Dark matter–dark energy exchanges in an expanding universe

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Abstract

We revisit Aydiner's 'Chaotic universe model' (2018) that considers a hypothetical coupling between dark matter and dark energy in a spatially flat, Friedman-Robertson-Walker universe. In this model, dark matter and dark energy are perfect fluids able to transform into each other through a specified coupling function. We extend Aydiner's analysis by dropping his assumption of a constant Hubble parameter, an assumption that is valid only on short time scales. In this more general setting, and using a dynamical systems approach, we explore the effect of coupling on the dark matter/energy densities. We show that the cyclic dynamics between the two dark elements persists only when $w_{de} > -1$. For other values of $w_{\rm de}$, either dark matter vanishes and dark energy becomes constant or both disappear completely.

Keywords: dark matter, dark energy, dynamical systems

1 Introduction

In cosmology, the study of the formation and evolution of the universe, one often assumes the *cosmological principle* or that every location (homogeneity) and direction (isotropy) at which one looks at the universe will give the same observations over large scales [1]. This assumption approximates current observations [2] and simplifies cosmological models that mathematically depict our universe and its evolution [3].

One of these models involves an interaction between dark matter (DM) and dark energy (DE) [4]. Although such a phenomenon seems to be purely hypothetical in our observable universe, some kind of coupling between them and other fluids (*i.e.* matter and radiation) might have happened in its early stages. Furthermore, a more realistic model could arise by understanding the dynamics of such a universe [4].

Regarding the analysis of a cosmological model such as this, it would be useful to study it in a geometric point of view which involves deriving differential equations that characterizes the evolution of the relevant variables in the model and treating these equations as a dynamical system. In doing so, one can look into the behaviour of the solutions for these equations without completely solving them by utilizing an abstract construct called a phase space in which each point and trajectory would represent initial conditions and evolution from these conditions [5]. This becomes an advantage as one can infer the behaviour of the universe that the model depicts without the need for assumptions that one needs to simplify and solve the equations.

In this paper, we will first derive the coupling equations for dark matter and dark energy presented in [4] and provide their non-dimensionalized forms. We will then treat them as a dynamical system without assuming a constant Hubble parameter and analyze the stability of its fixed points. After which, we will look into its phase space for a graphical representation of the system's states under certain conditions.

2 Linear coupling between dark energy and dark matter

Consider a spatially flat universe which is homogeneous and isotropic on large scales. The most general metric, an expression that gives a sense of distance in an arbitrary space, that incorporates this is the Robertson-Walker metric [1] and a spacetime it defines would have a line element given by

$$ds^2 = -dt^2 + a(t)^2 \sum_{i=1}^{3} (dx^i)^2 \tag{2.1}$$

where $a(t)$ is a dimensionless scale factor for the three-dimensional spatial components of x, y, and z [4]. Plugging this metric into Einstein's equations to relate the scale factor with the energy-momentum of the universe and applying geometric units (*i.e.*, $G = 1$, $c = 1$), we can derive the following equations [6]

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi}{3}\rho \tag{2.2}$$

$$\frac{\ddot{a}}{a} = -\frac{8\pi}{6}(\rho + 3P) \tag{2.3}$$

where ρ is the energy density and P is the pressure. Equations (2.2) and (2.3) are called the Friedmann equations. Defining the Hubble parameter as $H = \frac{\dot{a}}{a}$, its time derivative can be expressed as $\dot{H} = \frac{\ddot{a}}{a} - H^2$. With these considerations, the Friedmann equations lead to

$$H^2 = \frac{\kappa^2}{3}\rho, \qquad \dot{H} = -\frac{\kappa^2}{2}(\rho + P) \tag{2.4}$$

by letting $\kappa^2 = 8\pi$. An equation for energy conservation, the continuity equation, which must be satisfied by ρ and P can also obtained as

$$\dot{\rho} = -3(\rho + P)\frac{\dot{a}}{a} = -3H(1 + w)\rho \tag{2.5}$$

where ρ and P are related by the equation of state (EoS)

$$P = w\rho \tag{2.6}$$

as perfect fluids in cosmology often follows [6]. Here, w is a dimensionless constant called the EoS parameter.

