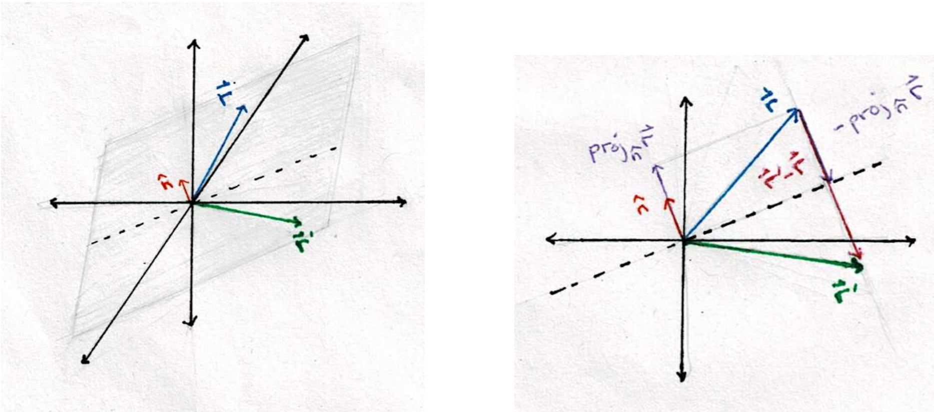
Problem 1 (3.2)

Construct the vector **r**' obtained by reflecting vector **r** in the plane whose unit normal vector is $\hat{\mathbf{n}}$. Without any calculations, using only geometric arguments, determine the eigenvalues and eigenvectors of the corresponding transformation matrix $\mathcal{A}$. If $\hat{\mathbf{n}} = (n_1, n_2, n_3)$, show that $\mathcal{A}$ has elements $a_{ij} = \delta_{ij} - 2n_in_j$ and is an improper orthogonal matrix.



(a) Full diagram

(b) Side view of 1a

Figure 1: Reflecting **r** in a plane (plane of reflection is shaded in pencil gray).

Solution:
Before we start, let's recall the geometric interpretation of eigenvectors and eigenvalues. When we apply a linear transformation A on an eigenvector **v**, it does not change direction but is now affected by a scale factor called an eigenvalue (often denoted by λ). Note that the eigenvector must have a corresponding non-zero eigenvalue for this to be applicable. The transformed vector $\mathbf{v}'$ as shown in Eq. (1) points in the direction of the eigenvector while its length scales according to the effect of λ on **v**.

$$\mathbf{v}' = A\mathbf{v} = \lambda\mathbf{v} \tag{1}$$

Now, going back to the problem, if we consider a good number of configurations for **r** in a 3D Cartesian space with a plane of reflection oriented like in Figure 1a, we might notice that there are two geometrically different kinds of vectors: those that are reflected in the plane when the linear transformation $\mathcal{A}$ is applied to them and those that are not. Note that the transformed vector **r**' can be expressed as

$$\mathbf{r}' = \mathcal{A}\mathbf{r} \tag{2}$$

where **r** is an eigenvector. First, let's focus on the vectors that are not affected by $\mathcal{A}$. These vectors lie on the plane of reflection itself. To span the plane, we need two linearly independent vectors. So, we find that we have two eigenvectors lying on the plane and any other vector on the plane can be expressed as a linear combination of these two. We need to search for another eigenvector to span the whole 3D space. This is where the other kind of vectors come in. Among these vectors, we have those that lie on the line perpendicular to the plane which have their direction totally reflected. We choose this as the orientation of the third eigenvector since the direction of vectors with other orientations can only be partially reflected (not all of their components are reversed) by $\mathcal{A}$. This suggests that these vectors can be linear combination of vectors not affected by $\mathcal{A}$ and one totally reflected after the transformation.

So, we have the direction of our eigenvectors, how about the eigenvalues? Well, note that a reflection does not change the length of the original vector, only its direction. Since the length does not change, the transformed vector must be scaled by 1. Thus, $\lambda = 1$ but only for the eigenvectors lying on the plane since they retain their original direction after the transformation. As for the eigenvector perpendicular to these two and has its direction totally flipped after the transformation, $\lambda = -1$ since this is the only way to reverse its direction (note again that an eigenvector does not change direction after we apply a transformation). To summarize, we have Table 1.

λ	Multiplicity	Eigenvector
1	2	two linearly independent vectors that lie on the plane of reflection
-1	1	vector $\perp$ to the the two eigenvectors lying on the plane

Table 1: Eigenvalues and eigenvectors of the corresponding transformation matrix $\mathcal{A}$

To find the elements of the matrix $\mathcal{A}$, note that the projection of $\mathbf{r}$ onto $\hat{\mathbf{n}}$ is

$$\mathrm{proj}_{\hat{\mathbf{n}}}\mathbf{r} = \frac{\hat{\mathbf{n}}\cdot\mathbf{r}}{\hat{\mathbf{n}}\cdot\hat{\mathbf{n}}}\hat{\mathbf{n}} = \frac{\hat{\mathbf{n}}\cdot\mathbf{r}}{|\hat{\mathbf{n}}|^2}\hat{\mathbf{n}} = (\hat{\mathbf{n}}\cdot\mathbf{r})\hat{\mathbf{n}} = \hat{\mathbf{n}}^T\mathbf{r}\hat{\mathbf{n}} \tag{3}$$

where $\hat{\mathbf{n}}\cdot\hat{\mathbf{n}} = |\hat{\mathbf{n}}|^2$ and $|\hat{\mathbf{n}}| = 1$. Also, $\mathbf{a}\cdot\mathbf{b} = \mathbf{a}\mathbf{b}^T$ in matrix form. Looking at Figure 1b, we see that

$$\frac{1}{2}\left(\mathbf{r}' - \mathbf{r}\right) = -\,\mathrm{proj}_{\hat{\mathbf{n}}}\mathbf{r} \tag{4}$$

since the reflected vector must be in the same distance away from the plane of reflection as the original vector. Rearranging this equation, we get

