- What does this imply?

Consider the components of our metric. For instance, we can infer that

Recall condition for orthogonality discussed in pg. 1

$$g_{00} = g(\partial_0, \partial_0) < 0$$ since we assumed that k^a is timelike

$$g_{0i} = g(\partial_0, \partial_i) = 0$$ since $\partial_0 \perp \partial_i$; $\{\frac{\partial}{\partial x_i}\}$ are tangent vectors to Σ which are orthogonal to k^a

$$= g_{ab} \underbrace{\left(\frac{\partial}{\partial x^0}\right)^a}_{k^a} \underbrace{\left(\frac{\partial}{\partial x^i}\right)^b}_{t^b} = 0$$

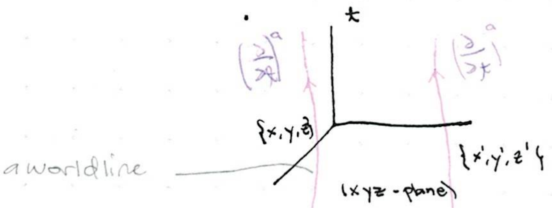
Because of this, we can write the line element of our static spacetime to be

$$ds^2 = g_{00}(x^\mu)(dx^0)^2 + g_{ij}(x^\mu)\,dx^i dx^j + \cancelto{0}{g_{0i}\,dx^0 dx^i}$$

In other words, with the use of symmetries and the way we adopted our coordinates to match these symmetries, we can reduce our line element generally of the form $ds^2 = g_{\mu\nu}\,dx^\mu dx^\nu$ to the form above which has no cross terms.

Sept. 28, 2021 $\boxed{2}$ As a recall:

- Direction of the Killing flow represents direction where symmetry is present



The status of the xyz coordinates is that of a label of the integral curve. Note that we have set

$$k^a = \left(\frac{\partial}{\partial t}\right)^a$$

- Now, since we have assumed our time coordinate to be along the killing flow, the components of the killing flow are $k^a = (1,0,0,0)$
  - If we have chosen $k^a = \left(\frac{\partial}{\partial \lambda}\right)^a$ where $t = t(\lambda)$, then, with the use of chain rule $\left(\frac{\partial}{\partial t}\right)^a = \frac{d\lambda}{dt}\left(\frac{\partial}{\partial \lambda}\right)^a$, we obtain $k^a = \left(\left(\frac{d\lambda}{dt}\right)^{-1}, 0, 0, 0\right)$
    - i.e., if we choose our time coord to be along some arbitrary parameter instead, then the time component of k^a will not necessarily be constant
    - This is called straightening
  - Note that when we derived $ds^2 = g_{00}(x^\mu)(dx^0)^2 + g_{ij}(x^\mu)\,dx^i dx^j$, the coord's are still just geometrically defined. Yes, we aligned t along k^a but we have not incorporated the nature of k^a being constant
- With $k^a = (1,0,0,0)$, we can now evaluate the lie derivative

$$\mathcal{L}_k g = k^\mu \partial_\mu g_{\alpha\beta} + \cancelto{0}{g_{\alpha\mu} k^\mu{}_{,\beta}} + \cancelto{0}{g_{\mu\beta} k^\mu{}_{,\alpha}} = 0$$

which leads to: $$k^\mu \partial_\mu g_{\alpha\beta} = k^0 \partial_0 g_{\alpha\beta} + \cancelto{0}{k^1} \partial_1 g_{\alpha\beta} + \dots = \boxed{\frac{\partial g_{\alpha\beta}}{\partial t} = 0}$$

This implies the temporal invariance of the metric

- Note that

$$g_{\alpha\mu} k^\mu{}_{,\beta} = g_{\alpha 0} \cancelto{0}{\frac{\partial k^0}{\partial x^\beta}} + g_{\alpha 1} \cancelto{0}{\frac{\partial k^1}{\partial x^\beta}} + \dots + g_{\alpha i} \cancelto{0}{\frac{\partial k^i}{\partial x^\beta}} + \dots = 0$$

