Physics 232

Mar 2, 2021 ▶

- Grading system

  LE 1 - Ch 4 & 5 : 20% PS: 30%

  LE 2 - Ch 6 & 7 : 20% Class Notes: 10%

  LE3 - Ch 8, 9, & 10 : 20%

- W/ strict deadlines in PS and other submissions
- Exams will be conducted real time w/ zoom monitoring (class hours)
- Each chapter has PS, will come from Jackson, submission: 1 wk after finishing chapter
- jabesogas@up.edu.ph - for questions regarding PS

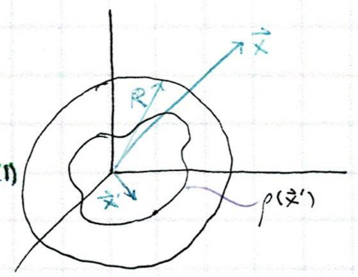
Mar 4, 2021 ▶ Ch 4: Multipoles, electrostatics of macroscopic media, dielectrics

▶ **Multipole expansion (4.1)**

↳ we assume no boundary sources

- The potential is

$$\boxed{\Phi(\vec{x}) = \frac{1}{4\pi\epsilon_0}\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{4\pi}{2l+1}\,q_{lm}\,\frac{Y_{lm}(\theta,\phi)}{r^{l+1}}} \quad (1)$$



▲ Consider a localized distribution of charge described by $\rho(\vec{x})$

↑ non-vanishing only inside the sphere

This expansion in spherical harmonics is called the multipole exansion of $\Phi(\vec{x})$; $l=0$ - monopole term $l=1$ - dipole term, etc

- The expansion can be obtained from the known result

$$\Phi(\vec{x}) = \frac{1}{4\pi\epsilon_0}\int\frac{\rho(\vec{x}')}{|\vec{x}-\vec{x}'|}\,d^3x' \quad (2)$$

using the expansion:

$$\frac{1}{|\vec{x}-\vec{x}'|} = 4\pi\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{1}{2l+1}\frac{r_<^l}{r_>^{l+1}}Y^*_{lm}(\theta',\phi')\,Y_{lm}(\theta,\phi) \quad (3)$$

Since we are interested with the potential outside the charge distribution, $r_< = r'$ and $r_> = r$. Then, substituting $1/|\vec{x}-\vec{x}'|$ into Eq (2)

$$\Phi(x) = \frac{1}{4\pi\epsilon_0}\int\rho(\vec{x}')\,4\pi\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{1}{2l+1}\frac{r'^l}{r^{l+1}}Y^*_{lm}(\theta',\phi')\,Y_{lm}(\theta,\phi)\,d^3x'$$

$$= \frac{1}{\epsilon_0}\sum_{l,m}\frac{1}{2l+1}\underbrace{\left[\int\rho(\vec{x}')\,r'^l\,Y^*_{lm}(\theta',\phi')\,d^3x'\right]}_{}\frac{Y_{lm}(\theta,\phi)}{r^{l+1}} \quad (4)$$

q_{lm} → coefficients called multipole moments

Mar 6, 2021 ▶

- To see the physical interpretation of them, we exhibit the first few explicitly in terms of cartesian coord's:

★ $Y_{00}(\theta,\phi) = \frac{1}{\sqrt{4\pi}}$

$$q_{00} = \int \cancel{Y^*_{00}(\theta',\phi')}\,\cancel{r'^0}\,\rho(\vec{x}')\,d^3x' = \frac{1}{\sqrt{4\pi}}\int\rho(\vec{x})\,d^3x' = \frac{q}{\sqrt{4\pi}} \quad (5)$$

(total charge; $\frac{1}{\sqrt{4\pi}}$, 1)

★ $Y_{11}(\theta,\phi) = -\sqrt{\frac{3}{8\pi}}\sin\theta\, e^{i\phi}$

$$q_{11} = \int Y^*_{11}(\theta',\phi')\,r'^1\rho(\vec{x}')\,d^3x'$$

$$= -\int\sqrt{\frac{3}{8\pi}}\sin\theta'\,r'\,\rho(\vec{x}')\,d^3x'\,e^{-i\phi'}$$

$$= -\sqrt{\frac{3}{8\pi}}\int\sin\theta'(\cos\phi' - i\sin\phi')\,r'\rho(\vec{x}')\,d^3x'$$

$$= -\sqrt{\frac{3}{8\pi}}\int(r'\sin\theta'\cos\phi' - ir'\sin\theta'\sin\phi')\,\rho(\vec{x}')\,d^3x'$$

$$= -\sqrt{\frac{3}{8\pi}}\int(x' - iy')\,\rho(\vec{x}')\,d^3x'$$

$$= -\sqrt{\frac{3}{8\pi}}\left(\underbrace{\int x'\rho(\vec{x}')\,d^3x'}_{p_x} - i\underbrace{\int y'\rho(\vec{x}')\,d^3x'}_{p_y}\right)$$

$$q_{11} = -\sqrt{\frac{3}{8\pi}}\,(p_x - ip_y) \quad (6)$$

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★ $Y_{10} = \sqrt{\frac{3}{4\pi}} \cos\theta$ $\qquad q_{10} = \int Y^*_{10}(\theta',\phi')\, r' \rho(\vec{x}')\, d^3x' = \int \sqrt{\frac{3}{4\pi}} \cos\theta\, r' \rho(\vec{x}')\, d^3x' = \sqrt{\frac{3}{4\pi}} \int z' \rho(\vec{x}')\, d^3x' = \sqrt{\frac{3}{4\pi}}\, p_z$ (7)

★ $Y_{22} = \frac{1}{4}\sqrt{\frac{15}{2\pi}} \sin^2\theta\, e^{2i\phi}$

$$q_{22} = \int Y^*_{22}(\theta',\phi')\, r'^2 \rho(\vec{x}')\, d^3x'$$

$$= \int \frac{1}{4}\sqrt{\frac{15}{2\pi}} \sin^2\theta'\, e^{-2i\phi'}\, r'^2 \rho(\vec{x}')\, d^3x'$$

$$= \int \frac{1}{4}\sqrt{\frac{15}{2\pi}} \left(\sin\theta'\, e^{-i\phi'} r'\right)^2 \rho(\vec{x}')\, d^3x'$$

$$= \frac{1}{4}\sqrt{\frac{15}{2\pi}} \int (x' - iy')^2 \rho(\vec{x}')\, d^3x'$$

$$= \frac{1}{4}\sqrt{\frac{15}{2\pi}} \int \left(x'^2 - 2i x'y' - y'^2\right) \rho(\vec{x}')\, d^3x'$$

$$= \frac{1}{4}\sqrt{\frac{15}{2\pi}} \left[\int x'^2 \rho(\vec{x}')\, d^3x' - 2i \int x'y' \rho(\vec{x}')\, d^3x' - \int y'^2 \rho(\vec{x}')\, d^3x'\right]$$

$$q_{22} = \frac{1}{4}\sqrt{\frac{15}{2\pi}} \left(Q_{11} - 2i Q_{12} - Q_{22}\right) \cdot \frac{1}{3} \qquad (8)$$

Similar calculation yields

★ $Y_{21} = -\sqrt{\frac{15}{8\pi}} \sin\theta \cos\theta\, e^{i\phi}$ $\qquad q_{21} = -\sqrt{\frac{15}{8\pi}} \int z'(x' - iy') \rho(\vec{x}')\, d^3x' = -\frac{1}{3}\sqrt{\frac{15}{8\pi}} \left(Q_{13} - i Q_{23}\right)$ (9)

★ $Y_{20} = \sqrt{\frac{5}{4\pi}} \left(\frac{3}{2}\cos^2\theta - \frac{1}{2}\right)$ $\qquad q_{20} = \frac{1}{2}\sqrt{\frac{5}{4\pi}} \int \left(3z'^2 - r'^2\right) \rho(\vec{x}')\, d^3x' = \frac{1}{2}\sqrt{\frac{5}{4\pi}}\, Q_{33}$ (10)

- Only the moments w/ $m \geq 0$ have been given since for a real charge density, the moments w/ $m < 0$ are related through:

$$q_{l,-m} = (-1)^m q^*_{lm} \qquad (11)$$

In the above equations (Eq. (5) - (10))

q : the total charge

$\vec{p} = \int \vec{x}' \rho(\vec{x}')\, d^3x'$: electric dipole moment (12)

$Q_{ij} = \int \left(3x_i' x_j' - r'^2 \delta_{ij}\right) \rho(\vec{x}')\, d^3x'$: traceless quadraple moment tensor (13)

Explicitly, we can write the potential as follows:

$$\Phi(\vec{x}) = \frac{1}{4\pi\epsilon_0}\left[4\pi q_{00} \frac{Y_{00}(\theta,\phi)}{r} + \frac{4\pi}{3r^2}\left(q_{1,-1} Y_{1,-1}(\theta,\phi) + q_{10} Y_{10}(\theta,\phi) + q_{11} Y_{11}(\theta,\phi)\right) + \frac{4\pi}{5r^3}\left(q_{2,-2} Y_{2,-2}(\theta,\phi) + q_{2,-1} Y_{2,-1}(\theta,\phi) + q_{20} Y_{20}(\theta,\phi) + q_{21} Y_{21}(\theta,\phi) + q_{22} Y_{22}(\theta,\phi)\right) + \ldots\right] \qquad (14)$$

The first term:

$$4\pi q_{00} Y_{00}(\theta,\phi) = \cancel{4\pi} \frac{q}{\cancel{\sqrt{4\pi}}} \cdot \frac{1}{\sqrt{4\pi}} = q \qquad (15)$$

The second term:

$$q_{1,-1} Y_{1,-1}(\theta,\phi) + q_{10} Y_{10}(\theta,\phi) + q_{11} Y_{11}(\theta,\phi)$$

$$= \left(-\sqrt{\frac{3}{8\pi}}\right)(p_x + i p_y)(-1)\left(-\sqrt{\frac{3}{8\pi}}\right) \sin\theta\, e^{-i\phi} (-1) + \left(\sqrt{\frac{3}{4\pi}}\, p_z\right)\left(\sqrt{\frac{3}{4\pi}} \cos\theta\right)$$

$$+ \left[-\sqrt{\frac{3}{8\pi}} (p_x - i p_y)\right]\left(-\sqrt{\frac{3}{8\pi}} \sin\theta\, e^{i\phi}\right)$$

$$= \frac{3}{8\pi}(p_x + i p_y) \sin\theta\, e^{-i\phi} + \frac{3}{4\pi} p_z \cos\theta + \frac{3}{8\pi}(p_x - i p_y) \sin\theta\, e^{i\phi}$$

$$= \frac{3}{8\pi} P_x \sin\theta \left(e^{-i\phi} + e^{i\phi}\right) + \frac{3}{4\pi} P_z \cos\theta + \frac{3}{8\pi}\, i P_y \sin\theta \left(e^{-i\phi} - e^{i\phi}\right)$$

$$= \frac{3}{\cancel{8}4\pi} P_x \sin\theta \cdot \cancel{2}\cos\phi + \frac{3}{4\pi} P_z \cos\theta + \frac{3}{\cancel{8}4\pi}\, i P_y \sin\theta\, (-\cancel{2}i) \sin\phi$$

$$= \frac{3}{4\pi}\left(P_x \sin\theta \cos\phi + P_z \cos\theta + P_y \sin\theta \sin\phi\right) \frac{r}{r}$$

$$= \frac{3}{4\pi r}\left(P_x\, r\sin\theta\cos\phi + P_z\, r\cos\theta + P_y\, r\sin\theta\sin\phi\right)$$

$$= \frac{3}{4\pi r}\left(P_x x + P_y y + P_z z\right)$$

$$= \frac{3}{4\pi r}\, \vec{p} \cdot \vec{x} \qquad (16)$$

The third term:

$$q_{2,-2} Y_{2,-2} + q_{2,-1} Y_{2,-1} + q_{20} Y_{20} + q_{21} Y_{21} + q_{22} Y_{22}$$

