Problem 1 (3.3)

A thin, flat, conducting, circular disc of radius R is located in the $x-y$ plane with its center at the origin, and is maintained at a fixed potential V. With the information that the charge density on a disc at fixed potential is proportional to $\left(R^2-\rho^2\right)^{-1/2}$ where ρ is the distance out from the center of the disc,

(a) show that for $r > R$, the potential is

$$\Phi(r,\theta,\phi) = \frac{2V}{\pi}\frac{R}{r}\sum_{l=0}^{\infty}\frac{(-1)^l}{2l+1}\left(\frac{R}{r}\right)^{2l} P_{2l}(\cos\theta) \tag{1}$$

Solution:
First, let us reassign the variable ρ to r as they both pertain to a radial distance with ρ being a special case of r when $\cos\theta = 0$ (at the xy plane). Thus, the surface charge density is

$$\sigma(\mathbf{x}) = \sigma(r) = \frac{k}{\sqrt{R^2-r^2}} \tag{2}$$

with k as the proportionality constant. Note that $\sigma(\mathbf{x})$ is solely dependent on r. With azimuthal symmetry present in the problem, we can infer that the volume charge density is

$$\rho(\mathbf{x}) = f(r)\delta(\cos\theta - \cos\theta_0) = f(r)\delta(\cos\theta - \cos\frac{\pi}{2}) = f(r)\delta(\cos\theta) \tag{3}$$

If we take an infinitesimal charge along the 2D surface of the disk, we have

$$\mathrm{d}q = \sigma(\mathbf{x})\mathrm{d}A = \sigma(r)r\mathrm{d}r\mathrm{d}\phi \tag{4}$$

In 3D, this infinitesimal charge is given by

$$\mathrm{d}q = \rho(\mathbf{x})\mathrm{d}V = f(r)\delta(\cos\theta)r^2\mathrm{d}r\mathrm{d}(\cos\theta)\mathrm{d}\phi = r^2 f(r)\delta(\cos\theta)\mathrm{d}r\mathrm{d}(\cos\theta)\mathrm{d}\phi \tag{5}$$

To equate these two expressions, we should first make the dimensionality of the two to be equal by noting that the disk is on the xy plane. So, from Eq. (5), we have

$$\mathrm{d}q_{\mathrm{2D}} = r^2 f(r)\left(\int_{-1}^{1}\delta(\cos\theta)\mathrm{d}(\cos\theta)\right)\mathrm{d}r\mathrm{d}\phi = r^2 f(r)\mathrm{d}r\mathrm{d}\phi \tag{6}$$

Comparing Eqs. (4) and (6), we find that

$$f(r) = \frac{\sigma(r)}{r} \tag{7}$$

Thus, the volume charge density is

$$\rho(\mathbf{x}) = \frac{\sigma(r)}{r}\delta(\cos\theta) \tag{8}$$

Next, we want to find k. The potential in terms of the charge density is given in Eq. (1.17) of Jackson:

$$\Phi(\mathbf{x}) = \frac{1}{4\pi\epsilon_0}\int\frac{\rho\left(\mathbf{x}'\right)}{|\mathbf{x}-\mathbf{x}'|}d^3x' \tag{9}$$

Since $\Phi(\mathbf{x}) = V$ on the disk, let us evaluate the potential at $r = 0$, $\cos\theta = 0$:

$$\begin{aligned}
\Phi(r=0, \cos\theta=0, \phi) &= \frac{1}{4\pi\epsilon_0}\int_0^{2\pi}\int_{-1}^{1}\int_0^{R} \frac{\frac{\sigma(r')}{r'}\delta(\cos\theta')}{\sqrt{(0-r')^2+(0-\cos\theta')^2+(\phi-\phi')^2}} r'^2 \mathrm{d}r' \mathrm{d}(\cos\theta')\mathrm{d}\phi' \\
&= \frac{1}{4\pi\epsilon_0}\left(\int_0^{2\pi}\mathrm{d}\phi'\right)\left(\int_0^{R} r'\sigma(r')\left(\int_{-1}^{1}\frac{\delta(\cos\theta')}{\sqrt{r'^2+\cos^2\theta'+(\phi'-\phi')^2}}\mathrm{d}(\cos\theta')\right)\mathrm{d}r'\right) \\
&= \frac{2\pi}{4\pi\epsilon_0}\int_0^{R}\frac{r'\sigma(r')}{\sqrt{r'^2}}\mathrm{d}r' \\
&= \frac{1}{2\epsilon_0}\int_0^{R}\frac{k}{\sqrt{R^2-r'^2}}\mathrm{d}r' \\
V &= \frac{1}{2\epsilon_0}\cdot k\frac{\pi}{2}
\end{aligned} \tag{10}$$

