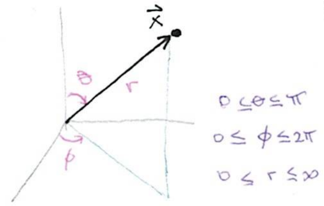
Physics 231

Boundary Value Problems in Electrostatics (chapter 3)

Nov. 24, 2020

Lecture Notes

▲ The Laplace Equation in Spherical coord's



In spherical coord's, the Laplace eqn is expressed as

$$\frac{1}{r}\frac{\partial^2}{\partial r^2}(r\,\Phi) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\,\frac{\partial\Phi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\Phi}{\partial\phi^2} = 0$$

Assume a separable soln :

$$\Phi = \frac{u(r)}{r}\,P(\theta)\,Q(\phi)$$

Then, substituting this to the Laplace eqn

$$\left[PQ\frac{d^2u}{dr^2} + \frac{uQ}{r^2\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dP}{d\theta}\right) + \frac{uP}{r^2\sin^2\theta}\frac{d^2Q}{d\phi^2} = 0\right]\frac{r^2\sin^2\theta}{uPQ}$$

$$\underbrace{r^2\sin^2\theta\left[\frac{1}{u}\frac{d^2u}{dr^2} + \frac{1}{P}\frac{1}{r^2\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dP}{d\theta}\right)\right]}_{\text{depends on } r \text{ and } \theta} + \underbrace{\frac{1}{Q}\frac{d^2Q}{d\phi^2}}_{\text{depends on } \phi \text{ alone}} = 0$$

$$\frac{1}{Q}\frac{d^2Q}{d\phi^2} = -m^2, \quad m = \text{const}$$

$$\hookrightarrow \frac{d^2Q}{d\phi^2} + m^2 Q = 0, \quad Q = e^{im\phi} \leftarrow \text{soln of this 2nd}$$

Substituting this relation to the Laplace eqn

$$r^2\sin^2\theta\left[\frac{1}{u}\frac{d^2u}{dr^2} + \frac{1}{P}\frac{1}{r^2\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dP}{d\theta}\right)\right] - m^2 = 0$$

Nov. 26, 2020

Lecture Notes

▶ The Laplace Eqn in Spherical Coord's (Cont.)

For Q to be single-valued, Q must be periodic : $Q(\phi) = Q(\phi + 2\pi)$

This condition emminates from the percieved isotropy of space. Meaning, there is no preferred direction. This implies that m is an integer $\Rightarrow m = 0, \pm 1, \pm 2 \ldots$

The P-part satisfies the differential eqn

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dP}{d\theta}\right) + \left[l(l+1) - \frac{m^2}{\sin^2\theta}\right]P = 0$$

The radial part $u(r)$ satisfies

$$\frac{d^2u}{dr^2} - \frac{l(l+1)}{r^2}u = 0$$

where l is some constant to be determined. The gen. radial solution is given by

$$u(r) = A r^{l+1} + B r^{-l}$$

★ Legendre equation and Legendre polynomials

Solve for the P-part. Do a change of variables $\theta \rightarrow x = \cos\theta$

$\sin^2\theta + \cos^2\theta = 1$
$\sin^2\theta = 1 - \cos^2\theta$
$= 1 - x^2$

$$\frac{d}{d\theta} = \frac{dx}{d\theta}\frac{d}{dx} = -\sin\theta\frac{d}{dx} \quad \Big| \quad \sin\theta\frac{d}{d\theta} = \sin\theta\left(-\sin\theta\frac{d}{dx}\right)$$

$$= -\sin^2\theta\frac{d}{dx} = -(1-x^2)\frac{d}{dx}$$

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Then, 11/26/20

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d}{d\theta}\right) = \frac{1}{\sin\theta}\left(-\sin\theta\frac{d}{dx}\right)\left(-(1-x^2)\frac{d}{dx}\right)$$
$$= \frac{d}{dx}\left[(1-x^2)\frac{d}{dx}\right]$$

Then,
$$\frac{d}{dx}\left[(1-x^2)\frac{dQ}{dx}\right] + \left[l(l+1) - \frac{m^2}{(1-x^2)}\right]P = 0$$ Generalized Legendre eqn

