surface S.

* Applying this theorem to Eq. (2),

$$\left(\oint_S \vec{E}\cdot\vec{n}\,da = \int_V \nabla\cdot\vec{E}\,d^3x\right) = \frac{1}{\epsilon_0}\int_V \rho(x)\,d^3x$$

$$\int_V (\nabla\cdot\vec{E} - \rho/\epsilon_0)\,d^3x = 0$$

▲ The only way for this to be true is if the integrand is zero since V and S are arbitrary.

for an arbitrary vol. V. Equating the integrand to zero, we can get

$$\boxed{\nabla\cdot\vec{E} = \rho/\epsilon_0}$$

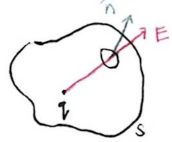
Sept. 22, 2020

▶ Lecture Notes

▶ Gauss's Law

Consider a point charge in a closed surface.

Case 1: If the charge is inside the surface,



Calculate the surface integral:

$$\oint_S \vec{E}\cdot d\vec{a} = \oint_S \frac{1}{4\pi\epsilon_0}\left(\frac{q}{r^2}\hat{r}\right)\cdot(\hat{n}\,da)$$

$$= \oint_S \frac{1}{4\pi\epsilon_0} q \underbrace{\frac{\hat{r}\cdot\hat{n}}{r^2}da}_{=d\Omega}$$

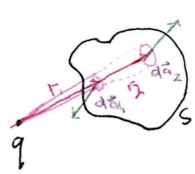
The component of da that is ⊥ to $\hat{r}$

$$= \oint_S \frac{q}{4\pi\epsilon_0}\,d\Omega$$

$$= \frac{q}{4\pi\epsilon_0}\underbrace{\oint_S d\Omega}_{=4\pi}$$

$$\oint_S \vec{E}\cdot d\vec{a} = \frac{q}{\epsilon_0}$$

Case 2: If the charge is outside the surface



$d\vec{a}_1$ & $d\vec{a}_2$ - direction of the surface area is normal to the surface and outside for a closed surface

▲ $d\vec{a}_1$ & $d\vec{a}_2$ subtend the same solid angle at q

$$d\Omega_1 = d\Omega_2 = d\Omega$$

Then,

$$\vec{E}_1\cdot d\vec{a}_1 + \vec{E}_2\cdot d\vec{a}_2 = \frac{q}{4\pi\epsilon_0}\underbrace{\frac{\hat{r}}{r_1^2}\cdot d\vec{a}_1}_{=d\Omega_1} + \frac{q}{4\pi\epsilon_0}\underbrace{\frac{\hat{r}}{r_2^2}\cdot d\vec{a}_2}_{=d\Omega_2}$$

Let $q>0$. Look how $d\vec{a}_2$ and $\hat{r}$ points in the generally same direction. Thus, their dot product is positive. On the other hand, $d\vec{a}_1$ and $\hat{r}$ points in the generally opp. direction so their dot product is negative.

$$q>0: \quad d\Omega_2 > 0, \quad d\Omega_1 < 0$$

Thus,

$$\vec{E}_1\cdot d\vec{a}_1 + \vec{E}_2\cdot d\vec{a}_2 = \frac{q}{4\pi\epsilon_0}\,(\cancelto{0}{-d\Omega + d\Omega})$$

$$= 0$$

Therefore, we see from both cases that

$$\oint_S \vec{E}\cdot d\vec{a} = \begin{cases} \frac{q}{\epsilon_0}, & \text{if } q \text{ is inside } S \\ 0, & \text{if } q \text{ is outside } S \end{cases}$$

For a charge distribution

Gauss's law

$$\boxed{\oint_S \vec{E}\cdot d\vec{a} = \frac{Q_{enc}}{\epsilon_0}}$$

total charge enclosed by surface S

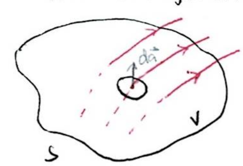
For a discrete distribution,

$$Q_{enc} = \sum_i q_i, \quad q_i \text{ is inside surface } S$$

For a continuous distribution,

$$Q_{enc} = \int_V \rho(\vec{x})\, d^3x, \quad \text{where } V \text{ is the vol. bounded by } S$$

▶ Differential form of Gauss's law



$$\boxed{\oint_S \vec{A}\cdot d\vec{a} = \int_V \vec{\nabla}\cdot\vec{A}\, d^3x}$$

Divergence Theorem

From Gauss's law, we that for a cont. distribution

$$\oint_S \vec{E}\cdot d\vec{a} = \frac{1}{\epsilon_0}\int_V \rho(\vec{x})\, dx^3$$

By divergence theorem,

$$\boxed{\nabla\cdot\vec{E} = \frac{1}{\epsilon_0}\rho(\vec{x})} \qquad (3)$$

▶ Scalar potential

☆ Note that Eq. (3) is not enough to specify completely the components of $\vec{E}$. We still need to specify the curl of $\vec{E}$ to do this (as Eq. (3) gives the divergence of $\vec{E}$ and you need $\vec{\nabla}\cdot\vec{E}$ & $\vec{\nabla}\times\vec{E}$ to completely specify $\vec{E}$)

