Ch. 6: Maxwell Equations, Macroscopic Electromagnetism, Conservation Laws

6.1: Maxwell's Displacement Current; Maxwell Equations

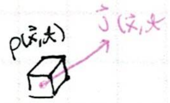
Summary of what we have learned so far:

(1)

$$\vec{\nabla}\cdot\vec{D} = \rho \qquad \text{Coulomb's law} \qquad\qquad \vec{\nabla}\times\vec{E} + \frac{\partial\vec{B}}{\partial t} = 0 \qquad \text{Faraday's law}$$

$$\vec{\nabla}\times\vec{H} = \vec{J} \;\; (\nabla\cdot\vec{J}=0) \qquad \text{Ampere's law} \qquad\qquad \vec{\nabla}\cdot\vec{B} = 0 \qquad \text{Absence of free magnetic poles}$$

Ampere's law is not consistent w/ Faraday's law. How do we correct Ampere's law?



The conservation of charges requires that ρ and $\vec{J}$ satisfy the continuity equation:

(2) $$\vec{\nabla}\cdot\vec{J} + \frac{\partial\rho}{\partial t} = 0$$ consequence of the law that charge can't be created or destroyed

$$\vec{\nabla}\cdot\vec{J} = 0 \longrightarrow \frac{\partial\rho}{\partial t} = 0 \qquad \text{constant in time} \qquad (3)$$

Then,

$$\vec{\nabla}\cdot\vec{J} + \frac{\partial}{\partial t} = \vec{\nabla}\cdot\vec{J} + \frac{\partial}{\partial t}\vec{\nabla}\cdot\vec{D}$$

$$= \underbrace{\vec{\nabla}\cdot\vec{J} + \vec{\nabla}\cdot\frac{\partial\vec{D}}{\partial t}}_{\text{divergence less}}$$

$$= \nabla\cdot\left(\vec{J} + \frac{\partial\vec{D}}{\partial t}\right) = 0 \qquad (4)$$

Maxwell: In the presence of time varying fields, $\vec{J} \rightarrow \vec{J} + \partial_t\vec{D}$.
Ampere's law

$$\vec{\nabla}\times\vec{H} = \vec{J} + \frac{\partial\vec{D}}{\partial t} \qquad (5)$$

This expression reduces to the original Ampere's law for steady-state current i.e. $\partial_t\vec{D} = 0$. Then, we have a set of 4 equations

$$\vec{\nabla}\cdot\vec{D} = \rho \qquad\qquad \vec{\nabla}\times\vec{E} + \frac{\partial\vec{B}}{\partial t} = 0$$

$$\vec{\nabla}\times\vec{H} = \vec{J} + \frac{\partial\vec{D}}{\partial t} \qquad\qquad \vec{\nabla}\cdot\vec{B} = 0 \qquad (6)$$

which are the Maxwell's equations. Everything about classical electromagnetism are already encoded in these equations.

6.2: Vector and Scalar potentials

Since $\vec{\nabla}\cdot\vec{B} = 0$, $\vec{B} = \vec{\nabla}\times\vec{A}$

$$\nabla\times E + \frac{\partial B}{\partial t} = 0 \longrightarrow \nabla\times E + \frac{\partial}{\partial t}\nabla\times A = \nabla\times\left(E + \frac{\partial A}{\partial t}\right) = 0 \qquad (7)$$

Since $\nabla\times\nabla f = 0$, we can write $E + \partial_t A = -\nabla\Phi$ or $E = -\nabla\Phi - \partial_t\vec{A}$
where Φ is the electric potential due to the distrib. of charges while A is the electric potential due to the changing magnetic field.
Consider the Maxwell's equations in vaccum $[\vec{D} = \epsilon_0\vec{E},\ \vec{B} = \mu_0\vec{H}]$:
Then,

$$\nabla\cdot D = \rho \longrightarrow \nabla\cdot E = \rho/\epsilon_0$$

$$\nabla\times H = \vec{J} + \frac{\partial D}{\partial t} \longrightarrow \vec{\nabla}\times\vec{B} - \frac{1}{\mu_0\epsilon_0}\frac{\partial E}{\partial t} = \mu_0\vec{J} \qquad (8)$$

Note that $\sqrt{\mu_0 \epsilon_0} = c$ (speed of light).