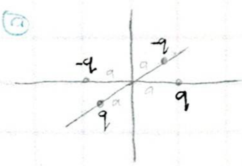
$$= \left(\frac{1}{4}\sqrt{\frac{15}{2\pi}}\right)\left(Q_{11} - 2i\, Q_{12} - Q_{22}\right)^* (-1)^{-2} \left(\frac{1}{4}\sqrt{\frac{15}{2\pi}}\right)(-1)^{+2} \sin^2\theta\, e^{-2i\phi} + \ldots \qquad (17)$$

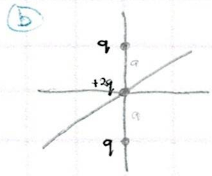
Simplification leads to the expression:

$$\Phi(\vec{x}) = \frac{1}{4\pi\epsilon_0}\Big[\underbrace{\frac{q}{r}}_{\text{monopole}} + \underbrace{\frac{\vec{p}\cdot\vec{x}}{r^3}}_{\text{dipole}} + \underbrace{\frac{1}{2}\sum_{ij} Q_{ij}\frac{x_i x_j}{r^5}}_{\text{quadrupole}} + \ldots\Big] \qquad (18)$$

- Problems:

4.1) Calculate the multipole moments q_{lm} of the charge distribution shown as parts a and b





Solution: Write down the charge density

$$\rho(\vec{x}) = q\,\delta(x-a)\,\delta(y)\,\delta(z) + q\,\delta(x)\,\delta(y-a)\,\delta(z) - q\,\delta(x+a)\,\delta(y)\,\delta(z)$$
$$- q\,\delta(x)\,\delta(y+a)\,\delta(z) \qquad (19)$$

The total charge is: $\int \rho(\vec{x}')\, d^3x' = 2q - 2q = 0$ (20)

Thus

$$q_{00} = \int Y^*_{00}(\theta', \phi')\, \rho(\vec{x}')\, d^3x' = 0 \qquad (21)$$

$$q_{11} = -\sqrt{\frac{3}{8\pi}} \int (x' - iy')\, \rho(\vec{x}')\, d^3x'$$

$$= -\sqrt{\frac{3}{8\pi}} \int (x' - iy')\Big[\delta(x-a)\delta(y)\delta(z) + \delta(x)\delta(y-a)\delta(z) - \delta(x+a)\delta(y)\delta(z) -$$
$$\delta(x)\delta(y+a)\delta(z)\Big]\, dx'\, dy'\, dz'$$

$$= -\sqrt{\frac{3}{8\pi}}\Big[a - (-a) - ia + (-ia)\Big]\, q$$

$$q_{11} = -2\sqrt{\frac{3}{8\pi}}\,(1-i)\, q a \qquad (22)$$

$$q_{1,-1} = (-1)^{1} q_{11}^{*} = (-1)^{1}\left(-\sqrt{\frac{3}{8\pi}}\right) 2(1-i)\, a = 2\sqrt{\frac{3}{8\pi}}(1+i)\, q\, a \qquad (23)$$

$$q_{1,0} = \sqrt{\frac{3}{4\pi}} \int z' \rho(\vec{x}') \, d^3x' \xrightarrow{\;0\;} = 0 \qquad (24)$$

Then,

$$\sum_{m=-1}^{l=1} \frac{1}{2(1)+1} q_{1m} \frac{Y_{lm}(\theta,\phi)}{r^{l+1}} = \frac{1}{3r^2}\left[q_{1,-1} Y_{1,-1}(\theta,\phi) + q_{11} Y_{1,1}(\theta,\phi)\right]$$

$$= \frac{1}{3r^2}\left[2\sqrt{\frac{3}{8\pi}}(1+i)\, q\left(-\sqrt{\frac{3}{8\pi}}\right)(-1)^{1} \sin\theta\, e^{-i\phi} a + (-2)\sqrt{\frac{3}{8\pi}}(1-i)\, q\left(-\sqrt{\frac{3}{8\pi}}\right)\sin\theta\, e^{i\phi} a\right]$$

$$= \frac{1}{\cancel{3}r^2} \cdot \frac{\cancel{2}\cancel{3}q}{\cancel{8}_{4}\pi}\left[(1+i)\sin\theta\, e^{-i\phi} + (1-i)\sin\theta\, e^{i\phi}\right] a$$

$$= \frac{aq}{4r^2\pi}\left[2\sin\theta\cos\phi + 2\sin\theta\sin\phi\right]$$

$$= \frac{aq}{_{2}\cancel{4}r^2\pi}\cancel{2}\left[\frac{x}{r} + \frac{y}{r}\right]$$

$$= \frac{aq}{2\pi r^3}\left[x + y\right] \qquad (25)$$

Compute the quadupole moments:

$$\left[\text{Note: } Q_{ij} = \int (3x_i' x_j' - r'^2 \delta_{ij})\, \rho(\vec{x}')\, d^3x', \quad x_1 = x,\ x_2 = y,\ x_3 = z\right]$$

$$Q_{11} = \int (3x'^2 - r'^2)\, \rho(\vec{x}')\, d^3x'$$

$$= \int (3x_1'^2 - x_1'^2 - x_2'^2 - x_3'^2)\, \rho(\vec{x}')\, d^3x'$$

$$= \int (2x'^2 - y'^2 - z'^2)\, \rho(\vec{x}')\, d^3x'$$

$$= 2\int x'^2 \rho(\vec{x}')\, d^3x' - \int y'^2 \rho(\vec{x}')\, d^3x' - \int z'^2 \rho(\vec{x}')\, d^3x'$$

$$= 2\left(qa^2 - qa^2\right) \xrightarrow{\;0\;} - \left(qa^2 - qa^2\right) \xrightarrow{\;0\;} - 0$$

$$Q_{11} = 0 \qquad (26)$$

$$Q_{22} = \int (2y'^2 - x'^2 - z'^2)\, \rho(\vec{x}')\, dx' = 0 \qquad (27)$$

$$Q_{33} = \int (2z'^2 - x'^2 - y'^2)\, \rho(\vec{x}')\, dx' = 0 \qquad (28)$$

$$Q_{12} = \int (3x'y')\, \rho(\vec{x}')\, dx' = 3\int x'y' \rho(\vec{x}')\, dx' = 0 \qquad Q_{21} = Q_{12} = 0 \qquad (29)$$

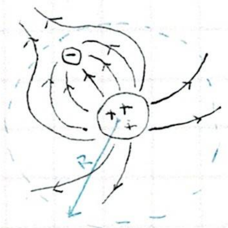
$$Q_{13} = \int (3x'z')\, \rho(\vec{x}')\, dx' = 0 \qquad Q_{13} = Q_{31} = 0 \qquad (30)$$

$$Q_{23} = \int 3y'z'\, \rho(\vec{x}')\, dx' = 0 \qquad Q_{32} = Q_{23} = 0 \qquad (31)$$

Thus, the quadrupole moment vanishes far from the origin and the configuration of charges behaves like a dipole.

Mar 4, 2021 ▸ • We consider a result that is useful in elucidating the basic difference bet. electric and magnetic dipoles. Consider a localized charge distribution $\rho(\vec x)$ that gives rise to an electric field $\vec E(\vec x)$ throughout space. We wish to calculate the integral of $\vec E$ over the volume of the sphere of radius R. We choose the origin to the outer of the sphere

> Case 1: The sphere contains all charges



$$\int_{r<R} \vec E(\vec x)\, d^3x = -\int_{r<R} \nabla \Phi \, d^3x \underset{\text{volume} \to \text{surface}}{=} -\int_{r=R} R^2 d\Omega\, \Phi(\vec x)\, \hat n \qquad (32)$$

where $\hat n = \frac{\vec x}{R}$, is the outwardly directed normal. Substituting Eq. (2), we have

$$\int_{r<R} \vec E(\vec x)\, d^3x = -\frac{R^2}{4\pi\epsilon_0}\int d^3x'\, \rho(\vec x') \int_{r=R} d\Omega \frac{\hat n}{|\vec x - \vec x'|} \qquad (33)$$

To perform the angular integration, write $\hat n$ in terms of the spherical angle (θ, ϕ) as

$$\hat n = \hat\imath \sin\theta\cos\phi + \hat\jmath \sin\theta \sin\phi + \hat k \cos\theta \qquad (34)$$

Note that:

$$Y_{11} = -\sqrt{\frac{3}{8\pi}} \sin\theta\, e^{i\phi} \qquad (35)$$

$$Y_{1,-1} = (-1)^{-1} Y_{11} = +\sqrt{\frac{3}{8\pi}} \sin\theta\, e^{-i\phi} \qquad (36)$$

Subtracting Eqs.

$$Y_{11} - Y_{1,-1} = -\sqrt{\frac{3}{8\pi}} \sin\theta \left(e^{i\phi} + e^{-i\phi}\right) = -2\sqrt{\frac{3}{8\pi}} \sin\theta\cos\phi \qquad (37)$$

$$\hookrightarrow \sin\theta\cos\phi = -\frac{1}{2}\sqrt{\frac{8\pi}{3}} \left[Y_{11}(\theta,\phi) - Y_{1,-1}(\theta,\phi)\right] \qquad (38)$$

Adding them leads to

$$Y_{11} + Y_{1,-1} = -\sqrt{\frac{3}{8\pi}} \sin\theta \left(e^{i\phi} - e^{-i\phi}\right) = -2i\sqrt{\frac{3}{8\pi}} \sin\theta\sin\phi \qquad (39)$$

$$\hookrightarrow \sin\theta\sin\phi = \frac{i}{2}\sqrt{\frac{8\pi}{3}} \left[Y_{11}(\theta,\phi) + Y_{1,-1}(\theta,\phi)\right] \qquad (40)$$