☆ In general, in the presence of a charge distribution $\rho(\vec{x})$, the electric field is

$$\vec{E}(\vec{x}) = \frac{1}{4\pi\epsilon_0}\int \rho(\vec{x}')\frac{\vec{x}-\vec{x}'}{|\vec{x}-\vec{x}'|^3}\, d^3x'$$

$$d\vec{E} = \frac{1}{4\pi\epsilon_0}\frac{\rho(\vec{x})\, d^3x'}{|\vec{x}-\vec{x}'|^2}\frac{\vec{x}-\vec{x}'}{|\vec{x}-\vec{x}'|}$$

$$\vec{\nabla}\frac{1}{|\vec{x}-\vec{x}'|} = \vec{\nabla}\left[(x_1-x_1')^2 + (x_2-x_2')^2 + (x_3-x_3')^2\right]^{-1/2}$$

$$= \left(\hat{i}\frac{\partial}{\partial x_1} + \hat{j}\frac{\partial}{\partial x_2} + \hat{k}\frac{\partial}{\partial x_3}\right)\left[\quad\cdots\quad\right]^{-1/2}$$

Then,

$$\vec{E}(\vec{x}) = \frac{1}{4\pi\epsilon_0}\int \rho(\vec{x}')\,\nabla\frac{1}{|\vec{x}-\vec{x}'|}\, d^3x'$$

$$= -\vec{\nabla}\left[\frac{1}{4\pi\epsilon_0}\int\frac{\rho(\vec{x}')}{|\vec{x}-\vec{x}'|}\, d^3x'\right]$$

$$= -\vec{\nabla}\,\Phi(\vec{x})$$

Then,

$$\vec{\nabla} \times \vec{E} = -\vec{\nabla} \times \vec{\nabla}\Phi$$
$$= 0 \quad \text{as long as } \Phi \text{ is differentiable}$$

Therefore, the ff. two eqns completely specify E given that $\rho(\vec{x})$ is known

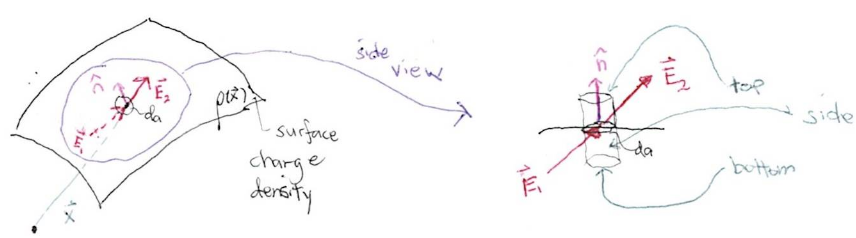
$$\vec{\nabla} \cdot \vec{E} = \frac{\rho(\vec{x})}{\epsilon_0}, \qquad \vec{\nabla} \times \vec{E} = 0$$

The fxn $\Phi(x)$ is known as the scalar potential

Oct. 1, 2020

- Lecture Notes
- Discontinuities in Electric field

☆ There is a discontinuity in



Applying Gauss's law over the surface

$$\int \vec{E} \cdot d\vec{a} = \frac{Q_{enc}}{\epsilon_0}$$

Note that

$$\int \vec{E} \cdot d\vec{a} = \int \vec{E} \cdot d\vec{a}_{side} + \int \vec{E} \cdot d\vec{a}_{top} + \int \vec{E} \cdot d\vec{a}_{bottom}$$

Take the limit as the top & bottom surface approach the surface charge. Then,

$$\int \vec{E} \cdot d\vec{a}_{side} \to 0$$

$$\int \vec{E} \cdot d\vec{a}_{top} \to \vec{E}_2 \cdot d\vec{a} = \vec{E}_2 \cdot \hat{n}\, da = E_2^{\perp}\, da$$ component of $\vec{E}_2$ ⊥ to the surface

$$\int E \cdot \vec{a}_{bottom} \to \vec{E}_1 \cdot d\vec{a} = \vec{E}_1 (-\hat{n})\, da = -E_1^{\perp}\, da$$ component of $\vec{E}_1$ ⊥ to the surface

Also, $Q_{enc} = \sigma(\vec{x})\, da$. Thus,

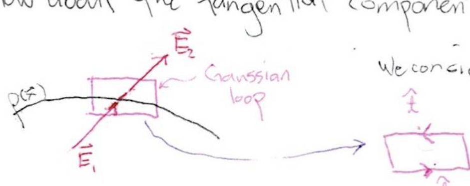
since da is arbitrary

$$\int \vec{E} \cdot da = E_2^{\perp} \cancel{da} - E_1^{\perp} \cancel{da} = \frac{\rho(\vec{x})}{\epsilon_0} \cancel{da}$$

$$\boxed{E_2^{\perp} - E_1^{\perp} = \frac{\rho(\vec{x})}{\epsilon_0}}$$

The normal component of $\vec{E}$ across a surface charge distribution is discontinuous

How about the tangential component of the E field?