From this, we find

$$k = \frac{4\epsilon_0 V}{\pi} \tag{11}$$

Updating the form of $\rho(\mathbf{x})$, we have

$$\rho(\mathbf{x}) = \frac{4\epsilon_0 V}{\pi r\sqrt{R^2-r^2}}\delta(\cos\theta) \tag{12}$$

The factor $1/|\mathbf{x}-\mathbf{x}'|$ for problems with azimuthal symmetry is given in Eq. 3.38 of Jackson

$$\frac{1}{|\mathbf{x}-\mathbf{x}'|} = \sum_{l=0}^{\infty}\frac{r_<^l}{r_>^{l+1}}P_l(\cos\gamma) \tag{13}$$

where $r_<$ ($r_>$) is the smaller (larger) of $|\mathbf{x}|$ and $|\mathbf{x}'|$, and γ is the angle between $\mathbf{x}$ and $\mathbf{x}'$,. We let $r = r_>$ and $r' = r_<$ to indicate that the region we are interested in is $r > R$ where the potential is valid at $r \to \infty$. Substituting this and $\rho(\mathbf{x})$ to Eq. (9),

$$\begin{aligned}
\Phi(\mathbf{x}) &= \frac{1}{4\pi\epsilon_0}\int_0^{2\pi}\int_{-1}^{1}\int_0^{R}\frac{4\epsilon_0 V}{\pi r'\sqrt{R^2-r'^2}}\delta(\cos\theta')\left(\sum_{l=0}^{\infty}\frac{r'^l}{r^{l+1}}P_l(\cos\gamma)\right) r'^2\mathrm{d}r'\mathrm{d}(\cos\theta')\mathrm{d}\phi' \\
&= \frac{V}{\pi^2}\left(\int_0^{2\pi}\mathrm{d}\phi'\right)\left(\int_{-1}^{1}\delta(\cos\theta')\mathrm{d}(\cos\theta')\right)\left(\sum_{l=0}^{\infty} r^{-(l+1)}P_l(\cos\gamma)\int_0^{R}\frac{r'^l}{\sqrt{R^2-r'^2}}r'\mathrm{d}r'\right) \\
\Phi(\mathbf{x}) &= \frac{2\pi V}{\pi^2}\sum_{l=0}^{\infty} r^{-(l+1)}P_l(\cos\gamma)\int_0^{R}\frac{r'^{l+1}}{\sqrt{R^2-r'^2}}\mathrm{d}r'
\end{aligned} \tag{14}$$

Let us consider the potential at the z-axis. For this case, $\gamma = \pi/2$. Note that

$$P_l(\cos\frac{\pi}{2}) = P_l(0) = P_{2n}(0) + \cancelto{0}{P_{2n+1}(0)} = (-1)^n\frac{(2n)!}{2^{2n}(n!)^2} \tag{15}$$

Then, we have

$$\Phi(r=z, \cos\theta=0, \phi) = \frac{2V}{\pi}\sum_{l=0}^{\infty} r^{-(l+1)}(-1)^n\frac{(2n)!}{2^{2n}(n!)^2}\int_0^{R}\frac{r'^{l+1}}{\sqrt{R^2-r'^2}}\mathrm{d}r' \tag{16}$$

for $l = 2n$. The general solution of the potential for problems with azimuthal symmetry is (Eq. 3.33 of Jackson)

$$\Phi(r,\theta,\phi) = \sum_{l=0}^{\infty} \left[A_l r^l + B_l r^{-(l+1)}\right] P_l(\cos\theta) \tag{17}$$

If we consider the potential on the positive z-axis, then this simplifies to

$$\Phi(r=z,\theta,\phi) = \sum_{l=0}^{\infty} \left[A_l r^l + B_l r^{-(l+1)}\right] \tag{18}$$

Comparing Eqs. (16) and (18),

$$\begin{aligned} A_l &= 0 \\ B_l &= \frac{2V}{\pi}\sum_{l=0}^{\infty}(-1)^n \frac{(2n)!}{2^{2n}(n!)^2}\int_0^R \frac{r'^{l+1}}{\sqrt{R^2-r'^2}}\mathrm{d}r' \end{aligned} \tag{19}$$