Now, let dark matter and dark energy be the only relevant fluids for this universe and allow them to change into each other over time. With the assumption of being perfect fluids, their interaction is taken from (2.5) and given as [4]

$$\dot{\rho}_{\mathrm{de}} + 3H(\rho_{\mathrm{de}} + P_{\mathrm{de}}) = -Q \tag{2.7}$$

$$\dot{\rho}_{\mathrm{dm}} + 3H(\rho_{\mathrm{dm}} + P_{\mathrm{dm}}) = Q \tag{2.8}$$

where Q is an arbitrary coupling function. The sign of Q in (2.7) and (2.8) must be opposite for conversion between the two fluids to take place. Note that the total conservation of energy is satisfied

$$\dot{\rho}_{\mathrm{tot}} + 3H(\rho_{\mathrm{tot}} + P_{\mathrm{tot}}) = 0 \tag{2.9}$$

where $\rho_{\mathrm{tot}} = \rho_{\mathrm{de}} + \rho_{\mathrm{dm}}$ and $P_{\mathrm{tot}} = P_{\mathrm{de}} + P_{\mathrm{dm}}$. The Friedmann equations are also subjected with a change of variables as $\rho \to \rho_{\mathrm{tot}}$ and $P \to P_{\mathrm{tot}}$. Letting Q have a simple form of $Q = \gamma\rho_{\mathrm{de}}\rho_{\mathrm{dm}}$ and arranging (2.7) and (2.8), we obtain the following differential equations,

$$\frac{\mathrm{d}\rho_{\mathrm{de}}}{\mathrm{d}t} = -3H(1 + w_{\mathrm{de}})\rho_{\mathrm{de}} - \gamma\rho_{\mathrm{de}}\rho_{\mathrm{dm}} \tag{2.10}$$

$$\frac{\mathrm{d}\rho_{\mathrm{dm}}}{\mathrm{d}t} = -3H(1 + w_{\mathrm{dm}})\rho_{\mathrm{dm}} + \gamma\rho_{\mathrm{de}}\rho_{\mathrm{dm}} \tag{2.11}$$

where the pressures of the two fluids are set as $P_{\mathrm{de}} = w_{\mathrm{de}}\rho_{\mathrm{de}}$ and $P_{\mathrm{dm}} = w_{\mathrm{dm}}\rho_{\mathrm{dm}}$ using (2.6). From (2.4), ρ should have the same dimensions as H^2 in geometric units. With this, non-dimensionalized forms of (2.10) and (2.11) can be derived as

$$\frac{\mathrm{d}\tilde{\rho}_{\mathrm{de}}}{\mathrm{d}\tau} = -3\tilde{H}(1 + w_{de})\tilde{\rho}_{\mathrm{de}} - \tilde{\gamma}\tilde{\rho}_{\mathrm{de}}\tilde{\rho}_{\mathrm{dm}} = \tilde{\rho}_{\mathrm{de}}\left[-(1 + w_{de})\kappa\sqrt{3(\tilde{\rho}_{\mathrm{de}} + \tilde{\rho}_{\mathrm{dm}})} - \tilde{\gamma}\tilde{\rho}_{\mathrm{dm}}\right] \tag{2.12}$$

$$\frac{\mathrm{d}\tilde{\rho}_{\mathrm{dm}}}{\mathrm{d}\tau} = -3\tilde{H}(1 + w_{dm})\tilde{\rho}_{\mathrm{dm}} + \tilde{\gamma}\tilde{\rho}_{\mathrm{de}}\tilde{\rho}_{\mathrm{dm}} = \tilde{\rho}_{\mathrm{dm}}\left[-(1 + w_{dm})\kappa\sqrt{3(\tilde{\rho}_{\mathrm{de}} + \tilde{\rho}_{\mathrm{dm}})} + \tilde{\gamma}\tilde{\rho}_{\mathrm{de}}\right] \tag{2.13}$$