We consider $\oint_{loop} \vec{E} \cdot d\vec{l}$

Then

$$\oint_{loop} \vec{E}\cdot d\vec{l} \rightarrow \int \vec{E}\cdot d\vec{l}_{left} + \int \vec{E}\cdot d\vec{l}_{right} + \int \vec{E}\cdot d\vec{l}_{left\ normal} + \int \vec{E}\cdot d\vec{l}_{right\ normal}$$

Take the limit as the normal sides approaches zero. Then

$$\int \vec{E}\cdot d\vec{l}_{left\ normal} \longrightarrow 0 \longleftarrow \int \vec{E}\cdot d\vec{l}_{right\ normal}$$

$$\int \vec{E}\cdot d\vec{l}_{left} \longrightarrow \hat{E}_2 \cdot \hat{x}\ dl = E_2^{\parallel}\ dl$$

$$\int \vec{E}\cdot d\vec{l}_{right} \longrightarrow \vec{E}_1 \cdot (-\hat{x})\ dl = -E_1^{\parallel}\ dl$$

But recall that for any closed loop

$$\oint \vec{E}\cdot d\vec{l} = 0$$

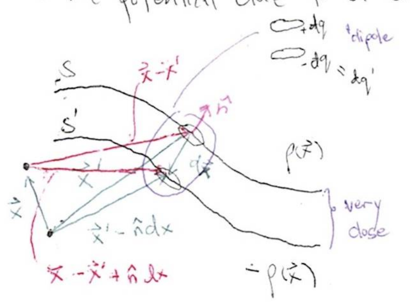
Thus,

$$\oint \vec{E}\cdot d\vec{l} = E_2^{\parallel} \cancel{dl} - E_1^{\parallel} \cancel{dl} = 0$$

$$\boxed{E_2^{\parallel} = E_1^{\parallel}}$$

The tangential component of $\vec{E}$ across a surface charge distribution is continuous

- The potential due to a dipole layer distribution

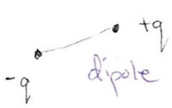


Obtain the electric potential at $\hat{x}$ in the limit as two closed surface approach each other under the ff conditions:

- $d(\vec{x}) \to 0$, $\rho(\vec{x}) \to 0$ such that the limit,

$$\lim_{d(\vec{x})\to 0} d(\vec{x})\,\rho(\vec{x}) = D(\vec{x}) \quad \text{exists}$$

Recall:



The resulting configuration in the limit is dipole layer distribution. Computing the electric potential at $\vec{x}$:

$$\Phi(\vec{x}) = \frac{1}{4\pi\epsilon_0}\int_S \frac{\overbrace{dq}^{\rho(\vec{x}')\,da'}}{|\vec{x}-\vec{x}'|} + \frac{1}{4\pi\epsilon_0}\int_{S'} \frac{dq'}{|\vec{x}-\vec{x}'+\hat{n}\,dx|}$$

$$= \frac{1}{4\pi\epsilon_0}\int_S \frac{\rho(\vec{x})\,da'}{|\vec{x}-\vec{x}'|} - \frac{1}{4\pi\epsilon_0}\int_{S'} \frac{\rho(\vec{x})\,da'}{|\vec{x}-\vec{x}'+\hat{n}\,dx|}$$

Evaluate this for arbitrarily small dx. We recall the Taylor expansion in 3D

$$f(\vec{r}+\vec{a}) = f(\vec{r}) + (\vec{a}\cdot\vec{\nabla})f(\vec{r}) + \frac{1}{2}\vec{a}\cdot[\vec{a}\cdot\vec{\nabla}(\nabla f(\vec{r}))] + \dots$$

Then,

$$\frac{1}{|\vec{x}-\vec{x}'+\underbrace{\hat{n}\,dx}_{\text{small}}|} = \frac{1}{|\vec{x}-\vec{x}'|} + (\hat{n}\,dx)\cdot\nabla\left(\frac{1}{|\vec{x}-\vec{x}'|}\right) + \underbrace{O(dx^2)}_{\text{small compared to the second term}}$$

$$\frac{1}{|\vec{x}-\vec{x}'+\hat{n}\,dx|} \approx \frac{1}{|\vec{x}-\vec{x}'|} + \hat{n}\,dx \cdot \vec{\nabla}_{\vec{x}}\left(\frac{1}{|\vec{x}-\vec{x}'|}\right)$$

$$= \frac{1}{|\vec{x}-\vec{x}'|} - n\,dx \cdot \nabla_{\vec{x}'}\left(\frac{1}{|\vec{x}-\vec{x}'|}\right)$$

