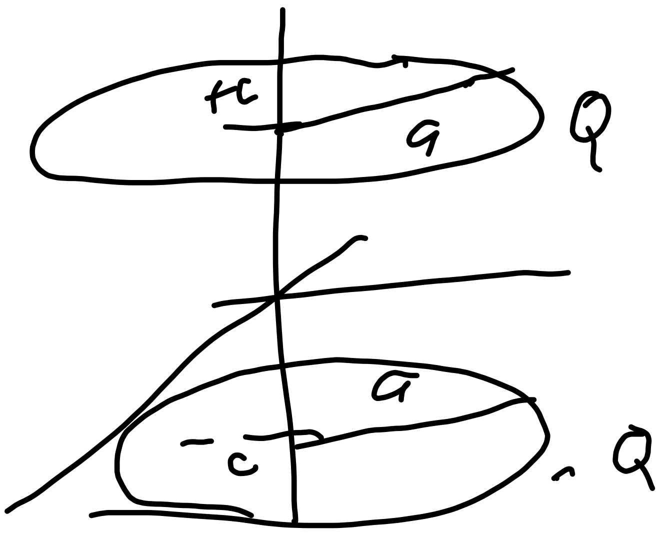
Physics 231 First Long Examination

1. (a) Obtain the scalar potential at a distance d from the center of and perpendicular to a square sheet with a constant dipole moment density D. (b) How does the potential behave for arbitrarily large d?

2. Write down the electric charge density corresponding to the charge distribution depicted in the figure below.

3. A point charge Q is placed near outside a conducting hollow sphere that is maintained at a potential $V > 0$. Use the method of images to determine the potential inside and outside the sphere.

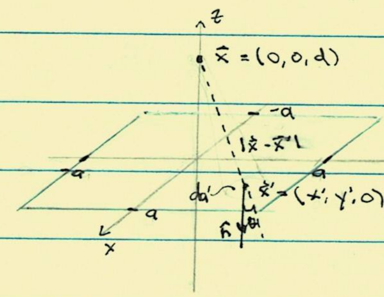
4. Solve problem 2.14 found in page-89 of Jackson's book (3rd ed).

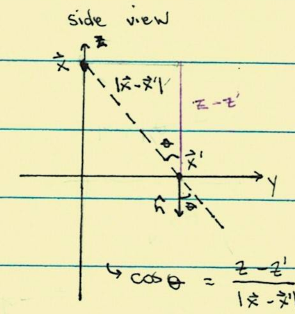


Physics 231 - LE 1

Angeles, Janna May Z. (2015-09948)

Problem 1





We assume that the square sheet has a region of $-a \le x, y \le a$ and is placed at $z=0$.

(a) The potential due to a dipole layer can be written as

$$\Phi(\vec{x}) = \frac{-1}{4\pi\epsilon_0}\int_S D(\vec{x}')\, d\Omega' \qquad (1)$$

where

$$d\Omega' = \frac{\cos\theta\, da'}{|\vec{x}-\vec{x}'|^2} \qquad (2)$$

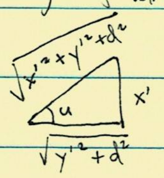
Note that, for this case, $\cos\theta = \frac{+(z-z')}{|\vec{x}-\vec{x}'|}$. Using Cartesian coordinates, Eq. (2) becomes

$$d\Omega' = +\frac{z-z'}{|\vec{x}-\vec{x}'|^3}dx'dy' = +\frac{(d-0)\,dx'dy'}{[(0-x')^2+(0-y')^2+(d-0)^2]^{3/2}} \qquad (3)$$

Substituting Eq. (3) into Eq. (1) and noting that $D(\vec{x}) = D$ is a constant, we have

$$\Phi(\vec{x}) = -\frac{Dd}{4\pi\epsilon_0}\int_{-a}^{a}\int_{-a}^{a}\frac{dx'\,dy'}{(x'^2+y'^2+d^2)^{3/2}} \qquad (4)$$

We let $x' = \sqrt{y'^2+d^2}\tan u$ which results to $dx' = \sqrt{y'^2+d^2}\sec^2 u\, du$. Applying this change of variables



$$\sin u = \frac{x'}{\sqrt{x'^2+y'^2+d^2}}$$

$$\Phi(\vec{x}) = \frac{-Dd}{4\pi\epsilon_0}\int_{-a}^{a}\left(\int\frac{\sqrt{y'^2+d^2}\,\sec^2 u\,du}{[\tan^2 u\,(y'^2+d^2)+y'^2+d^2]^{3/2}}\right)dy'$$

$$= \frac{-Dd}{4\pi\epsilon_0}\int_{-a}^{a}\left(\int\frac{\sqrt{y'^2+d^2}\,\sec^2 u\,du}{(y'^2+d^2)^{3/2}\,(\underbrace{\tan^2 u+1}_{\sec^2 u})^{3/2}}\right)dy'$$

$$= \frac{-Dd}{4\pi\epsilon_0}\int_{-a}^{a}\left(\frac{1}{y'^2+d^2}\int\frac{du}{\sec u}\right)dy'$$

$$= \frac{-Dd}{4\pi\epsilon_0}\int_{-a}^{a}\frac{1}{y'^2+d^2}\left(\sin\Big[\overbrace{\tan^{-1}\Big(\frac{x'}{\sqrt{y'^2+d^2}}\Big)}^{u}\Big]\Bigg|_{x'=-a}^{x'=a}\right)dy'$$