Also, note that

$$Y_{10} = \sqrt{\frac{3}{4\pi}} \cos\theta \quad \to \quad \cos\theta = \sqrt{\frac{4\pi}{3}}\, Y_{10}(\theta,\phi) \qquad (41)$$

Then, Eq. (34) becomes

$$\hat n = \hat\imath\left(-\frac{1}{2}\sqrt{\frac{8\pi}{3}}\right)\left[Y_{11}(\theta,\phi) - Y_{1,-1}(\theta,\phi)\right] + \hat\jmath\, \frac{i}{2}\sqrt{\frac{8\pi}{3}}\left[Y_{11}(\theta,\phi) + Y_{1,-1}(\theta,\phi)\right]$$
$$+ \hat k \sqrt{\frac{8\pi}{3}}\, Y_{10}(\theta,\phi) \qquad (42)$$

Recall that:

$$\frac{1}{|\vec x - \vec x'|} = \sum_{l=0}^{\infty} \frac{r_<^l}{r_>^{l+1}} P_l(\cos\gamma) = 4\pi \sum_{l=0}^{\infty}\sum_{m=-l}^{l} \frac{1}{2l+1} \frac{r_<^l}{r_>^{l+1}} Y^*_{lm}(\theta',\phi')\, Y_{lm}(\theta,\phi) \qquad (43)$$

Then,

$$\int_{r=R} d\Omega \frac{\hat n}{|\vec x - \vec x'|} = \int_{r=R} d\Omega\, 4\pi \sum_{l=0}^{\infty}\sum_{m=-l}^{l} \frac{r_<^l}{r_>^{l+1}} Y^*_{lm}(\theta',\phi')\, Y_{lm}(\theta,\phi)$$
$$\times\left[\left(-\frac{1}{2}\sqrt{\frac{8\pi}{3}}\right)\left[Y_{11}(\theta,\phi) - Y_{1,-1}(\theta,\phi)\right]\hat\imath + \left(\frac{i}{2}\sqrt{\frac{8\pi}{3}}\right)\left[Y_{11}(\theta,\phi) - Y_{1,-1}(\theta,\phi)\right]\hat\jmath \right.$$
$$\left. + \sqrt{\frac{4\pi}{3}}\, Y_{10}(\theta,\phi)\,\hat k\right] \qquad (44)$$

Due to the orthogonality of the Y_{lm}, only the $l=1$ term survives:

$$\int_{r=R} d\Omega \frac{\hat{n}}{|\vec{x}-\vec{x}'|} = \int_{r=R} d\Omega\, 4\pi \sum_{m=-1}^{1} \frac{r_<}{r_>^2} Y^*_{1m}(\theta',\phi')\, Y_{1m}(\theta,\phi) \times [\dots] \tag{45}$$

Note that we can simplify the summation by extracting them in terms of spherical angles:

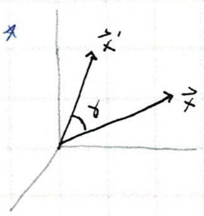
$$\sum_{m=-1}^{1} Y_{1m}(\theta',\phi')\, Y_{1m}(\theta,\phi) = Y^*_{1,-1}(\theta',\phi')\, Y_{1,-1}(\theta,\phi) + Y^*_{10}(\theta',\phi')\, Y_{10}(\theta,\phi) + Y^*_{11}(\theta',\phi')\, Y_{11}(\theta,\phi)$$

$$= \left(\sqrt{\frac{3}{8\pi}} \sin\theta' e^{i\phi}\right)\left(\sqrt{\frac{3}{8\pi}} \sin\theta\, e^{-i\phi}\right) + \left(\sqrt{\frac{3}{4\pi}} \cos\theta'\right)\left(\sqrt{\frac{3}{4\pi}} \cos\theta\right)$$

$$+ \left(-\sqrt{\frac{3}{8\pi}} \sin\theta' e^{-i\phi}\right)\left(-\sqrt{\frac{3}{8\pi}} \sin\theta\, e^{i\phi}\right)$$

$$= \frac{3}{8\pi} \sin\theta' \sin\theta \cdot \left(e^{i(\phi'-\phi)} + e^{-i(\phi'-\phi)}\right) + \frac{3}{4\pi} \cos\theta' \cos\theta$$

$$= \frac{3}{\cancel{8}_{4}\pi} \sin\theta' \sin\theta \left(\cancel{2} \cos(\phi'-\phi)\right) + \frac{3}{4\pi} \cos\theta' \cos\theta$$



$$= \frac{3}{4\pi}\left(\sin\theta' \sin\theta \cos(\phi-\phi') + \cos\theta' \cos\theta\right)$$

$$= \frac{3}{4\pi} \cos\gamma \tag{46}$$

Evaluate each term of Eq. (45)

$$> \int d\Omega\, Y^*_{1,-1}(\theta',\phi')\, Y_{1,-1}(\theta,\phi)\, \hat{n} = Y^*_{1,-1}(\theta',\phi') \int d\Omega\, (-1)\, Y_{1,1}(\theta,\phi) \times$$

$$\left\{ -\frac{1}{2}\sqrt{\frac{8\pi}{3}} \left[Y_{11}(\theta,\phi) - Y_{1,-1}(\theta,\phi)\right] \hat{\imath} + \frac{i}{2}\sqrt{\frac{8\pi}{3}} \left[Y_{11}(\theta,\phi) + Y_{1,-1}(\theta,\phi)\right] \hat{\jmath} + \sqrt{\frac{4\pi}{3}}\, Y_{10}(\theta,\phi)\, \hat{k} \right\}$$

$$= Y^*_{1,-1}(\theta',\phi') \int d\Omega\, (-1) \left\{ \left(-\frac{1}{2}\sqrt{\frac{8\pi}{3}}\right) \left[Y_{1,1}(\theta,\phi)\right]^2 \hat{\imath} + \frac{i}{2}\sqrt{\frac{8\pi}{3}} \left[Y_{1,1}(\theta,\phi)\right]^2 \hat{\jmath} \right\}$$

$$= Y^*_{1,-1}(\theta',\phi')\, (-1) \left(\frac{1}{2}\sqrt{\frac{8\pi}{3}}\right) (-\hat{\imath} + i\hat{\jmath})$$

$$= \frac{1}{2}\sqrt{\frac{8\pi}{3}}\, Y^*_{1,-1}(\theta',\phi')\, (\hat{\imath} - i\hat{\jmath}) \tag{47}$$

$$> \int d\Omega\, Y^*_{10}(\theta',\phi')\, Y_{10}(\theta,\phi)\, \hat{n} = \int d\Omega\, Y^*_{10}(\theta',\phi')\, Y_{10}(\theta',\phi') \sqrt{\frac{4\pi}{3}}\, Y_{10}(\theta,\phi)\, \hat{k} = \sqrt{\frac{4\pi}{3}}\, Y^*_{10}(\theta',\phi')\, \hat{k} \quad (4$$

$$> \int d\Omega\, Y^*_{11}(\theta',\phi')\, Y_{11}(\theta,\phi)\, \hat{n} = \int d\Omega\, Y^*_{11}(\theta',\phi') \left[(-1)\, Y^*_{1,-1}(\theta,\phi)\right] \hat{n}$$

$$= -Y^*_{11}(\theta',\phi') \left[-\frac{1}{2}\sqrt{\frac{8\pi}{3}} (-1)\, \hat{\imath} + \frac{i}{2}\sqrt{\frac{8\pi}{3}}\, \hat{\jmath} \right]$$

$$= \frac{1}{2}\sqrt{\frac{8\pi}{3}} (-\hat{\imath} - i\hat{\jmath})\, Y^*_{11}(\theta',\phi') \tag{49}$$

Adding everything:

$$\int_{r=R} d\Omega \frac{\hat{n}}{|\vec{x}-\vec{x}'|} = \frac{4\pi}{3}\left[\frac{1}{2}\sqrt{\frac{8\pi}{3}}\, Y^*_{1,-1}(\theta',\phi')\, (\hat{\imath} - i\hat{\jmath}) + \frac{1}{2}\sqrt{\frac{8\pi}{3}}\, Y^*_{11}(\theta',\phi')\, (-\hat{\imath} - i\hat{\jmath}) \right.$$

$$\left. + \sqrt{\frac{4\pi}{3}}\, Y^*_{10}(\theta',\phi')\, \hat{k} \right] \frac{r_<}{r_>^2}$$

$$= \frac{4\pi}{3}\left[\frac{1}{2}\cancel{\sqrt{\frac{8\pi}{3}}}\, \cancel{\sqrt{\frac{3}{8\pi}}} \sin\theta' e^{i\phi'} (\hat{\imath} - i\hat{\jmath}) + \frac{1}{2}\cancel{\sqrt{\frac{8\pi}{3}}} (-1) \cancel{\sqrt{\frac{3}{8\pi}}} \sin\theta' e^{-i\phi'} (-\hat{\imath} - i\hat{\jmath}) \right.$$

$$\left. + \cancel{\sqrt{\frac{4\pi}{3}}}\, \cancel{\sqrt{\frac{3}{4\pi}}} \cos\theta'\, \hat{k} \right] \frac{r_<}{r_>^2}$$

$$= \frac{4\pi}{3}\left\{\frac{1}{2}\sin\theta'\left[\overbrace{\left(e^{i\phi'}+e^{-i\phi'}\right)}^{2\cos\phi'}\hat{i} - i\overbrace{\left(e^{i\phi'}-e^{-i\phi'}\right)}^{2i\sin\phi'}\hat{j}\right] + \cos\theta'\hat{k}\right\}\frac{r_<}{r_>^2}$$

$$= \frac{4\pi}{3}\left[\sin\theta'\cos\phi'\,\hat{i} + \sin\theta'\sin\phi'\,\hat{j} + \cos\theta'\,\hat{k}\right]\frac{r_<}{r_>^2}$$

$$= \frac{4\pi}{3}\,\hat{n}'\,\frac{r_<}{r_>^2} \qquad (50)$$

Thus, Eq. (38) becomes

$$\int_{r<R} \vec{E}(\vec{x})\,d^3x = -\frac{R^2}{\cancel{4\pi}\epsilon_0}\int d^3x'\,\rho(\vec{x}')\,\frac{r_<}{r_>^2}\,\hat{n}'\,\rho(\vec{x})\,\frac{\cancel{4\pi}}{3} = -\frac{R^2}{3\epsilon_0}\int d^3x'\,\frac{r_<}{r_>^2}\,\hat{n}'\rho(\vec{x}') \qquad (51)$$

where $(r_<, r_>) = (r', R)$ or (R, r') depending on which of r' and R is greater.

