Physics 226

Sept. 21, 2021 ▸ $\boxed{0}$ Review of Differential Geometry

▸ Non-Euclidean Geometries

- Consider Euclidean coord's (ξ^1, ξ^2) and some other coord. sys. (x^1, x^2). What is the distance bet. two neighbouring points in the new sys.?

we explicitly evaluate $d\xi^1$ & $d\xi^2$ in the terms of the new coord's

$\boxed{0}$ Orientation

- Bo ~~mher~~'s book for cosmology part
- Perturbations of Schwarzchild - (ideal) basis of LISSA
- Difference bet. a tensor and a its components
- Connection bet. geodesics and the equivalence priciple
- Lie derivatives
- Killing vectors → symmetries
- ~~Metrics~~ Metrics in manifolds → sense of distance and angle
  - ↳ isomorphism (one to one corres.) vectors & dual vectors
  - ↳ geodesics - extremal path
- Auto-parallel path - path by parallel transport
- Connection coefficients - imposes a parallel transport rule on a manifold
  - ↳ Levi-Civita connection - preserves the length of the vector as you parallel transport it
  - ↳ affine connections
- Anolonomy ← curvature
- Physical aspects of curvature - measured by the difference bet. the relative acceleration of diff. objects
  - ↳ given by geodesic deviation eqn
- Physical measurments in curved spacetime
  ↳ go to an orthonormal frame (local frame of the observer)/rest frame, and represent the geometric ~~component~~ object into the orthonormal basis. The components of the represented object are what are measured by the observer

* Special relativity in General frames

- Symmetry arguments to reduce the ansatz
  - Apply symmetry (vacuum soln, time symmetry, spherical sym.)
  - Get reduced metric
  - Plug in to Einstein tensor

Sept. 23, 2021 ▸ $\boxed{1}$ Review of Differential Geometry (cont.)

- GR: All gravitational phenomena are the result of stuff moving in spacetime
  - tensors → matter (in other words)
  - spacetime ↳ Curved Lorentzian Manifold of events
- Tensors: fancy name for the geometric objects on the manifold
  - geometric: does not depend on coordinate sys.
- Manifolds: smooth in context of physics
  : topological space that is locally coordinatizable (locally $\mathbb{R}^n$)
  - topological space: set on which you have open sets

1 Review of the Schwarzschild Solution

- The Schwarzschild soln / geometry takes the form of

$$ds^2 = -\left(1 - \frac{2M}{r}\right)dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 d\Omega^2$$

$d\Omega^2 = d\theta^2 + \sin^2\theta \, d\phi^2$ (solid angle)

in Schwarzschild coord's $\{t, r, \theta, \phi\}$.

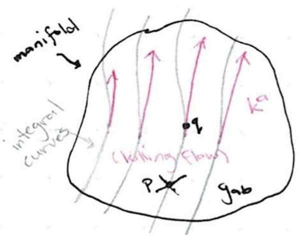
(no matter fields) $G_{ab} = 8\pi T_{ab} \rightarrow R_{ab} = 0$

spherically symmetric star

- unique, static, spherically symmetric soln to the vacuum Einstein equation
- peculiar since it's non-trivial even if it's in vacuum
- reasonable approx. for spherically symmetric stars

Uses the convention: $(- + + +)$

- Stationary spacetime $\longrightarrow$ $\exists\, k^a$ (killing vector) ① where $g_{ab}k^a k^b < 0$ (time like) ②
  - (special case) $\hookrightarrow$ static spacetime $\xrightarrow{\text{add'l condition}}$ k^a must be hyper surface - orthogonal ③



- Killing vector : If I differentiate the metric wrt the Killing flow, it doesn't change; vector field that corres. to symmetry

$$\underbrace{\mathcal{L}_k}_{\text{Lie derivative}} g = 0 \quad \longleftrightarrow \quad \nabla_{(a} k_{b)} = 0 \ \text{(Killing's eqn)}$$

* Symmetrization symbol

$$\nabla_{(a} k_{b)} := \frac{1}{2}\left(\nabla_a k_b + \nabla_b k_a\right) = 0$$