after substituting in H and introducing the Hubble constant H_0 which is the value of the Hubble parameter at the present time. Here, $\tilde{\rho}_{\mathrm{de}} = \frac{\rho_{\mathrm{de}}}{H_0^2}$, $\tilde{\rho}_{\mathrm{dm}} = \frac{\rho_{\mathrm{dm}}}{H_0^2}$, $\tau = H_0 t$, $\tilde{H} = \frac{H}{H_0}$, and $\tilde{\gamma} = H_0\gamma$. This is the dynamical system consisting of state variables $(\rho_{\mathrm{de}}, \rho_{\mathrm{dm}})$ that describes the exchange of dark matter and dark energy under the linear coupling function Q [4]. However, the analysis in [4] assumed $\tilde{H}$ to be constant. This simplifies the dynamical system, but is only valid in an approximate sense. In what follows, we shall analyze the nature of the dark matter-dark energy exchange without recourse to this approximation.

3 Stability Analysis

To determine the fixed points of the system given by Eqs. (2.12) and (2.13) (*i.e.* points where the variables involved stay constant over time), we set

$$\frac{\mathrm{d}\tilde{\rho}_{\mathrm{de}}}{\mathrm{d}\tau} = 0 = \frac{\mathrm{d}\tilde{\rho}_{\mathrm{dm}}}{\mathrm{d}\tau} \tag{3.1}$$

The possible fixed points can be shown to be

$$(\tilde{\rho}_{\mathrm{de}_0}, \tilde{\rho}_{\mathrm{dm}_0}) \in \left\{ (0,0), \left(-\frac{3\kappa^2(w_{\mathrm{de}} - w_{\mathrm{dm}})(1+w_{\mathrm{dm}})}{\tilde{\gamma}^2}, \frac{3\kappa^2(w_{\mathrm{de}} - w_{\mathrm{dm}})(1+w_{\mathrm{de}})}{\tilde{\gamma}^2} \right) \right\} \tag{3.2}$$

Note that the last ordered pair can only be a fixed point if $\tilde{\gamma} > 0$ and $w_{\mathrm{de}} - w_{\mathrm{dm}} < 0$ which can be shown by substituting it back to the differential equations. To determine the stability of these fixed points, Eqs. (2.12) and (2.13) can be linearized around them, yielding the corresponding Jacobian matrix given by

$$J(\tilde{\rho}_{\mathrm{de}_0}, \tilde{\rho}_{\mathrm{dm}_0}) = \begin{pmatrix} -\gamma\tilde{\rho}_{\mathrm{dm}_0} - \sigma(3\tilde{\rho}_{\mathrm{de}_0} + 2\tilde{\rho}_{\mathrm{dm}_0})(1+w_{\mathrm{de}}) & -\gamma\tilde{\rho}_{\mathrm{de}_0} - \sigma\tilde{\rho}_{\mathrm{de}_0}(1+w_{\mathrm{de}}) \\ \gamma\tilde{\rho}_{\mathrm{dm}_0} - \sigma\tilde{\rho}_{\mathrm{dm}_0}(1+w_{\mathrm{dm}}) & \gamma\tilde{\rho}_{\mathrm{de}_0} - \sigma(2\tilde{\rho}_{\mathrm{de}_0} + 3\tilde{\rho}_{\mathrm{dm}_0})(1+w_{\mathrm{dm}}) \end{pmatrix} \tag{3.3}$$

where $\sigma = \dfrac{\kappa\sqrt{3}}{2\sqrt{\tilde{\rho}_{\mathrm{de}_0} + \tilde{\rho}_{\mathrm{dm}_0}}}$. We note that the Jacobian matrix vanishes at $(0,0)$, indicating that linearization is insufficient to determine the stability at this point. Looking at the other fixed point however, we can compute the eigenvalues of the Jacobian matrix. These are

