Angeles, Janna May Z.
2015-09948

Physics 232 - PS2

Problem 1 (5.4 in Jackson)

A magnetic induction $\vec{B}$ in a current-free region in a uniform medium is cylidrically symmetric with components $B_z(\rho,z)$ and $B_\rho(\rho,z)$ and with a known $B_z(0,z)$ on the axis of symmetry. The magnitude of the axial field varies slowly in z.

a) Show that near the axis, the axial and radial components of magnetic induction are approximately:

$$B_z(\rho,z) \approx B_z(0,z) - \left(\frac{\rho^2}{4}\right)\left[\frac{\partial^2 B_z(0,z)}{\partial z^2}\right]$$

$$B_\rho(\rho,z) \approx -\left(\frac{\rho}{2}\right)\left[\frac{\partial B_z(0,z)}{\partial z}\right] + \left(\frac{\rho^3}{16}\right)\left[\frac{\partial^3 B_z(0,z)}{\partial z^3}\right]$$

Solution:

Since the region is free of current, $\vec{B}$ can be written in terms of a magnetic scalar potential Φ_M. Thus,

$$\vec{\nabla}\cdot\vec{B} = 0 \longrightarrow \vec{\nabla}\cdot(-\vec{\nabla}\Phi_M) = 0 \longrightarrow \vec{\nabla}^2\Phi_M = 0 \qquad (1)$$

In cylindrical coordinates,

$$\nabla^2\Phi_M = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\Phi_M}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2\Phi_M}{\partial\phi^2} + \frac{\partial^2\Phi_M}{\partial z^2} = 0 \qquad (2)$$

Solving this Laplace equation gives

$$\Phi_M(\rho,\phi,z) = \sum_{m=0}^{\infty}\int dk\, e^{-kz} J_m(k\rho)\, A_m(k)\, e^{im\phi} \qquad (3)$$

assuming that the potential is finite in the current-free region and that it vanishes as $z\to\infty$. Since the system is symmetric along ϕ, terms other than the $m=0$ term do not contribute. Thus, we have

$$\Phi_M(\rho,z) = \int dk\, A_0(k)\, J_0(k\rho)\, e^{-kz} \qquad (4)$$

Note that, near the axis ($\rho\approx0$), we can taylor expand the Bessel function of the first kind of order zero as

$$J_0(x) = \sum_{s=0}^{\infty}\frac{(-1)^s}{2^{2s}(s!)^2}x^{2s} \qquad (5)$$

Then, Eq. (4) becomes

$$\Phi_M(\rho,z) = \int dk\, A_0(k)\, e^{-kz}\sum_{s=0}^{\infty}\frac{(-1)^s}{2^{2s}(s!)^2}(k\rho)^{2s} = \int dk\, A_0(k)\, e^{-kz}\sum_{s=0}^{\infty}\frac{(-1)^s}{4^s(s!)^2}k^{2s}\rho^{2s} \qquad (6)$$

We can infer that the powers of k are equivalent to the derivatives of the exponential term $\left[\partial_z e^{-kz} = -k e^{-kz}\right]$.
Thus, we have

$$\Phi_M(\rho,z) = \sum_{s=0}\frac{(-1)^s}{4^s(s!)^2}\rho^{2s}\int dk\, A_0(k)\, \cancelto{0}{(-1)^{2s}}\frac{\partial^{2s}}{\partial z^{2s}}e^{-kz}$$

$$= \sum_{s=0}^{\infty}\frac{(-1)^s}{4^s(s!)^2}\rho^{2s}\frac{\partial^{2s}}{\partial z^{2s}}\int dk\, A_0(k)\, e^{-kz} \qquad (7)$$

$$\Phi_M(\rho,z) = \int dk\, A_0(k)\, e^{-kz} + \sum_{s=1}^{\infty}\frac{(-1)^s}{4^s(s!)^2}\rho^{2s}\frac{\partial^{2s}}{\partial z^{2s}}\int dk\, A_0(k)\, e^{-kz}$$

Then, we can see that

$$\Phi_M(0, z) = \int dk\, A_0(k)\, e^{-kz} \tag{8}$$

Thus, Eq. (7) becomes

$$\Phi_M(\rho, z) = \sum_{s=0}^{\infty} \frac{(-1)^s}{4^s (s!)^2} \rho^{2s} \frac{\partial^{2s}}{\partial z^{2s}} \Phi_M(0, z) \tag{9}$$

Using Eq. (1),

$$B(\rho, z) = -\nabla \Phi_M(\rho, z) = -\left[\frac{\partial \Phi_M}{\partial \rho} \hat{\rho} + \frac{\partial \Phi_M}{\partial z} \hat{z} \right] \tag{10}$$

Thus, $B_z(\rho, z) = -\partial_z \Phi_M$ and $B_\rho(\rho, z) = -\partial_\rho \Phi_M$. Note that

$$B_z(0, z) = -\frac{\partial}{\partial z} \Phi_M(0, z) = -\frac{\partial}{\partial z} \int dk\, A_0(k)\, e^{-kz} = \int dk\, A_0(k)\, k\, e^{-kz} \tag{11}$$

Then, we have

$$B_z(\rho, z) = -\frac{\partial}{\partial z} \sum_{s=0}^{\infty} \frac{(-1)^s}{4^s (s!)^2} \rho^{2s} \frac{\partial^{2s}}{\partial z^{2s}} \Phi_M(0, z) = \sum_{s=0}^{\infty} \frac{(-1)^s}{4^s (s!)^2} \rho^{2s} \frac{\partial^{2s}}{\partial z^{2s}} \left[-\frac{\partial}{\partial z} \Phi_M(0, z) \right] = \sum_{s=0}^{\infty} \frac{(-1)^s}{4^s (s!)^2} \rho^{2s} \frac{\partial^{2s}}{\partial z^{2s}} B_z(0, z) \tag{12}$$

and

$$B_\rho(\rho, z) = -\frac{\partial}{\partial \rho} \sum_{s=0}^{\infty} \frac{(-1)^s}{4^s (s!)^2} \rho^{2s} \frac{\partial^{2s}}{\partial z^{2s}} \Phi_M(0, z) = -\sum_{s=0}^{\infty} \frac{(-1)^s}{4^s (s!)^2} \frac{\partial (\rho^{2s})}{\partial \rho} \frac{\partial^{2s}}{\partial z^{2s}} \left[-\int B_z(0, z)\, dz \right] = \sum_{s=0}^{\infty} \frac{(-1)^s}{4^s (s!)^2} 2s \rho^{2s-1} \frac{\partial^{2s-1}}{\partial z^{2s-1}} B_z(0, z) \tag{13}$$

by using Eq. (7) and arranging it in terms of an integral. Shifting the index by $s \to s+1$, $B_\rho(\rho, z)$ becomes

$$B_\rho(\rho, z) = \sum_{s+1=0}^{\infty} \frac{(-1)^{s+1}}{4^{s+1} (s+1)!} 2(s+1) \rho^{2(s+1)-1} \frac{\partial^{2(s+1)-1}}{\partial z^{2(s+1)-1}} B_z(0, z)$$

$$= \sum_{s=-1}^{\infty} \frac{(-1)^{s+1}}{4^s \cdot \cancel{4}^{2}} \frac{\cancel{2(s+1)}}{(s+1)^2 (s!)^2} \rho^{2s+1} \frac{\partial^{2s+1}}{\partial z^{2s+1}} B_z(0, z) \tag{14}$$

$$B_\rho(\rho, z) = \sum_{s=0}^{\infty} \frac{(-1)^{s+1}}{4^s (s!)^2} \frac{\rho^{2s+1}}{2(s+1)} \frac{\partial^{2s+1}}{\partial z^{2s+1}} B_z(0, z)$$

since the $s=-1$ term of the summation is undefined. Writing out Eqs. (12) and (14),

$$B_z(\rho, z) = B_z(0, z) + \frac{-1}{4(1!)} \rho^2 \frac{\partial}{\partial z^2} B_z(0, z) + \ldots$$

$$B_\rho(\rho, z) = \frac{-1}{4(1!)^2} 2\rho \frac{\partial}{\partial z} B_z(0, z) + \frac{(-1)^2}{4^2 (2!)^2} 4\rho^3 \frac{\partial^3}{\partial z^3} B_z(0, z) + \ldots \tag{15}$$

which are just the given.

b) What are the magnitudes of the neglected terms, or equivalently, what is the criterion defining "near the axis"?

<u>Solution:</u>

From Eqs. (12) and (14), we can infer that the nth term of $\vec{B}(\rho, z)$ in its series expansion is

$$n^{th} \text{ term} = \frac{(-1)^{\frac{n}{2}}}{2^n} \frac{\rho^n}{(\frac{n}{2}!)^2} \frac{\partial^n}{\partial z^n} B_z(0, z) \tag{16}$$

Note that only the series expansion of $B_z(\rho, z)$ contributes to this term. If we ignore the constants, the ratio of this term with the next one is

$$\frac{(n+2)^{th} \text{ term}}{n^{th} \text{ term}} \approx \frac{\rho^{n+2}}{\rho^n} \frac{\partial_z^{n+2} B_z(0, z)}{\partial_z^n B_z(0, z)} \tag{17}$$

The next term is the $(n+2)^{th}$ term since this is the term that is included in the series expansion of B_z. The $(n+1)^{th}$ term, on the other hand, is included in the series expansion of B_ρ. We want only the terms included in B_z since we want to know when does the series converge. To answer this, note that the ratio between adjacent terms must be much less than 1 for the series to converge. Thus, we have

$$\rho^2 \frac{\partial_z^{n+2} B_z(0,z)}{\partial_z^n B_z(0,z)} \ll 1 \qquad (18)$$

$$\rightarrow \rho \ll \sqrt{\frac{\partial_z^n B_z(0,z)}{\partial_z^{n+2} B_z(0,z)}} \qquad (19)$$

as the criterion for being near the axis.

Angeles, Janna May Z
2015-09948

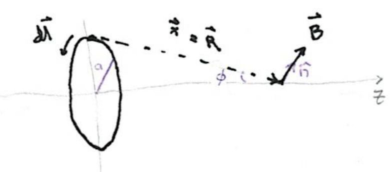
Problem 2 (5.7 in Jackson)

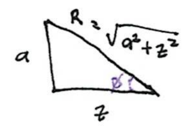
A compact circular coil of radius a, carrying a current I (perhaps N turns, each with current I/N), lies in the xy plane with its center at the origin

a) By elementary means [Eq. (5.4)], find the magnetic induction at any point on the z-axis.

