Final Exam

6.4: Green Functions for the Wave Function

The wave equations

$$\nabla^2 \Phi - \frac{1}{c^2}\frac{\partial^2 \Phi}{\partial t^2} = -\frac{\rho}{\epsilon_0} \quad (1) \qquad \nabla^2 \vec{A} - \frac{1}{c^2}\frac{\partial^2 \vec{A}}{\partial t^2} = -\mu_0 \vec{J} \quad (2)$$

all have the basic structure of

$$\nabla^2 \psi - \frac{1}{\underset{\text{Velocity propagation in medium}}{c^2}}\frac{\partial^2 \psi}{\partial t^2} = -4\pi \underbrace{f(\vec{x},t)}_{\text{known source distribution}} \quad (3)$$

To solve this, it would be useful to find the Green's function, just like in electrostatics. If there are no boundary surfaces, solution by Fourier transform is best so that we can get rid of the explicit time dependence. Suppose that $\psi(\vec{x},t)$ and $f(\vec{x},t)$ have the Fourier integral representations

$$\psi(\vec{x},t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \psi(\vec{x},\omega)\, e^{-i\omega t}\, d\omega, \qquad f(\vec{x},t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(\vec{x},\omega)\, e^{-i\omega t}\, d\omega \quad (4)$$

with inverse transformations

$$\psi(\vec{x},\omega) = \int_{-\infty}^{\infty} \psi(\vec{x},t)\, e^{i\omega t}\, dt, \qquad f(\vec{x},\omega) = \int_{-\infty}^{\infty} f(\vec{x},t)\, e^{i\omega t}\, dt \quad (5)$$

Substituting in Eq. (4) into (3), we have

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega\, e^{-i\omega t}\, \nabla^2 \psi(\vec{x},t) - \frac{1}{c^2}\frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega\, \psi(\vec{x},\omega)\left(\frac{\partial^2}{\partial t^2} e^{-i\omega t}\right)$$

$$= -4\pi \frac{1}{2\pi}\int_{-\infty}^{\infty} f(\vec{x},\omega)\, e^{-i\omega t}\, d\omega \quad (6)$$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega\, e^{-i\omega t}\left[\nabla^2 \psi(\vec{x},\omega) + \overset{k^2}{\frac{\omega^2}{c^2}}\psi(\vec{x},\omega) + 4\pi f(\vec{x},\omega)\right] = 0$$

which is valid if the integrand is zero. Thus, we can infer that $\psi(\vec{x},\omega)$ satisfies

$$\nabla^2 \psi(\vec{x},\omega) + k^2 \psi(\vec{x},\omega) + 4\pi f(\vec{x},\omega) = 0$$

$$(\nabla^2 + k^2)\,\psi(\vec{x},\omega) = -4\pi f(\vec{x},\omega) \quad (7)$$

This is the inhomogeneous Helmholtz wave equation for each value of ω. This equation is an elliptic partial differential equation (like the Poisson equation ← $k=0$ for Helmholtz eqn). The Green function $G(\vec{x},\vec{x}')$ approriate to Eq. (7) satisfies

$$(\nabla^2 + k^2)\, G_k(\vec{x},\vec{x}') = -4\pi\,\delta(\vec{x}-\vec{x}') \quad (8)$$

We also assume that the Green's function in this unrestricted region is:

- translationally invariant $G_k(\vec{x},\vec{x}') = G_k(\vec{x}-\vec{x}')$
- rotationally symmetric $G_k(\vec{x},\vec{x}') = G_k(|\vec{x}-\vec{x}'|)$

$$\Downarrow$$

$$G_k(\vec{x},\vec{x}') = G_k(|\vec{x}-\vec{x}'|) = G_k(|\vec{R}|) = G_k(R)$$

Since we assumed that there are no boundary surfaces, $G_k(\vec{x},\vec{x}')$ can only depend on $\vec{R} = \vec{x}-\vec{x}'$ and must be spherically symmetric (dependent only on $R = |\vec{R}|$). Note that the Laplacian in spherical coordinates is given by

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2} \quad (9)$$

We see here that only the derivative with respect to the radial direction is non-vanishing in the term $\nabla^2 G_k(R)$. Thus, $G_k(R)$ satisfies

$$\frac{1}{R}\frac{d^2}{dR^2}\left(R\,G_k(R)\right) + k^2 G_k(R) = -4\pi\,\delta(\vec{R}) \quad (10)$$

Everywhere except $R=0$,

$$\frac{1}{R}\frac{d^2}{dR^2}\left(R\,G_k(R)\right) + k^2 G_k(R) = 0 \quad (11)$$

$$\frac{d^2}{dR^2}\left(R\, G_k(R)\right) + k^2\left(R\, G_k(R)\right) = 0 \tag{12}$$

is satisfied by $R\,G_k(R)$. This is a homogeneous equation so it can be solved with the characteristic equation $m^2 + k^2 = 0$. Thus, the solution for Eq. (12) is

$$R\, G_k(R) = A e^{ikR} + B e^{-ikR} \tag{13}$$

How do we determine A and B? Note that the delta function now influences Eq. (10) as $R \to 0$. In this limit, the equation reduces to the Poisson equation since $1 \gg kR = (\omega/c) R$. Thus, with $\nabla^2 G_k(R) = -4\pi\, \delta(\vec{R})$,

$$\lim_{kR\to 0} G_k(R) = \frac{1}{|\vec{x}-\vec{x}'|} + \underbrace{\cancel{F(\vec{x},\vec{x}')}}_{0 \text{ for simplicity}} = \frac{1}{R} \tag{14}$$

Substituting this into Eq. (13), with $1 \gg kR$,

$$R\,\frac{1}{R} \approx A e^0 + B e^0 \quad \rightarrow \quad A + B = 1 \tag{15}$$

Therefore, the general solution for the Green's function is

$$G_k(R) = A\, G_k^{(+)}(R) + B\, G_k^{(-)}(R) \tag{16}$$

where

$$G_k^{\pm} = \frac{1}{R} e^{\pm ikR} \tag{17}$$

Each term in Eq. (16) has a time dependence with the convention in Eq. (4).

- First term - outgoing spherical wave
- Second term - incoming spherical wave

The specific choice of A and B depends on temporal boundary conditions. To understand the different time behaviors associated $G_k^{(+)}$ and $G_k^{(-)}$, we need to construct the corres. time-dep. Green's functions that satisfy

$$\left(\nabla_x^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right) G^{(\pm)}(\vec{x},t;\vec{x}',t') = -4\pi\,\delta(\vec{x}-\vec{x}')\,\delta(t-t') \tag{18}$$

With the use of Eq. (5), we see that the Fourier transform of the source term for Eq. (7) is

$$f(\vec{x},\omega) = \int_{-\infty}^{\infty} \left(-4\pi\,\delta(\vec{x}-\vec{x}')\,\delta(t-t')\right) e^{i\omega t}\, dt$$

$$= 4\pi\,\delta(\vec{x}-\vec{x}')\, e^{i\omega t'} \tag{19}$$

The solutions are therefore $G_k^{(\pm)}(R)\, e^{i\omega t'}$. From Eq. (4), the time-dependent Green functions are

$$G^{(\pm)}(\vec{x},t;\vec{x}',t') = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega\, e^{-i\omega t}\, G_k^{(\pm)}(R)\, e^{i\omega t'}$$

$$G^{(\pm)}(\underbrace{|\vec{x}-\vec{x}'|}_{R}, \underbrace{t-t'}_{\tau}) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega\, G_k^{(\pm)}(R)\, e^{-i\omega \overbrace{(t-t')}^{\tau}} \tag{20}$$

$$G^{(\pm)}(R,\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega\, \frac{e^{\pm ikR}}{R}\, e^{-i\omega\tau}$$

Note that the infinite-space Green function depends only on R and τ. For a non-dispersive medium where $k = \omega/c$, Eq. (20) is just the delta function defined as

$$\delta(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\omega x}\, d\omega \tag{21}$$

Thus, the Green functions are

$$G^{(\pm)}(R,\tau) = \frac{1}{R}\frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega\, e^{-i\omega\left(\tau \mp \frac{R}{c}\right)} = \frac{1}{R}\,\delta\left(\tau \mp \frac{R}{c}\right) \quad (22)$$

or, more explicitly,

$$G^{(\pm)}(\vec{x},t;\vec{x}',t') = \frac{1}{|\vec{x}-\vec{x}'|}\,\delta\left(t' - \left(t \mp \frac{|\vec{x}-\vec{x}'|}{c}\right)\right) \quad (23)$$

The delta function requires $t' = t \mp R/c$ to contribute and R/c is always non-negative.

- $G^{(+)}$ - only $t' \leq t$ contributes; sources only affect the wave fxn after they act
  - retarded Green function
- $G^{(-)}$ - advanced Green function
  - give effects which preceed their causes

Particular integrals of Eq. (3) are

$$\psi^{(\pm)}(\vec{x},t) = \iint G^{(\pm)}(\vec{x},t;\vec{x}',t')\, f(\vec{x}',t')\, d^3x'\,dt' \quad (24)$$

To specify a definite physical problem, solns of the homogeneous eqn may be added to either of these. Consider:

$f(\vec{x}',t)$ - localized in time and space
- diff. from zero only for a finite interval of time around $t=0$

There appears to be two limiting situations.

Here, the source of ψ_{in} turns on and generates waves of its own

1. Assume that at time $t \to -\infty$, there exists a $\psi_{in}(\vec{x},t)$ that satisfies the homogeneous equation. The complete soln for all times is

$$\psi(\vec{x},t) = \psi_{in}(\vec{x},t) + \underbrace{\iint G^{(+)}(\vec{x},t;\vec{x}',t')}_{\text{guarantees that only } \psi_{in} \text{ exists before the source has been activated}}\, f(\vec{x}',t')\, d^3x'\,dt \quad (25)$$

2. At remotely late times ($t \to \infty$), the wave is given as $\psi_{out}(\vec{x},t)$ which also satisfies the homogeneous equation. Then, the complete soln for all times is

$$\psi(\vec{x},t) = \psi_{out}(\vec{x},t) + \underbrace{\iint G^{(-)}(\vec{x},t;\vec{x}',t')}_{\text{ensures that no signal from the source exists after it has been shut off}}\, f(\vec{x}',t')\, d^3x'\,dt' \quad (26)$$

The most common physical situation is described by Eq. (25) with $\psi_{in}=0$. Substituting in Eq. (23) into Eq. (25), we see that

$$\psi(\vec{x},t) = \overset{0}{\psi_{in}(\vec{x},t)} + \iint \frac{1}{|\vec{x}-\vec{x}'|}\,\delta\left(t' - \left(t - \frac{|\vec{x}-\vec{x}'|}{c}\right)\right) f(\vec{x},t')\, d^3x'\,dt'$$

$$= \int \frac{[f(\vec{x}',t')]_{ret}}{|\vec{x}-\vec{x}'|}\, d^3x' \quad (27)$$

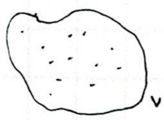
where

$$[f(\vec{x}',t')]_{ret} = \int \delta\left(t' - \left(t - \frac{|\vec{x}-\vec{x}'|}{c}\right)\right) f(\vec{x}',t)\, dt' \quad (28)$$

6.7: Poynting's Theorem and Conservation of Energy and Momentum for a System of Charged Particles and Electromagnetic Fields

The forms of the laws of conservation of energy and momentum are important results to establish for the EM field. We begin by considering conservation of energy, often called Poynting's theorem.

Consider several point charges $q_1, \dots, q_N$ located at $\vec{x}_1, \dots, \vec{x}_N$ and moving with velocities $v_1, \dots, v_N$ in an external EM field. The rate of work done on these charges due to the EM field is



$$\sum_{i=1}^{N} \vec{F}_i \cdot \vec{v}_i = \sum_{i=1}^{N} q_i \vec{E}(\vec{x}_i) \cdot \vec{v}_i = \int d^3x \left(\sum_{i=1}^{N} q_i \vec{v}_i \delta(\vec{x} - \vec{x}_i) \right) \cdot \vec{E}(\vec{x}) \quad (1)$$

Thus, in terms of current density $\vec{J}$,

represents conversion from EM energy to mech. or thermal energy

$$\text{rate of work done} = \int_V \vec{J} \cdot \vec{E} \, d^3x \quad (2)$$

To exhibit this conservation law explicitly, we use the Ampere-Maxwell law given by

$$\vec{\nabla} \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t} \quad (3)$$

to eliminate $\vec{J}$ in Eq. (2):

$$\begin{aligned} \int_V \vec{J} \cdot \vec{E} \, d^3x &= \int_V \left(\vec{\nabla} \times \vec{H} - \frac{\partial \vec{D}}{\partial t} \right) \cdot \vec{E} \, d^3x \\ &= \int_V \left(\vec{E} \cdot (\vec{\nabla} \times \vec{H}) - \vec{E} \cdot \frac{\partial D}{\partial t} \right) d^3x \end{aligned} \quad (4)$$

If we apply the vector identity $\nabla \cdot (\vec{E} \times \vec{H}) = \vec{H} \cdot (\vec{\nabla} \times \vec{E}) - \vec{E} \cdot (\vec{\nabla} \times \vec{H})$ and use Faraday's law given by

$$\vec{\nabla} \times \vec{E} = -\frac{\partial B}{\partial t}, \quad (5)$$

Eq. (4) becomes

$$\begin{aligned} \int_V \vec{J} \cdot \vec{E} \, d^3x &= -\int_V \left[\vec{\nabla} \cdot (\vec{E} \times \vec{H}) - \vec{H} \cdot (\vec{\nabla} \times \vec{E}) + E \cdot \frac{\partial D}{\partial t} \right] d^3x \\ &= -\int_V \left[\vec{\nabla} \cdot (\vec{E} \times \vec{H}) + \vec{E} \cdot \frac{\partial \vec{D}}{\partial t} + \vec{H} \cdot \frac{\partial \vec{B}}{\partial t} \right] d^3x \end{aligned} \quad (6)$$

We now make assumptions:

- macroscopic medium is linear without dispersion in electric and magnetic properties, meaning $\vec{B} \propto \vec{H}$ and $\vec{D} \propto \vec{E}$
- the sum of

$$W = \frac{1}{2} \int \vec{E} \cdot \vec{D} \, d^3x \qquad \text{and} \qquad W = \frac{1}{2} \int \vec{H} \cdot \vec{B} \, d^3x$$

represents the total EM energy density, even for time varying fields.

