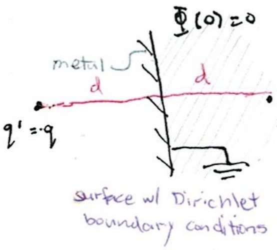
Physics 231

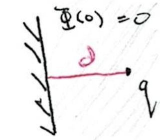
Boundary Value Problem in Electrostatics
(chapter 2)

Oct. 13, 2020

- Lecture Notes
- Method of Images
  - why do this work?



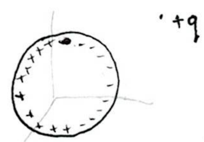
- Under the Dirichlet b.c., a unique soln exists



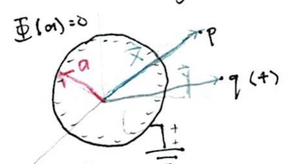
- The two systems are equivalent over the indicated region and the soln in one is also the soln in the other because they satisfy same b.c.

- If you have difficulty ~~is~~ in solving a problem, find a simpler system that satisfies the same boundary conditions

*** Point charge in the prescence of a grounded sphere**

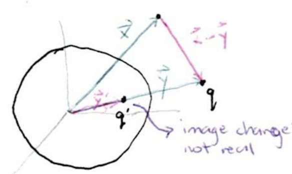


not grounded; a harder problem to solve



The sphere becomes negatively charged while the positive charges goes to the ground. However, the (−) charges are not uniformly distributed. They are more concentrated near the external positive charge

We replace this system with a simpler system with an image charge



$$\Phi(\vec{x}) = \frac{q}{4\pi\epsilon_0|\vec{x}-\vec{y}|} + \frac{q'}{4\pi\epsilon_0|\vec{x}-\vec{y}'|}$$

q and q' must be chosen such that the b.c. of the original problem is reproduced, i.e. $\Phi(a)=0$

Let $\hat{n} = \frac{\vec{x}}{|\vec{x}|} = \frac{\vec{x}}{x}$, $\hat{n}' = \frac{\vec{y}}{|\vec{y}|} = \frac{\vec{y}}{y}$. Then,

$$\Phi(\vec{x}) = \frac{q}{4\pi\epsilon_0|x\hat{n}-y\hat{n}'|} + \frac{q'}{4\pi\epsilon_0|x\hat{n}-y'\hat{n}'|}$$

At $\vec{x}=a$

$$\Phi(\vec{x}=a) = \frac{q}{4\pi\epsilon_0|a\hat{n}-y\hat{n}'|} + \frac{q'}{4\pi\epsilon_0|a\hat{n}-y'\hat{n}'|} = \frac{q/a}{4\pi\epsilon_0|\hat{n}-\frac{y}{a}\hat{n}'|} + \frac{q'/y'}{4\pi\epsilon_0|\hat{n}-\frac{a}{y'}\hat{n}'|}$$

By inspection, choose the ff. values

$$\frac{q}{a} = -\frac{q'}{y'}, \quad \frac{y}{a} = \frac{a}{y'}$$

Thus, we have

$$\Phi(a) = \frac{q/a}{4\pi\epsilon_0|\hat{n}-\frac{y}{a}\hat{n}'|} + \frac{(-q/a)}{4\pi\epsilon_0|\hat{n}-\frac{y}{a}\hat{n}'|} = 0$$

So we know now that

$$\boxed{y' = \frac{a^2}{y}}, \quad \boxed{q' = -\frac{y'}{a}q = -\frac{a^2/y}{a}q = -\frac{a}{y}q}$$

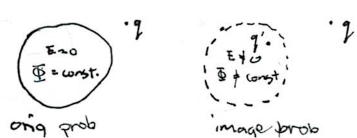
For the charge q and the charge q', the electric potential is given by

$$\Phi(\vec{x}) = \frac{1}{4\pi\epsilon_0}\frac{q}{|\vec{x}-\vec{y}|} + \frac{1}{4\pi\epsilon_0}\frac{(-a/y)q}{|\vec{x}-\frac{a^2}{y}\frac{\vec{y}}{y}|} \quad \frac{(-a/y)q}{|\vec{x}-\frac{a^2}{y}\frac{\vec{y}}{y}|} = -\frac{aq}{y|\frac{1}{y^2}(y^2\vec{x}-a^2\vec{y})|} = -\frac{aq}{\frac{y}{y}|y^2\vec{x}^2-a^2\vec{y}|}$$

$$= \frac{1}{4\pi\epsilon_0}\frac{q}{|\vec{x}-\vec{y}|} - \frac{aq}{4\pi\epsilon_0|y^2\vec{x}^2-a^2\vec{y}|}$$

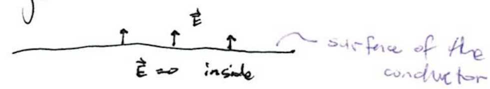
Note that

$$\Phi^{orig}_{outside} = \Phi^{image}_{outside} \Rightarrow \Phi^{orig}_{inside} \neq \Phi^{image}_{inside} \Rightarrow$$