In the same manner, $g_{\mu\beta} k^\mu{}_{,\alpha} = 0$

Note that x^μ where $\mu = 0,1,2,3$ incorporates the time component $x^0 = t$

- Since we now know that the metric does not depend on time, the line element can further be reduced to

$$ds^2 = g_{00}(x^i)\,dt^2 + g_{ij}(x^k)\,dx^i dx^j$$

- Spherical symmetry
  - ↳ another symmetry property of the Schwarzschild soln
  - ↳ means that the spatial part of the metric is foliated by 2-spheres
  - ↳ $\{x^i\} \longrightarrow \{r, \theta, \phi\}$

no longer dep. on the angles θ & ϕ

- Since the metric should be rotation invariant, it can be written as

$$ds^2 = A(r)\, dt^2 + B(r)\, dr^2 + C(r)\, d\Omega^2$$

- We can choose a diff. radial component such that

$$R^2 = C(r) \longrightarrow r = C^{-1}(R^2)$$

$$2R\, dR - (C^{-1})'\, dr = 0 \longleftarrow dr = 2R/(C^{-1})' dR$$

Thus, we can write the metric in the form of:

$$ds^2 = F(R)\, dt^2 + G(R)\, dR^2 + R^2\, d\Omega^2$$

where $F(R) = A(r(R))$ and $G(R) = 4R^2/[(C^{-1})']^2\, B(r(R))$.

Let $R = r$ and we have:

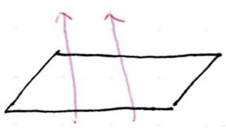
$$ds^2 = F(r)\, dt^2 + G(r)\, dr^2 + r^2\, d\Omega^2$$

- This is the standard metric for a static spacetime that is spherically sym.
- Notice that there are only two free functions here, $F(r)$ & $G(r)$
- Also, notice that we started from geometry (and applied geometric conditions) before we integrated a coordinate system in our manifold. However, the nature of the underlying geometry is still unknown since fxns F & G are still arbitrary
  - i.e. the metric derived here is just a geometric construct that satisfies staticity & spherical symmetry; no physical meaning to coord's yet

- In GR, we are free to choose which coordinate system to use. So, after solving Einstein's field eqns and obtain some soln expressed in a coordinate sys., we still need to interpret these coordinates with the use of the metric.

- Thusing the standard form, we know that time lies along the Killing flow. So, we have the meaning of the time coord. If we want to know the meaning of the radial component r, we look at t = const. surfaces (also, w/ θ & ϕ const)

t = const.
θ, ϕ = const.



We have the arc length as the only changing parameter. So, the metric in this case is:

$$dl^2 = G(r)\, dr^2$$

- In Euclidean space, the distance of points w/ the same radial coord. is just given by $dl_{Euclidean} = dr^2 = r_2 - r_1$.

We see here that the change in r when t, θ, ϕ = const. is no longer a simple difference like in Euclidean space. The distance is now given by

$$l = \int_{r_1}^{r_2} \sqrt{G(r)}\, dr$$

- What does r mean then? We can take t = const. & r = const.. The metric becomes $ds^2 = r^2 d\Omega^2$, the geometry of a 2-sphere. Thus, the area of t, r = const. surfaces will be:

$$A_{2\text{-sphere}} = 4\pi r^2$$

Euclidean
Schwarzschild
any geometry described by the standard form metric

We then obtain:

$$r = \sqrt{\frac{\text{Area}}{4\pi}} \quad \longleftarrow \text{areal radius}$$

This is the geometric meaning of our r-coordinate

- There are geometries (like wormholes) where $r=0$ does not exist but r still retains the meaning of the areal radius
- If we have an arbitrary metric,

$$ds^2 = A\,dt^2 + B\,dr^2 + C(r)\,d\Omega^2$$

the areal radius here is not r but $R = \sqrt{C(r)}$. We need $C(r) = r^2$ for r to be the areal radius

Important! • Geometry (form of the metric) supplies the meaning of the coordinates.

• Recall the Schwarzchild soln mentioned in earlier lectures. The r-coord. there is the areal radius due to the r^2 factor of $d\Omega^2$.

• M: Schwarzchild mass; no physical meaning at the moment

- If we have an obj. far away, moving at a circular geodesic of a central obj. where $r = \text{const}$, then its angular velocity is: $\left(\frac{d\phi}{dt}\right)^2 = \boxed{\Omega^2 = \frac{M}{r^3}}$ Kepler's law

We have here something that's geometric leading to a physical law. So, we think of M as a mass similar to Newtonian mass

▸ Singularities of the Schwarzchild soln

Looking back at the line element, we clearly see that there are two values of r in which ds^2 blows up (becomes undefined): $r=0$ & $r=2M$

• Singularities: points where the particular soln becomes ill-behaved

• Is this ill-behavior due to the choice of coord's or has a physical consequence?