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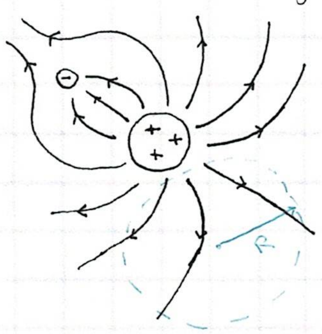
- If a sphere completely encloses within its radius R the charge density, then $r_< = r'$ and $r_> = R$. Thus, we have from Eq. (51):

$$\boxed{\int_{r<R}\vec{E}(\vec{x})\,d^3x =} -\frac{R^2}{3\epsilon_0}\int d^3x'\,\frac{r'}{\cancel{R^2}}\,\hat{n}'\rho(\vec{x}) = -\frac{1}{3\epsilon_0}\int \vec{x}'\rho(\vec{x}')\,d^3x' = \boxed{-\frac{\vec{p}}{3\epsilon_0}} \qquad (52)$$

where $\vec{p}$ is the electric dipole moment of the charge distribution wrt the center of the sphere. This volume integral is independent of the size of the sphere.

> Case 2: All charges are outside of the sphere



With the same approach, we will choose the origin of coor'ds at the center of the sphere. The charges are outside the sphere. We have the volume integral of the electric field, (from Eq. (51))

$$\int_{r<R}\vec{E}(\vec{x})\,d^3x = -\frac{R^2}{3\epsilon_0}\int d^3x'\,\frac{R}{r'^2}\,\hat{n}'\,\rho(\vec{x}')$$

$$= -\frac{R^3}{3\epsilon_0}\int d^3x'\,\frac{\hat{n}'}{r'^2}\,\rho(\vec{x}') \qquad (53)$$

where $r_> = r'$ and $r_< = R$ in this case.

- Recall Coulomb's law ($\vec{E}$ field of a given $\rho(\vec{x})$)

$$\vec{E}(\vec{x}) = \frac{1}{4\pi\epsilon_0}\int \rho(\vec{x}')\,\frac{\vec{x}-\vec{x}'}{|\vec{x}-\vec{x}'|^3}\,d^3x' \qquad (54)$$

Getting $\vec{E}(\vec{x}=0)$,

$$\vec{E}(0) = \frac{1}{4\pi\epsilon_0}\int \rho(\vec{x}')\,\frac{-\vec{x}'}{\underbrace{|-\vec{x}'|^3}_{r'}}\,d^3x' = \frac{1}{4\pi\epsilon_0}\int \rho(\vec{x}')\,\frac{\vec{x}'}{r'}\,\frac{1}{r'^2}\,d^3x' = \frac{1}{4\pi\epsilon_0}\int d^3x'\,\frac{\hat{n}'}{r'^2}\,\rho(\vec{x}') \qquad (55)$$

Substituting Eq (55) to Eq (53),

$$\boxed{\int_{r<R}\vec{E}(\vec{x})\,d^3x = \frac{4\pi R^3}{3}\,\vec{E}(0)} \qquad (56)$$

In other words, the ave. value of the electric field over a spherical volume containing no charge is the value of the field at the center of the sphere.

- What is the consequence of this?

The electric field is given by: $\vec{E}(\vec{x}) = -\nabla\Phi(\vec{x})$ (57)

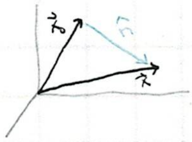
Each of the multipole moment contributes to the electric field components

$$E_r = \frac{l+1}{(2l+1)\epsilon_0}\,q_{lm}\,\frac{Y_{lm}(\theta,\phi)}{r^{l+2}} \qquad (58)$$

$$E_\theta = -\frac{1}{(2l+1)\epsilon_0}\,q_{lm}\,\frac{1}{r^{l+1}}\,\frac{\partial}{\partial\theta}\,Y_{lm}(\theta,\phi) \qquad (59)$$

$$E_\phi = -\frac{1}{2l+1} q_{lm} \frac{1}{r^{l+1}} \frac{im}{\sin\theta} Y_{lm}(\theta,\phi) \tag{60}$$

Note that $\partial Y_{lm}/\partial\theta$ and $Y_{lm}/\sin\theta$ can be expressed as linear combinations of other Y_{lm}'s



From this, the electric field due to a dipole $\vec{p}$ at the point $\vec{x}_0$ is

$$\vec{E}(\vec{x}) = \frac{3\hat{n}(\vec{p}\cdot\hat{n}) - \vec{p}}{4\pi\epsilon_0 |\vec{x}-\vec{x}'|^3}, \qquad \hat{n}: \text{unit vector directed from } \vec{x}_0 \text{ to } \vec{x} \tag{61}$$

- What is wrong with this?

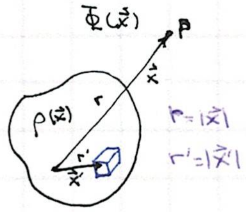
If we integrate this over a sphere centered at $\vec{x} = \vec{x}_0$, we arrive at the ambiguous result due to singularity at $x = x_0$. Also, the integral is conventionally equal to zero which is inconsistent with Eq. (52).

So, to account for the non-vanishing integral, we can modify the electric field of a dipole to be

$$\boxed{\vec{E}(\vec{x}) = \frac{1}{4\pi\epsilon_0}\left[\frac{3\hat{n}(\vec{p}\cdot\hat{n}) - \vec{p}}{|\vec{x}-\vec{x}'|^3} - \frac{4\pi}{3}\vec{p}\,\delta(\vec{x}-\vec{x}_0)\right]} \tag{62}$$

The added delta fxn does not contribute to the field (due to the singularity) away from the site of the dipole.

▶ Another approach to multipole expansion and examples



The potential at point $\vec{x}$ is given by

$$\Phi(\vec{x}) = \frac{1}{4\pi\epsilon_0}\int_{V'} \frac{\rho(\vec{x}')}{|\vec{x}'-\vec{x}|} d^3x' \tag{63}$$

Same as Eq. (2)

Assume that $|\vec{x}'|/|\vec{x}| \ll 1$. This allows the use of the binomial expansion of $1/|\vec{x}-\vec{x}'|$. Recall that:

$$|\vec{x}-\vec{x}'| = \sqrt{(\vec{x}-\vec{x}')\cdot(\vec{x}-\vec{x}')} = \sqrt{\vec{x}\cdot\vec{x} - 2\vec{x}\cdot\vec{x}' + \vec{x}'\cdot\vec{x}'} = \sqrt{r^2 - 2\vec{x}\cdot\vec{x}' + r'^2} = r\sqrt{1 + \frac{-2\vec{x}\cdot\vec{x}' + r'^2}{r^2}} \tag{64}$$

Under the given assumption, $\frac{|2\vec{x}\cdot\vec{x} - r'^2|}{r^2} \ll 1$, enabling us to use the binomial expansion given by

$$(1+x)^s = \sum_{n=0}^{\infty} \frac{s!}{n!(s-n)!} x^n \overset{|x|<1}{=} 1 + \frac{s}{1!}x + \frac{s(s-1)}{2!}x^2 + \frac{s(s-1)(s-2)}{3!} + \dots \tag{65}$$

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▶ Now, let $\epsilon = (-2\vec{x}\cdot\vec{x}' + r'^2)/r^2 = -2\frac{\vec{x}}{r^2}\cdot\vec{x}' + \frac{r'^2}{r^2} = -2\frac{\hat{x}\cdot\vec{x}'}{r} + \frac{r'^2}{r^2}$. Then

$$\frac{1}{|\vec{x}-\vec{x}'|} = \frac{1}{r(1+\epsilon)^{1/2}} = \frac{1}{r}(1+\epsilon)^{-1/2} \tag{66}$$

Comparing Eq (66) with Eq. (65), we see that $s = -1/2$. So,

$$\frac{1}{|\vec{x}-\vec{x}'|} = \frac{1}{r}\left(1 - \frac{\epsilon}{2} + \frac{(-1/2)(-1/2-1)}{2}\epsilon^2 + \frac{(-1/2)(-1/2-1)(-1/2-2)}{3!}\epsilon^3 + \dots\right)$$

$$= \frac{1}{r}\left(1 - \frac{\epsilon}{2} + \frac{3}{8}\epsilon^2 - \frac{5}{16}\epsilon^3 + \dots\right)$$

$$= \frac{1}{r}\left[1 - \frac{1}{2}\left(-2\frac{\hat{x}\cdot\vec{x}'}{r} + \frac{r'^2}{r^2}\right) + \frac{3}{8}\left(-2\frac{\hat{x}\cdot\vec{x}'}{r} + \frac{r'^2}{r^2}\right)^2 + \mathcal{O}\left(\frac{r'^3}{r^3}\right)\right]$$

$$= \frac{1}{r}\left[1 + \frac{\hat{x}\cdot\vec{x}'}{r} - \frac{1}{2}\frac{r'^2}{r^2} + \frac{3}{8}\left(4\frac{(\hat{x}\cdot\vec{x}')^2}{r^2} + \underbrace{2(-2)\frac{\hat{x}\cdot\vec{x}'}{r}\cdot\frac{r'^2}{r^2}}_{\mathcal{O}(\frac{1}{r^3})} + \mathcal{O}\left(\frac{r'^4}{r^4}\right)\right)\right]$$

$$= \frac{1}{r}\left[1 + \frac{\hat{x}\cdot\vec{x}'}{r} - \frac{1}{2}\frac{r'^2}{r^2} + \frac{3}{2}\frac{(\hat{x}\cdot\vec{x}')^2}{r^2} + \mathcal{O}\left(\frac{1}{r^3}\right)\right]$$

$$\frac{1}{|\vec{x}-\vec{x}'|} = \frac{1}{r}\left[1 + \frac{\hat{x}\cdot\vec{x}'}{r} + \frac{1}{2r^2}\left(3(\hat{x}\cdot\vec{x}')^2 - r'^2\right) + \mathcal{O}\left(\frac{r'^3}{r^3}\right)\right] \tag{67}$$

* $d^3x' = dV'$

Recall that $|\vec{x}'|/|\vec{x}| = r'/r \ll 1$. Then,

$$\Phi(\vec{x}) = \frac{1}{4\pi\epsilon_0}\int_{V'} \rho(\vec{x}')\, \frac{1}{r}\left[1 + \frac{\hat{x}\cdot\vec{x}'}{r} + \frac{1}{2r^2}\left(3(\hat{x}\cdot\vec{x}')^2 - r'^2\right) + O\left(\frac{r'^3}{r^3}\right)\right] dV'$$

$$= \underbrace{\frac{1}{4\pi\epsilon_0 r}\int_{V'} \rho(\vec{x}')\, dV'}_{\text{monopole term}} + \underbrace{\frac{1}{4\pi\epsilon_0}\frac{\hat{x}}{r^2}\cdot\int_{V'} \rho(\vec{x}')\,\vec{x}'\, dV'}_{\text{dipole term}}$$

$$+ \underbrace{\frac{1}{4\pi\epsilon_0}\frac{1}{2r^3}\int_{V'} \rho(\vec{x})\left(3(\hat{x}\cdot\vec{x})^2 - r'^2\right) dV'}_{\text{quadrupole term}} + \ldots \qquad (68)$$

- For the monople term,

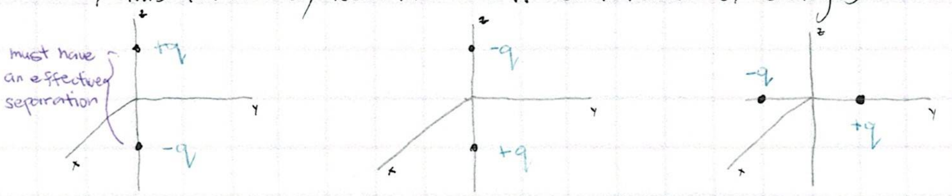
$$\Phi_{monople}(\vec{x}) = \frac{1}{4\pi\epsilon_0 r}\int_{V'} \rho(\vec{x}')\, dV' = \frac{q}{4\pi\epsilon_0 r} \qquad (69)$$

where $q = \int_{V'} \rho(\vec{x}')\, dV'$ is the total charge in V'. This means that far from the source of charges, the distribution of charge appears like a point charge at the origin if $q \neq 0$.