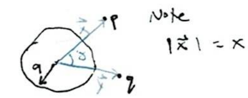


- The region where the image charge is located is where the image problem will not match w/ the orig. problem
- what is the charge distribution on the surface

Recall:



$\vec{E}$ just outside the surface is $\vec{E} = \frac{\rho}{\epsilon_0}\hat{n}$. Then,



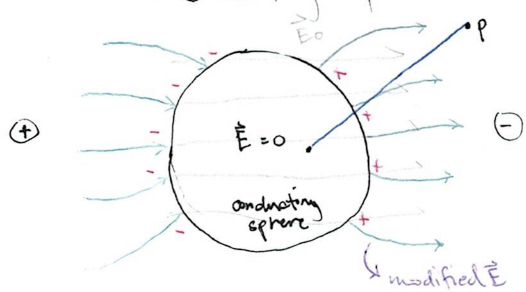
$$\rho = \epsilon_0 E = \epsilon_0(-\hat{n}\cdot\nabla\Phi) = -\epsilon_0 \left.\frac{\partial\Phi}{\partial x}\right|_{x=a}$$

Thus,

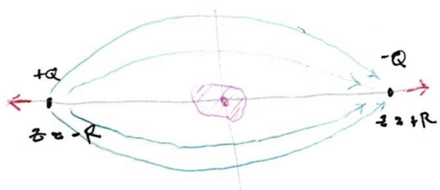
$$\rho = -\frac{q}{4\pi a^2}\left(\frac{a}{y}\right)\frac{1-\frac{a^2}{y^2}}{\left(1+\frac{a^2}{y^2}-2\frac{a}{y}\cos\gamma\right)^{3/2}}$$ angle bet. x & y

Oct. 15, 2020

- Lecture Notes
- Conducting Sphere in a uniform electric field by the method of images



Key idea: A uniform electric field can be thought of as being produced by appropriate + and − charges at infinity



Notice that as R ↑ & Q increases to compensate for the increase in R, the $\vec{E}$ in the neighborhood of the origin approaches a uniform value

The electric field at the origin is $\vec{E} = \frac{Q}{4\pi\epsilon_0 R^2} + \frac{Q}{4\pi\epsilon_0 R^2} = \frac{2Q}{4\pi\epsilon_0 R^2}$

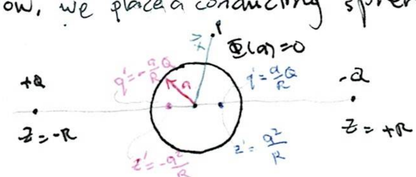
Note that at the origin: $E_0 \simeq \frac{2Q}{4\pi\epsilon_0 R^2}$

Take the limit as $R, Q \to \infty$ under the

$$\lim_{Q,R\to 0}\frac{Q}{R^2} = \text{constant} \longrightarrow \lim_{Q,R\to 0}\frac{2Q}{4\pi\epsilon_0 R^2} = E_0 = \text{constant}$$

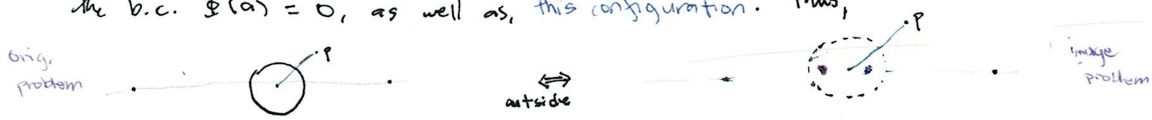
* Now, we place a conducting sphere at the origin

Legend:
- Image charge config. due to −Q
- Image charge config. due to +Q

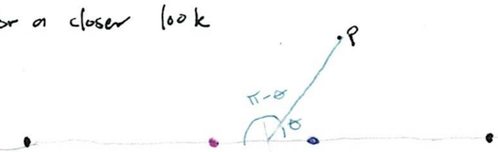


Replace the surface by a sys. of charges, real and image charges, that reproduce the given b.c. ← method of images

Consider +Q and −Q respectively. This configuration of real & image charges reproduces the b.c. $\Phi(a) = 0$, as well as, this configuration. Thus,



