Problem 1 (1.2)

Consider the system in Figure 1 in the case in which all the masses are equal($m_1 = m_2 = m_3 = m$) and the system is released from rest with $x_2 = 0$ and $x_3 = l$.

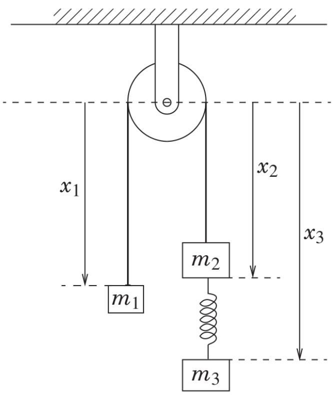

Figure 1: A diagram of the problem (Source: Analytic Mechanics by N. Lemos)

(a) Determine the equations of motion.

Solution:
In this scenario, we will assume that the pulley and the string that connects masses m_1 and m_2 have negligible mass and that the string is inextensible. Also, we assume that the spring constant is k. Looking at Example 1.12 in [NL], we can infer that the Atwood's machine in the system imposes a constraint given by

$$x_1 + x_2 = l_0 \tag{1}$$

where the constant l_0 is dependent on the radius of the pulley and the length of the string. Because of this constraint, this system only has 2 degrees of freedom so we have 2 independent variables which we choose to be x_2 and x_3 (considering that we were given initial conditions for x_2 and x_3, this choice of variables should be better than using x_1 with x_3). Then, we calculate for the kinetic energy of the system given by

$$\begin{aligned} T &= \frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}m_2\dot{x}_2^2 + \frac{1}{2}m_3\dot{x}_3^2 \\ &= \frac{1}{2}m(-\dot{x}_2)^2 + \frac{1}{2}m\dot{x_2}^2 + \frac{1}{2}m\dot{x_3}^2 \\ T &= m\dot{x}_2^2 + \frac{1}{2}m\dot{x}_3^2 \end{aligned} \tag{2}$$

since the masses are equal and $x_1 = l_0 - x_2$ from Eq. (1) which leads to $\dot{x}_1 = -\dot{x}_2$.

As for the potential energy, we set the zero level of the gravitational potential at the center of the pulley. Also, recall that a spring's potential energy is $V_{\text{spring}} = (1/2)kx$ where x is the displacement of the spring. From the fact that $x_2(0) = 0$ and $x_3(0) = l$, we know that $x_3 - x_2 = l$ when the spring is at its initial position. This relation changes once the spring is displaced by x in which we obtain $x_3 - x_2 = l + x$. Thus, the displacement of the spring in this system is given by $x = x_3 - x_2 - l$. Therefore, we have

$$\begin{aligned} V &= V_{\text{grav}} + V_{\text{spring}} \\ &= -m_1 g x_1 - m_2 g x_2 - m_3 g x_3 + \frac{1}{2}k\left(x_3 - x_2 - l\right)^2 \\ &= -mg(l_0 - x_2) - mgx_2 - mgx_3 + \frac{1}{2}k\left(x_3 - x_2 - l\right)^2 \\ V &= -mgl_0 - mgx_3 + \frac{1}{2}k\left(x_3 - x_2 - l\right)^2 \end{aligned} \tag{3}$$

By substituting in Eqs. (2) and (3), the Lagrangian is given by

$$L = T - V = m\dot{x}_2^2 + \frac{1}{2}m\dot{x}_3^2 + mgx_3 - \frac{1}{2}k\left(x_3 - x_2 - l\right)^2 + mgl_0 \tag{4}$$

Now, Lagrange's equations are calculated using the formula

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \frac{\partial L}{\partial q_k} = 0, \quad k = 1, \ldots, n \tag{5}$$

where the generalised coordinates q_k for this system are $q_1 = x_2$ and $q_2 = x_3$. We solve the following partial derivatives:

$$\frac{\partial L}{\partial \dot{x}_2} = 2m\dot{x}_2 \tag{6}$$

$$\frac{\partial L}{\partial x_2} = k\left(x_3 - x_2 - l\right) \tag{7}$$

$$\frac{\partial L}{\partial \dot{x}_3} = m\dot{x}_3 \tag{8}$$

$$\frac{\partial L}{\partial x_3} = mg - k\left(x_3 - x_2 - l\right) \tag{9}$$

and substitute them into Eq. (5) to obtain

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{dt}}\left(\frac{\partial L}{\partial \dot{x}_2}\right) - \frac{\partial L}{\partial x_2} &= \frac{\mathrm{d}}{\mathrm{dt}}\left(2m\dot{x}_2\right) - k\left(x_3 - x_2 - l\right) \\ 0 &= 2m\ddot{x}_2 - k\left(x_3 - x_2 - l\right) \end{aligned} \tag{10}$$

and

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{dt}}\left(\frac{\partial L}{\partial \dot{x}_3}\right) - \frac{\partial L}{\partial x_3} &= \frac{\mathrm{d}}{\mathrm{dt}}\left(m\dot{x}_3\right) - mg + k\left(x_3 - x_2 - l\right) \\ 0 &= m\ddot{x}_3 - mg + k\left(x_3 - x_2 - l\right) \end{aligned} \tag{11}$$

Therefore, the equations of motion for this system are

$$\boxed{\ddot{x}_2 = k\left(x_3 - x_2 - l\right)} \tag{12}$$

$$\boxed{\ddot{x}_3 = g - \frac{k}{m}\left(x_3 - x_2 - l\right)} \tag{13}$$

(b) Solve the equations of motion to show that

$$x_2(t) = \frac{2mg}{9k}(\cos\omega t - 1) + \frac{1}{6}gt^2, \quad \omega = \sqrt{\frac{3k}{2m}} \tag{14}$$

Prove that $\dot{x}_2 > 0$ for all $t > 0$ and conclude that the string always remains taut.

Solution:
Recall that $x_2(0) = 0$, $x_3(0) = l$ and $\dot{x}_2(0) = 0 = \dot{x}_3(0)$ (since the system is released from rest). Using Mathematica to solve Eqs. (12) and (13) with these initial conditions as we see in Figure 2, we show that Eq. (14) results from solving the said equations of motion. Note that $\omega > 0$. ■

Taking the time derivative of $x_2(t)$, we get

$$\dot{x}_2(t) = -\frac{2mg}{9k}\omega\sin\omega t + \frac{1}{3}gt = -\frac{g}{3\omega^{\cancel{2}}}\cancel{\omega}\sin\omega t + \frac{1}{3}gt = \frac{g}{3\omega}(\omega t - \sin\omega t) \tag{15}$$