Solution:

We consider a single loop in the coil. The forces related to this loop can be sketched as follows





With the symmetry along the loop, only the z component contributes to $\vec{B}$. Then, applying the Biot-Savart law, we have

$$B_z = kI\int \frac{(d\vec{l}\times\vec{x})_z}{|\vec{x}|^3} = \frac{\mu_0 I}{4\pi}\int\frac{(d\vec{l}\times\vec{R})_z}{R^3} = \frac{\mu_0 I}{4\pi}\int\frac{|d\vec{l}|\,R\sin\phi}{R^3} = \frac{\mu_0 I}{4\pi}\frac{\sin\phi}{R^2}\int dl = \frac{\mu_0 I}{4\pi}\frac{\sin\phi}{R^2}(2\pi a) \quad (1)$$

By geometry, we obtain

$$B_z = \frac{\mu_0 I}{4\pi R^2}\frac{a}{R}(2\pi a) = \frac{\mu_0 I a^2}{2(a^2+R^2)^{3/2}} \quad (2)$$

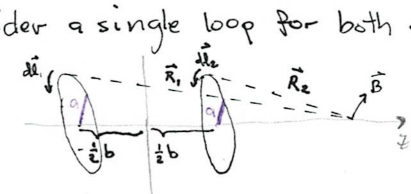
b) An identical coil with the same magnitude and sense of the current is located on the same axis, parallel to, and a distance b above the first coil. With the coordinate origin relocated at the point midway bet. the centers of the two coils, determine the magnetic induction on the axis near the origin as an expansion in powers of z, up to z^4 inclusive:

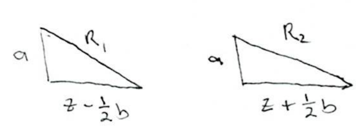
$$B_z = \left(\frac{\mu_0 I a^2}{d^3}\right)\left[1 + \frac{3(b^2-a^2)z^2}{2d^4} + \frac{15(b^4-6b^2a^2+2a^4)z^4}{16d^8} + \dots\right]$$

where $d^2 = a^2 + \frac{b^2}{4}$.

Solution:

We again consider a single loop for both coils. Then, the diagram of the problem can be sketched as follows:





Then, from part a), we can infer that

$$B_z = \frac{\mu_0 I}{4\pi}\left(\int_{\text{loop 1}}\frac{(d\vec{l}_1\times\vec{R}_1)_z}{R_1^3} + \int_{\text{loop 2}}\frac{(d\vec{l}_2\times\vec{R}_2)_z}{R_2^3}\right) = \frac{\mu_0 I}{4\pi}\left(\frac{2\pi a^2}{R_1^3} + \frac{2\pi a^2}{R_2^3}\right) \quad (3)$$

By geometry,

$$B_z = \frac{\mu_0 I a^2}{2}\left[\frac{1}{\left(a^2+\left(z-\frac{b}{2}\right)^2\right)^{3/2}} + \frac{1}{\left(a^2+\left(z+\frac{b}{2}\right)^2\right)^{3/2}}\right] \qquad (4)$$

Note that $a^2 + \left(z^2 \pm \frac{b}{2}\right)^2 = a^2 + z^2 \pm bz + \frac{b^2}{4} = d^2 + z^2 \pm bz$ where $d^2 = a^2 + \frac{b^2}{4}$. Then, Eq. (4) becomes

$$B_z = \frac{\mu_0 I a^2}{2}\left\{\left[a^2 + \left(z-\frac{b}{2}\right)^2\right]^{-3/2} + \left[a^2 + \left(z+\frac{b}{2}\right)^2\right]^{-3/2}\right\}$$

$$= \frac{\mu_0 I a^2}{2}\left\{\left(d^2+z^2-bz\right)^{-3/2} + \left(d^2+z^2+bz\right)^{-3/2}\right\} \qquad (5)$$

$$B_z = \frac{\mu_0 I a}{2d^3}\left\{\left(1+\left(\frac{z}{d}\right)^2 - b\frac{z}{d^2}\right)^{-\frac{3}{2}} + \left(1+\left(\frac{z}{d}\right)^2 + b\frac{z}{d^2}\right)^{-3/2}\right\}$$

as the magnetic induction on the axis near the origin. Let $\gamma = \frac{z}{d^2}$. Then, we have

$$B_z = \frac{\mu_0 I a}{2d^3}\left\{\left[1-b\gamma+\left(a^2+\frac{b^2}{4}\right)\gamma^2\right]^{-3/2} + \left[1+b\gamma+\left(a^2+\frac{b^2}{4}\right)\gamma^2\right]^{-3/2}\right\} \qquad (6)$$

Using Mathematica, the Taylor series expansion for both terms are

$$\left[1-b\gamma+\left(a^2+\frac{b^2}{4}\right)\gamma^2\right]^{-3/2} = 1+\frac{3}{2}b\gamma - \frac{3}{2}(a^2-b^2)\gamma^2 - \left(\frac{15a^2b}{4}-\frac{5b^3}{4}\right)\gamma^3 + \frac{15}{16}\left(2a^4-6a^2b^2+b^4\right)\gamma^4 + \dots \qquad (7)$$

$$\left[1+b\gamma+\left(a^2+\frac{b^2}{4}\right)\gamma^2\right]^{-3/2} = 1-\frac{3}{2}b\gamma - \frac{3}{2}(a^2-b^2)\gamma^2 + \left(\frac{15a^2b}{4}-\frac{5b^3}{4}\right)\gamma^3 + \frac{15}{16}\left(2a^4-6a^2b^2+b^4\right)\gamma^4 + \dots \qquad (8)$$

If we plug these into Eq. (6), we see that odd terms of γ cancel each other. Thus, we have

$$B_z = \frac{\mu_0 I a^2}{2d^3}\left[2+2\frac{3}{2}(b^2-a^2)\gamma^2 + 2\frac{15}{16}\left(b^4-6b^2a^2+2a^4\right)\gamma^4 + \dots\right]$$

$$= \frac{\mu_0 I a^2}{d^3}\left[1+\frac{3}{2}(b^2-a^2)\gamma^2 + \frac{15}{16}\left(b^4-6b^2a^2+2a^4\right)\gamma^4 + \dots\right] \qquad (9)$$

$$B_z = \frac{\mu_0 I a^2}{d^3}\left[1+\frac{3}{2}(b^2-a^2)\frac{z^2}{d^4} + \frac{15}{16}\left(b^4-6b^2a^2+2a^4\right)\frac{z^4}{d^8} + \dots\right]$$

which is just the given.

c) Show that, off-axis near the origin, the axial and radial components, correct to second order in the coordinates, take the form

$$B_z(0,z) = \sigma_0 + \sigma_2\left(z^2 - \frac{\rho^2}{2}\right); \qquad B_\rho = -\sigma_2 z\rho$$

Solution:

From Eq. (9), we can infer that

$$\sigma_0 = \frac{\mu_0 I a^2}{d^3}, \qquad \sigma_2 = \frac{\mu_0 I a^2}{d^3}\frac{3}{2}(b^2-a^2)\frac{1}{d^4} = \sigma_0\frac{3}{2d^4}(b^2-a^2) \qquad (10)$$

Using $B_z(\rho,z)$ and $B_\rho(\rho,z)$ of problem 1 [5.4 of Jackson], we obtain

$$B_z(\rho,z) = B_z(0,z) - \frac{\rho^2}{4}\left[\frac{\partial^2 B_z(0,z)}{\partial z^2}\right] + \dots$$

$$= \sigma_0 + \sigma_2 z^2 + \dots - \frac{\rho^2}{4}\left[\frac{\partial^2}{\partial z^2}(\sigma_0 + \sigma_2 z^2)\right] + \dots \qquad (11)$$

$$= \sigma_0 + \sigma_2 z^2 + \dots - \frac{\rho^2}{4}\left[2\sigma_2 + \dots\right] + \dots$$

$$B_z(\rho,z) = \sigma_0 + \sigma_2\left(z^2 - \frac{\rho^2}{2}\right) + \dots$$

and

$$B_\rho(\rho,z) = -\frac{\rho}{2}\left[\frac{\partial B_z(0,z)}{\partial z}\right] + \frac{\rho^3}{16}\left[\frac{\partial^3 B_z(0,z)}{\partial z^3}\right] + \dots$$

$$= -\frac{\rho}{2}\left[\frac{\partial}{\partial z}(\sigma_0 + \sigma_2 z^2 + \dots)\right] + \dots \qquad (12)$$

$$= -\frac{\rho}{2}\cdot 2\sigma_2 z + \dots$$

$$B_\rho(\rho,z) = -\sigma_2 z\rho + \dots$$

which is just the given when we consider only up to the second order of the coordinates.

d) For the two coils in part b), show that the magnetic induction on the z-axis for large $|z|$ is given by the expansion in inverse odd powers of $|z|$ obtained from the small z expansion of part b) by the formal substitution, $d \to |z|$.

Solution:

From Eq. (4), the magnetic induction on the z-axis for large $|z|$ is given by

$$B_z = \frac{\mu_0 I a}{2}\left\{\left[a^2 + \left(z - \frac{b}{2}\right)^2\right]^{-3/2} + \left[a^2 + \left(z + \frac{b}{2}\right)^2\right]^{-3/2}\right\}$$

$$= \frac{\mu_0 I a}{2}\left\{\left[a^2 + z^2 - bz + \frac{b^2}{4}\right]^{-3/2} + \left[a^2 + z^2 + bz + \frac{b^2}{4}\right]^{-3/2}\right\} \qquad (13)$$

$$= \frac{\mu_0 I a}{2z^3}\left\{\left[1 - b\frac{1}{z} + \left(a^2 + \frac{b^2}{4}\right)\frac{1}{z^2}\right]^{-3/2} + \left[1 + b\frac{1}{z} + \left(a^2 + \frac{b^2}{4}\right)\frac{1}{z^2}\right]^{-3/2}\right\}$$

$$B_z = \frac{\mu_0 I a}{2z^3}\left\{\left[1 - bz^{-1} + \left(a^2 + \frac{b^2}{4}\right)z^{-2}\right]^{-3/2} + \left[1 + bz^{-1} + \left(a^2 + \frac{b^2}{4}\right)z^{-2}\right]^{-3/2}\right\}$$

Now, substituting $d \to |z|$, we find that the small z expansion of the magnetic expansion in Eq. (5) leads to

$$B_z = \frac{\mu_0 I a^2}{2|z|^3}\left\{\left[1 - b\frac{z}{|z|^2} + \left(a^2 + \frac{b^2}{4}\right)\frac{z^2}{|z|^4}\right]^{-3/2} + \left[1 + b\frac{z}{|z|^2} + \left(a^2 + \frac{b^2}{4}\right)\frac{z^2}{|z|^4}\right]^{-3/2}\right\}$$

$$= \frac{\mu_0 I a^2}{2z^3}\left\{\left[1 - bz^{-1} + \left(a^2 + \frac{b^2}{4}\right)z^{-2}\right]^{-3/2} + \left[1 + bz^{-1} + \left(a^2 + \frac{b^2}{4}\right)z^{-2}\right]^{-3/2}\right\} \qquad (14)$$

Notice that Eqs. (13) and (14) are just the same. Thus, we have shown B_z for large z is given by its small z

expansion by applying the substitution $d \to |z|$.

e) If $b = a$, the two coils are known as a pair of Helmholtz coils. For this choice of geometry, the second terms in the expansions of parts b) and d) are absent ($\sigma_2 = 0$ in part c)). The field near the origin is then very uniform, what is the maximum permitted value of $\frac{|z|}{a}$ if the axial field is to be uniform to one part in 10^4, one part in 10^2?