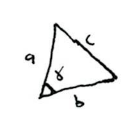
- For a closer look



Recall: Cosine law

$$c = \sqrt{a^2+b^2-2ab\cos\gamma}$$

$$\cos(\pi-\gamma) = -\cos\gamma$$



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The potential due to $+Q$ 10/15/20

$$\Phi_+ = \frac{Q}{4\pi\epsilon_0 (R^2 + r^2 - 2rR\cos(\pi-\theta))^{1/2}} = \frac{Q}{4\pi\epsilon_0 (R^2+r^2+2rR\cos\theta)^{1/2}}$$

and due to $-Q$

$$\Phi_- = \frac{-Q}{4\pi\epsilon_0 (R^2 + r^2 - 2rR\cos\theta)}$$

On the other hand, the potential due to the image charge of $+Q$ $(-Qa/R)$

$$\Phi'_+ = \frac{-Qa/R}{4\pi\epsilon_0 \left(r^2 + \frac{a^4}{R^2} - 2r\frac{a^2}{R}\cos(\pi-\theta)\right)^{1/2}} = -\frac{Qa/R}{4\pi\epsilon_0 \left(r^2 + \frac{a^4}{R^2} + 2r\frac{a^2}{R}\cos\theta\right)^{1/2}}$$

and due to the image charge of $-Q$ $(+Qa/R)$

$$\Phi'_- = \frac{Qa/R}{4\pi\epsilon_0 \left(r^2 + \frac{a^4}{R^2} + 2r\frac{a^2}{R}\cos\theta\right)^{1/2}}$$

Thus, we have

$$\Phi = \Phi_+ + \Phi_- + \Phi'_+ + \Phi'_-$$

$$= \frac{Q}{4\pi\epsilon_0 (r^2+R^2+2rR\cos\theta)^{1/2}} - \frac{Q}{4\pi\epsilon_0 (r^2+R^2+2R\cos\theta)^{1/2}} - \frac{Qa}{4\pi\epsilon_0 R\left(r^2+\frac{a^4}{R^2}+2\frac{a^2}{R}\cos\theta\right)^{1/2}} + \frac{Qa}{4\pi\epsilon_0 R\left(r^2+\frac{a^4}{R^2}+2\frac{a^2}{R}\cos\theta\right)^{1/2}}$$

Take the limit as $R, Q \to \infty$ to recover the original problem. Consider Φ_+:

$$\Phi_+ = \frac{Q}{4\pi\epsilon_0 (r^2+R^2+2rR\cos\theta)^{1/2}} = \frac{Q}{4\pi\epsilon_0 R\left(\frac{r^2}{R^2}+1+2\frac{r}{R}\cos\theta\right)^{1/2}} \approx \frac{Q}{4\pi\epsilon_0 R\left(1+2\frac{r}{R}\cos\theta\right)^{1/2}}$$

after factoring out R^2 in the denominator. Note that $r \ll R \to \frac{r}{R} \ll 1$

Also, $\frac{1}{(1+x)^{1/2}} = 1 - \frac{1}{2}x + O(x^2)$. Thus, as $O\left(\frac{r^2}{R^2}\right)$ becomes negligible since $R\to\infty$, we have

$$\Phi_+ = \frac{Q}{4\pi\epsilon_0 R}\left(1 - \frac{1}{2}\cdot 2\frac{r}{R}\cos\theta\right) = \frac{Q}{4\pi\epsilon_0 R}\left(1 - \frac{r}{R}\cos\theta\right)$$

In the same manner,

$$\Phi_- = -\frac{Q}{4\pi\epsilon_0 R}\left(1 + \frac{r}{R}\cos\theta\right)$$

Then,
$$\Phi_+ + \Phi_- = \frac{Q}{4\pi\epsilon_0 R}\left(\cancel{1} - \frac{r}{R}\cos\theta\right) + \frac{-Q}{4\pi\epsilon_0 R}\left(\cancel{1} + \frac{r}{R}\cos\theta\right) = -\frac{Q}{2\pi\epsilon_0 R^2} r\cos\theta$$

Now, we consider Φ'_+:

$$\Phi'_+ = \frac{-aQ}{4\pi\epsilon_0 R\left(r^2+\frac{a^4}{R^2}+2\frac{a^2}{R}r\cos\theta\right)^{1/2}} = -\frac{aQ}{4\pi\epsilon_0 Rr\left(1+\frac{a^4}{R^2r^2}+\frac{2a^2}{Rr}\cos\theta\right)^{1/2}} \approx \frac{-aQ}{4\pi\epsilon_0 Rr\left(1+\frac{2a^2}{Rr}\cos\theta\right)^{1/2}}$$

after factoring out r^2. Note that $a < r$, $\frac{a^4}{R^2r^2} \ll \frac{a^2}{Rr}$, $\frac{a^2}{Rr} \ll 1$. Then,

$$\Phi'_+ = \frac{-aQ}{4\pi\epsilon_0 Rr}\left(1 - \frac{1}{2}\, 2\frac{a^2}{Rr}\cos\theta\right) =$$

In the same manner,

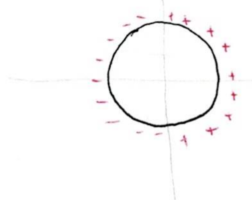
$$\Phi'_+ + \Phi'_- = -\frac{aQ}{4\pi\epsilon_0 Rr}\left(\cancel{1} - \frac{a^2}{Rr}\cos\theta\right) + \frac{aQ}{4\pi\epsilon_0 Rr}\left(\cancel{1} + \frac{a^2}{Rr}\cos\theta\right) = \frac{Q}{2\pi\epsilon_0 R^2}\frac{a^3}{r^2}\cos\theta$$