Then,

$$\Phi(\vec{x}) = \frac{1}{4\pi\epsilon_0}\int_S \cancel{\frac{\rho(\vec{x})}{|\vec{x}-\vec{x}'|}}\, da'$$

$$= \frac{1}{4\pi\epsilon_0}\int \rho(\vec{x}')\left[\cancel{\frac{1}{|\vec{x}-\vec{x}'|}} - \hat{n}\,dx \cdot \vec{\nabla}'\left(\frac{1}{|\vec{x}-\vec{x}'|}\right)\right] da'$$

Thus,

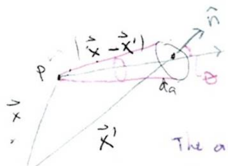
S and S' are essentially on top of each other so that one can integrate either surface and get the same result

$$\Phi(\vec{x}) = \frac{1}{4\pi\epsilon_0}\int d(\vec{x}')\,\rho(\vec{x}')\,\hat{n}\cdot\vec{\nabla}'\left(\frac{1}{|\vec{x}-\vec{x}'|}\right) da' + O(dx^2)$$

Take the limit as $d(\vec{x}) \to 0$, $d(\vec{x})\,\rho(\vec{x}) \to D(\vec{x}')$, $O(d\vec{x}) \to 0$,

$$\Phi(\vec{x}) = \frac{1}{4\pi\epsilon_0}\int \rho(\vec{x})\,\hat{n}\cdot\vec{\nabla}'\left(\frac{1}{|\vec{x}-\vec{x}'|}\right) da'$$

Now, note that



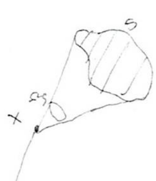
The area da substends a differential solid angle $d\Omega$

$$\hat{n}\cdot\nabla\left(\frac{1}{|\vec{x}-\vec{x}'|}\right) = \underbrace{-\hat{n}'\cdot\frac{\vec{x}-\vec{x}'}{|\vec{x}-\vec{x}'|^2}\,da}_{d\Omega}$$

(Recall: $d\Omega = \frac{d\vec{A}\cdot\hat{n}}{r^2}$)

Thus,

$$\Phi(\vec{x}) = \frac{1}{4\pi\epsilon_0}\int_S D(\vec{x}')\,d\Omega$$



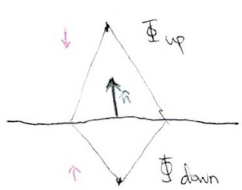
If it happens that $D(\vec{x}') = D_0 = \text{constant}$,

the solid angle subtended by surface S at $\vec{x}$

$$\Phi(\vec{x}) = -\frac{D_0}{4\pi\epsilon_0}\int d\Omega = -\frac{D_0\,\Omega_S}{4\pi\epsilon_0}$$

▶ Discontinuity of the potential across a dipole distribution

What is the diff. in the potential just above & below the surface?



$$\Phi_{down} = -\frac{1}{4\pi}D_0\Omega_S = -\frac{1}{4\pi}D_0\,2\pi = -\frac{1}{2\epsilon_0}D_0$$

$$\Phi_{up} = -(-1)\frac{1}{4\pi}D_0\Omega_S = \frac{1}{4\pi}D_0\,2\pi = \frac{1}{2\epsilon_0}D_0$$

$$\boxed{\Phi_{up} - \Phi_{down}} = \frac{1}{2\epsilon_0}D_0 - \left(-\frac{1}{2\epsilon_0}D_0\right) \boxed{= \frac{D_0}{\epsilon_0}}$$

the electric potential is discontinuous a dipole surface distribution

Oct. 6, 2020

▶ Lecture Notes

▷ Poisson & Laplace Eqns

$$\underbrace{\nabla\cdot\vec{E} = \frac{\rho}{\epsilon_0} \qquad \nabla\times\vec{E} = 0}_{\text{uniquely det. } \vec{E}}$$

Given the curl of $\vec{E}$, we have $\vec{E} = -\nabla\Phi$. Then

$$\nabla\cdot\vec{E} = \nabla\cdot(-\nabla\Phi) = -\nabla^2\Phi$$

$$\boxed{\nabla^2\Phi = \frac{\rho}{\epsilon_0}}$$ Poisson Eqn

describes the scalar potential in the presence of charges

In the abscence of charges in a given region, i.e. $\rho = 0$,

$$\nabla^2\Phi = 0$$ Laplace Eqn

A soln to Poisson Eqn is given by

$$\Phi(\vec{x}) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\vec{x}')}{|\vec{x}-\vec{x}'|}\, d^3x'$$

That this is a soln to Poisson Eqn follows from the fact that:

$$\nabla^2\left(\frac{1}{|\vec{x}-\vec{x}'|}\right) = -4\pi\,\delta(\vec{x}-\vec{x}')$$