* Note that the determinant of the metric (gives the volume element of the geometry the metric describes) must not vanish anywhere. If it does, we know there is something wrong w/ the metric

$g \to \eta$
$\Gamma \sim \partial g \to 0$

• Principle of Equivalence: The metric g can always be set to η by a local set of coord's (Riemann coord's) and ∂g can be set to zero
Christoffel symbols

- It's not really the metric or its first derivative which is measurable. It's the 2nd derivative which is measurable (physical) thru tidal waves & geodesic deviation

• $\partial\partial g$: pertains to curvature (Riemann curvature); integrated in Riemann tensor

- The non-vanishing components of the Riemann tensor for the Schwarzchild metric are

$$R^t{}_{rtr} = -2\frac{m}{r^3}\left(1-\frac{2m}{r}\right)^{-1} \qquad R^r{}_{\theta r\theta} = \frac{1}{\sin^2\theta} R^r{}_{\phi r\phi} = -\frac{m}{r^3}$$

$$R^t{}_{\theta t\theta} = \frac{1}{\sin^2\theta} R^t{}_{\phi t\phi} = \frac{M}{r^3} \qquad R^\theta{}_{\phi\theta\phi} = 2\frac{m}{r^3}\sin^2\theta$$

$\partial\partial g \sim$ Riemann tensor
matter ↙ ↘
(10) Ricci tensor   Weyl curvature (10)

* There are 20 independent comp. of a Riemann tensor

* Ricci tensor: linked to a matter field

* Weyl curvature: sources the curvature when the spacetime is Ricci flat (vacuum)
: reason why we have gravitational waves

* $G_{\alpha\beta} = R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R = 8\pi\, T_{\alpha\beta}$
↳ the trace of the Riemann tensor is specified by the matter content

* Non-linearity of GR gives rise to non-trivial solns even in vacuum (Weyl $\neq 0$ even with Ricci $= 0$)

We can't use $R^{t}{}_{rtr}$, $R^{t}{}_{\phi t\theta}$, etc. to determine whether the singularities are physical even if they correspond to curvature because they are still components (dependent on the coord. sys)

- Instead, we make use of (scalar) curvature invariants which don't change under coordinate transformations [no free indices]
- We obtain them thru contractions; combining the curvature components into scalars

contraction of Ricci
- Ricci scalar: $R^{\mu\nu} R_{\mu\nu} = 0$ — in Schwarzschild

contraction of Riemann
- Kretschmann scalar: $R_{\mu\nu\alpha\beta} R^{\mu\nu\alpha\beta} = K_1 = \frac{48M^2}{r^6}$

dual of K_1
- Chern-Pontryagin scalar: $\epsilon_{\mu\nu}{}^{\rho\sigma} R^{\mu\nu\alpha\beta} R_{\rho\sigma\alpha\beta} = K_2$

⋮ (you can obtain other scalars)

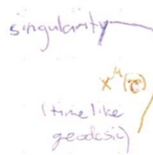
* Since we are in vaccum, $G_{\alpha\beta} = 0$. Thus, by multiplying the metric on both sides
$$g^{\alpha\beta}(R_{\alpha\beta} - \tfrac{1}{2} g_{\alpha\beta} R) = R - \tfrac{1}{2} 4R = -R = 0 \rightarrow R = 0$$

- Levi-Civita tensor: $\epsilon_{\mu,\rho\sigma} = \sqrt{-g}\, \underbrace{[\mu, \rho\sigma]}_{\text{completely anti-symmetric}}$
- This means that
- This means that if I do $\{t, r, \theta, \phi\} \rightarrow \{t, x, y, z\}$, we still have the same value of K_1:
$$K_1 = \frac{48M^2}{r^6} \rightarrow K_1 = \frac{48M^2}{r(t,x,y,z)^6}$$
- By substituting in the singularities into K_1, we see that $r = 2M$ is no longer singular but $r=0$ still is. So we have
  - $r = 2M$: Apparent / coordinate singularity (due to inappropriate choice of coord's)
  - $r = 0$: curvature singularity (physical)

Sept. 30, 2021

Note that that this check on K_1 does not always work and we have to turn to higher order scalars.