Therefore,

$$\Phi = \frac{Q}{2\pi\epsilon_0 R^2} r\cos\theta + \frac{Q}{2\pi\epsilon_0 R^2}\frac{a^3}{r^2}\cos\theta = -\frac{Q}{2\pi\epsilon_0 R^2}\left(r - \frac{a^3}{r^2}\right)\cos\theta$$

Taking the limit of Φ as $R, Q \to \infty$, we have

$$\lim_{R,Q\to\infty} \Phi = - \lim_{Q,R\to\infty} \underbrace{\frac{Q}{2\pi\epsilon_0 R^2}}_{E_0} \left(r - \frac{a^3}{r^2}\right)\cos\theta = -E_0\left(r - \frac{a^3}{r^2}\right)\cos\theta$$



Calculating for the charge density:

$$\rho = -\epsilon_0 \frac{\partial\Phi}{\partial r}\Big|_{r=a} = 3\epsilon_0 E_0 \cos\theta$$

$$\frac{\partial\Phi}{\partial r}\Big|_{r=a} = -E_0\left(1 + 2\frac{a^3}{r^3}\right)\cos\theta\Big|_{r=a}$$

$$= -E_0\left(1 + 2\cancelto{1}{\frac{a^3}{a^3}}\right)\cos\theta = -3E_0\cos\theta$$

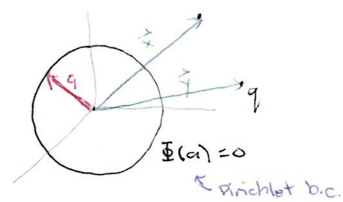
Oct. 22, 2020

- Green's function for the sphere

Recall: In

$$\Phi(\vec{x}) = \frac{1}{4\pi\epsilon_0}\int_V \rho(\vec{x}')\, G(\vec{x},\vec{x}')\, d^3x' + \frac{1}{4\pi}\oint_S \left[G(\vec{x},\vec{x}')\frac{\partial\Phi(\vec{x}')}{\partial n'} - \Phi(\vec{x}')\frac{\partial G(\vec{x},\vec{x}')}{\partial n'}\right] da'$$

Recall: The problem of the grounded sphere.



We found that:

$$\Phi(\vec{x}) = \frac{q}{4\pi\epsilon_0|\vec{x}-\vec{y}|} - \frac{aq}{4\pi\epsilon_0\, y\left|\vec{x} - \frac{a^2}{y^2}\vec{y}\right|}$$

$$= \frac{q}{4\pi\epsilon_0}\left[\frac{1}{|\vec{x}-\vec{y}|} - \frac{a}{y\left|\vec{x} - \frac{a^2}{y^2}\vec{y}\right|}\right]$$

We want $\vec{y} \to \vec{x}'$. So, $\Phi(\vec{x})$ can be rewritten as

$$\Phi(\vec{x}) = \frac{1}{4\pi\epsilon_0}\int_V \rho(\vec{x}')\, G(\vec{x},\vec{x}')\, d^3x', \qquad \rho(\vec{x}') = q\,\delta(\vec{x}'-\vec{y})$$

$$G(\vec{x},\vec{x}') = \frac{1}{|\vec{x}-\vec{x}'|} - \frac{a}{x'\left|\vec{x} - \frac{a^2}{x'^2}\vec{x}'\right|}$$

Then,

$$\Phi(\vec{x}) = \frac{1}{4\pi\epsilon_0}\int_V q\,\delta(\vec{x}'-\vec{x})\left[\frac{1}{|\vec{x}-\vec{x}'|} - \frac{a}{x'\left|\vec{x} - \frac{a^2}{x'^2}\vec{x}'\right|}\right] d^3x'$$

The volume

$$= \frac{1}{4\pi\epsilon_0}\, q\left[\frac{1}{|\vec{x}-\vec{y}|} - \quad y \quad x\right.$$

At $\vec{x}' = a$

$$G(\vec{x}, a) = \frac{1}{|\vec{x}-\vec{x}'|} - \frac{a}{\cancel{a}\left|\vec{x} - \frac{a^2}{a^2}\vec{x}'\right|} = 0$$

For the grounded sphere, at $\vec{x}' = a$,

$$\Phi(\vec{x}) = \ldots + \frac{1}{4\pi}\oint_S \left[\cancelto{0}{G(\vec{x},\vec{x}')}\frac{\partial\Phi}{\partial n'} - \cancelto{0}{\Phi(\vec{x}')}\frac{\partial G(\vec{x},\vec{x}')}{\partial n'}\right] da$$

$$= \frac{1}{4\pi\epsilon_0}\int_V \rho(\vec{x}')$$

This is the Green's fxn of a sphere satisfying a Dirichlet b.c.