- For the dipole term,

$$\Phi_{dipole}(\vec{x}) = \frac{1}{4\pi\epsilon_0 r^2}\hat{x}\cdot\int_{V'} \rho(\vec{x}')\,\vec{x}'\, dV' = \frac{\vec{x}\cdot\vec{p}}{4\pi\epsilon_0 r^3} \qquad (70)$$

where $\vec{p} = \int_{V'} \rho(\vec{x}')\,\vec{x}'\, dV'$ is the dipole moment. This term contributes when there is an effective separation of positive and negative charges in the volume. Far from the source, this term may look like the ff. distribution of charges

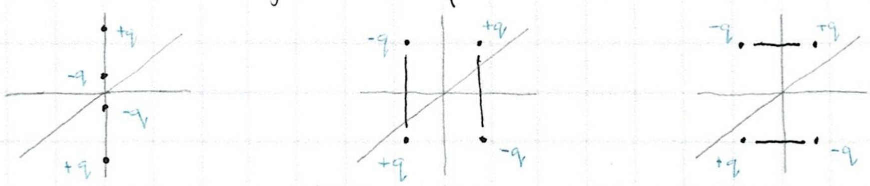


and so on...

- For the quadrupole term,

$$\Phi_{quadrupole}(\vec{x}) = \frac{1}{8\pi\epsilon_0}\frac{1}{r^3}\int_{V'} \rho(\vec{x}')\left(3(\hat{x}\cdot\vec{x}')^2 - r'^2\right) dV' \qquad (71)$$

Note that a quadrupole is a sys. of two dipoles. It could arise from one of the possible effective configuration of dipoles



and so on...

We can rewrite the quadruple term as follows,

$$3(\hat{x}\cdot\vec{x}')^2 - r'^2 = 3\left(\frac{\vec{x}}{r}\cdot\vec{x}'\right)^2 - r'^2$$

$$= \frac{3}{r^2}(\vec{x}\cdot\vec{x}')^2 - r'^2$$

$$= \frac{3}{r^2}\left(\sum_i x_i x_i'\right)\left(\sum_j x_j x_j'\right) - r'^2$$

$$= \frac{3}{r^2}\sum_i\sum_j x_i x_j x_i' x_j' - r'^2$$

$$= \frac{1}{r^2}\left[3\sum_{i,j} x_i x_j x_i' x_j' - r^2 r'^2\right]$$

$$= \frac{1}{r^2}\left[3\sum_{i,j} x_i x_j x_i' x_j' - \sum_i x_i^2\, r'^2\right]$$

$$= \frac{1}{r^2}\left[3\sum_{i,j} x_i x_j x_i' x_j' - \sum_{i,j} x_i x_j \delta_{ij}\, r'^2\right]$$

$$3(\hat{x}\cdot\vec{x}')^2 - r'^2 = \frac{1}{r^2}\sum_{ij} x_i x_j (3x'_i x'_j - \delta_{ij} r'^2) \quad (72)$$

Thus, Eq. (71) becomes

$$\Phi_{quadrupole}(\vec{x}) = \frac{1}{4\pi\epsilon_0}\cdot\frac{1}{2r^3}\int_{V'} \rho(\vec{x}')\frac{1}{r^2}\sum_{ij} x_i x_j (3x'_i x'_j - \delta_{ij} r'^2)\, dv'$$

$$= \frac{1}{4\pi\epsilon_0}\frac{1}{2r^5}\sum_{ij} x_i x_j \int_{V'} \rho(\vec{x}')(3x'_i x'_j - \delta_{ij} r'^2)\, dv'$$

$$= \frac{1}{4\pi\epsilon_0}\frac{1}{2r^5}\sum_{ij} x_i x_j Q_{ij} \quad (73)$$

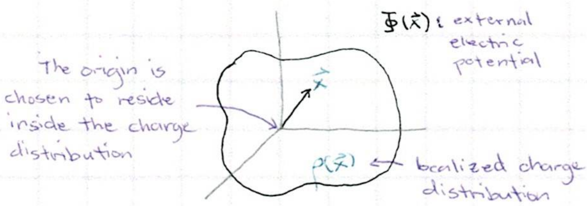
where $Q_{ij} = \int_{V'} \rho(\vec{x}')(3x'_i x'_j - \delta_{ij} r'^2)$ is the quadrupole moment tensor

▶ 4.2: Multipole expansion of the energy of a charge distribution in an external field

Let a localized charge distribution, described by $\rho(\vec{x})$, be placed in an external electric potential $\Phi(\vec{x})$. The electrostatic energy of the sys. is

$$\boxed{W = \int \rho(\vec{x})\,\Phi(\vec{x})\, d^3x} \quad (74)$$

We choose the origin as indicated in the figure. Assume that $\Phi(\vec{x})$ varies slowly in the region where the charge is distributed. Then, $\Phi(\vec{x})$ can be Taylor expanded around the origin



$$\Phi(\vec{x}) = \Phi(0) - \vec{x}\cdot\vec{\nabla}\Phi(0) + \frac{1}{2}\sum_i\sum_j \frac{\partial^2\Phi(0)}{\partial x_i \partial x_j} + \ldots \quad (75)$$

The point $\vec{x}$ resides inside the charge distribution

Fact: $\vec{E} = -\vec{\nabla}\Phi \rightarrow \vec{\nabla}\Phi(0) = -\vec{E}(0)$ (76)

$$\frac{\partial^2\Phi(0)}{\partial x_i \partial x_j} = \left[\frac{\partial}{\partial x_i}\frac{\partial\Phi(0)}{\partial x_j}\right] = \frac{\partial}{\partial x_i}\left[(-1)E_j(0)\right] = -\frac{\partial E_j(0)}{\partial x_j} \quad (77)$$

Then,

$$\Phi(\vec{x}) = \Phi(0) - \vec{x}\cdot\vec{E}(0) - \frac{1}{2}\sum_{ij} x_i x_j \frac{\partial E_j(0)}{\partial x_i} + \ldots \quad (78)$$

Fact: $\vec{\nabla}\cdot\vec{E} = 0$

Then, we can add the term $\frac{1}{6}r^2\vec{\nabla}\cdot\vec{E} = 0$ without changing the sum. Rewriting this term:

$$\frac{1}{6}r^2\vec{\nabla}\cdot\vec{E}(0) = \frac{1}{6}r^2\sum_i \frac{\partial E_i(0)}{\partial x_i} = \frac{1}{6}r^2\sum_i\sum_j \frac{\partial E_j(0)}{\partial x_i}\delta_{ij} \quad (79)$$

Then,

$$\Phi(\vec{x}) = \Phi(0) - \vec{x}\cdot\vec{E}(0) - \frac{1}{2}\sum_{ij} x_i x_j \frac{\partial E_j(0)}{\partial x_i} + \frac{1}{6}r^2\sum_{ij}\delta_{ij}\frac{\partial E_j(0)}{\partial x_i}$$

$$= \Phi(0) - \vec{x}\cdot\vec{E}(0) - \frac{1}{6}\sum_{ij}(3x_i x_j - r^2\delta_{ij})\frac{\partial E_j(0)}{\partial x_i} + \ldots \quad (80)$$

Substituting back in W

$$W = \int \rho(\vec{x})\left[\Phi(0) - \vec{x}\cdot\vec{E}(0) - \frac{1}{6}\sum_{ij}(3x_i x_j - r^2\delta_{ij})\frac{\partial E_j}{\partial x_i}(0) + \ldots\right] d^3x$$

$$= \Phi(0)\overbrace{\int \rho(\vec{x})\, d^3x}^{q} - \overbrace{\int \vec{x}\rho(\vec{x})\, d^3x}^{\vec{p}}\cdot\vec{E}(0) - \frac{1}{6}\sum_{ij}\overbrace{\int \rho(\vec{x})(3x_i x_j - r^2\delta_{ij})\, d^3x}^{Q_{ij}}\cdot\frac{\partial E_j}{\partial x_i}(0) + \ldots$$

$$W = \Phi(0)q - \vec{p}\cdot\vec{E}(0) - \frac{1}{6}\sum_{ij} Q_{ij}\frac{\partial E_j}{\partial x_i}(0) + \ldots \quad (81)$$

energy distribution from the total charge q

charge q interacts w/ the electric potential

energy from the electric dipole $\vec{p}$

electric dipole $\vec{p}$ interacts with the electric field

energy from the electric quadrupole Q_{ij}

the quadrupole interacts w/ the electric field gradient

The quadrupole interaction is of particular in nuclear physics.

"Atomic nuclei can possess electric quadrupole moments, and magnitudes and signs reflect the nature of the forces between neutrons & protons, as well as the shapes of the nuclei themselves."

The dipole - dipole interaction can be obtained from Eq. (81) and the electric field due to a dipole established in the previous notes,

$$E(\vec{x}) = \frac{1}{4\pi\epsilon_0}\left[\frac{3\hat{n}(\vec{p}\cdot\hat{n}) - \vec{p}}{|\vec{x}-\vec{x}_0|^3} - \frac{4\pi}{3}\vec{p}\,\delta(\vec{x}-\vec{x}_0)\right] \tag{82}$$

- We consider the case of the two dipoles $\vec{p}_1$ and $\vec{p}_2$

Then, $\vec{E}(0)$ due to $\vec{p}_2$ is

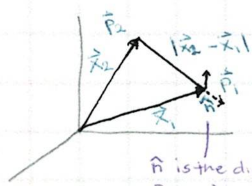
$$\vec{E}(0) = \frac{1}{4\pi\epsilon_0}\left[\frac{3\hat{n}(\vec{p}_2\cdot\hat{n}) - \vec{p}_2}{|x_0|^3}\right] \tag{83}$$

* the dirac delta term vanishes because the field point is away from its location

Then, $$W = -\vec{p}_1\cdot\vec{E}(0) = \frac{1}{4\pi\epsilon_0}\left[\frac{\vec{p}_1\cdot\vec{p}_2 - 3(\vec{p}_2\cdot\hat{n})(\vec{p}_1\cdot\hat{n})}{|x_0|^3}\right] \tag{84}$$

where $|\vec{x}_0|$ is the distance separating the two dipoles.