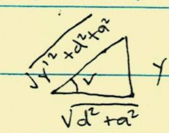
$$= \frac{-Dd}{4\pi\epsilon_0}\int_{-a}^{a}\frac{1}{y'^2+d^2}\left(\frac{x'}{\sqrt{x'^2+y'^2+d^2}}\Bigg|_{x'=-a}^{x'=a}\right)dy'$$

$$\Phi(\vec{x}) = -\frac{Dd}{4\pi\epsilon_0}\int_{-a}^{a}\frac{1}{y'^2+d^2}\cdot\frac{2a}{\sqrt{a^2+y'^2+d^2}}\,dy' \qquad (5)$$

Physics 231 - LE 1

Angeles, Janna May Z. (2015-09918)

By applying another change of variables where we let $y' = \sqrt{d^2+a^2}\tan v$ $\rightarrow dy' = \sqrt{d^2+a^2}\sec^2 v\, dv$, Eq. (5) becomes



$\sin v = \dfrac{y'}{\sqrt{y'^2+d^2+a^2}}$

$$\Phi(\vec{x}) = \frac{-2aDd}{4\pi\epsilon_0}\int \frac{1}{(d^2+a^2)\tan^2 v + d^2}\,\frac{\sqrt{d^2+a^2}\,\sec^2 v\, dv}{\sqrt{(d^2+a^2)\tan^2 v + d^2+a^2}}$$

$$= \frac{-2aDd}{4\pi\epsilon_0}\int \frac{1}{(d^2+a^2)\tan^2 v + d^2}\,\frac{\cancel{\sqrt{d^2+a^2}}\,\sec^2 v}{\cancel{\sqrt{d^2+a^2}}\,\sqrt{\sec^2 v}}\,dv \qquad (6)$$

Note that: $$(d^2+a^2)\tan^2 v + d^2 = d^2\overbrace{(\tan^2 v+1)}^{\sec^2 v} + a^2\tan^2 v = \sec^2 v\,(d^2 + a^2\sin^2 v) \qquad (7)$$

Substituting this to Eq. (6),

$$\Phi(\vec{x}) = \frac{-2aDd}{4\pi\epsilon_0}\int \frac{\sec v\, dv}{\sec^2 v\,(d^2+a^2\sin^2 v)} = \frac{2aDd}{4\pi\epsilon_0}\int \frac{\cos v}{d^2+a^2\sin^2 v}\,dv \qquad (8)$$

Applying $w = \sin v \rightarrow dw = \cos v\, dv$, Eq. (8) becomes

$$\Phi(\vec{x}) = -\frac{2aDd}{4\pi\epsilon_0}\int \frac{dw}{d^2+a^2w^2} \qquad (9)$$

which can be written as (where $\alpha = \frac{a}{d}w \rightarrow d\alpha = \frac{a}{d}dw$)

$$\Phi(\vec{x}) = \frac{-2aDd}{4\pi\epsilon_0}\int \frac{(d/a)\,d\alpha}{d^2+a^2\left(\frac{d}{a}\alpha\right)^2} = \frac{2aDd}{4\pi\epsilon_0}\int \frac{d/a}{d^2(1+\alpha^2)}\,d\alpha = \frac{2D}{4\pi\epsilon_0}\int \frac{d\alpha}{1+\alpha^2} \qquad (10)$$

Note that: $$\int \frac{dx}{1+x^2} = \tan^{-1}x + C \qquad (11)$$

Thus, Eq. (10) becomes

$$\Phi(\vec{x}) = \frac{-2D}{4\pi\epsilon_0}\tan^{-1}\left\{\frac{a}{d}\sin\left[\overbrace{\tan^{-1}\left(\frac{y'}{\sqrt{d^2+a^2}}\right)}^{v}\right]\right\}\Bigg|_{y'=-a}^{y'=a} = \frac{2D}{4\pi\epsilon_0}\tan^{-1}\left(\frac{a}{d}\,\frac{y'}{\sqrt{y'^2+d^2+a^2}}\right)\Bigg|_{y'=-a}^{y'=a} \qquad (12)$$