$$\begin{aligned}
\mathbf{r}' &= \mathbf{r} - 2\,\mathrm{proj}_{\hat{\mathbf{n}}}\mathbf{r} \\
&= \mathbf{r} - 2\hat{\mathbf{n}}^T\mathbf{r}\hat{\mathbf{n}} \\
&= \mathbf{r} - 2\hat{\mathbf{n}}\hat{\mathbf{n}}^T\mathbf{r} \\
\mathbf{r}' &= \left(\mathbf{I} - 2\hat{\mathbf{n}}\hat{\mathbf{n}}^T\right)\mathbf{r}
\end{aligned} \tag{5}$$

after factoring out $\mathbf{r}$ where $\mathbf{I}$ is a 3D identity matrix. Comparing Eq. (2) and Eq. (5), we can notice that

$$\mathcal{A} = \mathbf{I} - 2\hat{\mathbf{n}}\hat{\mathbf{n}}^T \tag{6}$$

which, in index notation, is expressed as

$$a_{ij} = \delta_{ij} - 2n_in_j \tag{7}$$

given that $\hat{\mathbf{n}} = (n_1, n_2, n_3)$. Using this to write out the matrix $\mathcal{A}$, we have

$$\mathcal{A} = \begin{pmatrix} 1-2n_1n_1 & 2n_1n_2 & -2n_1n_3 \\ -2n_2n_1 & 1-2n_2n_2 & -2n_2n_3 \\ -2n_3n_1 & -2n_3n_2 & 1-2n_3n_3 \end{pmatrix} \tag{8}$$

We can calculate the determinant with Mathematica as shown in the following figure:

```
In[3]:= Det[{{1 - 2 n1 n1, -2 n1 n2, -2 n1 n3}, {-2 n2 n1, 1 - 2 n2 n2, -2 n2 n3}, {-2 n3 n1, -2 n3 n2, 1 - 2 n3 n3}}] //
         Simplify
Out[3]= 1 - 2 n1^2 - 2 n2^2 - 2 n3^2
```

Figure 2: Mathematica code to calculate the determinant of $\mathcal{A}$.

Further simplifying the expression of the determinant, we get

$$\det(\mathcal{A}) = 1 - 2\left(n_1^2 + n_2^2 + n_3^2\right) = 1 - 2\left(\hat{\mathbf{n}} \cdot \hat{\mathbf{n}}\right) = 1 - 2|\hat{\mathbf{n}}|^2 = -1 \tag{9}$$

Since $\det(\mathcal{A}) = -1$, $\mathcal{A}$ is an improper orthogonal matrix. ■

**Acknowledgements: I am grateful for the insightful comments of Reinier Ramos, Patrok, and Lemuel Saret when solving this problem.*

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Problem 2 (3.6)

The translational displacement of a body can be represented by a vector and the linear velocity is the time derivative of the position vector. The angular velocity vector, however, is not, in general, the time derivative of an angular displacement vector. In order to prove this, note that if there exists such a vector $\mathbf{\Omega}$, its components $\Omega_x, \Omega_y, \Omega_z$ can be expressed in terms of the Euler angles and must be such that their time derivatives equal the corresponding components of the angular velocity:

$$\begin{aligned} \omega_x &= \frac{\partial \Omega_x}{\partial \theta}\dot{\theta} + \frac{\partial \Omega_x}{\partial \phi}\dot{\phi} + \frac{\partial \Omega_x}{\partial \psi}\dot{\psi}, \quad \omega_y = \frac{\partial \Omega_y}{\partial \theta}\dot{\theta} + \frac{\partial \Omega_y}{\partial \phi}\dot{\phi} + \frac{\partial \Omega_y}{\partial \psi}\dot{\psi} \\ \omega_z &= \frac{\partial \Omega_z}{\partial \theta}\dot{\theta} + \frac{\partial \Omega_z}{\partial \phi}\dot{\phi} + \frac{\partial \Omega_z}{\partial \psi}\dot{\psi} \end{aligned} \tag{1}$$

Using

$$\begin{aligned} \omega_x &= \dot{\theta}\cos\phi + \dot{\psi}\sin\theta\sin\phi \\ \omega_y &= \dot{\theta}\sin\phi - \dot{\psi}\sin\theta\cos\phi \\ \omega_z &= \dot{\phi} + \dot{\psi}\cos\theta, \end{aligned} \tag{2}$$

prove that there can be no vector $(\Omega_x, \Omega_y, \Omega_z)$ such that these equations are satisfied.

Solution:
If we were given $\dfrac{\mathrm{d}\varphi}{\mathrm{d}x} = f(x,y)$ and $\dfrac{\mathrm{d}\varphi}{\mathrm{d}y} = g(x,y)$ where

$$\mathrm{d}\varphi(x,y) = f(x,y)\mathrm{d}x + g(x,y)\mathrm{d}y, \tag{3}$$

then we can infer that

$$\varphi(x,y) = \int f(x,y)\mathrm{d}x + C(y) \tag{4}$$

where $C(y)$ is a constant of the integration (Note that this only has to be a constant in x so it can be a function in y or a constant in both variables.) and that

$$g(x,y) = \frac{\mathrm{d}\varphi}{\mathrm{d}y} = \frac{\partial}{\partial y}\left(\int f(x,y)\mathrm{d}x\right) + \frac{\partial C(y)}{\partial y} \tag{5}$$

Now, comparing ω_x in Eq. (1) with Eq. (2), we see that

$$\frac{\partial \Omega_x}{\partial \phi} = 0, \quad \frac{\partial \Omega_x}{\partial \theta} = \cos\phi, \quad \frac{\partial \Omega_x}{\partial \psi} = \sin\theta\,\sin\phi \tag{6}$$

Then, by applying Eq. (4), we have

$$\Omega_x(\phi,\theta,\psi) = \int \cancelto{0}{\frac{\partial \Omega_x}{\partial \phi}}\,\mathrm{d}\phi + C(\theta,\psi) = C(\theta,\psi) \tag{7}$$

This says that Ω_x should only be a function of θ and ψ. By applying Eq. (5), we know that

$$\frac{\partial \Omega_x}{\partial \theta} = \frac{\partial C(\theta,\psi)}{\partial \theta} \tag{8}$$

However, integrating this equation with $\partial\Omega_x/\partial\theta$ given in Eq. (6), we see that $C(\theta, \psi)$, and Ω_x for that matter, is actually dependent on ϕ which is a contradiction. Therefore, we cannot find a Ω_x that can satisfy Eq. (1). In the same manner, we cannot find a Ω_y that will satisfy Eq. (1).