The only way that $\dot{x}_2(t)$ becomes negative is when $\sin\omega t$ becomes bigger than ωt. However, we see in Figure 3 where we plot $\sin(x)$ and x that this does not happen. Thus, we can say that $\dot{x}_2 > 0$ for all $t > 0$. Because of this, oscillations due to the spring does not affect the direction of $\dot{x}_2$. Thus, the string (assumed to be inextensible) remains taut for all $t > 0$.

```
solns = DSolve[{-k (-l - x2[t] + x3[t]) + 2 m x2''[t] == 0, -g m + k (-l - x2[t] + x3[t]) + m x3''[t] == 0, x2[0] == 0,
   x3[0] == l, x2'[0] == 0, x3'[0] == 0}, {x2[t], x3[t]}, t]
```

$$\left\{\left\{x2[t] \to \frac{e^{-\frac{i\sqrt{\frac{3}{2}}\sqrt{k}\,t}{\sqrt{m}}}\, g \left(2m - 4 e^{\frac{i\sqrt{\frac{3}{2}}\sqrt{k}\,t}{\sqrt{m}}} m + 2 e^{\frac{i\sqrt{6}\sqrt{k}\,t}{\sqrt{m}}} m + 3 e^{\frac{i\sqrt{\frac{3}{2}}\sqrt{k}\,t}{\sqrt{m}}} k\, t^2\right)}{18\,k},\right.\right.$$

$$\left.\left. x3[t] \to -\frac{1}{18\,k} e^{-\frac{i\sqrt{\frac{3}{2}}\sqrt{k}\,t}{\sqrt{m}}} \left(-18 e^{\frac{i\sqrt{\frac{3}{2}}\sqrt{k}\,t}{\sqrt{m}}} k\, l + 4 g m - 8 e^{\frac{i\sqrt{\frac{3}{2}}\sqrt{k}\,t}{\sqrt{m}}} g m + 4 e^{\frac{i\sqrt{6}\sqrt{k}\,t}{\sqrt{m}}} g m - 3 e^{\frac{i\sqrt{\frac{3}{2}}\sqrt{k}\,t}{\sqrt{m}}} g k\, t^2\right)\right\}\right\}$$

```
x2[t] == (x2[t] /. solns[[1]] // ExpToTrig // Simplify // Expand) /. (Sqrt[3/2] Sqrt[k] t)/Sqrt[m] → ω t
```

$$x2[t] == -\frac{2\,g\,m}{9\,k} + \frac{g\,t^2}{6} + \frac{2\,g\,m\,\mathrm{Cos}[t\,\omega]}{9\,k}$$

Figure 2

```
Plot[{x, Sin[x]}, {x, 0, 5}, PlotTheme → "Scientific",
 PlotLegends → Placed[LineLegend[{Style["x", FontSize → 18], Style["sin(x)", FontSize → 18]},
   LegendFunction → (Framed[#, RoundingRadius → 5, FrameStyle → {Gray, Thin}] &), LegendMargins → 5], {After, Top}],
 FrameLabel → {Style["x", FontSize → 20, FontWeight → Bold], None}, FrameTicksStyle → 15]
```

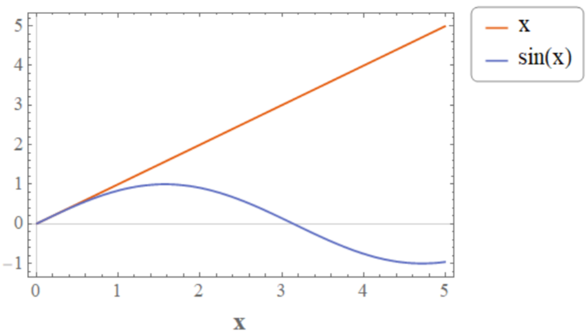

Figure 3: Plot of $\sin x$ and x

Acknowledgements: I am grateful for the insightful comments of Christian Buco and Lemuel Saret when solving this problem.

References

Lemos, N., *Analytical Mechanics*, Chapter 1

Problem 2 (1.5)

Consider the so-called swinging Atwood's machine shown in Fig. 1 in which M moves only vertically (Tufillaro, Abbott & Griffiths, 1984). Using the coordinates indicated in the figure, show that the Lagrangian is given by

$$L = \frac{m+M}{2}\dot{r}^2 + \frac{m}{2}r^2\dot{\theta}^2 - gr(M - m\cos\theta) \tag{1}$$

and write down Lagrange's equations.

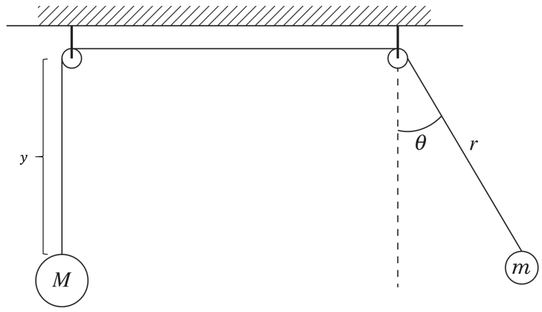

Figure 1: A diagram of the problem (Source: modified from Analytic Mechanics by N. Lemos)

Solution:
For this scenario, we first consider a single pendulum with the same variables as the swinging part in the figure. Here, we infer that $x = r\cos\theta$ which leads to $\dot{x} = \dot{r}\cos\theta - r\sin\theta$. Also, we have $\dot{y} = \dot{r}\sin\theta + r\cos\theta$ by differentiating $y = r\sin\theta$. From these, we obtain the kinetic energy of a single pendulum

$$\begin{aligned} T &= \frac{1}{2}m\dot{x}^2 + \frac{1}{2}m\dot{y}^2 \\ &= \frac{1}{2}m\left(\dot{r}^2\cos^2\theta + r^2\dot{\theta}^2\sin^2\theta \cancel{- 2\dot{r}\dot{\theta}r\cos\theta\sin\theta} + \dot{r}^2\sin^2\theta + r^2\dot{\theta}^2\cos^2\theta + \cancel{2\dot{r}\dot{\theta}r\cos\theta\sin\theta}\right) \\ T &= \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right) \end{aligned} \tag{2}$$

Now, we will treat the swinging part of the system as a single pendulum. Before that, we introduce a variable y to denote the vertical distance from the center of the first pulley to the top of mass M. We can then express the constraint on this system as

$$r + y = l_0 \tag{3}$$

where the constant l_0 is dependent on the radius of the pulley and the length of the string. Using Eq. (2) and substituting in Eq. (3), we have

$$\begin{aligned} T &= \frac{1}{2}M\dot{y}^2 + \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right) \\ &= \frac{1}{2}M(-\dot{r})^2 + \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right) \\ T &= \frac{1}{2}M\dot{r}^2 + \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right) \end{aligned} \tag{4}$$

As for the potential energy, we set the zero level of the gravitational potential at the center of the pulleys. Then, we get

$$
\begin{aligned}
V &= V_{\text{grav}} \\
&= -Mg(l_0 - r) - mgr\cos\theta \\
V &= Mgr - mgr\cos\theta - Mgl_0
\end{aligned} \tag{5}
$$

Thus, the Lagrangian calculated using Eqs. (4) and (5) is

$$
\begin{aligned}
L &= T - V \\
&= \frac{1}{2}M\dot{r}^2 + \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\dot{\theta}^2 - Mgr + mgr\cos\theta + Mgl_0 \\
L &= \frac{1}{2}(m+M)\dot{r}^2 + \frac{1}{2}mr^2\dot{\theta}^2 - gr(M - m\cos\theta) + Mgl_0
\end{aligned} \tag{6}
$$

which shows that the Lagrangian of the swinging Atwood's machine is Eq. (1) up to a constant Mgl_0. ∎