Substituting these coefficients into the general solution in Eq. (17), we have

$$\begin{aligned} \Phi(r,\theta,\phi) &= \frac{2V}{\pi}\sum_{l=0}^{\infty}(-1)^n \frac{(2n)!}{2^{2n}(n!)^2}\left(\int_0^R \frac{r'^{l+1}}{\sqrt{R^2-r'^2}}\mathrm{d}r'\right) r^{-(l+1)}P_l(\cos\theta) \\ &= \frac{2V}{\pi}\sum_{l=0}^{\infty}(-1)^n \frac{(2n)!}{2^{2n}(n!)^2} R^{2n+1}\frac{2^{2n}(n!)^2}{(2n+1)!} r^{-(l+1)}P_l(\cos\theta) \end{aligned} \tag{20}$$

by using the relation

$$\int_0^R \frac{r'^{l+1}}{\sqrt{R^2-r'^2}}\mathrm{d}r' = R^{2n+1}\frac{2^{2n}(n!)^2}{(2n+1)!} \tag{21}$$

from integral tables. As the resulting expression of the potential is valid for $l = 2n$, we convert l to n and get

$$\Phi(r,\theta,\phi) = \frac{2V}{\pi}\sum_{l=0}^{\infty}\frac{(-1)^n}{(2n+1)}\frac{R^{2n+1}}{r^{2n+1}}P_{2n}(\cos\theta) \tag{22}$$

which shows the given for $r > R$.

(b) find the potential for $r < R$

Solution:
At the boundary, $\Phi(r,\theta,\phi)$ for $r < R$ and $r > R$ must be equal. Thus,

$$A_l R^l = B_l R^{-(l+1)} \quad \rightarrow \quad A_{2n}R^{2n} = B_{2n}R^{-(2n+1)} \tag{23}$$

with $l = 2n$. Substituting in the coefficients in Eq. (19) and using the relation in Eq. (21),

$$A_{2n}R^{2n} = \frac{2V}{\pi}(-1)^n\frac{(2n)!}{2^{2n}(n!)^2}R^{2n+1}\frac{2^{2n}(n!)^2}{(2n+1)!}R^{-(2n+1)} = \frac{2V}{\pi}\sum_{l=0}^{\infty}\frac{(-1)^n}{(2n+1)} \tag{24}$$

Thus,

$$A_{2n} = \frac{2V}{\pi}\frac{(-1)^n}{(2n+1)}R^{-2n} \tag{25}$$

Note that for $r > R$, $B_l = 0$ since the potential must be valid at $r = 0$. Then, from the general potential, the potential at $r < R$ must be

$$\Phi(r, \theta, \phi) = \frac{2V}{\pi} \sum_{l=0}^{\infty} \frac{(-1)^n}{(2n+1)} \frac{r^{2n}}{R^{2n}} P_{2n}(\cos\theta) \tag{26}$$

(c) What is the capacitance of the disc?

Solution:
The total charge on the disk is

$$\begin{aligned} Q &= \int \sigma(\mathbf{x}) \mathrm{d}A \\ &= \frac{4\epsilon_0 V}{\pi} \int_0^{2\pi} \int_0^R \frac{1}{\sqrt{R^2 - r^2}} r \mathrm{d}r \mathrm{d}\phi \\ &= \frac{4\epsilon_0 V}{\pi} \left(\int_0^{2\pi} \mathrm{d}\phi \right) \left(\int_0^R \frac{r}{\sqrt{R^2 - r^2}} \mathrm{d}r \right) \\ &= \frac{4\epsilon_0 V}{\pi} \cdot 2\pi \cdot R \\ Q &= 8\epsilon_0 V R \end{aligned} \tag{27}$$

with $u = R^2 - r^2$ for the r-integral. Thus, the capacitance is

$$C = \frac{Q}{V} = \frac{8\epsilon_0 V R}{V} = 8\epsilon_0 R \tag{28}$$

Problem 2 (3.5)

A hollow sphere of inner radius a has the potential specified on its surface to be $\Phi = V(\theta, \phi)$. Prove the equivalence of the two forms of solution for the potential inside the sphere:

(a)

$$\Phi(\mathbf{x}) = \frac{a\left(a^2 - r^2\right)}{4\pi} \int \frac{V\left(\theta', \phi'\right)}{\left(r^2 + a^2 - 2ar\cos\gamma\right)^{3/2}} d\Omega' \tag{1}$$

where $\cos\gamma = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos\left(\phi - \phi'\right)$.