Then,

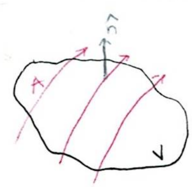
$$\nabla^2\Phi(\vec{x}) = \nabla^2 \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\vec{x}')}{|\vec{x}-\vec{x}'|}\, d^3x'$$

$$= \frac{1}{4\pi\epsilon_0}\int \nabla^2\left(\frac{1}{|\vec{x}-\vec{x}'|}\right)\rho(\vec{x}')\, d^3x$$

$$= \frac{1}{4\pi\epsilon_0}\int -4\pi\,\delta(\vec{x}-\vec{x}')\,\rho(\vec{x}')\, d^3x'$$

$$\nabla^2\Phi(\vec{x}') = -\frac{\rho(\vec{x})}{\epsilon_0}$$

▷ Green's theorem



V is bounded by the surface S

A vector field A in V is def. as

$$\int_V \nabla\cdot\vec{A}\, d^3x = \oint_S \vec{A}\cdot\hat{n}\, da$$

Let $A = \phi\nabla\psi$ where ϕ and ψ are arbitrary scalar fields. Then,

$$\vec{\nabla}\cdot(\phi\nabla\psi) = \phi\nabla^2\psi + (\nabla\phi)\cdot(\nabla\psi)$$

$$\phi\,\nabla\psi\cdot\hat{n} = \phi\,\hat{n}\cdot\nabla\psi, \qquad \hat{n}\cdot\vec{\nabla} = \frac{\partial}{\partial n}$$

→ directional derivative: the partial derivative in the direction of $\hat{n}$ / rate of change in a given direction

Then, the Green's theorem implies

$$\int_V (\phi\nabla^2\psi + \nabla\phi\cdot\nabla\psi)\, d^3x' = \oint_S \phi\frac{\partial\psi}{\partial n}\, da'$$

Interchanging $\psi \leftrightarrow \phi$

$$\int_V (\psi\nabla^2\phi + \nabla\psi\cdot\nabla\phi)\, d^3x' = \oint_S \psi\frac{\partial\phi}{\partial n}\, da'$$

Subtracting these eqns, $[\nabla\phi\cdot\nabla\psi - \nabla\psi\cdot\nabla\phi = 0]$

$$\boxed{\int_V (\phi\nabla^2\psi - \psi\nabla^2\phi)\, d^3x' = \oint_S \left(\phi\frac{\partial\psi}{\partial n} - \psi\frac{\partial\phi}{\partial n}\right) da'}$$ Green's 2nd identity / Green's theorem

Let $\psi = \frac{1}{R} = \frac{1}{|\vec{x}-\vec{x}'|}$, $\phi = \Phi(\vec{x})$. We have, for the integrand of the LHS,

$$\phi\nabla^2\psi - \psi\nabla^2\phi = \Phi\,\nabla^2\left(\frac{1}{|\vec{x}-\vec{x}'|}\right) - \frac{1}{|\vec{x}-\vec{x}'|}\nabla^2\Phi$$

$$= \Phi(\vec{x}')(-4\pi)\,\delta(\vec{x}-\vec{x}') - \frac{1}{|\vec{x}-\vec{x}'|}\,\frac{-\rho(\vec{x}')}{\epsilon_0}$$

$$\phi \nabla^2 \psi - \psi \nabla^2 \phi = -4\pi \, \Phi(\vec{x}')\, \delta(\vec{x}-\vec{x}') + \frac{\rho(\vec{x}')}{\epsilon\, |\vec{x}-\vec{x}'|}$$

Then,

$$\underbrace{\int_V \left[-4\pi\, \Phi(\vec{x}')\, \delta(\vec{x}-\vec{x}') + \frac{\rho(\vec{x}')}{\epsilon_0 |\vec{x}-\vec{x}'|} \right] d^3x' = -4\pi\, \Phi(\vec{x}) + \int_V \frac{\rho(\vec{x}')}{\epsilon_0 |\vec{x}-\vec{x}'|} d^3x'}_{\mathrm{I}}$$

$$\mathrm{I} = \oint \left[\Phi(\vec{x}') \frac{\partial}{\partial n'}\left(\frac{1}{R}\right) - \frac{1}{R}\frac{\partial \Phi}{\partial n'} \right] da'$$

Solve for $\Phi(\vec{x})$,

$$\Phi(\vec{x}) = \underbrace{\frac{1}{4\pi\epsilon_0}\int_V \frac{\rho(\vec{x}')}{|\vec{x}-\vec{x}'|} d^3x'}_{\text{due to charge distribution}} + \underbrace{\frac{1}{4\pi}\oint_S \left[\frac{1}{R}\frac{\partial \Phi}{\partial n} - \Phi(\vec{x}')\frac{\partial}{\partial n'}\left(\frac{1}{R}\right) \right] da'}_{\text{boundary values on the surface bounding the volume V}}$$

if $\vec{x}$ lies inside the volume V.