The solns are known as the associated Legendre functions.
Consider the solns for $m=0$. For this case, $Q(\phi) =$ constant. This situation corres. to the presence of azimuthal symmetry (no dependence on ϕ)

$$\frac{d}{dx}\left[(1-x^2)\frac{dP}{dx}\right] + l(l+1)P = 0$$

Assume that the soln is single-valued, finite, and continuous in the interval $[-1,1]$.
This is because these conditions ~~the~~ seem to reflect experimental

Assume a soln of the form:

$$P(x) = x^\alpha \sum_{j=0} a_j x^j$$

where α and a_j are constants to be determined. We need to know

$$\frac{dP(x)}{dx} = \sum_{j=0}^{\infty} a_j(\alpha+j)x^{j-1+\alpha}, \qquad \frac{d^2P}{dx^2} = \sum_{j=0}^{\infty} a_j(\alpha+j)(\alpha+j-1)x^{j+\alpha-2}$$

and

$$\frac{d}{dx}\left[x^2\frac{dP}{dx}\right] = \frac{d}{dx}\left[\sum_{j=0}^{\infty} a_j(\alpha+j)x^{j+1+\alpha}\right] = \sum_{j=0}^{\infty} a_j(\alpha+j)(j+1+\alpha)x^{j+\alpha}$$

Substitute back and collect equal powers of x,

$$\sum_{j=0}^{\infty}(\alpha+j)(\alpha+j-1)a_j x^{\alpha+j-2} - \sum_{j=0}^{\infty}\left[(\alpha+j)(\alpha+j+1) - l(l+1)\right]x^{\alpha+j}a_j = 0$$

Since x is arbitrary and the RHS is zero, each coeff. of x must vanish. Cancelling out the α exponent

$$\sum_{j=0}^{\infty}(\alpha+j)(\alpha+j-1)a_j x^{j-2} - \sum_{j=0}^{\infty}\left[(\alpha+j)(\alpha+j+1) - l(l+1)\right]a_j x^j = 0$$

The first term can be

$$\alpha(\alpha-1)a_0x^{-2} + (\alpha+1)(\alpha+1-1)a_1x^{1-2} + (\alpha+2)(\alpha+2-1)x^{2-2}a_2$$
$$+ (\alpha+3)(\alpha+3-1)x^{3-2}a_3 + \ldots = \sum_{j=0}^{\infty}(\alpha+2+j)(\alpha+1+j)x^j a_{2+j}$$

Then,
$$\alpha(\alpha-1)a_0x^{-2} + (\alpha+1)\alpha a_1x^{-1} + \sum_{j=0}^{\infty}(\alpha+2+j)(\alpha+1+j)a_{2+j}x^j$$
$$- \sum_{j=0}^{\infty}\left[(\alpha+j)(\alpha+j+1) - l(l+1)\right]a_j x^j = 0$$

We can now combine terms of x^j:

$$0 = \alpha(\alpha-1)a_0x^2 + (\alpha+1)\alpha a_1x^{-1} + \sum_{j=0}^{\infty}\left\{(\alpha+2+j)(\alpha+1+j)a_{2+j} - \left[(\alpha+j)(\alpha+j+1) - l(l+1)\right]a_j\right\}x_j$$

Thus,

$$\alpha(\alpha-1)a_0 = 0$$
$$\alpha(\alpha+1)a_1 = 0$$
$$(\alpha+2+j)(\alpha+1+j)a_{2+j} - \left[(\alpha+j)(\alpha+j+1) - l(l+1)\right]a_j = 0$$

$$\boxed{a_{j+2} = \frac{(\alpha+j)(\alpha+j+1) - l(l+1)}{(\alpha+j+1)(\alpha+j+2)}a_j}$$

Given $a_0 \neq 0$, $a_2, a_4, a_6, \ldots$ are determined. Given $a_1 = 0$, $a_3, a_4, a_5, a_7, \ldots$ are determined

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- If we consider $a_0 \neq 0$, $a_1 = 0$

$$\alpha(\alpha-1)a_0 = 0 \longrightarrow \alpha(\alpha-1) = 0$$

There are two possible solutions to this: $\alpha = 0$ or $\alpha = 1$

Recall: The assumed solution is

$$\sum_{j=0}^{\infty} a_j x^{\alpha+j} \longrightarrow \sum_{k=0}^{\infty} a_{2k} x^{2k+\alpha}$$

When $\alpha = 0$ [$\alpha = 1$], we get even [odd] solutions.

