Problem 1 (1.1)

Use Gauss's theorem [and (1.21) if necessary] to prove the following:

(a) Any excess charge placed on a conductor must lie entirely on its surface.

Proof:
First, we note that $\mathbf{E} = 0$ inside a conductor since an electric field inside would move the free charges so the situation would no longer be considered as electrostatics. From this, it follows that there are no charges inside the conductor using the differential form of Gauss' law:

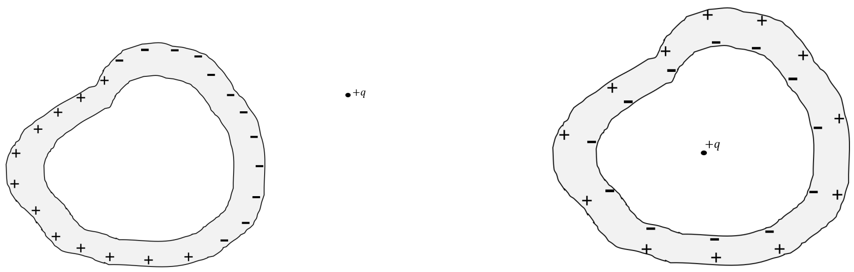
$$\cancelto{0}{\nabla \cdot \mathbf{E}} = \frac{\rho}{\epsilon_0} \qquad \longrightarrow \qquad \rho = 0 \tag{1}$$

Since there can be no charges inside a conductor, the only place excess charges can reside is at the conductor's surface. ■

(b) A closed, hollow conductor shields its interior from fields due to charges outside, but does not shield its exterior from the fields due to charges placed inside it.

Proof:
Let's first consider a closed, hollow, and uncharged conductor. If we place a charge of $+q$ close to it, induced charges will arise at the conductor's surface in such a way that the placed charge will attract negative charges to the near side while the positive charges will be repelled to the far side (Fig. 1a). Note that the induced charges are at the surface because no charges can lie inside the conductor as mentioned in part (a). A similar but opposite scenario happens for $-q$. We can see here that any external fields will not be able to penetrate the conductor's surface due to induced charges that cancel them out with their own fields. Therefore, a closed, hollow conductor shields its interior from fields due to charges outside.

Now, we consider a closed, and hollow conductor with a charge $+q$ inside. This charge will then attract the negative charges to the inner side of the conductor's surface while the positive charges will be repelled to the outer side (Fig. 1b). A similar but opposite scenario happens for $-q$. The effect of these induced charges is that the net electric field inside cancels out. On the other hand, the exterior of the conductor experiences a field due to the induced positive charges on the outer side. We see here that the conductor's exterior is effectively affected by inside charges through induced charges. Therefore, a closed, hollow conductor does not shield its exterior from the fields due to charges placed inside it. ■



(a) (b)

Figure 1

(c) The electric field at the surface of a conductor is normal to the surface and has a magnitude σ/ϵ_0, where σ is the charge density per unit area on the surface.

Proof:
Let's first consider some surface with a charge distribution σ. We draw a Gaussian pillbox small enough that when we take a sufficiently close look at it, the scenario looks like the case of an infinite plane with σ as its charge density per unit area. Note that A is the area of the top/bottom surfaces of the pillbox. A diagram of the problem is shown Figure 2. Recall that Gauss's law for some arbitrary surface S is given by

$$\oint_S \mathbf{E} \cdot \mathrm{d}\mathbf{a} = \frac{Q_{\text{enc}}}{\epsilon_0} \tag{2}$$

Now, the total charge enclosed by the pillbox is $Q_{\text{enc}} = \sigma A$. Consider the normal component of $\mathbf{E}_{\text{above}}$ and $\mathbf{E}_{\text{below}}$. Then, we have

$$\oint_S \mathbf{E} \cdot \mathrm{d}\mathbf{a} = \int \mathbf{E} \cdot \mathrm{d}\mathbf{a}_{\text{top}} + \int \mathbf{E} \cdot \mathrm{d}\mathbf{a}_{\text{side}} + \int \mathbf{E} \cdot \mathrm{d}\mathbf{a}_{\text{bottom}} \tag{3}$$