(b)

$$\Phi(\mathbf{x}) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} A_{lm} \left(\frac{r}{a}\right)^l Y_{lm}(\theta, \phi) \tag{2}$$

where $A_{lm} = \int d\Omega' Y_{lm}^{*}\left(\theta', \phi'\right) V\left(\theta', \phi'\right)$.

Solution:

In spherical coordinates, the appropriate Green's function is (Eq. 2.17 of Jackson)

$$G\left(\mathbf{x}, \mathbf{x}'\right) = \frac{1}{\left(r^2 + r'^2 - 2rr'\cos\gamma\right)^{1/2}} - \frac{1}{\left(\frac{r^2 r'^2}{a^2} + a^2 - 2rr'\cos\gamma\right)^{1/2}} \tag{3}$$

where γ is the angle between $\mathbf{x}$ and $\mathbf{x}'$. Since the derivative of G is radially outward when we consider the region interior of the sphere, we have

$$\left.\frac{\partial G}{\partial n'}\right|_{r'=a} = \frac{\left(r^2 - a^2\right)}{a\left(r^2 + a^2 - 2ar\cos\gamma\right)^{3/2}} \tag{4}$$

Now, the general solution to the potential in terms of the Green's function is

$$\Phi(\mathbf{x}) = \frac{1}{4\pi\epsilon_0} \int_V \rho\left(\mathbf{x}'\right) G\left(\mathbf{x}, \mathbf{x}'\right) d^3x' + \frac{1}{4\pi} \oint_S \left[G\left(\mathbf{x}, \mathbf{x}'\right) \frac{\partial \Phi}{\partial n'} - \Phi\left(\mathbf{x}'\right) \frac{\partial G\left(\mathbf{x}, \mathbf{x}'\right)}{\partial n'}\right] da' \tag{5}$$

Using Dirichlet boundary conditions and the fact that there is no charge density, this simplifies to

$$\Phi(\mathbf{x}) = -\frac{1}{4\pi} \oint_S \Phi\left(\mathbf{x}'\right) \frac{\partial G\left(\mathbf{x}, \mathbf{x}'\right)}{\partial n'} da' \tag{6}$$

Substituting in Eq. (4),

$$\begin{aligned}\Phi(\mathbf{x}) &= -\frac{1}{4\pi} \int \Phi\left(a, \theta', \phi'\right) \frac{\left(r^2 - a^2\right)}{a\left(r^2 + a^2 - 2ar\cos\gamma\right)^{3/2}} a^2 \mathrm{d}\Omega' \\ &= \frac{a(a^2 - r^2)}{4\pi} \int \frac{V\left(\theta', \phi'\right)}{\left(r^2 + a^2 - 2ar\cos\gamma\right)^{3/2}} \mathrm{d}\Omega'\end{aligned} \tag{7}$$

which shows (a) in the given.

We can also solve the potential inside the sphere through the Laplace equation. The general solution of the potential in this case is (Eq. (3.61) of Jackson)

$$\Phi(r, \theta, \phi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \left[A_{lm} r^l + B_{lm} r^{-(l+1)}\right] Y_{lm}(\theta, \phi) \tag{8}$$

Since Φ must be finite at $r=0$, B_{lm} must vanish. Thus, the form of the potential becomes

$$\Phi(r,\theta,\phi)=\sum_{l=0}^{\infty}\sum_{m=-l}^{l}A_{lm}r^{l}Y_{lm}(\theta,\phi) \tag{9}$$

At $r=a$, the potential on the sphere's surface results to

$$\Phi(r=a,\theta,\phi)=\sum_{l=0}^{\infty}\sum_{m=-l}^{l}A_{lm}a^{l}Y_{lm}(\theta,\phi)=V(\theta,\phi) \tag{10}$$

Multiplying both sides by $Y^{*}_{l'm'}(\theta',\phi')\sin\theta'\mathrm{d}\theta'\mathrm{d}\phi'$ and integrating, Eq. (10) becomes

$$\int_0^{2\pi}\int_0^{\pi}V(\theta',\phi')Y^{*}_{l'm'}(\theta',\phi')\sin\theta'\mathrm{d}\theta'\mathrm{d}\phi'=\sum_{l=0}^{\infty}\sum_{m=-l}^{l}A_{lm}a^{l}\int_0^{2\pi}\int_0^{\pi}Y_{lm}(\theta,\phi)Y^{*}_{l'm'}(\theta',\phi')\sin\theta'\mathrm{d}\theta'\mathrm{d}\phi' \tag{11}$$