- Now, if we consider $a_1 \neq 0$, $a_0 = 0$

$$\alpha(\alpha+1)a_1 = 0 \longrightarrow \alpha(\alpha+1) = 0, \quad \alpha = 0 \text{ or } \alpha = -1$$

Then, the assumed soln now becomes

$$\sum_{j=0}^{\infty} a_j x^{\alpha+j} \longrightarrow \sum_{k=0}^{\infty} a_{2k+1} x^{\alpha+2k+1}$$

When $\alpha = 0$ [$\alpha = -1$], we get odd [even] solutions.

The differential eqn is second order so that there are at most two linearly independent solutions. Since we can obtain even/odd solns when we choose either $a_0 \neq 1$ or $a_1 \neq 1$,

If the series solution does not terminate so that it is an infinite series, does the solution converge in the entire interval $[-1, 1]$?

Going back to our recurrence relation; For any x in the interval $[-1, 1]$, the contributions will be by the large j-terms. This is especially true ~~for~~ near ± 1.

For large j,

$$a_{j+2} \sim \frac{j^2}{j^2} a_j \rightarrow \text{i.e. } a_{j+2} \sim a_j$$

The solution assymptotically behaves as $\sum_j^{\infty} x^j$, $|x| \to 1$. However, this infinite series does not converge at the boundaries ($|x| = 1$).

Therefore, this solution cannot be accepted on the basis that ~~the~~ it diverges at the boundaries. For the solution to be accepted, the series must terminate. So, the solution must be a polynomial.

Recall the recurrence relation:

$$a_{j+2} = \frac{(\alpha+j)(\alpha+j+1) - l(l+1)}{(\alpha+j+1)(\alpha+j+2)} a_j$$

Let $a_0 \neq 0$, $a_1 = 0$

Case: $\alpha = 0$, $j = 2k$, $k = 0, 1, 2, \ldots$ $\Rightarrow$ $a_{2k+2} = \dfrac{2k(2k+1) - l(l+1)}{(2k+1)(2k+2)} a_{2k}$

Let $l = 0$: $a_{2k+2} = \dfrac{2k(2k+1)}{(2k+1)(2k+2)} a_{2k}$

Let $k = 0$: $a_2 = \frac{0}{1 \cdot 2} a_0 = 0$

$k = 1$: $a_{2+2} = a_4 = \frac{2(2+1)}{(2+1)(2+2)} \cancel{a_2}^{\,0} = 0$

$a_6 = 0$

$\vdots$

thus, we deduce the solution $P_0(x) = a_0 = 1$

Dec. 1, 2020

▶ Boundary problems with azimuthal symmetry

In the presence of azimuthal symmetry, the soln to the Laplace eqn is independent of ϕ. This corres. to $m = 0$ in the separated soln

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12/1/20

The gen. soln is given by

$$\Phi(r,\theta) = \sum_{l=0}^{\infty} \left[A_l r^l + B_l r^{-(l+1)} \right] P_l(\cos\theta)$$

The coeff. A_l and B_l are to be determined and dictated by the problem

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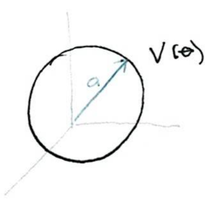


Two cases:

Inside the sphere / outside the sphere, soln is required to be continuous and finite everywhere

Assume that there are no charges inside and outside the sphere

- Inside: $\nabla^2 \Phi = 0$, no charges inside

Recall the soln,

$$\Phi(r,\theta) = \sum_{l=0}^{\infty} \left[A_l r^l + \underbrace{\cancel{B_l r^{-(l+1)}}}_{\text{diverges at } r=0} \right] P_l(\cos\theta)$$

must vanish to maintain the continuity of the solution inside and outside of the sphere

Thus,

$$\Rightarrow \Phi_{in}(r,\theta) = \sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta)$$

- Outside: $\nabla^2 \Phi = 0$, no charges outside $(a < r < \infty)$

This time,

must vanish to maintain continuity

$$\Phi(r,\theta) = \sum_{l=0}^{\infty} \left[\underbrace{\cancel{A_l r^l}}_{\text{diverges at } r\to\infty} + B_l r^{-(l+1)} \right] P_l(\cos\theta)$$

Thus,

$$\Phi_{out}(r,\theta) = \sum_{l=0}^{\infty} \frac{B_l}{r^{l+1}} P_l(\cos\theta)$$

* What if we have the ff. configuration?