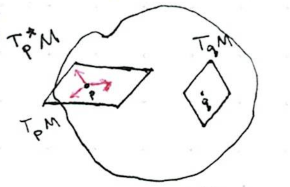
* Anti - symmetrization symbol

$$\nabla_{[a} k_{b]} := \frac{1}{2}\left(\nabla_a k_b - \nabla_b k_a\right)$$

- If a vector field satisfies the Killing's eqn, then that Killing flow is a symmetry of your given spacetime

- Lie derivative : $\mathcal{L}_v T$ - use vector field v to bring tensor T at a diff. point connected by the curves

cotangent space corres. to T_pM



- Here, we can't simply compare objects in T_pM with T_qM (like vector subtraction between the two tangent spaces). We can establish a connection to do so. Another way is with the use of an arbitrary vector field which gives rise to integral curves.

an arbitrary choice for the sake of comparing geometric objects

$\Phi_\lambda : M \rightarrow M$

- The flow Φ generated by v^a can be thought of as a mapping of the manifold onto itself. For example, it can map a point to another point lying on the same integral curve with a distance parameter λ away from the orig. point. Thus, since p and q lie on the same curve, the object living in their corres. tangent spaces are related through the said curve. The comparison of these objects makes use of pull backs & push forwards.

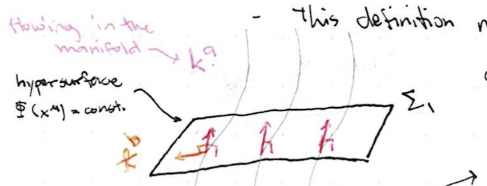
- Hyper surface - orthogonality

The dual of a vector is given by: $k_a = g_{ab}k^b$

$\hookrightarrow$ There exists two scalar functions Φ, f (where $\Phi, f : M \rightarrow \mathbb{R}$) such that k_a (dual of the Killing vector field) can be expressed as $k_a = -f \nabla_a \Phi$

$\nabla_a \Phi$ $\rightarrow$ gradient one form / covariant derivative of a scalar

$\nabla_a \Phi = (\partial_\alpha \Phi)\, dx^\alpha$

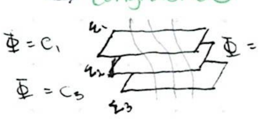


- This definition means that for k_a, $\exists$ hyper surfaces $\Phi(x^\mu) = $ const. s.t. the congruence of the worldlines / integral curves generated by k^a is perpendicular to each point in these hyper surfaces. We can then say that k^a is hypersurface - orthogonal

$$\frac{dx^\mu}{d\lambda} = k^\mu(x)$$

$\leftarrow$ Each worldline in the congruence is a solution to this eqn

$x^\mu(\lambda) : \{t(\lambda), r(\lambda), \theta(\lambda), \phi(\lambda)\}$



} Foliation : splitting of a manifold into non-intersecting parts

- Orthogonality : If I take an arbitrary tangent vector t^b lying on a specific hyper surface Σ, it is orthogonal to k^a if $g_{ab}k^a t^b = 0$ (inner product bet. k^a & t^b using the metric vanishes)

: We demand this for all points in all our hypersurfaces described by const. Φ

Note: $\nabla\Phi \rightarrow$ the normal vector corres to Φ in usual connotation

- implies that k^a is the normal vector wrt the hypersurface

- implies that Killing vector fields generate foliation in your manifold

$\Phi = c_1$

Consider one of the hypersurfaces Σ_1 and imagine a curve σ_1 lying on it.
In an arbitrary coord. sys., σ is described by $\sigma : x^\mu(s)$ — arc length parameter along the curve
We can compute how Φ changes along the curve:

$$\frac{d\Phi}{ds} = 0$$ — since the curve lives on the hypersurface

Note that this can be written as:

$$\boxed{0 = \frac{d\Phi(x^\mu(s))}{ds} \overset{\text{chain rule}}{=} \frac{d\Phi}{dx^\mu}\frac{dx^\mu}{ds} = \Phi_{,\mu}\, t^\mu}$$

expression for arbitrary tangent vector

From this, we can see that if

$$k_\mu = -f\,\Phi_{,\mu} \xrightarrow{\text{then}} k_\mu t^\mu = 0$$

- The reverse is harder to prove: We can write $k_\mu = -f\Phi_{,\mu}$ if it is orthogonal to all tangent vectors