We can drop this constant term since it has no effect in the calculation of Lagrange's equations using the formula

$$
\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \frac{\partial L}{\partial q_k} = 0, \quad k = 1, \ldots, n \tag{7}
$$

where the generalised coordinates q_k for this system are $q_1 = r$ and $q_2 = \theta$. Solving for the following partial derivatives:

$$
\frac{\partial L}{\partial \dot{r}} = (m+M)\dot{r} \tag{8}
$$

$$
\frac{\partial L}{\partial r} = mr\dot{\theta}^2 - g(M - m\cos\theta) \tag{9}
$$

$$
\frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta} \tag{10}
$$

$$
\frac{\partial L}{\partial \theta} = mgr\sin\theta \tag{11}
$$

We then obtain

$$
\frac{\mathrm{d}}{\mathrm{dt}}\left(\frac{\partial L}{\partial \dot{r}}\right) - \frac{\partial L}{\partial r} = \frac{\mathrm{d}}{\mathrm{dt}}\left((m+M)\dot{r}\right) - k\left(x_3 - x_2 - l\right) \tag{12}
$$

$$
\boxed{0 = (m+M)\ddot{r} - k\left(x_3 - x_2 - l\right)} \tag{13}
$$

and

$$
\frac{\mathrm{d}}{\mathrm{dt}}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = \frac{\mathrm{d}}{\mathrm{dt}}\left(mr^2\dot{\theta}\right) - mgr\sin\theta \tag{14}
$$

$$
\boxed{0 = mr^2\ddot{\theta} - mgr\sin\theta} \tag{15}
$$

as our Lagrange's equations.

References

Swinging Atwood's machine, https://www.wikiwand.com/en/Swinging_Atwood%27s_machine

Problem 3 (1.7)

A projectile is fired near the surface of the Earth. Assuming the force of air resistance is proportional to the velocity, obtain the projectile's equation of motion using the dissipation function $\mathcal{F} = \lambda v^2/2$.

Solution:
Let the projectile has a mass of m. The kinetic energy of the projectile is given by

$$T = \frac{1}{2}mv^2 = \frac{1}{2}m\left(\sqrt{v_x^2 + v_y^2}\right)^2 = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) \tag{1}$$

Setting the zero level of the gravitational potential energy at the ground, we obtain

$$V = mgy \tag{2}$$

for the projectile. Thus, in setting up the Lagrangian from Eqs. (1) and (2), we get

$$L = T - V = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2\right) - mgy \tag{3}$$

Lagrange's equations when there is a dissipative force in the system are calculated using the formula

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \frac{\partial L}{\partial q_k} + \frac{\partial F}{\partial \dot{q}_k} = 0, \quad k = 1, \ldots, n \tag{4}$$

where the generalised coordinates q_k for this case are $q_1 = x$ and $q_2 = y$. Solving for the following partial derivatives,

$$\frac{\partial L}{\partial \dot{x}} = m\dot{x} \tag{5}$$

$$\frac{\partial L}{\partial x} = 0 \tag{6}$$

$$\frac{\partial F}{\partial \dot{x}} = \lambda\dot{x} \tag{7}$$

$$\frac{\partial L}{\partial \dot{y}} = -m\dot{y} \tag{8}$$

$$\frac{\partial L}{\partial y} = -mg \tag{9}$$

$$\frac{\partial F}{\partial \dot{y}} = \lambda\dot{y} \tag{10}$$

we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} + \frac{\partial F}{\partial \dot{x}} = \frac{\mathrm{d}}{\mathrm{d}t}(m\dot{x}) - 0 + \lambda\dot{x} = m\ddot{x} + \lambda\dot{x} \tag{11}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{y}}\right) - \frac{\partial L}{\partial y} + \frac{\partial F}{\partial \dot{y}} = \frac{\mathrm{d}}{\mathrm{d}t}(m\dot{y}) + mg + \lambda\dot{y} = m\ddot{y} + mg + \lambda\dot{y} \tag{12}$$

Thus, the projectile's equations of motion are

$$m\ddot{x} + \lambda\dot{x} = 0 \quad \longrightarrow \quad \boxed{\ddot{x} = -\frac{\lambda}{m}\dot{x}} \tag{13}$$

$$m\ddot{y} + mg + \lambda\dot{y} = 0 \quad \longrightarrow \quad \boxed{\ddot{y} = -\left(g + \frac{\lambda}{m}\dot{y}\right)} \tag{14}$$

Problem 4 (1.8)

Certain dissipative systems admit a Lagrangian formulation that dispenses with the Rayleigh dissipation function. Consider a projectile in the constant gravitational field $\mathbf{g} = -g\hat{\mathbf{y}}$ and assume that the force of air resistance is proportional to the velocity.

(a) Show that the equations of motion generated by the Lagrangian

$$L = \exp\left(\frac{\lambda t}{m}\right)\left[\frac{m}{2}\left(\dot{x}^2 + \dot{y}^2\right) - mgy\right] \tag{1}$$

coincide with those obtained in the preceding problem.

Solution:
Note that Lagrange's equations are calculated using the formula

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \frac{\partial L}{\partial q_k} = 0, \quad k = 1, \dots, n \tag{2}$$

where the generalised coordinates q_k for this system are $q_1 = x$ and $q_2 = y$. Solving for the following partial derivatives:

$$\frac{\partial L}{\partial \dot{x}} = e^{\frac{\lambda t}{m}} \cdot m\dot{x} \tag{3}$$

$$\frac{\partial L}{\partial x} = 0 \tag{4}$$

$$\frac{\partial L}{\partial \dot{y}} = e^{\frac{\lambda t}{m}} \cdot m\dot{y} \tag{5}$$

$$\frac{\partial L}{\partial y} = e^{\frac{\lambda t}{m}} \cdot (-mg) \tag{6}$$

which we substitute into (2), we obtain

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} &= \frac{\mathrm{d}}{\mathrm{d}t}\left(e^{\frac{\lambda t}{m}} \cdot m\dot{x}\right) \\ 0 &= e^{\frac{\lambda t}{m}} \cdot m\ddot{x} + m\dot{x} \cdot \frac{\lambda}{m} e^{\frac{\lambda t}{m}} \\ 0 &= m\ddot{x} + \lambda\dot{x} \end{aligned} \tag{7}$$

and

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{y}}\right) - \frac{\partial L}{\partial y} &= \frac{\mathrm{d}}{\mathrm{d}t}\left(e^{\frac{\lambda t}{m}} \cdot m\dot{y}\right) + mg\, e^{\frac{\lambda t}{m}} \\ 0 &= e^{\frac{\lambda t}{m}} \cdot m\ddot{y} + m\dot{y} \cdot \frac{\lambda}{m} e^{\frac{\lambda t}{m}} + mg\, e^{\frac{\lambda t}{m}} \\ 0 &= m\ddot{y} + \lambda\dot{y} + mg \end{aligned} \tag{8}$$

Thus, the equations of motion for this system are

$$\boxed{\ddot{x} = -\frac{\lambda}{m}\dot{x}} \tag{9}$$

$$\boxed{\ddot{y} = -g - \frac{\lambda}{m}\dot{y}} \tag{10}$$

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2015-09948

(b) Solve the equations of motion for x and y assuming the projectile is fired from the origin with velocity of magnitude v_0 making angle θ_0 with the horizontal.