Note that $\tan^{-1}x = \tan^{-1}(-x)$. Thus,

$$\Phi(\vec{x}) = \frac{-2D}{4\pi\epsilon_0}\left[\tan^{-1}\left(\frac{a}{d}\,\frac{a}{\sqrt{a^2+d^2+a^2}}\right) - \tan^{-1}\left(\frac{a}{d}\,\frac{-a}{\sqrt{(-a)^2+d^2+a^2}}\right)\right] = \frac{2D}{4\pi\epsilon_0}\cdot 2\tan^{-1}\left(\frac{a^2}{d\sqrt{2a^2+d^2}}\right) \qquad (13)$$

Therefore, the potential is given by

$$\boxed{\Phi(\vec{x}) = -\frac{D}{\pi\epsilon_0}\tan^{-1}\left(\frac{a^2}{d\sqrt{2a^2+d^2}}\right)} \qquad (14)$$

Physics 231 - LE1
Angeles, Janna May Z.

(b) The series expansion of $\tan^{-1} x$ is

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots, \qquad |x| < 1 \tag{15}$$

Now, for arbitrarily large d (d > a), we can express (14) as

$$\Phi(\vec{x}) \approx \frac{-D}{\pi\epsilon_0} \tan^{-1}\left(\frac{a^2}{d\sqrt{d^2}}\right) = \frac{-D}{\pi\epsilon_0} \tan^{-1}\left(\frac{a^2}{d^2}\right) \tag{16}$$

Since $a/d < 1$, we can use the expansion in Eq. (15) to write Eq. (16) as

$$\boxed{\Phi(\vec{x}) \approx \frac{-Da^2}{\pi\epsilon_0 d^2}} \tag{17}$$

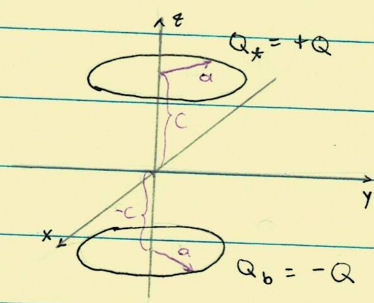
in which we ignore higher order terms of $\left(\frac{a}{d}\right)^2$ because of their negligible value.

Physics 231 - LE1

Angeles, Janna May Z. (2015-09948)

Problem 2

A diagram of the problem (note that charges +Q and -Q are uniformly distributed on the top and bottom rings, respectively.)



In general, the electric charge density function is given by

$$\rho(\vec{x}) = Q\,\delta(\vec{x} - \vec{x}_0) \tag{1}$$

where Q is the total charge and $\vec{x}_0$ is the location of the source. When there are more than one uniformly charged object in the system, we can just add the contribution of each object to calculate the total charge density. Thus, for this problem, we have

$$\rho(\vec{x}) = Q_t\,\delta(\vec{x} - \vec{x}_t) + Q_b\,\delta(\vec{x} - \vec{x}_b) \tag{2}$$

Note that there is symmetry along ϕ so to take advantage of this symmetry, we will use cylindrical coordinates. In this coordinate system, the Dirac delta function is expressed as

$$\delta(\vec{x} - \vec{x}_0) = \frac{1}{r}\,\delta(r - r_0)\,\delta(\theta - \theta_0)\,\delta(z - z_0) \tag{3}$$

Projecting out the ϕ-integral due to the azimuthal symmetry, the denominator in Eq. (3) becomes:

$$\int_0^{2\pi} r\,d\phi = 2\pi r \tag{4}$$

Consequently, Eq. (3) is now written as

$$\delta(\vec{x} - \vec{x}_0) = \frac{1}{2\pi r}\,\delta(r - r_0)\,\delta(z - z_0) \tag{5}$$

Thus, for the top and bottom rings respectively, we have

$$\delta(\vec{x} - \vec{x}_t) = \frac{1}{2\pi r}\,\delta(r-a)\,\delta(z-c) \quad (6) \qquad \delta(\vec{x} - \vec{x}_b) = \frac{1}{2\pi r}\,\delta(r-a)\,\delta(z-(-c)) \quad (7)$$

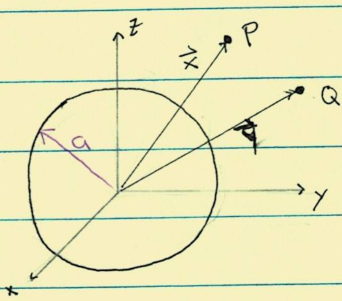
Therefore, Eq. (2) becomes

$$\boxed{\rho(\vec{x})} = \frac{Q}{2\pi r}\,\delta(r-a)\,\delta(z-c) - \frac{Q}{2\pi r}\,\delta(r-a)\,\delta(z+c) \boxed{= \frac{Q}{2\pi r}\,\delta(r-a)\left[\delta(z-c) - \delta(z+c)\right]} \tag{8}$$

Physics 231 - LE 1
Angeles, Janna May Z (2015-09948)

