I. Position and Momentum

In three dimensions:

(a) What is the action of the momentum operator $\hat{\mathbf{p}}$ on the wavefunction $\langle \mathbf{p}|\psi\rangle$?

Solution:
In momentum space, the action of $\hat{\mathbf{p}}$ on $\langle \mathbf{p}|\psi\rangle$ is

$$\langle \mathbf{p}|\hat{\mathbf{p}}|\psi\rangle = p\,\langle \mathbf{p}|\psi\rangle = p\psi(\mathbf{p}) \tag{1}$$

where p is the magnitude of $\mathbf{p}$.

(b) What is the action of the momentum operator $\hat{\mathbf{p}}$ on the wavefunction $\langle \mathbf{x}|\psi\rangle$?

Solution:
In position space, the action of $\hat{\mathbf{p}}$ on $\langle \mathbf{x}|\psi\rangle$ is

$$\langle \mathbf{x}|\hat{\mathbf{p}}|\psi\rangle = \frac{\hbar}{i}\nabla\,\langle \mathbf{x}|\psi\rangle = \frac{\hbar}{i}\nabla\psi(\mathbf{x}) \tag{2}$$

where $\nabla = \frac{\partial}{\partial x}\hat{\boldsymbol{\imath}} + \frac{\partial}{\partial y}\hat{\boldsymbol{\jmath}} + \frac{\partial}{\partial z}\hat{\boldsymbol{k}}$.

II. Translation

On an arbitrary state $|\psi\rangle$:

(a) How does a finite translation $\mathscr{T}(\mathbf{l})$ affect $\langle \hat{\mathbf{x}}\rangle_\psi$?

Solution:
We can use the result of problem 1.29 in Sakurai in which it is given that

$$[\hat{x}_i, G(\hat{\mathbf{p}})] = i\hbar\frac{\partial G}{\partial \hat{p}_i} \tag{3}$$

or, equivalently,

$$[\hat{\mathbf{x}}, G(\hat{\mathbf{p}})] = i\hbar\nabla_{\hat{\mathbf{p}}}G(\hat{\mathbf{p}}) \tag{4}$$

where $\hat{\mathbf{p}} = \hat{p}_x\hat{\boldsymbol{\imath}} + \hat{p}_y\hat{\boldsymbol{\jmath}} + \hat{p}_z\hat{\boldsymbol{k}}$ and $\nabla_{\hat{\mathbf{p}}} = \frac{\partial}{\partial \hat{p}_x}\hat{\boldsymbol{\imath}} + \frac{\partial}{\partial \hat{p}_y}\hat{\boldsymbol{\jmath}} + \frac{\partial}{\partial \hat{p}_z}\hat{\boldsymbol{k}}$. Now, the finite translation operator is

$$\mathscr{T}(\mathbf{l}) = \exp\left(\frac{-i\hat{\mathbf{p}}\cdot\mathbf{l}}{\hbar}\right) \tag{5}$$

where $\mathbf{l} = l_x\hat{\boldsymbol{\imath}} + l_y\hat{\boldsymbol{\jmath}} + l_z\hat{\boldsymbol{k}}$ is a displacement vector. Applying Eq. (4), we have

$$[\hat{\mathbf{x}}, \mathscr{T}(\mathbf{l})] = i\hbar\nabla_{\hat{\mathbf{p}}}\mathscr{T}(\mathbf{l}) = i\hbar\nabla_{\hat{\mathbf{p}}}\left(e^{\frac{-i\hat{\mathbf{p}}\cdot\mathbf{l}}{\hbar}}\right) \tag{6}$$

Note that

$$\begin{aligned}
\nabla\left(e^{\frac{-i\hat{\mathbf{p}}\cdot\mathbf{l}}{\hbar}}\right) &= i\hbar\left[\frac{\partial}{\partial \hat{p}_x}\left(e^{\frac{-i\hat{\mathbf{p}}\cdot\mathbf{l}}{\hbar}}\right)\hat{\boldsymbol{\imath}} + \frac{\partial}{\partial \hat{p}_y}\left(e^{\frac{-i\hat{\mathbf{p}}\cdot\mathbf{l}}{\hbar}}\right)\hat{\boldsymbol{\jmath}} + \frac{\partial}{\partial \hat{p}_z}\left(e^{\frac{-i\hat{\mathbf{p}}\cdot\mathbf{l}}{\hbar}}\right)\hat{\boldsymbol{k}}\right] \\
&= i\cancel{\hbar}\left(-\frac{i}{\cancel{\hbar}}\right)e^{\frac{-i\hat{\mathbf{p}}\cdot\mathbf{l}}{\hbar}}\left(l_x\hat{\boldsymbol{\imath}} + l_y\hat{\boldsymbol{\jmath}} + l_z\hat{\boldsymbol{k}}\right) \\
\nabla\left(e^{\frac{-i\hat{\mathbf{p}}\cdot\mathbf{l}}{\hbar}}\right) &= \mathbf{l}\,e^{\frac{-i\hat{\mathbf{p}}\cdot\mathbf{l}}{\hbar}}
\end{aligned} \tag{7}$$

where $\hat{\mathbf{p}} \cdot \mathbf{l} = \hat{p}_x l_x + \hat{p}_y l_y + \hat{p}_z l_z$. So,

$$[\hat{\mathbf{x}}, \mathscr{T}(\mathbf{l})] = \mathbf{l}\,\mathscr{T}(\mathbf{l}) \tag{8}$$

The commutator of the position and finite translation operator is also defined as

$$[\hat{\mathbf{x}}, \mathscr{T}(\mathbf{l})] \equiv \hat{\mathbf{x}}\mathscr{T}(\mathbf{l}) - \mathscr{T}(\mathbf{l})\hat{\mathbf{x}} \tag{9}$$