On the other hand, the reasoning why there is no Ω_z that satisfies Eq. (1) is a little bit different from the other components. The initial steps are still the same. We compare ω_z in Eq. (1) with Eq. (2) to see that

$$\frac{\partial\Omega_z}{\partial\phi} = 1, \quad \frac{\partial\Omega_z}{\partial\theta} = 0, \quad \frac{\partial\Omega_z}{\partial\psi} = \cos\theta \tag{9}$$

Then, we apply Eq. (4) to obtain

$$\Omega_z(\phi, \theta, \psi) = \int \cancelto{0}{\frac{\partial\Omega_z}{\partial\theta}}\, \mathrm{d}\theta + C(\phi, \psi) = C(\phi, \psi) \tag{10}$$

which says that Ω_z should only be a function of ϕ and ψ. By applying Eq. (5), we know that

$$\frac{\partial\Omega_z}{\partial\phi} = \frac{\partial C(\phi, \psi)}{\partial\phi} \tag{11}$$

Now, we integrate this to get

$$C(\phi, \psi) = \int \cancelto{1}{\frac{\partial\Omega_z}{\partial\phi}}\, \mathrm{d}\phi + C(\psi) = \phi + C(\psi) \tag{12}$$

Equating Eqs. (10) and (11), we infer that Ω_z must be

$$\Omega_z(\phi, \theta, \psi) = \phi + C(\psi) \tag{13}$$

However, from Eq. (9), we can also infer that

$$\Omega_z(\phi, \theta, \psi) = \phi + \sin\theta \tag{14}$$

Thus, there is a contradiction in the form of $\Omega_z(\phi, \theta, \psi)$. Therefore, the vector $\mathbf{\Omega}$ with components $\Omega_x, \Omega_y, \Omega_z$ does not exist. ■

**Acknowledgements: I am grateful for the insightful comments of Reinier Ramos when solving this problem.*

References

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Problem 3 (3.9)

A particle is fired vertically upward from the surface of the Earth with initial speed v_0, reaches its maximum height and returns to the ground. Show that the Coriolis deflection when it hits the ground has the opposite sense and is four times bigger than the deviation for a particle dropped from the same maximum height.

Solution:
The angular velocity is given by $\boldsymbol{\omega} = -\omega\cos\lambda\hat{\boldsymbol{\imath}} + \omega\sin\lambda\hat{\boldsymbol{k}}$. Let's consider the velocity of a particle with an arbitrary path near the surface of the Earth given as $\dot{\mathbf{r}}(t) = \dot{x}(t)\hat{\boldsymbol{\imath}} + \dot{y}(t)\hat{\boldsymbol{\jmath}} + \dot{z}(t)\hat{\boldsymbol{k}}$. Then, taking the cross product of $\dot{\mathbf{r}}$ and ω, we have

$$\begin{aligned}
\dot{\mathbf{r}} \times \boldsymbol{\omega} &= \begin{vmatrix} \hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \hat{\boldsymbol{k}} \\ \dot{x} & \dot{y} & \dot{z} \\ \omega_x & 0 & \omega_z \end{vmatrix} \\
&= \dot{y}\omega_z\hat{\boldsymbol{\imath}} + (\dot{z}\omega_x - \dot{x}\omega_z)\,\hat{\boldsymbol{\jmath}} - \dot{y}\omega_x\hat{\boldsymbol{k}} \\
\dot{\mathbf{r}} \times \boldsymbol{\omega} &= \dot{y}\omega\sin\lambda\hat{\boldsymbol{\imath}} - (\dot{z}\omega\cos\lambda + \dot{x}\omega\sin\lambda)\,\hat{\boldsymbol{\jmath}} + \dot{y}\omega\cos\lambda\hat{\boldsymbol{k}}
\end{aligned} \tag{1}$$

Ignoring centripetal and Euler forces, the particle's equation of motion is given by

$$\begin{aligned}
m\ddot{\mathbf{r}} &= \mathbf{F} + \mathbf{F}_{\text{coriolis}} \\
&= m\mathbf{g} + 2m\,\dot{\mathbf{r}} \times \boldsymbol{\omega} \\
m\ddot{\mathbf{r}} &= -mg\hat{\boldsymbol{k}} + 2m\left(\dot{y}\omega\sin\lambda\hat{\boldsymbol{\imath}} - (\dot{z}\omega\cos\lambda + \dot{x}\omega\sin\lambda)\,\hat{\boldsymbol{\jmath}} + \dot{y}\omega\cos\lambda\hat{\boldsymbol{k}}\right)
\end{aligned} \tag{2}$$

after substituting in Eq. (1). Note that the particle's acceleration can be expressed as $\ddot{\mathbf{r}}(t) = \ddot{x}(t)\hat{\boldsymbol{\imath}} + \ddot{y}(t)\hat{\boldsymbol{\jmath}} + \ddot{z}(t)\hat{\boldsymbol{k}}$. Comparing this with Eq. (2), we find that

$$\ddot{x} = 2\dot{y}\omega\sin\lambda \tag{3}$$

$$\ddot{y} = -2\omega\,(\dot{z}\cos\lambda + \dot{x}\sin\lambda) \tag{4}$$

$$\ddot{z} = -g + 2\dot{y}\omega\cos\lambda \tag{5}$$

We integrate $\ddot{x}(t)$ to find $x(t)$ as follows:

$$\begin{aligned}
\frac{\mathrm{d}\dot{x}}{\mathrm{d}t} &= 2\dot{y}\omega\sin\lambda \\
\int_{\dot{x}(0)}^{\dot{x}(t)} \mathrm{d}\dot{x}(t') &= \int_0^t \big(2\dot{y}(t')\omega\sin\lambda\big)\,\mathrm{d}t' \\
\dot{x}(t) - \dot{x}(0) &= 2\omega\sin\lambda\int_0^t \dot{y}(t')\mathrm{d}t' \\
\Rightarrow \quad \dot{x}(t) &= \dot{x}(0) + 2\omega\,(y(t) - y(0))\sin\lambda
\end{aligned} \tag{6}$$

$$\begin{aligned}
\frac{\mathrm{d}x}{\mathrm{d}t} &= \dot{x}(0) + 2\omega\,(y(t) - y(0))\sin\lambda \\
\int_{x(0)}^{x(t)} \mathrm{d}z(t') &= \int_0^t (\dot{x}(0) + 2\omega\,(y(t) - y(0))\sin\lambda)\,\mathrm{d}t' \\
x(t) - x(0) &= \dot{x}(0)t - 2\omega y(0)t\sin\lambda + 2\omega\sin\lambda\int_0^t y(t')\mathrm{d}t' \\
\Rightarrow \quad x(t) &= x(0) + \dot{x}(0)t - 2\omega y(0)t\sin\lambda + 2\omega\sin\lambda\int_0^t y(t')\mathrm{d}t'
\end{aligned} \tag{7}$$