Solution:
Using Mathematica as we see in Figure 1, solving the equations of motion leads to the following solutions

$$\boxed{x(t) = \frac{m}{\lambda}\left(1 - e^{-\frac{\lambda t}{m}}\right) v_0 \cos\theta_0} \tag{11}$$

$$\boxed{y(t) = -\frac{m}{\lambda^2}\left\{g\left[m\left(1 - e^{-\frac{\lambda t}{m}}\right) - t\lambda\right] + \left(1 - e^{-\frac{\lambda t}{m}}\right) v_0 \sin\theta_0\right\}} \tag{12}$$

```
solns = DSolve[{e^(tλ/m) λ x'[t] + e^(tλ/m) m x''[t] == 0, e^(tλ/m) g m + e^(tλ/m) λ y'[t] + e^(tλ/m) m y''[t] == 0, x[0] == 0, y[0] == 0,
   x'[0] == v0 Cos[θ0], y'[0] == v0 Sin[θ0]}, {x[t], y[t]}, t]

{{x[t] → (e^(-tλ/m) (-1 + e^(tλ/m)) m v0 Cos[θ0])/λ, y[t] → -1/λ^2 e^(-tλ/m) m (g m - e^(tλ/m) g m + e^(tλ/m) g t λ + v0 λ Sin[θ0] - e^(tλ/m) v0 λ Sin[θ0])}}
```

Figure 1

(c) Eliminate time to get the equation for the trajectory of the projectile.

Solution:
Using Mathematica as we see in Figure 2, we obtain an expression for t from $x(t)$.

$$t = \frac{m}{\lambda}\ln\left(\frac{mv_0\cos\theta_0}{mv_0\cos\theta_0 - \lambda x}\right) = -\frac{m}{\lambda}\ln\left(1 - \frac{\lambda x}{mv_0\cos\theta_0}\right) \tag{13}$$

Then, we substitute this t into $y(t)$ to the equation for the trajectory of the projectile.

$$\boxed{y = \frac{m^2}{\lambda^2} g \ln\left(1 - \frac{\lambda x}{mv_0\cos\theta_0}\right) + \frac{m}{\lambda}\frac{gx}{v_0\cos\theta_0} + x\tan\theta_0} \tag{14}$$

**Acknowledgements: I am grateful for the insightful comments of Matthew Banaag and Lemuel Saret when solving this problem.*

We extract x(t) from the list,

```
x == x[t] /. solns[[1, 1]] // Simplify
```

$$x + \frac{\left(-1 + e^{-\frac{t\lambda}{m}}\right) m\, v0\, \mathrm{Cos}[\theta 0]}{\lambda} == 0$$

Then, isolating t in this equation, we get

```
Solve[%, t]
```

$$\left\{\left\{t \to \mathrm{ConditionalExpression}\left[\frac{m \left(2\, i\, \pi\, C[1] + \mathrm{Log}\left[\frac{m\, v0\, \mathrm{Cos}[\theta 0]}{-x\,\lambda + m\, v0\, \mathrm{Cos}[\theta 0]}\right]\right)}{\lambda},\ C[1] \in \mathbb{Z}\right]\right\}\right\}$$

Since we don' t need to consider other multiples of 2 π, we let C[1] = 0. Extracting y (t) from the list,

```
y[t] /. solns[[1, 2]] // Simplify
```

$$\frac{m \left(g \left(m - e^{-\frac{t\lambda}{m}} m - t\,\lambda\right) + \left(1 - e^{-\frac{t\lambda}{m}}\right) v0\, \lambda\, \mathrm{Sin}[\theta 0]\right)}{\lambda^2}$$

Rearranging t and substituting it into y,

$$y == \% \,/.\, t \to -\frac{m \left(\mathrm{Log}\left[1 - \frac{x\,\lambda}{m\, v0\, \mathrm{Cos}[\theta 0]}\right]\right)}{\lambda} \ // \ \mathrm{Simplify}$$

$$y\,\lambda == \frac{g\, m^2\, \mathrm{Log}\left[1 - \frac{x\,\lambda\, \mathrm{Sec}[\theta 0]}{m\, v0}\right]}{\lambda} + \frac{g\, m\, x\, \mathrm{Sec}[\theta 0]}{v0} + x\,\lambda\, \mathrm{Tan}[\theta 0]$$

Figure 2

Problem 5 (1.10)

In Weber's electrodynamics, the force between two charged particles in motion is directed along the line connecting them and has a magnitude

$$F = \frac{q_1 q_2}{r^2}\left[1 + \frac{r\ddot{r}}{c^2} - \frac{\dot{r}^2}{2c^2}\right] \tag{1}$$

where r denotes the distance between the particles (of charges q_1 and q_2) and c is the speed of light in vacuum. Find the generalized potential $U(r, \dot{r})$ associated with this force. Set up the Lagrangian and Lagrange's equations for a charge in the presence of another charge held fixed at the origin of the coordinate system

Solution:
The magnitude of the force expressed in Eq. (1) can be written as

$$F = q_1 q_2\left[\frac{1}{r^2} + \frac{1}{c^2}\frac{\ddot{r}}{r} - \frac{1}{2c^2}\frac{\dot{r}^2}{r^2}\right] = q_1 q_2\left[\frac{1}{r^2} + \frac{1}{2c^2}\left(\frac{2\ddot{r}}{r} - \frac{\dot{r}^2}{r^2}\right)\right] \tag{2}$$

Now, note that

$$\frac{\mathrm{d}\left(\dot{r}^2\right)}{\mathrm{d}r} = \frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{\mathrm{d}r}{\mathrm{d}t}\right)^2 = 2\left(\frac{\mathrm{d}r}{\mathrm{d}t}\right)\cdot\frac{\mathrm{d}}{\mathrm{d}r}\frac{\mathrm{d}r}{\mathrm{d}t} = 2\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathrm{d}r}{\mathrm{d}t}\right)\cdot\cancelto{1}{\frac{\mathrm{d}r}{\mathrm{d}r}} = 2\frac{\mathrm{d}^2 r}{\mathrm{d}t^2} = 2\ddot{r} \tag{3}$$

by chain rule. Applying this to $\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{\dot{r}^2}{r}\right)$, we get