Using the normalization and orthogonality condition (Eq. 3.55 of Jackson)

$$\int_0^{2\pi}d\phi\int_0^{\pi}\sin\theta d\theta Y^{*}_{l'm'}(\theta,\phi)Y_{lm}(\theta,\phi)=\delta_{l'l}\delta_{m'm}, \tag{12}$$

Eq. (11) results to

$$\begin{aligned}\int_0^{2\pi}\int_0^{\pi}V(\theta',\phi')Y^{*}_{l'm'}(\theta',\phi')\sin\theta'\mathrm{d}\theta'\mathrm{d}\phi' &=\sum_{l=0}^{\infty}\sum_{m=-l}^{l}A_{lm}a^{l}\delta_{l'l}\delta_{m'm}\\ &=A_{l'm'}a^{l'}\end{aligned} \tag{13}$$

Since l' and m' are just dummy indices, we can convert them to l and m respectively and we get,

$$A_{lm}=\frac{1}{a^{l}}\int V(\theta',\phi')Y^{*}_{lm}(\theta',\phi')\mathrm{d}\Omega' \tag{14}$$

Substituting this to Eq. (9),

$$\Phi(r,\theta,\phi)=\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{r^{l}}{a^{l}}Y_{lm}(\theta,\phi)\int V(\theta',\phi')Y^{*}_{lm}(\theta',\phi')\mathrm{d}\Omega' \tag{15}$$

which shows given in (b) provided that we define a new $A_{lm}=\int V(\theta',\phi')Y^{*}_{lm}(\theta',\phi')\mathrm{d}\Omega'$. As we have shown that the two given expressions of different formulations refer to the potential in the same region (inside of the sphere), we have indirectly proven their equivalence.

Problem 3 (3.12)

An infinite, thin, plane sheet of conducting material has a circular hole of radius a cut in it. A thin, flat disc of the same material and slightly smaller radius lies in the plane, filling the hole, but separated from the sheet by a very narrow insulating ring. The disc is maintained at a fixed potential V, while the infinite sheet is kept at zero potential.

(a) Using appropriate cylindrical coordinates, find an integral expression involving Bessel functions for the potential at any point above the plane.

Solution:
The general form of the potential in cylindrical coordinates for $z \geq 0$ is given in Eq. (3.106) of Jackson:

$$\Phi(\rho, \phi, z) = \sum_{m=0}^{\infty} \int_0^{\infty} dk e^{-kz} J_m(k\rho) \left[A_m(k) \sin m\phi + B_m(k) \cos m\phi\right] \tag{1}$$

Since there is azimuthal symmetry present in the problem, $m = 0$. Thus, we have

$$\Phi(\rho, \phi, z) = \frac{1}{2} \int_0^{\infty} dk e^{-kz} J_0(k\rho) B_0(k) \tag{2}$$

Note that we need to use $(1/2)B_0(k)$ in the series when $m = 0$. At the disk ($\rho < a$, $z = 0$), this results to

$$\Phi(\rho < a, \phi, z = 0) = \frac{1}{2} \int_0^{\infty} dk J_0(k\rho) B_0(k) = V \tag{3}$$

The coefficient $B_0(k)$ can be obtained by evaluating Eq. (3.109) of Jackson with $m = 0$,$\rho < a$ and $V(\rho, \phi) = V$:

$$B_0(k) = \frac{k}{\pi} \int_0^a J_0(k\rho)\rho \left(V \int_0^{2\pi} \mathrm{d}\phi\right) \mathrm{d}\rho = 2kV \int_0^a J_0(k\rho)\rho \mathrm{d}\rho \tag{4}$$

Now, we consider a differential equation for J_0 given by

$$J_0''(x) + \frac{1}{x} J_0'(x) + J_0(x) = 0 \tag{5}$$

Rearranging this and integrating both sides with respect to x from $x = 0$ to $x = a$, we have

$$\int_0^a x J_0(x) \mathrm{d}x = -\int_0^a x J_0''(x) \mathrm{d}x - \int_0^a J_0'(x) \mathrm{d}x \tag{6}$$

