Problem 1

Calculate the density matrix for a statistical mixture in the states $|0\rangle$ and $|1\rangle$ with probability $p_0 = \frac{3}{4}$ and $p_1 = \frac{1}{4}$.

Solution:
For this system, the density operator is given by

$$\hat{\rho} = \frac{3}{4}|0\rangle\langle 0| + \frac{1}{4}|1\rangle\langle 1| \tag{1}$$

where we let the basis set to be $\{|\psi_1\rangle, |\psi_2\rangle\} = \{|0\rangle, |1\rangle\}$. With the elements of the density matrix calculated by

$$\rho_{kj} = \langle\psi_k|\hat{\rho}|\psi_j\rangle, \tag{2}$$

we obtain the following:

$$\begin{aligned}
\rho_{11} &= \langle\psi_1|\hat{\rho}|\psi_1\rangle \\
&= \langle 0|\left(\frac{3}{4}|0\rangle\langle 0| + \frac{1}{4}|1\rangle\langle 1|\right)|0\rangle \\
&= \frac{3}{4}\langle 0|0\rangle\langle 0|0\rangle + \frac{1}{4}\langle 0|1\rangle\langle 1|0\rangle \\
\rho_{00} &= \frac{3}{4}
\end{aligned} \tag{3}$$

$$\begin{aligned}
\rho_{12} &= \langle\psi_1|\hat{\rho}|\psi_2\rangle \\
&= \langle 0|\left(\frac{3}{4}|0\rangle\langle 0| + \frac{1}{4}|1\rangle\langle 1|\right)|1\rangle \\
&= \frac{3}{4}\langle 0|0\rangle\langle 0|1\rangle + \frac{1}{4}\langle 0|1\rangle\langle 1|1\rangle \\
\rho_{01} &= 0
\end{aligned} \tag{4}$$

$$\begin{aligned}
\rho_{21} &= \langle\psi_2|\hat{\rho}|\psi_1\rangle \\
&= \langle 1|\left(\frac{3}{4}|0\rangle\langle 0| + \frac{1}{4}|1\rangle\langle 1|\right)|0\rangle \\
&= \frac{3}{4}\langle 1|0\rangle\langle 0|0\rangle + \frac{1}{4}\langle 1|1\rangle\langle 1|0\rangle \\
\rho_{10} &= 0
\end{aligned} \tag{5}$$

$$\begin{aligned}
\rho_{22} &= \langle\psi_2|\hat{\rho}|\psi_2\rangle \\
&= \langle 1|\left(\frac{3}{4}|0\rangle\langle 0| + \frac{1}{4}|1\rangle\langle 1|\right)|1\rangle \\
&= \frac{3}{4}\langle 1|0\rangle\langle 0|1\rangle + \frac{1}{4}\langle 1|1\rangle\langle 1|1\rangle \\
\rho_{11} &= \frac{1}{4}
\end{aligned} \tag{6}$$

where we took advantage of the fact that $\langle n|n\rangle = 1$ and $\langle m|n\rangle = 0$ in which m, n are integers. Therefore, our density matrix is

$$\rho = \begin{pmatrix} \rho_{00} & \rho_{01} \\ \rho_{10} & \rho_{11} \end{pmatrix} = \begin{pmatrix} \frac{3}{4} & 0 \\ 0 & \frac{1}{4} \end{pmatrix} \tag{7}$$

Problem 2

What is the density matrix for a statistical mixture of the superposition of states $|\psi_A\rangle$ and $|\psi_B\rangle$ with probabilities $p_A = p_B = \frac{1}{2}$

Solution:
For this system, the density matrix is calculated to be

$$\hat{\rho} = \sum_i p_i |\psi_i\rangle \langle\psi_i| = \frac{1}{2} |\psi_A\rangle \langle\psi_A| + \frac{1}{2} |\psi_B\rangle \langle\psi_B| \tag{8}$$

Let us first consider a specific case where our basis set is $\{|\psi_1\rangle, |\psi_2\rangle\} = \{|0\rangle, |1\rangle\}$. Then, the superposition of states $|\psi_A\rangle$ and $|\psi_B\rangle$ can be written as

$$|\psi_A\rangle = \alpha^{(A)} |0\rangle + \beta^{(A)} |1\rangle \tag{9}$$
$$|\psi_B\rangle = \alpha^{(B)} |0\rangle + \beta^{(B)} |1\rangle \tag{10}$$

Then, the first element of the density matrix is calculated to be

$$\begin{aligned}
\rho_{11} &= \langle 0|\hat{\rho}|0\rangle \\
&= \langle 0| \left[\frac{1}{2}\left(\alpha^{(A)}|0\rangle + \beta^{(A)}|1\rangle\right)\left(\alpha^{(A)*}\langle 0| + \beta^{(A)*}\langle 1|\right)\right. \\
&\qquad \left. + \frac{1}{2}\left(\alpha^{(B)}|0\rangle + \beta^{(B)}|1\rangle\right)\left(\alpha^{(B)*}\langle 0| + \beta^{(B)*}\langle 1|\right)\right]|0\rangle \\
&= \frac{1}{2}\alpha^{(A)*}\alpha^{(A)}\langle 0|0\rangle\langle 0|0\rangle + \frac{1}{2}\alpha^{(B)*}\alpha^{(B)}\langle 1|1\rangle\langle 1|1\rangle \\
&= \frac{1}{2}\left(\left|\alpha^{(A)}\right|^2 + \left|\alpha^{(B)}\right|^2\right) \\
\rho_{11} &= \frac{1}{2}\sum_i \left|\alpha^{(i)}\right|^2
\end{aligned} \tag{11}$$