In general, when we have the following situation below:



$\hat{n}$ is the direction from $\vec{x}_1$ to $\vec{x}_2$

Then, the interaction energy is

$$W = \frac{1}{4\pi\epsilon_0}\,\frac{\vec{p}_1\cdot\vec{p}_2 - 3(\vec{p}_2\cdot\hat{n})(\vec{p}_1\cdot\hat{n})}{|\vec{x}_1-\vec{x}_2|^3} \tag{85}$$

Notice that the change $\hat{n}\to-\hat{n}$ does not affect W. So it does not matter whether $\hat{n}$ is defined from $\vec{x}_1$ to $\vec{x}_2$ or from $\vec{x}_2$ to $\vec{x}_1$.

Mar 12, 2021 ▶ ▶ 4.3: Elementary treatment of electrostatics with ponderable media

- Ponderable: having a detectable amount of matter
- Ponderable media: a substance with substantial amount of matter that do not conduct electricity; generally an insulator. Polymers are good examples.
  - > Vacuum is not a ponderable medium (but a perfect insulator)
  - > Air is sufficiently tenuous that its treatment may not be diff. from vaccum. So, even if it is also an insulator, it is also a non-ponderable media
  - > Pure water is a perfect insulator and is an example of ponderable medium.
  - > We will consider insulators that are dielectrics. A dielectric is a ponderable media whose charge distribution can be slightly deformed by an external electric field.
  - > Such deformation may lead to separation of positive and negative charges in the presence of external electric field
  - > A dielectric acquires an effective dipole moment when applied with an electric field
  - > All dielectric are insulators. But not all insulators are dielectrics. Vacuum is a perfect insulator but not a dielectric because it is empty of any matter
- There is a need to distinguich electrostatics in the absence / presence of ponderable media.

Absence	Presence
$\vec{E}$ is due to isolated or distribution of charges in free space	$\vec{E}$ is due to isolated or distribution of charges AND dipoles arising from polarization of the medium
$\vec{E}$ is referred to as microscopic field, $\vec{E}_{micro}$ $[\nabla\times\vec{E}_{micro} = 0 \to \vec{E}_{micro} = -\nabla\Phi_{micro}]$	$\vec{E}$ is referred to as macroscopic field, $\vec{E}_{macro}$ $[\nabla\times\vec{E}_{macro} = 0 \to \vec{E}_{macro} = -\nabla\Phi_{macro}]$

Mar 16, 2021

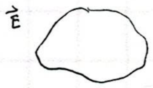
- If an electric field is applied to a medium made of a large number of atoms/molecules, the charges bound in each molecule will respond to the electric field and the molecular charge density will be distorted.
- In simple substances, when there is no applied fields, the multipole moments are all zero, at least when averaged over many molecules
- In the presence of an applied field, the moments will be induced. The dominant contribution will come from the dipole moments. There is thus produced, in the medium, an electric polarization $\vec{P}$ (dipole moment per unit volume) given by

$$\vec{P}(\vec{x}) = \sum_i N_i \langle p_i \rangle \tag{86}$$

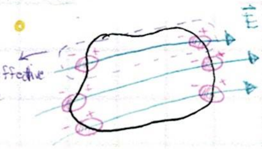
where $\vec{P}(\vec{x})$: electric polarization

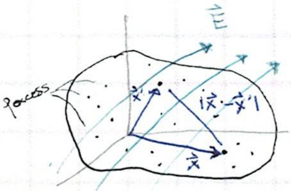
$\vec{p}_i(\vec{x})$: dipole moment of the ith molecule

N_i : ave. no. per unit vol. of the ith type molecule at point $\vec{x}$

- 

Without an ext. electric field, there could be free charges in the medium, ρ_{free}. Electric field is only due to these charges

- 



In the presence of external electric field, there is an effective induced dipole moment. The free charge density ρ_{free} in small volume ΔV is the sum of net charges coming from each molecule and the excess charges in the volume. Let there be some different kinds of molecules in ΔV, then let $\langle e_i \rangle$ be the ave. net charge from each molecules. We have

$$\rho_{free}(\vec{x}) = \sum_i N_i \langle e_i \rangle + \rho_{excess} \tag{87}$$

Assuming that no higher poles are induced beyond the dipole (e.g. no quadrupole and higher), the macroscopic potential at $\vec{x}$ due to volume ΔV is

$$\Delta\Phi(\vec{x},\vec{x}') = \frac{1}{4\pi\epsilon_0}\left[\underbrace{\frac{\rho(\vec{x}')\Delta V}{|\vec{x}-\vec{x}'|}}_{\text{monopole contrib.}} + \underbrace{\frac{\vec{P}(\vec{x}')\cdot(\vec{x}-\vec{x}')}{|\vec{x}-\vec{x}'|^3}\Delta V}_{\text{dipole contrib.}}\right] \tag{88}$$

where $\vec{x}$ is outside volume ΔV. In the macroscopic limit, $\Delta V \to dV$ and $\Sigma \to \int$. Integrating over all space,

$$\Phi(\vec{x}) = \sum_{\vec{x}'} \Delta\Phi(\vec{x},\vec{x}') = \frac{1}{4\pi\epsilon_0}\int \left[\frac{\rho(\vec{x}')\Delta V}{|\vec{x}-\vec{x}'|} + \vec{P}(\vec{x}')\cdot\nabla'\left(\frac{1}{|\vec{x}-\vec{x}'|}\right)\right] d^3x' \tag{89}$$

★ Note that:

$$\nabla'\cdot\left[\frac{\vec{P}(\vec{x}')}{|\vec{x}-\vec{x}'|}\right] = \frac{\nabla'\cdot\vec{P}(\vec{x}')}{|\vec{x}-\vec{x}'|} + \vec{P}(\vec{x}')\cdot\nabla'\left(\frac{1}{|\vec{x}-\vec{x}'|}\right)$$

Rearranging this,

$$\vec{P}(\vec{x}')\cdot\nabla'\left(\frac{1}{|\vec{x}-\vec{x}'|}\right) = \nabla'\cdot\left[\frac{\vec{P}(\vec{x}')}{|\vec{x}-\vec{x}'|}\right] - \frac{\nabla'\cdot\vec{P}(\vec{x}')}{|\vec{x}-\vec{x}'|}$$

★ Divergence theorem

$$\int_V (\nabla\cdot\vec{F})\, dV = \oint_S (\vec{F}\cdot\hat{n})\, dS$$

where we used $\nabla'\left(\frac{1}{|\vec{x}-\vec{x}'|}\right) = \frac{\vec{x}-\vec{x}'}{|\vec{x}-\vec{x}'|^3}$. Observe that thru IBP,

$$\int d^3x'\, \vec{P}(\vec{x}')\cdot\nabla'\left(\frac{1}{|\vec{x}-\vec{x}'|}\right) = \int d^3x'\, \nabla'\left(\frac{\vec{P}(\vec{x}')}{|\vec{x}-\vec{x}'|}\right) - \int d^3x' \frac{\nabla'\cdot\vec{P}(\vec{x}')}{|\vec{x}-\vec{x}'|}$$

(divergence theorem)

$$= \oint dS\, \cancelto{0}{\frac{\vec{P}(\vec{x}')\cdot\hat{n}}{|\vec{x}-\vec{x}'|}} - \int d^3x' \frac{\nabla'\cdot\vec{P}(\vec{x}')}{|\vec{x}-\vec{x}'|} \tag{90}$$

The surface integral vanishes outside since there is no media (i.e. the polarization $\vec{P}=0$). Also, the surface is infinitely large. Substituting in Eq. (90) to Eq. (89),

$$\Phi(\vec{x}) = \frac{1}{4\pi\epsilon_0}\int d^3x' \frac{1}{|\vec{x}-\vec{x}'|}\left[\rho(\vec{x}') - \nabla'\cdot\vec{P}(\vec{x}')\right] \tag{91}$$

We know that

$$\vec{E}(\vec{x}) = -\nabla\Phi(\vec{x}) = -\nabla\frac{1}{4\pi\epsilon_0}\int d^3x' \frac{1}{|\vec{x}-\vec{x}'|}\left[\rho(\vec{x}') - \nabla'\cdot\vec{P}(\vec{x}')\right]$$

Using the relation used earlier,

$$\vec{E}(\vec{x}) = -\frac{1}{4\pi\epsilon_0}\int d^3x'\, \nabla\left(\frac{1}{|\vec{x}-\vec{x}'|}\right)\left[\rho(\vec{x}') - \nabla'\cdot\vec{P}(\vec{x}')\right] \tag{92}$$

Note that the operator ∇ did not act on the terms inside the brackets since they are in the terms of charge points $\vec{x}'$. Getting the divergence of the macroscopic electric field $\vec{E}(\vec{x})$,

$$\nabla\cdot\vec{E}(\vec{x}) = -\frac{1}{4\pi\epsilon_0}\int d^3x'\, \underbrace{\nabla^2\left(\frac{1}{|\vec{x}-\vec{x}'|}\right)}_{-4\pi\delta(\vec{x}-\vec{x}')}\left[\rho(\vec{x}') - \nabla'\cdot\vec{P}(\vec{x}')\right]$$

$$= \frac{1}{\epsilon_0}\int d^3x'\, \delta(\vec{x}-\vec{x}')\left[\rho(\vec{x}') - \nabla'\cdot\vec{P}(\vec{x}')\right]$$

$$\nabla\cdot\vec{E}(\vec{x}) = \frac{1}{\epsilon_0}\left[\rho(\vec{x}) - \nabla\cdot\vec{P}(\vec{x})\right] \tag{93}$$

Defining the electric displacement $\vec{D}$ as $\vec{D}(\vec{x}) \equiv \epsilon_0\vec{E}(\vec{x}) + \vec{P}(\vec{x})$. Then,

$$\nabla\cdot\vec{D}(\vec{x}) = \frac{1}{\epsilon_0}\ \nabla\cdot\vec{E}(\vec{x}) + \nabla\cdot\vec{P}(\vec{x}) = \rho(\vec{x}) \tag{94}$$

- If the medium is isotropic, the induced polarization $\vec{P}$ is parallel to $\vec{E}$ with a coefficient of proportionality that is indep. of direction

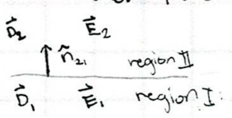
$$\vec{P} = \epsilon_0\chi_e\vec{E} \tag{95}$$

The constant χ_e is called the electric susceptibility of the medium. Then, the displacement $\vec{D}$ is

$$\vec{D} = \epsilon_0\vec{E}+\vec{P} = \epsilon_0\vec{E} + \epsilon_0\chi_e\vec{E} = \epsilon_0(1+\chi_e)\vec{E} = \epsilon\vec{E} \tag{96}$$

where $\epsilon = \epsilon_0(1+\chi_e)$ is called the electric permittivity, $\epsilon/\epsilon_0 = 1+\chi_e$ is called the dielectric constant / relative electric permittivity.