Taking the limit as the width of the pillbox's sides approaches zero,

$$\begin{aligned}
\oint_S \mathbf{E} \cdot \mathrm{d}\mathbf{a} &= \int \mathbf{E} \cdot \mathrm{d}\mathbf{a}_{\text{top}} + \cancelto{0}{\int \mathbf{E} \cdot \mathrm{d}\mathbf{a}_{\text{side}}} + \int \mathbf{E} \cdot \mathrm{d}\mathbf{a}_{\text{bottom}} \\
\frac{Q_{\text{enc}}}{\epsilon_0} &= \int \mathbf{E}_{\text{above}} \cdot \hat{\boldsymbol{n}}\, \mathrm{d}a_{\text{top}} + \int \mathbf{E}_{\text{below}} \cdot \hat{\boldsymbol{n}}\, \mathrm{d}a_{\text{bottom}} \\
\frac{\sigma A}{\epsilon_0} &= \int |\mathbf{E}^{\perp}_{\text{above}}||\hat{\boldsymbol{n}}| \cos(0)\, \mathrm{d}a_{\text{top}} - \int |\mathbf{E}^{\perp}_{\text{below}}||\hat{\boldsymbol{n}}| \cos(\pi)\, \mathrm{d}a_{\text{top}} \\
&= \int E^{\perp}_{\text{above}}\, \mathrm{d}a_{\text{top}} - \int E^{\perp}_{\text{below}}\, \mathrm{d}a_{\text{bottom}}
\end{aligned} \tag{4}$$

Since $E^{\perp}_{\text{above}}$ and $\int E^{\perp}_{\text{below}}$ are just constant with regards to the area occupied by the pillbox, we can take them out of the integral. Thus,

$$\frac{\sigma A}{\epsilon_0} = E^{\perp}_{\text{above}} \int \mathrm{d}a_{\text{top}} - E^{\perp}_{\text{below}} \int \mathrm{d}a_{\text{bottom}} = E^{\perp}_{\text{above}} A - E^{\perp}_{\text{below}} A \tag{5}$$

Dividing the whole equation by A, we get

$$E^{\perp}_{\text{above}} - E^{\perp}_{\text{below}} = \frac{\sigma}{\epsilon_0} \tag{6}$$

As for the tangential component, we have

$$\oint \mathbf{E} \cdot \mathrm{d}\mathbf{l} = \int \mathbf{E} \cdot \mathrm{d}\mathbf{l}_{\text{top}} + \int \mathbf{E} \cdot \mathrm{d}\mathbf{l}_{\text{left}} + \int \mathbf{E} \cdot \mathrm{d}\mathbf{l}_{\text{bottom}} + \int \mathbf{E} \cdot \mathrm{d}\mathbf{l}_{\text{right}} \tag{7}$$

Taking the limit as the distance between the top and bottom parts of the loop approaches zero, we have

$$\begin{aligned}
\oint \mathbf{E}\cdot \mathrm{d}\mathbf{l} &= \int \mathbf{E}\cdot \mathrm{d}\mathbf{l}_{\text{top}} + \cancelto{0}{\int \mathbf{E}\cdot \mathrm{d}\mathbf{l}_{\text{left}}} + \int \mathbf{E}\cdot \mathrm{d}\mathbf{l}_{\text{bottom}} + \cancelto{0}{\int \mathbf{E}\cdot \mathrm{d}\mathbf{l}_{\text{right}}} \\
0 &= \int \mathbf{E}_{\text{above}}\cdot \hat{\boldsymbol{t}}\,\mathrm{d}l_{\text{top}} - \int \mathbf{E}_{\text{below}}\cdot \hat{\boldsymbol{t}}\,\mathrm{d}l_{\text{bottom}} \\
&= \int |\mathbf{E}^{\|}_{\text{above}}||\hat{\boldsymbol{t}}|\cos(0)\,\mathrm{d}l_{\text{top}} - \int |\mathbf{E}^{\|}_{\text{below}}||\hat{\boldsymbol{t}}|\cos(\pi)\,\mathrm{d}l_{\text{bottom}} \\
&= \int E^{\|}_{\text{above}}\,\mathrm{d}l_{\text{top}} - \int E^{\|}_{\text{below}}\,\mathrm{d}l_{\text{bottom}} \\
&= E^{\|}_{\text{above}} \int \mathrm{d}l_{\text{top}} - E^{\|}_{\text{below}} \int \mathrm{d}l_{\text{bottom}} \\
&= E^{\|}_{\text{above}}\, l - E^{\|}_{\text{below}}\, l
\end{aligned} \tag{8}$$