Using integration by parts, we obtain

$$\begin{aligned} \int_0^a x J_0(x) \mathrm{d}x &= -\left(x J_0'(x)\big|_0^a - \int_0^a J_0'(x) \mathrm{d}x \right) - \int_0^a J_0'(x) \mathrm{d}x \\ &= -a J_0'(a) \\ \int_0^a x J_0(x) \mathrm{d}x &= a J_1(a) \end{aligned} \tag{7}$$

after applying $J_0(x) = -J_1(x)$ from $J_n'(x) = (n/x)J_n(x) - J_{n+1}(x)$. With this, we find that

$$\int_0^a J_0(k\rho)\rho \mathrm{d}\rho = \int_0^{ka} \frac{u}{k} J_0(u) \frac{\mathrm{d}u}{k} = \frac{a J_1(ka)}{k} \tag{8}$$

Thus, we have

$$B_0(k) = 2aVJ_1(ka) \tag{9}$$

Substituting this back to Eq. (2), we get

$$\Phi(\rho, \phi, z) = \frac{1}{2}\int_0^\infty dke^{-kz}J_0(k\rho)(2aVJ_1(ka)) = aV\int_0^\infty e^{-kz}J_0(k\rho)J_1(ka)\mathrm{d}k \tag{10}$$

as the integral expression for the potential at any point above the plane.

(b) Show that the potential a perpendicular distance z above the center of the disc is

$$\Phi_0(z) = V\left(1 - \frac{z}{\sqrt{a^2+z^2}}\right)$$

Solution:
Substituting Eq. (4) into Eq. (2) with $\rho = 0$, Eq. (2) becomes

$$\begin{aligned}
\Phi(\rho = 0, \phi, z) &= \frac{1}{2}\int_0^\infty dke^{-kz}J_0(0)\left(2kV\int_0^a J_0(k\rho')\rho'\mathrm{d}\rho'\right) \\
&= VJ_0(0)\int_0^a \rho'\left(\int_0^\infty ke^{-kz}J_0(k\rho')\mathrm{d}k\right)\mathrm{d}\rho' \\
&= VJ_0(0)\int_0^a \rho'\left(\int_0^\infty \left(-\frac{\partial}{\partial z}e^{-kz}\right)J_0(k\rho')\mathrm{d}k\right)\mathrm{d}\rho' \\
\Phi(\rho = 0, \phi, z) &= VJ_0(0)\int_0^a \rho'\left(-\frac{\partial}{\partial z}\int_0^\infty e^{-kz}J_0(k\rho')\mathrm{d}k\right)\mathrm{d}\rho'
\end{aligned} \tag{11}$$

Using Mathematica, we know that

$$\int_0^\infty e^{-ay}J_0(bx)\mathrm{d}x = \frac{1}{\sqrt{b^2+y^2}} \tag{12}$$

Then, we have

$$\begin{aligned}
\Phi(\rho = 0, \phi, z) &= -VJ_0(0)\int_0^a \rho'\left(\frac{\partial}{\partial z}\frac{1}{\sqrt{\rho'^2+z^2}}\right)\mathrm{d}\rho' \\
&= -VJ_0(0)\int_0^a \rho'\left(-\frac{2z}{2\sqrt{\rho'^2+z^2}}\right)\mathrm{d}\rho' \\
&= zVJ_0(0)\int_0^a \frac{\rho'}{2\sqrt{\rho'^2+z^2}}\mathrm{d}\rho' \\
&= zV\cancelto{1}{J_0(0)}\left[\frac{1}{\sqrt{z^2}} - \frac{1}{\sqrt{a^2+z^2}}\right] \\
\Phi(\rho = 0, \phi, z) &= V\left[1 - \frac{z}{\sqrt{a^2+z^2}}\right]
\end{aligned} \tag{13}$$

(c) Show that the potential a perpendicular distance z above the edge of the disc is

$$\Phi_a(z) = \frac{V}{2}\left[1 - \frac{kz}{\pi a}K(k)\right]$$

where $k = 2a/\left(z^2+4a^2\right)^{1/2}$, and $K(k)$ is the complete elliptic integral of the first kind.