- If the dielectric is not only isotropic but also uniform, then ϵ is indep. of position
- All problems in that medium are reduced to those of preceding chapters, except that the electric field produced by the given charges are reduced by a factor ϵ_0/ϵ
- When there are two media, at the interface, the boundary conditions are.

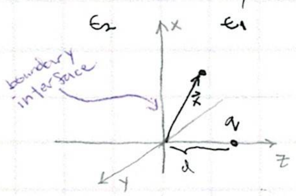


$$\begin{cases}(\vec{D}_2 - \vec{D}_1)\cdot\hat{n}_{21} = \sigma \\ (\vec{E}_2 - \vec{E}_1)\times\hat{n}_{21} = 0\end{cases} \tag{97}$$

where $\hat{n}_{21}$ is the normal to the surface, directed from region I to region II and σ is the macroscopic surface charge density on the boundary surface.

Mar 18, 2021

4.4: Boundary value problems with dielectrics



Assume that there is no macroscopic surface charge density on the interface, $\sigma = 0$. Also, assume that the two media are both isometric and uniform; What is the electric field everywhere in the medium?

Describe the system in terms of cylindrical coor'ds. Solve using the method of images

Region $z>0$: medium 1 $\rightarrow$ $\epsilon_1\nabla\cdot\vec{E}_1 = \rho = q\,\delta(x)\,\delta(y)\,\delta(z-d)$

Region $z<0$: medium 2 $\rightarrow$ $\epsilon_2\nabla\cdot\vec{E}_2 = 0$

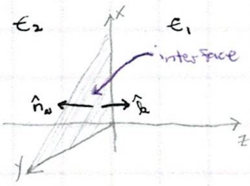
At the interface:

$$(\vec{D}_2 - \vec{D}_1)\cdot \hat{n}_{21} = \sigma = 0$$ ← there is no macroscopic charge density on the surface (98)

$$(\vec{E}_2 - \vec{E}_1)\times \hat{n}_{21} = 0 \qquad (99)$$

Since the media are isotropic & uniform

$$\vec{D}_2 = \epsilon_2 \vec{E}_2, \qquad \vec{D}_1 = \epsilon_1 \vec{E}_1 \qquad (100)$$



Note $\hat{n}_{21} = -\hat{k}$ on the interface.

$$(\vec{D}_2 - \vec{D}_1)\cdot \hat{n}_{21} = 0$$

$$(\epsilon_2 \vec{E}_2 - \epsilon_1 \vec{E}_1)\cdot(-\hat{k}) = 0$$

$$-\epsilon_2 \vec{E}_2 \cdot \hat{k} + \epsilon_1 \vec{E}_1 \cdot \hat{k} = 0$$

$$-\epsilon_2 E_{2,z} + \epsilon_1 E_{1,z} = 0$$

$$\epsilon_2 E_{2,z} = \epsilon_1 E_{1,z}$$ at the interface (101)

Also, we have

$$(\vec{E}_2 - \vec{E}_1)\times \hat{n}_{21} = 0$$

$$(\vec{E}_2 - \vec{E}_1)\times(-\hat{k}) = 0$$

$$\vec{E}_2 \times \hat{k} = \vec{E}_1 \times \hat{k}$$

$$(E_{2,x}\hat{i} + E_{2,y}\hat{j} + E_{2,z}\hat{k})\times\hat{k} = (E_{1,x}\hat{i} + E_{1,y}\hat{j} + E_{1,z}\hat{k})\times\hat{k}$$

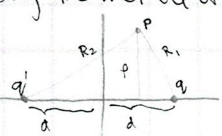
$$E_{2,x}(\hat{i}\times\hat{k}) + E_{2,y}(\hat{j}\times\hat{k}) = E_{1,x}(\hat{i}\times\hat{k}) + E_{1,y}(\hat{j}\times\hat{k}) \qquad (102)$$

Since $\hat{i}\times\hat{k}$ and $\hat{j}\times\hat{k}$ are linearly independent, then

$$\vec{E}_{2,x} = \vec{E}_{1,x}, \quad (103) \qquad \vec{E}_{2,y} = \vec{E}_{1,y}$$ at the interface (104)

OL Lecture Notes ▶ The electric field is derivable from an electric potential Φ everywhere due to $\nabla\times\vec{E} = 0$

Using the method of images, (Region $z>0$: medium 1)



$$\Phi = \frac{1}{4\pi\epsilon_1}\frac{q}{R_1} + \frac{1}{4\pi\epsilon_1}\frac{q'}{R_2} \qquad (105)$$

↳ we use this not ϵ_2, since we replaced the mirrored zone w/ the medium of the real zone

Note that

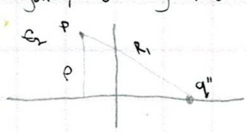
$$R_1 = \sqrt{\rho^2 + (d-z)^2}, \qquad R_2 = \sqrt{\rho^2 + (d+z)^2} \qquad (106)$$

Region $z<0$: medium 2

In this region, $\rho = 0$, i.e. there are no free charges

The potential is just the solution to the Laplace equation in $z<0$.

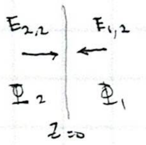
Again, using the method of images,



The charge q in medium 1 appears as a different charge q'' to an observer in medium 2

$$\Phi = \frac{1}{4\pi\epsilon_2}\frac{q''}{R_1} \qquad (107)$$

The charges q' and q'' are to be determined by imposing the boundary conditions. For the z-component,



$$\lim_{z\to 0^-} \epsilon_2 E_{2,z} = \lim_{z\to 0^+} \epsilon_1 E_{1,z}$$

$$\lim_{z\to 0^-} \epsilon_2 (-1)\frac{\partial \Phi_2}{\partial z} = \lim_{z\to 0^+} \epsilon_1 (-1)\frac{\partial \Phi_1}{\partial z}$$

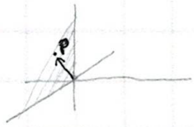
| $\vec{E} = -\nabla\Phi$
Then, $E_z = -\dfrac{\partial \Phi}{\partial z}$

Then,
$$\lim_{z\to 0^-} \cancel{\epsilon_2} \frac{1}{4\pi \cancel{\epsilon_2}} \frac{\partial}{\partial z}\left(\frac{q''}{R_1}\right) = \lim_{z\to 0^+} \cancel{\epsilon_1} \frac{1}{4\pi\cancel{\epsilon_1}} \frac{\partial}{\partial z}\left(\frac{q}{R_1} + \frac{q'}{R_2}\right) \tag{108}$$

We know that
$$\frac{\partial}{\partial z}\left(\frac{1}{R_1}\right)\Big|_{z=0} = -\frac{\partial}{\partial z}\left(\frac{1}{R_2}\right)\Big|_{z=0} = \frac{d}{(\rho^2+d^2)^{3/2}} \tag{109}$$

Then
$$\frac{q'' d}{(\rho^2+d^2)^{3/2}} = \frac{q d}{(\rho^2+d^2)^{3/2}} + \frac{q'(-1)d}{(\rho^2+d^2)^{3/2}} \rightarrow q - q' = q'' \tag{110}$$

Note that Eqs. (103) and (104) are equivalent boundary conditions on the given system



P lies on the xy plane on the interface. Now, align P to the x-axis

Then, using Eq. (103)
$$E_{2,x} = E_{1,x} \longrightarrow E_{2,\rho} = E_{1,\rho} \quad \text{at the interface} \tag{111}$$

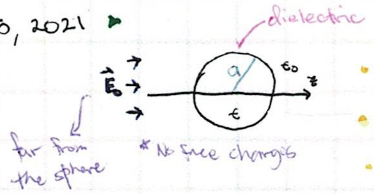
Thus,
$$\lim_{z\to 0^-} E_{2,\rho} = \lim_{z\to 0^+} E_{1,\rho}$$
$$\lim_{z\to 0^-} (-1)\frac{\partial \Phi_2}{\partial \rho} = \lim_{z\to 0^+} (-1)\frac{\partial \Phi_1}{\partial \rho}$$
$$\lim_{z\to 0^-} \frac{1}{4\pi\epsilon_2}\frac{\partial}{\partial\rho}\frac{q''}{R_1} = \lim_{z\to 0^+} \frac{1}{4\pi\epsilon_1}\frac{\partial}{\partial\rho}\left(\frac{q}{R_1} + \frac{q'}{R_2}\right) \tag{112}$$

Note that
$$\frac{\partial}{\partial\rho}\left(\frac{1}{R_1}\right)\Big|_{z=0} = \frac{\partial}{\partial\rho}\left(\frac{1}{R_2}\right)\Big|_{z=0} = -\frac{\rho}{(\rho^2+d^2)^{3/2}} \tag{113}$$

Then,
$$\frac{q''}{\epsilon_2}(-1)\frac{\rho}{(\rho^2+d^2)^{3/2}} = \frac{1}{\epsilon_1}\left(\frac{(-1)\rho q}{(\rho^2+d^2)^{3/2}} + \frac{(-1)\rho q'}{(\rho^2+d^2)^{3/2}}\right) \rightarrow \frac{q''}{\epsilon_2} = \frac{1}{\epsilon_1}(q+q') \tag{114}$$

Solving for q' and q'' using Eqs. (110) & (114)
$$q' = -\left(\frac{\epsilon_2-\epsilon_1}{\epsilon_2+\epsilon_1}\right)q \qquad q'' = \left(\frac{2\epsilon_2}{\epsilon_2+\epsilon_1}\right)q \tag{115}$$

Mar 23, 2021

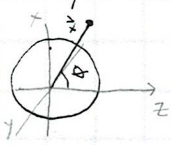


Consider this electrostatics problem where a dielectric is placed in an initially uniform electric field

- In the absence of the sphere, we have a uniform electric field everywhere.
- The dielectric becomes polarized and modifies the electric field
- The goal is to obtain the electric field everywhere (inside and outside of sphere).