Dividing the whole equation by l, we get

$$E^{\|}_{\text{above}} - E^{\|}_{\text{below}} = 0 \tag{9}$$

Thus, from Eqs. 6 and 6, we get

$$\mathbf{E}_{\text{above}} - \mathbf{E}_{\text{below}} = \frac{\sigma}{\epsilon_0}\hat{\boldsymbol{n}} \tag{10}$$

However, $E = 0$ inside a conductor. So, $\mathbf{E}_{\text{below}} = 0$ for this case which leads to

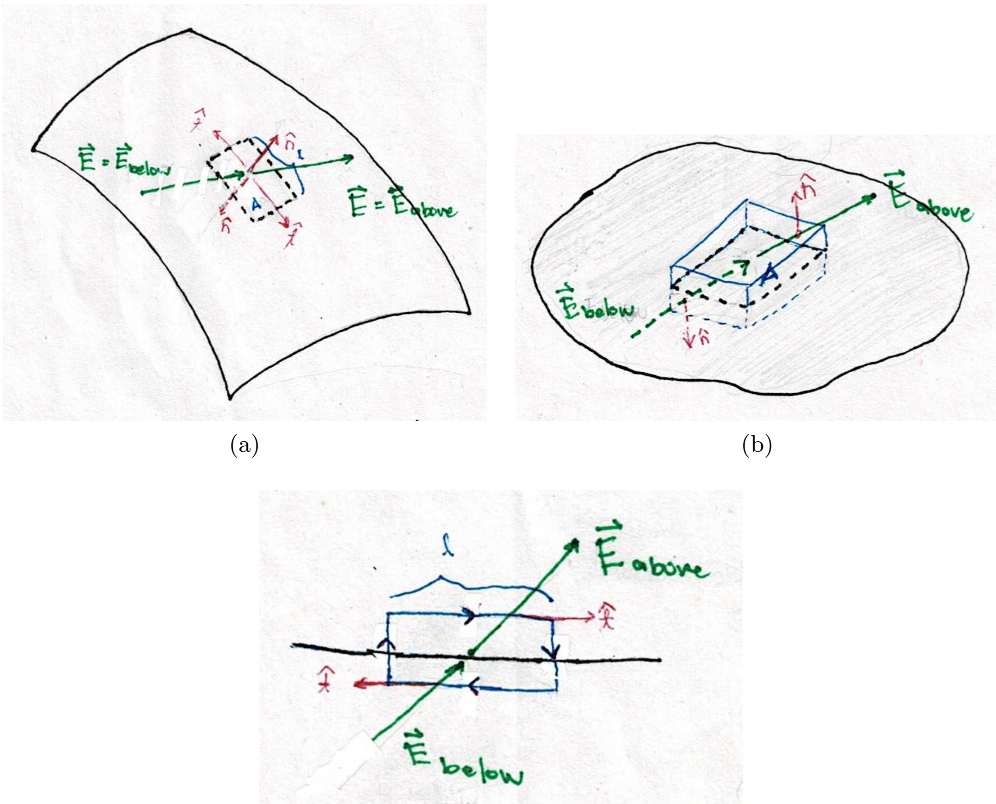
$$\mathbf{E}_{\text{above}} = \frac{\sigma}{\epsilon_0}\hat{\boldsymbol{n}} \tag{11}$$