Then, defining the total energy density as

$$u = \frac{1}{2} (\vec{E} \cdot \vec{D} + \vec{H} \cdot \vec{B}), \quad (7)$$

Eq. (6) can be written as

$$\begin{aligned} -\int_V \vec{J} \cdot \vec{E} \, d^3x &\approx \int_V \left[\vec{E} \cdot \frac{\partial \vec{E}}{\partial t} + \vec{B} \cdot \frac{\partial \vec{B}}{\partial t} + \nabla \cdot (\vec{E} \times \vec{H}) \right] d^3x \\ &= \int_V \left[\frac{1}{2} \frac{\partial}{\partial t} E^2 + \frac{1}{2} \frac{\partial}{\partial t} B^2 + \nabla \cdot (\vec{E} \times \vec{H}) \right] d^3x \end{aligned}$$

$$-\int_V \vec{J}\cdot\vec{E}\, d^3x = \int_V \left[\frac{\partial}{\partial t}\left(\frac{1}{2}(E^2+B^2)\right) + \vec{\nabla}\cdot\vec{E}\times\vec{H}\right] d^3x$$

$$= \int_V \left[\frac{\partial u}{\partial t} + \vec{\nabla}\cdot(\vec{E}\times\vec{H})\right] d^3x \qquad (8)$$

Since the volume is arbitrary, this can be cast into its differential form as

$$\frac{\partial u}{\partial t} + \nabla\cdot\vec{S} = -\vec{J}\cdot\vec{E} \qquad (9)$$

$[\vec{S}] = \left[\frac{\text{energy}}{\text{area}} \times \text{time}\right]$

where $\vec{S}$ is the Poynting vector that represents energy flow and is denoted by

$$\vec{S} = \vec{E}\times\vec{H} \qquad (10)$$

From relativistic considerations, this is unique. The physical interpretation of Eq. (8) or (9) is that the time rate of change of EM energy within a certain volume, plus the energy flowing out through the boundary surfaces of the volume per unit time, is equal to the negative of the total work done by the fields on the sources within the volume. This is the statement of conservation of energy.

- The assumptions restrict this simple version of Poynting's theorem to vacuum macroscopic / microscopic fields. Even for linear media, there is dispersion. The more realistic situation of linear dispersive media is discussed in section 6.8.
- Since matter is ultimately composed of charged particles, we can think of this rate of conversion as a rate of increase of energy of the charged particles per unit volume. Then, we can interpret Poynting's theorem for the microscopic fields $(\vec{E},\vec{B})$ as a statement of conservation of energy of the combined system of particles and fields. If we assume that no particles move out of the volume, we have

total energy of the particles within the volume V

$$\frac{dE_{mech}}{dt} = \int_V \vec{J}\cdot\vec{E}\, d^3x$$

Then, Poynting's theorem expresses the energy conservation for the combined system as

$$\int_V \vec{J}\cdot\vec{E}\, d^3x + \frac{d}{dt}\underbrace{\int_V u\, d^3x}_{E_{field}} = -\int_V \vec{\nabla}\cdot\vec{S}\, d^3x$$

$$\frac{dE_{mech}}{dt} + \frac{dE_{field}}{dt} = -\oint_S \vec{S}\cdot d\vec{a}$$

divergence theorem

$$\frac{dE}{dt} = -\oint_S \vec{S}\cdot\hat{n}\, da \qquad (11)$$

where $E = E_{mech} + E_{field}$.

- The conservation of linear momentum can be similarly considered. The EM force on a charged particle is

$$\vec{F} = q(\vec{E} + \vec{v}\times\vec{B}) \qquad (12)$$

If the sum of all the momenta of all the particles in the volume V is $\vec{P}_{mech}$, then, with the use of Newton's second law, we have

$$\sum_i m_i \ddot{x}_i = \sum_i F_i$$

$$\frac{d}{dt}\left(\sum_i m_i \dot{\vec{x}}_i\right) = \sum_i q_i \vec{E}(\vec{x}_i) + \vec{v}_i \times \vec{B}(\vec{x}_i)$$

$$\frac{d\vec{P}_{mech}}{dt} = \int_V (\rho\vec{E} + \vec{J}\times\vec{B})\, d^3x \qquad (13)$$

where we have converted the sum over particles to an integral over charge and current densities for convenience in manipulation. In the same manner as for Poynting's theorem, we use Maxwell's equations to eliminate ρ and $\vec{J}$ from Eq. (13):

$$\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\epsilon_0} \quad\longrightarrow\quad \rho = \epsilon_0\,\vec{\nabla}\cdot\vec{E} \tag{14}$$

$$\vec{\nabla}\times\vec{B} = \mu_0\vec{J} + \mu_0\epsilon_0\frac{\partial\vec{E}}{\partial t} \quad\longrightarrow\quad \vec{J} = \frac{1}{\mu_0}\vec{\nabla}\times\vec{B} - \epsilon_0\frac{\partial\vec{E}}{\partial t} \tag{15}$$

Substituting these into Eq. (13), the integrand becomes

$$\rho\vec{E} + \vec{J}\times\vec{B} = \epsilon_0(\vec{\nabla}\cdot\vec{E})\vec{E} + \frac{1}{\mu_0}(\vec{\nabla}\times\vec{B})\times\vec{B} - \epsilon_0\frac{\partial\vec{E}}{\partial t}\times\vec{B}$$

$\vec{a}\times\vec{b} = -\vec{b}\times\vec{a}$

$$= \epsilon_0\left[\vec{E}(\vec{\nabla}\cdot\vec{E}) + \vec{B}\times\frac{\partial\vec{E}}{\partial t} - \underbrace{\frac{1}{\mu_0\epsilon_0}}_{c^2}\vec{B}\times(\vec{\nabla}\times\vec{B})\right] \tag{16}$$

Note that

$$\frac{\partial}{\partial t}(\vec{E}\times\vec{B}) = \vec{E}\times\frac{\partial\vec{B}}{\partial t} + \frac{\partial\vec{E}}{\partial t}\times\vec{B} = \vec{E}\times\frac{\partial\vec{B}}{\partial t} - \vec{B}\times\frac{\partial\vec{E}}{\partial t}$$

$$\vec{B}\times\frac{\partial\vec{E}}{\partial t} = -\frac{\partial}{\partial t}(\vec{E}\times\vec{B}) + \vec{E}\times\frac{\partial\vec{B}}{\partial t} \tag{17}$$

No magnetic monopoles: $\vec{\nabla}\cdot\vec{B} = 0$

Also, since $c^2\vec{B}(\vec{\nabla}\cdot\vec{B}) = 0$, we can add this to the square bracket without a problem. Thus, by applying these, we have

$$\frac{d\vec{P}_{mech}}{dt} = \epsilon_0\int_V\left[\vec{E}(\vec{\nabla}\cdot\vec{E}) - \frac{\partial}{\partial t}(\vec{E}\times\vec{B}) + \vec{E}\times\frac{\partial\vec{B}}{\partial t} - \frac{1}{\mu_0\epsilon_0}\vec{B}\times(\vec{\nabla}\times\vec{B}) + c^2\vec{B}(\vec{\nabla}\cdot\vec{B})\right]d^3x$$

use Faraday's law given in Eq. (5)

$$= \epsilon_0\int_V\left[\vec{E}(\vec{\nabla}\cdot\vec{E}) - \vec{E}\times(\vec{\nabla}\times\vec{E}) - c^2\vec{B}\times(\vec{\nabla}\times\vec{B}) + c^2\vec{B}(\vec{\nabla}\cdot\vec{B})\right]d^3x - \epsilon_0\int_V\frac{\partial}{\partial t}(\vec{E}\times\vec{B})d^3x$$

$$\frac{dP_{mech}}{dt} + \frac{d}{dt}\int_V\epsilon_0(\vec{E}\times\vec{B})\,d^3x = \epsilon_0\int_V\left[\vec{E}(\vec{\nabla}\cdot\vec{E}) - \vec{E}\times(\vec{\nabla}\times\vec{E}) + c^2\vec{B}(\vec{\nabla}\cdot\vec{B}) - c^2\vec{B}\times(\vec{\nabla}\times\vec{B})\right]d^3x \tag{18}$$

We define

total EM momentum in volume V

$$\vec{P}_{field} = \epsilon_0\int_V\vec{E}\times\vec{B}\,d^3x = \epsilon_0\int_V\vec{E}\times\mu_0\vec{H}\,d^3x = \frac{1}{c^2}\int_V\overbrace{\vec{E}\times\vec{H}}^{\vec{S}}\,d^3x \tag{19}$$

Let the linear momentum density be

$$\vec{g} = \frac{1}{c^2}(\vec{E}\times\vec{H}) \tag{20}$$

From Eq. (19), we can turn Eq. (18) into

$$\frac{d\vec{P}_{mech}}{dt} + \frac{d\vec{P}_{field}}{dt} = \epsilon_0\int_V\left[\vec{E}(\vec{\nabla}\cdot\vec{E}) - \vec{E}\times(\vec{\nabla}\times\vec{E}) + c^2\vec{B}(\vec{\nabla}\cdot\vec{B}) - c^2\vec{B}\times(\vec{\nabla}\times\vec{B})\right]d^3x \tag{21}$$

To establish Eq. (21) as the conservation law for momentum, we must convert the volume integral on the right into a surface integral of the normal component of something that can be identified as momentum flow. Let the Cartesian coordinates be denoted by x_α, $\alpha = 1,2,3$. The $\alpha = 1$ component of the electric part of the integrand in Eq. (21) is given explicitly by

$$\left[\vec{E}(\vec{\nabla}\cdot\vec{E}) - \vec{E}\times(\vec{\nabla}\times\vec{E})\right]_1 = E_1\left(\frac{\partial E_1}{\partial x_1} + \frac{\partial E_2}{\partial x_2} + \frac{\partial E_3}{\partial x_3}\right) - E_2\left(\frac{\partial E_2}{\partial x_1} - \frac{\partial E_1}{\partial x_2}\right) + E_3\left(\frac{\partial E_1}{\partial x_3} - \frac{\partial E_3}{\partial x_1}\right)$$

$$= \frac{1}{2}\frac{\partial}{\partial x_1}(E_1^2) + \frac{\partial}{\partial x_2}(E_1E_2) + \frac{\partial}{\partial x_3}(E_1E_3) - \frac{1}{2}\frac{\partial}{\partial x_1}E_2^2 - \frac{1}{2}\frac{\partial}{\partial x_1}E_3^2 \tag{22}$$

$$= \frac{\partial}{\partial x_1}(E_1^2) + \frac{\partial}{\partial x_2}(E_1E_2) + \frac{\partial}{\partial x_3}(E_1E_3) - \frac{1}{2}\frac{\partial}{\partial x_1}(E_1^2 + E_2^2 + E_3^2)$$

This means that we can write the α^{th} component as

$$\left[\vec{E}(\vec{\nabla}\cdot\vec{E}) - \vec{E}\times(\vec{\nabla}\times\vec{E})\right]_\alpha = \sum_\beta \frac{\partial}{\partial x_\beta}\left[E_\alpha E_\beta - \frac{1}{2}(\vec{E}\cdot\vec{E})\delta_{\alpha\beta}\right] \quad (28)$$

and have the form of a divergence of a second rank tensor on the right hand side. With the definition of a Maxwell stress tensor $T_{\alpha\beta}$ as

$$T_{\alpha\beta} = \epsilon_0\left[E_\alpha E_\beta + c^2 B_\alpha B_\beta - \frac{1}{2}(\vec{E}\cdot\vec{E} + c^2\vec{B}\cdot\vec{B})\delta_{\alpha\beta}\right] \quad (24)$$

we can write Eq. (21) in component form as

$$\begin{aligned}\frac{d}{dt}(\vec{P}_{mech} + \vec{P}_{field})_\alpha &= \epsilon_0\int_V\left\{\sum_\beta \frac{\partial}{\partial x_\beta}\left[E_\alpha E_\beta - \frac{1}{2}(\vec{E}\cdot\vec{E})\delta_{\alpha\beta}\right] + \sum_\beta \frac{\partial}{\partial x_\beta}\left[c^2 B_\alpha B_\beta - \frac{1}{2}c^2(\vec{B}\cdot\vec{B})\delta_{\alpha\beta}\right]\right\} \times d^3x \\ &= \sum_\beta\int_V \frac{\partial}{\partial x_\beta}\epsilon_0\left[E_\alpha E_\beta + c^2 B_\alpha B_\beta - \frac{1}{2}(\vec{E}\cdot\vec{E} + c^2\vec{B}\cdot\vec{B})\delta_{\alpha\beta}\right] d^3x \\ &= \sum_\beta\int_V \frac{\partial}{\partial x_\beta} T_{\alpha\beta}\, d^3x \quad (25)\end{aligned}$$

where we applied the same method as that of Eq. (28) to the components of the magnetic part of the integrand in Eq. (21). Applying the divergence theorem to Eq. (25),

$$\frac{d}{dt}(\vec{P}_{mech} + \vec{P}_{field})_\alpha = \oint_S \sum_\beta T_{\alpha\beta}\, n_\beta\, da \quad (26)$$

β^{th} component of the outward normal vector to closed surface S

If Eq. (26) represents momentum conservation, then $\sum_\beta T_{\alpha\beta} n_\beta$ is the α^{th} component of the flow per unit area of momentum across surface S into volume V (i.e. it's the force per unit area transmitted across S and acting on combined sys. of particles & fields inside V. Therefore, this can be used to calculate the forces acting on material objects in EM fields by enclosing the objects with a boundary surface S and adding up the total EM force acc. to the RHS of Eq. (26)

- The conservation of angular momentum of the combined sys. of particles and fields can be treated in the same way as that of the linear momentum
- While a treatment using macroscopic Maxwell equations leads to $\vec{g} = \vec{D}\times\vec{B}$, the generally accepted for a medium at rest is

$$\vec{g} = \frac{1}{c^2}(\vec{E}\times\vec{H}) = \mu_0\epsilon_0(\vec{E}\times\vec{H}) = \frac{1}{c^2}\vec{S}$$

which is the EM momentum associated with the fields.