Problem 3

A diagram of the problem ▸



We can first treat this problem like that of the case of a point charge in the presence of a grounded conducting sphere. By method of images, the potential for this case is given by

$$\Phi(\vec{x}) = \frac{1}{4\pi\epsilon_0}\left[\frac{Q}{|\vec{x}-\vec{y}|} + \frac{q'}{|\vec{x} - \vec{y}'|}\right] = \frac{1}{4\pi\epsilon_0}\left[\frac{Q}{|\vec{x}-\vec{y}|} - \frac{aQ}{y\left|\vec{x} - \frac{a^2}{y^2}\vec{y}\right|}\right] \quad (1)$$

where $q' = -(a/y)Q$ $\left[\vec{y}' = \frac{a^2}{y^2}\vec{y}\right]$ and vanishes $(\Phi(|\vec{x}|=a) = 0)$ at $|\vec{x}| = a$. To go back to the original problem, we remove the connection grounding the sphere and add to the sphere's surface a fixed potential $V > 0$. We want to add a second image charge q'' to figure out the changes in the calculated potential. Since q' balanced out the electrostatic forces due to Q, q'' should distribute uniformly on the sphere's surface. For points outside, this can be percieved as if q'' is a point charge at the origin. Thus, by linear superposition,

$$\Phi(\vec{x}) = \frac{1}{4\pi\epsilon_0}\left[\frac{Q}{|\vec{x}-\vec{y}|} - \frac{aQ}{y\left|\vec{x}-\frac{a^2}{y^2}\vec{y}\right|} + \frac{q''}{|\vec{x}-\vec{y}''|}\right] \quad (2)$$

where $\vec{y}'' = \vec{0}$. At $|\vec{x}| = a$

$$\Phi(|\vec{x}|=a) = \frac{1}{4\pi\epsilon_0}\frac{q''}{a} \quad (3)$$

as the first two terms cancel each other out at the surface. With $\Phi(|\vec{x}|=a) = V$,

$$V = \frac{1}{4\pi\epsilon_0}\frac{q''}{a} \longrightarrow q'' = 4\pi\epsilon_0 Va \quad (4)$$

Substituting this into Eq. (2), the potential outside the sphere is given by

$$\Phi_{out}(\vec{x}) = \frac{1}{4\pi\epsilon_0}\left[\frac{Q}{|\vec{x}-\vec{y}|} - \frac{aQ}{y\left|\vec{x}-\frac{a^2}{y^2}\vec{y}\right|}\right] + \frac{Va}{|\vec{x}|} \quad (5)$$

As for the potential inside, note again that the sphere in this scenario is hollow and conductive. Thus, there can be no electric field inside the

Physics 231 - LE 1
Angeles, Janna May Z.

sphere since any net charges in conductors must lie on their surfaces. As a result, the potential is constant inside (from $\vec{E} = -\nabla \Phi$)

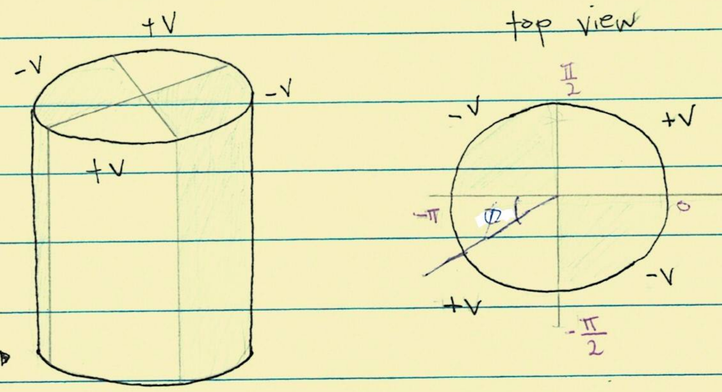
$$\Phi_{in}(\vec{x}) = \text{constant} \tag{6}$$

Physics 231- LE 1

Angeles, Janna May Z. (2015-09998)

Problem 4

A diagram of the problem (The angle ϕ runs in the interval $-\pi \leq \phi < \pi$. This interval is chosen for convenience when solving for the coeff. of Fourier series).



Since this 3D system is just a variant of a 2D problem, we can use the general solution of the potential for such problems given as

$$\Phi(\rho,\phi) = a_0 + b_0 \ln\rho + \sum_{n=1}^{\infty} a_n \rho^n \sin(n\phi+\alpha_n) + \sum_{n=1}^{\infty} b_n \rho^{-n} \sin(n\phi+\beta_n) \quad (1)$$

We impose the condition that $\Phi(\rho,\phi)$ must be finite at $\rho=0$ as we are evaluating the potential inside the cylinder. Thus, the second and the last term must vanish, leaving us with

$$\Phi(\rho,\phi) = a_0 + \sum_{n=1}^{\infty} a_n \rho^n \sin(n\phi+\alpha_n) \quad (2)$$