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{\dot{r}^2}{r}\right) = \frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{1}{r}\cdot\dot{r}^2\right) = \frac{1}{r}\cdot\frac{\mathrm{d}\left(\dot{r}^2\right)}{\mathrm{d}r} + \dot{r}^2\cdot\left(-\frac{1}{r^2}\right) = \frac{2\ddot{r}}{r} - \frac{\dot{r}^2}{r^2} \tag{4}$$

by product rule. Here, we see that Eq. (2) can further be written as

$$F = q_1 q_2\left[\frac{1}{r^2} + \frac{1}{2c^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{\dot{r}^2}{r}\right)\right] \tag{5}$$

Solving for the potential, we have

$$\begin{aligned}
U &= -\int_0^{r'} F(r)\mathrm{d}r' \\
&= -\int_0^{r'} q_1 q_2\left[\frac{1}{r^2} + \frac{1}{2c^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{\dot{r}^2}{r}\right)\right]\mathrm{d}r' \\
&= -q_1 q_2\left[\int_0^{r'}\frac{1}{r^2}\mathrm{d}r' + \frac{1}{2c^2}\int_0^{r'}\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{\dot{r}^2}{r}\right)\mathrm{d}r'\right] \\
&= -q_1 q_2\left[-\frac{1}{r'} + \frac{1}{2c^2}\frac{\mathrm{d}}{\mathrm{d}r}\int_0^{r'}\left(\frac{\dot{r}^2}{r}\right)\mathrm{d}r'\right] \\
U &= -q_1 q_2\left(-\frac{1}{r'} + \frac{1}{2c^2}\frac{\dot{r}^2}{r}\right)
\end{aligned} \tag{6}$$

Footnote: In general, velocity-dependent forces cannot be expressed as a negative gradient of some potential. Luckily, as stated in the references, Weber's potential can still be obtained through this relation.

by using the first fundamental theorem of calculus. Here, we assume that r and $\dot{r}$ are smooth and continuous functions such that we can interchange the differentiation and integration. Since r' is just a dummy variable, we replace it with r in Eq. (6) to get the generalized potential

$$\boxed{U(r,\dot{r}) = q_1 q_2 \left(\frac{1}{r} - \frac{1}{2c^2}\frac{\dot{r}^2}{r}\right)} \tag{7}$$

Let's assume that q_2 is the charge that is held fixed in the origin of a 2D coordinate system. In polar coordinates, the kinetic energy of the given system is

$$T = \frac{1}{2} m_1 \left(\dot{r}^2 + r^2\dot{\theta}^2\right) \tag{8}$$

Thus, our Lagrangian is given by

$$\boxed{L = T - U = \frac{1}{2} m_1 \left(\dot{r}^2 + r^2\dot{\theta}^2\right) - q_1 q_2 \left(\frac{1}{r} - \frac{1}{2c^2}\frac{\dot{r}^2}{r}\right)} \tag{9}$$

We can calculate Lagrange's equations using the formula

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \frac{\partial L}{\partial q_k} = 0, \quad k = 1, \ldots, n \tag{10}$$

where the generalised coordinates q_k for this system are $q_1 = r$ and $q_2 = \theta$. We then solve for the following partial derivatives

$$\frac{\partial L}{\partial \dot{r}} = m_1\dot{r} + \frac{q_1 q_2 \dot{r}}{c^2 r} \tag{11}$$

$$\frac{\partial L}{\partial r} = \frac{q_1 q_2}{r^2}\left(1 - \frac{\dot{r}^2}{2c^2}\right) + m_1 r\dot{\theta}^2 \tag{12}$$

$$\frac{\partial L}{\partial \dot{\theta}} = m_1 r^2\dot{\theta} \tag{13}$$

$$\frac{\partial L}{\partial \theta} = 0 \tag{14}$$

and use Eq. (10) to obtain this system's Lagrange's equations.

$$\frac{\mathrm{d}}{\mathrm{dt}}\left(\frac{\partial L}{\partial \dot{r}}\right) - \frac{\partial L}{\partial r} = \frac{\mathrm{d}}{\mathrm{dt}}\left(m_1\dot{r} + \frac{q_1 q_2 \dot{r}}{c^2 r}\right) - \frac{q_1 q_2}{r^2}\left(1 - \frac{\dot{r}^2}{2c^2}\right) - m_1 r\dot{\theta}^2 \tag{15}$$

$$0 = m_1\ddot{r} + \frac{q_1 q_2 \ddot{r}}{c^2 r} - \frac{q_1 q_2 \dot{r}^2}{c^2 r^2} - \frac{q_1 q_2}{r^2}\left(1 - \frac{\dot{r}^2}{2c^2}\right) - m_1 r\dot{\theta}^2 \tag{16}$$

$$\boxed{0 = \ddot{r}\left(m_1 + \frac{q_1 q_2}{c^2 r}\right) - \frac{q_1 q_2}{2c^2}\frac{\dot{r}^2}{r^2} - \frac{q_1 q_2}{r^2} - m_1 r\dot{\theta}^2} \tag{17}$$

and

$$\frac{\mathrm{d}}{\mathrm{dt}}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = \frac{\mathrm{d}}{\mathrm{dt}}\left(m_1 r^2\dot{\theta}\right) - 0 \tag{18}$$

$$\boxed{0 = m_1 r^2\ddot{\theta} + 2m_1 r\dot{r}\dot{\theta}} \tag{19}$$

**Acknowledgements: I am grateful for the insightful comments of Allen Garcia when solving this problem.*

References

Weber electrodynamics, https://en.wikipedia.org/wiki/Weber_electrodynamics

Assis, A., *Weber's Electrodynamics*, Chapter 3

Problem 6 (1.14)

The system depicted in Figure 1 is such that the the string is inextensible and its mass, as well as that of the pulleys, is negligible. Mass m_1 moves on a frictionless horizontal table whereas m_2 moves only vertically. Show that, up to a constant, the Lagrangian for the system in terms of coordinate x is

$$L = \left(\frac{m_1\left(h^2+x^2\right)+m_2x^2}{h^2+x^2}\right)\frac{\dot{x}^2}{2} - m_2g\sqrt{h^2+x^2} \tag{1}$$

Set up Lagrange's equation for x.