Note that $\mathscr{T}^\dagger(\mathbf{l})\mathscr{T}(\mathbf{l}) = \left(\mathscr{T}^*(\mathbf{l})\right)^T \mathscr{T}(\mathbf{l}) = \exp\left(\frac{i\hat{\mathbf{p}}\cdot\mathbf{l}}{\hbar}\right) \cdot \exp\left(\frac{-i\hat{\mathbf{p}}\cdot\mathbf{l}}{\hbar}\right) = 1$. Multiplying $\mathscr{T}^\dagger(\mathbf{l})$ to Eq. (9),

$$\mathscr{T}^\dagger(\mathbf{l})[\hat{\mathbf{x}}, \mathscr{T}(\mathbf{l})] = \mathscr{T}^\dagger(\mathbf{l})\,\hat{\mathbf{x}}\mathscr{T}(\mathbf{l}) - \cancelto{1}{\mathscr{T}^\dagger(\mathbf{l})\mathscr{T}(\mathbf{l})}\,\hat{\mathbf{x}} = \mathscr{T}^\dagger(\mathbf{l})\,\hat{\mathbf{x}}\mathscr{T}(\mathbf{l}) - \hat{\mathbf{x}} \tag{10}$$

Also, from Eq. (8),

$$\mathscr{T}^\dagger(\mathbf{l})[\hat{\mathbf{x}}, \mathscr{T}(\mathbf{l})] = \mathscr{T}^\dagger(\mathbf{l})\,\mathbf{l}\mathscr{T}(\mathbf{l}) = \cancelto{1}{\mathscr{T}^\dagger(\mathbf{l})\mathscr{T}(\mathbf{l})}\,\mathbf{l} = \mathbf{l} \tag{11}$$

in which we can interchange $\mathbf{l}$ and $\mathscr{T}(\mathbf{l})$ since $\mathbf{l}$ itself is not an operator. Equating these relations, we get

$$\mathscr{T}^\dagger(\mathbf{l})\,\hat{\mathbf{x}}\mathscr{T}(\mathbf{l}) - \hat{\mathbf{x}} = \mathbf{l} \longrightarrow \mathscr{T}^\dagger(\mathbf{l})\,\hat{\mathbf{x}}\mathscr{T}(\mathbf{l}) = \hat{\mathbf{x}} + \mathbf{l} \tag{12}$$

Then, if $\mathscr{T}(\mathbf{l})$ acts on an arbitrary state $|\psi\rangle$ as $|\psi'\rangle = \mathscr{T}(\mathbf{l})\,|\psi\rangle$, the effect of $\mathscr{T}(\mathbf{l})$ on $\langle\hat{\mathbf{x}}\rangle_\psi$ is given by

$$\langle\hat{\mathbf{x}}\rangle_{\psi'} = \langle\psi|\mathscr{T}^\dagger(\mathbf{l})\hat{\mathbf{x}}\mathscr{T}(\mathbf{l})|\psi\rangle = \langle\psi|\hat{\mathbf{x}} + \mathbf{l}|\psi\rangle = \langle\psi|\hat{\mathbf{x}}|\psi\rangle + \mathbf{l} = \langle\hat{\mathbf{x}}\rangle_\psi + \mathbf{l} \tag{13}$$

(b) How does a finite translation $\mathscr{T}(\mathbf{l})$ affect $\langle\mathbf{p}\rangle_\psi$?

Solution:
We can use the relation in Sakurai (1.6.45)

$$[\hat{\mathbf{p}}, \mathscr{T}(\mathrm{d}\mathbf{x}')] = 0 \tag{14}$$

for the infinitesimal translation operator $\mathscr{T}(\mathrm{d}\mathbf{x}')$. Extending this to a finite translation, we have

$$[\hat{\mathbf{p}}, \mathscr{T}(\mathbf{l})] = 0 \tag{15}$$

which we expect since $\mathscr{T}(\mathbf{l})$ is still function of the momentum operator like $\mathscr{T}(\mathrm{d}\mathbf{x}')$ and $[\hat{p}_i, \hat{p}_j] = 0$. The commutator of the momentum and finite translation operator is also defined as

$$[\hat{\mathbf{p}}, \mathscr{T}(\mathbf{l})] \equiv \hat{\mathbf{p}}\mathscr{T}(\mathbf{l}) - \mathscr{T}(\mathbf{l})\hat{\mathbf{p}} \tag{16}$$

In the same manner as part (a), we multiply this commutation by $\mathscr{T}^\dagger(\mathbf{l})$ to get

$$\mathscr{T}^\dagger(\mathbf{l})[\hat{\mathbf{p}}, \mathscr{T}(\mathbf{l})] = \mathscr{T}^\dagger(\mathbf{l})\,\hat{\mathbf{p}}\mathscr{T}(\mathbf{l}) - \cancelto{1}{\mathscr{T}^\dagger(\mathbf{l})\mathscr{T}(\mathbf{l})}\,\hat{\mathbf{p}} = \mathscr{T}^\dagger(\mathbf{l})\,\hat{\mathbf{p}}\mathscr{T}(\mathbf{l}) - \hat{\mathbf{p}} \tag{17}$$

Equating Eqs. (15) and (16), we have

$$\mathscr{T}^\dagger(\mathbf{l})\,\hat{\mathbf{p}}\mathscr{T}(\mathbf{l}) - \hat{\mathbf{p}} = 0 \longrightarrow \mathscr{T}^\dagger(\mathbf{l})\,\hat{\mathbf{p}}\mathscr{T}(\mathbf{l}) = \hat{\mathbf{p}} \tag{18}$$

Then, the effect of $\mathscr{T}(\mathbf{l})$ on $\langle\mathbf{p}\rangle_\psi$ is given by

$$\langle\hat{\mathbf{p}}\rangle_{\psi'} = \langle\psi|\mathscr{T}^\dagger(\mathbf{l})\hat{\mathbf{p}}\mathscr{T}(\mathbf{l})|\psi\rangle = \langle\psi|\hat{\mathbf{p}}|\psi\rangle = \langle\hat{\mathbf{p}}\rangle_\psi \tag{19}$$

References

Quantum Theory I (MIT) Lecture 7 Notes,
`https://ocw.mit.edu/courses/physics/8-321-quantum-theory-i-fall-2017/lecture-notes/`
Sakurai, *Modern Quantum Mechanics*, Chapter 1

III. Spin Precession

(a) Consider the precession of a spin-1/2 system in an external magnetic field $\mathbf{B} = B\hat{\mathbf{z}}$. Solve the Heisenberg equations of motion to obtain $\mathbf{S}(t)$.

