Problem 1 (2.3)

For a harmonic oscillator of mass m and angular frequency ω, if $x_{\text{ph}}(t)$ is the solution to the equation of motion and $x(t) = x_{\text{ph}}(t) + \eta(t)$ with $\eta(0) = \eta(T) = 0$ proceeding as in the previous problem,...

(a) ... show that $S[x] = S[x_{\text{ph}}] + (m/2)\int_0^T \left(\dot{\eta}^2 - \omega^2\eta^2\right) \mathrm{d}t$.

Solution:
Note that the potential for a harmonic oscillator is $V(x) = (1/2)kx^2$ where $k = m\omega^2$. Then, the Lagrangian of the system is

$$L = T - V = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2 \tag{1}$$

Also, the equation of motion for this problem is given by

$$m\ddot{x} = -\frac{\mathrm{d}V(x)}{\mathrm{d}x} = -\frac{1}{2}k\frac{\mathrm{d}x^2}{\mathrm{d}x} = -kx \tag{2}$$

The action is given by

$$\begin{aligned}
S[x] &= \int_0^T L(x(t), \dot{x}(t), t)\mathrm{d}t \\
&= \int_0^T \left(\frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2\right)\mathrm{d}t \\
&= \int_0^T \left\{\frac{1}{2}m\left[\frac{\mathrm{d}}{\mathrm{d}t}(x_{\text{ph}} + \eta)\right]^2 - \frac{1}{2}k(x_{\text{ph}} + \eta)^2\right\}\mathrm{d}t \\
&= \int_0^T \left[\frac{1}{2}m\left(\dot{x}_{\text{ph}} + \dot{\eta}\right)^2 - \frac{1}{2}k(x_{\text{ph}} + \eta)^2\right]\mathrm{d}t \\
&= \int_0^T \left(\frac{1}{2}m\dot{x}_{\text{ph}}^2 + \frac{1}{2}m\dot{\eta}^2 + m\dot{x}_{\text{ph}}\dot{\eta} - \frac{1}{2}kx_{\text{ph}}^2 - \frac{1}{2}k\eta^2 + mx_{\text{ph}}\eta\right)\mathrm{d}t \\
&= \int_0^T \left(\frac{1}{2}m\dot{x}_{\text{ph}}^2 - \frac{1}{2}kx_{\text{ph}}^2\right)\mathrm{d}t + \int_0^T \left(\frac{1}{2}m\dot{\eta}^2 - \frac{1}{2}k\eta^2\right)\mathrm{d}t \\
&\qquad + \int_0^T \left(m\dot{x}_{\text{ph}}\dot{\eta} + kx_{\text{ph}}\eta\right)\mathrm{d}t
\end{aligned} \tag{3}$$

Using IBP,

$$\begin{aligned}
u &= m\dot{x}_{\text{ph}}, \quad \mathrm{d}u = m\mathrm{d}(\dot{x}_{\text{ph}}) = m\frac{\partial}{\partial t}(\dot{x}_{\text{ph}})\mathrm{d}t = m\frac{\mathrm{d}}{\mathrm{d}t}(\dot{x}_{\text{ph}})\mathrm{d}t \\
\mathrm{d}v &= \dot{\eta}\mathrm{d}t, \quad v = \int \dot{\eta}\mathrm{d}t = \int \frac{\mathrm{d}\eta}{\mathrm{d}t}\mathrm{d}t = \frac{\mathrm{d}}{\mathrm{d}t}\int \eta\mathrm{d}t = \eta
\end{aligned} \tag{4}$$

the first term of the third integral in the action becomes

$$\int_0^T m\dot{x}_{\text{ph}}\dot{\eta}\mathrm{d}t = \cancelto{0}{m\dot{x}_{\text{ph}}\eta|_0^T} - \int_0^T \eta m\ddot{x}_{\text{ph}}\mathrm{d}t \tag{5}$$

Then, the third integral becomes

$$\int_0^T \left(m\dot{x}_{\text{ph}}\dot{\eta} + m x_{\text{ph}}\eta\right) \mathrm{d}t = \int_0^T \left(m\ddot{x}_{\text{ph}}\eta + m x_{\text{ph}}\eta\right) \mathrm{d}t = \int_0^T \cancelto{0}{\left(m\ddot{x}_{\text{ph}} + m x_{\text{ph}}\right)} \eta \mathrm{d}t \tag{6}$$

after applying the relation in Eq. (2). Therefore, the action can be expressed as

$$\begin{aligned} S[x] &= \int_0^T \left(\frac{1}{2}m\dot{x}_{\text{ph}}^2 - \frac{1}{2}kx_{\text{ph}}^2\right) \mathrm{d}t + \int_0^T \left(\frac{1}{2}m\dot{\eta}^2 - \frac{1}{2}k\eta^2\right) \mathrm{d}t \\ &= S[x_{\text{ph}}] + \frac{m}{2}\int_0^T \left(\dot{\eta}^2 - \omega^2\eta^2\right) \mathrm{d}t \end{aligned} \tag{7}$$