Solution:

With $b = a$, we have

$$d^2 = a^2 + \frac{b^2}{4} = a^2 + \frac{a^2}{4} = \frac{5}{4}a^2 \to d = \frac{a}{2}\sqrt{5} \qquad (15)$$

We will use the small z expansion of B_z in part b) [Eq. (9)]. Thus, we obtain

$$B_z = \frac{\mu_0 I a^2}{\left(\frac{a}{2}\sqrt{5}\right)^3}\left[1 + \frac{3}{2}(a^2 - a^2)\frac{z^2}{\left(\frac{a}{2}\sqrt{5}\right)^4} + \frac{15}{16}\underbrace{(a^4 - 6a^4 + 2a^4)}_{-3a^4}\frac{z^4}{\left(\frac{a}{2}\sqrt{5}\right)^8} + \dots\right]$$

$$= \frac{8\sqrt{5}}{25a^3}\mu_0 I a^2\left[1 - \frac{144}{125}\frac{z^4}{a^4} + \dots\right] \qquad (16)$$

Considering the $(|z|/a)^4$ term to be the small correction to the field and neglecting higher order terms, the non-uniformity of the axial field is given to be

$$\frac{\delta B_z}{B_z} \approx \frac{144}{125}\left(\frac{z}{a}\right)^4 \qquad (17)$$

If the field is to be uniform to one part in 10^4, then $|z|/a$ must be

$$\frac{144}{125}\left(\frac{z}{a}\right)^4 < 10^{-4} \longrightarrow \frac{z}{a} < \sqrt[4]{\frac{125}{144}10^{-4}} = 0.097 \qquad (18)$$

On the other hand, if the field is to be uniform to one part in 10^2, then $|z|/a$ must be

$$\frac{144}{125}\left(\frac{z}{a}\right)^4 < 10^{-2} \longrightarrow \frac{z}{a} < \sqrt[4]{\frac{125}{144}10^{-2}} = 0.305 \qquad (19)$$

Problem 3 (5.8 in Jackson)

A localized cylindrically symmetric current distribution is such that the current flows only in the azimuthal direction; the current density is a function only of r and θ (or ρ and z): $\vec{J} = \hat{\phi}\, J(r,\theta)$. The distribution is "hollow" in the sense that there is a current-free region near the origin, as well as outside.

a) Show that the magnetic field can be derived from the azimuthal component of the vector potential, with a multipole expansion

$$A_\phi(r,\theta) = -\frac{\mu_0}{4\pi}\sum_L m_L\, r^L P_L^1(\cos\theta)$$

in the interior and

$$A_\phi(r,\theta) = -\frac{\mu_0}{4\pi}\sum_L \mu_L\, r^{-L-1} P_L^1(\cos\theta)$$

outside the current distribution.

b) Show that the internal and external multipole moments are

$$m_L = -\frac{1}{L(L+1)}\int d^3x\, r^{-L-1} P_L^1(\cos\theta)\, J(r,\theta)$$

and

$$\mu_L = -\frac{1}{L(L+1)}\int d^3x\, r^{L} P_L^1(\cos\theta)\, J(r,\theta)$$

Solution:

In general, the vector potential has the form

$$\vec{A}(\vec{x}) = \frac{\mu_0}{4\pi}\int \frac{\vec{J}(\vec{x}')}{|\vec{x}-\vec{x}'|}\, d^3x' \tag{1}$$

With the addition theorem for spherical harmonics, we know that

$$\frac{1}{|\vec{x}-\vec{x}'|} = 4\pi\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{1}{2l+1}\frac{r_<^l}{r_>^{l+1}}\, Y_l^{m*}(\theta',\phi')\, Y_l^m(\theta,\phi) \tag{2}$$

where $r_{<[>]}$ is the smaller [larger] of r and r'. Thus, Eq. (1) becomes

$$\vec{A}(\vec{x}) = \frac{\mu_0}{4\pi}\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{4\pi}{2l+1}\int \vec{J}(r',\theta',\phi')\frac{r_<^l}{r_>^{l+1}}\, Y_l^{m*}(\theta',\phi')\, Y_l^m(\theta,\phi)\, r'^2\sin\theta'\, dr'\, d\phi'\, d\theta' \tag{3}$$

Substituting in the given current density, we have

$$\vec{A}(\vec{x}) = \frac{\mu_0}{4\pi}\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{4\pi}{2l+1}\int \hat{\phi}\, J(r',\theta')\frac{r_<^l}{r_>^{l+1}}\, Y_l^{m*}(\theta',0)\, Y_l^m(\theta,0)\, e^{im(\phi-\phi')}\, r'^2\, dr'\, d\phi'\, d(\cos\theta') \tag{4}$$

after also writting out explicitly the ϕ-dependence of the spherical harmonics. Note that spherical harmonics has the properties

$$Y_l^{m*}(\theta,\phi) = (-1)^m\, Y_l^{-m}(\theta,\phi), \qquad Y_l^m(\theta,0) = Y_l^{m*}(\theta,0) \tag{5}$$

Then, we obtain

$$\vec{A}(\vec{x}) = \frac{\mu_0}{4\pi}\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{4\pi}{2l+1}\int \hat{\phi}' J(r',\theta') \frac{r_<^l}{r_>^{l+1}} Y_l^m(\theta',0)\, Y_l^m(\theta,0) \cos\left[m(\phi-\phi')\right] r'^2 dr' d\phi' d(\cos\theta') \qquad (6)$$

$$= \frac{\mu_0}{4\pi}\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{4\pi}{2l+1}\left(\int_{r',\cos\theta'} J(r',\theta') \frac{r_<^l}{r_>^{l+1}} Y_l^m(\theta',0)\, Y_l^m(\theta,0)\, r'^2 dr' d(\cos\theta')\right)\left(\underbrace{\int_0^{2\pi}\hat{\phi}' \cos\left[m(\phi-\phi')\right] d\phi'}_{I_\phi}\right)$$

Let us consider the ϕ-integral I_ϕ. Note that the basis vectors in spherical coordinates are given by

$$\hat{r} = (\sin\theta\cos\phi, \sin\theta, \cos\theta) \qquad \hat{\theta} = (\cos\theta\cos\phi, \cos\theta\sin\phi, -\sin\theta) \qquad \hat{\phi} = (-\sin\phi, \cos\phi, 0) \qquad (7)$$

Also, note that an arbitrary vector $\vec{v}$ can be written out as

$$\vec{v} = (\hat{r}\cdot\vec{v})\,\hat{r} + (\hat{\theta}\cdot\vec{v})\,\hat{\theta} + (\hat{\phi}\cdot\vec{v})\,\hat{\phi} \qquad (8)$$

in terms of the spherical basis vectors. Then, we can calculate $\hat{\phi}'$ as

$$\hat{\phi}' = (\hat{r}\cdot\hat{\phi}')\,\hat{r} + (\hat{\theta}\cdot\hat{\phi}')\,\hat{\theta} + (\hat{\phi}\cdot\hat{\phi}')\,\hat{\phi}$$

$$= \left[-\sin\phi'\cos\phi\sin\theta + \cos\phi'\sin\phi\sin\theta\right]\hat{r} + \left[-\sin\phi'\cos\theta\cos\phi + \cos\theta\sin\phi\cos\phi'\right]\hat{\theta} + \left[\sin\phi\sin\phi' + \cos\phi\cos\phi'\right]\hat{\phi}$$

$$= \sin(\phi-\phi')\sin\theta\,\hat{r} + \sin(\phi-\phi')\cos\theta\,\hat{\theta} + \cos(\phi-\phi')\,\hat{\phi} \qquad (9)$$

$$\hat{\phi}' = \sin(\phi-\phi')\left[\sin\theta\,\hat{r} + \cos\theta\,\hat{\theta}\right] + \cos(\phi-\phi')\,\hat{\phi}$$

Thus, the integral I_ϕ becomes

$$I_\phi = \int_0^{2\pi}\left\{\sin(\phi-\phi')\left[\sin\theta\,\hat{r} + \cos\theta\,\hat{\theta}\right] + \cos(\phi-\phi')\,\hat{\phi}'\right\}\cos\left[m(\phi-\phi')\right] d\phi' \qquad (10)$$

$$= \left(\int_0^{2\pi}\sin(\phi-\phi')\cos\left[m(\phi-\phi')\right] d\phi\right)\sin\theta\,\hat{r} + \left(\int_0^{2\pi}\sin(\phi-\phi')\cos\left[m(\phi-\phi')\right] d\phi'\right)\cos\theta\,\hat{\theta} + \left(\int_0^{2\pi}\cos(\phi-\phi')\cos\left[m(\phi-\phi')\right] d\phi'\right)\hat{\phi}$$

Let $u = \phi-\phi'$ $[du = -d\phi']$. Then,

$$I_\phi = \left(-\int_\phi^{\phi-2\pi}\sin u\cos mu\, du\right)\sin\theta\,\hat{r} + \left(-\int_\phi^{\phi-2\pi}\sin u\cos mu\right)\cos\theta\,\hat{\theta} + \left(-\int_\phi^{\phi-2\pi}\cos u\cos mu\, du\right)\hat{\phi} \qquad (11)$$