A Two concentric spheres

* We know that there is azimuthal symmetry if the potential doesn't depend on ϕ

$0 \le r < a_1$: Since the origin is involved, the 2^{nd} term must vanish

$$\Phi_1(r,\theta) = \sum_{l=0}^{\infty} A_l^{(1)} r^l P_l(\cos\theta)$$

$a_1 < r < a_2$: Both terms of the gen soln must contribute. Thus,

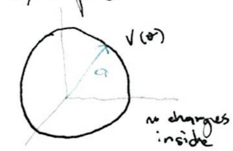
$$\Phi_2(r,\theta) = \sum_{l=0}^{\infty} \left[A_l^{(2)} r^l + B_l^{(2)} r^{-(l+1)} \right] P_l(\cos\theta)$$

$a_2 < r$: Since $r \to \infty$, the 1^{st} term must vanish,

$$\Phi_3(r,\theta) = \sum_{l=0}^{\infty} \frac{B_l}{r^{l+1}} P_l(\cos\theta)$$

To determine the coefficients, the continuity of Φ and its derivative across r must be imposed

* Example:



Recall that inside, the soln is

$$\Phi(r,\theta) = \sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta)$$

Given $V(\theta)$, what are the coeff A_l's? On the surface of the sphere, we have the soln,

$$\Phi(a,\theta) = \sum_{l=0}^{\infty} A_l a^l P_l(\cos\theta) = V(\theta)$$

Note that:

$$\int_{-1}^{1} P_l(x) P_m(x)\, dx = \frac{2}{2l+1} \delta_{ml} = \int_{-1}^{1} P_l(\cos\theta) P_m(\cos\theta)\, d(\cos\theta)$$

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Thus, 12/1/20

$$\int_{-1}^{1} \left[\sum_{l=0}^{\infty} A_l a^l P_l(\cos\theta) = V(\theta) \right] P_m(\cos\theta)\, d(\cos\theta)$$

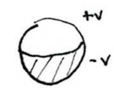
$$\sum_{l=0}^{\infty} A_l a^l \int_{-1}^{1} P_l(\cos\theta) P_m(\cos\theta)\, d(\cos\theta) = \int_{-1}^{1} V(\theta) P_m(\cos\theta)\, d(\cos\theta)$$

$$\underbrace{\delta_{ml} \sum_{l=0}^{\infty} \frac{2}{2l+1} A_l a^l}_{\frac{2}{2m+1}} = \int_{-1\to 2\pi}^{1\to 0} V(\theta) P_m(\cos\theta)\, (-1) \sin\theta\, d\theta$$

And we obtain:

$$A_m = \frac{2m+1}{2a^m}$$

* Example: Let $V(\theta) = \begin{cases} +V, & 0 \le \theta < \frac{\pi}{2} \\ -V, & \end{cases}$



The coefficients are given by

$$A_l = \frac{2l+1}{2a^l} \left[\int_0^{\pi/2} V P_l(\cos\theta) \sin\theta\, d\theta + \int_{\pi/2}^{\pi} -V P_l(\cos\theta) \sin\theta\, d\theta \right]$$

Note that: $\int_0^{\pi/2} P_l(\cos\theta) \sin\theta\, d\theta = -\int_0^{\pi/2} P_l(\cos\theta)\, d(\cos\theta)$

$$= -\int_1^0 P_l(x)\, dx$$

$$= \int_0^1 P(x)\, dx$$

and

$$\int_{\pi/2}^{\pi} P_l(\cos\theta) \sin\theta\, d\theta = -\int_{\pi/2}^{\pi} P_l(\cos\theta)\, d(\cos\theta) = -\int_0^{-1} P_l(x)\,dx = \int_{-1}^{0} P_l(x)\, dx$$

Thus,

$$A_l = \frac{2l+1}{2a^l} V \left[\int_0^1 P_l(x)\, dx - \int_{-1}^{0} P_l(x)\, dx \right]$$