We do the same for $\ddot{y}(t)$ as follows:

$$\begin{aligned}
\frac{\mathrm{d}\dot{y}}{\mathrm{d}t} &= -2\omega\left(\dot{z}\cos\lambda + \dot{x}\sin\lambda\right) \\
\int_{\dot{y}(0)}^{\dot{y}(t)} \mathrm{d}\dot{y}(t') &= -\int_0^t 2\omega\left(\dot{z}(t')\cos\lambda + \dot{x}(t')\sin\lambda\right)\mathrm{d}t' \\
\dot{y}(t) - \dot{y}(0) &= -2\omega\cos\lambda\int_0^t \dot{z}(t')\mathrm{d}t' - 2\omega\sin\lambda\int_0^t \dot{x}(t')\mathrm{d}t' \\
\Rightarrow \quad \dot{y}(t) &= \dot{y}(0) - 2\omega\left(z(t) - z(0)\right)\cos\lambda - 2\omega\left(x(t) - x(0)\right)\sin\lambda
\end{aligned} \tag{8}$$

$$\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}t} &= \dot{y}(0) - 2\omega\left(z(t) - z(0)\right)\cos\lambda - 2\omega\left(x(t) - x(0)\right)\sin\lambda \\
\int_{y(0)}^{y(t)} \mathrm{d}y(t') &= \int_0^t \left(\dot{y}(0) - 2\omega\left(z(t) - z(0)\right)\cos\lambda - 2\omega\left(x(t) - x(0)\right)\sin\lambda\right)\mathrm{d}t' \\
y(t) - y(0) &= \dot{y}(0)t + 2\omega z(0)t\cos\lambda + 2\omega x(0)t\cos\lambda - 2\omega\cos\lambda\int_0^t z(t')\mathrm{d}t' - 2\omega\sin\lambda\int_0^t x(t')\mathrm{d}t' \\
\Rightarrow \quad y(t) &= y(0) + \dot{y}(0)t + 2\omega z(0)t\cos\lambda + 2\omega x(0)t\cos\lambda - 2\omega\cos\lambda\int_0^t z(t')\mathrm{d}t' - 2\omega\sin\lambda\int_0^t x(t')\mathrm{d}t'
\end{aligned} \tag{9}$$

In same manner as $x(t)$ and $y(t)$, we find $z(t)$ as follows:

$$\begin{aligned}
\frac{\mathrm{d}\dot{z}}{\mathrm{d}t} &= -g + 2\dot{y}\omega\cos\lambda \\
\int_{\dot{z}(0)}^{\dot{z}(t)} \mathrm{d}\dot{z}(t') &= \int_0^t \left(-g + 2\dot{y}(t')\omega\cos\lambda\right)\mathrm{d}t' \\
\dot{z}(t) - \dot{z}(0) &= -gt + 2\omega\cos\lambda\int_0^t \dot{y}(t')\mathrm{d}t' \\
\Rightarrow \quad \dot{z}(t) &= \dot{z}(0) - gt + 2\omega\left(y(t) - y(0)\right)\cos\lambda
\end{aligned} \tag{10}$$

$$\begin{aligned}
\frac{\mathrm{d}z}{\mathrm{d}t} &= \dot{z}(0) - gt + 2\omega\left(y(t) - y(0)\right)\cos\lambda \\
\int_{z(0)}^{z(t)} \mathrm{d}z(t') &= \int_0^t \left(\dot{z}(0) - gt + 2\omega\left(y(t) - y(0)\right)\cos\lambda\right)\mathrm{d}t' \\
z(t) - z(0) &= \dot{z}(0)t - \frac{1}{2}gt^2 + 2\omega y(0)t\cos\lambda + 2\omega\cos\lambda\int_0^t y(t')\mathrm{d}t' \\
\Rightarrow \quad z(t) &= z(0) + \dot{z}(0)t - \frac{1}{2}gt^2 + 2\omega y(0)t\cos\lambda + 2\omega\cos\lambda\int_0^t y(t')\mathrm{d}t'
\end{aligned} \tag{11}$$

Now, we have two situations in this problem as shown in Figure 1. To solve for the components of the particle's position $x(t)$, $y(t)$, and $z(t)$ for both of these situations, we will use the method of iteration (or perturbation theory) in which we first solve for the components when $\omega = 0$ (the term due to the actual force $\mathbf{F}$). Then, we will substitute these solutions to the RHS of $x(t)$, $y(t)$, and $z(t)$ when we now include terms proportional to ω (a kind of first-order corrections due to $\mathbf{F}_{\text{coriolis}}$). We can continue on to include terms proportional to higher powers of ω but then we would need to include the other forces that we neglected earlier so we do not bother with these terms.