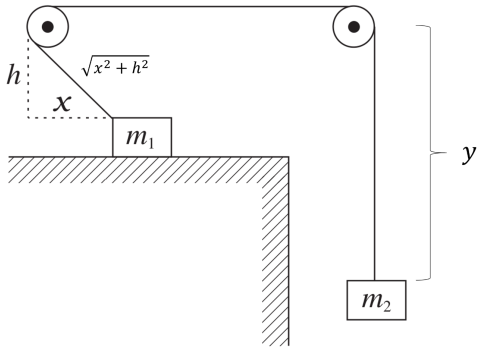

Figure 1: A diagram of the problem (Source: modified from Analytic Mechanics by N. Lemos)

Solution:
Given the diagram of the problem, we let y be the distance from the center of the pulley to the top of mass m_2. Then, the constraint imposed is

$$\sqrt{x^2+h^2}+y^2 = l_0 \tag{2}$$

by noting that the depicted system is just a modified Atwood's machine. The kinetic energy of the system is given by

$$\begin{aligned}
T &= \frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2 \\
&= \frac{1}{2}m_1\left(\sqrt{v_{1x}^2+v_{1y}^2}\right)^2 + \frac{1}{2}m_2\left(\sqrt{v_{2x}^2+v_{2y}^2}\right)^2 \\
&= \frac{1}{2}m_1(\dot{x}^2+0) + \frac{1}{2}m_2(0+\dot{y}^2) \\
&= \frac{1}{2}m_1\dot{x}^2 + \frac{1}{2}m_2\left(-\frac{x}{\sqrt{x^2+h^2}}\dot{x}\right)^2 \\
&= \frac{\dot{x}^2}{2}\left(m_1 + m_2\frac{x^2}{x^2+h^2}\right) \\
T &= \frac{\dot{x}^2}{2}\left(\frac{m_1\left(x^2+h^2\right)+m_2x^2}{x^2+h^2}\right)
\end{aligned} \tag{3}$$

Setting the zero level of the gravitational potential at the center of the pulleys, the potential energy of the system is

$$\begin{aligned} V &= -m_1gh - m_2gy \\ &= -m_1gh - m_2g\left(l_0 - \sqrt{x^2+h^2}\right) \\ V &= m_2g\sqrt{x^2+h^2} - m_1gh - m_2gl_0 \end{aligned} \tag{4}$$

Thus, the Lagrangian is

$$L = T - V = \frac{\dot{x}^2}{2}\left(\frac{m_1\left(x^2+h^2\right)+m_2x^2}{x^2+h^2}\right) - m_2g\sqrt{x^2+h^2} + g(m_1h + m_2l_0) \tag{5}$$

which is similar to Eq. (1) up to the constant $g(m_1h + m_2l_0)$. ■

We can calculate Lagrange's equations using the formula

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \frac{\partial L}{\partial q_k} = 0, \quad k = 1, \ldots, n \tag{6}$$

where the generalised coordinate for this system is $q_1 = x$. Using the code in Figure 2 to do this in Mathematica from the given the kinetic and potential energy of the system, as well as the list of generalised coordinates,

```
LagrangianEquations[T_, V_, genCoords_List] :=
    Module[{L = T - V},
    (D[D[L, D[#, t]], t] - D[L, #] == 0) & /@ genCoords
    ]
```

Figure 2: (modified from a notebook made by Jeff Adams)

the Lagrange's equation for this system is

$$\boxed{\frac{gm_2x}{\sqrt{h^2+x^2}} + \frac{(m_2+2m_1x)\dot{x}^2}{2\left(h^2+x^2\right)} - \frac{x\left(m_2x+m_1\left(h^2+x^2\right)\right)\dot{x}^2}{\left(h^2+x^2\right)^2} + \frac{\left(m_2x+m_1\left(h^2+x^2\right)\right)\ddot{x}}{h^2+x^2} = 0} \tag{7}$$

This is taken from the code shown in Figure 3 after combining the second and fourth terms.

```
genCoords = {x[t]};
T = 1/2 x'[t]^2 ((m1 (h^2 + x[t]^2) + m2 x[t]) / (h^2 + x[t]^2));
V = m2 g Sqrt[h^2 + x[t]^2];
```

From these, Lagrange' s equations are calculated as

```
LEqns = LagrangianEquations[T, V, genCoords]
```

$$\left\{\frac{g\,m2\,x[t]}{\sqrt{h^2+x[t]^2}} - \frac{(m2+2\,m1\,x[t])\,x'[t]^2}{2\left(h^2+x[t]^2\right)} - \frac{x[t]\,\left(m2\,x[t]+m1\left(h^2+x[t]^2\right)\right)x'[t]^2}{\left(h^2+x[t]^2\right)^2} + \frac{x'[t]\,(m2\,x'[t]+2\,m1\,x[t]\,x'[t])}{h^2+x[t]^2} + \frac{\left(m2\,x[t]+m1\left(h^2+x[t]^2\right)\right)x''[t]}{h^2+x[t]^2} == 0\right\}$$

Figure 3

References

Adams, J., *Lagrangian Equations Notebook*, https://library.wolfram.com/infocenter/Demos/4656/

Problem 7 (1.17)

Show that the Lagrangian

$$L = \frac{m}{2}\left(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\dot{\phi}^2\sin^2\theta\right) - \frac{eg}{c}\dot{\phi}\cos\theta \tag{1}$$

describes a charged particle in the magnetic field $\mathbf{B} = g\mathbf{r}/r^3$ 3 of a magnetic monopole and find Lagrange's equations. Hint: check that the vector potential for a magnetic monopole in spherical coordinates has components $A_r = 0$, $A_\theta = 0$, $A_\phi = g(1-\cos\theta)/r\sin\theta$.

Solution:
In spherical coordinates, the velocity of a charged particle in a magnetic field is $\mathbf{v} = \dot{r}\hat{\boldsymbol{r}} + r\dot{\theta}\hat{\boldsymbol{\theta}} + r\sin\theta\dot{\phi}\hat{\boldsymbol{\phi}}$. Then, its kinetic energy is given by

$$T = \frac{1}{2}mv^2 = \frac{1}{2}m\left(\sqrt{(\dot{r})^2 + \left(r\dot{\theta}\right)^2 + \left(r\sin\theta\dot{\phi}\right)^2}\right)^2 = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\dot{\phi}^2\sin^2\theta\right) \tag{2}$$

where m is the mass of the charged particle. Now, from Section 1.6 of [NL], its potential energy is given by

$$U = -\frac{e}{c}\mathbf{v}\cdot\mathbf{A} \tag{3}$$

since there is no electric field in the given scenario that will contribute to a scalar potential. Note that the magnetic field is related to the vector potential $\mathbf{B} = \nabla\times\mathbf{A}$. We need to obtain an expression for $\mathbf{A}$ that is related to $\mathbf{B} = g\mathbf{r}/r^3$. Going by the hint, let's consider a vector potential expressed in spherical coordinates as

$$\mathbf{A} = -\frac{g\cos\theta}{r\sin\theta}\hat{\boldsymbol{\phi}} \tag{4}$$

To check if this is valid, we compute $\mathbf{B} = \nabla\times\mathbf{A}$ as follows:

$$\begin{aligned}
\mathbf{B} &= \frac{1}{r^2\sin\theta}\begin{vmatrix} \hat{\boldsymbol{r}} & r\hat{\boldsymbol{\theta}} & r\sin\theta\hat{\boldsymbol{\phi}} \\ \partial_r & \partial_\theta & \partial_\phi \\ A_r & rA_\theta & r\sin\theta A_\phi \end{vmatrix} \\
&= \frac{1}{r^2\sin\theta}\begin{vmatrix} \hat{\boldsymbol{r}} & r\hat{\boldsymbol{\theta}} & r\sin\theta\hat{\boldsymbol{\phi}} \\ \partial_r & \partial_\theta & \partial_\phi \\ 0 & 0 & \cancel{r\sin\theta}\cdot\left(-g\frac{\cos\theta}{\cancel{r\sin\theta}}\right) \end{vmatrix} \\
&= \frac{1}{r^2\sin\theta}\,\partial_\theta\left(-g\cos\theta\right)\hat{\mathbf{r}} \\
&= \frac{g\cancel{\sin\theta}}{r^2\cancel{\sin\theta}}\frac{\mathbf{r}}{r} \\
\mathbf{B} &= \frac{g}{r^3}\mathbf{r}
\end{aligned} \tag{5}$$