When l is odd, $P_l(x)$ is odd in x $\Rightarrow$ $\int P_l(x)\, dx$ is even

$$\int_0^1 P_l(x)\, dx + \int_0^{-1} P_l(x)\, dx = 2\int_0^1 P_l(x)\, dx$$

When l is even, $P_l(x)$ is even in x $\Rightarrow$ $\int P_l(x)\, dx$ is odd

$$\int_0^1 P_l(x)\, dx - \int_{-1}^{0} P_l(x)\, dx = 0$$

and we get

$A_l = 0$ when l is even $\qquad$ $A_l \neq 0$ when l is odd

$$\int_0^1 P_{2n+1}(x)\, dx = -\frac{1}{2} \frac{\frac{2n+1-1}{2} (2n+1-2)!!}{2\left(\frac{2n+1+1}{2}\right)!!} = \left(-\frac{1}{2}\right)^n \frac{(2n-1)!!}{2(n+1)!}$$

Thus,

$$\Phi(r,\theta) = V\left[\frac{3}{2}\left(\frac{r}{a}\right) P_1(\cos\theta) - \frac{7}{8}\left(\frac{r}{a}\right)^3 P_3(\cos\theta) + \frac{11}{16}\left(\frac{r}{a}\right)^5 P_5(\cos\theta) - \dots \right]$$

- For outside of the sphere, just make the replacement

$$\left(\frac{r}{a}\right)^l \longrightarrow \left(\frac{a}{r}\right)^{l+1}$$

- Solving problems w/

The general soln in the presence of azimuthal symmetry is again

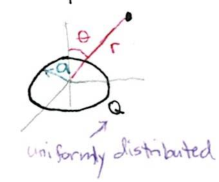
$$\Phi(r,\theta) = \sum_{l=0}^{\infty} \left[A_l r^l + B_l r^{-(l+1)} \right] P_l(\cos\theta)$$

Along the positive z axis, $\theta = 0$, $\cos\theta = 1$

$$\Phi(r,\theta) \rightarrow \Phi(r,0) = \sum_{l=0}^{\infty} \left[A_l r^l + B_l r^{-(l+1)} \right] \cancel{P_l(1)}^{\,1}$$

$$= \sum_{l=0}^{\infty} \left[A_l r^l + B_l r^{-(l+1)} \right]$$

☆ Example: Consider a ring of radius a



The potential along the positive z-axis is

$$\Phi(r,\theta=0) = \frac{Q}{4\pi\epsilon_0 \sqrt{r^2+a^2}}$$

There are two cases.

Case 1: $r > a$

$$\Phi(r,\theta=0) = \frac{Q}{4\pi\epsilon_0} \frac{1}{r\left(1+\frac{a^2}{r^2}\right)}, \quad \frac{a}{r} < 1$$

Applying binomial expansion,

$$\Phi(r,\theta=0) = \frac{Q}{4\pi\epsilon_0} \frac{1}{r} \sum_{k=0}^{\infty} \binom{-1/2}{k} \left(\frac{a}{r}\right)^{2k}$$

$$= \frac{Q}{4\pi\epsilon_0} \sum_{k=0}^{\infty} \binom{-1/2}{k} \frac{a^{2k}}{r^{2k+1}}$$

← compare this to $\Phi(r,0)$ earlier. It seems $A_l = 0$ for all l, $B_l \neq 0$ for even l, $B_l = 0$ for odd l

looks like $r^{-(l+1)}$

Thus,

$$\Phi(r,\theta) = \frac{Q}{4\pi\epsilon_0} \sum_{k=0}^{\infty} \binom{-1/2}{k} \frac{a^{2k}}{r^{2k+1}} P_{2k}(\cos\theta)$$

Case 2: $r < a$

$$\Phi(r,\theta=0) = \frac{Q}{4\pi\epsilon_0} \frac{1}{a\left(1+\frac{r^2}{a^2}\right)^{1/2}}$$

$$= \frac{Q}{4\pi\epsilon_0 a} \sum_{k=0}^{\infty} \binom{-1/2}{k} \left(\frac{r^2}{a^2}\right)^k$$

$$= \frac{Q}{4\pi\epsilon_0 a} \sum_{k=0}^{\infty} \binom{-1/2}{k} \frac{r^{2k}}{a^{2k}}$$

looks like r^l

← comparing this to $\Phi(r,0)$ earlier. It seems $B_l = 0$ for all l, $A_l \neq 0$ for even l, $A_l = 0$ for odd l