(a) Situation A: thrown upwards with inital speed v_0

(b) Situation B: falling from rest

Figure 1: Diagram of the problem

Note that, for situation A, the initial conditions are $x_A(0) = y_A(0) = z_A(0) = 0$, $\dot{x}_A(0) = 0 = \dot{y}_A(0)$, and $\dot{z}_A(0) = v_0$. Thus, the components of the particle's position $x(t)$, $y(t)$, and $z(t)$ from Eqs. (7), (9) and (11) become

$$x_A(t_A) = 2\omega \sin\lambda \int_0^{t_A} y_A(t'_A)\mathrm{d}t' \tag{12}$$

$$y_A(t_A) = -2\omega\cos\lambda \int_0^{t_A} z_A(t'_A)\mathrm{d}t'_A - 2\omega\sin\lambda \int_0^{t_A} x_A(t'_A)\mathrm{d}t'_A \tag{13}$$

$$z_A(t_A) = v_0 t_A - \frac{1}{2}g t_A^2 + 2\omega\cos\lambda \int_0^{t_A} y_A(t'_A)\mathrm{d}t'_A \tag{14}$$

Using the method of iterations, the solutions for the 0^{th} order of ω are

$$x_A^{(0)}(t_A) = 0 \tag{15}$$
$$y_A^{(0)}(t_A) = 0 \tag{16}$$
$$z_A^{(0)}(t_A) = v_0 t_A - \frac{1}{2}g t_A^2 \tag{17}$$

This is the first iteration. For the second iteration, the solutions for the 1^{st} order of ω are

$$x_A^{(1)}(t_A) = 2\omega\sin\lambda \int_0^{t_A} \cancelto{0}{y_A^{(0)}(t'_A)}\ \mathrm{d}t'_A = 0 \tag{18}$$

$$\begin{aligned}
y_A^{(1)}(t_A) &= -2\omega\cos\lambda \int_0^{t_A} z_A^{(0)}(t'_A)\mathrm{d}t'_A - 2\omega\sin\lambda \int_0^{t_A} \cancelto{0}{x_A^{(0)}(t'_A)}\ \mathrm{d}t'_A \\
&= -2\omega\cos\lambda \int_0^{t_A} \left(v_0 t'_A - \frac{1}{2}g t_A'^2\right)\mathrm{d}t'_A \\
&= -2\omega v_0 \frac{t_A^2}{2}\cos\lambda + 2\omega\frac{g t_A^3}{2\cdot 3}\cos\lambda \\
y_A^{(1)}(t_A) &= -\omega v_0 t_A^2\cos\lambda + \omega g\frac{t_A^3}{3}\cos\lambda
\end{aligned} \tag{19}$$

$$z_A^{(1)}(t_A) = v_0 t - \frac{1}{2} g t_A^2 + 2\omega \cos\lambda \int_0^{t_A} \cancelto{0}{y_A^{(0)}(t'_A)} \, \mathrm{d}t'_A = v_0 t_A - \frac{1}{2} g t_A^2 \tag{20}$$

Taking the difference of the solutions from the second iteration with the first one, we see that the only deviation is in the y-direction as given

$$\Delta y_A = -\omega v_0 t_A^2 \cos\lambda + \frac{1}{3}\omega g t_A^3 \cos\lambda \tag{21}$$

This is the deflection due to the Coriolis force for situation A.

On the other hand, for situation B, the initial conditions are $x_B(0) = 0 = y_B(0)$, $z_B(0) = h$, and $\dot{x}_B(0) = \dot{y}_B(0) = \dot{z}_B(0) = 0$. The components of the particle's position $x(t)$, $y(t)$, and $z(t)$ from Eqs. (7), (9) and (11) become

$$x_B(t_B) = 2\omega \sin\lambda \int_0^{t_B} y_B(t'_B) \mathrm{d}t'_B \tag{22}$$

$$y_B(t_B) = 2\omega h t_B \cos\lambda - 2\omega \cos\lambda \int_0^{t_B} z_B(t'_B)\mathrm{d}t'_B - 2\omega \sin\lambda \int_0^{t_B} x_B(t'_B)\mathrm{d}t'_B \tag{23}$$

$$z_B(t_B) = h - \frac{1}{2} g t_B^2 + 2\omega \cos\lambda \int_0^{t_B} y_B(t'_B)\mathrm{d}t'_B \tag{24}$$

In the same manner as that of situation A, the first iteration results to

$$x_B^{(0)}(t_B) = 0 \tag{25}$$

$$y_B^{(0)}(t_B) = 0 \tag{26}$$

$$z_B^{(0)}(t_B) = h - \frac{1}{2} g t_B^2 \tag{27}$$

and the second iteration results to

$$x_B^{(1)}(t_B) = 2\omega \sin\lambda \int_0^{t_B} \cancelto{0}{y_B^{(0)}(t'_B)} \, \mathrm{d}t'_B = 0 \tag{28}$$

$$\begin{aligned}
y_B^{(1)}(t_B) &= 2\omega h t_B \cos\lambda - 2\omega \cos\lambda \int_0^{t_B} z_B^{(0)}(t'_B)\mathrm{d}t'_B - 2\omega \sin\lambda \int_0^{t_B} \cancelto{0}{x_B^{(0)}(t'_B)} \, \mathrm{d}t'_B \\
&= 2\omega h t_B \cos\lambda - 2\omega \cos\lambda \int_0^{t_B} \left(h - \frac{1}{2} g t'^2_B\right) \mathrm{d}t'_B \\
&= 2\omega h t_B \cos\lambda - 2\omega h t_B \cos\lambda + 2\omega \frac{g t_B^3}{2 \cdot 3} \cos\lambda \\
y_B^{(1)}(t_B) &= \frac{1}{3} \omega g t_B^3 \cos\lambda
\end{aligned} \tag{29}$$

$$z_B^{(1)}(t_B) = h - \frac{1}{2} g t_B^2 + 2\omega \cos\lambda \int_0^{t_B} \cancelto{0}{y_B^{(0)}(t'_B)} \, \mathrm{d}t'_B = h - \frac{1}{2} g t_B^2 \tag{30}$$