Solution:
The Heisenberg equation of motion is given by

$$\frac{\mathrm{d}A^{(H)}}{\mathrm{d}t} = \frac{1}{i\hbar}[A^{(H)}, H] \tag{20}$$

where $A^{(H)}$ is an arbitrary operator in the Heisenberg picture. We also have the following relations:

$$\mathscr{U}(t,0) = e^{-iHt/\hbar} \tag{21}$$
$$A^{(H)}(t) = \mathscr{U}^\dagger(t,0)A^{(S)}\mathscr{U}(t,0) \tag{22}$$
$$A^{(H)}(0) = A^{(S)} = A \tag{23}$$

where $A^{(S)}$ is the same arbitrary operator but in the Schrödinger picture and $\mathscr{U}(t,0)$ is the time evolution operator. Note as well that

$$[S_y, S_z] = i\hbar S_x, \qquad [S_z, S_x] = i\hbar S_y, \qquad [S_j, S_j] = 0 \tag{24}$$

where $\mathbf{S}$ is the spin operator with components S_j $(j = x, y, z)$. Now, the Hamiltonian for the given system is

$$H = \omega S_z, \qquad \omega = \frac{|e|B}{m_e c} \tag{25}$$

For the S_x component, the Heisenberg EOM leads to

$$\begin{aligned}
\frac{\mathrm{d}S_x^{(H)}}{\mathrm{d}t} &= \frac{1}{i\hbar}[S_x^{(H)}, H] \\
&= \frac{1}{i\hbar}\left[e^{i\omega S_z t/\hbar}\, S_x^{(H)}(0)\, e^{-i\omega S_z t/\hbar}, \omega S_z\right] \\
&= \frac{\omega}{i\hbar}e^{i\omega S_z t/\hbar}[S_x, S_z]e^{-i\omega S_z t/\hbar} \\
&= \frac{\omega}{i\hbar}e^{i\omega S_z t/\hbar}(-i\hbar S_y)e^{-i\omega S_z t/\hbar} \\
\frac{\mathrm{d}S_x^{(H)}}{\mathrm{d}t} &= -\omega S_y^{(H)}
\end{aligned} \tag{26}$$

Taking the second time derivative of S_x,

$$\begin{aligned}\frac{\mathrm{d}^2 S_x^{(H)}}{\mathrm{d}t^2} &= \frac{1}{i\hbar}\left[\frac{\mathrm{d}S_x^{(H)}}{\mathrm{d}t}, H\right]\\ &= \frac{1}{i\hbar}\left[-\omega S_y^{(H)}, \omega S_z\right]\\ &= -\frac{\omega^2}{i\hbar}\left[e^{i\omega S_z t/\hbar}\, S_y^{(H)}(0)\, e^{-i\omega S_z t/\hbar}, S_z\right]\\ &= -\frac{\omega^2}{i\hbar} e^{i\omega S_z t/\hbar}\left[S_y, S_z\right] e^{-i\omega S_z t/\hbar}\\ &= -\frac{\omega^2}{i\hbar} e^{i\omega S_z t/\hbar}(i\hbar S_x) e^{-i\omega S_z t/\hbar}\\ \ddot{S}_x^{(H)} &= -\omega^2 S_x^{(H)}\end{aligned} \tag{27}$$

Notice that Eq. (27) is a homogeneous differential equation which we can solve by applying a characteristic equation

$$m^2 = -\omega^2 m \longrightarrow m = \pm i\omega \tag{28}$$

Thus, the general solution of Eq. (27) has the form

$$S_x^{(H)}(t) = C_1 \cos\omega t + C_2 \sin\omega t \tag{29}$$

To solve for the constants, let's calculate $S_x^{(H)}(0)$ and $\dot{S}_x^{(H)}(0)$:

$$S_x^{(H)}(0) = C_1 \cancelto{1}{\cos(0)} + C_2 \cancelto{0}{\sin(0)} \longrightarrow C_1 = S_x^{(H)}(0) \tag{30}$$

$$\dot{S}_x^{(H)}(0) = -C_1 \cancelto{0}{\sin(0)} + C_2 \cancelto{1}{\cos(0)} \longrightarrow C_2 = \dot{S}_x^{(H)}(0) \tag{31}$$

Then, Eq. (29) becomes

$$S_x^{(H)}(t) = S_x^{(H)}(0)\cos\omega t + \dot{S}_x^{(H)}(0)\sin\omega t \tag{32}$$

In the same manner as $S_x^{(H)}$ and using the relation in Eq. (24), we can obtain

$$\frac{\mathrm{d}^2 S_y^{(H)}}{\mathrm{d}t^2} = -\omega^2 S_y^{(H)} \tag{33}$$

for $S_y^{(H)}$. Thus, like $S_x^{(H)}$, this has the solution

$$S_y^{(H)}(t) = S_y^{(H)}(0)\cos\omega t + \dot{S}_y^{(H)}(0)\sin\omega t \tag{34}$$

As for $S_z^{(H)}$, we have

$$\begin{aligned}\frac{\mathrm{d}S_z^{(H)}}{\mathrm{d}t} &= \frac{1}{i\hbar}[S_z^{(H)}, H]\\ &= \frac{1}{i\hbar}\left[e^{i\omega S_z t/\hbar}\, S_z^{(H)}(0)\, e^{-i\omega S_z t/\hbar}, \omega S_z\right]\\ &= \frac{\omega}{i\hbar} e^{i\omega S_z t/\hbar} \cancelto{0}{[S_z, S_z]} e^{-i\omega S_z t/\hbar}\\ \frac{\mathrm{d}S_z^{(H)}}{\mathrm{d}t} &= 0\end{aligned} \tag{35}$$

This leads to the solution

$$S_z^{(H)}(t) = C \quad \longrightarrow \quad S_z^{(H)}(t) = S_z^{(H)}(0) \tag{36}$$

where C is just some constant. We applied the initial condition that when $t = 0$, $S_z^{(H)}(0) = C$. Note that

$$\mathbf{S}(t) = \mathbf{S}^{(H)}(t) = \left(S_x^{(H)}(t), S_y^{(H)}(t), S_z^{(H)}(t)\right) \tag{37}$$

Then, from Eqs. (32), (34), and (36), $\mathbf{S}(t)$ is

$$\mathbf{S}(t) = \left(S_x^{(H)}(0)\cos\omega t + \dot{S}_x^{(H)}(0)\sin\omega t, S_y^{(H)}(0)\cos\omega t + \dot{S}_y^{(H)}(0)\sin\omega t, S_z^{(H)}(0)\right) \tag{38}$$

(b) Let the initial state be an eigenstate of S_x with eigenvalue $\hbar/2$. Show that the expectation value $\langle \mathbf{S}(t)\rangle$ precesses about the z-axis.

