Problem 1

Consider the following distribution:

$$W(n) = (1-p)^{n-1}p \tag{1}$$

This is the probability of the first occurrence of success requires $n = 1, 2, 3\ldots$ independent trials. Here p is the success probability of each trial.

1.1 Obtain the mean.

Solution:
For this problem, it would be useful to note the sum of the infinite geometric series given by

$$\sum_{n=0}^{\infty} ax^n = \frac{a}{1-x} \tag{2}$$

which is valid when $|x| < 1$. If we let $a = 1$ and start the index at 1, we have

$$\sum_{n=1}^{\infty} x^{n-1} = \frac{1}{1-x} \quad \longleftrightarrow \quad \sum_{n=1}^{\infty} x^n = \frac{x}{1-x} \tag{3}$$

which we can apply to this problem with the change of variables

$$x = 1-p \quad \longleftrightarrow \quad 1-x = p \tag{4}$$

Note that this is valid since p lies in the interval $0 < p < 1$ as it is the success probability of each trial so $|1-p| < 1$. With these, we can evaluate the following summation:

$$\sum_{n=1}^{\infty} p(1-p)^{n-1} = \frac{p}{1-(1-p)} = 1 \tag{5}$$

in which we see that the given probability is normalized. To calculate for the mean, we consider the derivative of Eq. (3) with respect to x:

$$\begin{aligned} \frac{\partial}{\partial x}\left(\sum_{n=1}^{\infty} x^n\right) &= \frac{\partial}{\partial x}\left(\frac{x}{1-x}\right) \\ \sum_{n=1}^{\infty} nx^{n-1} &= \frac{1}{(1-x)^2} \\ \sum_{n=1}^{\infty} nx^n &= \frac{x}{(1-x)^2} \end{aligned} \tag{6}$$

Then, we can infer that the mean of $W(n)$ is given by

$$\begin{aligned} \overline{n} &= \frac{\sum_{n=1}^{\infty} np(1-p)^{n-1}}{\sum_{n=1}^{\infty} p(1-p)^{n-1}} \\ &= \frac{p}{1-p}\sum_{n=1}^{\infty} n(1-p)^n \\ &= \frac{p}{1-p}\frac{1-p}{p^2} \\ \overline{n} &= \frac{1}{p} \end{aligned} \tag{7}$$

1.2 Derive the variance.

Solution:
For the variance, we can apply the same method used in the previous part and consider the derivative of Eq. (6) with respect to x:

$$\begin{aligned}
\frac{\partial}{\partial x}\left(\sum_{n=1}^{\infty} n x^n\right) &= \frac{\partial}{\partial x}\left(\frac{x}{(1-x)^2}\right) \\
\sum_{n=1}^{\infty} n^2 x^{n-1} &= \frac{1}{(1-x)^2} + \frac{2x}{(1-x)^3} \\
\sum_{n=1}^{\infty} n^2 x^n &= x\left[\frac{1}{(1-x)^2} + \frac{2x}{(1-x)^3}\right]
\end{aligned} \tag{8}$$

Then, $\overline{n^2}$ is given by

$$\begin{aligned}
\overline{n^2} &= \frac{\sum_{n=1}^{\infty} n^2 p(1-p)^{n-1}}{\sum_{n=1}^{\infty} p(1-p)^{n-1}} \\
&= \frac{p}{1-p}\sum_{n=1}^{\infty} n^2 (1-p)^n \\
&= \frac{p}{1-p}(1-p)\left[\frac{1}{p^2} + \frac{2(1-p)}{p^3}\right] \\
&= \frac{1}{p} + \frac{2-2p}{p^2} \\
\overline{n^2} &= \frac{2-p}{p^2}
\end{aligned} \tag{9}$$

after applying Eqs. (4) and (5). Therefore, the variance of $W(n)$ is

$$\sigma^2 = \overline{n^2} - \overline{n}^2 = \frac{2-p}{p^2} - \frac{1}{p^2} = \frac{1-p}{p^2} \tag{10}$$