Note that

$$\int_V \Phi(\vec{x}')\, \delta(\vec{x}-\vec{x}')\, d^3x' = \begin{cases} \Phi(\vec{x}), & \vec{x} \text{ is inside} \\ 0, & \vec{x} \text{ is outside} \end{cases}$$

In the expression for $\Phi(\vec{x})$, the scalar potential is revealed as arising from the charge distribution and the values of the potential on the surface boundary of the volume V

Remarks:

1) Assume that all charges are localized, i.e. charges are not present at infinity. Then, the scalar potential will vanish at infinity

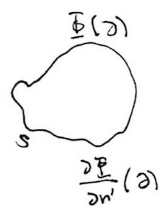
If we let S → infinity, the surface contribution will vanish. Thus, we have

$$\Phi(\vec{x}) = \frac{1}{4\pi\epsilon_0}\int_V \frac{\rho(\vec{x}')}{|\vec{x}-\vec{x}'|} d^3x'$$

2) For a charge-free volume, i.e. $\rho(\vec{x}) = 0$ inside V, $\Phi(\vec{x})$ is terms of its value and its derivative on the surface

$$\Phi(\vec{x}) = \frac{1}{4\pi}\oint_S \left[\frac{1}{R}\frac{\partial \Phi}{\partial n'} - \Phi \frac{\partial}{\partial n'}\left(\frac{1}{R}\right) \right] da'$$

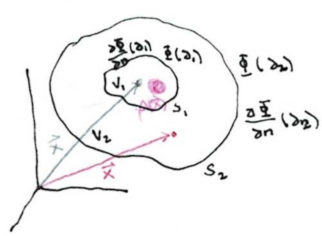
∂: on the surface



when $\Phi(\partial)$ is given → Dirichlet boundary condition

When $\frac{\partial \Phi}{\partial n'}(\partial)$ is given → Neumann boundary condition

* what if there is an inner surface?



For the case of $\vec{x}$,

$$\Phi(x) = \frac{1}{4\pi\epsilon_0}\int_{V_1} \frac{\rho(\vec{x}')}{|\vec{x}-\vec{x}'|} d^3x' + \frac{1}{4\pi}\oint_{\partial_1} \left[\frac{1}{R}\frac{\partial \Phi}{\partial n'} - \Phi(\vec{x}')\frac{\partial}{\partial n'}\left(\frac{1}{R}\right) \right] da_1$$

↕ gives the same result but take note of the conditions

$$\Phi(\vec{x}) = \frac{1}{4\pi\epsilon_0}\int_{V_2} \frac{\rho(\vec{x}')}{|\vec{x}-\vec{x}'|} d^3x' + \frac{1}{4\pi}\oint_{S_2} \left[\frac{1}{R}\frac{\partial \Phi}{\partial n'} - \Phi(\vec{x}')\frac{\partial}{\partial n'}\left(\frac{1}{R}\right) \right] da_2$$

For the case of $\vec{x}$, it is not inside V_1 so $\Phi(x) = 0$ on S_1 [i.e. S_1 has no relation to $\Phi(\vec{x})$]

$$\Phi(x) = \frac{1}{4\pi\epsilon_0}\int_{V_2} \frac{\rho(\vec{x}')}{|\vec{x}-\vec{x}'|} d^3x' + \frac{1}{4\pi}\oint_{S_2} \left[\frac{1}{R}\frac{\partial \Phi}{\partial n'} - \Phi(\vec{x}')\frac{\partial}{\partial n'}\left(\frac{1}{R}\right) \right] da_2$$

* What if we have a charge distribution outside of the considered volume?

10/6/20

$$\Phi(x) = \frac{1}{4\pi\epsilon_0}\int_{V_1} \frac{\rho(\vec{x}')}{|\vec{x}-\vec{x}'|} d^3x' + \frac{1}{4\pi}\oint_{S_1}\left[\frac{1}{R}\frac{\partial\Phi}{\partial n'} - \Phi(\vec{x}')\frac{\partial}{\partial n'}\frac{1}{R}\right] da'$$

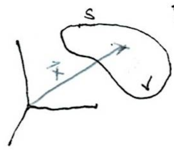
only this part will contribute

what happens to the contributions of this part? } The contributions are taken care of by the boundary conditions

Oct 8, 2020

▶ Lecture Notes

▶ Boundary Conditions



Recall: $\Phi(\vec{x}) = \dfrac{1}{4\pi\epsilon_0}\displaystyle\int_V \frac{\rho(\vec{x})}{|\vec{x}-\vec{x}'|} d^3x'$ ◂ usual scalar potential

$$+ \frac{1}{4\pi}\oint\left[\frac{1}{R}\frac{\partial\Phi}{\partial\eta} - \Phi\frac{\partial}{\partial\eta}\left(\frac{1}{R}\right)\right] da'$$ ◂ if $\vec{x}$ is inside volume V ◂

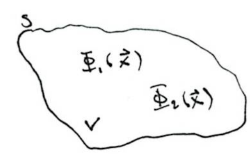
from the above expression

★ Does it follow that we need to impose simultaneous conditions on Φ and $\frac{\partial\Phi}{\partial\eta}$? i.e. Do we need to specify Φ and $\frac{\partial\Phi}{\partial\eta}$ on the surface to uniquely solve the Poisson eqn? $\longrightarrow \nabla^2\Phi = -\frac{\rho}{\epsilon_0}$

Let $\Phi(\vec{x}')$

$$\nabla^2\Phi = -\frac{\rho}{\epsilon_0}$$

Assume the existence of two solns Φ_1 & Φ_2 satisfying the same b.c.