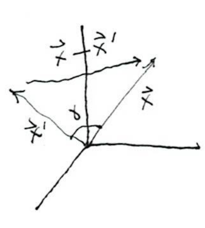
Thus,

$$\Phi(r,\theta) = \frac{Q}{4\pi\epsilon_0} \sum_{k=0}^{\infty} \binom{-1/2}{k} \frac{r^{2k}}{a^{2k+1}} P_{2k}(\cos\theta)$$

Dec, 3, 2020

▸ Solving problems of azimuthal symmetry (cont.)

Example: For $\vec{x} \neq \vec{x}'$, the following expansion holds



$$\frac{1}{|\vec{x}-\vec{x}'|} = \sum_{l=0}^{\infty} \frac{r_<^l}{r_>^{l+1}} P_l(\cos\gamma)$$

where $r_>$ ($r_<$) is the larger (smaller) of $|\vec{x}|$, $|\vec{x}'|$ and γ is the angle bet. them

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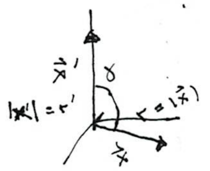
The key is the fact that 12/3/20

$$\vec{\nabla}^2\left(\frac{1}{|\vec{x}-\vec{x}'|}\right) = 4\pi\,\delta(\vec{x}-\vec{x}')$$

If $\vec{x} \neq \vec{x}'$, $\delta(\vec{x}-\vec{x}) = 0 \rightarrow \vec{\nabla}^2\left(\frac{1}{|\vec{x}-\vec{x}'|}\right.$

Reduce the problem into a problem of solving the Laplace eqn w/ azimuthal symmetry. We do this, by aligning $\vec{x}'$ along the positive z-axis



• W/ $\frac{1}{|\vec{x}-\vec{x}'|}$ being a solution to the eqn, we can write the expansion

$$\frac{1}{|\vec{x}-\vec{x}'|} = \sum_{l=0}^{\infty}\left(A_l r^l + B_l r^{-(l+1)}\right) P_l(\cos\theta)$$

where the A_l's and B_l's are to be determined. Aligning $\vec{x}$ along the z-axis

$$\frac{1}{|\vec{x}-\vec{x}'|} = \sum_{l=0}^{\infty}\left(A_l r^l + B_l r^{-(l+1)}\right)$$

There are two cases:

$|\vec{x}| > |\vec{x}'|$

$r > r'$

$|\vec{x}| < |\vec{x}'|$

$r < r'$

Case 1: $r > r'$

$$\frac{1}{\vec{x}-\vec{x}'} = \frac{1}{r-r'} = \frac{1}{r}\frac{1}{\left(1-\frac{r'}{r}\right)}, \quad \frac{r'}{r} < 1$$

$$= \frac{1}{r}\sum_{l=0}^{\infty}\frac{r'^l}{r^l}$$

$$= \sum_{l=0}^{\infty}\frac{r'^l}{r^{l+1}}$$

← comparing this to $\frac{1}{\vec{x}-\vec{x}'}$, $A_l = 0$ for all l, $B_l = 1$ for all l

→ similar to 2nd term of B_l

$$\frac{1}{|\vec{x}-\vec{x}'|} = \sum_{l=0}^{\infty}\frac{r'^l}{r^{l+1}} \longrightarrow \frac{1}{|\vec{x}-\vec{x}'|} = \sum_{l=0}^{\infty}\frac{r'^l}{r^{l+1}}P_l(\cos\gamma)$$

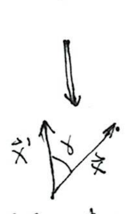
Case 2: $r' > r$

$$\frac{1}{|\vec{x}-\vec{x}'|} = \frac{1}{r'-r}$$

$$= \frac{1}{r'}\frac{1}{1-\frac{r}{r'}}, \quad \frac{r}{r'} < 1$$

$$= \frac{1}{r'}\sum_{l=0}^{\infty}\frac{r^l}{r'^l} = \sum_{l=0}^{\infty}\frac{r^l}{r'^{l+1}}$$

comparing this to $\frac{1}{\vec{x}-\vec{x}'}$, $A_l = 1$ for all



$$\frac{1}{|\vec{x}-\vec{x}'|} = \sum_{l=0}^{\infty}\frac{r^l}{r'^{(l+1)}}P_l(\cos\gamma)$$