Note that

$$\int_a^{a-2\pi}\sin(mx)\cos(nx)\, dx = 0$$

$$\int_a^{a-2\pi}\cos(mx)\cos(nx) = \begin{cases} 0 & m\neq n \\ -\pi & m=n \end{cases} \qquad (12)$$

which is valid for any interval as long as it is a full period. Applying this to Eq. (11), we have

$$I_\phi = 0\,\hat{r} + 0\,\hat{\theta} + \left[(\pi\,\delta_{m,1}) + (\pi\,\delta_{m,-1})\right]\hat{\phi} = \pi\left(\delta_{m,1}+\delta_{m,-1}\right)\hat{\phi} \qquad (13)$$

The $\hat{\phi}$ term is of that form since the cosine function is even $[\cos -x = \cos x]$ so there are two values of m in which the integral is non-zero. Substituting this back to Eq. (6), we have

$$\vec{A}(\vec{x}) = \frac{\mu_0}{4\pi}\sum_{l=0}^{\infty}\frac{4\pi}{2l+1}\hat{\phi}\int J(r',\phi') \frac{r_<^l}{r_>^{l+1}}\left[Y_l^1(\theta',0)\,Y_l^1(\theta',0) + Y_l^{-1}(\theta',0)\,Y_l^{-1}(\theta',0)\right]\pi\, r'^2 dr' d(\cos\theta') \qquad (14)$$

Applying Eq. (5), this becomes

$$\vec{A}(\vec{x}) = \frac{\mu_0}{4\pi} \sum_{l=0}^{\infty} \frac{4\pi}{2l+1} \hat{\phi} \int J(r',\theta') \frac{r_<^l}{r_>^{l+1}} \left[Y_l^1(\theta',0)\, Y_l^1(\theta,0) + (-1)^{-1} Y_l^1(\theta',0)\, (-1)^{-1} Y_l^1(\theta,0) \right] \pi r'^2\, dr'\, d(\cos\theta')$$

$$= \frac{\mu_0}{4\pi} \sum_{l=0}^{\infty} \frac{4\pi}{2l+1} \hat{\phi} \int J(r',\theta')\, Y_l^1(\theta',0)\, Y_l^1(\theta,0)\; 2\pi r'^2\, dr'\, d(\cos\theta') \qquad (15)$$

Note that the spherical harmonics in terms of Legendre polynomials are given by

$$Y_l^m(\theta,\phi) = \sqrt{\frac{2l+1}{4\pi} \frac{(l-m)!}{(l+m)!}}\; P_l^m(\cos\theta)\, \cos(m\phi) \qquad (16)$$

Then, Eq. (15) becomes

$$\vec{A}(\vec{x}) = \frac{\mu_0}{4\pi} \sum_{l=0}^{\infty} \frac{\cancel{4\pi}}{\cancel{2l+1}} \hat{\phi} \int J(r',\theta') \left(\frac{\cancel{2l+1}}{\cancel{4\pi}} \frac{1}{l(l+1)} \right) P_l^1(\cos\theta')\, P_l^1(\cos\theta)\, 2\pi r'^2\, dr'\, d(\cos\theta') \qquad (17)$$

We see here that only the $\hat{\phi}$ component of $\vec{A}(\vec{x})$ is non-zero so we focus on A_ϕ which is no longer dependent on ϕ. Thus, we have

$$A_\phi(r,\theta) = \frac{\mu_0}{4\pi} \sum_{l=0}^{\infty} \frac{P_l^1(\cos\theta)}{l(l+1)} \int J(r',\theta') \frac{r_<^l}{r_>^{l+1}} P_l^1(\cos\theta')\, \underbrace{2\pi r'^2\, dr'\, d(\cos\theta')}_{d^3x'} \qquad (18)$$

In the interior region where $r < r'$, this results to

$$A_\phi^{inner}(r,\theta) = \frac{\mu_0}{4\pi} \sum_{l=0}^{\infty} \frac{P_l^1(\cos\theta)}{l(l+1)}\, r^l \int J(r',\theta') \frac{P_l^1(\cos\theta')}{r'^{\,l+1}}\, d^3x' \qquad (19)$$

while outside the distribution where $r > r'$, Eq. (18) results to

$$A_\phi^{outer}(r,\theta) = \frac{\mu_0}{4\pi} \sum_{l=0}^{\infty} \frac{P_l^1(\cos\theta)}{l(l+1)} \frac{1}{r^{l+1}} \int J(r',\theta')\, r'^l\, P_l^1(\cos\theta)\, d^3x' \qquad (20)$$

Let the internal multipole be

$$m_l = -\frac{1}{l(l+1)} \int J(r',\theta') \frac{P_l^1(\cos\theta')}{r'^{\,l+1}}\, d^3x' \qquad (21)$$

and the external multipole be

$$\mu_l = -\frac{1}{l(l+1)} \int J(r',\theta')\, r'^l\, P_l^1(\cos\theta')\, d^3x' \qquad (22)$$

Then, Eq. (19) becomes

$$A_\phi^{inner}(r,\theta) = -\frac{\mu_0}{4\pi} \sum_{l=0}^{\infty} m_l\, r^l\, P_l^1(\cos\theta) \qquad (23)$$

while Eq. (20) becomes

$$A_\phi^{outer}(r,\theta) = -\frac{\mu_0}{4\pi} \sum_{l=0}^{\infty} \mu_l \frac{P_l^1(\cos\theta)}{r^{l+1}} \qquad (24)$$

which are just the given with $l = L$.

Angeles, Janna May Z.
2015 - 09948

Problem 4) (5.18 in Jackson)

A circular loop of wire having a radius a and carrying a current I is located in vacuum with its center a distance d away from a semi-infinite slab of permeability μ. Find the force acting on the loop when

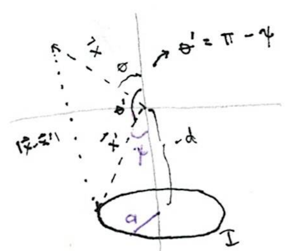
a) the plane of the loop is parallel to the face of the slab

Solution:

In general, the vector potential is of the form

$$\vec{A}(\vec{x}) = \frac{\mu_0}{4\pi}\int \frac{\vec{J}(\vec{x}')}{|\vec{x}-\vec{x}'|}\, d^3x' \tag{1}$$

Let us consider a circular loop of wire in the following orientation:



$$\rightarrow \cos\psi = \frac{-d}{\sqrt{d^2+a^2}}$$

$$\psi = \cos^{-1}\frac{-d}{\sqrt{d^2+a^2}}$$

Then,

$$|\vec{x}-\vec{x}'| = \sqrt{(x-x')^2+(y-y')^2+(z-z')^2} \tag{2}$$

$$= \sqrt{r^2+r'^2-2rr'\left[\cos\theta\cos\theta'+\sin\theta\sin\theta'\cos(\phi-\phi')\right]}$$

For simplicity, we let $\phi = 0$. Then, we have

$$|\vec{x}-\vec{x}'| = \sqrt{r^2+r'^2-2rr'\left[\cos\theta\cos\theta'+\sin\theta\sin\theta'\cos\phi'\right]} \tag{3}$$

The current density for this case is given by

$$J_\phi = I\,\delta\left(\theta'-(\pi-\psi)\right)\frac{\delta\left(r'-\sqrt{d^2+a^2}\right)}{\sqrt{d^2+a^2}} \tag{4}$$

which we can infer that only ϕ component of $\vec{A}(\vec{x})$ is non-zero. With the guide of Section 5.5 of Jackson, we know that

$$A_\phi(r,\theta) = \frac{\mu_0}{4\pi}\frac{1}{\sqrt{d^2+a^2}}\int \frac{\cos\phi'\,\delta(\theta'-(\pi-\psi))\,\delta(r'-\sqrt{d^2+a^2})}{\sqrt{r^2+r'^2-2rr'\left[\cos\theta\cos\theta'+\sin\theta\sin\theta'\cos\phi'\right]}}\, r'^2\sin\theta'\,dr'\,d\phi'\,d\theta' \tag{5}$$

Let us focus on the numerator in terms of θ

$$\int \delta(\theta'-(\pi-\psi))\sin\theta'\,d\theta = \sin(\pi-\psi) = \sin\overset{0}{\cancel{\pi}}\cos\psi - \overset{-1}{\cancel{\cos\pi}}\sin\psi = \sin\psi = \frac{a}{\sqrt{d^2+a^2}} \tag{6}$$

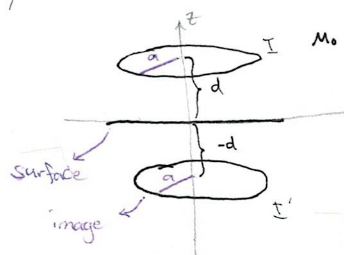
Applying this into A_ϕ, we have

$$A_\phi(r,\theta) = \frac{\mu_0}{4\pi}\,\frac{a}{d^2+a^2}\int_0^{2\pi}\frac{\cos\phi'\,(d^2+a^2)\,d\phi'}{\sqrt{r^2+d^2+a^2-2r\sqrt{d^2+a^2}\left[\frac{-d}{\sqrt{d^2+a^2}}\cos\theta+\sin\theta\,\frac{a}{\sqrt{d^2+a^2}}\cos\phi'\right]}} \quad (7)$$

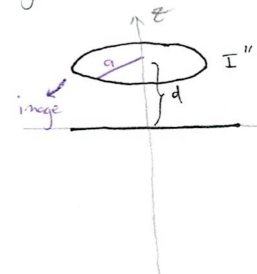
after integrating out the r-integral and θ-integral. Simplifying this, we get

$$A_\phi(r,\theta) = \frac{\mu_0}{4\pi}\,a\int_0^{2\pi}\frac{\cos\phi'\,d\phi'}{\sqrt{r^2+d^2+a^2+2rd\cos\theta-2ra\sin\theta\cos\phi'}} \quad (8)$$

Going back to the original problem, we will use the method of images so that we can take advantage of the boundary conditions. Let us consider the following configuration:



$z>0,\ \mu_0$



$z<0,\ \mu$

In the region of $z>0$, we can infer from Eq. (8) that

$$A_\phi^{upper}(r,\theta) = \frac{\mu_0 a}{4\pi}\int_0^{2\pi}\left[\frac{I}{\sqrt{r^2+d^2+a^2-2r(d\cos\theta+a\sin\theta\cos\phi')}} + \frac{I'}{\sqrt{r^2+d^2+a^2-2r(-d\cos\theta+a\sin\theta\cos\phi')}}\right]\cos\phi'\,d\phi' \quad (9)$$