Since there are no free charges, the electric potential inside & outside of the sphere are solutions to the Laplace equation ($\vec{\nabla}^2\Phi = 0$)

In cylindrical coords,



$$\Phi(\vec{x}) = \sum_{l=0}^{\infty}\left[A_l r^l + B_l r^{-(l+1)}\right] \underbrace{P_l(\cos\theta)}_{\text{Legendre Polynomials}} \tag{116}$$

The solution must be regular everywhere. Inside, $\Phi(\vec{x})$ must be finite there. Since $r^{-(l+1)}$ blows up at the origin, B_l must vanish ($B_l = 0$ for all l).
$$\Phi_{IN}(\vec{x}) = \sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta) \tag{117}$$

Outside,
$$\Phi_{OUT}(\vec{x}) = \sum_{l=0}^{\infty}\left[C_l r^l + D_l r^{-(l+1)}\right] P_l(\cos\theta) \tag{118}$$

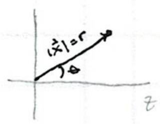
The coefficients are obtained by imposing boundary conditions:

① b.c. on the surface of the sphere

② b.c. at infinity

Far from the sphere, there is a uniform electric field $\vec{E}_0$. Thus, at this region, $\Phi_{out} \to -E_0 z$.
(Note: $\vec{E} = -\vec{\nabla}\Phi = -\vec{\nabla}(-E_0 z) = E_0 \hat{z}$). Then,

$$\Phi_{out} \to -E_0 z = -E_0 r\cos\theta \qquad (119)$$



Comparing this with Eq. (118), we see $B_\ell = 0$ if $\ell \neq 1$ and $B_1 = -E_0$.
Note that $P_1(x) = x \to P_1(\cos\theta) = \cos\theta$. Thus, $\Phi_{out} \to B_1 r P_1(\cos\theta)$

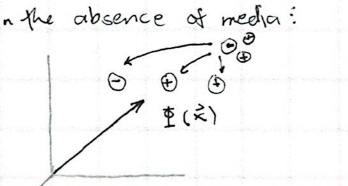
Electrostatic Energy in Dielectric Media (4.7)

The electrostatic energy of a system of charges in free space has been shown to be given by

$$W = \frac{1}{2}\int \rho(\vec{x})\,\Phi(\vec{x})\,d^3x \qquad (120)$$

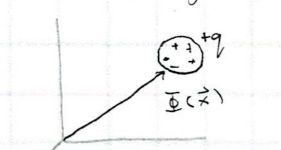
where $\rho(\vec{x})$ has been assembling bit by bit in free space in the presence of the pre-existing $\Phi(\vec{x})$
However, the above expression does not hold for dielectric media. The reason is that work is done not only to bring real (macroscopic) charge into position, but also to produce a certain state of polarization in the medium

- In the absence of media:



elementary charges or bits of charges are brought in from infinity

- When media is brought from infinity



A chunk of medium is brought from infinity. Work is done against the charge and induced dipole

- Let there be a pre-existing charge density $\rho(\vec{x})$ in all space together with a pre-existing potential $\Phi(\vec{x})$ due to $\rho(\vec{x})$. Now, we change the density $\rho(\vec{x})$ by some small amount $\delta\rho$. Then, there will be a work done to accomplish this. It is given by

$$\delta W = \int \delta\rho(\vec{x})\,\Phi(\vec{x})\,d^3x \qquad (121)$$

where $\Phi(\vec{x})$ is the potential due to the charge density $\rho(\vec{x})$ already present.
Since $\vec{\nabla}\cdot\vec{D} = \rho$, we can calculate the change $\delta\rho$ to a change in the displacement of $\delta\vec{D}$

$$\delta\rho = \vec{\nabla}\cdot\delta\vec{D} \qquad (122)$$

With $\vec{E} = -\vec{\nabla}\Phi$;
and IBP:
$\vec{\nabla}\cdot[(\delta\vec{D})\Phi(\vec{x})]$
$= (\vec{\nabla}\cdot\delta\vec{D})\Phi(\vec{x}) + \delta\vec{D}\cdot\nabla\Phi(\vec{x})$

$$\begin{aligned}
\delta W &= \int (\vec{\nabla}\cdot\delta\vec{D})\,\Phi(\vec{x}) \\
&= \int \left\{\nabla\cdot[\delta\vec{D}\,\Phi(\vec{x})] - (\delta\vec{D})\cdot\nabla\Phi(\vec{x})\right\} d^3x \\
&= \int \vec{\nabla}\cdot[\delta\vec{D}\,\Phi(\vec{x})]\,d^3x + \int \delta\vec{D}\cdot(-\nabla\Phi(\vec{x}))\,d^3x \\
&= \int_S \overbrace{[(\delta\vec{D})\Phi(\vec{x})]\cdot\hat{n}\,dS}^{\to 0 \text{ for localized } \rho(\vec{x})} + \int(\delta\vec{D})\cdot\vec{E}\,d^3x
\end{aligned}$$

$$\delta W = \int (\delta\vec{D})\cdot\vec{E}\,d^3x \qquad (123)$$

Formally, the total electrostatic energy is obtained by allowing $\vec{D}$ to be brought from an initial value $\vec{D} = 0$ to its final value $\vec{D}$

$$W = \int d^3x \int_0^{\vec{D}} \vec{E}\cdot\delta\vec{D} \qquad (124)$$

If the medium is linear, so that $\vec{D} = \epsilon\vec{E}$,

$$\delta(\vec{E}\cdot\vec{D}) = (\delta\vec{E})\cdot\vec{D} + \vec{E}\cdot\delta\vec{D} = 2\vec{E}\cdot\delta\vec{D} \quad\longrightarrow\quad \vec{E}\cdot\delta\vec{D} = \frac{1}{2}\delta(\vec{E}\cdot\vec{D}) \qquad (125)$$

Then,

$$W = \int d^3x \int_0^{\vec{D}} \vec{E}\cdot\delta\vec{D} = \int d^3x \int_0^{\vec{D}} \frac{1}{2}\delta(\vec{E}\cdot\vec{D}) = \int \frac{1}{2}\vec{E}\cdot\vec{D}\,d^3x \qquad (126)$$

With $\vec{E} = -\nabla\Phi$ and $\vec{\nabla}\cdot\vec{D} = \rho$, we have

$$W = -\frac{1}{2}\int (\nabla\Phi)\cdot\vec{D}\, d^3x$$

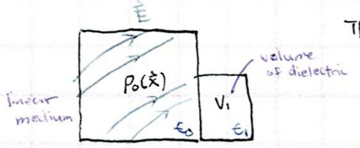
$$= -\frac{1}{2}\int \left[\overset{0}{\vec{\nabla}\cdot(\Phi\vec{D})} - (\vec{\nabla}\cdot\vec{D})\Phi \right] d^3x$$

$$= -\frac{1}{2}\int (\vec{\nabla}\cdot\vec{D})\,\Phi(\vec{x})\, d^3x$$

$$W = -\frac{1}{2}\int \rho(\vec{x})\,\Phi(\vec{x})\, d^3x \qquad (127)$$

We have reproduced the result in free space. Thus, the original expression is valid macroscopically only if the behavior of the medium is linear. Otherwise, the energy of a final configuration must be calculated from Eq. (124) and it generally depends on the past history of the system (hysteresis effects).

- Suppose that initially the electric field $\vec{E}_0$ due to a certain distribution of charges $\rho_0(\vec{x})$ exists in a medium of electric susceptibility ϵ_0, which may be a function of position.



The initial electrostatic energy is

$$W_0 = \frac{1}{2}\int \vec{E}_0\cdot\underbrace{\vec{D}_0}_{\epsilon_0\vec{E}_0}\, d^3x \qquad (128)$$

With the sources fixed in space, a medium of electric permitivity ϵ is introduced. This will change the electric field from $\vec{E}_0$ to $\vec{E}$. The energy has now the value

$$W_1 = \frac{1}{2}\int \vec{E}\cdot\vec{D}\, d^3x \qquad (129)$$

where $\vec{D} = \epsilon_1\vec{E}$. The diff. in energy is

$$W = \frac{1}{2}\int(\vec{E}\cdot\vec{D} - \vec{E}_0\cdot\vec{D}_0)\,d^3x = \frac{1}{2}\int(\vec{E}\cdot\vec{D}_0 - \vec{D}\cdot\vec{E}_0)\,d^3x + \overbrace{\frac{1}{2}\int(\vec{E}+\vec{E}_0)\cdot(\vec{D}+\vec{D}_0)\,d^3x}^{I_2} \qquad (130)$$

The second term can be shown to vanish as follows:

Since $\vec{\nabla}\times(\vec{E}+\vec{E}_0) = 0$, we can write $\vec{E}+\vec{E}_0 = -\vec{\nabla}\Phi$. Then, the second integral becomes

$$I_2 = -\frac{1}{2}\int \vec{\nabla}\Phi\cdot(\vec{D}-\vec{D}_0)\,d^3x \qquad (131)$$

Using IBP $\left[\vec{\nabla}\cdot(\Phi(\vec{D}-\vec{D}_0)) = \vec{\nabla}\Phi\cdot\vec{D}-\vec{D}_0 + (\vec{\nabla}\cdot(\vec{D}-\vec{D}_0))\Phi\right]$,

$$I_2 = -\frac{1}{2}\int\left[-(\vec{\nabla}\cdot(\vec{D}-\vec{D}_0))\Phi + \overset{0}{\vec{\nabla}\cdot(\Phi(\vec{D}-\vec{D}_0))}\right]d^3x \qquad (132)$$

Now, $\vec{\nabla}\cdot(\vec{D}-\vec{D}_0) = \vec{\nabla}\cdot\vec{D} - \vec{\nabla}\cdot\vec{D}_0 = \rho - \rho = 0$ because ρ is assumed unaltered by the insertion of the dielectric. Hence, $I_2 = 0$

Thus, the energy change is:

$$W = \frac{1}{2}\int(\vec{E}\cdot\vec{D}_0 - \vec{D}-\vec{E}_0)\,d^3x \qquad (133)$$

Using $\vec{D} = \epsilon_1\vec{E}$ & $\vec{D}_0 = \epsilon\vec{E}_0$,

$$W = \frac{1}{2}\int_{V_1}(\vec{E}\cdot\epsilon_0\vec{E}_0 - \epsilon_1\vec{E}\cdot\vec{E}_0)\,d^3x = -\frac{1}{2}\int_{V_1}(\epsilon_1-\epsilon_0)\,\vec{E}\cdot\vec{E}_0\,d^3x \qquad (134)$$

Given that the polarization is $\vec{P} = (\epsilon_1-\epsilon_0)\vec{E}$, we have

$$W = -\frac{1}{2}\int_{V_1}(\vec{P}\cdot\vec{E}_0)\,d^3x \qquad (135)$$

Then, we have the energy density of a dielectric placed in a field $\vec{E}_0$ whose sources are fixed:

$$w = \frac{1}{2}\vec{P}\cdot\vec{E}_0 \qquad (136)$$

This is the energy density of a polarizable dielectric in an external electric field, rather than a permanent dipole