Show that indeed: $\nabla^2 G(\vec{x},\vec{x}') = -4\pi\delta(\vec{x}-\vec{x}')$

We know that

$$\nabla^2\left(\frac{1}{|\vec{x}-\vec{x}'|}\right) = -4\pi\,\delta(\vec{x}-\vec{x}')$$

Recall

$$G(\vec{x},\vec{x}') = \frac{1}{|\vec{x}-\vec{x}'|} + F(\vec{x},\vec{x}'), \qquad \nabla^2 F(\vec{x},\vec{x}') = 0$$

For the case of the sphere, we have

$$F(\vec{x},\vec{x}') = -\frac{a}{x'\left|\vec{x}-\frac{a^2}{x'^2}\vec{x}'\right|}$$

So we need to show that this $F(\vec{x},\vec{x}')$ does satisfy $\nabla^2 F = 0$. Note that

$$\left|\vec{x}-\frac{a^2}{x'^2}\vec{x}'\right| = \left[\left(\vec{x}-\frac{a^2}{x'^2}\vec{x}'\right)\cdot\left(\vec{x}-\frac{a^2}{x'^2}\vec{x}'\right)\right]^{1/2}$$

$$= \left(\vec{x}^2 - 2\frac{a^2}{x'^2}\vec{x}\cdot\vec{x}' + \frac{a^4}{x'^4}\vec{x}'^2\right)^{1/2} \qquad \vec{x}\cdot\vec{x}' = x x'\cos\gamma$$

Then,

$$F(\vec{x},\vec{x}) = \frac{1}{\left(\frac{x^2}{a^2}x'^2 + a^2 - 2xx'\cos\gamma\right)^{1/2}}$$

To simplify the calculation of $\nabla^2 F$, align the z-axis along the $\vec{x}$ vector

Then,

$$\nabla^2 F = \left[\frac{1}{x'^2}\frac{\partial}{\partial x'}\left(x'^2\frac{\partial}{\partial x'}\right) + \frac{1}{x'^2\sin\gamma}\frac{\partial}{\partial\gamma}\left(\sin\gamma\frac{\partial}{\partial\gamma}\right)\right]F$$

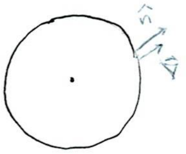
$=$

$= 0$

which shows that

Then, in general, the potential outside the sphere is given by

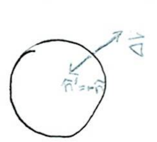
$$\Phi(\vec{x}) = \frac{1}{4\pi\epsilon_0}\int_V \rho(\vec{x}')\,G(\vec{x},\vec{x}')\,d^3x' - \frac{1}{4\pi}\int_S \Phi(\vec{x}')\frac{\partial G(\vec{x},\vec{x}')}{\partial n'}\,d^2a'$$



The directional derivative is along the radial direction:

$$\left.\frac{\partial G}{\partial n'}\right|_{x=a} = \hat{n}\cdot\vec{\nabla}G\Big|_{x'=a} = \left.\frac{\partial G}{\partial x'}\right|_{x'=a} = -\frac{x^2-a^2}{a(x^2+a^2-2ax\cos\gamma)^{3/2}}$$

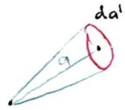
What about the charge when we solve for $\Phi(\vec{x})$ inside the sphere?



$$\frac{\partial G}{\partial n'} = \hat{n}'\cdot\nabla G = -\hat{n}\cdot\vec{\nabla}G = -\frac{\partial G}{\partial x'}$$

$da' = a^2\, d\Omega'$

$d\Omega' =$



In the absence of charge distribution $\rho(\vec{x})$, i.e. $\rho(\vec{x}) = 0$,

$$\Phi(\vec{x}) = -\frac{1}{4\pi}\int_S \Phi(\vec{x}')\frac{\partial G(\vec{x},\vec{x}')}{\partial n'}\,d^3x'$$

$$= -\frac{1}{4\pi}\int \Phi(\vec{x}')\frac{\partial G(\vec{x},\vec{x}')}{\partial n'}\,a^2\,d\Omega'$$

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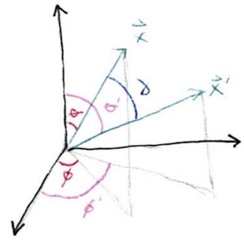


Hence, for the exterior

10/22/20

$$\Phi(\vec{x}) = -\frac{1}{4\pi}\int_S \Phi(a,\theta',\phi')\,\frac{(-1)\,(x^2-a^2)\,a^2}{a\,(x^2+a^2-2ax\cos\gamma)^{3/2}}$$

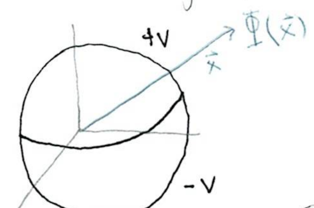
$$= \frac{1}{4\pi}\int_S \Phi(a,\theta',\phi')\,\frac{a\,(x^2-a^2)\;d\Omega'}{(x^2+a^2-2ax\cos\gamma)^{3/2}}, \qquad d\Omega' = \sin\, d\theta'\,d\phi'$$