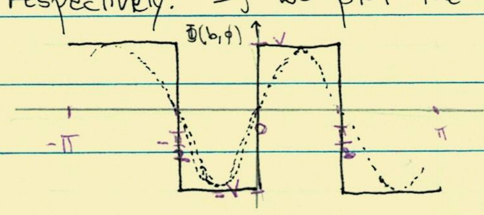
With the trigonometric identity : $\sin(x+y) = \sin x \cos y + \cos x \sin y$

$$a_n \sin(n\phi+\alpha_n) = a_n \cos\alpha_n \sin n\phi + a_n \sin\alpha_n \cos n\phi \quad (3)$$

Substituting Eq. (3) into Eq. (2), we get

$$\Phi(\rho,\phi) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos\alpha_n \sin n\phi + a_n \sin\alpha_n \cos n\phi\right)\rho^n \quad (4)$$

Note that this looks like a Fourier series where $a_n \cos\alpha_n$ and $a_n \sin\alpha_n$ are coefficients of $\sin n\phi$ and $\cos n\phi$, respectively. If we plot the potential at the surface as shown :



then, this suggests that the potential is similar to a sine wave which indicates that a_0 and $a_n \sin\alpha_n$ must vanish. Updating Eq. (4), we have

$$\Phi(\rho,\phi) = \sum_{n=1}^{\infty} a_n \cos\alpha_n \rho^n \sin n\phi = \sum_{n=1}^{\infty} B_n(\rho) \sin n\phi \quad (5)$$

where we let $B_n(\rho) = a_n \cos\alpha_n \rho^n$ to imitate the form of a Fourier sine series. At $\rho = b$, Eq. (5)

Physics 231 - LE 1

Angeles, Janna May Z.

results to:

$$\Phi(\rho = b, \phi) = \sum_{n=1}^{\infty} B_n(b) \sin n\phi \tag{6}$$

Now, recall the Fourier sine series generally has the form

$$f(x) = \sum_{n=1}^{\infty} b_n \sin(nx) \tag{7}$$

whose coefficients b_n are given by

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \sin(nx)\, dx \tag{8}$$

Applying this to Eq.(6), the coefficients $B_n(b)$ can be calculated by

$$B_n(b) = \frac{1}{\pi}\int_{-\pi}^{\pi} \Phi(b, \phi) \sin n\phi \, d\phi$$

$$= \frac{1}{\pi}\left(\int_{-\pi}^{-\frac{\pi}{2}} V \sin n\phi \, d\phi + \int_{-\frac{\pi}{2}}^{0} (-V) \sin n\phi \, d\phi + \int_{0}^{\frac{\pi}{2}} V \sin n\phi \, d\phi + \int_{\frac{\pi}{2}}^{\pi} (-V) \sin n\phi \, d\phi\right)$$

$$= \frac{V}{\pi}\left[-\frac{\cos n\phi}{n}\Big|_{-\pi}^{-\frac{\pi}{2}} + \frac{\cos n\phi}{n}\Big|_{-\frac{\pi}{2}}^{0} - \frac{\cos n\phi}{n}\Big|_{0}^{\frac{\pi}{2}} + \frac{\cos n\phi}{n}\Big|_{\frac{\pi}{2}}^{\pi}\right]$$

$$= \frac{V}{n\pi}\left[-\left(\cos\frac{n\pi}{2} - \cos n\pi\right) + \left(1 - \cos\frac{n\pi}{2}\right) - \left(\cos\frac{n\pi}{2} - 1\right) + \left(\cos n\pi - \cos\frac{n\pi}{2}\right)\right]$$

$$B_n(b) = \frac{2V}{n\pi}\left(1 + \cos n\pi - 2\cos\frac{n\pi}{2}\right) \tag{9}$$

Let's solve $1 + \cos n\pi - 2\cos\frac{n\pi}{2}$ for specific values of n:

$n=1: \; 1 + \cos\pi - 2\cos\frac{\pi}{2} = 1 + (-1) - 0$

$n=2: \; 1 + \cos 2\pi - 2\cos\pi = 1 + 1 - 2(-1) = 4$

$n=3: \; 1 + \cos 3\pi - 2\cos\frac{3\pi}{2} = 1 + (-1) - 0 = 0$

$n=4: \; 1 + \cos 4\pi - 2\cos 2\pi = 1 + 1 - 2(1) = 0$

$n=5: \; 1 + \cos 5\pi - 2\cos\frac{5\pi}{2} = 1 + (-1) - 0 = 0$

$n=6: \; 1 + \cos 6\pi - 2\cos 3\pi = 1 + 1 - 2(-1) = 4$

We see here that n needs to be an even number that results to a whole odd number when divided by 2 for a non-zero value of $1 + \cos n\pi - 2\cos\frac{n\pi}{2}$ (specifically, resulting to 4).