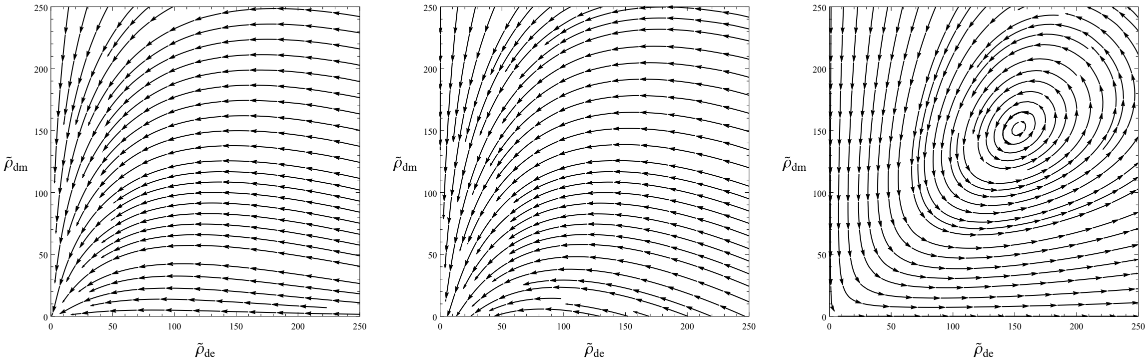
$$\lambda_1 = -\lambda_2 = -\frac{3\kappa^2(w_{\mathrm{de}} - w_{\mathrm{dm}})\sqrt{(1+w_{\mathrm{de}})(1+w_{\mathrm{dm}})}}{\tilde{\gamma}\sqrt{2}} \tag{3.4}$$

Note again that these eigenvalues are subjected under the constraints mentioned earlier. We also constrain the values of the EoS parameters to be $w_{\mathrm{de}} < -\frac{1}{3}$, which is a requirement for introducing accelerated expansion in the model, and $0 < w_{\mathrm{dm}} \ll 1$. Notice that we do not know much about the nature of dark matter. However, there has been a proposition that it could have a non-zero pressure [7] which implies that its EoS parameter is also non-zero [4]. Thus, it would be reasonable to set it as a small but positive number.

These conditions can be used in analyzing the nature of the eigenvalues expressed in Eq. (3.4). For $-1 < w_{\mathrm{de}} < -1/3$, the eigenvalues would be real as $(1+w_{\mathrm{de}})$ becomes positive and $(1+w_{\mathrm{dm}})$ is always so. Since they would have opposite signs, the considered fixed point will become a saddle (*i.e.* an attractor in one direction and a repulsor in another). However, this behaviour is not shown in Figure 1a since the fixed point would lie in the region of negative $\tilde{\rho}_{\mathrm{dm}}$ which does not physically make sense. The figure does show the behaviour around the fixed point (0,0) which, for this case, acts as an attractor. This implies that dark matter and dark energy in the universe would both vanish but dark matter will first increase gradually before disappearing.

In the case of $w_{\mathrm{de}} = -1$ which represents a universe where dark energy is the cosmological constant, the part of $\tilde{\rho}_{\mathrm{dm}} = 0$ less than the value of the second fixed point in Eq. (3.3) would act as an attractor while the part greater than it would become a repulsor. This shows that the densities of dark matter and dark energy for this universe can also exhibit the same behaviour as that when $-1 < w_{\mathrm{de}} < -1/3$ as shown in Figure 1b. There are, however, key differences in the phase plots of the two cases. Now, it is possible to have a universe with constant dark energy and no dark matter which results from the characteristic of $\tilde{\rho}_{\mathrm{dm}} = 0$ mentioned before. Also, notice in the figure that the more dark energy the universe initially has, the less it will have in the future. Dark matter would still vanish for all initial conditions but would increase a greater deal than the previous case before does so.