In the region of $z<0$, there is no loop so there is no current. Thus, we only need an image loop with current I'' at $z=d$. Therefore, the vector potential on this region is (Note: this region is inside the slab)

$$A_\phi^{lower}(r,\theta) = \frac{\mu a}{4\pi}\int_0^{2\pi}\frac{I''}{\sqrt{r^2+d^2+a^2-2r(d\cos\theta+a\sin\theta\cos\phi')}}\cos\phi'\,d\phi' \quad (10)$$

We now apply boundary conditions to match the vector potential of both regions at the boundary (located at $\theta=\frac{\pi}{2}$).

$$\left[\vec{B}_{upper}\cdot\hat{n} = B_{lower}\cdot\hat{n}\right]_{\text{on surface}}$$

$$\left[(\vec{\nabla}\times\vec{A}_{upper})\cdot\hat{\theta}\right]_{\theta=\frac{\pi}{2}} = \left[\vec{\nabla}\times\vec{A}_{lower}\cdot\hat{\theta}\right]_{\theta=\frac{\pi}{2}} \quad (11)$$

Note that

$$\vec{\nabla}\times\vec{A} = \frac{1}{r\sin\theta}\left(\frac{\partial}{\partial\theta}(A_\phi\sin\theta)-\frac{\partial A_\theta}{\partial\phi}\right)\hat{r} + \frac{1}{r}\left(\frac{1}{\sin\theta}\frac{\partial A_r}{\partial\phi}-\frac{\partial}{\partial r}(rA_\phi)\right)\hat{\theta} + \frac{1}{r}\left(\frac{\partial}{\partial r}(rA_\theta)-\frac{\partial A_r}{\partial\theta}\right)\hat{\phi} \quad (12)$$

Only $\hat{\theta}$ component will survive and only A_ϕ is non-zero. Thus,

$$\left[\frac{\partial}{\partial r}\left(r A_\phi^{upper}\right)\right]_{\theta=\frac{\pi}{2}} = \left[\frac{\partial}{\partial r}\left(r A_\phi^{lower}\right)\right]_{\theta=\frac{\pi}{2}} \quad (13)$$

Interchanging the derivative and the integral, Eq. (13) after substituting in Eqs. (9) and (10) becomes

$$\frac{\mu_0 a}{4\pi}\int_0^{2\pi} d\phi' \cos\phi' \left\{ \frac{\partial}{\partial r}\left[\frac{r I}{\sqrt{r^2+d^2+a^2-2r(d\cos\theta + a\sin\theta\cos\phi')}} + \frac{r I'}{\sqrt{r^2+d^2+a^2-2r(-d\cos\theta+a\sin\theta\cos\phi')}}\right]\right\}_{\theta=\frac{\pi}{2}}$$

$$= \frac{\mu a}{4\pi}\int_0^{2\pi} d\phi' \cos\phi' \left\{ \frac{\partial}{\partial r}\left[\frac{r I''}{\sqrt{r^2+d^2+a^2-2r(d\cos\theta+a\sin\theta\cos\phi')}}\right]\right\}_{\theta=\frac{\pi}{2}} \quad (14)$$

With the use of Mathematica, this can be evaluated into

$$\frac{\mu_0 a}{4\pi}\int_0^{2\pi} d\phi' \cos\phi' \left[I \frac{a^2+d^2-dr\cos\theta - ar\cos\phi'\sin\theta}{(a^2+d^2+r^2-2dr\cos\theta-2ar\cos\phi'\sin\theta)^{3/2}} + I' \frac{a^2+d^2+dr\cos\theta - ar\cos\phi'\sin\theta}{(a^2+d^2+r^2+2dr\cos\theta-2ar\cos\phi'\sin\theta)^{3/2}}\right]_{\theta=\frac{\pi}{2}}$$

$$= \frac{\mu a}{4\pi}\int_0^{2\pi} d\phi' \cos\phi' \left[I'' \frac{a^2+d^2-dr\cos\theta-ar\cos\phi'\sin\theta}{(a^2+d^2+r^2-2dr\cos\theta-2ar\cos\phi'\sin\theta)^{3/2}}\right]_{\theta=\frac{\pi}{2}} \quad (15)$$

Evaluating it on the surface

$$\frac{\mu_0 a}{4\pi}\int_0^{2\pi} d\phi' \cos\phi' \left(I \frac{a^2+d^2-ar\cos\phi'}{(a^2+d^2+r^2-2ar\cos\phi')^{3/2}} + I' \frac{a^2+d^2-ar\cos\phi'}{(a^2+d^2+r^2-2ar\cos\phi')^{3/2}}\right)$$

$$= \frac{\mu a}{4\pi}\int_0^{2\pi} d\phi' \cos\phi' \, I'' \frac{a^2+d^2-ar\cos\phi}{a^2+d^2+r^2-2ar\cos\phi'} \quad (16)$$

Notice that we can cancel out the integrals on both sides. We are left with

$$\frac{\mu_0 \cancel{a}}{\cancel{4\pi}}(I+I') = \frac{\mu \cancel{a}}{\cancel{4\pi}} I'' \rightarrow \mu_0(I+I') = \mu I'' \quad (17)$$

As for the other boundary condition, we have (note that there is no free current on the surface)

$$\left[\hat{n}\times\vec{H}_{upper} = \hat{n}\times\vec{H}_{lower}\right]_{on\ surface}$$

$$\left[\frac{1}{\mu_0}\hat{\theta}\times(\vec{\nabla}\times\vec{A}_{upper})\right]_{\theta=\frac{\pi}{2}} = \left[\frac{1}{\mu}\hat{\theta}\times(\vec{\nabla}\times\vec{A}_{lower})\right]_{\theta=\frac{\pi}{2}} \quad (18)$$

Recall $\vec{\nabla}\times\vec{A}$ in Eq. (12). Then,

$$\hat{\theta}\times(\underbrace{\vec{\nabla}\times\vec{A}}_{\vec{\nabla}}) = \begin{vmatrix} \hat{r} & r\hat{\theta} & r\sin\theta\hat{\phi} \\ 0 & 1 & 0 \\ \nabla_r & \nabla_\theta & \cancel{\nabla_\phi}^{\,0} \end{vmatrix} = \frac{1}{r\cancel{\sin\theta}}\left(\frac{\partial}{\partial\theta}(A_\phi\sin\theta) - \frac{\partial \cancel{A_\theta}^{\,0}}{\partial\phi}\right) r\cancel{\sin\theta}\,\hat{\phi} \quad (19)$$

Thus, Eq. (18) becomes

$$\left[\frac{1}{\mu_0}\frac{\partial}{\partial\theta}(A_\phi^{upper}\sin\theta)\right]_{\theta=\frac{\pi}{2}} = \left[\frac{1}{\mu}\frac{\partial}{\partial\theta}(A_\phi^{lower}\sin\theta)\right]_{\theta=\frac{\pi}{2}} \quad (20)$$

Interchanging the derivative and the integral, Eq. (20) after substituting in Eqs. (9) and (10) becomes

$$\frac{1}{\cancel{\mu_0}}\frac{\cancel{\mu_0} a}{4\pi}\int_0^{2\pi} d\phi' \cos\phi' \left\{ \frac{\partial}{\partial\theta}\left[\frac{\sin\theta\, I}{\sqrt{r^2+d^2+a^2-2r(d\cos\theta+a\sin\theta\cos\phi')}} + \frac{\sin\theta\, I'}{\sqrt{r^2+d^2+a^2-2r(-d\cos\theta+a\sin\theta\cos\phi')}}\right]\right\}_{\theta=\frac{\pi}{2}}$$

$$= \frac{1}{\cancel{\mu}}\frac{\cancel{\mu} a}{4\pi}\int_0^{2\pi} d\phi' \cos\phi' \left\{ \frac{\partial}{\partial\theta}\left[\frac{\sin\theta\, I''}{\sqrt{r^2+d^2+a^2-2r(d\cos\theta+a\sin\theta\cos\phi')}}\right]\right\}_{\theta=\frac{\pi}{2}} \quad (21)$$

With the use of Mathematica, this can be evaluated to

$$\int_0^{2\pi} d\phi' \cos\phi' \left[I \frac{-2dr\cos^2\theta - dr\sin^2\theta + \cos\theta\,(a^2+d^2+r^2-ar\cos\phi'\sin\theta)}{(a^2+d^2+r^2-2dr\cos\theta-2ar\cos\phi'\sin\theta)^{3/2}} + I' \frac{2dr\cos^2\theta + dr\sin^2\theta + \cos\theta\,(a^2+d^2+r^2-ar\cos\phi'\sin\theta)}{(a^2+d^2+r^2+2dr\cos\theta-2ar\cos\phi'\sin\theta)^{3/2}} \right]_{\theta=\frac{\pi}{2}}$$

$$= \int_0^{2\pi} d\phi' \cos\phi' \left[I'' \frac{-2dr\cos^2\theta - dr\sin^2\theta + \cos\theta\,(a^2+d^2+r^2\,ar\cos\phi'\sin\theta)}{a^2+d^2+r^2-2dr\cos\theta-2ar\cos\phi'\sin\theta} \right]_{\theta=\frac{\pi}{2}} \tag{22}$$

Evaluating this on the surface,

$$\int_0^{2\pi} d\phi' \cos\phi' \left(I \frac{-dr}{(a^2+d^2+r^2-2ar\cos\phi')^{3/2}} + I' \frac{dr}{(a^2+d^2+r^2-2ar\cos\phi')^{3/2}} \right)$$

$$= \int_0^{2\pi} d\phi' \cos\phi' \; I'' \frac{-dr}{(a^2+d^2+r^2-2ar\cos\phi')^{3/2}} \tag{23}$$

Notice that we can cancel out the integrals on both sides. We are left with

$$-I + I' = -I'' \quad \rightarrow \quad I - I' = I'' \tag{24}$$

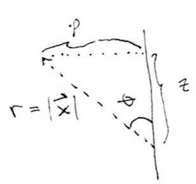
To calculate for the force acting on the loop, we only need the vector potential on the $z>0$ region. So, we combine the results in applying boundary conditions:

$$\mu_0(I+I') = \mu(I-I') \quad \rightarrow \quad I' = \frac{\mu-\mu_0}{\mu+\mu_0} I \tag{25}$$

and substitute this to $A_\phi^{upper}(r,\theta)$ to get

$$A_\phi^{upper}(r,\theta) = \frac{\mu_0 a}{4\pi} I \int_0^{2\pi} \left[\frac{1}{\sqrt{r^2+d^2+a^2-2r(d\cos\theta + a\sin\theta\cos\phi')}} + \left(\frac{\mu-\mu_0}{\mu+\mu_0}\right) \frac{1}{\sqrt{r^2+d^2+a^2-2r(-d\cos\theta+a\sin\theta\cos\phi)}} \right] \cos\phi'\, d\phi' \tag{26}$$