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(b) Expanding η in the Fourier series $\eta(t) = \sum_{n=1}^{\infty} C_n \sin(n\pi t/T)$(why is this possible?), show that $S[x] = S[x_{\text{ph}}] + (mT/4)\sum_{n=1}^{\infty}\left(\frac{n^2\pi^2}{T^2} - \omega^2\right) C_n^2$ and conclude that the action is a minimum for the physical path if $T < \pi/\omega$. An elementary treatment of this problem can be found in Moriconi (2017), where it is shown that the condition $T < \pi/\omega$ is also necessary for a minimum.

Solution:
We can expand η as a sine Fourier series since the sine function satisfies the boundary conditions of η. Taking a time derivative of this expansion, we have

$$\dot{\eta} = \sum_{n=1}^{\infty} C_n \frac{\mathrm{d}}{\mathrm{d}t}\left[\sin\left(\frac{n\pi t}{T}\right)\right] = \sum_{n=1}^{\infty} C_n \frac{n\pi}{T}\cos\left(\frac{n\pi t}{T}\right) \tag{8}$$

Plugging η and $\dot{\eta}$, we obtain

$$\begin{aligned} \int_0^T \left(\dot{\eta}^2 - \omega^2\eta^2\right) \mathrm{d}t &= \sum_{n=1}^{\infty} C_n^2 \left[\frac{n^2\pi^2}{T^2}\int_0^T \cos^2\left(\frac{n\pi t}{T}\right) \mathrm{d}t - \omega^2 \int_0^T \sin^2\left(\frac{n\pi t}{T}\right) \mathrm{d}t\right] \\ &= \sum_{n=1}^{\infty} C_n^2 \left\{\frac{n^2\pi^2}{2T^2}\int_0^T \left[1 + \cos\left(\frac{2n\pi t}{T}\right)\right] \mathrm{d}t - \frac{\omega^2}{2}\int_0^T \left[1 - \cos\left(\frac{2n\pi t}{T}\right)\right] \mathrm{d}t\right\} \\ &= \sum_{n=1}^{\infty} C_n^2 \left\{\frac{n^2\pi^2}{2T^2}\left[t + \sin\left(\frac{n\pi t}{T}\right)\right]_0^T - \frac{\omega^2}{2}\left[t - \sin\left(\frac{n\pi t}{T}\right)\right]_0^T\right\} \\ &= \sum_{n=1}^{\infty} C_n^2 \left[\frac{n^2\pi^2}{2T^2}T - \frac{\omega^2}{2}T\right] \\ \int_0^T \left(\dot{\eta}^2 - \omega^2\eta^2\right) \mathrm{d}t &= \frac{T}{2}\sum_{n=1}^{\infty} C_n^2 \left[\frac{n^2\pi^2}{T^2} - \omega^2\right] \end{aligned} \tag{9}$$

Then, the action becomes

$$\begin{aligned} S[x] &= S[x_{\text{ph}}] + \frac{m}{2}\left(\frac{T}{2}\sum_{n=1}^{\infty} C_n^2 \left[\frac{n^2\pi^2}{T^2} - \omega^2\right]\right) \\ &= S[x_{\text{ph}}] + \frac{mT}{4}\sum_{n=1}^{\infty} C_n^2 \left[\frac{n^2\pi^2}{T^2} - \omega^2\right] \end{aligned} \tag{10}$$

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Since we want the action to be a minimum of the physical path, we have

$$\begin{aligned} S[x] - S[x_{\text{ph}}] &> 0 \\ \frac{mT}{4}\sum_{n=1}^{\infty} C_n^2 \left[\frac{n^2\pi^2}{T^2} - \omega^2\right] &> 0 \end{aligned} \tag{11}$$

With $n = 1$ as the lowest value of n, we have

$$\begin{aligned} \frac{(1)^2\pi^2}{T^2} - \omega^2 > 0 \\ \frac{\pi^2}{T^2} > \omega^2 \longrightarrow \boxed{T < \frac{\pi}{\omega}} \end{aligned} \tag{12}$$

as the condition for the action to be a minimum of the physical path.

References

Moriconi, M., *Condition for minimal harmonic oscillator action*, American Journal of Physics 85, 633 (2017), `https://doi.org/10.1119/1.4984778`