6.8: Poynting's Theorem in Linear Dispersive Media

In the previous section, Poynting's theorem was derived with restriction to linear media with no dispersion or losses (i.e. $\vec{D} = \epsilon \vec{E}$ and $\vec{B} = \mu \vec{H}$), with ϵ and μ frequency independent. However, actual media (except vacuum) exhibit dispersion (i.e. frequency-dependent μ and ϵ). To discuss dispersion, we need to make a Fourier decomposition in time of both $\vec{E}$ and $\vec{D}$ (and $\vec{B}$ and $\vec{H}$). Thus, with

$$\vec{E}(\vec{x},t) = \int_{-\infty}^{\infty} d\omega\, \vec{E}(\vec{x},\omega)\, e^{-i\omega t}, \qquad \vec{D}(\vec{x},t) = \int_{-\infty}^{\infty} d\omega\, \vec{D}(\vec{x},\omega)\, e^{-i\omega t} \tag{1}$$

the assumption of linearity (and, for simplicity, isotropy) implies that

complex and frequency-dependent susceptibility

$$\vec{D}(\vec{x},\omega) = \epsilon(\omega)\, \vec{E}(\vec{x},\omega) \tag{2}$$

Similarly, with

$$\vec{B}(\vec{x},t) = \int_{-\infty}^{\infty} d\omega\, \vec{B}(\vec{x},\omega)\, e^{-i\omega t}, \qquad \vec{H}(\vec{x},t) = \int_{-\infty}^{\infty} d\omega\, \vec{H}(\vec{x},\omega)\, e^{-i\omega t} \tag{3}$$

the said assumption implies that

$$\vec{B}(\vec{x},\omega) = \mu(\omega)\, \vec{H}(\vec{x},\omega) \tag{4}$$

The reality of the fields implies that

$$\vec{E}(\vec{x},-\omega) = \vec{E}^*(\vec{x},\omega) \qquad \vec{D}(\vec{x},-\omega) = \vec{D}^*(\vec{x},\omega) \qquad [\mu]\,\epsilon(-\omega) = [\mu]\,\epsilon^*(\omega) \tag{5}$$

The presence of dispersion carries with it a temporally non-local connection bet. $\vec{D}(\vec{x},t)$ and $\vec{E}(\vec{x},t)$. As a consequence, $\vec{E}\cdot\partial_t\vec{D}$ in Sec. 6.7 Eq. (6) is not simply $\partial_t(\vec{E}\cdot\vec{D}/2)$. We write out this term with Fourier integrals that are implicitly spatially-dependent:

$$\begin{aligned}
\vec{E}(\vec{x},t)\cdot\frac{\partial \vec{D}(\vec{x},t)}{\partial t} &= \vec{E}^*(\vec{x},t)\cdot\frac{\partial}{\partial t}\left(\int_{-\infty}^{\infty} d\omega\, \vec{D}(\vec{x},\omega)\, e^{-i\omega t}\right) \\
&= \left(\int_{-\infty}^{\infty} d\omega'\, \vec{E}^*(\vec{x},\omega)\, e^{+i\omega' t}\right)\cdot\left(-i\omega\int_{-\infty}^{\infty} d\omega\, \epsilon(\omega)\,\vec{E}(\vec{x},\omega)\, e^{-i\omega t}\right) \\
&= \int d\omega \int d\omega' [-i\omega\,\epsilon(\omega)]\; \vec{E}^*(\vec{x},\omega')\cdot\vec{E}(\vec{x},\omega)\, e^{-i(\omega-\omega')t}
\end{aligned} \tag{6}$$

Alternatively, we could have taken the complex conjugate of $\vec{D}$ instead and interchanged ω and ω':

$$\begin{aligned}
\vec{E}(\vec{x},t)\cdot\frac{\partial \vec{D}(\vec{x},t)}{\partial t} &= \left(\int_{-\infty}^{\infty} d\omega\, \vec{E}(\vec{x},\omega)\, e^{-i\omega t}\right)\cdot\frac{\partial D^*(\vec{x},t)}{\partial t} \\
&= \left(\int_{-\infty}^{\infty} d\omega\, \vec{E}(\vec{x},\omega)\, e^{-i\omega t}\right)\cdot\frac{\partial}{\partial t}\left(\int_{-\infty}^{\infty} d\omega'\, D^*(\vec{x},\omega)\, e^{i\omega' t}\right) \\
&= \left(\int_{-\infty}^{\infty} d\omega\, \vec{E}(\vec{x},\omega)\, e^{-i\omega t}\right)\cdot\left(i\omega'\int_{-\infty}^{\infty} d\omega'\, \epsilon^*(\omega')\,\vec{E}^*(\vec{x},\omega')\, e^{i\omega' t}\right) \\
&= \int d\omega \int d\omega' [i\omega'\,\epsilon^*(\omega')]\; \vec{E}^*(\vec{x},\omega')\cdot\vec{E}(\vec{x},\omega)\, e^{-i(\omega-\omega')t}
\end{aligned} \tag{7}$$

Averaging the two expressions,

$$\vec{E}(\vec{x},t)\cdot\frac{\partial \vec{D}(\vec{x},t)}{\partial t} = \frac{1}{2}\int d\omega \int d\omega' \left[i\omega'\,\epsilon^*(\omega') - i\omega\,\epsilon(\omega)\right]\; \vec{E}^*(\vec{x},\omega')\cdot\vec{E}(\vec{x},\omega)\, e^{-i(\omega-\omega')t} \tag{8}$$

Now, suppose that the fields are dominated near a given frequency when compared to the characteristic frequency interval over which $\epsilon(\omega)$ changes appreciably. We may then assume that

$$\omega' \epsilon^*(\omega') \approx \omega \epsilon^*(\omega) + (\omega' - \omega) \frac{d}{d\omega} \left(\omega \epsilon^*(\omega)\right) + \ldots \qquad (9)$$

Then, the term inside the square brackets of Eq. (8) becomes

$$[\ldots] = i\omega \epsilon^*(\omega) + i(\omega' - \omega) \frac{d}{d\omega}\left(\omega \epsilon^*(\omega)\right) + \ldots - i\omega\epsilon(\omega)$$

$$= i\omega \left[\epsilon^*(\omega) - \epsilon(\omega)\right] - i(\omega - \omega') \frac{d}{d\omega}\left(\omega \epsilon^*(\omega)\right) + \ldots$$

$$= i\omega \left[-2i\, \mathrm{Im}\left(\epsilon(\omega)\right)\right] - i(\omega - \omega') \frac{d}{d\omega}\left(\omega \epsilon^*(\omega)\right) + \ldots$$

$$\mathrm{Im}(z) = \frac{z - z^*}{2i}$$

$$= 2\omega\, \mathrm{Im}\left(\epsilon(\omega)\right) - i(\omega - \omega') \frac{d}{d\omega}\left(\omega \epsilon^*(\omega)\right) + \ldots \qquad (10)$$

Substituting this back into Eq. (8), we have

$$\vec{E} \cdot \frac{\partial \vec{D}}{\partial t} = \frac{1}{2}\int d\omega \int d\omega'\; 2\omega\, \mathrm{Im}(\epsilon(\omega))\, \vec{E}^*(\vec{x},\omega') \cdot \vec{E}(\vec{x},\omega)\, e^{-i(\omega-\omega')t}$$

$$+ \frac{1}{2}\int d\omega \int d\omega' \left[-i(\omega-\omega') \frac{d}{d\omega}\left(\omega \epsilon^*(\omega)\right)\right] \vec{E}^*(\vec{x},\omega') \cdot \vec{E}(\vec{x},\omega)\, e^{-i(\omega-\omega')t}$$

$$= \int d\omega \int d\omega'\; \vec{E}^*(\vec{x},\omega') \cdot \vec{E}(\vec{x},\omega)\; \omega\, \mathrm{Im}(\epsilon(\omega))\, e^{-i(\omega-\omega')t}$$

$$\frac{\partial}{\partial t} e^{-i(\omega-\omega')t} = -i(\omega-\omega') e^{-i(\omega-\omega')t}$$

$$+ \frac{1}{2}\frac{\partial}{\partial t}\int d\omega \int d\omega'\; \vec{E}^*(\vec{x},\omega) \cdot \vec{E}(\vec{x},\omega) \frac{d}{d\omega}\left[\omega \epsilon^*(\omega)\right] e^{-i(\omega-\omega')t} \qquad (11)$$

In the same manner, we can obtain a similar expression for $\vec{H} \cdot \partial_t \vec{B}$. Writing this out, we have

$$\vec{H}(\vec{x},t) \cdot \frac{\partial \vec{B}(\vec{x},t)}{\partial t} = \vec{H}^*(\vec{x},t) \cdot \frac{\partial}{\partial t}\left(\int_{-\infty}^{\infty} d\omega\; \vec{B}(\vec{x},\omega)\, e^{-i\omega t}\right)$$

$$= \left(\int_{-\infty}^{\infty} d\omega'\; \vec{H}^*(\vec{x},\omega)\, e^{i\omega' t}\right) \cdot \left(-i\omega \int_{-\infty}^{\infty} d\omega\; \mu(\omega)\, \vec{H}(\vec{x},\omega)\, e^{-i\omega t}\right)$$

$$= \int d\omega \int d\omega' \left[-i\omega\mu(\omega)\right] \vec{H}^*(\vec{x},\omega') \cdot \vec{H}(\vec{x},\omega)\, e^{-i(\omega-\omega')t} \qquad (12)$$

Alternatively, we can have

$$\vec{H}(\vec{x},t) \cdot \frac{\partial \vec{B}(\vec{x},t)}{\partial t} = \left(\int_{-\infty}^{\infty} d\omega\; \vec{H}(\vec{x},\omega)\, e^{-i\omega t}\right) \cdot \frac{\partial \vec{B}^*(\vec{x},t)}{\partial t}$$

$$= \left(\int_{-\infty}^{\infty} d\omega\; \vec{H}(\vec{x},\omega)\, e^{-i\omega t}\right) \cdot \frac{\partial}{\partial t}\left(\int_{-\infty}^{\infty} d\omega'\; \vec{B}^*(\vec{x},\omega)\, e^{i\omega' t}\right)$$

$$= \left(\int_{-\infty}^{\infty} d\omega\; \vec{H}(\vec{x},\omega)\, e^{-i\omega t}\right) \cdot \left(i\omega' \int_{-\infty}^{\infty} d\omega'\; \mu^*(\omega)\, \vec{H}^*(\vec{x},\omega)\, e^{i\omega' t}\right)$$

$$= \int d\omega \int d\omega' \left[i\omega' \mu^*(\omega')\right] \vec{H}^*(\vec{x},\omega') \cdot \vec{H}(\vec{x},\omega)\, e^{-i(\omega-\omega')t} \qquad (13)$$

Averaging these,

$$\vec{H}(\vec{x},t) \cdot \frac{\partial \vec{B}(\vec{x},t)}{\partial t} = \frac{1}{2}\int d\omega \int d\omega' \left[i\omega' \mu^*(\omega') - i\omega\mu(\omega)\right] \vec{H}^*(\vec{x},\omega) \cdot \vec{H}(\vec{x},\omega)\, e^{-i(\omega-\omega')t} \qquad (14)$$

Applying the assumption to obtain Eq. (9), we have

$$\omega' \mu^*(\omega') \approx \omega \mu^*(\omega) + (\omega' - \omega) \frac{d}{d\omega}\left(\omega \mu^*(\omega)\right) + \ldots \qquad (15)$$

Then, Eq. (14) becomes

$$\vec{H}\cdot\frac{\partial \vec{B}}{\partial t} = \frac{1}{2}\int d\omega\int d\omega'\; 2\,\mathrm{Im}\left(\mu(\omega)\right)\,\vec{H}^*(\vec{x},\omega')\cdot\vec{H}(\vec{x},\omega)\,e^{-i(\omega-\omega')t}$$

$$+\frac{1}{2}\int d\omega\int d\omega'\left[-i(\omega-\omega')\frac{d}{d\omega}\left(\omega\mu^*(\omega)\right)\right]\vec{H}^*(\vec{x},\omega')\cdot\vec{H}(\vec{x},\omega)\,e^{-i(\omega-\omega')t}$$

$$=\int d\omega\int d\omega'\;\vec{H}^*(\vec{x},\omega')\cdot\vec{H}(\vec{x},\omega)\;\omega\,\mathrm{Im}\left(\mu(\omega)\right)e^{-i(\omega-\omega')t}$$

$$+\frac{1}{2}\frac{\partial}{\partial t}\int d\omega\int d\omega'\;\vec{H}^*(\vec{x},\omega')\cdot\vec{H}(\vec{x},\omega)\,\frac{d}{d\omega}\left[\omega\,\mu^*(\omega)\right]e^{-i(\omega-\omega')t} \qquad (16)$$