$$\nabla \cdot E = \frac{\rho}{\epsilon_0} \rightarrow \nabla \cdot \left(-\nabla\Phi - \frac{\partial A}{\partial t}\right) = \frac{\rho}{\epsilon_0} \rightarrow \nabla^2 \Phi + \frac{\partial}{\partial t}(\vec{\nabla}\cdot\vec{A}) = -\frac{\rho}{\epsilon_0} \qquad (9)$$

$$\nabla \times B - \frac{1}{c^2}\frac{\partial \vec{E}}{\partial t} = \mu_0 \vec{J} \rightarrow \nabla \times \nabla A - \frac{1}{c^2}\frac{\partial}{\partial t}\left(-\nabla\Phi - \frac{\partial \vec{A}}{\partial t}\right) = \mu_0 \vec{J}$$

$$-\nabla^2 A + \nabla(\nabla \cdot A) - \frac{1}{c^2}\nabla\frac{\partial \Phi}{\partial t} + \frac{1}{c^2}\frac{\partial^2 A}{\partial t^2} = \mu_0 \vec{J}$$

$$\rightarrow \vec{\nabla}^2 \vec{A} - \frac{1}{c^2}\frac{\partial^2 \vec{A}}{\partial t^2} - \vec{\nabla}\left(\vec{\nabla}\cdot\vec{A} + \frac{1}{c^2}\frac{\partial \Phi}{\partial t}\right) = -\mu_0 \vec{J} \qquad (10)$$

Φ and $\vec{A}$ are to be determined are the solutions to the above equations. But the equations are coupled. We wish to uncouple them. We do this by applying gauge transformation.

Since $\vec{B} = \vec{\nabla}\times\vec{A}$, $A \rightarrow \vec{A}' = A + \nabla\Lambda$.

$$B' = \nabla \times A' = \nabla \times A + \nabla\Lambda = \nabla \times \vec{A} + \cancel{(\nabla \times \nabla\Lambda)}^{0} = B \qquad (10)$$

This shows that $\vec{B}$ is unaffected by the transformation of $\vec{A}$.

How about the electric field? $[\vec{E} = -\nabla\Phi - \partial_t \vec{A}]$ Let $\Phi \rightarrow \Phi'$, $\vec{A} \rightarrow \vec{A}'$

$$\vec{E}' = -\nabla\Phi' - \frac{\partial A'}{\partial t} = -\nabla\Phi - \frac{\partial}{\partial t}(\nabla + \nabla\Lambda) = -\nabla\Phi - \frac{\partial A}{\partial t} - \nabla\frac{\partial\Lambda}{\partial t} \qquad (11)$$

If we impose $\vec{E}' = \vec{E}\,(= -\nabla\Phi - \partial_t A)$, then we see that

$$\Phi \rightarrow \Phi' = \Phi - \frac{\partial\Lambda}{\partial t} \qquad (12)$$

- The simultaneous transformations

$$A \rightarrow A' = A + \nabla\Lambda \qquad \Phi \rightarrow \Phi' = \Phi - \partial_t\Lambda \qquad (13)$$

preserves $\vec{E}$ and $\vec{B}$. This pair of expressions is called gauge transformation.

- E & B are the physical quantities. A & Φ are just auxiliary tools to solve for E and B and they can undergo gauge transformations w/o changing the physical fields
  - → This allows us to choose A & Φ arbitrarily to simplify our differential eqns
- The freedom afforded by gauge transformation allows us to choose A and Φ such that their equations uncouple. Thus, choose A & Φ such that

$$\nabla \cdot A + \frac{1}{c^2}\frac{\partial\Phi}{\partial t} = 0 \quad \text{or} \quad \vec{\nabla}\cdot\vec{A} = -\frac{1}{c^2}\frac{\partial\Phi}{\partial t} \qquad (14)$$

Then,

$$\nabla^2\Phi - \frac{1}{c^2}\frac{\partial^2\Phi}{\partial t^2} = -\frac{\rho}{\epsilon_0}, \qquad \nabla^2 A - \frac{1}{c^2}\frac{\partial^2 A}{\partial t^2} = -\mu_0 \vec{J} \qquad (15)$$

These are homogeneous wave equations which describe propagating waves w/ speed c. A & Φ are the propagating waves w/ speed c

These are wave equations with sources

- Gauge transformations (Lorenz gauge, Coulomb gauge) [6.3]

We show that there exists a gauge transformation that satisfies the Lorenz condition. Let Φ and $\vec{A}$ be solutions to equations and but they do not satisfy the Lorenz condition