Note that $\cos\gamma = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\phi-\phi')$. For the interior problem,

$$(x^2-a^2) \longrightarrow -(x^2-a^2) = (a^2-x^2)$$

Oct. 29, 2020

- Lecture Notes
- Conducting sphere w/ hemispheres at diff. potentials



Outside the sphere, there is no charge distribution.

$$\rho(\vec{x}') = 0, \quad \vec{x}' \text{ is ext. to the sphere}$$

Recall: the gen. soln is the potential

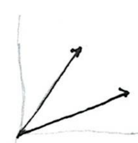
$$\Phi(\vec{x}) = \frac{1}{4\pi\epsilon_0}\int_V \rho(\vec{x}')\,G(\vec{x},\vec{x}')\,d^3x' + \frac{1}{4\pi}\oint_S\left[G(\vec{x},\vec{x}')\frac{\partial\Phi}{\partial n'} - \Phi(\vec{x}')\frac{\partial G(\vec{x},\vec{x}')}{\partial n'}\right]da'$$

The volume V is the space ext. to the sphere. The bounding surface S is the surface of the sphere.

The potential is specified on the surface w/c implies Dirichlet b.c.

$$G_D(\vec{x},\vec{x}') = 0, \quad \text{for all } \vec{x}' \text{ on boundary surface } S$$

Then,



$$\Phi(\vec{x}) = -\frac{1}{4\pi}\oint_S \Phi(\vec{x}')\,\frac{\partial G(\vec{x},\vec{x}')}{\partial n'}\,da'$$

Recall: The Dirichlet Green's fxn for the sphere is

$$G(\vec{x},\vec{x}') = \frac{1}{|\vec{x}-\vec{x}'|} - \frac{a}{x'\left|\vec{x}-\vec{x}'\frac{a^2}{x'^2}\right|}, \qquad |\vec{x}-\vec{x}'| = \sqrt{(\vec{x}-\vec{x}')\cdot(\vec{x}-\vec{x}')}$$

Thus,

$$\vec{x} = x\sin\theta\cos\phi\,\hat{i} + x\sin\theta\sin\phi\,\hat{j} + x\cos\theta\,\hat{k}$$

$$\vec{x}' = x'\sin\theta'\cos\phi'\,\hat{i} + x'\sin\theta'\sin\phi'\,\hat{j} + x'\cos\theta'\,\hat{k}$$

$$\vec{x}-\vec{x}' = (x\sin\theta\cos\phi - x\sin\theta\cos\phi)\,\hat{i} + (x\sin\theta\sin\phi - x\sin\theta\sin\phi)\,\hat{j} + (x\cos\theta - x'\cos\theta')\,\hat{k}$$

$$(\vec{x}-\vec{x}')\cdot(\vec{x}-\vec{x}') = (x\sin\theta\cos\phi - x'\sin\theta'\cos\phi')^2 + (x\sin\theta\sin\phi - x'\sin\theta'\sin\phi')^2 + (x\cos\theta - x'\cos\theta')^2$$

$$= x^2\sin^2\theta\cos^2\theta - 2xx'\sin\theta\cos\phi\sin\theta'\cos\phi' + x'^2\sin^2\theta'\cos^2\phi'$$
$$+ x^2\sin^2\theta\sin^2\theta - 2xx'\sin\theta\sin\phi\sin\theta'\sin\phi' + x'^2\sin^2\theta'\sin\phi'$$
$$+ x^2\cos^2\theta - 2xx'\cos\theta\cos\theta' + x'^2\cos^2\theta'$$

$$= x^2\sin^2\theta(\cos^2\phi + \sin^2\phi) - xx'\sin\theta\sin\theta'(\cos\phi\cos\phi' + \sin\phi\sin\phi')$$
$$+ x'^2\sin^2\theta'(\cos^2\phi' + \sin^2\phi') + x^2\cos^2\theta - 2xx'\cos\theta\cos\theta' +$$

$$= x^2 + x'^2 - 2xx'\sin\theta\sin\theta'\cos(\phi-\phi') - 2xx'\cos\theta\cos\theta'$$

$$= x^2 + x'^2 - 2xx'\left[\sin\theta\sin\theta'\cos(\phi-\phi') + \cos\theta\cos\theta'\right]$$

Note: 10/29/20

$(\vec{x}-\vec{x}')\cdot(\vec{x}\cdot\vec{x}') = \vec{x}\cdot\vec{x} - \vec{x}\cdot\vec{x}' - \vec{x}'\cdot\vec{x} + \vec{x}'\cdot\vec{x}' = x^2 + x'^2 - 2xx'\cos\gamma$