Solution:
The eigenstate of S_x with eigenvalue $\hbar/2$ is given by

$$|\psi(t=0)\rangle = \frac{1}{\sqrt{2}}|+\rangle + \frac{1}{\sqrt{2}}|-\rangle \tag{39}$$

Note that the expectation value $\langle \mathbf{S}(t)\rangle$ is given by

$$\langle \mathbf{S}(t)\rangle = \left(\langle S_x(t)\rangle, \langle S_y(t)\rangle, \langle S_z(t)\rangle\right) \tag{40}$$

where $|+\rangle = (1,0)$ and $|+\rangle = (0,1)$. For $\langle S_x(t)\rangle$, we have

$$\begin{aligned}
\langle S_x(t)\rangle &= \langle \psi(t=0)|S_x^{(H)}(t)|\psi(t=0)\rangle \\
&= \left[\frac{1}{\sqrt{2}}\langle +| + \frac{1}{\sqrt{2}}\langle -|\right]\left(S_x^{(H)}(0)\cos\omega t + \dot{S}_x^{(H)}(0)\sin\omega t\right)\left[\frac{1}{\sqrt{2}}|+\rangle + \frac{1}{\sqrt{2}}|-\rangle\right] \\
&= \frac{1}{2}\langle +|S_x^{(H)}(0)|+\rangle\cos\omega t + \frac{1}{2}\langle -|S_x^{(H)}(0)|-\rangle\cos\omega t \\
&\qquad + \frac{1}{2}\langle +|\dot{S}_x^{(H)}(0)|+\rangle\sin\omega t + \frac{1}{2}\langle -|\dot{S}_x^{(H)}(0))|-\rangle\sin\omega t
\end{aligned} \tag{41}$$

Now, we know that the expectation value for the Heisenberg picture is just the same with the one in the Schrödinger picture as evident in this relation:

$$\begin{aligned}
\langle A\rangle &= \langle A^{(s)}\rangle \\
&= \langle \psi(t)|A^{(s)}|\psi(t)\rangle \\
&= \langle \mathscr{U}^\dagger(t,0)\psi(0)|A^{(S)}|\mathscr{U}(t,0)\psi(t)\rangle \\
&= \langle \psi(0)|\mathscr{U}^\dagger(t,0)A^{(S)}\mathscr{U}(t,0)|\psi(t)\rangle \\
&= \langle \psi(0)|A^{(H)}(t)|\psi(t)\rangle \\
\langle A\rangle &= \langle A(t)\rangle
\end{aligned} \tag{42}$$

We also know (from Sakurai Section 2.1) that the expectation value of S_x in the Schrödinger picture is $\langle S_x^{(S)}\rangle = \frac{\hbar}{2}\cos\omega t$. Comparing coefficients between the two pictures for $\langle S_x\rangle$, we obtain

$$\begin{aligned}
\cos\omega t: \quad & \frac{\hbar}{2} = \frac{1}{2}\left(\langle +|S_x^{(H)}(0)|+\rangle + \langle -|S_x^{(H)}(0)|-\rangle\right) \\
& \to \hbar = \langle +|S_x^{(H)}(0)|+\rangle + \langle -|S_x^{(H)}(0)|-\rangle
\end{aligned} \tag{43}$$

$$\begin{aligned} \sin\omega t: \qquad 0 &= \frac{1}{2}\left(\langle +|\dot{S}_x^{(H)}(0)|+\rangle + \langle -|\dot{S}_x^{(H)}(0)|-\rangle\right) \\ &\to \langle -|\dot{S}_x^{(H)}(0)|-\rangle = -\langle +|\dot{S}_x^{(H)}(0)|+\rangle \end{aligned} \tag{44}$$

Applying these to Eq. (41), we get

$$\langle S_x(t)\rangle = \frac{1}{2}\hbar\cos\omega t + \frac{1}{2}\underbrace{\cancel{\left(\langle +|\dot{S}_x^{(H)}(0)|+\rangle - \langle +|\dot{S}_x^{(H)}(0)|+\rangle\right)}}_{0}\sin\omega t = \frac{\hbar}{2}\cos\omega t \tag{45}$$

In the same manner as S_x, we find that for S_y

$$\langle -|S_y^{(H)}(0)|-\rangle = -\langle +|S_y^{(H)}(0)|+\rangle\,, \quad \hbar = \langle +|\dot{S}_y^{(H)}(0)|+\rangle + \langle -|\dot{S}_y^{(H)}(0)|-\rangle \tag{46}$$

Thus,

$$\langle S_y(t)\rangle = \frac{\hbar}{2}\sin\omega t \tag{47}$$

On the other hand,

$$\begin{aligned} \langle S_z(t)\rangle &= \langle \psi(t=0)|S_z^{(H)}(t)|\psi(t=0)\rangle \\ &= \left[\frac{1}{\sqrt{2}}\langle +| + \frac{1}{\sqrt{2}}\langle -|\right] S_z^{(H)}(0)\left[\frac{1}{\sqrt{2}}|+\rangle + \frac{1}{\sqrt{2}}|-\rangle\right] \\ &= \frac{1}{2}\langle +|S_z^{(H)}(0)|+\rangle + \frac{1}{2}\langle -|S_z^{(H)}(0)|-\rangle \end{aligned} \tag{48}$$

Now, the expectation value for S_z in the Schrodinger picture is zero. By comparing coefficients, we obtain

$$\begin{aligned} \text{constant}: \qquad 0 &= \frac{1}{2}\left(\langle +|S_z^{(H)}(0)|+\rangle + \langle -|S_z^{(H)}(0)|-\rangle\right) \\ &\to \langle -|S_z^{(H)}(0)|-\rangle = -\langle +|S_z^{(H)}(0)|+\rangle \end{aligned} \tag{49}$$