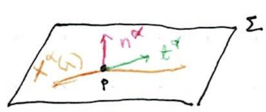
* A (true) singularity must "belong to the manifold", meaning it must be reachable (geodesics can reach the offending point at finite affine parameter (proper time)).



This means that for a spacetime to have a singularity there must exist a massive obj. whose world line stop at some finite proper time — geodesic incompleteness

- If the proper time to reach this singularity is infinite, this is an indication that the singularity is due to the bad choice of coord's

Hypersurfaces



Consider a generic hypersurface Σ parametrized by $\Phi(x^\mu) = 0$. Let the normal vector n^α be $\perp$ to Σ at point p. We already established that $n_\alpha = \Phi_{,\alpha}$ (pg. 2) and that any tangent vector t^a is $\perp$ to n_a. To verify this: Given the curve $x^\alpha(\lambda)$ and $t^\alpha = dx^\alpha/d\lambda$, we know
$$n_\alpha t^\alpha = \frac{d\Phi}{d\lambda} = 0$$
as shown in pg. 3. We go to the local inertial frame centered at p which is described by
$$ds^2 = -dt^2 + dx^i dx_i$$
We then rotate our spatial axis such that only one component of the normal vector's spatial part is non-vanishing: $n_\alpha = (n^0, n^1, 0, 0)$

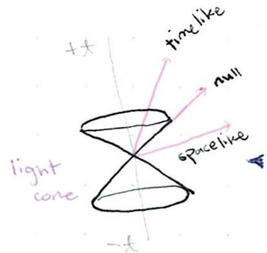
No loss of generality as no assumptions are made

As for our tangent vector, we still have: $t^\alpha = (t^0, t^1, t^2, t^3)$

Thus, due to the orthogonality of the two vectors, we have
$$n_\alpha t^\alpha = -n^0 t^0 + n^1 t^1 = 0 \Rightarrow \frac{t^0}{t^1} = \frac{n^1}{n^0}$$

With the relation between the comp. of t^α & n^α obtained, we have

$$t^\alpha = \left(t^1 \frac{n^1}{n^0}, t^1, t^2, t^3\right) = \frac{t^1}{n^0}(n^1, n^0, a, b) = \Lambda(n^1, n^0, a, b)$$



as the components of the tangent vector t^α. Notice its relation to $n_\alpha = (n_0, n_1, 0, 0)$

- what can we say about the norm of t^α?

Def'n: Σ_p is spacelike, timelike, or null at p depending on the norm of n

- $n_\alpha n^\alpha < 0 \longrightarrow n^\alpha$ timelike $\longrightarrow \Sigma$ spacelike
- $n_\alpha n^\alpha > 0 \longrightarrow n^\alpha$ spacelike $\longrightarrow \Sigma$ timelike
- $n_\alpha n^\alpha = 0 \longrightarrow n^\alpha$ null $\longrightarrow \Sigma$ null

This characterization is a local thing; meaning a surface can be spacelike at one point and be timelike in another

① $n_\alpha n^\alpha < 0$

Note that the norm of the normal vector is

$$g_{\alpha\beta} n_\alpha n^\beta = n_\alpha n^\alpha = -(n^0)^2 + (n^1)^2$$

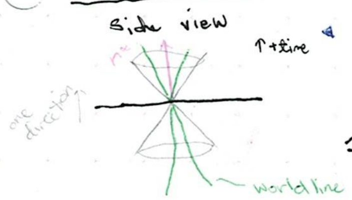
* causal curves: curves on & inside the light cone

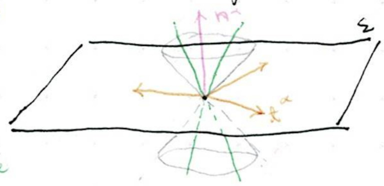
For the tangent vector, we have

$$g_{\alpha\beta} t_\alpha t^\beta = t_\alpha t^\alpha = \Lambda^2\left[-(n^1)^2 + (n^0)^2 + a^2 + b^2\right] = \Lambda^2\left[-n_\alpha n^\alpha + a^2 + b^2\right]$$

Because of the given condition, we know that $t_\alpha t^\alpha > 0$. Thus, t^α is spacelike

Side view



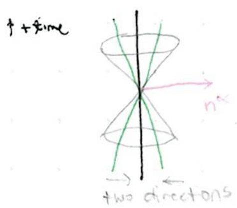


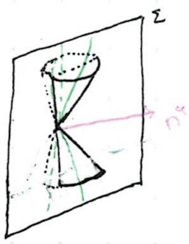
* Note that causal curves must move towards the positive time direction. Therefore, we can infer that space like hypersurfaces can only be crossed in one direction

② $n_\alpha n^\alpha > 0$

For this case, the norm of t^α depends on the values of a & b. Thus, t^α can either be spacelike, timelike, or null.