- Note that if ϵ and μ are real and frequency independent, we recover the simple connection between the time derivative terms in Sec. 6.7 Eq. (6) and $\partial_t u$ with u given by Sec. 6.7 Eq. (7).
- The first term in Eq. (11) represents the conversion of electrical energy into heat while the second term must be an effective energy density
- A more transparent expression, consistent with the assumption that the fields are dominated near a given frequency, by supposing that

$$\vec{E} = \tilde{\vec{E}}(t)\cos(\omega_0 t+\alpha), \qquad \vec{H} = \tilde{\vec{H}}(t)\cos(\omega_0 t+\beta) \qquad (17)$$

where $\tilde{\vec{E}}(t)$ and $\tilde{\vec{H}}(t)$ are slowly varying relative to both $1/\omega_0$ and the inverse of the frequency range over which $\epsilon(\omega)$ changes appreciably.
- If we substitute the Fourier transforms $\vec{E}(\vec{x},\omega)$ and $\vec{H}(\vec{x},\omega)$ and average both sides of the sum of Eqs. (11) and (16) over a period of the carrier frequency ω_0, we find

(18)

$$\left\langle \vec{E}\cdot\frac{\partial\vec{D}}{\partial t}+\vec{H}\cdot\frac{\partial\vec{B}}{\partial t}\right\rangle = \Bigg\langle \omega_0\,\mathrm{Im}\left(\epsilon(\omega_0)\right)\left(\int d\omega'\,\vec{E}^*(\vec{x},\omega')e^{i\omega' t}\right)\cdot\left(\int d\omega\,\vec{E}(\vec{x},\omega)e^{-i\omega t}\right)$$

$$+\frac{1}{2}\frac{\partial}{\partial t}\left(\int d\omega'\,\vec{E}^*(\vec{x},\omega')e^{i\omega' t}\right)\cdot\left(\int d\omega\,\vec{E}(\vec{x},\omega)e^{-i\omega t}\right)\left[\frac{d}{d\omega}\omega\epsilon^*(\omega)\right]_{\omega=\omega_0}$$

$$+\omega_0\,\mathrm{Im}\left(\mu(\omega_0)\right)\left(\int d\omega'\,\vec{H}^*(\vec{x},\omega')e^{i\omega' t}\right)\cdot\left(\int d\omega\,\vec{H}(\vec{x},\omega)e^{-i\omega t}\right)$$

$$+\frac{1}{2}\frac{\partial}{\partial t}\left(\int d\omega'\,\vec{H}^*(\vec{x},\omega')e^{i\omega' t}\right)\cdot\left(\int d\omega\,\vec{H}(\vec{x},\omega)e^{-i\omega t}\right)\left[\frac{d}{d\omega}\omega\mu^*(\omega)\right]_{\omega=\omega_0}\Bigg\rangle$$

Apply Eqs. (1), (2)

$$=\omega_0\,\mathrm{Im}\left(\epsilon(\omega_0)\right)\left\langle\vec{E}^*(\vec{x},t)\cdot\vec{E}(\vec{x},t)\right\rangle+\omega_0\,\mathrm{Im}\left(\mu(\omega_0)\right)\left\langle\vec{H}^*(\vec{x},t)\cdot\vec{H}(\vec{x},t)\right\rangle$$

$$+\frac{1}{2}\frac{\partial}{\partial t}\left\langle\vec{E}^*(\vec{x},t)\cdot\vec{E}(\vec{x},t)\right\rangle\mathrm{Re}\left[\frac{d}{d\omega}\omega\epsilon(\omega)\right]_{\omega=\omega_0}+\frac{1}{2}\frac{\partial}{\partial t}\left\langle\vec{H}^*(\vec{x},t)\cdot\vec{H}(\vec{x},t)\right\rangle\mathrm{Re}\left[\frac{d}{d\omega}\omega\mu(\omega)\right]_{\omega=\omega_0}$$

$\vec{E}=\vec{E}^*$, $\vec{H}=\vec{H}^*$

$$=\omega_0\,\mathrm{Im}\left(\epsilon(\omega_0)\right)\left\langle\vec{E}(\vec{x},t)\cdot\vec{E}(\vec{x},t)\right\rangle+\omega_0\,\mathrm{Im}\left(\mu(\omega_0)\right)\left\langle\vec{H}(\vec{x},t)\cdot\vec{H}(\vec{x},t)\right\rangle+\frac{\partial u_{eff}}{\partial t}$$

where

$$\frac{\partial u_{eff}}{\partial t}=\frac{1}{2}\left\langle\vec{E}(\vec{x},t)\cdot\vec{E}(\vec{x},t)\right\rangle\mathrm{Re}\left[\frac{d}{d\omega}\omega\epsilon(\omega)\right]_{\omega=\omega_0}+\frac{1}{2}\left\langle\vec{H}(\vec{x},t)\cdot\vec{H}(\vec{x},t)\right\rangle\mathrm{Re}\left[\frac{d}{d\omega}\omega\mu(\omega)\right]_{\omega=\omega_0} \qquad (19)$$

Poynting's theorem in these circumstances reads

compare with Eq. (9) in Sec. 6.7

$$\frac{\partial u_{eff}}{\partial t}+\vec{\nabla}\cdot\vec{S}=\underbrace{-\vec{J}\cdot\vec{E}}_{\text{describe explicit ohmic losses}}\;\underbrace{-\,\omega_0\,\mathrm{Im}\left(\epsilon(\omega_0)\right)\left\langle E(\vec{x},t)\cdot E(\vec{x},t)\right\rangle-\omega_0\,\mathrm{Im}\left(\mu(\omega_0)\right)\left\langle\vec{H}(\vec{x},t)\cdot\vec{H}(\vec{x},t)\right\rangle}_{\text{absorptive dissipation in the medium}}$$

- This exhibits the local conservation of EM energy in realistic situations where, as well as $\vec{\nabla}\cdot\vec{S}\neq 0$, there may be losses from heating the medium ($\mathrm{Im}\,\epsilon\neq 0$, $\mathrm{Im}\,\mu\neq 0$) leading to a slow decay of energy in fields.

7.1: Plane in a Non-conducting Medium

A basic feature of the Maxwell equations for the EM field is the existence of traveling wave solns which represent the transport of energy from one point to another. The most simplest and most fundamental EM waves are transverse, plane waves. To see how such solns can be obtained in simple non-conducting media described by spatially constant μ and ϵ, we consider, in the absence of sources, Maxwell eqns in infinite media:

$$\vec{\nabla}\cdot\vec{B}=0 \qquad \vec{\nabla}\times\vec{E}+\frac{\partial\vec{B}}{\partial t}=0$$
$$\vec{\nabla}\cdot\vec{D}=0 \qquad \vec{\nabla}\times\vec{H}-\frac{\partial\vec{D}}{\partial t}=0 \tag{1}$$

Assuming solns w/ harmonic time dependence $e^{-i\omega t}$, from which we can build an arbitrary Fourier superposition, eqns for amplitudes $\vec{E}(\vec{x},\omega)$ turn to

$$\vec{\nabla}\cdot\frac{1}{2\pi}\int d\omega\, e^{-i\omega t}\vec{B}(\vec{x},\omega)=\frac{1}{2\pi}\int d\omega\, e^{-i\omega t}\left(\vec{\nabla}\cdot\vec{B}(\vec{x},\omega)\right)=0$$

$$\vec{\nabla}\cdot\vec{B}(\vec{x},\omega)=0 \tag{2}$$

$$\vec{\nabla}\cdot\frac{1}{2\pi}\int d\omega\, e^{-i\omega t}\vec{D}(\vec{x},\omega)=\frac{1}{2\pi}\int d\omega\, e^{-i\omega t}\left(\vec{\nabla}\cdot\vec{D}(\vec{x},\omega)\right)=0$$

$$\vec{\nabla}\cdot\vec{D}(\vec{x},\omega)=0 \tag{3}$$

$$\vec{\nabla}\times\frac{1}{2\pi}\int d\omega\, e^{-i\omega t}\vec{E}(\vec{x},\omega)+\frac{1}{2\pi}\int d\omega\,\frac{\partial}{\partial t}\left(e^{-i\omega t}\right)\vec{B}(\vec{x},\omega)=0$$

$$\frac{1}{2\pi}\int d\omega\, e^{-i\omega t}\left[\vec{\nabla}\times\vec{E}(\vec{x},\omega)-i\omega\vec{B}(\vec{x},\omega)\right]=0$$

$$\vec{\nabla}\times\vec{E}(\vec{x},\omega)-i\omega\vec{B}(\vec{x},\omega)=0 \tag{4}$$

$$\vec{\nabla}\times\frac{1}{2\pi}\int d\omega\, e^{-i\omega t}\vec{H}(\vec{x},\omega)-\frac{1}{2\pi}\int d\omega\,\frac{\partial}{\partial t}\left(e^{-i\omega t}\right)\vec{D}(\vec{x},\omega)=0$$

$$\frac{1}{2\pi}\int d\omega\, e^{-i\omega t}\left[\vec{\nabla}\times\vec{H}(\vec{x},\omega)+i\omega\vec{D}(\vec{x},\omega)\right]=0$$

$$\nabla\times\vec{H}(\vec{x},\omega)+i\omega\vec{D}(\vec{x},\omega)=0 \tag{5}$$

For uniform isotropic linear media, we have $\vec{D}=\epsilon\vec{E}$, $\vec{B}=\mu H$ where ϵ and μ may generally be complex functions of ω. We assume that they are real and positive (no losses). Then, from Eqs. (4) and (5), we have

$$\vec{\nabla}\times\vec{E}(\vec{x},\omega)-i\omega\vec{B}(\vec{x},\omega)=0$$
$$\vec{\nabla}\times\frac{1}{\mu}\vec{B}(\vec{x},\omega)+i\omega\epsilon\vec{E}(\vec{x},\omega)=0 \Rightarrow \vec{\nabla}\times\vec{B}(\vec{x},\omega)+i\omega\mu\epsilon\vec{E}(\vec{x},\omega)=0 \tag{6}$$

while from Eqs (2) and (3), we have

$$\vec{\nabla}\cdot\vec{B}(\vec{x},\omega)=0 \qquad \vec{\nabla}\cdot\vec{E}(\vec{x},\omega)=0 \tag{7}$$

By combining the expressions in Eq. (6), we obtain

$\vec{\nabla}\times(\vec{\nabla}\times\vec{v}) = \vec{\nabla}(\vec{\nabla}\cdot\vec{v}) - \vec{\nabla}^2\vec{v}$

$$\vec{\nabla}\times\left(\vec{\nabla}\times\vec{E}(\vec{x},\omega)\right) - i\omega(\vec{\nabla}\times\vec{B}) = \vec{\nabla}(\cancelto{0}{\vec{\nabla}\cdot\vec{E}}) - \vec{\nabla}^2\vec{E} - i\omega\left(-i\omega\mu\epsilon\vec{E}\right) = 0$$

$$\nabla^2\vec{E}(\vec{x},\omega) + \omega^2\mu\epsilon\,\vec{E}(\vec{x},\omega) = 0 \qquad (8)$$

by taking the curl of both sides and applying Eq. (7). In the same manner, we also obtain

$$\vec{\nabla}\times\left(\vec{\nabla}\times\vec{B}(\vec{x},\omega)\right) + i\omega\mu\epsilon\left(\nabla\times\vec{E}(\vec{x},\omega)\right) = \vec{\nabla}(\cancelto{0}{\vec{\nabla}\cdot\vec{B}}) - \nabla^2\vec{B} + i\omega\mu\epsilon(i\omega\vec{B}) = 0$$

$$\vec{\nabla}^2\vec{B}(\vec{x},\omega) + \omega^2\mu\epsilon\,\vec{B}(\vec{x},\omega) = 0 \qquad (9)$$

Consider as a possible solution a plane wave traveling in the x direction $[\vec{E}(x,t),\ \vec{B}(x,t) \propto e^{i(kx-\omega t)}]$. From Eqs. (8) and (9), we find that the wave number k is related to the frequency ω by

$$\frac{\partial^2}{\partial x^2} e^{i(kx-\omega t)} + \omega^2\mu\epsilon\, e^{i(kx-\omega t)} = (ik)^2 e^{i(kx-\omega t)} + \omega^2\mu\epsilon\, e^{i(kx-\omega t)} = 0$$

$$k = \omega\sqrt{\mu\epsilon} \qquad (10)$$

The phase velocity of the wave is

$$v = \frac{\omega}{k} = \frac{\cancel{\omega}}{\cancel{\omega}\sqrt{\mu\epsilon}} = \overbrace{\frac{1}{\sqrt{\mu_0\epsilon_0}}}^{c}\underbrace{\sqrt{\frac{\mu_0\epsilon_0}{\mu\epsilon}}}_{1/n} = \frac{c}{n} \qquad (11)$$

index of refraction

where n is usually a function of frequency. The primordial solution in 1D is

$$u(x,t) = a e^{i(kx-\omega t)} + b e^{-i(kx+\omega t)} \qquad (12)$$

Using $w = kv$ from Eq. (11), we get

$$u_k(x,t) = a e^{ik(x-vt)} + b e^{-ik(x+vt)} \qquad (13)$$

- If the medium is non-dispersive (μ, ϵ independent on ω), the Fourier superposition theorem given by

$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} A(k)\, e^{ikx}\, dk \qquad (14)$$

where

$$A(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ikx} f(x)\, dx \qquad (15)$$

can be used to construct a general soln of the form

$$u(x,t) = f(x-vt) + g(x-vt) \qquad (16)$$

where $f(z)$ and $g(z)$ are arbitrary fxns. This eqn represents waves traveling in the positive and negative x directions with speeds equal to the phase velocity v.