■

References

Griffiths, D., *Introduction to Electrodynamics* (3rd ed.), Chapter 2

Zangwill, A., *Modern Electrodynamics* (1st ed.), Chapter 3



(a) (b) (c)

Figure 2: (a) A diagram of the problem; (b) A closer look at the area of interest; (c) A side view of the area of interest;

Problem 2 (1.3)

Using Dirac delta functions in the appropriate coordinates, express the following charge distributions as three dimensional charge densities $\rho(\mathbf{x})$:

(a) In spherical coordinates, a charge Q uniformly distributed over a spherical shell of radius R.
Solution:
In general, the charge density function is given by

$$\rho(\mathbf{x}) = Q\delta(\mathbf{x}-\mathbf{x}_0) \tag{1}$$

where Q is the total charge. In spherical coordinates, the Dirac delta function is given by

$$\delta(\mathbf{x}-\mathbf{x}_0) = \frac{1}{r^2\sin\theta}\delta(r-r_0)\,\delta(\theta-\theta_0)\,\delta(\phi-\phi_0) \tag{2}$$

Since there is symmetry along θ and ϕ in this scenario, we must project out the θ - integral and ϕ - integral. The denominator in Eq. (2) becomes

$$\int_0^\pi\int_0^{2\pi} r^2\sin\theta\,\mathrm{d}\phi\mathrm{d}\theta = r^2\left(\int_0^\pi \sin\theta\mathrm{d}\theta\right)\left(\int_0^{2\pi}\mathrm{d}\phi\right) = r^2\cdot 2\cdot 2\pi = 4\pi r^2 \tag{3}$$

Consequently, Eq. (2) is now written as

$$\delta(\mathbf{x}-\mathbf{x}_0) = \frac{1}{4\pi r^2}\delta(r-r_0) \tag{4}$$

Note that $r_0 = R$ for this scenario. Substituting Eq. (4) into Eq. (1), the charge distribution is

$$\boxed{\rho(\mathbf{x}) = \frac{Q}{4\pi r^2}\delta(r-R)} \tag{5}$$

(b) In cylindrical coordinates, a charge λ per unit length uniformly distributed over a cylindrical surface of radius λ.
Solution:
Here, we assume that the cylindrical surface has length L oriented at the z-axis with a total charge Q. The center of the bottom part of the surface is placed at the origin. In cylindrical coordinates, the Dirac delta function is given by

$$\delta(\mathbf{x}-\mathbf{x}_0) = \frac{1}{r}\delta(r-r_0)\,\delta(\phi-\phi_0)\,\delta(z-z_0) \tag{6}$$

In this scenario, there is symmetry along ϕ. There is also symmetry along z but only on $z=0$ to $z=L$. Projecting out the ϕ - integral and z - integral, the denominator in Eq. (6) becomes

$$\int_0^L\int_0^{2\pi} r\,\mathrm{d}\phi\mathrm{d}z = r\left(\int_0^L\mathrm{d}z\right)\left(\int_0^{2\pi}\mathrm{d}\phi\right) = r\cdot 2\pi\cdot L = 2\pi r L \tag{7}$$

Consequently, Eq. (6) is now written as

$$\delta(\mathbf{x}-\mathbf{x}_0) = \frac{1}{2\pi r L}\delta(r-r_0) \tag{8}$$

Note that $r_0 = b$ for this scenario. Substituting Eq. (8) into Eq. (1), the charge distribution is given by

$$\rho(\mathbf{x}) = \frac{Q}{2\pi r L}\delta\left(r - b\right) \tag{9}$$

We also note that the total charge Q is related to the charge-per-unit-length λ by $Q = \lambda L$. Applying $\lambda = Q/L$, we have

$$\boxed{\rho(\mathbf{x}) = \frac{\lambda}{2\pi r}\delta\left(r - b\right)} \tag{10}$$