Comparing the two expressions:

$$\cos\gamma = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\phi-\phi')$$

Then, $|\vec{x}-\vec{x}'| = (x^2 + x'^2 - 2xx'\cos\gamma)^{1/2}$

$$\left|\vec{x} - \frac{a^2}{x'^2}\vec{x}'\right| = \sqrt{\left(\vec{x} - \frac{a^2}{x'^2}\vec{x}'\right)\cdot\left(\vec{x} - \frac{a^2}{x'^2}\vec{x}\right)}$$

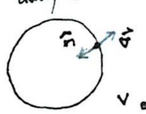
$$= \sqrt{x^2 - 2\frac{a^2}{x^2}\vec{x}'\cdot\vec{x} + \frac{a^4}{x'^4}x'^2}$$

$$= \sqrt{x^2 - 2\frac{a^2}{x'^2}xx'\cos\gamma + \frac{a^4}{x'^4}x'^2}$$

$$= \frac{a}{x'}\left(\frac{x'^2}{a^2}x^2 - 2xx'\cos\gamma + a^2\right)^{1/2}$$

The Dirichlet Green's function is now

$\hat{n}$ always points away from V



V ext. to the sphere

$$G_D(\vec{x},\vec{x}') = \frac{1}{(x^2+x'^2-2xx'\cos\gamma)^{1/2}} - \frac{\cancel{a}}{\cancel{x'}\frac{a}{\cancel{x}}\left(x'^2\frac{x^2}{a^2} - 2xx'\cos\gamma + a^2\right)^{1/2}}$$

The directional derivative is

$$\frac{\partial G}{\partial n'} = \hat{n}\cdot\nabla G = -\frac{\partial G}{\partial x'} = -\left[-\frac{1}{2}\frac{2x'-2x\cos\gamma}{(x^2+x'^2-2xx'\cos\gamma)^{3/2}} + \frac{1}{2}\frac{2x'(x^2/a^2) - 2x\cos\gamma}{\left(\frac{x'^2}{a^2}x^2 - 2xx'\cos\gamma + a^2\right)^{3/2}}\right]$$

Thus, $\frac{\partial G}{\partial n'} = -\frac{x^2-a^2}{a(x^2-2xa\cos\gamma+a^2)^{3/2}}$ ← after evaluating $x'=a$

The potential is now

$$\Phi(\vec{x}) = \frac{1}{4\pi}\oint_S \Phi(a,\theta',\phi')\frac{(x^2-a^2)\,a\sin\theta'\,d\theta'\,d\phi'}{(x^2+a^2-2xa\cos\gamma)^{3/2}}$$

$$= \frac{(x^2-a^2)a}{4\pi}\oint_S \Phi(a,\theta',\phi')\frac{\sin\theta'\,d\theta'\,d\phi'}{(x^2+a^2-2xa\cos\gamma)^{3/2}}$$

For the given problem: $\Phi(a,\theta',\phi') = \begin{cases} V, & 0\le\theta'\le\pi/2 \\ -V, & \pi/2\le\theta'\le\pi\end{cases}$

Then,

$$\Phi(\vec{x}') = \frac{(x^2-a^2)a}{4\pi}\int_0^{2\pi}d\phi'\left[\int_0^{\pi/2}d\theta'\frac{V\sin\theta'}{(x^2+a^2-2xa\cos\gamma)^{3/2}} + \int_{\pi/2}^{\pi}d\theta'\frac{(-V)\sin\theta'}{(x^2+a^2-2xa\cos\gamma)^{3/2}}\right]$$

Let $u=\cos\theta'$, $du = \sin\theta'\,d\theta'$

$$\Phi(\vec{x}) = \frac{a(x^2-a^2)}{4\pi}V\int_0^{2\pi}\left[\int_1^0\frac{du}{(x^2+a^2-2xa\cos\gamma)^{3/2}} - \int_0^1\frac{du}{(x^2+a^2-2xa\cos\gamma)^{3/2}}\right]d\phi'$$

$$= \frac{a(x^2-a^2)}{4\pi}V\int_0^{2\pi}d\phi'\left[\int_0^1\frac{d(\cos\theta')}{(a^2+x^2-2ax\cos\gamma)^{3/2}} - \int_{-1}^0\frac{d(\cos\theta')}{(a^2+x^2-2ax\cos\gamma)^{3/2}}\right]$$

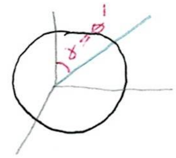
We want to combine the 2nd integral so let $\theta' \to \pi-\theta'$, $\phi'\to\phi'+\pi$,

$\cos\theta' \to \cos(\pi-\theta') = -\cos\theta'$, $\cos\gamma \to \cos\theta\cos(\pi-\theta') + \sin\theta\sin(\pi-\theta')\cos(\phi'+\pi) = -\cos\gamma$

$\int_{-1}^0 d\theta' \to -\int_1^0 d\theta'$

$$\Phi(\vec{x},\theta,\phi) = \frac{Va(x^2-a^2)}{4\pi}\int_0^{2\pi}d\phi'\int_0^1 d(\cos\theta')\left[(a^2+x^2-2ax\cos\gamma)^{3/2} - (a^2+x^2+2ax\cos\gamma)^{3/2}\right]$$