Thus, n must be $n = 2(2m+1) = 4m+2$ where $m = 0, 1, 2, \ldots$. Applying this to Eq.(9)

$$B_n(b) = \frac{2V}{n\pi}\underbrace{\left(1 + \cos n\pi - 2\cos\frac{n\pi}{2}\right)}_{4} = \frac{8V}{n\pi} \tag{10}$$

Physics 231 - LE1
Angeles, Janna May Z. (2015-09948)

Equating $B_n(b) = a_n \cos\alpha_n b^n$ and $B_n(b) = 8V/n\pi$ from Eq. (10),

$$a_n \cos\alpha_n b^n = \frac{8V}{n\pi} \longrightarrow a_n \cos\alpha_n = \frac{8V}{n\pi}\frac{1}{b^n} \tag{11}$$

Substituting Eq. (11) to Eq. (5),

$$\Phi(\rho,\phi) = \sum_{n=2,6,10,\ldots} a_n \cos\alpha_n \rho^n \sin n\phi$$

$$= \sum_{n=2,6,10,\ldots} \frac{8V}{n\pi}\frac{\rho^n}{b^n}\sin n\phi$$

$$\Phi(\rho,\phi) = \frac{4V}{\pi}\sum_{m=0}^{\infty}\left(\frac{\rho}{b}\right)^{4m+2}\frac{\sin(4m+2)\phi}{2m+1} \tag{12}$$

Since m is just a dummy variable, we can exchange it with n. So, we have

$$\boxed{\Phi(\rho,\phi) = \frac{4V}{\pi}\sum_{n=0}^{\infty}\left(\frac{\rho}{b}\right)^{4n+2}\frac{\sin(4n+2)\phi}{2n+1}} \tag{13}$$

(b) Recall Euler's equation : $e^{i\theta} = \cos\theta + i\sin\theta$. Note here that $\sin\theta = \mathrm{Im}[e^{i\theta}]$.
If we apply this to Eq. (13), we obtain

$$\Phi(\rho,\phi) = \frac{4V}{\pi}\,\mathrm{Im}\left[\sum_{n=0}^{\infty}\left(\frac{\rho}{b}\right)^{4n+2}\frac{e^{i(4n+2)\phi}}{2n+1}\right] = \frac{4V}{\pi}\,\mathrm{Im}\left[\sum_{n=0}^{\infty}\left[\left(\frac{\rho}{b}\right)^2 e^{i2\phi}\right]^{2n+1}\cdot\frac{1}{2n+1}\right] \tag{14}$$

Let $z = \left(\frac{\rho}{b}\right)^2 e^{i2\phi}$. Then,

$$\Phi(\rho,\phi) = \frac{4V}{\pi}\,\mathrm{Im}\left[\sum_{n=0}^{\infty}\frac{z^{2n+1}}{2n+1}\right] = \frac{2V}{\pi}\,\mathrm{Im}\left[\ln\frac{1+z}{1-z}\right] \tag{15}$$

after noting that : $$\ln\frac{1+x}{1-x} = 2\sum_{n=0}^{\infty}\frac{x^{2n+1}}{2n+1} \tag{16}$$

Also, note that : $$\ln w = \frac{1}{2}\ln(a^2+b^2) + i\tan^{-1}\frac{b}{a}, \quad w = a+ib \tag{17}$$

We consider $(1+z)/(1-z)$ where $z = \left(\frac{\rho}{b}\right)^2 e^{i2\phi}$.

$$\frac{1+z}{1-z} = \frac{1+\left(\frac{\rho}{b}\right)^2 e^{i2\phi}}{1-\left(\frac{\rho}{b}\right)^2 e^{i2\phi}} = \frac{1+\left(\frac{\rho}{b}\right)^2(\cos2\phi + i\sin2\phi)}{1-\left(\frac{\rho}{b}\right)^2(\cos2\phi - i\sin2\phi)} = \frac{1+\left(\frac{\rho}{b}\right)^2\cos2\phi + i\sin2\phi\left(\frac{\rho}{b}\right)^2}{1-\left(\frac{\rho}{b}\right)^2\cos2\phi - i\sin2\phi\left(\frac{\rho}{b}\right)^2} \tag{18}$$

Let $c_+ = 1+\left(\frac{\rho}{b}\right)^2\cos2\phi$, $c_- = 1-\left(\frac{\rho}{b}\right)^2\cos2\phi$, and $d = \sin2\phi\left(\frac{\rho}{b}\right)^2$. Then, by rationalising

NO.: 4/6
DATE:

Physics 231 - LE 1

Angeles, Janna May Z. (2015-09948)

$$\frac{1+z}{1-z} = \frac{c_+ + id}{c_- - id} \cdot \frac{c_- + id}{c_- + id} = \frac{c_+c_- - d^2 + ic_-d + ic_+d}{c_-^2 + d^2} = \frac{c_+c_- - d^2}{c_-^2+d^2} + i\,\frac{(c_+ + c_-)d}{c_-^2 + d^2} \quad (19)$$