(c) In cylindrical coordinates, a charge Q spread uniformly over a flat circular disk of negligible thickness and radius R.
Solution:
Here, we assume that the center of the disk is placed at the origin. The Dirac delta function is still expressed as that of Eq. (6) in the previous scenario. Also, there is still a symmetry along ϕ. For this case, however, there is now symmetry along r instead of z but only when $r \leq R$. Projecting out the ϕ - integral and r - integral, the denominator in Eq. (6) becomes

$$\int_0^R \int_0^{2\pi} r\, \mathrm{d}\phi \mathrm{d}r = \left(\int_0^R r \mathrm{d}r\right)\left(\int_0^{2\pi} \mathrm{d}\phi\right) = \frac{r^2}{2} \cdot 2\pi = \pi r^2 \tag{11}$$

Consequently, Eq. (6) is now written as

$$\delta(\mathbf{x} - \mathbf{x}_0) = \frac{1}{\pi r^2}\delta\left(z - z_0\right) \tag{12}$$

Note that $z_0 = 0$ for this scenario. Substituting Eq. (12) into Eq. (1), the charge distribution is given by

$$\boxed{\rho(\mathbf{x}) = \frac{Q}{\pi r^2}\delta\left(z\right)} \tag{13}$$

**Acknowledgements: I am grateful for the insightful comments of Adrielle Cusi, Lemuel Saret, and Dylan Salcedo when solving this problem.*

References

The Dirac Delta in Curvilinear Coordinates, `http://www.fen.bilkent.edu.tr/~ercelebi/mp03`

Physics 231 lecture notes (dated 9/17/20)

Problem 3 (1.7*)

Two long, cylindrical conductors of radii a_1 and a_2 are parallel and separated by a distance d, which is large compared with their radius. Show that the capacitance per unit length is given approximately by

$$C = \pi\epsilon_0 \left(\ln \frac{d}{a}\right)^{-1} \tag{1}$$

where a is the geometrical mean of the two radii.

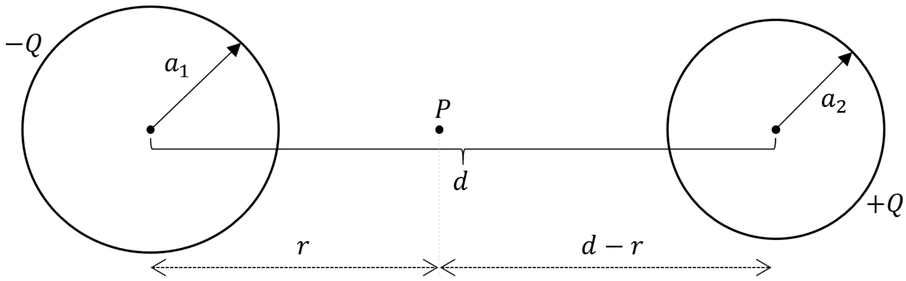


Figure 1: A diagram of the problem.

Solution:
We want to find the electric field between the the two cylindrical conductors to calculate the potential difference in the same region to get the capacitance. To do this, let's assume that the origin be at the center of the first conductor and the direction to the right be the positive $\hat{\mathbf{r}}$. The conductors, which have have the same length L, are oriented as shown in Figure 1. Also, let the first conductor have a $-Q$ charge and the second with a $+Q$ charge. the electric field on P due to the first conductor,

$$\begin{aligned}\oint_S \mathbf{E}_1 \cdot \mathrm{d}\mathbf{a} &= \frac{Q_{\text{enc}}}{\epsilon_0} \\ E_1 \int \mathrm{d}a &= \frac{-Q}{\epsilon_0} \\ E_1(2\pi r L) &= -\frac{Q}{\epsilon_0}\end{aligned} \tag{2}$$

we obtain

$$\mathbf{E}_1 = -\frac{Q}{2\pi\epsilon_0 L}\frac{\hat{\mathbf{r}}}{r} \tag{3}$$

In the same manner, the electric field on P due to the second conductor is given by

$$\mathbf{E}_2 = \frac{Q}{2\pi\epsilon_0 L}\frac{-\hat{\mathbf{r}}}{d-r} = -\frac{Q}{2\pi\epsilon_0 L}\frac{\hat{\mathbf{r}}}{d-r} \tag{4}$$