For $w_{\mathrm{de}} < -1$, it is clear that both of the eigenvalues would always be purely imaginary since $(1+w_{\mathrm{de}})$ becomes negative. This means that initial values of $\tilde{\rho}_{\mathrm{de}}$ and $\tilde{\rho}_{\mathrm{dm}}$ near the considered fixed point will evolve in a cyclic manner as shown in Figure 1c. Dark matter will turn into dark energy and vice versa without the influence of any external forces. On the other hand, the fixed point at (0,0) is shown to be a saddle for this case.



(a) $w_{\rm de} = -2/3$ (b) $w_{\rm de} = -1$ (c) $w_{\rm de} = -2$

Figure 1: Phase plots of the dynamical system with $\tilde{\gamma} = 1$, $w_{\rm dm} = 1/100$ for different values of $w_{\rm de}$. (Note that other values of $0 < w_{\rm dm} \ll 1$ will not give any additional information. Thus, only cases of different $w_{\rm de}$ were shown.

Although it would not be obvious in Figure 1 since we did not consider negative densities, note that the position of the second fixed point in Eq. (3.3) shifts according to the EoS parameters as it can be located at a more positive $\tilde{\rho}_{\rm de}$ and $\tilde{\rho}_{\rm dm}$ given a more negative value of $w_{\rm de}$.

4 Discussion

With a changing Hubble parameter, we observe that dark energy with an EoS parameter of $w_{\rm de} < -1$, the range of values for $w_{\rm de}$ that Aydiner considered, will always have a cyclic relationship with dark matter which coincides with his analysis. For this case, the fixed point at (0,0) was found to be a saddle point which also agrees with the original analysis. These mean that the densities of dark matter/energy as dynamical variables will, more or less, exhibit the same behaviour under a varying Hubble parameter compared to a constant one that Aydiner assumed. However, we found a shift in the position of the fixed point which exhibits the said cyclic behaviour. Its position moves towards increasingly positive values of $\tilde{\rho}_{\rm de}$ and $\tilde{\rho}_{\rm dm}$ for increasingly negative values of $w_{\rm de}$ which would not be seen in Aydiner's stability analysis since the fixed point that he found is constant with respect to $w_{\rm de}$. Furthermore, we found out that other forms of dark energy, which he did not consider in his analysis (*i.e.* $-1 < w_{\rm de} < -1/3$ and $w_{\rm de} = -1$), will lead to a universe where either dark matter vanishes and dark energy becomes constant or both disappear completely.

5 Conclusions

We have analyzed the dynamics of a proposed DE-DM exchange model without the assumption of a constant or very slowly changing Hubble parameter. We showed that the fixed point exhibiting cyclic behavior between dark matter and dark energy shifts in position according to the value of the EoS parameters. We have also shown that this cyclic relationship persists only when $w_{\rm de} < -1$. For other values of $w_{\rm de}$, either dark matter vanishes and dark energy becomes constant or both disappear completely. It would be interesting to extend this study to more general coupling functions $Q = \gamma\rho_{\rm de}^{m}\rho_{\rm dm}^{n}$.

References

[1] B. Ryden, *Introduction to Cosmology* (Pearson Education, Inc., New York, 2017).

[2] R. Maartens, Is the Universe homogeneous?, arXiv:1104.1300v2 (2011).

[3] J. Uzan, The big-bang theory: construction, evolution and status, arXiv:1606.06112v1 (2016).

[4] E. Aydiner, Chaotic universe model, *Sci Rep* **8** (2018).

[5] S. Strogatz, *Nonlinear Dynamics and Chaos: With Applications to Physics, Biology, Chemistry, and Engineering* (Westview Press, Boulder, CO, 2015).

[6] S. Carroll, *Spacetime and Geometry: An Introduction in General Relativity* (Addison-Wesley, San Francisco, CA, 2004).

[7] T. Harko and F. Lobo, Could pressureless dark matter have pressure?, arXiv:1104.2674v2 (2012).