Using Eq. (17), Eq. (13) becomes

$$\Phi(\rho,\phi) = \frac{2V}{\pi}\tan^{-1}\left[\frac{\mathrm{Im}\left(\frac{1+z}{1-z}\right)}{\mathrm{Re}\left(\frac{1+z}{1-z}\right)}\right] \quad (20)$$

From Eq. (19), we know that

$$\frac{\mathrm{Im}\left(\frac{1+z}{1-z}\right)}{\mathrm{Re}\left(\frac{1+z}{1-z}\right)} = \frac{(c_+ + c_-)d}{c_-^2+d^2}\cdot\frac{c_-^2+d^2}{c_+c_- - d^2} = \frac{2\sin 2\phi\,\left(\frac{\rho}{b}\right)^2}{1-\left(\frac{\rho}{b}\right)^4\cos^2 2\phi - \left(\frac{\rho}{b}\right)^4\sin^2 2\phi} = \frac{2\left(\frac{\rho}{b}\right)^2\sin 2\phi}{1-\left(\frac{\rho}{b}\right)^4(\cos^2 2\phi + \sin^2 2\phi)} \quad (21)$$

Multiplying Eq. (21) by b^4/b^4,

$$\frac{\mathrm{Im}\left(\frac{1+z}{1-z}\right)}{\mathrm{Re}\left(\frac{1+z}{1-z}\right)} = \frac{2\rho^2b^2\sin 2\phi}{b^4-\rho^4} \quad (22)$$

Substituting Eq. (22) into Eq. (20), we obtain

$$\boxed{\Phi(\rho,\phi) = \frac{2V}{\pi}\tan^{-1}\left(\frac{2\rho^2b^2\sin 2\phi}{b^4-\rho^4}\right)} \quad (23)$$

(c) An equipotential is a curve where $\Phi(\rho,\phi)$ = constant = Φ_0. Applying this to Eq. (23), equipotentials in this situation has the form

$$\Phi_0 = \frac{2V}{\pi}\tan^{-1}\left(\frac{2\rho^2b^2\sin 2\phi}{b^4-\rho^4}\right) \quad (24)$$

Rearranging this,

$$\tan\frac{\pi\Phi_0}{2V} = \frac{2\left(\frac{\rho}{b}\right)^2\sin 2\phi}{1-\left(\frac{\rho}{b}\right)^4} \quad (25)$$

$$\tan\frac{\pi\Phi_0}{2V} - \tan\frac{\pi\Phi_0}{2V}\left(\frac{\rho}{b}\right)^4 = 2\sin 2\phi\left(\frac{\rho}{b}\right)^2 \quad (26)$$

$$\left[\left(\frac{\rho}{b}\right)^2\right]^2 + 2\,\frac{\sin 2\phi}{\tan\frac{\pi\Phi_0}{2V}}\left(\frac{\rho}{b}\right)^2 - 1 = 0 \quad (27)$$

Using the quadratic formula,

$$\left(\frac{\rho}{b}\right)^2 = \frac{-2\frac{\sin 2\phi}{\tan\frac{\pi\Phi_0}{2V}} \pm \sqrt{4\frac{\sin^2 2\phi}{\tan^2\frac{\pi\Phi_0}{2V}} - 4(1)(-1)}}{2} = -\frac{\sin 2\phi}{\tan\frac{\pi\Phi_0}{2V}} \pm \sqrt{1+\frac{\sin^2 2\phi}{\tan^2\frac{\pi\Phi_0}{2V}}} \quad (28)$$

Physics 231 - LE 1

Angeles, Janna May Z. (2015-09948)

We choose the positive solution since the the negative one will give an imaginary solution. Thus,

$$\left(\frac{\rho}{b}\right)^2 = -\frac{\sin 2\phi}{\tan\frac{\pi\Phi_0}{2V}} + \sqrt{1 + \frac{\sin^2 2\phi}{\tan^2\frac{\pi\Phi_0}{2V}}} \tag{29}$$

From this, we obtain

$$\rho(\phi) = b\sqrt{-\frac{\sin 2\phi}{\tan\frac{\pi\Phi_0}{2V}} + \sqrt{1 + \frac{\sin^2 2\phi}{\tan^2\frac{\pi\Phi_0}{2V}}}} \tag{30}$$

as the equipotential's form. For the field lines, we note that

$$\vec{E} = -\nabla\Phi(\rho,\phi) = -\left(\frac{\partial\Phi}{\partial\rho}\hat{\rho} + \frac{1}{\rho}\frac{\partial\Phi}{\partial\phi}\hat{\phi}\right) \tag{31}$$

in polar coordinates. With $\Phi(\rho,\phi)$ given in Eq. (23), the field lines have the form of