In the same manner, we have

$$\rho_{22} = \frac{1}{2}\sum_i \left|\beta^{(i)}\right|^2 \tag{12}$$

On the other hand, the second element of the matrix is given by

$$\begin{aligned}
\rho_{12} &= \langle 0|\hat{\rho}|1\rangle \\
&= \langle 0| \left[\frac{1}{2}\left(\alpha^{(A)}|0\rangle + \beta^{(A)}|1\rangle\right)\left(\alpha^{(A)*}\langle 0| + \beta^{(A)*}\langle 1|\right)\right. \\
&\qquad \left. + \frac{1}{2}\left(\alpha^{(B)}|0\rangle + \beta^{(B)}|1\rangle\right)\left(\alpha^{(B)*}\langle 0| + \beta^{(B)*}\langle 1|\right)\right]|1\rangle \\
&= \frac{1}{2}\beta^{(A)*}\alpha^{(A)}\langle 1|1\rangle\langle 0|0\rangle + \frac{1}{2}\beta^{(B)*}\alpha^{(B)}\langle 1|1\rangle\langle 0|0\rangle \\
\rho_{12} &= \frac{1}{2}\sum_i \beta^{(i)*}\alpha^{(i)}
\end{aligned} \tag{13}$$

Using the same method, we have

$$\rho_{21} = \frac{1}{2}\sum_i \alpha^{(i)*}\beta^{(i)} \tag{14}$$

Here, we see a pattern regarding the diagonal and off-diagonal terms of the density matrix. We can use this for the more general case where our basis set is

$$|\psi_i\rangle = \sum_j c_j^{(i)} |\phi_j\rangle \tag{15}$$

Consider m, n between the interval of 1 to N. We will assume that $\langle\phi_n|\phi_n\rangle = 1$ and $\langle\phi_m|\phi_n\rangle = 0$. With this, the superposition of states $|\psi_A\rangle$ and $|\psi_B\rangle$ is now written as

$$|\psi_1\rangle = c_1^{(1)} |\phi_1\rangle + \ldots + c_N^{(1)} |\phi_N\rangle \tag{16}$$

$$|\psi_2\rangle = c_1^{(2)} |\phi_1\rangle + \ldots + c_N^{(2)} |\phi_N\rangle \tag{17}$$

Then, we can infer that the off-diagonal terms of the density matrix can be calculated to be

$$\begin{aligned}
\rho_{mn} &= \langle\phi_m|\hat{\rho}|\phi_n\rangle \\
&= \langle\phi_m| \left[\frac{1}{2}\left(c_1^{(A)} |\phi_1\rangle + \ldots + c_N^{(A)} |\phi_N\rangle\right)\left(c_1^{(A)*} \langle\phi_1| + \ldots + c_N^{(A)*} \langle\phi_N|\right)\right. \\
&\qquad \left. + \frac{1}{2}\left(c_1^{(B)} |\phi_1\rangle + \ldots + c_N^{(B)} |\phi_N\rangle\right)\left(c_1^{(B)*} \langle\phi_1| + \ldots + c_N^{(B)*} \langle\phi_N|\right)\right] |\phi_n\rangle \\
&= \frac{1}{2} c_n^{(A)*} c_m^{(A)} \langle\phi_n|\phi_n\rangle \langle\phi_m|\phi_m\rangle + \frac{1}{2} c_n^{(B)*} c_m^{(B)} \langle\phi_n|\phi_n\rangle \langle\phi_m|\phi_m\rangle \\
\rho_{mn} &= \frac{1}{2}\sum_i c_n^{(i)*} c_m^{(i)}
\end{aligned} \tag{18}$$

We can test this using the off-diagonal terms in the density matrix of the specific case earlier in which we can see that they have the same form. In other words, this general result leads to the specific case given the same initial parameters. For the diagonal terms, we have

$$\begin{aligned}
\rho_{nn} &= \langle\phi_n|\hat{\rho}|\phi_n\rangle \\
&= \langle\phi_n| \left[\frac{1}{2}\left(c_1^{(A)} |\phi_1\rangle + \ldots + c_N^{(A)} |\phi_N\rangle\right)\left(c_1^{(A)*} \langle\phi_1| + c_N^{(A)*} \langle\phi_N|\right)\right. \\
&\qquad \left. + \frac{1}{2}\left(c_1^{(B)} |\phi_1\rangle + c_N^{(B)} |\phi_N\rangle\right)\left(c_1^{(B)*} \langle\phi_1| + \ldots + c_N^{(B)*} \langle\phi_N|\right)\right] |\phi_n\rangle \\
&= \frac{1}{2} c_n^{(A)*} c_n^{(A)} \langle\phi_n|\phi_n\rangle \langle\phi_n|\phi_n\rangle + \frac{1}{2} c_n^{(B)*} c_n^{(B)} \langle\phi_n|\phi_n\rangle \langle\phi_n|\phi_n\rangle \\
\rho_{nn} &= \frac{1}{2}\sum_i \left|c_n^{(i)}\right|^2
\end{aligned} \tag{19}$$

which also leads to the results in the specific case.