Again, taking the difference of the solutions from the second iteration with the first one, we obtain the Coriolis deflection for situation B given as

$$\Delta y_B = \frac{1}{3} \omega g t_B^3 \cos\lambda \tag{31}$$

Eyeballing the particle's path in situation A, we can infer that the time elapsed for situation A can be expressed as $t_A = 2t_{\text{top}}$ where t_{top} is the time the particle need to reach the maximum height of its path. Also, by differentiating Eq. (20), we know that

$$0 = \dot{z}_A(t_{\text{top}}) = v_0 - gt_{top} \longrightarrow t_{top} = \frac{v_0}{g} \tag{32}$$

since the velocity of a particle thrown upwards at the highest point of its path is zero. Comparing both situations, we can also infer that the time elapsed for situation A is twice that of situation B ($t_A = 2t_B$). Equating both expressions of t_A, we get

$$t_{\text{top}} = t_B \tag{33}$$

Using this in calculating the ratio between the two deflections, we have

$$\begin{aligned}
\frac{\Delta y_A}{\Delta y_B} &= \frac{-\omega v_0 t_A^2 \cos\lambda + \frac{1}{3}\omega g t_A^3 \cos\lambda}{\frac{1}{3}\omega g t_B^3 \cos\lambda} \\
&= -3\frac{v_0}{g}\frac{t_A^2}{t_B^3} + \frac{t_A^3}{t_B^3} \\
&= -3\frac{v_0}{g}\frac{2^2 t_{\text{top}}^2}{t_{\text{top}}^3} + \frac{2^3 t_{\text{top}}^3}{t_{\text{top}}^3} \\
&= -3 \cdot 2^2 \frac{v_0}{g}\frac{1}{t_{\text{top}}} + 2^3 \\
&= -12\frac{v_0}{g}\frac{g}{v_0} + 8 \\
\frac{\Delta y_A}{\Delta y_B} &= -4
\end{aligned} \tag{34}$$

This shows that Coriolis deflection for situation A is in the opposite direction and is four times bigger than the deviation for situation B.

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Problem 4 (3.11)

At $t = 0$ a projectile is fired horizontally near the surface of the Earth with speed v.

(a) Neglecting gravity, show that, to a good approximation, the horizontal deviation of the projectile due to the Coriolis force is

$$d_H = \omega v |\sin\lambda| t^2 \tag{1}$$

where λ is the latitude.

Solution:
The angular velocity is given by $\boldsymbol{\omega} = -\omega\cos\lambda\hat{\boldsymbol{\imath}} + \omega\sin\lambda\hat{\boldsymbol{k}}$. Let's consider the velocity of a particle with an arbitrary path near the surface of the Earth given as $\dot{\mathbf{r}}(t) = \dot{x}(t)\hat{\boldsymbol{\imath}} + \dot{y}(t)\hat{\boldsymbol{\jmath}} + \dot{z}(t)\hat{\boldsymbol{k}}$. Then, taking the cross product of $\dot{\mathbf{r}}$ and ω, we have

$$\begin{aligned} \dot{\mathbf{r}} \times \boldsymbol{\omega} &= \begin{vmatrix} \hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \hat{\boldsymbol{k}} \\ \dot{x} & \dot{y} & \dot{z} \\ \omega_x & 0 & \omega_z \end{vmatrix} \\ &= \dot{y}\omega_z\hat{\boldsymbol{\imath}} + (\dot{z}\omega_x - \dot{x}\omega_z)\,\hat{\boldsymbol{\jmath}} - \dot{y}\omega_x\hat{\boldsymbol{k}} \\ \dot{\mathbf{r}} \times \boldsymbol{\omega} &= \dot{y}\omega\sin\lambda\hat{\boldsymbol{\imath}} - (\dot{z}\omega\cos\lambda + \dot{x}\omega\sin\lambda)\,\hat{\boldsymbol{\jmath}} + \dot{y}\omega\cos\lambda\hat{\boldsymbol{k}} \end{aligned} \tag{2}$$

Ignoring centripetal and Euler forces, the particle's equation of motion is given by

$$\begin{aligned} m\ddot{\mathbf{r}} &= \mathbf{F} + \mathbf{F}_{\text{coriolis}} \\ &= m\mathbf{g} + 2m\,\dot{\mathbf{r}} \times \boldsymbol{\omega} \\ m\ddot{\mathbf{r}} &= -mg\hat{\boldsymbol{k}} + 2m\left(\dot{y}\omega\sin\lambda\hat{\boldsymbol{\imath}} - (\dot{z}\omega\cos\lambda + \dot{x}\omega\sin\lambda)\,\hat{\boldsymbol{\jmath}} + \dot{y}\omega\cos\lambda\hat{\boldsymbol{k}}\right) \end{aligned} \tag{3}$$

after substituting in Eq. (2). Note that the particle's acceleration can be expressed as $\ddot{\mathbf{r}}(t) = \ddot{x}(t)\hat{\boldsymbol{\imath}} + \ddot{y}(t)\hat{\boldsymbol{\jmath}} + \ddot{z}(t)\hat{\boldsymbol{k}}$. Comparing this with Eq. (3), we find that

$$\ddot{x} = 2\dot{y}\omega\sin\lambda \tag{4}$$

$$\ddot{y} = -2\omega\,(\dot{z}\cos\lambda + \dot{x}\sin\lambda) \tag{5}$$

$$\ddot{z} = -g + 2\dot{y}\omega\cos\lambda \tag{6}$$

We can integrate these in the same manner as problem 3 in this problem set to obtain

$$x(t) = x(0) + \dot{x}(0)t - 2\omega y(0)t\sin\lambda + 2\omega\sin\lambda\int_0^t y(t')\mathrm{d}t' \tag{7}$$

$$\begin{aligned} y(t) = y(0) + \dot{y}(0)t + 2\omega z(0)t\cos\lambda + 2\omega x(0)t\cos\lambda \\ - 2\omega\cos\lambda\int_0^t z(t')\mathrm{d}t' - 2\omega\sin\lambda\int_0^t x(t')\mathrm{d}t' \end{aligned} \tag{8}$$

$$z(t) = z(0) + \dot{z}(0)t - \frac{1}{2}gt^2 + 2\omega y(0)t\cos\lambda + 2\omega\cos\lambda\int_0^t y(t')\mathrm{d}t' \tag{9}$$

To solve for the components of the particle's position $x(t)$, $y(t)$, and $z(t)$, we will use the method of iteration (or perturbation theory) in which we first solve for the components when $\omega = 0$ (the term due to the actual force $\mathbf{F}$). Then, we will substitute these solutions to the RHS of $x(t)$, $y(t)$, and $z(t)$ when we now include terms proportional to ω (a kind of first-order corrections due to $\mathbf{F}_{\text{coriolis}}$). We can continue on to include terms proportional to higher powers of ω but then we would need to include the other forces that we neglected earlier so we do not bother with these terms. For the given situation in this problem, the initial conditions are $x(0) = 0 = y(0)$, $z(0) = h$, $\dot{y}(0) = 0 = \dot{z}(0)$, and $\dot{x}(0) = \mp v$ (- if particle moves to the North; + if particle moves to the South). Thus, the components of the particle's position $x(t)$, $y(t)$, and $z(t)$ from Eqs. (7), (8) and (9) when $\omega = 0$ are

$$x^{(0)}(t) = x(0) + \dot{x}(0)t = \mp vt \tag{10}$$

$$y^{(0)}(t) = y(0) + \dot{y}(0)t = 0 \tag{11}$$

$$z^{(0)}(t) = z(0) + \dot{z}(0)t - \frac{1}{2}gt^2 = h - \frac{1}{2}gt^2 \tag{12}$$