Then,

$$\langle S_z(t)\rangle = \frac{1}{2}\underbrace{\cancel{\left(\langle +|\dot{S}_z^{(H)}(0)|+\rangle - \langle +|\dot{S}_z^{(H)}(0)|+\rangle\right)}}_{0} = 0 \tag{50}$$

Therefore, Eq. (40) becomes

$$\langle \mathbf{S}(t)\rangle = \left(\frac{\hbar}{2}\cos\omega t, \frac{\hbar}{2}\sin\omega t, 0\right) \tag{51}$$

which shows that $\mathbf{S}(t)$ precess in the xy-plane or about z-axis. ■

References

Sakurai, *Modern Quantum Mechanics*, Chapter 2

IV. Coherent State

The energy eigenkets of the harmonic oscillator $\langle n\rangle$ are stationary states under the harmonic oscillator Hamiltonian. In one dimension:

(a) Construct the state $|\lambda\rangle$ that satisfies $a|\lambda\rangle = \lambda|\lambda\rangle$ in the basis of harmonic oscillator energy eigenkets $|n\rangle$

Solution:
We can construct a coherent state using the finite displacement operator in 1D since it is a displaced oscillator eigenstate. Thus, we have

$$|\lambda\rangle = \mathscr{T}(l)|0\rangle = e^{-i\hat{p}l/\hbar}|0\rangle \tag{52}$$

where $|0\rangle$ is the ground state of the oscillator. We can check if this satisfies $a|\lambda\rangle = \lambda|\lambda\rangle$. Note that annihilation/lowering operator a and creation/raising operator $a^\dagger$ is given by

$$a = \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x} + \frac{i\hat{p}}{m\omega}\right), \quad a^\dagger = \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x} - \frac{i\hat{p}}{m\omega}\right) \tag{53}$$

where $\omega = \sqrt{k/m}$. Then, by adding or subtracting a and $a^\dagger$, we can infer the following relations

$$a^\dagger + a = x\sqrt{\frac{2m\omega}{\hbar}}, \qquad a^\dagger - a = -i\hat{p}\sqrt{\frac{m\omega\hbar}{2}} \tag{54}$$

Thus, $-i\hat{p} = \sqrt{m\omega\hbar/2}\left(a - a^\dagger\right)$. Applying this to Eq. (52),

$$|\lambda\rangle = e^{-i\hat{p}l/\hbar}|0\rangle = e^{l\sqrt{m\omega/2\hbar}\left(a-a^\dagger\right)}|0\rangle = e^{\lambda\left(a-a^\dagger\right)}|0\rangle \tag{55}$$

where $\lambda = l\sqrt{m\omega/2\hbar}$ is real for this case (in general, λ is complex). Note that

$$\begin{aligned}
\left(e^{\lambda\left(a-a^\dagger\right)}\right)^\dagger a e^{\lambda\left(a-a^\dagger\right)} &= e^{\lambda a^\dagger - \lambda a} a e^{\lambda a - \lambda a^\dagger} \\
&= a + \left[\lambda a^\dagger - \lambda a, a\right] + \frac{1}{2!}\left[a, \left[\lambda a^\dagger - \lambda a, a\right]\right] + \ldots \\
&= a + \left(\lambda a^\dagger - \lambda a\right) a - a\left(\lambda a^\dagger - \lambda a\right) \\
&= a + \lambda a a - \lambda a^\dagger a - a\lambda a + a\lambda a^\dagger \\
&= a + \lambda \cancelto{0}{[a, a]} + \lambda \cancelto{1}{\left[a, a^\dagger\right]} \\
\left(e^{\lambda\left(a-a^\dagger\right)}\right)^\dagger a e^{\lambda\left(a-a^\dagger\right)} &= a + \lambda
\end{aligned} \tag{56}$$

by using the Baker-Campbell-Hausdorff formula: $e^A B e^{-A} = B + [A, B] + \frac{1}{2!}[A, [A, B]] + \ldots$.

Therefore, Eq. (55) becomes

$$
\begin{aligned}
a\left|\lambda\right\rangle &= e^{\lambda\left(a-a^\dagger\right)}\left(e^{\lambda\left(a-a^\dagger\right)}\right)^\dagger a\left|\lambda\right\rangle \\
&= e^{\lambda\left(a-a^\dagger\right)}\left(e^{\lambda\left(a-a^\dagger\right)}\right)^\dagger a e^{\lambda\left(a-a^\dagger\right)}\left|0\right\rangle \\
&= e^{\lambda\left(a-a^\dagger\right)}(a+\lambda)\left|0\right\rangle \\
&= e^{\lambda\left(a-a^\dagger\right)}\left(\cancelto{0}{a\left|0\right\rangle}+\lambda\left|0\right\rangle\right) \\
&= \lambda e^{\lambda\left(a-a^\dagger\right)}\left|0\right\rangle \\
a\left|\lambda\right\rangle &= \lambda\left|\lambda\right\rangle
\end{aligned}
\tag{57}
$$

(b) Show that the coherent state $|\lambda\rangle$ has expectation values $\langle x(t)\rangle$ and $\langle p(t)\rangle$ that have the same time development as a classical oscillator.