We switch to cylindrical coordinates for easier calculation of the force. Note that



$$z = r\cos\theta \qquad r = \sqrt{\rho^2+z^2}$$

$$\rho = r\sin\theta$$

Thus, Eq. (26) becomes

$$A_\phi^{upper}(\rho,z) = \frac{\mu_0 a}{4\pi} I \int_0^{2\pi} \left[\frac{1}{(\rho^2+z^2+d^2+a^2-2dz-2a\rho\cos\phi')^{1/2}} + \left(\frac{\mu-\mu_0}{\mu+\mu_0}\right) \frac{1}{(\rho^2+z^2+d^2+a^2+2dz-2a\rho\cos\phi')^{1/2}} \right] \cos\phi'\, d\phi' \tag{27}$$

Now, in general, the force on a loop due to $\vec{B}$ is given by

$$\vec{F} = I \oint d\vec{l} \times \vec{B} \tag{28}$$

Since $\vec{B}$ is constant around the wire, this becomes

$$\vec{F} = I\left[\oint dl\, \hat{\phi} \times \vec{B}_{upper}\right]_{on\ loop} = I\left[(\hat{\phi} \times \vec{B}_{upper}) \underbrace{\oint dl}_{2\pi a}\right]_{on\ loop} = 2\pi a I (\hat{\phi} \times \vec{B}_{upper})_{on\ loop} = 2\pi a I (\hat{\phi} \times (\vec{\nabla} \times \vec{A}_{upper}))_{on\ loop} \quad (29)$$

In cylindrical coordinates, we have

$$\vec{\nabla} \times \vec{A} = \left(\frac{1}{\rho}\frac{\partial A_z}{\partial \phi} - \frac{\partial A_\phi}{\partial z}\right)\hat{\rho} + \left(\frac{\partial A_\rho}{\partial z} - \frac{\partial A_z}{\partial \rho}\right)\hat{\phi} + \frac{1}{\rho}\left(\frac{\partial(\rho A_\phi)}{\partial \rho} - \frac{\partial A_\rho}{\partial \phi}\right)\hat{z} \quad (30)$$

Then, (note that only A_ϕ is non-zero)

$$\hat{\phi} \times (\underbrace{\vec{\nabla} \times \vec{A}}_{\vec{v}}) = \begin{vmatrix} \hat{\rho} & \rho\hat{\phi} & \hat{z} \\ 0 & 1 & 0 \\ v_\rho & v_\phi \to 0 & v_z \end{vmatrix} = v_z \hat{\rho} - v_\rho \hat{z} = \frac{1}{\rho}\frac{\partial(\rho A_\phi)}{\partial \rho}\hat{\rho} + \frac{\partial A_\phi}{\partial z}\hat{z} \quad (31)$$

We will drop the term along ρ since $\vec{F}$ is radially symmetric as we are considering a loop. Thus, Eq. (29) becomes

$$\vec{F} = 2\pi a I \left(\frac{\partial A_\phi}{\partial z}\hat{z}\right)_{\rho=a,\ z=d} \quad (32)$$

Interchanging the derivative and the integral, Eq. (32) after substituting in Eq (27) becomes

$$\vec{F} = 2\pi a I\, \hat{z} \left\{ \frac{\mu_0 a}{4\pi} I \int_0^{2\pi} \frac{\partial}{\partial z}\left[\frac{1}{(\rho^2 + z^2 + d^2 + a^2 - 2dz - 2a\rho\cos\phi')^{1/2}} + \left(\frac{\mu - \mu_0}{\mu + \mu_0}\right)\frac{1}{(\rho^2 + z^2 + d^2 + a^2 + 2dz - 2a\rho\cos\phi')^{1/2}}\right]\cos\phi'\, d\phi' \right\}_{\rho=a,\ z=d} \quad (33)$$

With the use of Mathematica, this can be evaluated to

$$\vec{F} = \hat{z}\frac{\mu_0 a^2}{2} I^2 \int_0^{2\pi} d\phi' \cos\phi' \left[\frac{d-z}{(a^2 + (d-z)^2 + \rho^2 - 2a\rho\cos\phi')^{3/2}} + \left(\frac{\mu - \mu_0}{\mu + \mu_0}\right)\frac{-(d+z)}{(a^2 + (d+z)^2 + \rho^2 - 2a\rho\cos\phi')^{3/2}}\right]_{\rho=a,\ z=d} \quad (34)$$

Evaluating along the loop, we have

$$\vec{F} = \hat{z}\frac{\mu_0 a^2 I^2}{2}\int_0^{2\pi} d\phi' \cos\phi' \left[0 + \left(\frac{\mu - \mu_0}{\mu + \mu_0}\right)\frac{-2d}{(a^2 + (2d)^2 + a^2 - 2a^2\cos\phi')^{3/2}}\right]$$

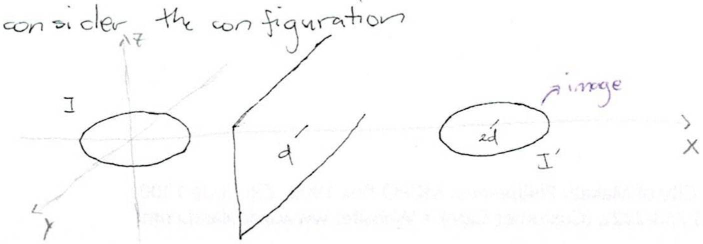
$$= -\hat{z}\mu_0 a^2 I^2 d \left(\frac{\mu - \mu_0}{\mu + \mu_0}\right)\int_0^{2\pi} d\phi' \cos\phi' \frac{1}{(4d^2 + 2a^2(1 - \cos\phi'))^{3/2}} \quad (35)$$

as the force acting on the loop when the plane of the loop is parallel to the face of the slab.

b) the plane of the loop is perpendicular to the face of the slab

Solution:

Let us consider the configuration



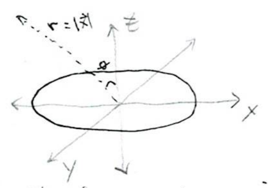
We will again use the method of images. From part a), we know that the real loop has the vector potential given by

$$A_\phi^{real}(r,\theta) = \frac{\mu_0 a}{4\pi}\int_0^{2\pi} \frac{I\cos\phi'\, d\phi'}{\sqrt{r^2+a^2-2ar\sin\theta\cos\phi'}} \qquad (36)$$

It would be more useful to shift to Cartesian coordinates so that we can easily apply this to the image loop. Thus,

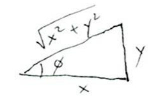
$$\vec{A}_{real}(\vec{x}) = A_\phi^{real}\,\hat{\phi}$$
$$= \frac{\mu_0 a}{4\pi} I\left(\int_0^{2\pi} \frac{\cos\phi'\, d\phi'}{\left(x^2+y^2+z^2+a^2-2\sqrt{x^2+y^2}\cos\phi'\right)^{1/2}}\right)\frac{-y\hat{i}+x\hat{j}}{\sqrt{x^2+y^2}} \qquad (37)$$

where we note that



$$r^2 = x^2+y^2+z^2$$
$$r\sin\theta = \sqrt{x^2+y^2}$$

$$\hat{\phi} = (-\sin\phi, \cos\phi, 0)$$
$$= \left(-\frac{y}{\sqrt{x^2+y^2}}, \frac{x}{\sqrt{x^2+y^2}}, 0\right)$$



As for the image loop, it has the same form as Eq. (37) but shifted to $x-2d$. Thus,

$$\vec{A}_{image}(\vec{x}) = \frac{\mu_0 a}{4\pi} I'\left(\int_0^{2\pi} \frac{\cos\phi'\, d\phi'}{\left[(x-2d)^2+y^2+z^2+a^2-2\sqrt{(x-2d)^2+y^2}\cos\phi'\right]^{1/2}}\right)\frac{-y\hat{i}+(x-2d)\hat{j}}{\sqrt{(x-2d)^2+y^2}} \qquad (38)$$

We could use the result from part a) for the image current

$$I' = \frac{\mu-\mu_0}{\mu+\mu_0} I$$

Thus,

$$\vec{A}_{image}(\vec{x}) = \frac{\mu_0 a}{4\pi} I\left(\frac{\mu-\mu_0}{\mu+\mu_0}\right)\left(\int_0^{2\pi} \frac{\cos\phi'\, d\phi'}{\left[(x-2d)^2+y^2+z^2+a^2-2\sqrt{(x-2d)^2+y^2}\cos\phi'\right]^{1/2}}\right)\frac{-y\hat{i}+(x-2d)\hat{j}}{\sqrt{(x-2d)^2+y^2}} \qquad (39)$$

When solving for the force, note that the force on the loop does not contribute since its components will cancel out. This is evident in part a) where the first term of the integral in Eq. (35), which corresponds to the force on the loop, vanishes. Thus, we only need the vector potential of the image loop to calculate for the force acting on the real loop. We again shift coordinates to cylindrical to simplify the calculations:

$$\vec{A}_{image}(\vec{x}) = \frac{\mu_0 a}{4\pi} I\left(\frac{\mu-\mu_0}{\mu+\mu_0}\right)\left(\int_0^{2\pi} \frac{\cos\phi'\, d\phi'}{\left(\rho^2+4d^2-4\rho\cos\phi\, d+z^2+a^2-2a\sqrt{\rho^2+4d^2-4\rho\cos\phi\, d}\,\cos\phi'\right)^{1/2}}\right)\left(\frac{-2d\sin\phi\,\hat{\rho}}{\sqrt{\rho^2+4d^2-4\rho\cos\phi\, d}} + \frac{(\rho-2d\cos\phi)\hat{\phi}}{\sqrt{\rho^2+4d^2-4\rho\cos\phi\, d}}\right) \qquad (40)$$

where $x=\rho\cos\phi$ and $y=\rho\sin\phi$. The force on the loop due to $\vec{B}$ is again given by Eq. (28). However, unlike in part a), the magnetic field is no longer constant around the loop due to the difference in orientation of the slab. Thus,

$$\vec{F} = I\left[\oint dl\,\hat{\phi}\times\vec{B}\right]_{\text{on loop}} = I\int_0^a dr\int_0^{2\pi} d\phi\,(\hat{\phi}\times\vec{B})_{\text{on loop}} = Ia\int_0^{2\pi} d\phi\,\left(\hat{\phi}\times(\nabla\times\vec{A}_{image})\right)_{\text{on loop}} \qquad (41)$$