This shows that the magnetic field of a magnetic monopole can be written in terms of the vector potential in Eq. (4). Note that

$$\mathbf{v}\cdot\mathbf{A} = (\dot{r}\hat{\boldsymbol{r}} + r\dot{\theta}\hat{\boldsymbol{\theta}} + r\sin\theta\dot{\phi}\hat{\boldsymbol{\phi}})\cdot\left(-g\frac{\cos\theta}{r\sin\theta}\hat{\boldsymbol{\phi}}\right) = -g\cancel{r\sin\theta}\dot{\phi}\frac{\cos\theta}{\cancel{r\sin\theta}} = -g\cos\theta\dot{\phi} \tag{6}$$

Thus, the potential energy is

$$U = \frac{e}{c} g \cos\theta \dot{\phi} \tag{7}$$

and the Lagrangian is

$$L = T - U = \frac{1}{2} m \left(\dot{r}^2 + r^2 \dot{\theta}^2 + r^2 \dot{\phi}^2 \sin^2\theta \right) - \frac{e}{c} g \cos\theta \dot{\phi} \tag{8}$$

■

We can calculate Lagrange's equations using the formula

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \frac{\partial L}{\partial q_k} = 0, \quad k = 1, \ldots, n \tag{9}$$

where the generalised coordinates q_k for this system are $q_1 = r$, $q_2 = \theta$, and $q_3 = \phi$. Using the code in Figure 1 to do this in Mathematica from the given the kinetic and potential energy of the system, as well as the list of generalised coordinates,

```
LagrangianEquations[T_, V_, genCoords_List] :=
    Module[{L = T - V},
    (D[D[L, D[#, t]], t] - D[L, #] == 0) & /@ genCoords
    ]
```

Figure 1: (modified from a notebook made by Jeff Adams)

the Lagrange's equations for this system are

$$\boxed{m\ddot{r} - mr\dot{\theta}^2 - mr\sin^2\theta\,\dot{\phi}^2 = 0} \tag{10}$$

$$\boxed{mr^2\ddot{\theta} + 2mr\dot{r}\dot{\theta} - mr^2\cos\theta\sin\theta\,\dot{\phi}^2 = 0} \tag{11}$$

$$\boxed{mr^2\sin^2\theta\,\ddot{\phi} + 2mr\sin^2\theta\,\dot{r}\dot{\phi} + 2mr^2\cos\theta\sin\theta\,\dot{\theta}\dot{\phi} = 0} \tag{12}$$

which are taken from the code shown in Figure 2.

Here, we list down the generalized coordinates we used for our system. Also, we express the kinetic energy T and potential energy V in terms of these coordinates:

```
genCoords = {r[t], θ[t], ϕ[t]};
T = (1/2) m (r'[t]^2 + r[t]^2 θ'[t]^2 + r[t]^2 Sin[θ[t]]^2 ϕ'[t]^2);
V = (e g / c) Cos[θ] ϕ'[t];
```

From these, Lagrange' s equations are calculated as

```
LEqns = LagrangianEquations[T, V, genCoords]
{-1/2 m (2 r[t] θ'[t]^2 + 2 r[t] Sin[θ[t]]^2 ϕ'[t]^2) + m r''[t] == 0,
 2 m r[t] r'[t] θ'[t] - m Cos[θ[t]] r[t]^2 Sin[θ[t]] ϕ'[t]^2 + m r[t]^2 θ''[t] == 0,
 2 m r[t] Sin[θ[t]]^2 r'[t] ϕ'[t] + 2 m Cos[θ[t]] r[t]^2 Sin[θ[t]] θ'[t] ϕ'[t] + m r[t]^2 Sin[θ[t]]^2 ϕ''[t] == 0}
```

Figure 2

**Acknowledgements: I am grateful for the insightful comments of Christian Buco when solving this problem.*

References

Adams, J., *Lagrangian Equations Notebook*, https://library.wolfram.com/infocenter/Demos/4656/

Problem 8 (1.20)

In Kepler's problem, show that for elliptic orbits the relation between r and t can be put in the form

$$t = \sqrt{\frac{m}{2}} \int_{r_{\min}}^{r} \frac{dr}{\sqrt{E + \frac{\kappa}{r} - \frac{l^2}{2mr^2}}} = \sqrt{\frac{m}{2\kappa}} \int_{r_{\min}}^{r} \frac{rdr}{\sqrt{r - \frac{r^2}{2a} - \frac{a(1-e^2)}{2}}} \tag{1}$$

where a is the semi-major axis, e is the eccentricity and at $t = 0$ the planet passes the perihelion.

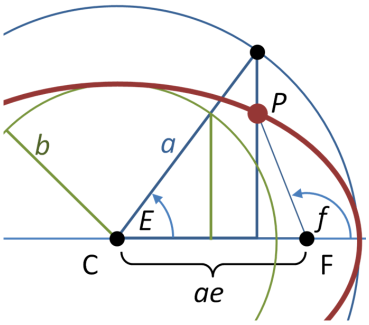

Figure 1: A diagram of the eccentric anomaly E and and true anomaly f in an ellipse. Here, C is the center and F is a focus on the ellipse (Source: `https://commons.wikimedia.org/wiki/File:Eccentric_and_true_anomaly.PNG`). In the context of this problem, the Sun is at F, the planet that elliptically orbits it is denoted by point P, $E = \psi$, $f = \phi$ and r is the distance from F to P (Sun to planet).

Solution:
From Section 1.7 of [NL], we have a relation r and ϕ given by

$$\frac{1}{r} = \frac{m\kappa}{l^2} (1 + e \cos \phi) \tag{2}$$

where $\kappa = GmM$, m and M are the masses of the planet and the Sun respectively, e is the eccentricity, and G is the gravitational constant. From this, we can calculate the $r_{\min}$ and $r_{\max}$ of the planet's orbit. Note that $r_{\min}$ [$r_{\max}$] is the distance between the Sun and the orbit's perihelion [aphelion].

$$r_{\min} = \frac{l^2}{m\kappa} \frac{1}{1 + e \cos \phi_{\min}} = \frac{l^2}{m\kappa} \frac{1}{1 + e \cancelto{1}{\cos 0}} \tag{3}$$

$$r_{\max} = \frac{l^2}{m\kappa} \frac{1}{1 + e \cos \phi_{\max}} = \frac{l^2}{m\kappa} \frac{1}{1 + e \cancelto{-1}{\cos \pi}} \tag{4}$$