Solution:
By solving the Heisenberg equations of motion for a harmonic oscillator, we know that $\hat{x}(t)$ and $\hat{p}(t)$ are given by

$$\hat{x}(t) = \hat{x}(0)\cos\omega t + \frac{\hat{p}(0)}{m\omega}\sin\omega t \tag{58}$$

$$\hat{p}(t) = -m\omega\hat{x}(0)\sin\omega t + \hat{p}(0)\cos\omega t \tag{59}$$

Recall the state in Eq. (52). Then, $\langle\hat{x}(t)\rangle$ is calculated as

$$
\begin{aligned}
\langle\hat{x}(t)\rangle &= \langle\lambda|\hat{x}(t)|\lambda\rangle \\
&= \left\langle 0\right| e^{i\hat{p}l/\hbar}\hat{x}(t)e^{-i\hat{p}l/\hbar}\left|0\right\rangle \\
&= \left\langle 0\right| e^{i\hat{p}l/\hbar}\left(\hat{x}(0)\cos\omega t + \frac{\hat{p}(0)}{m\omega}\sin\omega t\right)e^{-i\hat{p}l/\hbar}\left|0\right\rangle \\
\langle\hat{x}(t)\rangle &= \left\langle 0\right| e^{i\hat{p}l/\hbar}\hat{x}(0)e^{-i\hat{p}l/\hbar}\left|0\right\rangle\cos\omega t + \frac{1}{m\omega}\left\langle 0\right| e^{i\hat{p}l/\hbar}\hat{p}(0)e^{-i\hat{p}l/\hbar}\left|0\right\rangle\sin\omega t
\end{aligned}
\tag{60}
$$

Now, recall the effect of the finite translation operator on $\langle\hat{x}\rangle$ and $\langle\hat{p}\rangle$ as shown in the second problem in this problem set. Thus, in 1D, we have the following relations

$$e^{i\hat{p}l/\hbar}\hat{x}(0)e^{-i\hat{p}l/\hbar} = \hat{x}(0)+l, \qquad e^{i\hat{p}l/\hbar}\hat{p}(0)e^{-i\hat{p}l/\hbar} = \hat{p}(0) \tag{61}$$

Applying these on $\langle\hat{x}(t)\rangle$, we get

$$
\begin{aligned}
\langle\hat{x}(t)\rangle &= \left\langle 0\right|(\hat{x}(0)+l)\left|0\right\rangle\cos\omega t + \frac{1}{m\omega}\left\langle 0\right|\hat{p}(0)\left|0\right\rangle\sin\omega t \\
&= \cancelto{0}{\left\langle 0\right|\hat{x}(0)\left|0\right\rangle}\cos\omega t + l\,\cancelto{1}{\langle 0|0\rangle}\cos\omega t + \frac{1}{m\omega}\cancelto{0}{\left\langle 0\right|\hat{p}(0)\left|0\right\rangle}\sin\omega t \\
\langle\hat{x}(t)\rangle &= l\cos\omega t
\end{aligned}
\tag{62}
$$

In the same manner for $\langle\hat{p}(t)\rangle$, we have

$$\begin{aligned}\langle\hat{p}(t)\rangle &= \langle\lambda|\hat{p}(t)|\lambda\rangle \\ &= \langle 0|\, e^{i\hat{p}l/\hbar}\hat{p}(t)e^{-i\hat{p}l/\hbar}\,|0\rangle \\ &= \langle 0|\, e^{i\hat{p}l/\hbar}\left(-m\omega\hat{x}(0)\sin\omega t+\hat{p}(0)\cos\omega t\right)e^{-i\hat{p}l/\hbar}\,|0\rangle \\ &= -m\omega\,\langle 0|\, e^{i\hat{p}l/\hbar}\hat{x}(0)e^{-i\hat{p}l/\hbar}\,|0\rangle\sin\omega t+\langle 0|\, e^{i\hat{p}l/\hbar}\hat{p}(0)e^{-i\hat{p}l/\hbar}\,|0\rangle\cos\omega t \\ &= -m\omega\,\langle 0|\left(\hat{x}(0)+l\right)|0\rangle\sin\omega t+\langle 0|\,\hat{p}(0)\,|0\rangle\cos\omega t \\ &= -m\omega\,\cancelto{0}{\langle 0|\,\hat{x}(0)\,|0\rangle}\sin\omega t-m\omega l\,\cancelto{1}{\langle 0|0\rangle}\sin\omega t+\cancelto{0}{\langle 0|\,\hat{p}(0)\,|0\rangle}\cos\omega t \\ \langle\hat{p}(t)\rangle &= -m\omega l\sin\omega t\end{aligned}\tag{63}$$

With the calculated form of $\langle\hat{x}(t)\rangle$ and $\langle\hat{p}(t)\rangle$, we have shown that they have the same time development as a classical oscillator.

(c) Evaluate that position-momentum uncertainty relation for this state at arbitrary times.

Solution:
Since we have proven that the state in Eq. (52) satisfies $a\,|\lambda\rangle=\lambda\,|\lambda\rangle$, we can use the following relations to evaluate the position-momentum uncertainty relation:

$$\langle\lambda|\, a^\dagger=\langle\lambda|\,\lambda^*=\langle\lambda|\,\lambda \tag{64}$$

$$\begin{aligned}\langle\lambda|\, a^\dagger\pm a\,|\lambda\rangle &= \langle\lambda|\, a^\dagger\pm a\,|\lambda\rangle \\ &= \left(\lambda^*\pm\lambda\right)\langle\lambda|\lambda\rangle \\ \langle\lambda|\, a^\dagger\pm a\,|\lambda\rangle &= \lambda\pm\lambda\end{aligned}\tag{65}$$

$$\begin{aligned}\langle\lambda|\, a^2\,|\lambda\rangle &= \lambda\,\langle\lambda|\, a\,|\lambda\rangle \\ &= \lambda^2\,\langle\lambda|\lambda\rangle \\ \langle\lambda|\, a^2\,|\lambda\rangle &= \lambda^2\end{aligned}\tag{66}$$

$$\begin{aligned}\langle\lambda|\left(a^\dagger\right)^2|\lambda\rangle &= \lambda^*\,\langle\lambda|\, a^\dagger\,|\lambda\rangle \\ &= \left(\lambda^*\right)^2\langle\lambda|\lambda\rangle \\ \langle\lambda|\left(a^\dagger\right)^2|\lambda\rangle &= \lambda^2\end{aligned}\tag{67}$$

$$\langle\lambda|\, a^\dagger a\,|\lambda\rangle=\lambda^*\lambda\,\langle\lambda|\lambda\rangle=\lambda^2 \tag{68}$$

$$\begin{aligned}\langle\lambda|\, aa^\dagger\,|\lambda\rangle &= \langle\lambda|\left(1+aa^\dagger\right)|\lambda\rangle \\ &= \langle\lambda|\lambda\rangle+\langle\lambda|\, a^\dagger a\,|\lambda\rangle \\ \langle\lambda|\, aa^\dagger\,|\lambda\rangle &= 1+\lambda^2\end{aligned}\tag{69}$$