Side view





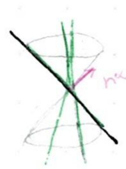
* In the same manner as in the first case, we can infer that timelike hypersurfaces can be crossed by two directions

③ $n_\alpha n^\alpha = 0$

Going back to the norm of t^α, we see that $t^\alpha t_\alpha \geq 0$ (spacelike or null).

If $a = b = 0$, then $t_\alpha t^\alpha = 0$. This value of a & b represent a unique tangent direction

Side view



* We can also infer that null hypersurfaces can only be crossed in one direction or not at all (causal curve that lies on Σ)

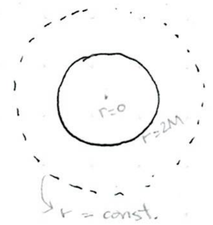
Note that the light cone touches the hypersurface only one line/tangent direction in which $a = b = 0$ (along the direction of the causal curve lying on Σ). Another way of looking at this is to think of null curves as generators of null hypersurfaces

- Constant-r hypersurfaces of Schwarzschild

Consider this kind of hypersurface characterized by $\Phi(x^\alpha) = 0$.
The norm of the normal vector is now given by

$$n_\alpha n^\alpha = g^{\alpha\beta} n_\alpha n_\beta = g^{rr}\left(\frac{d\Phi}{dr}\right)^2 = g^{rr} = 1 - \frac{2M}{r}$$

as Φ is only dep. on r so $n_\alpha = \Phi_{,\alpha} = (0, d\Phi/dr, 0, 0)$. From this, we can say that



$r > 2M$: Σ timelike (has this notion of a spatial coord as we can cross the surface in both directions)

$r = 2M$: Σ null

$r < 2M$: Σ spacelike

} has a diff. interpretation with $r > 2M$ as we can cross the surface in only one direction

* This change in the interpretation of the r - coord also coincides with the transition of the killing vector from timelike to spacelike.

- Recall : $k^\alpha = (1,0,0,0)$ is our killing vector which we required to be timelike. Calculating its norm,

$$k^\alpha k_\alpha = g_{\alpha\beta} k^\alpha k^\beta = g_{tt} = -\left(1 - \frac{2M}{r}\right)$$

From this, we can say that

$r > 2M \longrightarrow k^\alpha k_\alpha < 0 \longrightarrow k^\alpha$ is timelike | t - temporal r - radial

$r < 2M \longrightarrow k^\alpha k_\alpha > 0 \longrightarrow k^\alpha$ is spacelike | t - radial r - temporal

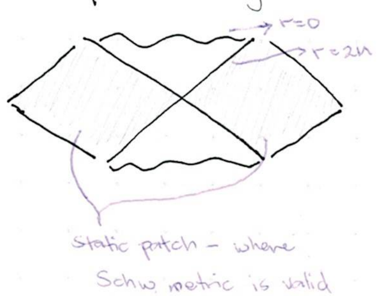
This means that the interpretation of the t and r coord's interchange when we cross $r = 2M$.

* Since $r = 0 < 2M$, this is not a point in space any more but a point in time. Reaching $r = 0$ is your future when we go past $r = 2M$ into the black hole. It is unavoidable

* Inside, it is the t = const. curves that are traversable in two directions (property of a spatial coord.)

- The assumptions made to derive the Schwarzschild geometry is not valid for $r \leq 2M$ (i.e. Schwarzschild metric is no longer applicable)

penrose diagram



* How do I know that $r = 2M$ is null while $r < 2M$ is spacelike when Schw. metric is not valid beyond the event horizon?

The reference did this naive, simple, but incorrect calculation of det. this. The results still stand, they are still the same even through the correct means (using a coord sys. that is valid past $r = 2M$ and extending the manifold)