The semi-major axis a of the orbit can be calculated from $a = (1/2)(r_{\min} + r_{\max})$. Substituting in Eqs. (3) and (4), we have

$$a = \frac{l^2}{2m\kappa} \left(\frac{1}{1+e} + \frac{1}{1-e} \right) = \frac{l^2}{2m\kappa} \frac{1 \cancel{+e} + 1 \cancel{-e}}{(1+e)(1-e)} = \frac{l^2}{m\kappa} \frac{1}{1-e^2} \tag{5}$$

which we rearrange to obtain a relation that will be useful later on

$$\frac{l^2}{m} = a\kappa \left(1 - e^2\right) \tag{6}$$

Now, eccentricity is defined by

$$e = \sqrt{1 + \frac{2El^2}{m\kappa^2}} \tag{7}$$

Isolating E, we get

$$\begin{aligned} e^2 &= 1 + \frac{2El^2}{m\kappa^2} \\ e^2 - 1 &= \frac{2El^2}{m\kappa^2} \\ -\frac{m}{l^2}\frac{\kappa^2\left(1-e^2\right)}{2} &= E \end{aligned} \tag{8}$$

Substituting in Eq. (6), the energy can be expressed as

$$E = -\frac{1}{a\cancel{\kappa}\,\cancel{\left(1-e^2\right)}}\frac{\kappa^{\cancel{2}}\,\cancel{\left(1-e^2\right)}}{2} = -\frac{\kappa}{2a} \tag{9}$$

Thus, by substituting Eqs. (6) and (9) the relation between r and t can be put in the form

$$\begin{aligned} t &= \sqrt{\frac{m}{2}}\int_{r_{\min}}^{r}\frac{dr}{\sqrt{E + \frac{\kappa}{r} - \frac{l^2}{2mr^2}}} \\ &= \sqrt{\frac{m}{2}}\int_{r_{\min}}^{r}\frac{dr}{\sqrt{-\frac{\kappa}{2a} + \frac{\kappa}{r} - \frac{a\kappa(1-e^2)}{2r^2}}} \\ &= \sqrt{\frac{m}{2}}\int_{r_{\min}}^{r}\frac{dr}{\frac{\sqrt{\kappa}}{r}\sqrt{-\frac{r^2}{2a} + r - \frac{a(1-e^2)}{2}}} \\ t &= \sqrt{\frac{m}{2\kappa}}\int_{r_{\min}}^{r}\frac{rdr}{\sqrt{r - \frac{r^2}{2a} - \frac{a(1-e^2)}{2}}} \end{aligned} \tag{10}$$

■

Show that, in terms of the angle ψ known as eccentric anomaly and defined by $r = a(1 - e\cos\psi)$, one has

$$\omega t = \psi - e\sin\psi \tag{11}$$

with $\omega = \sqrt{\kappa/ma^3}$. This last transcendental equation, which implicitly determines ψ as a function of t, is known as Kepler's equation.

Solution:
Taking the derivative of

$$r = a(1 - e\cos\psi) \tag{12}$$

with respect to time, we get

$$\dot{r} = -ae(-\sin\psi)\dot{\psi} = ae\sin\psi\dot{\psi} \tag{13}$$

We also have another relation for r given in Eq. (12). Substituting Eq. (6) in this relation, we have

$$\frac{1}{r} = \frac{1 + e\cos\phi}{a\left(1-e^2\right)} \tag{14}$$

Taking the derivative of this resulting equation with respect to time, we obtain

$$\frac{\dot{r}}{r^2} = \frac{e \sin\phi\,\dot{\phi}}{a\,(1-e^2)} \tag{15}$$

We need a relation for $\dot{\phi}$. Again, from Section 1.7 of [NL], we have $mr^2\dot{\phi} = l$. By rearranging Eq. (5) and substituting it to this relation, we get

$$\dot{\phi} = \frac{l}{m}\frac{1}{r^2} = \frac{\sqrt{ma\kappa\,(1-e^2)}}{mr^2} = \sqrt{\frac{a\kappa}{m}\cdot\frac{a^4}{a^4}}\frac{\sqrt{1-e^2}}{r^2} = \sqrt{\frac{\kappa}{ma^3}}\frac{a^2\sqrt{1-e^2}}{r^2} = \frac{\omega^2 a^2\sqrt{1-e^2}}{r^2} \tag{16}$$

where $\omega = \sqrt{\kappa/ma^3}$. Substituting this relation for $\dot{\phi}$ into Eq. (15), we get

$$\frac{\dot{r}}{r^2} = \frac{e\sin\phi}{a\,(1-e^2)}\cdot\frac{\omega^2 a^2\sqrt{1-e^2}}{r^2} \tag{17}$$

which we can simplify to

$$\dot{r} = \omega\frac{ae\sin\phi}{\sqrt{1-e^2}} \tag{18}$$

Note that we have now obtained two expressions for $\dot{r}$. Equating both of these expressions

$$ae\sin\psi\dot{\psi} = \omega\frac{ae\sin\phi}{\sqrt{1-e^2}} \tag{19}$$

and isolating $\dot{\psi}$, we get

$$\dot{\psi} = \omega\frac{\sin\phi}{\sin\psi\sqrt{1-e^2}} \tag{20}$$

We now need to obtain a relation between $\sin\psi$ and $\sin\phi$. To do this, let's put the center of the diagram in Figure 1 at the center of an xy-plane. We can then see that the planet's x and y positions are expressed as

$$x - ea = r\cos\phi \tag{21}$$
$$y = r\sin\phi \tag{22}$$

where $x = a\cos\psi$. The elliptical orbit is described by

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \tag{23}$$

where the semi-minor axis is defined by $b = a\sqrt{1-e^2}$. Isolating y from this equation, we can get

$$y = b\sqrt{1-\frac{x^2}{a^2}} = \sqrt{1-\frac{(a\cos\psi)^2}{a^2}} = b\sqrt{1-\cos^2\psi} = a\sqrt{1-e^2}\,\sin\psi \tag{24}$$

Substituting this to Eq. (22), we finally obtain the relation

$$r\sin\phi = a\sqrt{1-e^2}\,\sin\psi \tag{25}$$

Applying this relation to Eq. (20), we have

$$\dot{\psi} = \omega\frac{1}{\cancel{\sqrt{1-e^2}}}\cdot\frac{a\cancel{\sqrt{1-e^2}}}{r} = \frac{wa}{r} = \frac{w\cancel{a}}{\cancel{a}(1-e\cos\psi)} \tag{26}$$

by substituting in Eq. (12). Rearranging Eq. (26) and integrating,

$$\int_0^{\psi'} (1 - e\cos\psi)\mathrm{d}\psi = \int_0^{t'} \omega \mathrm{d}t \tag{27}$$

$$\psi' - e\sin\psi' = \omega t' \tag{28}$$

Since ψ' and t' are just dummy variables, we can replace them with ψ and t respectively to obtain

$$\psi - e\sin\psi = \omega t \tag{29}$$

■

References

Horn, W., *Kepler's Equation*, `http://www.csun.edu/~hcmth017/master/node15.html`