Using Eq. (54) as well, $\langle\hat{x}\rangle$ is given by

$$\langle\hat{x}\rangle=\langle\lambda|\,\hat{x}\,|\lambda\rangle=\sqrt{\frac{\hbar}{2m\omega}}\,\langle\lambda|\, a^\dagger+a\,|\lambda\rangle=\sqrt{\frac{\hbar}{2m\omega}}(\lambda+\lambda)\,\langle\lambda|\lambda\rangle=2\lambda\sqrt{\frac{\hbar}{2m\omega}} \tag{70}$$

In the same manner, we can calculate $\langle\hat{x}^2\rangle$:

$$\begin{aligned}\langle\hat{x}^2\rangle &= \langle\lambda|\,\hat{x}^2\,|\lambda\rangle \\ &= \frac{\hbar}{2m\omega}\,\langle\lambda|\left(a^\dagger+a\right)^2|\lambda\rangle \\ &= \frac{\hbar}{2m\omega}\,\langle\lambda|\left(\left(a^\dagger\right)^2+a^2+a^\dagger a+aa^\dagger\right)|\lambda\rangle \\ &= \frac{\hbar}{2m\omega}\left(\lambda^2+\lambda^2+\lambda^2+1+\lambda^2\right) \\ \langle\hat{x}^2\rangle &= \left(4\lambda^2+1\right)\frac{\hbar}{2m\omega}\end{aligned}\tag{71}$$

Thus, we have

$$(\Delta x)^2 \geq \langle \hat{x}^2 \rangle - \langle \hat{x} \rangle^2 = \left(4\lambda^2 + 1\right) \frac{\hbar}{2m\omega} - 2^2\lambda^2 \frac{\hbar}{2m\omega} = \frac{\hbar}{2m\omega} \tag{72}$$

We do the same for $\langle \hat{p} \rangle$

$$\langle \hat{p} \rangle = \langle \lambda | \hat{p} | \lambda \rangle = -i\sqrt{\frac{m\omega\hbar}{2}} \langle \lambda | a^\dagger - a | \lambda \rangle = \sqrt{\frac{m\omega\hbar}{2}} (\lambda - \lambda) \langle \lambda | \lambda \rangle = 0 \tag{73}$$

and $\langle \hat{p}^2 \rangle$

$$\begin{aligned} \langle \hat{p}^2 \rangle &= \langle \lambda | \hat{p}^2 | \lambda \rangle \\ &= i^2 \frac{m\omega\hbar}{2} \langle \lambda | \left(a^\dagger - a\right)^2 | \lambda \rangle \\ &= -\frac{m\omega\hbar}{2} \langle \lambda | \left(\left(a^\dagger\right)^2 + a^2 - a^\dagger a - a a^\dagger\right) | \lambda \rangle \\ &= -\frac{m\omega\hbar}{2} \left(\lambda^2 + \lambda^2 - \lambda^2 + 1 - \lambda^2\right) \\ \langle \hat{p}^2 \rangle &= \frac{m\omega\hbar}{2} \end{aligned} \tag{74}$$

to get

$$(\Delta p)^2 \geq \langle \hat{p}^2 \rangle - \langle \hat{p} \rangle^2 = \frac{m\omega\hbar}{2} - 0 = \frac{m\omega\hbar}{2} \tag{75}$$

Thus, we obtain

$$(\Delta x)^2 (\Delta p)^2 \geq \frac{\hbar}{2m\omega} \frac{m\omega\hbar}{2} = \frac{\hbar^2}{4} \tag{76}$$

or

$$\Delta x \Delta p \geq \frac{\hbar}{2} \tag{77}$$

as the position-momentum uncertainty relation for the state given in Eq. (52) at arbitrary times.

References

Harmonic oscillator position expectation value, `https://physics.stackexchange.com/questions/461794/harmonic-oscillator-position-expectation-value`

Chapter 5: Harmonic Oscillator and Coherent States, `http://homepage.univie.ac.at/reinhold.bertlmann/pdfs/T2_Skript_Ch_5.pdf`

IV. Probability Flux and Phase

Let $\psi(\mathbf{x}, t)$ be a wavefunction.

- ▶ Prove that the probability flux $\mathbf{j} = (\hbar/m)\Im\left[\psi^* \nabla \psi\right]$ satisfies a continuity equation

$$\frac{\partial \rho}{\partial t} = -\nabla \cdot \mathbf{j} \tag{78}$$

with $\rho = |\psi|^2$.

Solution:
Note that $\Im\left[\psi^*\nabla\psi\right]$ means taking the imaginary part of $\psi^*\nabla\psi$. With the given $\mathbf{j}$ and using the relation $\mathrm{Im}(z) = (1/2i)(z - z^*)$ where z is some complex number and z^* is its conjugate, $-\nabla\cdot\mathbf{j}$ can be expressed as

$$\begin{aligned} -\nabla\cdot\mathbf{j} &= -\nabla\cdot\frac{\hbar}{m}\frac{1}{2i}\left(\psi^*\nabla\psi - \psi\nabla\psi^*\right) \\ &= \frac{i\hbar}{2m}\left[\nabla\cdot\left(\psi^*\nabla\psi\right) - \nabla\cdot\left(\psi\nabla\psi^*\right)\right] \\ &= \frac{i\hbar}{2m}\left[\psi^*\left(\nabla\cdot\nabla\psi\right) + \cancel{\left(\nabla\psi\right)\cdot\left(\nabla\psi^*\right)} - \psi\left(\nabla\cdot\nabla\psi^*\right) - \cancel{\left(\nabla\psi^*\right)\cdot\left(\nabla\psi\right)}\right] \\ -\nabla\cdot\mathbf{j} &= \frac{i\hbar}{2m}\left(\psi^*\nabla^2\psi - \psi\nabla^2\psi^*\right) \end{aligned} \tag{79}$$

after applying $\nabla\cdot(u\mathbf{A}) = u\,(\nabla\cdot\mathbf{A}) + \mathbf{A}\cdot(\nabla u)$. On the hand, we can express $(\partial\rho/\partial t)$ as

$$\frac{\partial\rho}{\partial t} = \frac{\partial|\psi|^2}{\partial t} = \frac{\partial\left(\sqrt{\psi\psi^*}\right)^2}{\partial t} = \psi^*\frac{\partial\psi}{\partial t} + \psi\frac{\partial\psi^*}{\partial t} \tag{80}$$