Recall the curl of A in cylindrical coordinates in Eq. (30). Then,

$$\hat{\phi} \times (\vec{\nabla} \times \vec{A}) = \begin{vmatrix} \hat{\rho} & \rho\hat{\phi} & \hat{z} \\ 0 & 1 & 0 \\ v_\rho & v_\phi & v_z \end{vmatrix} = v_z\hat{\rho} - v_\rho\hat{z} = \frac{1}{\rho}\left(\frac{\partial\rho A_\phi^{image}}{\partial\rho} - \frac{\partial A_\rho^{image}}{\partial\phi}\right)\hat{\rho} - \left(-\frac{\partial A_\phi}{\partial z}\right)\hat{z} \quad (42)$$

Recall as well that $\hat{\rho} = \cos\phi\,\hat{i} + \sin\phi\,\hat{j}$. Due to the symmetry in this case, only the x component of $\vec{F}$ does not cancel out. Thus,

$$\vec{F} = \hat{i}\, I a \left[\int_0^{2\pi} d\phi \cos\phi \cdot \frac{1}{\rho}\left(\frac{\partial(\rho A_\phi^{image})}{\partial\rho} - \frac{\partial A_\rho^{image}}{\partial\phi}\right)\cos\phi\right]_{\substack{\rho=a \\ z=0}} \quad (43)$$

where A_ϕ^{image} and A_ρ^{image} is given by Eq. (40). The integrals are complicated so we stop here

c) Determine the limiting form of your answer to parts a and b when $d \gg a$. Can you obtain these limiting values in some simple and direct way?

Solution:

For the answer in part a), we factor out $4d^2$ from the square root term for the limiting case

$$\vec{F} = -\hat{z}\,\mu_0 a^2 I^2 d\left(\frac{\mu-\mu_0}{\mu+\mu_0}\right)\int_0^{2\pi} d\phi' \cos\phi' \frac{1}{2^3 d^3\left(1+\frac{1}{2}\left(\frac{a}{d}\right)^2(1-\cos\phi')\right)^{3/2}}$$

$$= -\hat{z}\,\mu_0 I^2\left(\frac{a}{d}\right)^2\frac{1}{8}\left(\frac{\mu-\mu_0}{\mu+\mu_0}\right)\int_0^{2\pi}\frac{d\phi'\cos\phi'}{\left(1+\frac{1}{2}\left(\frac{a}{d}\right)^2(1-\cos\phi')\right)^{3/2}} \quad (44)$$

Let $\alpha = \left(\frac{a}{d}\right)^2$. Using the binomial expansion

$$(1+x)^n = 1 + nx + \ldots, \quad (45)$$

we have

$$\left(1+\frac{1}{2}(1-\cos\phi')\,\alpha\right)^{-3/2} = 1 - \frac{3}{4}(1-\cos\phi')\left(\frac{a}{d}\right)^2 + \ldots \quad (46)$$

keeping only the first two terms since higher order terms of α are negligible with $\frac{a}{d} \ll 1$, Eq. (44) becomes

$$\vec{F} = -\hat{z}\,\mu_0 I^2\left(\frac{a}{d}\right)^2\frac{1}{8}\,\frac{\mu-\mu_0}{\mu+\mu_0}\int_0^{2\pi}\left[1-\frac{3}{4}(1-\cos\phi')\left(\frac{a}{d}\right)^2\right]\cos\phi'\,d\phi'$$

$$= -\hat{z}\,\mu_0 I^2\left(\frac{a}{d}\right)^2\frac{1}{8}\,\frac{\mu-\mu_0}{\mu+\mu_0}\left(\frac{3\pi}{4}\left(\frac{a}{d}\right)^2\right) \quad (47)$$

$$\vec{F} = -\hat{z}\,\mu_0 I\,\frac{3\pi}{32}\,\frac{\mu-\mu_0}{\mu+\mu_0}\left(\frac{a}{d}\right)^4$$

As for the answer in part b), note that we are evaluating $\vec{F}$ at $z=0$. Thus, we have

$$\vec{A}_{image}(\vec{x}) = \frac{\mu_0 a}{4\pi} I\,\frac{\mu-\mu_0}{\mu+\mu_0}\int_0^{2\pi}\frac{\cos\phi'\,d\phi'}{\left(\rho^2+4d^2-4\rho\cos\phi\, d+a^2-2a\sqrt{\rho^2+4d^2-4\rho\cos\phi\, d}\,\cos\phi'\right)^{1/2}}\;\frac{(-2d\sin\phi)\,\hat{\rho}+(\rho-\cos\phi\, 2d)\,\hat{\phi}}{\sqrt{\rho^2+4d^2-4\rho\cos\phi\, d}} \quad (48)$$

Factoring out $4d^2$ in the denominator,

$$\vec{A}_{image}(\vec{x}) = \frac{\mu_0 a}{4\pi} I \frac{\mu-\mu_0}{\mu+\mu_0} \int_0^{2\pi} \frac{\cos\phi'\, d\phi'}{2d\left(1 + \underbrace{\cancel{\frac{1}{4}\left(\frac{a}{d}\right)^2} + \cancel{\frac{1}{4}\frac{\rho^2}{d^2}}}_{\text{negligible}} - \frac{\rho}{d}\cos\phi - \frac{4ad}{4d^2}\sqrt{1 + \underbrace{\cancel{\frac{1}{2}\frac{\rho^2}{d^2}} - \cancel{\frac{\rho}{d}}}_{\text{negligible}}\cos\phi}\,\cos\phi'\right)^{1/2}}$$

$$\times \frac{\cancel{2d}\left[(-\sin\phi)\hat{\rho} + \left(\frac{1}{2}\frac{\rho}{d} - \cos\phi\right)\hat{\phi}\right]}{\cancel{3d}\sqrt{1 + \underbrace{\cancel{\frac{\rho^2}{4d^2}}}_{\text{negligible}} - \frac{\rho}{d}\cos\phi}}$$

$$= \frac{\mu_0 a}{4\pi \cdot 2d} I \frac{\mu-\mu_0}{\mu+\mu_0}\left(\int_0^{2\pi} \frac{\cos\phi'\, d\phi'}{\left(1 - \frac{\rho}{d}\cos\phi - \frac{a}{d}\cos\phi'\right)^{1/2}}\right) \frac{-\sin\phi\,\hat{\rho} + \left(\frac{1}{2}\frac{\rho}{d} - \cos\phi\right)\hat{\phi}}{\sqrt{1 - \frac{\rho}{d}\cos\phi}} \qquad 48$$

Using the expansion in Eq. (45),

$$\left[1 - \left(\frac{\rho}{d}\cos\theta + \frac{a}{d}\cos\phi'\right)\right]^{-1/2} = 1 + \left(-\frac{1}{2}\right)\left[-\left(\frac{\rho}{d}\cos\theta + \frac{a}{d}\cos\phi'\right)\right] + \dots \qquad (49)$$

$$\left[1 - \frac{\rho}{d}\cos\phi\right]^{-1/2} = 1 + \left(-\frac{1}{2}\right)\left(-\frac{\rho}{d}\cos\phi\right) + \dots \qquad (50)$$

Keeping only the first two terms,

$$\vec{A}_{image}(\vec{x}) = \frac{\mu_0 a}{4\pi \cdot 2d} I \frac{\mu-\mu_0}{\mu+\mu_0}\left(\int_0^{2\pi}\left[1 + \frac{1}{2}\left(\frac{\rho}{d}\cos\theta + \frac{a}{d}\cos\phi'\right)\right]\cos\phi'\, d\phi'\right)\left(1 + \frac{1}{2}\frac{\rho}{d}\cos\phi\right)\left(-\sin\phi\,\hat{\rho} + \left(\frac{1}{2}\frac{\rho}{d} - \cos\phi\right)\hat{\phi}\right)$$

$$= \frac{\mu_0 a}{4\pi \cdot 2d} I \frac{\mu-\mu_0}{\mu+\mu_0}\left(\frac{a\pi}{2d}\right)\left(1 + \frac{1}{2}\frac{\rho}{d}\cos\phi\right)\left(-\sin\phi\,\hat{\rho} + \left(\frac{1}{2}\frac{\rho}{d} - \cos\phi\right)\hat{\phi}\right) \qquad (51)$$

Substituting this in Eq. (43), we have

$$\vec{F} = \hat{1}\, I^2 a \frac{\mu_0}{16}\left(\frac{a}{d}\right)^2 \frac{\mu-\mu_0}{\mu+\mu_0}\left[\int_0^{2\pi} d\phi \cos\phi \; \frac{1}{\rho}\left(\frac{\partial}{\partial\rho}\left[\rho\left(\frac{1}{2}\frac{\rho}{d} - \cos\phi\right)\left(1 + \frac{1}{2}\frac{\rho}{d}\cos\phi\right)\right] - \frac{\partial}{\partial\phi}\left[-\sin\phi\left(1 + \frac{1}{2}\frac{\rho}{d}\cos\phi\right)\right]\right)\right]$$

$$= \hat{1}\, I^2 a \frac{\mu_0}{16}\frac{a^2}{d^2}\frac{\mu-\mu_0}{\mu+\mu_0}\left[\int_0^{2\pi} d\phi \cos\phi \; \frac{1}{\rho}\left(\frac{\rho}{d} - \cos\phi + \frac{3\rho^2\cos\phi}{4d^2} - \frac{\rho\cos^2\phi}{d} + \frac{\cos\phi\sin\phi}{2d}\right)\right]$$

$$= \hat{1}\, I^2 a \frac{\mu_0}{16}\frac{a^2}{d^2}\frac{\mu-\mu_0}{\mu+\mu_0}\left(\frac{\pi}{d}\right) \qquad (52)$$

$$\vec{F} = \hat{1}\, \mu_0 I^2 \frac{\pi}{16}\frac{\mu-\mu_0}{\mu+\mu_0}\left(\frac{a}{d}\right)^3$$

Note that most of the integrals and derivatives in this section are evaluated using Mathematica

PS 2

#2

Taylor series expansion shown in Eq. (7):

In[27]:= $\text{Series}\left[\left(1-b\gamma+\left(a^2+\frac{b^2}{4}\right)\gamma^2\right)^{-3/2},\ \{\gamma, 0, 4\}\right]$

Out[27]= $1+\frac{3b\gamma}{2}-\frac{3}{2}\left(a^2-b^2\right)\gamma^2+\left(-\frac{15a^2b}{4}+\frac{5b^3}{4}\right)\gamma^3+\frac{15}{16}\left(2a^4-6a^2b^2+b^4\right)\gamma^4+O[\gamma]^5$