- If the medium is dispersive (μ, ϵ dependent on ω), Eq. (12) still holds but the gen. soln in Eq. (16) does not since dispersion produces modifications. The wave changes shape as it propagates.

- Consider an EM plane wave of frequency ω and wave vector $\vec{k} = k\hat{n}$. We require:
  1. satisfies Helmholtz wave eqn in Eqs. (8) and (9) → kinematic constraint
  2. satisfies all of Maxwell's equations → dynamic constraint

With the convention that the physical $\vec{E}$ and $\vec{B}$ are obtained by taking the real parts of complex quantities, we write the plane wave fields as

$$\vec{E}(\vec{x},t) = \vec{\tilde{E}}\, e^{ik\hat{n}\cdot\vec{x} - i\omega t} \qquad \vec{B}(\vec{x},t) = \vec{\tilde{B}}\, e^{ik\hat{n}\cdot\vec{x} - i\omega t} \tag{17}$$

where $\vec{\tilde{E}}$, $\vec{\tilde{B}}$, $\hat{n}$ are constant vectors. Each component of $\vec{E}$ and $\vec{B}$ satisfies Eqs. (8) and (9) provided

$$k\hat{n}\cdot k\hat{n} = k^2(\hat{n}\cdot\hat{n}) = \mu\epsilon\omega^2 \tag{18}$$

To recover Eq. (10), it is necessary that $\hat{n}$ be a unit vector such that $\hat{n}\cdot\hat{n} = 1$. With the wave eqn satisfied, there only remains the fixing of the vectorial properties so that Maxwell equations in Eq. (1) are valid. The divergence equations demand that

$$\left(\frac{\partial}{\partial x}\hat{\imath} + \frac{\partial}{\partial y}\hat{\jmath} + \frac{\partial}{\partial z}\hat{k}\right)\cdot\left(\tilde{E}_x\hat{\imath} + \tilde{E}_y\hat{\jmath} + \tilde{E}_z\hat{k}\right)e^{ik(n_x\hat{\imath} + n_y\hat{\jmath} + n_z\hat{k})\cdot\vec{x} - i\omega t} = 0$$

$$\left[\tilde{E}_x(ikn_x) + \tilde{E}_y(ikn_y) + \tilde{E}_z(ikn_z)\right]e^{ik\hat{n}\cdot\vec{x} - i\omega t} = 0$$

$$\tilde{E}_x n_x + \tilde{E}_y n_y + \tilde{E}_z n_z = 0$$

$$\hat{n}\cdot\vec{\tilde{E}} = \tag{19}$$

In the same manner,

$$\left(\frac{\partial}{\partial x}\hat{\imath} + \frac{\partial}{\partial y}\hat{\jmath} + \frac{\partial}{\partial z}\hat{k}\right)\cdot\left(\tilde{B}_x\hat{\imath} + \tilde{B}_y\hat{\jmath} + \tilde{B}_z\hat{k}\right)e^{ik(n_x\hat{\imath} + n_y\hat{\jmath} + n_z\hat{k})\cdot\vec{x} - i\omega t} = 0$$

$$\left[\tilde{B}_x(ikn_x) + \tilde{B}_y(ikn_y) + \tilde{B}_z(ikn_z)\right]e^{ik\hat{n}\cdot\vec{x} - i\omega t} = 0$$

$$\tilde{B}_x n_x + \tilde{B}_y n_y + \tilde{B}_z n_z = 0$$

$$\hat{n}\cdot\vec{\tilde{B}} = 0 \tag{20}$$

This means that $\vec{E}$ and $\vec{B}$ are $\perp$ to the direction of propagation $\hat{n}$. Such a wave is called a transverse wave. The curl equations provide further restrictions:

$$\nabla\times\vec{E} + \frac{\partial}{\partial t}\vec{B} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ E_x & E_y & E_z \end{vmatrix} + \frac{\partial}{\partial t}\vec{\tilde{B}}\, e^{(ik\hat{n}\cdot\vec{x} - i\omega t)}$$

$$0 = \left[\left(\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z}\right)\hat{\imath} + \left(\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x}\right)\hat{\jmath} + \left(\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y}\right)\hat{k}\right] - i\omega\vec{\tilde{B}}\, e^{(\dots)}$$

$$0 = \left[(\tilde{E}_z n_y - \tilde{E}_y n_z)\hat{\imath} + (\tilde{E}_x n_z - \tilde{E}_z n_x)\hat{\jmath} + (\tilde{E}_y n_x - \tilde{E}_x n_y)\right] ik\, e^{(\dots)}$$

$$- i\omega\vec{\tilde{B}}\, e^{(\dots)}$$

$$0 = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ n_x & n_y & n_z \\ \tilde{E}_x & \tilde{E}_y & \tilde{E}_z \end{vmatrix} ik\, e^{(\dots)} - i\omega\vec{\tilde{B}}\, e^{(\dots)}$$

$$0 = (\hat{n}\times\vec{\tilde{E}})\frac{k}{\omega} - \vec{\tilde{B}}$$

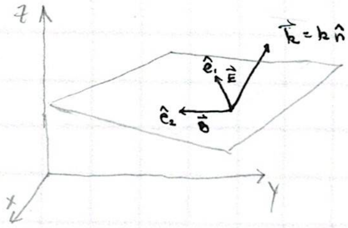
$$\hookrightarrow \vec{\tilde{B}} = \sqrt{\mu\epsilon}\,(\hat{n}\times\vec{\tilde{E}}) \tag{21}$$

The factor $\sqrt{\mu\epsilon}$ can be written $\sqrt{\mu\epsilon} = n/c$ from Eq. (11). Thus, we see that $c\vec{B}$ and $\vec{E}$, which have the same dimensions, have the same magnitude for plane EM waves in free space and differ by the index of refraction in ponderable media. In engineering lit., the magnetic field $\vec{H}$ is often displayed parallel, instead of $\vec{B}$, with $\vec{E}$. In this case, Eq. becomes

$$\mu\vec{H} = \sqrt{\mu\epsilon}\,(\hat{n}\times\vec{E}) \rightarrow \vec{H} = \sqrt{\frac{\epsilon}{\mu}}\,(\hat{n}\times\tilde{\vec{E}}) = \frac{1}{Z}\,(\hat{n}\times\tilde{\vec{E}}) \qquad (22)$$

where $Z = \sqrt{\mu/\epsilon}$ is an impedance.

If $\hat{n}$ is real, Eq. (21) implies that $\tilde{\vec{E}}$ and $\tilde{\vec{B}}$ have the same phase. It is then useful to introduce a set of real mutually orthogonal unit vectors $(\hat{e}_1, \hat{e}_2, \hat{n})$ as shown in the Figure. In terms of these unit vectors, the field strengths $\tilde{\vec{E}}$ and $\tilde{\vec{B}}$ are



$\hat{n}\times\hat{e}_1 = \hat{e}_2$

$$\tilde{\vec{E}} = \hat{e}_1 E_0, \qquad \tilde{\vec{B}} = \hat{e}_2\sqrt{\mu\epsilon}\,E_0 \qquad (23)$$

with the use of Eq. (21). This can also be expressed as

$\hat{n}\times\hat{e}_2 = -\hat{e}_1$

$$\vec{E} = \hat{e}_2 E_0' \qquad \vec{B} = -\hat{e}_1\sqrt{\mu\epsilon}\,E_0' \qquad (24)$$

where E_0 and E_0' are constants, possibly complex. The wave described by Eq. (17) and Eq. (23) or (24) is a transverse wave propagating in the direction of $\hat{n}$. It represents a time-averaged flux given by the real part of the complex Poynting vector

$$\langle\vec{S}\rangle = \frac{1}{2}\langle\vec{E}\times\vec{H}^*\rangle \qquad (25)$$

The energy flow (energy per unit area per unit time) is given by

$\hat{e}_1\times\frac{\sqrt{\mu\epsilon}}{\mu}\hat{e}_2 = \frac{\sqrt{\mu\epsilon}}{\mu}(\hat{e}_1\times\hat{e}_2)$

$e^{i(k\hat{n}\cdot\vec{x}-\omega t)}$ can be written as $\cos(k\hat{n}\cdot\vec{x}-\omega t)$

$$\langle\vec{S}\rangle = \frac{1}{2}\left\langle\tilde{\vec{E}}\,e^{i(k\hat{n}\cdot\vec{x}-\omega t)}\times\frac{1}{\mu}\tilde{\vec{B}}\,e^{i(k\hat{n}\cdot\vec{x}-\omega t)}\right\rangle \approx \frac{1}{2}(\hat{e}_1\times\hat{e}_2)E_0^2\,\langle\cancelto{1}{\cos^2(k\hat{n}\cdot\vec{x}-\omega t)}\rangle\,\frac{\sqrt{\mu\epsilon}}{\mu}$$

$$\langle\vec{S}\rangle = \frac{1}{2}\sqrt{\frac{\epsilon}{\mu}}\,|E_0|^2\,\hat{n} \qquad (26)$$

The time-averaged density $\langle u\rangle$ is correspondingly

$$\langle u\rangle = \frac{1}{2}\left\langle\frac{1}{2}(\vec{D}\cdot\vec{E}^* + \vec{B}^*\cdot\vec{H})\right\rangle = \frac{1}{4}\left\langle\epsilon\vec{E}\cdot\vec{E}^* + \vec{B}^*\cdot\frac{1}{\mu}\vec{B}\right\rangle$$

which leads to

$$\langle u\rangle = \frac{1}{4}\left\langle\left(\epsilon\tilde{E}^2 + \frac{1}{\mu}\tilde{B}^2\right)e^{2i(k\hat{n}\cdot\vec{x}-\omega t)}\right\rangle \approx \frac{1}{4}\overbrace{\left(\epsilon E_0^2 + \frac{\cancel{\mu}\epsilon}{\cancel{\mu}}E_0^2\right)}^{2\epsilon E_0^2}\langle\cancelto{1}{\cos^2(k\hat{n}\cdot\vec{x}-\omega t)}\rangle$$

$$\langle u\rangle = \frac{1}{2}\epsilon|E_0|^2 \qquad (27)$$

The ratio of the magnitude of Eq. (26) to Eq. (27) shows that the speed of energy flow is

$$v = \frac{|\langle\vec{S}\rangle|}{\langle u\rangle} = \frac{\frac{1}{2}\sqrt{\frac{\epsilon}{\mu}}\,|E_0|^2}{\frac{1}{2}\epsilon\,|E_0|^2} = \frac{1}{\sqrt{\mu\epsilon}}$$

as expected from Eq. (11)

In the discussion that follows Eq. (21), we assumed that $\hat{n}$ was a real unit vector. This does not yield the most general possible solution for a plane wave. Suppose $\hat{n}$ is complex:

$$\hat{n} = \hat{n}_R + i\,\hat{n}_I \qquad (28)$$

Then, the exponential in Eq. (17) becomes

$$e^{ik\hat{n}\cdot\vec{x}-i\omega t} = e^{-k\hat{n}_I\cdot\vec{x}}\, e^{ik\hat{n}_R\cdot\vec{x}-i\omega t} \tag{29}$$

- The wave possesses exponential growth or decay in some directions → inhomogeneous plane wave
- Surfaces of constant amplitude and phase are still planes but $\nparallel$
- Eqs. (19), (20), and (21) still hold
- The requirement $\hat{n}\cdot\hat{n}=1$ leads to

$$(\hat{n}_R + i\hat{n}_I)\cdot(n_R + i\hat{n}_I) = n_R^2 - n_I^2 = 1 \tag{30}$$

which means that
$$\hat{n}_R \cdot \hat{n}_I = 0 \tag{31}$$

This shows that $\hat{n}_R$ and $\hat{n}_I$ are orthogonal. The coordinate axes can be oriented so that $\hat{n}_R \to x$ direction and $\hat{n}_I \to y$ direction. Eq. (30) can be satisfied generally by

$$\hat{n} = \hat{i}\cosh\theta + i\hat{j}\sinh\theta \tag{32}$$

where θ is a real constant. The most general vector $\vec{\tilde{E}}$ satisfying $\hat{n}\cdot\vec{\tilde{E}}=0$ is then

$$\vec{\tilde{E}} = (i\hat{i}\sinh\theta - \hat{j}\cosh\theta)A + \hat{k}A' \tag{33}$$

where A and A' are complex constants. For $\theta\neq 0$, $\vec{\tilde{E}}$ in general has components in the direction(s) of $\hat{n}$. If $\theta=0$, Eqs. (23) and (24) are recovered.

Inhomogeneous plane waves form a general basis for the treatment of boundary-value problems for waves and are especially useful in the soln of diffraction in 2D.