Using the Schrödinger equation given by

$$-\frac{\hbar^2}{2m}\nabla^2\psi + V\psi = i\hbar\frac{\partial\psi}{\partial t} \tag{81}$$

for a potential with no imaginary components and its conjugate given by

$$-\frac{\hbar^2}{2m}\nabla^2\psi^* + V\psi^* = -i\hbar\frac{\partial\psi^*}{\partial t} \tag{82}$$

Eq. (80) becomes

$$\begin{aligned} \frac{\partial\rho}{\partial t} &= \psi^*\left(-\frac{\hbar}{2im}\nabla^2\psi + \frac{1}{i\hbar}V\psi\right) + \psi\left(\frac{\hbar}{2im}\nabla^2\psi^* - \frac{1}{i\hbar}V\psi^*\right) \\ &= \frac{\hbar}{2im}\left(\psi\nabla^2\psi^* - \psi^*\nabla^2\psi\right) + \frac{1}{i\hbar}V\cancelto{0}{\left(\psi^*\psi - \psi\psi^*\right)} \\ \frac{\partial\rho}{\partial t} &= \frac{i\hbar}{2m}\left(\psi^*\nabla^2\psi - \psi\nabla^2\psi^*\right) \end{aligned} \tag{83}$$

Since Eqs. (79) and (83) have the same RHS, we can equate them which results to the continuity equation in Eq. (78). ■

- ▶ Write the wavefunction as the product of an amplitude and complex phase factor

$$\psi(\mathbf{x},t) = \sqrt{\rho(\mathbf{x},t)} \times e^{iS(\mathbf{x},t)/\hbar} \tag{84}$$

to show that $\mathbf{j} = \rho\boldsymbol{\nabla}S/m$.

Solution:
We will impose the condition that $S(\mathbf{x},t)$ is real. The conjugate of the given wavefunction in Eq. (84) is

$$\psi^*(\mathbf{x},t) = \sqrt{\rho(\mathbf{x},t)} \times e^{-iS(\mathbf{x},t)/\hbar} \tag{85}$$

Taking the gradient of ψ, we have

$$\nabla\psi = \sqrt{\rho}\,\nabla\left(e^{iS/\hbar}\right) + e^{iS/\hbar}\,\nabla\left(\sqrt{\rho}\right) \tag{86}$$

by product rule. Note that

$$\begin{aligned}\nabla\left(e^{iS/\hbar}\right) &= \frac{\partial}{\partial x}\left(e^{iS/\hbar}\right)\hat{\boldsymbol{\imath}} + \frac{\partial}{\partial y}\left(e^{iS/\hbar}\right)\hat{\boldsymbol{\jmath}} + \frac{\partial}{\partial z}\left(e^{iS/\hbar}\right)\hat{\boldsymbol{k}} \\ &= e^{iS/\hbar}\cdot\frac{i}{\hbar}\frac{\partial S}{\partial x}\hat{\boldsymbol{\imath}} + e^{iS/\hbar}\cdot\frac{i}{\hbar}\frac{\partial S}{\partial y}\hat{\boldsymbol{\jmath}} + e^{iS/\hbar}\cdot\frac{i}{\hbar}\frac{\partial S}{\partial z}\hat{\boldsymbol{k}} \\ \nabla\left(e^{iS/\hbar}\right) &= \frac{i}{\hbar}e^{iS/\hbar}\,\nabla S\end{aligned} \tag{87}$$

Then,

$$\nabla\psi = \sqrt{\rho}\,\frac{i}{\hbar}e^{iS/\hbar}\,\nabla S + e^{iS/\hbar}\,\nabla\left(\sqrt{\rho}\right) = e^{iS/\hbar}\left[\frac{i\sqrt{\rho}}{\hbar}\,\nabla S + \nabla\left(\sqrt{\rho}\right)\right] \tag{88}$$

Recall the given expression of $\mathbf{j}$ in (a). With Eqs. (85) and (88), $\mathbf{j}$ becomes

$$\begin{aligned}\mathbf{j} &= \frac{\hbar}{m}\Im\left[\psi^*\nabla\psi\right] \\ &= \frac{\hbar}{m}\Im\left[\sqrt{\rho}\,\cancel{e^{-iS/\hbar}}\cdot\cancel{e^{iS/\hbar}}\left[\frac{i\sqrt{\rho}}{\hbar}\,\nabla S + \nabla\left(\sqrt{\rho}\right)\right]\right] \\ &= \frac{\hbar}{m}\Im\left[\frac{i\rho}{\hbar}\,\nabla S + \sqrt{\rho}\,\nabla\left(\sqrt{\rho}\right)\right] \\ \mathbf{j} &= \frac{\rho}{m}\,\nabla S\end{aligned} \tag{89}$$

■

References

Sakurai, *Modern Quantum Mechanics*, Chapter 2, Section 4

Branson, J., *Probability Conservation Equation*, `https://quantummechanics.ucsd.edu/ph130a/130_notes/node127.html#derive:flux`

IV. WKB Approximation

Describe the limit of validity of the WKB approximation in terms of the two length scales: spatial variation of the wavefunction and spatial variation of the potential.

Solution:

For the limit of validity of the WKB approximation in terms of the spatial variation of the wavefunction, the amplitude and wavelength of wavefunction must be slowly varying but its phase must be rapidly changing for the approximation to be valid. In terms of the spatial variation of the potential, these variations must have a length-scale larger that the wave length of the wavefunction (i.e. the potention should also be slowly varying) for the approximation to be valid.

References

Griffiths, D., *Introduction to Quantum Mechanics*, Chapter 8

WKB Approximation in 3D , http://physics.gmu.edu/~pnikolic/PHYS684/lectures/WKB.pdf

Fowler, M., *The WKB Approximation* , https://phys.libretexts.org/Bookshelves/Quantum_Mechanics/Book%3A_Quantum_Mechanics_(Fowler)/08%3A_Approximate_Methods/8.02%3A_The_WKB_Approximation