$$\Phi_1(\partial) = \Phi_2(\partial)$$

$$\frac{\partial\Phi_1}{\partial\eta}(\partial) = \frac{\partial\Phi}{\partial\eta}(\partial)$$

but $\Phi_1(\vec{x}) \neq \Phi_2(\vec{x})$ in the volume V

Let $u = \Phi_1 - \Phi_2$ where u satisfies the boundary condition

$$u(\partial) = \Phi_1(\partial) - \Phi_2(\partial) \longrightarrow u(\partial) = 0$$

or

$$\frac{\partial u}{\partial n}(\partial) = \frac{\partial\Phi_1}{\partial\eta}(\partial) - \frac{\partial\Phi_2}{\partial\eta}(\partial) \longrightarrow \frac{\partial u}{\partial n} = 0$$

Recall: Green's 2nd identity

$$\int_V (\phi\nabla^2\psi + \nabla\phi\cdot\nabla\psi)\, d^3x = \oint_S \phi\frac{\partial\psi}{\partial\eta} da$$

Let $\phi = \psi = u$. Then

$$\int_V (u\cancelto{0}{\nabla^2 u} + \nabla u\cdot\nabla u)\, d^3x = \oint_S u\frac{\partial u}{\partial\eta} da$$

Note that $\nabla^2 u = \nabla^2\Phi_1 - \nabla^2\Phi_2 = -\frac{\rho_0}{\epsilon_0} + -\frac{\rho_0}{\epsilon_0} = 0$. Thus,

$$\int_V \underbrace{\nabla\vec{u}\cdot\nabla u}_{|\nabla u|^2}\, d^3x = \oint u\frac{\partial u}{\partial\eta} da$$

When $u = 0$ on the surface, i.e. $\Phi_1(\partial) = \Phi_2(\partial)$,

this is on the surface $\oint_S u\frac{\partial u}{\partial n} da = 0 \longrightarrow \int_V |\nabla u|^2 d^3x = 0$

Since the volume is arbitrary and $|\nabla u|^2$ is positive definite. Then,

$$\int_V |\nabla u|^2 d^3x = 0 \quad \text{is true only if} \quad |\nabla u|^2 = 0$$

$$u = \text{constant}$$

$$\Phi_1 - \Phi_2 = \text{constant} \longrightarrow \Phi_1 = \Phi_2 + \text{constant}$$

 10/8/20

Therefore, if $\Phi_1(\partial) = \Phi_2(\partial)$, then Φ_1 & Φ_2 only differ by a constant

$$\vec{E} = -\vec{\nabla}\Phi$$

So that $E_1 = -\nabla\Phi_1$, $E_2 = -\nabla\Phi_2 = -\nabla\Phi_1$

$$\Rightarrow \vec{E}_1 = \vec{E}_2$$

This means that we get a unique soln to the Poisson eqn up to an additional constant if the potential is specified on the surface (Dirichlet conditions) w/o specifying $\frac{\partial\Phi}{\partial n}$ on the surface

If we have $\frac{\partial u}{\partial n}(\partial) = 0$

$$\int |\nabla u|^2 d^3x \rightarrow \nabla u = 0 \rightarrow u = \text{constant}$$

This implies that we obtain a unique soln on the Poissoin eqn by specifying the Neumann conditions w/o Dirichlet

$\Rightarrow$ Either we impose on the surface

Φ (Dirichlet conditions), $\frac{\partial\Phi}{\partial n}$ (Neumann conditions)

to obtain a unique soln up to a additive constant.

* Can we impose simultaneously Dirichlet & Neumann boundary conditions on a surface?

No! because the unique soln to the Poisson eqn satisfying a Dirichlet condition is, in general, inconsistent with an arbitrary Neumann condition

If Φ is imposed $\longrightarrow$ $\frac{\partial\Phi}{\partial n}$ is already determined on the surface

" $\frac{\partial\Phi}{\partial n}$ " " $\longrightarrow$ Φ " " "

▶ Formal Solution to Boundary Value Problems with Green's Function

We want to obtain a soln $\Phi(x)$ using Green's function. In 3D, a Green's function is a soln to the homogeneous

$$\nabla^2 G(\vec{x},\vec{x}') = -4\pi\,\delta(\vec{x}-\vec{x}')$$

Recall: $\nabla^2\left(\frac{1}{|\vec{x}-\vec{x}'|}\right) = -4\pi\,\delta(\vec{x},\vec{x}')$

Then, $\frac{1}{|\vec{x}-\vec{x}'|}$ is a Green's function.

* Is this the only Green's function we can construct?