7.2: Linear and Circular Polarization; Stokes Parameters

The plane wave in Eqs. (17) and (23) of Sec. 7.1 is a wave with its electric field vector always in the direction $\hat{e}_1$. Such a wave is said to be linearly polarized with polarization vector $\hat{e}_1$. Evidently, the wave described in Eq. (24) of Sec 7.1 is linearly polarized with polarization vector $\hat{e}_2$ and is linearly independant on the first. Thus

$$\vec{E}_1 = \hat{e}_1 E_1 e^{i\vec{k}\cdot\vec{x} - i\omega t}, \qquad \vec{E}_2 = \hat{e}_2 E_2 e^{i\vec{k}\cdot\vec{x} - i\omega t} \tag{1}$$

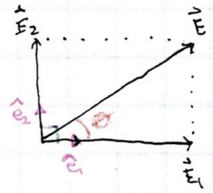
with

$$B_j = \sqrt{\mu\epsilon}\, \frac{\vec{k}\times\vec{E}_j}{k}, \qquad j = 1, 2 \tag{2}$$

can be combined to give the most homogeneous plane wave propagating in the direction $\vec{k} = k\hat{n}$

$$\vec{E}(\vec{x},t) = \vec{E}_1(\vec{x},t) + \vec{E}_2(\vec{x},t) = (\hat{e}_1 E_1 + \hat{e}_2 E_2)\, e^{i\vec{k}\cdot\vec{x} - i\omega t} \tag{3}$$

The amplitudes E_1 and E_2 are complex numbers, + allow the possibility of a phase diff. bet. waves of different linear polarization



- E_1 and E_2 have the same phase; Eq. (3) → linearly polarized wave w/ polarization vector making an angle $\theta = \tan^{-1}(E_2/E_1)$ with $\hat{e}_1$ and magnitude $E = \sqrt{E_1^2 + E_2^2}$
- E_1 and E_2 have different phases: Eq. (3) ⟶ elliptically polarized

To better understand this case, let us consider circular polarization (simplest case). Then, E_1 and E_2 have the same magnitude but differ in phase by 90°. The wave in Eq. (3) becomes

circular polarized wave: $\frac{E_1}{E_2} = \pm i$

$$\vec{E}(\vec{x},t) = \underbrace{E_0}_{\text{common real amplitude}}(\hat{e}_1 \pm i\hat{e}_2)\, e^{i\vec{k}\cdot\vec{x} - i\omega t} \tag{4}$$

This is because we choose the axes so that the wave propagates in $+z$ direction while $\hat{e}_1$ and $\hat{e}_2$ are in the x and y directions, respectively. Then, the components of the actual electric field, obtained by taking the real part Eq. (4), are

$$\vec{E}_x(\vec{x},t) = \hat{e}_1 \cdot \mathrm{Re}\left[E_0(\hat{e}_1 \pm i\hat{e}_2)\, e^{i\vec{k}\cdot\vec{x} - i\omega t}\right]$$

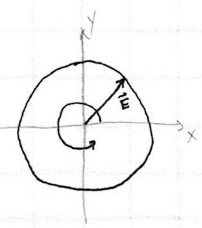
$$= \hat{e}_1 \cdot E_0\left(\hat{e}_1 \cos(\underbrace{\vec{k}\cdot\vec{x}}_{=k\hat{z}\cdot\vec{x}} - \omega t) \mp \hat{e}_2 \sin(\underbrace{\vec{k}\cdot\vec{x}}_{=k\hat{z}\cdot\vec{x}} - \omega t)\right)$$

$$= E_0 \cos(kz - \omega t) \tag{5}$$

In the same manner,

$$\vec{E}_y(\vec{x},t) = \hat{e}_2 \cdot E_0\left[\hat{e}_1 \cos(kz - \omega t) \mp \hat{e}_2 \sin(kz - \omega t)\right]$$

$$= \mp E_0 \sin(kz - \omega t) \tag{6}$$



- At a fixed point in space, the fields in Eqs. (5) and (6) are such that the electric vector is constant in magnitude but sweeps around in a circle at a frequency ω.
- $(\hat{e}_1 + i\hat{e}_2)$: rotation is counterclockwise when observer is facing into oncoming wave
  : left circularly polarized
  : positive helicity
  : positive projection of angular momentum on z axis
- $(\hat{e}_1 - i\hat{e}_2)$: rotation is clockwise
  : right circularly polarized
  : negative helicity

The two polarized waves in Eq. (4) form an equally acceptable set of basic fields for description of a gen. state of polarization. Let

$$\hat{e}_{\pm} = \frac{1}{\sqrt{2}}\left(\hat{e}_1 \pm i\hat{e}_2\right) \qquad (7)$$

be complex orthogonal unit vectors with properties

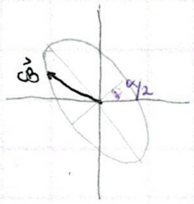
$$\hat{e}^*_{\pm} \cdot \hat{e}_{\mp} = 0 \qquad \hat{e}^*_{\pm} \cdot \hat{e}_3 = 0 \qquad \hat{e}^*_{\pm} \cdot \hat{e}_{\pm} = 1 \qquad (8)$$

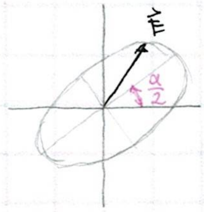
Then, a gen. representation, equiv. to Eq. (3), is

$$\vec{E}(\vec{x},t) = \vec{E}_+(\vec{x},t) + \vec{E}_-(\vec{x},t) = E_+\hat{e}_+ e^{i(\vec{k}\cdot\vec{x}-\omega t)} + E_-\hat{e}_- e^{i(\vec{k}\cdot\vec{x}-\omega t)} \qquad (9)$$

where E_+ and E_- are complex amplitudes.

- E_+ & E_- have different magnitudes but same phase: Eq. (9) → elliptically polarized wave w/ principal axes of the ellipse in the directions of $\hat{e}_1$ and $\hat{e}_2$
  : ratio of semi-major to semi minor axis is $\left|\frac{1+r}{1-r}\right|$ where $E_-/E_+ = r$





- E_+ & E_- have different magnitudes and phases: $E_-/E_+ = re^{i\alpha}$
  - the ellipse traced out by $\vec{E}$ has its axes rotated by an angle $(\alpha/2)$

- linearly polarized wave - $r = \pm 1$

The polarization content of a plane EM wave is known if it can be written in the form of either Eqs. (3) or (9) with known (E_1, E_2) or (E_+, E_-). In practice, the converse problem arises. Given that the wave is in the form of Eq. (17) Sec 7.1, how can we determine from observations on the beam the state of polarization in all its particulars? A useful vehicle for this are the four Stokes parameters proposed by G.G. Stokes in 1852. These parameters are

- quadratic in field strength
- det. through intensity measurements only
  ↳ in conjunction w/ a linear polarizer & a quarter-wave plate or equivalents
    ↳ det. completely the state of polarization of the wave

These parameters can be motivated by observing for wave propagating in the z-direction, the scalar products,

linear polarization: y direction — linear polarization: negative helicity

$$\hat{e}_1 \cdot \vec{E}, \qquad \hat{e}_2 \cdot \vec{E}, \qquad \hat{e}^*_+ \cdot \vec{E}, \qquad \hat{e}^*_- \cdot \vec{E} \qquad (10)$$

linear polarization: x direction — linear polarization: positive helicity

are the amplitudes of radiation. Note that for circular polarization, the complex conjugate of the appropriate polarization vector must be used, in accord with Eq. (8).

- squares of these amplitudes → measure of the intensity of each polarization type
- phase information → from cross products

Now, we define each of the scalar coeffs. in Eqs. (4) and (9) as a magnitude times a scale factor

$$E_1 = a_1 e^{i\delta_1} \qquad E_2 = a_2 e^{i\delta_2} \qquad E_+ = a_+ e^{i\delta_+} \qquad E_- = a_- e^{i\delta_-}$$

Then, the definitions of the Stoke's parameters with respect to linear polarization, in terms of the projected amplitudes in Eq. (10), are given by (11)

$$S_0 = |\hat{e}_1\cdot\vec{E}|^2 + |\hat{e}_2\cdot\vec{E}|^2 = \left|\hat{e}_1\cdot(E_1\hat{e}_1 + E_2\hat{e}_2)\,e^{i\vec{k}\cdot\vec{x}-i\omega t}\right|^2 + \left|\hat{e}_2\cdot E_1\hat{e}_1 + E_2\hat{e}_2\,e^{i\vec{k}\cdot\vec{x}-i\omega t}\right|^2$$

$$\hookrightarrow\ S_0 = \left|a_1 e^{i\delta_1} e^{i\vec{k}\cdot\vec{x}-i\omega t}\right|^2 + \left|a_2 e^{i\delta_2} e^{i\vec{k}\cdot\vec{x}-i\omega t}\right|^2 = a_1^2 + a_2^2$$

$$S_1 = |\hat{e}_1\cdot\vec{E}|^2 + |\hat{e}_2\cdot\vec{E}|^2 = \left|a_1 e^{i\delta_1} e^{i\vec{k}\cdot\vec{x}-i\omega t}\right|^2 - \left|a_2 e^{i\delta_2} e^{i\vec{k}\cdot\vec{x}-i\omega t}\right|^2 = a_1^2 - a_2^2$$

(12)

$$S_2 = 2\,\mathrm{Re}\left[(\hat{e}_1\cdot\vec{E})^*(\hat{e}_2\cdot\vec{E})\right] = 2\,\mathrm{Re}\left[\left(a_1 e^{i\delta_1} e^{i\vec{k}\cdot\vec{x}-i\omega t}\right)^*\left(a_2 e^{i\delta_2} e^{i\vec{k}\cdot\vec{x}-i\omega t}\right)\right]$$

$$\hookrightarrow\ S_2 = 2\,\mathrm{Re}\left[a_1 a_2 e^{i(\delta_2-\delta_1)}\right] = 2a_1a_2\cos(\delta_2-\delta_1)$$

$$S_3 = 2\,\mathrm{Im}\left[(\hat{e}_1\cdot\vec{E})^*(\hat{e}_2\cdot\vec{E})\right] = 2\,\mathrm{Im}\left[a_1 a_2 e^{i(\delta_2-\delta_1)}\right] = 2a_1a_2\sin(\delta_2-\delta_1)$$

If the circular polarization basis $(\hat{e}_+, \hat{e}_-)$ is used instead, we have

$$S_0 = |\hat{e}_+^*\cdot\vec{E}|^2 + |\hat{e}_-^*\cdot\vec{E}|^2 = \left|\hat{e}_+^*\cdot(E_+\hat{e}_+ + E_-\hat{e}_-)e^{i\vec{k}\cdot\vec{x}-i\omega t}\right| + \left|\hat{e}_-^*\cdot(E_+\hat{e}_+ - E_-\hat{e}_-)e^{i\vec{k}\cdot\vec{x}-i\omega t}\right|^2$$

$$S_0 = \left|a_+ e^{i\delta_+} e^{i\vec{k}\cdot\vec{x}-i\omega t}\right|^2 + \left|a_- e^{i\delta_-} e^{i\vec{k}\cdot\vec{x}-i\omega t}\right|^2 = a_+^2 + a_-^2$$

$$S_3 = |\hat{e}_+^*\cdot\vec{E}|^2 - |\hat{e}_-^*\cdot\vec{E}|^2 = \left|a_+ e^{i\delta_+} e^{i\vec{k}\cdot\vec{x}-i\omega t}\right|^2 - \left|a_- e^{i\delta_-} e^{i\vec{k}\cdot\vec{x}-i\omega t}\right|^2 = a_+^2 - a_-^2$$

(13)

$$S_1 = 2\,\mathrm{Re}\left[(\hat{e}_+^*\cdot\vec{E})^*(\hat{e}_-^*\cdot\vec{E})\right] = 2\,\mathrm{Re}\left[\left(a_+ e^{i\delta_+} e^{i\vec{k}\cdot\vec{x}-i\omega t}\right)^*\left(a_- e^{i\delta_-} e^{i\vec{k}\cdot\vec{x}-i\omega t}\right)\right]$$

$$S_1 = 2\,\mathrm{Re}\left[a_+a_- e^{i(\delta_- - \delta_+)}\right] = 2a_+a_-\cos(\delta_- - \delta_+)$$

$$S_2 = 2\,\mathrm{Im}\left[(\hat{e}_+^*\cdot\vec{E})^*(\hat{e}_-^*\cdot\vec{E})\right] = 2\,\mathrm{Im}\left[a_+a_- e^{i(\delta_- - \delta_+)}\right] = 2a_+a_-\sin(\delta_- - \delta_+)$$

These expressions show an interesting rearrangement of roles of the Stokes parameters with respect to the two bases.

- S_0 - measures relative intensity for both cases
- S_1 - gives the pre-ponderance of x - linear polarization over y - linear polarization
- S_2 & S_3 - give phase information in the linear basis
- S_3 - diff. in relative intensity of positive helicity w/ negative helicity
- S_1 & S_2 - give phase information in the circular basis

The four Stokes parameters are not indep., since they depend only on $a_1, a_2, \delta_2-\delta_1$. They satisfy

$$S_0^2 = S_1^2 + S_2^2 + S_3^2 \qquad (14)$$

For quasi monochromatic radiation, beams of radiation actually consist of a superposition of finite wave trains. By Fourier's theorem, they contain a range of frequencies and are not completely monochromatic. One way of viewing this is to say that the magnitudes and phases (a_i, δ_i) in Eq. (11) vary in time slowly compared to frequency ω. Then, the observable Stokes parameters become averages over a relatively long time interval. For example,

$$\langle S_2\rangle = 2\langle a_1a_2\cos(\delta_2-\delta_1)\rangle \qquad (15)$$

macroscopic time average

One consequence of the averaging process is that the Stokes parameters for a quasi - monochromatic beam satisfy

$$S_0^2 \geq S_1^2 + S_2^2 + S_3^2 \qquad (16)$$

instead of the equality in Eq. (14). "Natural light", even if monochromatic to a high degree, has $s_1 = s_2 = s_3 = 0$.

- Astrophysical example of use of Stokes parameters - study of optical and radio frequency radiation from the pulsar in the Crab nebula.