Taylor series expansion shown in Eq. (8) :

In[28]:= $\text{Series}\left[\left(1+b\gamma+\left(a^2+\frac{b^2}{4}\right)\gamma^2\right)^{-3/2},\ \{\gamma, 0, 4\}\right]$

Out[28]= $1-\frac{3b\gamma}{2}-\frac{3}{2}\left(a^2-b^2\right)\gamma^2+\left(\frac{15a^2b}{4}-\frac{5b^3}{4}\right)\gamma^3+\frac{15}{16}\left(2a^4-6a^2b^2+b^4\right)\gamma^4+O[\gamma]^5$

#3

Checking Eq. (12):

In[4]:= $\text{Integrate}\left[(\text{Cos}[\theta])^2,\ \{\theta, a-0, a-2\pi\}\right]$

Out[4]= $-\pi$

#4

Checking Eq. (3) :

```
x = r Cos[ϕ] Sin[θ];
y = r Sin[ϕ] Sin[θ];
z = r Cos[θ];
xprime = rprime Cos[ϕprime] Sin[θprime];
yprime = rprime Sin[ϕprime] Sin[θprime];
zprime = rprime Cos[θprime];
```

$\sqrt{(x-\text{xprime})^2+(y-\text{yprime})^2+(z-\text{zprime})^2}$ // FullSimplify

$\sqrt{\left(r^2+\text{rprime}^2-2\,r\,\text{rprime}\left(\text{Cos}[\theta]\,\text{Cos}[\theta\text{prime}]+\text{Cos}[\phi-\phi\text{prime}]\,\text{Sin}[\theta]\,\text{Sin}[\theta\text{prime}]\right)\right)}$

Derivatives for Eq. (15):

In[29]:=
$$D\left[\frac{r\,\mathrm{Ireal}}{\sqrt{r^2+d^2+a^2-2\,r\,\left(d\,\mathrm{Cos}[\theta]+a\,\mathrm{Sin}[\theta]\,\mathrm{Cos}[\phi\mathrm{prime}]\right)}},r\right]\;//\;\mathrm{FullSimplify}$$

Out[29]=
$$\frac{\mathrm{Ireal}\,\left(a^2+d^2-r\,\left(d\,\mathrm{Cos}[\theta]+a\,\mathrm{Cos}[\phi\mathrm{prime}]\,\mathrm{Sin}[\theta]\right)\right)}{\left(a^2+d^2+r^2-2\,r\,\left(d\,\mathrm{Cos}[\theta]+a\,\mathrm{Cos}[\phi\mathrm{prime}]\,\mathrm{Sin}[\theta]\right)\right)^{3/2}}$$

$$D\left[\frac{r\,\mathrm{Iprime}}{\sqrt{r^2+d^2+a^2-2\,r\,\left(-d\,\mathrm{Cos}[\theta]+a\,\mathrm{Sin}[\theta]\,\mathrm{Cos}[\phi\mathrm{prime}]\right)}},r\right]\;//\;\mathrm{FullSimplify}$$

$$\frac{\mathrm{Iprime}\,\left(a^2+d^2+d\,r\,\mathrm{Cos}[\theta]-a\,r\,\mathrm{Cos}[\phi\mathrm{prime}]\,\mathrm{Sin}[\theta]\right)}{\left(a^2+d^2+r^2+2\,d\,r\,\mathrm{Cos}[\theta]-2\,a\,r\,\mathrm{Cos}[\phi\mathrm{prime}]\,\mathrm{Sin}[\theta]\right)^{3/2}}$$

$$D\left[\frac{r\,\mathrm{Iprime2}}{\sqrt{r^2+d^2+a^2-2\,r\,\left(d\,\mathrm{Cos}[\theta]+a\,\mathrm{Sin}[\theta]\,\mathrm{Cos}[\phi\mathrm{prime}]\right)}},r\right]\;//\;\mathrm{FullSimplify}$$

$$\frac{\mathrm{Iprime2}\,\left(a^2+d^2-r\,\left(d\,\mathrm{Cos}[\theta]+a\,\mathrm{Cos}[\phi\mathrm{prime}]\,\mathrm{Sin}[\theta]\right)\right)}{\left(a^2+d^2+r^2-2\,r\,\left(d\,\mathrm{Cos}[\theta]+a\,\mathrm{Cos}[\phi\mathrm{prime}]\,\mathrm{Sin}[\theta]\right)\right)^{3/2}}$$

Derivatives for Eq. (22):

In[5]:=
$$D\left[\frac{\mathrm{Sin}[\theta]\;\mathrm{Iorig}}{\sqrt{r^2+d^2+a^2-2\,r\,\left(d\,\mathrm{Cos}[\theta]+a\,\mathrm{Sin}[\theta]\,\mathrm{Cos}[\phi\mathrm{prime}]\right)}},\theta\right]\;//\;\mathrm{FullSimplify}$$

Out[5]=
$$\left(\mathrm{Iorig}\,\left(-2\,d\,r\,\mathrm{Cos}[\theta]^2-d\,r\,\mathrm{Sin}[\theta]^2+\mathrm{Cos}[\theta]\,\left(a^2+d^2+r^2-a\,r\,\mathrm{Cos}[\phi\mathrm{prime}]\,\mathrm{Sin}[\theta]\right)\right)\right)\Big/\left(a^2+d^2+r^2-2\,r\,\left(d\,\mathrm{Cos}[\theta]+a\,\mathrm{Cos}[\phi\mathrm{prime}]\,\mathrm{Sin}[\theta]\right)\right)^{3/2}$$

In[6]:=
$$D\left[\frac{\mathrm{Sin}[\theta]\;\mathrm{Iprime}}{\sqrt{r^2+d^2+a^2-2\,r\,\left(-d\,\mathrm{Cos}[\theta]+a\,\mathrm{Sin}[\theta]\,\mathrm{Cos}[\phi\mathrm{prime}]\right)}},\theta\right]\;//\;\mathrm{FullSimplify}$$

Out[6]=
$$\left(\mathrm{Iprime}\,\left(2\,d\,r\,\mathrm{Cos}[\theta]^2+d\,r\,\mathrm{Sin}[\theta]^2+\mathrm{Cos}[\theta]\,\left(a^2+d^2+r^2-a\,r\,\mathrm{Cos}[\phi\mathrm{prime}]\,\mathrm{Sin}[\theta]\right)\right)\right)\Big/\left(a^2+d^2+r^2+2\,d\,r\,\mathrm{Cos}[\theta]-2\,a\,r\,\mathrm{Cos}[\phi\mathrm{prime}]\,\mathrm{Sin}[\theta]\right)^{3/2}$$

In[7]:=
$$D\left[\frac{\mathrm{Sin}[\theta]\;\mathrm{Iprime2}}{\sqrt{r^2+d^2+a^2-2\,r\,\left(d\,\mathrm{Cos}[\theta]+a\,\mathrm{Sin}[\theta]\,\mathrm{Cos}[\phi\mathrm{prime}]\right)}},\theta\right]\;//\;\mathrm{FullSimplify}$$

Out[7]=
$$\left(\mathrm{Iprime2}\,\left(-2\,d\,r\,\mathrm{Cos}[\theta]^2-d\,r\,\mathrm{Sin}[\theta]^2+\mathrm{Cos}[\theta]\,\left(a^2+d^2+r^2-a\,r\,\mathrm{Cos}[\phi\mathrm{prime}]\,\mathrm{Sin}[\theta]\right)\right)\right)\Big/\left(a^2+d^2+r^2-2\,r\,\left(d\,\mathrm{Cos}[\theta]+a\,\mathrm{Cos}[\phi\mathrm{prime}]\,\mathrm{Sin}[\theta]\right)\right)^{3/2}$$

Derivatives for Eq. (34):

In[10]:=
$$D\left[\frac{1}{\sqrt{\rho^2+z^2+d^2+a^2-2\,d\,z-2\,a\,\rho\;\mathrm{Cos}[\phi\mathrm{prime}]}},z\right]\;//\;\mathrm{FullSimplify}$$

Out[10]=
$$\frac{d-z}{\left(a^2+(d-z)^2+\rho^2-2\,a\,\rho\,\mathrm{Cos}[\phi\mathrm{prime}]\right)^{3/2}}$$

In[11]:=
$$D\left[\frac{1}{\sqrt{\rho^2+z^2+d^2+a^2+2\,d\,z-2\,a\,\rho\;\mathrm{Cos}[\phi\mathrm{prime}]}},z\right]\;//\;\mathrm{FullSimplify}$$

Out[11]=
$$\frac{-d-z}{\left(a^2+(d+z)^2+\rho^2-2\,a\,\rho\,\mathrm{Cos}[\phi\mathrm{prime}]\right)^{3/2}}$$

Integral in Eq. (47) where $\alpha = (a/d)^2$:

In[30]:=
```
Integrate[(1 - 3/4 (1 - Cos[ϕprime] α)) Cos[ϕprime], {ϕprime, 0, 2 π}]
```
Out[30]= $\frac{3 \pi \alpha}{4}$

Integral in Eq. (51):

In[20]:=
```
Integrate[(1 + 1/2 (ρ/d Cos[ϕ] + a/d Cos[ϕprime])) Cos[ϕprime], {ϕprime, 0, 2 π}]
```
Out[20]= $\frac{a \pi}{2 d}$

Derivatives for Eq. (52):

In[25]:=
```
D[ρ (1/2 ρ/d - Cos[ϕ]) (1 + 1/2 ρ/d Cos[ϕ]), ρ] // Simplify // Expand
```
Out[25]= $\frac{\rho}{d} - \mathrm{Cos}[\phi] + \frac{3 \rho^2 \mathrm{Cos}[\phi]}{4 d^2} - \frac{\rho \mathrm{Cos}[\phi]^2}{d}$

In[24]:=
```
D[-Sin[ϕ] (1 + 1/2 ρ/d Cos[ϕ]), ρ] // Simplify
```
Out[24]= $-\frac{\mathrm{Cos}[\phi] \mathrm{Sin}[\phi]}{2 d}$

Integral in Eq. (52):

In[26]:=
```
Integrate[1/ρ (ρ/d - Cos[ϕ] + (3 ρ^2 Cos[ϕ])/(4 d^2) - (ρ Cos[ϕ]^2)/d + (Cos[ϕ] Sin[ϕ])/(2 d)), {ϕ, 0, 2 π}]
```
Out[26]= $\frac{\pi}{d}$