$$\vec{E} = -\frac{2V}{\pi}\left[\frac{1}{1+\left(\frac{2\rho^2b^2\sin 2\phi}{b^4-\rho^4}\right)^2}\cdot 2b^2\sin 2\phi\left(\frac{(b^4-\rho^4)\cdot 2\rho + \rho^4(-4\rho^3)}{(b^4-\rho^4)^2}\right)\hat{\rho}\right.$$

$$\left. + \frac{1}{\rho}\frac{1}{1+\left(\frac{2\rho^2b^2\sin 2\phi}{b^4-\rho^4}\right)^2}\cdot\frac{2\rho^2b^2}{b^4-\rho^4}\cos 2\phi\cdot 2\,\hat{\phi}\right]$$

$$= -\frac{2V}{\pi}\frac{2b^2\rho}{1+\left(\frac{2\rho^2b^2\sin 2\phi}{b^4-\rho^4}\right)^2}\left[\sin 2\phi\,\frac{2b^4+2\rho^4}{(b^4-\rho^4)^2}\hat{\rho} + \frac{2\cos 2\phi}{b^4-\rho^4}\hat{\phi}\right]$$

$$\vec{E} = -\frac{8b^2V}{\pi}\frac{\rho}{(b^4-\rho^4)^2 + 2\rho^2b^2\sin 2\phi}\left[\sin 2\phi\,(b^4+\rho^4)\,\hat{\rho} + \cos 2\phi\,(b^4-\rho^4)\,\hat{\phi}\right] \tag{32}$$

in which we note that:

$$\frac{d}{dx}\left(\tan^{-1}x\right) = \frac{1}{1+x^2} \tag{33}$$

The field lines $\vec{E}$ and equipotentials $\rho(\phi)$ are plotted in Mathematica and are shown in page 6.

Physics 231 - LE 1
Angeles, Janna May Z. (2015 - 09948)

To plot the equipotentials and field lines, we first need to assume values for V an b :

In[1]:=
```
V = 3; b = 1;
```

As given in Eq. (30), equipotentials in this problem has the form of

In[2]:=
$$\rho[\phi\_, \Phi 0\_] = b \sqrt{\frac{-\text{Sin}[2\,\phi]}{\text{Tan}\left[\frac{\pi\,\Phi 0}{2\,V}\right]} + \sqrt{1 + \frac{(\text{Sin}[2\,\phi])^2}{\left(\text{Tan}\left[\frac{\pi\,\Phi 0}{2\,V}\right]\right)^2}}};$$

Plotting the curves with samples values of Φ_0,

In[3]:=
```
p1 = PolarPlot[Table[ρ[ϕ, Φ0], {Φ0, {-2, -1, -1/2, 1/2, 1, 2}}] // Evaluate,
    {ϕ, 0, 2 π}, RegionFunction → Function[{x, y}, x^2 + y^2 ≤ 1], PlotLegends →
     {"Φ0=-2", "Φ0=-1", "Φ0=-1/2", "Φ0=1/2", "Φ0=1", "Φ0=2"}, PlotStyle → Thick];
```

We also plot the top view of the cylinder,

In[11]:=
```
p2 = PolarPlot[1, {θ, 0, 2 Pi}, PlotTheme → "Minimal", PlotStyle → Black,
    AxesStyle → LightGray, PolarAxes → {True, False}, PolarAxesOrigin → {0, 1},
    PolarTicks → {{-Pi, -Pi/2, 0, Pi/2}, Automatic}, TicksStyle -> Directive[Black, 14]];
```

Since the calculated electric field is in polar coordinates, we need to convert the expression into Cartesian coordinates to be able to plot field lines :

In[5]:=
$$F[x\_, y\_] = \text{TransformedField}\left[\text{"Polar"} \to \text{"Cartesian"}, \left\{\frac{-8\,b^2\,V}{\pi} \frac{r\,\text{Sin}[2\,\phi]\,(b^4 + r^2)}{(b^4 - r^2)^2 + (2\,r^2\,b^2\,\text{Sin}[2\,\phi])^2}, \frac{-8\,b^2\,V}{\pi} \frac{r\,\text{Cos}[2\,\phi]\,(b^4 - r^2)}{(b^4 - r^2)^2 + (2\,r^2\,b^2\,\text{Sin}[2\,\phi])^2}\right\}, \{r, \phi\} \to \{x, y\}\right];$$

Here, we plot the field lines :

In[6]:=
```
p3 = StreamPlot[F[x, y], {x, -1, 1},
    {y, -1, 1}, RegionFunction → Function[{x, y}, x^2 + y^2 ≤ 1],
    PlotRange → 1, StreamPoints → 25, StreamStyle → Gray];
```

Combining these three plots, we get

In[13]:=
```
Show[{p2, p1, p3}, ImageSize → Medium]
```

Out[13]=