This is the first iteration. Using these, the second iteration (including terms proportional to ω) results to

$$x^{(1)}(t) = \mp vt + 2\omega \sin\lambda \int_0^t \cancelto{0}{y^{(0)}(t')}\, \mathrm{d}t' = \mp vt \tag{13}$$

$$\begin{aligned} y^{(1)}(t) &= 2\omega ht\cos\lambda - 2\omega\cos\lambda \int_0^t z^{(0)}(t')\mathrm{d}t' - 2\omega\sin\lambda \int_0^t x^{(0)}(t')\mathrm{d}t' \\ &= 2\omega ht\cos\lambda - 2\omega\cos\lambda \int_0^t \left(h - \frac{1}{2}gt'^2\right)\mathrm{d}t' - 2\omega\sin\lambda \int_0^t (\mp vt')\mathrm{d}t' \\ &= \cancel{2\omega ht\cos\lambda - 2\omega ht\cos\lambda} + 2\omega\frac{gt^3}{2\cdot 3}\cos\lambda \pm 2\omega v\frac{t^2}{2}\sin\lambda \\ y^{(1)}(t) &= \frac{1}{3}\omega gt^3\cos\lambda \pm \omega vt^2\sin\lambda \end{aligned} \tag{14}$$

$$z^{(1)}(t) = h - \frac{1}{2}gt^2 + 2\omega\cos\lambda \int_0^t \cancelto{0}{y^{(0)}(t')}\, \mathrm{d}t' = h - \frac{1}{2}gt^2 \tag{15}$$

Taking the difference of the solutions from the second iteration with the first one, we see that the only deviation is in the y-direction as given

$$\Delta y = \frac{1}{3}\omega gt^3\cos\lambda \pm \omega vt^2\sin\lambda \tag{16}$$

Note that we can express the second term as $\pm\omega vt^2\sin\lambda = \sqrt{(\omega vt^2\sin\lambda)^2} = |\omega vt^2\sin\lambda| = \omega vt^2|\sin\lambda|$ since ω, v, and t are positive variables. Neglecting the effect of gravity, the horizontal deviation of the projectile due to the Coriolis force is

$$d_H = \Delta y \approx \omega vt^2|\sin\lambda| \tag{17}$$

as given in Eq. (1). ■

(b) Show that the projectile is deflected to the right in the northern hemisphere and to the left in the southern hemisphere.

Solution:
Note that λ is positive [negative] in the northern [southern] hemisphere. Considering a

particle moving in either hemisphere and using the approximated deflection in part a, we have

$$d_H = \omega v t^2 |\sin(\pm\lambda)| = \omega v t^2 |(\pm \sin\lambda)| = \omega v t^2 \sin\lambda \tag{18}$$

We see that the deflection is positive for both hemispheres. Recall that this deflection is in the y-direction. For a particle moving to the North [South] in the northern [southern] hemisphere, this translates to a deflection to the right [left].

(c) To the same degree of approximation, express dH in terms of the projectile's distance D from the firing point at time t.

Solution:
The projectile's distance is just its path along the x-direction. Thus, we have

$$D = x^{(1)}(t) = \mp vt \longrightarrow t = \mp\frac{D}{v} \tag{19}$$

Substituting this to Eq. (17), we get

$$d_H = \omega v \left(\mp\frac{D}{v}\right)^2 |\sin\lambda| = \omega v \left(\frac{D^2}{v^2}\right) |\sin\lambda| = \frac{\omega D^2}{v} |\sin\lambda| \tag{20}$$

(d) How would the inclusion of the influence of gravity affect the result?

Solution:
From Eq. (16), we know that the deflection of the projectile (either moving to the North and South) with the inclusion of gravity is

$$d_H = \Delta y = \frac{1}{3}\omega g t^3 \cos\lambda + \omega v t^2 |\sin\lambda| \tag{21}$$

We see that the deflection is still positive for both hemispheres. So, a particle moving to the North [South] in the northern [southern] hemisphere is still deflected to the right [left]. Only in this case, the deflection is greater with the addition of the term related to gravity.

(e) What changes if the projectile is fired at an angle above the horizontal?

Solution:
For this situation, the initial conditions are $x(0) = 0 = y(0)$, $z(0) = h$, $\dot{y}(0) = 0$, $\dot{x}(0) = \mp v\cos\theta$, $\dot{z}(0) = \mp v \sin\theta$. Thus, the first iteration now results to

$$x^{(0)}(t) = \mp vt\cos\theta \tag{22}$$
$$y^{(0)}(t) = 0 \tag{23}$$
$$z^{(0)}(t) = h \mp vt\sin\theta - \frac{1}{2}gt^2 \tag{24}$$

and the second iteration results to

$$x^{(1)}(t) = \mp vt\cos\theta + 2\omega\sin\lambda \int_0^t \cancelto{0}{y^{(0)}(t')}\, \mathrm{d}t' = \mp vt\cos\theta \tag{25}$$