Perform the transformation in Eq. (13). We demand that

$$\nabla \cdot A' + \frac{1}{c^2}\frac{\partial\Phi}{\partial t} = 0 \qquad (16)$$

which becomes

$$\vec{\nabla}\cdot\vec{A} + \frac{1}{c^2}\frac{\partial\Phi}{\partial t} + \vec{\nabla}^2\Lambda - \frac{1}{c^2}\frac{\partial^2\Lambda}{\partial t^2} = 0 \quad (17)$$

Then,

$$\vec{\nabla}^2\Lambda - \frac{1}{c}\frac{\partial^2\Lambda}{\partial t} = -\left(\vec{\nabla}\cdot\vec{A} + \frac{1}{c^2}\frac{\partial\Phi}{\partial t}\right) \quad (18)$$

which can be solved in principle for Λ given A and Φ.

Note: The restricted gauge transformation $\vec{A} \to \vec{A} + \nabla\Lambda$, $\Phi \to \Phi - \partial_t\Lambda$ where $\nabla\Lambda - (1/c^2)\,\partial_t\Lambda = 0$ preserves the Lorenz condition, provided A and Φ satisfy the condition initially. All potentials in this restricted to belong to the Lorenz gauge

The Lorenz gauge is used for the following reasons

① It leads to the wave equations w/c treat Φ and A on equal footing

② It is a concept independent of the coordinate system chosen

Coulomb gauge

$$\nabla\cdot A = 0 \longrightarrow \nabla^2\Phi - \rho/\epsilon_0$$

w/ soln:

$$\Phi(x,t) = \frac{1}{4\pi\epsilon_0}\int\frac{\rho(\vec{x},t)}{|\vec{x}-\vec{x}'|}\,d^3x' \quad (19)$$

The scalar potential is just the instantaneous Coulomb potential due to charge density $\rho(x,t)$. The vector potential satisfies the inhomogeneous wave equation:

$$\nabla^2\vec{A} - \frac{1}{c^2}\frac{\partial^2 A}{\partial t^2} = -\mu_0\vec{J} + \frac{1}{c^2}\nabla\frac{\partial\Phi}{\partial t} \quad (20)$$

The current density $\vec{J}$ can be written as $\vec{J} = \vec{J}_\ell + \vec{J}_t$ where J_ℓ is the longitudinal/irrotational current ($\nabla\times J_\ell = 0$) while J_t is the transverse/solenoidal current ($\vec{\nabla}\cdot J_t = 0$)

(21) $$J_\ell = -\frac{1}{4\pi}\nabla\int\frac{\nabla\cdot J}{|x-x'|}\,d^3x' \qquad J_t = \frac{1}{4\pi}\nabla\times\nabla\times\int\frac{J}{|x-x'|}\,d^3x$$

From the continuity equation,

$$\nabla\cdot\vec{J} + \frac{\partial A}{\partial t} = 0 \longrightarrow \nabla\cdot(J_\ell + J_t) + \frac{\partial\rho}{\partial t} = 0 \longrightarrow \nabla\cdot J_\ell + \frac{\partial\rho}{\partial t} = 0 \quad (22)$$

which can be rewritten as

(23) $$\nabla\cdot J_\ell + \frac{\partial}{\partial t}\left(-\epsilon_0\nabla^2\Phi\right) = 0 \to \nabla\cdot\left(J_\ell - \epsilon_0\nabla\frac{\partial\Phi}{\partial t}\right) = 0 \to J_\ell = \epsilon_0\nabla\frac{\partial\Phi}{\partial t}$$

Note that $(1/c^2)\,\partial_t\nabla\Phi = \mu_0\vec{J}_\ell$. Then,

$$\nabla^2\vec{A} - \frac{1}{c^2}\frac{\partial^2\vec{A}}{\partial t^2} = \mu_0\vec{J}_t \quad (24)$$

The Coulomb gauge is used when there are no sources ($\rho = 0$). In this case, $\Phi = 0$ and $\vec{A}$ satisfies

$$\vec{\nabla}^2\vec{A} - \frac{1}{c^2}\frac{\partial^2\vec{A}}{\partial t^2} = 0 \quad (25)$$

The fields are given by

$$\left.\begin{aligned}\vec{E} &= -\vec{\nabla}\Phi - \frac{\partial\vec{A}}{\partial t} \longrightarrow \vec{E} = -\frac{\partial\vec{A}}{\partial t}\\ \vec{B} &= \vec{\nabla}\times\vec{A}\end{aligned}\right\} \quad (26)$$