With the principle of superposition, we can combine these two fields to get the total electric field experienced by point P to get

$$\mathbf{E}_{\text{tot}} = \mathbf{E}_1 + \mathbf{E}_2 = -\frac{Q}{2\pi\epsilon_0 L}\left(\frac{1}{r} + \frac{1}{d-r}\right)\hat{\mathbf{r}} \tag{5}$$

Recall that the potential difference can calculated using the relation

$$\Delta\Phi = \Phi(b) - \Phi(a) = -\int_a^b \mathbf{E} \cdot \mathrm{d}\mathbf{l} \tag{6}$$

*modified problem

Since we want $\Delta\Phi$ between the two conductors, we choose the radial path from a_1 to $d - a_2$. Thus, we have

$$
\begin{aligned}
\Delta\Phi &= -\int_{a_1}^{d-a_2} \mathbf{E}_{\text{tot}} \cdot \mathrm{d}\mathbf{r} \\
&= \int_{a_1}^{d-a_2} \frac{Q}{2\pi\epsilon_0 L}\left(\frac{1}{r} + \frac{1}{d-r}\right) \cancelto{1}{\hat{\mathbf{r}}\cdot\hat{\mathbf{r}}}\, \mathrm{d}r \\
&= \frac{Q}{2\pi\epsilon_0 L}\int_{a_1}^{d-a_2} \left(\frac{1}{r} + \frac{1}{d-r}\right) \mathrm{d}r \\
&= \frac{Q}{2\pi\epsilon_0 L}\left[\ln r - \ln(d-r)\right]_{a_1}^{d-a_2} \qquad (7) \\
&= \frac{Q}{2\pi\epsilon_0 L}\left[\ln(d-a_2) - \ln a_2 - \ln a_1 + \ln(d-a_1)\right] \\
&= \frac{Q}{2\pi\epsilon_0 L}\ln\left[\frac{(d-a_2)(d-a_1)}{a_2 a_1}\right] \\
\Delta\Phi &\approx \frac{Q}{2\pi\epsilon_0 L}\ln\left(\frac{d^2}{a_1 a_2}\right)
\end{aligned}
$$

as a_1 and a_2 are small compared to d that they are negligible in the term $(d-a_2)(d-a_1)$. Note that the capacitance is given by

$$
C = \frac{Q}{\Delta\Phi} \qquad (8)
$$

Substituting in Eq. (7), we have

$$
C = 2\pi\epsilon_0 L\left[\ln\left(\frac{d}{(a_1 a_2)^{1/2}}\right)^2\right]^{-1} = \pi\epsilon_0 L\left[\ln\left(\frac{d}{\sqrt{a_1 a_2}}\right)\right]^{-1} \qquad (9)
$$

Let the geometric mean be expressed as $a = \sqrt{a_1 a_2}$. Therefore, the capacitance per unit length is approximately

$$
\frac{C}{L} = \pi\epsilon_0\left[\ln\left(\frac{d}{a}\right)\right]^{-1} \qquad (10)
$$

References

Griffiths, D., *Introduction to Electrodynamics* (3rd ed.), Chapter 2

`http://pleclair.ua.edu/ph106/Homework/HW4_SOLN.pdf`

Problem 4 (1.18)

Consider the configuration of conductors of Problem 1.17, with all conductors except S_1 held at zero potential. Show that the potential $\Phi(\mathbf{x})$ anywhere in the volume V and on any of the surfaces S_i can be written

$$\Phi(\mathbf{x}) = \frac{1}{4\pi\epsilon_0} \oint_{S_1} \sigma_1\left(\mathbf{x}'\right) G\left(\mathbf{x}, \mathbf{x}'\right) \mathrm{d}a' \tag{1}$$

where $\sigma_1(\mathbf{x}')$ is the surface charge density on S_1 and $G(\mathbf{x}, \mathbf{x}')$ is the Green's function potential for a point charge in the presence of all the surfaces that are held at zero potential (but with S_1 absent). Show also that the electrostatic energy is

$$W = \frac{1}{8\pi\epsilon_0} \oint_{S_1} \mathrm{d}a \oint_{S_1} \mathrm{d}a' \sigma_1(\mathbf{x}) G\left(\mathbf{x}, \mathbf{x}'\right) \sigma_1\left(\mathbf{x}'\right) \tag{2}$$

where the integrals are only over the surface S_1.