Proof: Assume $k_a t^a = 0\ \forall$ tangent vectors t^a of Σ. This means that k^a is $\perp$ to Σ.
Since k^a is orthogonal to the hypersurface, there must exists three indep. tangent vectors. Then, We can write down 3 conditons:

$\mu = 0,1,2,3$ $\quad i = 1,2,3$ $\quad t^a_{(i)}\, k_a = 0$

In component form: $t^\mu_{(i)} k_\mu = 0$ (3 linear eqns w/ 4 unknowns)

- $k_\mu = (k_0, k_1, k_2, k_3) = k_0(1, k_i/k_0)$

This means that the orthogonality condition of k^a determines its components up to an over all scaling we can't specify.

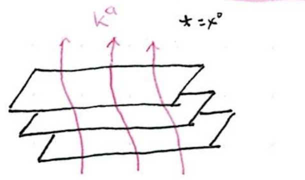
Thus, we have
$$\left.\begin{aligned} t^\mu_{(i)} k_\mu &= 0 \\ \text{and} \quad t^\mu_{(i)} \Phi_{,\mu} &= 0 \end{aligned}\right\}\ k_\mu \propto \Phi_{,\mu}$$

We can also say that "$-f$" is the ratio bet. k_μ and $\Phi_{,\mu}$ at point p:
$$f_p = -\frac{(k_a)_p}{(\nabla_a \Phi)_p}$$

from an earlier calculation. Then, this implies that the only difference bet. k_μ & $\Phi_{,\mu}$ is some scaling factor which we can call "$-f$". Therefore, we can express k_a as $k_a = -f\,\nabla_a \Phi$

- This is important since it turns out that there is a convenient set of coordinates for a static spacetime.

$\Phi = c$

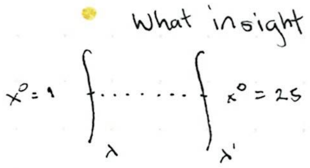


How do we choose our coordinate system?
- Choose one of the coordinates x^0 s.t.

$$k^a \propto \frac{\partial}{\partial x^0} = \frac{\partial}{\partial t}$$

In other words, x^0 should be a parameter that runs along the integral curve of k^a and that the other coordinates $\{x^i\}$ should not change on this curve.
- x^0: coord. along curve $\qquad \{x^i\}$: labels the curve
- Choose $\{x^i\}$ such that: $\quad \frac{\partial}{\partial x^i} \longrightarrow$ tangent to Σ

- What insight does the existence of the hypersurfaces give us?



Here, λ & λ' are parametrizations of the integral curves. We see that there is no need to have a common parametrization among these integral curves. as they are diff. curves from one another. However, we can use hypersurfaces to relate them. For example, if points among different integral curves share the same hypersurface, we could give them the same x^0

* Φ serves as some sort of "connection" between these diff. integral curves (could be w/ diff. coord. systems)

- What does this imply?

Consider the components of our metric. For instance, we can infer that

Recall condition for orthogonality discussed in pg. 1

$$g_{00} = g(\partial_0, \partial_0) < 0$$ since we assumed that k^a is timelike

$$g_{0i} = g(\partial_0, \partial_i) = 0$$ since $\partial_0 \perp \partial_i$; $\left\{\frac{\partial}{\partial x_i}\right\}$ are tangent vectors to Σ which are orthogonal to k^a

$$= g_{ab} \underbrace{\left(\frac{\partial}{\partial x^0}\right)^a}_{k^a} \underbrace{\left(\frac{\partial}{\partial x^i}\right)^b}_{t^b} = 0$$

Because of this, we can write the line element of our static spacetime to be

$$ds^2 = g_{00}(x^\mu)(dx^0)^2 + g_{ij}(x^\mu)\, dx^i dx^j + \cancel{g_{0i}\, dx^0 dx^i}^{\,0}$$

In other words, with the use of symmetries and the way we adopted our coordinates to match these symmetries, we can reduce our line element generally of the form $ds^2 = g_{\mu\nu} dx^\mu dx^\nu$ to the form above which has no cross terms.