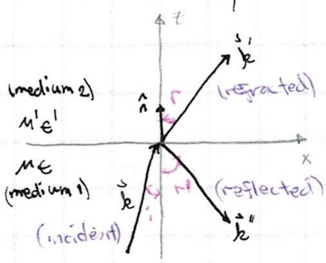
7.3: Reflection and Refraction of Electromagnetic Waves at a Plane Interface between Dielectrics

The reflection and refraction of light at a plane surface between two media of different dielectric properties are familiar phenomena. The various aspects of the phenomena divide themselves into two classes:

1. Kinematic properties → depend from the wave nature of the phenomena and the fact
   a) Angle of reflection equals angle of incidence
   b) Snell's law: $$\frac{\sin i}{\sin r} = \frac{n'}{n} \qquad (1)$$

   where i, r are the angles of incidence & refraction while n, n' are the corres. indices of refraction
2. Dynamic properties → depend on specific nature of EM-fields and their b.c.
   a) Intensities of reflected and refracted radiation
   b) Phase changes and polarization

The coordinate system and symbols appropriate to the problem are shown in ff. figure:



The media below and above $z=0$ have permeabilities and permittivities μ, ϵ and μ', ϵ', respectively. The indices of refraction defined by Eq. in Sec 4.1 are

$$n = \sqrt{\frac{\mu\epsilon}{\mu_0\epsilon_0}}, \qquad n' = \sqrt{\frac{\mu'\epsilon'}{\mu_0\epsilon_0}} \qquad (2)$$

A plane wave with wave vector $\vec{k}$ and frequency ω is incident from medium μ, ϵ. Acc. to Eqs. (1) and (2) in Sec 7.2, the three waves are:

Incident: $$\vec{E} = \vec{E}_0 e^{i\vec{k}\cdot\vec{x} - i\omega t} \qquad \vec{B} = \sqrt{\mu\epsilon}\,\frac{\vec{k}\times\vec{E}}{k} \qquad (3)$$

Refracted: $$\vec{E}' = \vec{E}_0' e^{i\vec{k}'\cdot\vec{x} - i\omega t} \qquad \vec{B}' = \sqrt{\mu'\epsilon'}\,\frac{\vec{k}'\times\vec{E}'}{k'} \qquad (4)$$

Reflected: $$\vec{E}'' = \vec{E}_0'' e^{i\vec{k}''\cdot\vec{x} - i\omega t} \qquad \vec{B}'' = \sqrt{\mu\epsilon}\,\frac{\vec{k}''\times\vec{E}''}{k''} \qquad (5)$$

The wave numbers have magnitudes

$$|\vec{k}| = |\vec{k}''| = k = \omega\sqrt{\mu\epsilon} \qquad (6)$$

$$|\vec{k}'| = k' = \omega\sqrt{\mu'\epsilon'} \qquad (7)$$

The existence of b.c. at $z=0$, which must be satisfied at all points on the plane at all times, implies that the spatial (and time) variation of all fields must be the same at $z=0$. Consequently, we must have the phase factors all equal at $z=0$:

$$(\vec{k}\cdot\vec{x})_{z=0} = (\vec{k}'\cdot\vec{x})_{z=0} = (\vec{k}''\cdot\vec{x})_{z=0} \qquad (8)$$

indep. of the nature of the boundary conditions. Eq. (8) contains the kinematic aspects of reflection and refraction. We see immediately that all three wave vectors must lie in a plane. Furthermore, in the notation of the figure earlier

$$k\sin i = k'\sin r = k''\sin r' \qquad (9)$$

Since $k'' = k$, we find $i = r'$ which is just a) of the kinematic properties. As for Snell's law, Eq. (1) gives the end result. To expand on it, we have

$$\frac{\sin i}{\sin r} = \frac{k'}{k} = \sqrt{\frac{\mu'\epsilon'}{\mu\epsilon}} = \frac{n'}{n} \qquad (10)$$

by applying Eq. (9).

The dynamic properties are contained in the b.c.:

1. normal components of $\vec{D}$ are continuous

$$(\vec{D}_1 - \vec{D}_2)\cdot\hat{n} = 0$$

$$(\vec{D} + \vec{D}'' - \vec{D}')\cdot\hat{n} = 0$$

$$(\epsilon\vec{E} + \epsilon\vec{E}'' - \epsilon'\vec{E}')\cdot\hat{n} = 0$$

The exponential terms cancel out since the phase factors are all equal at $z=0$ (Eq. 8)

$$\left[\epsilon\left(\vec{E}_0 e^{i\vec{k}\cdot\vec{x} - i\omega t} + \vec{E}_0'' e^{i\vec{k}''\cdot\vec{x} - i\omega t}\right) - \epsilon'\vec{E}_0 e^{i\vec{k}'\cdot\vec{x} - i\omega t}\right]\cdot\hat{n} = 0$$

$$\left[\epsilon(\vec{E}_0 + \vec{E}_0'') - \epsilon'\vec{E}_0'\right]\cdot\hat{n} = 0 \qquad (11)$$

2. normal components of $\vec{B}$ are continuous

$$(\vec{B}_1 - \vec{B}_2)\cdot\hat{n} = 0$$

$$\vec{B} + \vec{B}'' - \vec{B}'\cdot\hat{n} = 0$$

$$\left[\frac{\sqrt{\mu\epsilon}}{k}(\vec{k}\times\vec{E}) + \frac{\sqrt{\mu\epsilon}}{k''}(\vec{k}''\times\vec{E}'') - \frac{\sqrt{\mu'\epsilon'}}{k'}(\vec{k}'\times\vec{E}')\right]\cdot\hat{n}.$$

$$\omega\left(\vec{k}\times\vec{E}_0 e^{i\vec{k}\cdot\vec{x} - i\omega t}\right) + \omega\left(\vec{k}''\times\vec{E}_0'' e^{i\vec{k}''\cdot\vec{x} - i\omega t}\right) - \omega\left(\vec{k}'\times\vec{E}_0' e^{i\vec{k}'\cdot\vec{x} - i\omega t}\right)\cdot\hat{n} = 0$$

$$\left[\vec{k}\times\vec{E}_0 + \vec{k}''\times\vec{E}_0 - \vec{k}'\times\vec{E}_0'\right]\cdot\hat{n} = 0 \qquad (12)$$

3. tangential components of $\vec{E}$ are continuous

$$(\vec{E}_1 - \vec{E}_2)\times\hat{n} = 0$$

$$(\vec{E} + \vec{E}'' - \vec{E}')\times\hat{n} = 0$$

$$\left(\vec{E}_0 e^{i\vec{k}\cdot\vec{x} - i\omega t} + \vec{E}_0'' e^{i\vec{k}''\cdot\vec{x} - i\omega t} - \vec{E}_0' e^{i\vec{k}'\cdot\vec{x} - i\omega t}\right)\times\hat{n} = 0$$

$$(\vec{E}_0 + \vec{E}_0'' - \vec{E}_0')\times\hat{n} = 0 \qquad (13)$$

4. tangential components of $\vec{H}$ are continuous

$$(\vec{H}_1 - \vec{H}_2)\times\hat{n} = 0$$

$$\left(\frac{1}{\mu}\vec{B}_1 - \frac{1}{\mu'}\vec{B}_2\right)\times\hat{n} = 0$$

$$\left[\frac{1}{\mu}(\vec{B} + \vec{B}'') - \frac{1}{\mu'}\vec{B}'\right]\times\hat{n} = 0$$

$$\left\{\frac{1}{\mu}\left[\omega\left(\vec{k}\times\vec{E}_0 e^{i\vec{k}\cdot\vec{x} - i\omega t}\right) + \omega\left(\vec{k}''\times\vec{E}_0'' e^{i\vec{k}''\cdot\vec{x} - i\omega t}\right)\right] - \frac{1}{\mu}\omega\left(\vec{k}'\times\vec{E}_0' e^{i\vec{k}'\cdot\vec{x} - i\omega t}\right)\right\}\times\hat{n} = 0$$

$$\left[\frac{1}{\mu}\left(\vec{k}\times\vec{E}_0+\vec{k}''\times\vec{E}_0''\right)-\frac{1}{\mu'}\left(\vec{k}'\times\vec{E}_0'\right)\right]\times\hat{n}=0 \qquad (14)$$

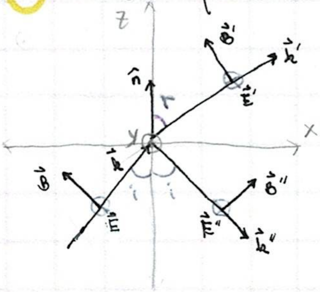
In applying these b.c., it is convenient to consider two separate situations.

1. Incident plane wave linearly polarized w/ polarization vector $\perp$ to plane of incidence

$\vec{E}_0=E_0(-\hat{j})$

$\vec{E}_0'=E_0'(-\hat{j})$

$\vec{E}_0''=E_0''(-\hat{j})$



All the electric fields are shown directed away from the viewer. The orientations of the $\vec{B}$ vectors are chosen to give a positive flow of energy in the direction of the wave vectors. Since the electric fields are ∥ to the surface, Eq. (11) gives no new information. As for the third and fourth conditions in Eqs. (13) and (14), we have

$$(\vec{E}_0+\vec{E}_0''-\vec{E}_0')\times\hat{n}=\left[E_0(\hat{j})+E_0''(\hat{j})-E_0'(\hat{j})\right]\times\cancel{\hat{n}}\hat{k}=-(E_0+E_0''-E_0')\,\hat{\imath}=0$$

$$\hookrightarrow\quad E_0+E_0''-E_0'=0 \qquad (15)$$

$\vec{a}\times\vec{b}\times\vec{c}=(\vec{a}\cdot\vec{c})\vec{b}-(\vec{a}\cdot\vec{b})\vec{c}$

and

$$\frac{1}{\mu}\left(\vec{k}\times\vec{E}_0\times\hat{n}+\vec{k}''\times\vec{E}_0''\times\hat{n}\right)-\frac{1}{\mu'}\vec{k}'\times\vec{E}_0'\times\hat{n}=0$$

The angle bet. the wave vectors k's and electric field E_0's is 90° as shown in the figure. Thus, their dot product is zero

$-\hat{n}$ (see last page)

$$\frac{1}{\mu}\left[(\vec{k}\cdot\hat{n})\vec{E}_0-\cancelto{0}{(\vec{k}\cdot\vec{E}_0)}\hat{n}+(\vec{k}''\cdot\hat{n})\vec{E}_0''-\cancelto{0}{(\vec{k}''\cdot\vec{E}_0'')}\hat{n}\right]-\frac{1}{\mu'}\left[(\vec{k}'\cdot\hat{n})\vec{E}_0'-\cancelto{0}{(\vec{k}'\cdot\vec{E}_0')}\hat{n}\right]=0$$

$$\frac{1}{\mu}\left[|\vec{k}||\hat{n}|\cos i\,E_0(-\hat{j})-|\vec{k}''||\hat{n}|\cos i\,E_0''(-\hat{j})\right]-\frac{1}{\mu'}|\vec{k}'||\hat{n}|\cos r\,E_0'(-\hat{j})=0$$

$$\left[\frac{1}{\mu}\left(kE_0\cos i-k''E_0''\cos i\right)-\frac{1}{\mu'}k'E_0'\cos r\right](-\hat{j})=0$$

Apply Eqs. (6) and (7)

$$\left[\frac{\omega\sqrt{\mu\epsilon}}{\mu}\left(E_0\cos i-E_0''\cos i\right)-\frac{\omega\sqrt{\mu'\epsilon'}}{\mu'}E_0'\cos r\right](-\hat{j})=0$$

$$\hookrightarrow\quad\sqrt{\frac{\epsilon}{\mu}}\,(E_0-E_0'')\cos i-\sqrt{\frac{\epsilon'}{\mu'}}\,E_0'\cos r=0 \qquad (16)$$

As for the second condition in Eq. (12),

$$\hat{n}\cdot(\vec{k}\times\vec{E}_0)+\hat{n}\cdot(\vec{k}''\times\vec{E}_0'')-\hat{n}\cdot(\vec{k}\times\vec{E}_0)=0$$

$$\vec{E}_0\cdot(\hat{n}\times\vec{k})+\vec{E}_0''\cdot(\hat{n}\times\vec{k}'')-\vec{E}_0'\cdot(\hat{n}\times\vec{k}')=0$$

$$E_0(-\hat{j})\cdot|\hat{n}||\vec{k}|\sin i\,(-\hat{j})+E_0''(-\hat{j})\cdot|\hat{n}||\vec{k}''|\sin i\,(-\hat{j})-E_0'(-\hat{j})\cdot|\hat{n}||\vec{k}'|\sin r\,(-\hat{j})=0$$

$$kE_0\sin i\,\cancelto{1}{(\hat{j}\cdot\hat{j})}+k''E_0''\sin i\,\cancelto{1}{(\hat{j}\cdot\hat{j})}-k'E_0'\sin r\,\cancelto{1}{(\hat{j}\cdot\hat{j})}=0$$