Let $F(\vec{x};\vec{x}')$ be a soln to the Laplace eqn

$$\nabla^2 F(\vec{x},\vec{x}') = 0$$

Then the gen soln to the Green's fxn eqn is given by

$$G(\vec{x},\vec{x}') = \frac{1}{|\vec{x}-\vec{x}'|} + F(\vec{x},\vec{x}')$$

since

$$\nabla^2 G(\vec{x},\vec{x}') = \nabla^2\left(\frac{1}{|\vec{x}-\vec{x}'|}\right) + \cancelto{0}{\nabla^2 F(\vec{x},\vec{x}')} = -4\pi\,\delta(\vec{x},\vec{x}')$$

Recall again Green's theorem:

Let $\phi = \Phi$, $\psi = G(\vec{x},\vec{x}')$

$$\int_V \left[\Phi(\vec{x}')\,\nabla^2 G(\vec{x},\vec{x}') - G(\vec{x},\vec{x}')\,\nabla^2\Phi(\vec{x}')\right] d^3x$$

$$= \int_V \left[\Phi(\vec{x}')(-4\pi\,\delta(\vec{x}-\vec{x}')) - G(\vec{x},\vec{x}')\left(-\frac{\rho(\vec{x})}{\epsilon_0}\right)\right] d^3x$$

$$= -4\pi\,\Phi(\vec{x}) + \frac{1}{\epsilon_0}\int_V G(\vec{x},\vec{x}')\,\rho(\vec{x}')\,d^3x$$

Thus,

$$-4\pi\,\Phi(\vec{x}) + \frac{1}{\epsilon_0}\int_V G(\vec{x},\vec{x}')\,\rho(\vec{x}')\,d^3x' = \oint_S \left[\Phi(\vec{x}')\frac{\partial G(\vec{x},\vec{x}')}{\partial n'} - G(\vec{x},\vec{x}')\frac{\partial \Phi(\vec{x}')}{\partial n'}\right] da'$$

Solving for $\Phi(\vec{x})$

$$\Phi(\vec{x}) = \frac{1}{4\pi\epsilon_0}\int G(\vec{x},\vec{x}')\,\rho(\vec{x}')\,d^3x' - \frac{1}{4\pi}\oint_S \left[G(\vec{x},\vec{x}')\frac{\partial \Phi(\vec{x}')}{\partial n'} - \Phi(\vec{x}')\frac{\partial G(\vec{x},\vec{x}')}{\partial n'}\right] da'$$

* Assume a Dirichlet boundary condition i.e. $\Phi(\vec{x})$ is specified on the surface. If we impose the condition

$$G(\vec{x};\vec{x}') = 0 \quad \text{for all } \vec{x}' \text{ on } S$$

and call the Green's function as $G_D(\vec{x},\vec{x}')$, then we have the soln

$$\boxed{\Phi(\vec{x}) = \frac{1}{4\pi\epsilon_0}\int_V G_D(\vec{x},\vec{x}')\,\rho(\vec{x}')\,d^3x' - \frac{1}{4\pi}\oint \Phi(\vec{x}')\frac{\partial G_D(\vec{x},\vec{x}')}{\partial n'}\,da'}$$

* For a Neumann boundary condition i.e. normal component of $\vec{E}$ $\left[\frac{\partial \Phi(\vec{x})}{\partial n'}\right]$ is specified on the surface, does it require

$$\frac{\partial G(\vec{x},\vec{x}')}{n} = 0 \quad \text{for all } \vec{x}' \text{ on } S\,?$$

No! because this condition leads to an inconsistency

Proof: Note that

$$\int_V \nabla^2 G(\vec{x},\vec{x}')\,d^3x = \int_V -4\pi\,\delta(\vec{x}-\vec{x}')\,d^3x' = -4\pi$$

Using Green's theorem,

$$\int_V \nabla'^2 G(\vec{x},\vec{x}')\,d^3x = \int_V \nabla'\cdot\nabla' G(\vec{x},\vec{x}')\,d^3x' \left(= \oint_S \hat{n}\cdot\nabla' G(\vec{x},\vec{x}')\,da' = \oint \frac{\partial G(\vec{x},\vec{x}')}{\partial n'}\,da'\right)$$

But we have

$$\oint_S \frac{\partial G(\vec{x},\vec{x}')}{\partial n'}\,da' = -4\pi$$

So if we impose $\frac{\partial G}{\partial n} = 0$, we get the contradiction $0 = -4\pi$.

The simplest possible condition is

$$\frac{\partial G_N(\vec{x},\vec{x}')}{\partial n'} = -\frac{4\pi}{S}$$

surface area

which we see from

$$\oint_S \frac{\partial G_N(\vec{x},\vec{x}')}{\partial n'}\,da' = \int -\frac{4\pi}{S}\,da = -\frac{4\pi}{S}\cdot S = -4\pi$$

Then, we have the soln

$$\boxed{\Phi(\vec{x}) = \frac{1}{S}\int_S \Phi(\vec{x}')\,da' + \frac{1}{4\pi\epsilon_0}\int_V \rho(\vec{x}')\,G_N(\vec{x},\vec{x}')\,d^3x' + \frac{1}{4\pi}\oint_S \frac{\partial \Phi(\vec{x}')}{\partial n'}\,G_N(\vec{x},\vec{x}')\,da'}$$