$$\begin{aligned}
y^{(1)}(t) &= 2\omega ht\cos\lambda - 2\omega\cos\lambda\int_0^t z^{(0)}(t')\mathrm{d}t' - 2\omega\sin\lambda\int_0^t x^{(0)}(t')\mathrm{d}t' \\
&= 2\omega ht\cos\lambda - 2\omega\cos\lambda\int_0^t\left(h\mp vt'\sin\theta - \frac{1}{2}gt'^2\right)\mathrm{d}t' \\
&\qquad\qquad\qquad\qquad - 2\omega\sin\lambda\int_0^t(\mp vt'\cos\theta)\mathrm{d}t' \\
&= \pm 2\omega v\frac{t^2}{2}\cos\lambda\sin\theta + 2\omega\frac{gt^3}{2\cdot 3}\cos\lambda \pm 2\omega v\frac{t^2}{2}\sin\lambda\cos\theta \\
&= \pm\omega vt^2\cos\lambda\sin\theta + \frac{1}{3}\omega gt^3\cos\lambda \pm \omega vt^2\sin\lambda\cos\theta \\
&= \frac{1}{3}\omega gt^3\cos\lambda \pm \omega vt^2\left(\cos\lambda\sin\theta + \sin\lambda\cos\theta\right) \\
y^{(1)}(t) &= \frac{1}{3}\omega gt^3\cos\lambda \pm \omega vt^2\sin(\lambda+\theta)
\end{aligned} \tag{26}$$

$$z^{(1)}(t) = h\mp vt\sin\theta - \frac{1}{2}gt^2 + 2\omega\cos\lambda\int_0^t \cancelto{0}{y^{(0)}(t')}\,\mathrm{d}t' = h\mp vt\sin\theta - \frac{1}{2}gt^2 \tag{27}$$

Then, in the same manner as in the previous situation, we get the horizontal deviation as

$$d_H = \Delta y = \frac{1}{3}\omega gt^3\cos\lambda + \omega vt^2|\sin(\lambda+\theta)| \tag{28}$$

for a projectile either moving to the North in the northern hemisphere or moving to the South in the southern hemisphere. Comparing this with Eq. (21), the deflection is now even greater by firing the projectile at an angle above the horizontal.

(f) It is told that during a World War I naval battle in the south Atlantic, British shells missed German ships by about 90 meters to the left because the British gunsights had been corrected to a latitude 50° north (for battles in Europe) instead of 50° south where the battle took place. Conclude that the British shells would have missed the targets by twice the deflection calculated above.

Solution:
The naval battle here refers to the Battle of Falklands during World War I. It seemed that the British fired at the German ships to the South. Thus, the shells travelled in the positive x-direction. Also, we can approximate $v \approx gt$ near the Earth's surface.

$$\begin{aligned}
\left|\frac{d_{H(S)}}{d_{H(N)}}\right| &= \left|\frac{\frac{1}{3}\omega gt^3\cos 50^\circ - \omega vt^2\sin 50^\circ}{\frac{1}{3}\omega gt^3\cos(-50^\circ) - \omega vt^2\sin(-50^\circ)}\right| \\
&\approx \left|\frac{\frac{1}{3}\omega vt^2\cos 50^\circ - \omega vt^2\sin 50^\circ}{\frac{1}{3}\omega vt^2\cos(-50^\circ) - \omega vt^2\sin(-50^\circ)}\right| \\
\left|\frac{d_{H(S)}}{d_{H(N)}}\right| &\approx \cancelto{1}{\frac{\omega vt^2}{\omega vt^2}}\left|\frac{\frac{1}{3}\cos 50^\circ - \sin 50^\circ}{\frac{1}{3}\cos 50^\circ + \sin 50^\circ}\right|
\end{aligned} \tag{29}$$

Using Mathematica to calculate this ratio as shown in the following figure:

```
Abs[N[(1/3 Cos[π Rationalize[N[50.` °]/π]] - Sin[π Rationalize[N[50.` °]/π]]) / (1/3 Cos[π Rationalize[N[50.` °]/π]] + Sin[π Rationalize[N[50.` °]/π]])]]

0.562866
```

Figure 1: Calculating $|d_{H(S)}/d_{H(N)}|$

Thus, we can find that

$$\left|\frac{d_{H(S)}}{d_{H(N)}}\right| \approx \frac{1}{2} \tag{30}$$

Note that $d_{H(N)} = 90\text{m}$ from the problem. Therefore, we can see that

$$d_{H(S)} \approx 2d_{H(N)} = 2(90\text{m}) = 180\text{m} \tag{31}$$

from which we can conclude that British shells would have missed the targets by twice the deflection for latitude 50° north when we use the latitude 50° south.

(g) Assuming the tale is true and $v = 700\,m/s$, how distant must the German ships have been?

Solution:
For the southward projectile with a latitude 50° north, the deflection is

$$\begin{aligned} d_H &= \frac{1}{3}\omega g t^3 \cos\lambda - \omega v t^2 \sin\lambda \\ &= \frac{1}{3}\omega v t^2 \cos\lambda - \omega v t^2 \sin\lambda \\ &= \frac{1}{3}\omega v t^2 \cos\lambda - \omega v t^2 \sin\lambda \\ d_H &= \omega v t^2 \left(\frac{1}{3}\cos\lambda - \sin\lambda\right) \end{aligned} \tag{32}$$

Calculating this in Mathematica as shown in the following figure:

Note that Dist = D. Solving for Dist, we have

In[2]:= $\text{Solve}\left[\text{dH} == \omega \frac{\text{Dist}^2}{\text{v}} \left(\frac{1}{3}\text{Cos}[\lambda] + \text{Sin}[\lambda]\right), \text{Dist}\right]$

Out[2]= $\left\{\left\{\text{Dist} \to -\frac{\sqrt{3}\sqrt{\text{dH}}}{\sqrt{\frac{\omega\,\text{Cos}[\lambda]}{\text{v}} + \frac{3\,\omega\,\text{Sin}[\lambda]}{\text{v}}}}\right\}, \left\{\text{Dist} \to \frac{\sqrt{3}\sqrt{\text{dH}}}{\sqrt{\frac{\omega\,\text{Cos}[\lambda]}{\text{v}} + \frac{3\,\omega\,\text{Sin}[\lambda]}{\text{v}}}}\right\}\right\}$

Now, we have the following values :

In[4]:= g = 9.8;
ω = 2 π / 86400;
v = 700;
$\lambda = \pi\,\text{Rationalize}\left[\frac{\text{N[50.` °]}}{\pi}\right];$
dH = 90;

We only choose the positive solution for Dist since the projectile is moving to the South (+ x direction). Thus, we get

In[9]:= Dist = N[Dist /. %2[[2, 1]]]

Out[9]= 29727.3

Figure 2: Calculating D

Thus, we have

$$D \approx 30km \tag{33}$$

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