But due to Snell's law, we know that (applying Eq. (9)) this leads to

$$E_0+E_0''-E_0'=0 \qquad (17)$$

which just duplicates Eq. (15). Now, we can find out that

$$n'^2=n'^2\cos^2 r+n'^2\sin^2 r=n'^2\cos^2 r+n^2\sin^2 i \qquad (18)$$

with Snell's law. We can arrange this to get

$$n'\cos r=\sqrt{n'^2-n^2\sin^2 i} \qquad (19)$$

Then, rearranging Eq. (15) into

$$E_0' = E_0 + E_0'' \tag{20}$$

and substituting this into Eq. (16), we obtain

$$\sqrt{\frac{\epsilon}{\mu}}\, E_0 \cos i - \sqrt{\frac{\epsilon}{\mu}}\, E_0'' \cos i - \sqrt{\frac{\epsilon'}{\mu'}}\, E_0 \cos r - \sqrt{\frac{\epsilon'}{\mu'}}\, E_0'' \cos r = 0$$

$$E_0\left(\sqrt{\frac{\epsilon}{\mu}} \cos i - \sqrt{\frac{\epsilon}{\mu}} \cos r\right) - \left(\sqrt{\frac{\epsilon'}{\mu'}} \cos i + \sqrt{\frac{\epsilon'}{\mu'}} \cos r\right) E_0'' = 0$$

$$\frac{E_0''}{E_0} = \frac{\sqrt{\frac{\epsilon}{\mu}} \cos i - \sqrt{\frac{\epsilon'}{\mu'}} \cos r}{\sqrt{\frac{\epsilon}{\mu}} \cos i + \sqrt{\frac{\epsilon'}{\mu'}} \cos r} \cdot \frac{\frac{\mu}{\sqrt{\mu_0 \epsilon_0}}}{\frac{\mu}{\sqrt{\mu_0 \epsilon_0}}}$$

$$= \frac{\sqrt{\frac{\mu\epsilon}{\mu_0\epsilon_0}} \cos i - \frac{\mu}{\mu'}\sqrt{\frac{\mu'\epsilon'}{\mu_0\epsilon_0}} \cos r}{\sqrt{\frac{\mu\epsilon}{\mu_0\epsilon_0}} \cos i + \frac{\mu}{\mu'}\sqrt{\frac{\mu'\epsilon'}{\mu_0\epsilon_0}} \cos r}$$

$$= \frac{n \cos i - \frac{\mu}{\mu'} n' \cos r}{n \cos i + \frac{\mu}{\mu'} n' \cos r}$$

$$\frac{E_0''}{E_0} = \frac{n \cos i - (\mu/\mu')\sqrt{n'^2 - n^2 \sin^2 i}}{n \cos i + (\mu/\mu')\sqrt{n'^2 - n^2 \sin^2 i}} \tag{21}$$

On the other hand, rearranging Eq. (15) into

$$E_0'' = E_0' - E_0 \tag{22}$$

and substituting this into Eq. (16), we obtain

$$\sqrt{\frac{\epsilon}{\mu}}\, E_0 \cos i - \sqrt{\frac{\epsilon}{\mu}}\, E_0' \cos i + \sqrt{\frac{\epsilon}{\mu}}\, E_0 \cos i - \sqrt{\frac{\epsilon'}{\mu'}}\, E_0' \cos r = 0$$

$$2\sqrt{\frac{\epsilon}{\mu}}\, E_0 \cos i - E_0'\left(\sqrt{\frac{\epsilon}{\mu}} \cos i + \sqrt{\frac{\epsilon'}{\mu'}} \cos r\right) = 0$$

$$\frac{E_0'}{E_0} = \frac{2\sqrt{\frac{\epsilon}{\mu}} \cos i}{\sqrt{\frac{\epsilon}{\mu}} \cos i + \sqrt{\frac{\epsilon'}{\mu'}} \cos r} \cdot \frac{\frac{\mu}{\sqrt{\mu_0 \epsilon_0}}}{\frac{\mu}{\sqrt{\mu_0 \epsilon_0}}}$$

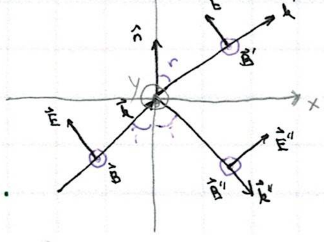
$$= \frac{2\sqrt{\frac{\mu\epsilon}{\mu_0\epsilon_0}} \cos i}{\sqrt{\frac{\mu\epsilon}{\mu_0\epsilon_0}} \cos i + \frac{\mu}{\mu'}\sqrt{\frac{\mu'\epsilon'}{\mu_0\epsilon_0}} \cos r}$$

$$= \frac{2n \cos i}{n \cos i + (\mu/\mu')\, n' \cos r}$$

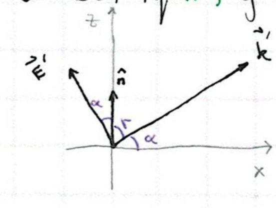
$$\frac{E_0'}{E_0} = \frac{2n \cos i}{n \cos i + (\mu/\mu')\sqrt{n'^2 - n^2 \sin^2 i}} \tag{23}$$

For optical frequencies, it is permitted that $\mu/\mu' = 1$

2 Incident plane wave linearly polarized w/ polarization vector $\parallel$ to plane of incidence



For this case, Eq. (12) gives no new information. Note that



◂ $\sin\alpha' = \cos r$

In the same manner,

$\sin\alpha = \sin\alpha'' = \cos i$

and vice versa ($\cos \to \sin$)

Then, the third condition in Eq. (13) gives

$$\vec{E}_0 \times \hat{n} + \vec{E}_0'' \times \hat{n} - \vec{E}_0' \times \hat{n} = 0$$

$$|\vec{E}_0||\hat{n}| \sin\alpha\,(-\hat{j}) + |\vec{E}_0''||\hat{n}| \sin\alpha''\,(+\hat{j}) - |\vec{E}_0'|\,|\hat{n}| \sin\alpha'\,(-\hat{j}) = 0$$

$$E_0 \cos i \;\hat{j} - E_0'' \cos i \;\hat{j} - E_0' \cos r\, \hat{j} = 0$$

$$\left[(E_0 - E_0'') \cos i - E_0' \cos r\right] \hat{j} = 0$$

$$\hookrightarrow \quad (E_0 - E_0'') \cos i - E_0' \cos r = 0 \qquad (24)$$

As for the fourth condition in Eq. (14), it leads to

$$\left[\frac{1}{\mu}\left(|\vec{k}||\vec{E}_0| \sin\frac{\pi}{2}\;\hat{j} + |\vec{k}''||\vec{E}_0''| \sin\frac{\pi}{2}\;\hat{j}\right) - \frac{1}{\mu'} |\vec{k}'||\vec{E}_0'| \sin\frac{\pi}{2}\;\hat{j}\right] \times \hat{n} = 0$$

$$\left[\frac{1}{\mu}\left(k E_0 + k'' E_0''\right) - \frac{1}{\mu'} k' E_0'\right] \hat{j} \cancel{\times (n \hat{k})}^{\,1} = 0$$

$$\frac{\omega\sqrt{\mu\epsilon}}{\mu} (E_0 + E_0'') - \frac{\omega\sqrt{\mu'\epsilon'}}{\mu'} E_0' = 0$$

$$\sqrt{\frac{\epsilon}{\mu}} (E_0 + E_0'') - \sqrt{\frac{\epsilon'}{\mu'}} E_0' = 0 \qquad (25)$$

On the other hand, the first condition in Eq. (11) just leads to

$$\epsilon\left(\vec{E}_0 \cdot \hat{n} + \vec{E}_0'' \cdot \hat{n}\right) - \epsilon' \vec{E}_0' \cdot n = 0$$

$$\epsilon\left(|\vec{E}_0||\hat{n}| \cos\alpha + |\vec{E}_0''||\hat{n}| \cos\alpha''\right) - \epsilon' |\vec{E}_0'||\hat{n}| \cos\alpha' = 0$$

$$\epsilon(E_0 \sin i + E_0'' \sin i) - \epsilon' E_0' \sin r = 0$$

$$\epsilon\left(E_0 \frac{k'}{k} \sin r + E_0'' \frac{k'}{k} \sin r\right) - \epsilon' E_0' \frac{k}{k'} \sin i = 0$$

$$\frac{\epsilon}{\omega\sqrt{\mu\epsilon}} (E_0 + E_0'') k' \sin r - \frac{\epsilon'}{\omega\sqrt{\mu'\epsilon'}} E_0' k \sin i = 0$$

$$\sqrt{\frac{\epsilon}{\mu}} (E_0 + E_0'') - \sqrt{\frac{\epsilon'}{\mu'}} E_0' = 0 \qquad (26)$$

which is just the duplicate of the expression in Eq. (25). We can rearrange Eq. (25) into

$$E_0' = \sqrt{\frac{\mu'\epsilon}{\epsilon'\mu}} (E_0 + E_0'')$$

Then, substituting this into Eq. (24), we have

$$E_0 \cos i - E_0'' \cos i - \sqrt{\frac{\mu'}{\epsilon'}}\sqrt{\frac{\epsilon}{\mu}} E_0 \cos r - \sqrt{\frac{\mu'}{\epsilon'}}\sqrt{\frac{\epsilon}{\mu}} E_0'' \cos r = 0$$

$$E_0\left(\cos i - \sqrt{\frac{\mu'}{\epsilon'}}\sqrt{\frac{\epsilon}{\mu}} \cos r\right) - E_0''\left(\cos i + \sqrt{\frac{\mu'}{\epsilon'}}\sqrt{\frac{\epsilon}{\mu}} \cos r\right) = 0$$

$$\frac{E_0''}{E_0} = \frac{\cos i - \sqrt{\frac{\mu'}{\epsilon'}}\sqrt{\frac{\epsilon}{\mu}} \cos r}{\cos i + \sqrt{\frac{\mu'}{\epsilon'}}\sqrt{\frac{\epsilon}{\mu}} \cos r} \cdot \frac{\frac{\mu}{\mu'}\frac{\mu'\epsilon'}{\mu_0\epsilon_0}}{\frac{\mu}{\mu'}\frac{\mu'\epsilon'}{\mu_0\epsilon_0}} = \frac{\frac{\mu}{\mu'} n'^2 \cos i - \sqrt{\frac{\mu\epsilon}{\mu'\epsilon'}}\frac{\mu'\epsilon'}{\mu_0\epsilon_0} \cos r}{\frac{\mu}{\mu'} n'^2 \cos i + \sqrt{\frac{\mu\epsilon}{\mu'\epsilon'}}\frac{\mu'\epsilon'}{\mu_0\epsilon_0} \cos r} \qquad (27)$$

Simplifying this further,

$$\frac{E_0''}{E_0} = \frac{\frac{\mu}{\mu'} n'^2 \cos i - \frac{n}{n'} n'^2 \cos r}{\frac{\mu}{\mu'} n'^2 \cos i + \frac{n}{n'} n'^2 \cos r}$$

$$= \frac{\frac{\mu}{\mu'} n'^2 \cos i - n\sqrt{n'^2 - n^2 \sin^2 i}}{\frac{\mu}{\mu'} n'^2 \cos i + n\sqrt{n'^2 - n^2 \sin^2 i}} \qquad (28)$$

after applying Eq. (19). On the other hand, rearranging Eq. (26) into

$$E_0'' = \sqrt{\frac{\epsilon'}{\mu'}}\sqrt{\frac{\mu}{\epsilon}}\, E_0' - E_0 \qquad (29)$$

Substituting this into Eq (25),

$$E_0 \cos i - \sqrt{\frac{\epsilon'}{\mu'}}\sqrt{\frac{\mu}{\epsilon}}\, E_0' \cos i + E_0 \cos i - E_0' \cos r = 0$$

$$2 E_0 \cos i - E_0' \left(\sqrt{\frac{\epsilon'}{\mu'}}\sqrt{\frac{\mu}{\epsilon}} \cos i + \cos r \right) = 0$$

$$\frac{E_0'}{E_0} = \frac{2 \cos i}{\sqrt{\frac{\epsilon'}{\mu'}}\sqrt{\frac{\mu}{\epsilon}} \cos i + \cos r} \cdot \frac{\sqrt{\frac{\mu\epsilon}{\mu_0\epsilon_0}}\sqrt{\frac{\mu'\epsilon'}{\mu_0\epsilon_0}}}{\sqrt{\frac{\mu\epsilon}{\mu_0\epsilon_0}}\sqrt{\frac{\mu'\epsilon'}{\mu_0\epsilon_0}}} = \frac{2 n n' \cos i}{\frac{\mu}{\mu'}\frac{\mu'\epsilon'}{\mu_0\epsilon_0} \cos i + n n' \cos r} \qquad (30)$$

Applying Eq. (19), we get

$$\frac{E_0'}{E_0} = \frac{2 n n' \cos i}{\frac{\mu}{\mu'} n'^2 \cos i + n\sqrt{n'^2 - n^2 \sin^2 i}} \qquad (31)$$

- Eqs. (23) and (31) are the relative amplitudes of the refracted fields for both cases while Eqs. (21) and (28) are the relative amplitudes of the reflected fields.
- For normal incidence ($i = 0$), both Eqs. (21) and (28) lead to

$$(21) \rightarrow \quad \frac{E_0''}{E_0} = \frac{n - (\mu/\mu')\, n'}{n + (\mu/\mu')\, n'} \longrightarrow -\frac{n' - n}{n' + n}$$

$$(28) \rightarrow \quad = \frac{(\mu/\mu')\, n'^2 - n n'}{(\mu/\mu')\, n'^2 + n n'} \longrightarrow \frac{n' - n}{n' + n} \quad \text{sign convention} \qquad (32)$$

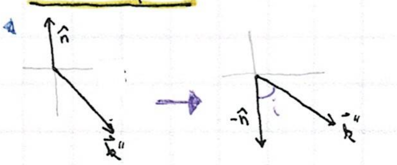
while both Eqs. (23) and (31) lead to

$$(23) \rightarrow \quad \frac{E_0'}{E_0} = \frac{2n}{n + (\mu/\mu')\, n'} \longrightarrow \frac{2n}{n' + n}$$

$$(31) \rightarrow \quad = \frac{2 n n'}{(\mu/\mu')\, n'^2 + n n'} \longrightarrow \frac{2n}{n' + n} \qquad (33)$$

if $\mu' = \mu$. For the reflected wave, the sign convention is that for polarization parallel to the plane of incidence (if $n' > n$, there is a phase reversal for the reflected wave)

For page ⑤



- To make use of the given angle i, $\hat{n}$ should be re-oriented to the opposite direction. Thus, $\hat{n} \rightarrow -\hat{n}$ in the cross product of $\vec{k}''$ and $\hat{n}$ in the equations leading to Eq. (16)