Solution:
Using Eq. (10) with $\rho = a$, we have

$$\Phi(\rho=a,\phi,z) = aV\int_0^\infty e^{-kz}J_0(ka)J_1(ka)\mathrm{d}k \tag{14}$$

From integral tables, we know that

$$\int_0^\infty e^{-px}J_1(ax)J_0(bx)dx = -\frac{pk}{2\pi a^2}K(k)+\frac{1}{2a}, \quad a=b \tag{15}$$

where $K(k)$ is the complete elliptic integral of the first kind and $k^2 = 4ab/(p^2+(a+b)^2)$. Then, Eq. (14) becomes

$$\Phi(\rho=a,\phi,z) = aV\left(\frac{1}{2a}-\frac{zk}{2\pi a^2}K(k)\right) = \frac{V}{2}\left(1-\frac{zk}{\pi a}K(k)\right) \tag{16}$$

with $k^2 = 4a^2/(z^2+4a^2)$ which shows the given in this scenario.

Problem 4 (3.19)

Consider a point charge q between two infinite parallel conducting planes held at zero potential. Let the planes be located at $z = 0$ and $z = L$ in a cylindrical coordinate system, with the charge on the z axis at $z = z_0, 0 < z_0 < L$. Use Green's reciprocation theorem of Problem 1.12 with problem 3.18 as the comparison problem.

(a) Show that the amount of induced charge on the plate at $z = L$ inside a circle of radius a whose center is on the z axis is given by

$$Q_L(a) = -\frac{q}{V}\Phi\left(z_0, 0\right) \tag{1}$$

where $\Phi\left(z_0, 0\right)$ is the potential of Problem 3.18 evaluated at $z = z_0, \rho = 0$ Find the total charge induced on the upper plate. Compare with the solution (in method and answer) of Problem 1.13 .

Solution:
The Green's reciprocation theorem is given by

$$\int_V \rho\Phi' d^3x + \int_S \sigma\Phi' da = \int_V \rho'\Phi d^3x + \int_S \sigma'\Phi da \tag{2}$$

We let the quantities in Problem 3.18 become the unprimed variables and the quantities in this problem be the primed variables. From the given scenarios, we have

$$\rho(\mathbf{x}) = 0 \tag{3}$$

$$\Phi(\mathbf{x}) = \begin{cases} 0, & z = 0 \\ 0, & z = L, r > a \\ V, & z = L, r < a \\ V\int_0^\infty d\lambda J_1(\lambda)J_0(\lambda r/a)\frac{\sinh(\lambda z/a)}{\sinh(\lambda L/a)}, & 0 < z < L \end{cases} \tag{4}$$

and

$$\rho'(\mathbf{x}) = \frac{q}{2\pi r}\delta(r-0)\delta(z-z_0) \tag{5}$$

$$\Phi'(\mathbf{x}) = 0, \quad z = 0 \,\text{and}\, z = L \tag{6}$$

Note that we do not know $\sigma(\mathbf{x})$ and $\sigma'(\mathbf{x})$, as well as $\Phi'(\mathbf{x})$ on the interval $0 < z < L$. Plugging these into the reciprocation theorem, we have

$$\begin{aligned} \int \sigma(r,\phi,0<z<L)\Phi'(r,\phi,0<z<L)da &= \int \rho'(r,\phi,0<z<L)\Phi(r,\phi,0<z<L)d^3x \\ &+ \int \sigma'(r,\phi,0<z<L)\Phi(r,\phi,0<z<L)da \\ &+ \int \sigma'(r<a,\phi,z=L)\Phi(r<a,\phi,z=L)da \end{aligned} \tag{7}$$

Since there should be no surface charge densities in the region $0 < z < L$, we have

$$\begin{aligned}
V\int \sigma'(r<a,\phi,z=L)da &= -\int \frac{q}{2\pi r}\delta(r-0)\delta(z-z_0)\left(V\int_0^\infty d\lambda J_1(\lambda)J_0(\lambda r/a)\frac{\sinh(\lambda z/a)}{\sinh(\lambda L/a)}\right)d^3x \\
&= -\frac{qV}{2\pi}\int_0^\infty \frac{J_1(\lambda)}{\sinh(\lambda L/a)}\left(\int_0^{2\pi}\int_0^L\int_0^a J_0(\lambda r/a)\sinh(\lambda z/a)\frac{\delta(r-0)\delta(z-z_0)}{r}r\mathrm{d}r\mathrm{d}z\mathrm{d}\theta\right)\mathrm{d}\lambda \\
&= -\frac{qV}{2\pi}2\pi\int_0^\infty \frac{J_1(\lambda)}{\sinh(\lambda L/a)}J_0(0)\sinh(\lambda z_0/a)\mathrm{d}\lambda \\
V\int \sigma'(r<a,\phi,z=L)da &= -qV\int_0^\infty J_1(\lambda)\frac{\sinh(\lambda z_0/a)}{\sinh(\lambda L/a)}\mathrm{d}\lambda
\end{aligned} \tag{8}$$