Combining the 2 cases, we have

$$\frac{1}{|\vec{x}-\vec{x}'|} = \sum_{l=0}^{\infty}\frac{r_<^l}{r_>^{l+1}}P_l(\cos\gamma)$$

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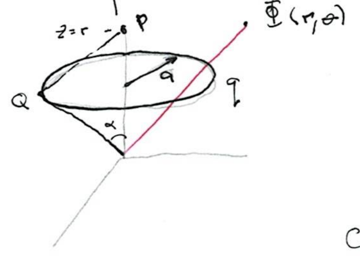
Example:



$$\Phi(z=r) = \frac{1}{4\pi\epsilon_0}\frac{q}{|PQ|}$$

Case I: $r > c$

$$\Phi(z=r) = \frac{q}{4\pi\epsilon_0}\sum_{l=0}^{\infty}\frac{c^l}{r^{l+1}}P_l(\cos\alpha)$$

Comparing this w/ $\frac{1}{|\vec{x}-\vec{x}'|}$, we get that

$A_l = 0$ for all l, $B_l = \frac{q}{4\pi\epsilon_0}P_l(\cos\alpha)$ for all l

$$\Phi(z=r) \longrightarrow \Phi(r,\theta) = \frac{q}{4\pi\epsilon_0}\sum_{l=0}^{\infty}\frac{c^l}{r^{l+1}}P_l(\cos\alpha)\,P_l\cdot$$

Case 2: $r < c$

$$\Phi(z=r) \longrightarrow \Phi(\quad) = \frac{q}{4\pi\epsilon_0}\sum_{l=0}^{\infty}\frac{r^l}{c^{l+1}}P_l(\cos\alpha)\,P_l\cos\theta$$

Combining the two cases

$$\Phi(r,\theta) = \frac{q}{4\pi\epsilon_0}\sum_{l=0}^{\infty}\frac{r_<^l}{r_>^{l+1}}P_l(\cos\alpha)\,P_l(\cos\theta)$$

where $r_<$ $[r_>]$ is the

- Solving problems w/o azimutal symmetry

W/ azimuthal symmetry: $m=0$

w/o azimuthal symmetry: $m \neq 0$

Then, the P-part of the soln must now given by

$$\frac{d}{dx}\left[(1-x^2)\frac{dP}{dx}\right] + \left[l(l+1) - \frac{m^2}{1-x^2}\right]P_l = 0$$

for arbitrary $l, m \neq 0$.

of soln in the interval $-1 \leq x \leq 1$:

$l = 0, 1, 2, \dots,$ $\quad m = -l, -(l-1), \dots, 0, \dots, (l-1)\, l$

The solutions are known as the associated Legendre function, $P_l^m(x)$

$$P_l^m(x) = (-1)^m(1-x^2)^{\frac{m}{2}}\frac{d^m}{dx^m}P_l\,(x^2-1)^l$$

$$= (-1)^m(1-x^2)^{m/2}\frac{d^{m+l}}{dx^{m+l}}(x^2-1)$$

$$P_l^{-m}(x) = \frac{(-1)^m(l-m)!}{l+m}$$

$$\int_{-1}^{1}P_l^m(x)\,P_{l'}^m(x)\,dx = \frac{l}{2l+1}$$

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Combining P and Q

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$$Y(\theta,\phi) = P(\theta)\, Q(\phi)$$

$$Y_{\ell}(\theta,\phi) = \sqrt{\frac{2\ell+1}{4\pi}\frac{(\ell-m)!}{(\ell+1)!}}\; P_{\ell}^{m}(\cos\theta$$

Spherical Harmonics

$$\int_{\substack{\text{over all}\\ \text{directions}}} d\Omega\, Y^{*}_{\ell',m'}(\theta,\phi)\, Y_{\ell m}(\theta,\phi) = \int$$