Solution:
In general, the potential is given as

$$\Phi(\mathbf{x}) = \frac{1}{4\pi\epsilon_0} \int_V \rho\left(\mathbf{x}'\right) G\left(\mathbf{x}, \mathbf{x}'\right) \mathrm{d}^3x' + \frac{1}{4\pi} \oint_S \left[G\left(\mathbf{x}, \mathbf{x}'\right) \frac{\partial \Phi\left(\mathbf{x}'\right)}{\partial n'} - \Phi\left(\mathbf{x}'\right) \frac{\partial G\left(\mathbf{x}, \mathbf{x}'\right)}{\partial n'}\right] \mathrm{d}a' \tag{3}$$

In this problem, we will use Dirichlet boundary conditions for convenience which entails that

$$G\left(\mathbf{x}, \mathbf{x}'\right) = 0 \quad \text{for } \mathbf{x}' \text{ on } S_2 \ldots S_N \tag{4}$$

where the Green's function in this problem does not include S_1 in its formulation as stated in the given. Imposing this condition, Eq. (3) becomes

$$\Phi(\mathbf{x}) = \frac{1}{4\pi\epsilon_0} \int_V \rho\left(\mathbf{x}'\right) G\left(\mathbf{x}, \mathbf{x}'\right) \mathrm{d}^3x' - \frac{1}{4\pi} \oint_{S_2 \ldots S_N} \Phi\left(\mathbf{x}'\right) \frac{\partial G}{\partial n'} \mathrm{d}a' \tag{5}$$

Note that the surface integral in the second term of Eq. (5) only includes surfaces from S_2 to S_N. These surfaces, however, have $\Phi = 0$ so the second term vanishes. On the other hand, the volume integral becomes a surface integral because any charge distribution would be situated at the conducting surfaces bounding the volume. But imposing the condition in Eq. (4) implies that the only charge distribution is at S_1, not at any other surface. So, ρ becomes σ_1. Therefore, Eq. (5) results to

$$\Phi(\mathbf{x}) = \frac{1}{4\pi\epsilon_0} \oint_{S_1} \sigma_1\left(\mathbf{x}'\right) G\left(\mathbf{x}, \mathbf{x}'\right) \mathrm{d}a' \tag{6}$$

which is exactly Eq. (1). ■

We also have the following relation for the electrostatic energy:

$$W = \frac{1}{2} \int \rho(\mathbf{x}) \Phi(\mathbf{x}) \mathrm{d}^3x \tag{7}$$

As mentioned before, the only charge distribution is at S_1 so the volume integral becomes a surface integral. Substituting Eq. (6), the relation becomes

$$\begin{aligned} W &= \frac{1}{2} \oint_{S_1} \sigma_1(\mathbf{x}) \left(\frac{1}{4\pi\epsilon_0} \oint_{S_1} \sigma_1\left(\mathbf{x}'\right) G\left(\mathbf{x}, \mathbf{x}'\right) da'\right) \mathrm{d}a \\ &= \frac{1}{8\pi\epsilon_0} \oint_{S_1} \sigma_1(\mathbf{x}) \mathrm{d}a \oint_{S_1} \sigma_1\left(\mathbf{x}'\right) G\left(\mathbf{x}, \mathbf{x}'\right) da' \end{aligned} \tag{8}$$

which can be arranged to Eq. (2). ■

**modified problem*

References

de Coulomb, C. A., Introduction to Electrostatics `http://www.phys.lsu.edu/~jarrell/COURSES/ELECTRODYNAMICS/Chap1/chap1.pdf`