Note that from Problem 3.18 (as shown in Eq. (4)), the unprimed potential when $r=0$ and $z=z_0$ is

$$\begin{aligned}
\Phi(r=0,\phi,z=z_0) &= V\int_0^\infty d\lambda J_1(\lambda)J_0(0)\frac{\sinh(\lambda z_0/a)}{\sinh(\lambda L/a)} \\
\Phi(0,z_0) &= V\int_0^\infty d\lambda J_1(\lambda)\frac{\sinh(\lambda z_0/a)}{\sinh(\lambda L/a)}
\end{aligned} \tag{9}$$

as it has no ϕ-dependence. With this, Eq. (8) becomes

$$\begin{aligned}
\int \sigma'(r<a,\phi,z=L)da &= -\frac{q}{V}V\int_0^\infty J_1(\lambda)\frac{\sinh(\lambda z_0/a)}{\sinh(\lambda L/a)}\mathrm{d}\lambda \\
Q_L(a) &= -\frac{q}{V}\Phi(0,z_0)
\end{aligned} \tag{10}$$

as LHS integral is just the total charge on the region where $z=L$ and $r<a$.

(b) Show that the induced charge density on the upper plate can be written as

$$\sigma(\rho) = -\frac{q}{2\pi}\int_0^\infty dk\frac{\sinh(kz_0)}{\sinh(kL)}kJ_0(k\rho)$$

Solution:
With the azimuthal symmetry, σ' should have no ϕ-dependence. Also, as there should be no discontinuity in σ' along r, Eq. (10) becomes

$$\int \sigma'(r,L)da = -q\int_0^\infty J_1(\lambda)\frac{\sinh(\lambda z_0/a)}{\sinh(\lambda L/a)}\mathrm{d}\lambda \tag{11}$$

We let $\lambda = ka$. With $\mathrm{d}a = r\mathrm{d}r\mathrm{d}\phi$, we have

$$\begin{aligned}
\left(\int_0^{2\pi}\mathrm{d}\phi\right)\left(\int_0^a \sigma'(r,L)r\mathrm{d}r\right) &= -q\int_0^\infty aJ_1(ka)\frac{\sinh(kz_0)}{\sinh(kL)}\mathrm{d}k \\
2\pi\frac{\partial}{\partial a}\left(\int_0^a \sigma'(r,L)r\mathrm{d}r\right) &= -q\frac{\partial}{\partial a}\int_0^\infty aJ_1(ka)\frac{\sinh(kz_0)}{\sinh(kL)}\mathrm{d}k \\
2\pi a\sigma'(a,L) &= -q\int_0^\infty \frac{\partial}{\partial a}\left[aJ_1(ka)\right]\frac{\sinh(kz_0)}{\sinh(kL)}\mathrm{d}k
\end{aligned} \tag{12}$$

Note that Bessel functions of the first kind has the following property:

$$\frac{d}{dx}\left(x^nJ_n(x)\right) = x^nJ_{n-1} \tag{13}$$

Then,

$$\frac{\partial}{\partial a}\left[aJ_1(ka)\right] = k\frac{\partial}{\partial \lambda}\left[\frac{\lambda}{k}J_1(\lambda)\right] = \frac{\partial}{\partial \lambda}\left[\lambda J_1(\lambda)\right] = \lambda J_0(\lambda) = kaJ_0(ka) \tag{14}$$

Thus, we have

$$2\pi a\sigma'(a,L) = -qa\int_0^\infty kJ_0(ka)\frac{\sinh(kz_0)}{\sinh(kL)}\mathrm{d}k \tag{15}$$

Rearranging this, we get

$$\sigma'(a,L) = -\frac{q}{2\pi}\int_0^\infty kJ_0(ka)\frac{\sinh(kz_0)}{\sinh(kL)}\mathrm{d}k \tag{16}$$

which we can generalize for all r in the upper plate ($z = L$):

$$\sigma'(r,L) = -\frac{q}{2\pi}\int_0^\infty kJ_0(kr)\frac{\sinh(kz_0)}{\sinh(kL)}\mathrm{d}k \tag{17}$$

which shows the given.