They are comple

$$\sum_{\ell=0}^{\infty}\sum_{m=\ell}^{\infty} Y^{*}_{\ell m}(\theta',\phi')\, Y_{\ell m}(\theta,\phi) = \delta(\phi-\phi)\,(\delta\cos\theta-\cos\theta')$$

The completes

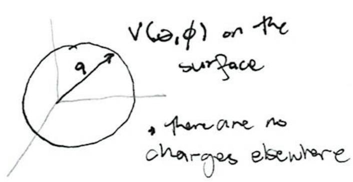
$$g(\theta,\phi) = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} A_{\ell m} Y_{\ell m}(\theta,\phi)$$

where $A_{\ell m} = \int_{\substack{\text{over all}\\ \text{directions}}}$

The general solution to the Laplace equation ($\nabla^2\Phi = 0$) in spherical coord's is given by

$$\Phi(r,\theta,\phi) = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\left\{\left[A_{\ell m} r^{\ell} + B_{\ell m} r^{-(\ell+1)}\right] Y_{\ell m}(\theta,\phi)\right\}$$

Consider the ff. scenario.



• Imposing regularity of solution, $B_{\ell m} = 0$. Thus

$$\Phi(r,\theta,\phi) = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} A_{\ell m} r^{\ell} Y_{\ell m}(\theta,\phi)$$

On the surface, $r = a$:

$$\Phi(a,\theta,\phi) = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} A_{\ell m} a^{\ell} Y_{\ell m}(\theta,\phi) = V(\theta,\phi)$$

Integrating,

$$\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} A_{\ell m} a^{\ell}\int d\Omega\, Y_{\ell m}(\theta,\phi)\, Y^{*}_{\ell' m'}(\theta,\phi) = \int d\Omega\, Y_{\ell' m'}(\theta,\phi)\, V(\theta,\phi)$$

Then,

$$A_{\ell' m'} a^{\ell'} = \int d\Omega\, Y^{*}_{\ell' m'}(\theta,\phi)\, Y(\theta,\phi)$$

$$A_{\ell m} = \frac{1}{a^{\ell}}\int d\Omega\, Y_{\ell m}(\theta,\phi)\, Y(\theta,\phi)$$

$$= \frac{1}{a^{\ell}}\int_0^{2\pi} d\phi\int_0^{\pi} d\theta\,\sin\theta\, Y^{*}_{\ell m}(\theta,\phi)\, Y(\theta,\phi)$$

Example:

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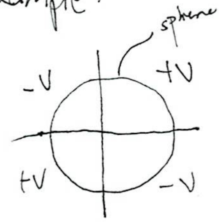

- Obtain the potential outside the sphere

$$\Phi(r,\theta,\phi) = \Sigma$$

The potential must remain finite at $r \to \infty$. Thus,

$$A_{lm} = 0 \text{ for all } l \text{ and } m$$

and we have

$$\Phi(r,\theta,\phi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{B_{lm}}{r^{l+1}} Y_{lm}(\theta,\phi)$$

At $r = a$

$$\Phi(r,\theta,\phi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{B_{lm}}{a^{l+1}} Y_{lm}(\theta,\phi) = V(\theta,\phi)$$

$$B_{lm} = a^{l+1} \int_0^{2a} d\phi \int_0^{\pi} d\theta \sin\theta \, Y^*_{lm}(\theta,\phi) \, V(\theta,\phi)$$

$$= a^{l+1} \int_0^{2a} d\phi \int_0^{\pi} d\theta \sin\theta \, P_l^m(\cos\theta) \, e^{-im\phi} \, V(\theta,\phi)$$

$$= a^{l+1} \left[\int_0^{\pi} d\theta \sin\theta \, P_l^m(\cos\theta)\right]\left[\int_0^{2\pi} d\phi \, e^{-im\phi} \, V(\phi)\right]$$

$$= a^{l+1} \left[\int_0^{\pi} P_l^m(\cos\theta) \sin\theta \, d\theta\right]\left[\int_0^{\pi/2} (-V) e^{-im\phi} d\phi + \int_{\frac{\pi}{2}}^{\pi} (+V) e^{-im\phi} d\phi + \int_{\pi}^{\frac{3\pi}{2}} (-V) e^{-im\phi} d\phi + \int_{\frac{3\pi}{2}}^{2\pi} (+V) e^{-im\phi} d\phi\right]$$

Exercise: Solve 3.2