Suppose we want to find the potential along the z-axis 10/29/20



$x = z, \quad \theta = 0, \quad \sin\theta = 0, \quad \cos\gamma = \cos\theta'$

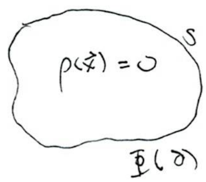
Then,

$$\Phi(x,\theta,\phi) = \frac{Va(z^2-a^2)}{4\pi}\int_0^{2\pi} d\phi' \int_0^1 d(\cos\theta')\left[(a^2+z^2-2az\cos\theta')^{3/2} - (a^2+z^2+2az\cos\theta')^{3/2}\right]$$

$$= V\left(1 - \frac{z^2-a}{z\sqrt{z^2+a^2}}\right)$$

Nov. 10, 2020

▶ Electric Potential in the absence of charged distributions



The Poisson eqn ($\vec{\nabla}^2 \Phi = \frac{1}{\epsilon_0}$) reduces to the Laplace eqn ($\vec{\nabla}^2 \Phi = 0$)

Solving the Laplace eqn leads to the

Consider fxns defined in the interval (a,b) in some variable ξ. A set of fxns $\{u_1(\xi), u_2(\xi) \ldots\}$ in (a,b) is called an orthogonal set if

$$\int_a^b u_n^*(\xi)\, u_m(\xi)\, d\xi = 0,$$

The set is called complete when it satisfies the closure relation

$$\sum_{n=1} u_n^*(\xi)\, u_n(\xi') = \delta(\xi-\xi')$$

Let $f(\xi)$ be a fxn defined in the interval (a,b). Note that $\delta(\xi-\xi') = \delta(\xi'-\xi)$.
Let $\xi \to \xi'$. Then,

$$\int_a^b f(\xi')\left[\sum_{n=1} u_n^*(\xi')\, u_n(\xi) = \delta(\xi'-\xi)\right] d\xi'$$

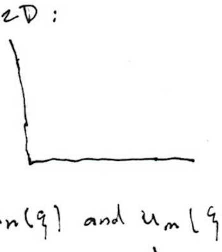
$$\rightarrow \sum_{n=1}\left[\int_a^b f(\xi')\, u_n^*(\xi')\, d\xi'\right] u_n(\xi) = \int_a^b f(\xi')\,\delta(\xi'-\xi)\, d\xi'$$

$$= f(\xi)$$

Note that $f(\xi)$ admits the expansion in 1D:

$$f(\xi) = \sum_{n=1} a_n u_n(\xi) \quad \text{where} \quad a_n = \int_a^b u_n^*(\xi')\, f(\xi')\, d\xi'$$

In 2D:



$u_n(\xi) : (a,b)$ and $v(\eta) : (c,d)$

If $u_n(\xi)$ and $u_m(\xi)$ are orthogonal and complete,

$$\int_a^b u_n^*(\xi)\, u_m(\xi)\, d\xi = \delta_{nm}$$

$$\sum^{\infty} u_n^*(\xi)\, u_n(\xi') = \delta(\xi-\xi')$$

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and
$$\int_c^d v_n^*(\eta)\, v_m(\eta)\, d\eta = \delta_{nm}$$

$$\sum_{n=1}^{\infty} v_n^*(\eta)\, v_n(\eta') = \delta(\eta-\eta')$$

Then the set of fxns $\{u_n(\xi)\, v_m(\eta) = \psi_{nm}(\xi,\eta)\}$ form a complete and orthogonal set in the region $(a,b)\times(c,d)$

orthogonality: $\displaystyle\int_a^b\int_c^d \psi_{nm}^*(\xi,\eta)\,\psi_{n'm'}(\xi,\eta)\, d\eta\, d\xi = \delta_{nn'}\,\delta_{mm'}$

completeness $\displaystyle\sum_{n,m=1}^{\infty} \psi_{nm}^*(\xi,\eta)\,\psi_{nm}(\xi',\eta') = \sum_{n,m=1}^{\infty} u_n^*(\xi)\, v_m(\eta)\, u_n(\xi')\, v_m(\eta')$

$$= \left(\sum_{n=1}^{\infty} u_n^*(\xi)\, u_n(\xi')\right)\left(\sum_{m=1}^{\infty} v_m^*(\eta)\, v_m(\eta')\right)$$

$$= \delta(\xi-\xi')\,\delta(\eta-\eta')$$

In 3D:

$u_n(\xi) : (a,b), \quad v_m(\eta) : (c,b), \quad S_\ell(\sigma) : f, g)$

Assuming these to be complete and orthogonal, then the set of fxns $\{\psi_{nm\ell}(\xi,\eta,\sigma) = u_n(\xi)\, v_m(\eta)\, S_\ell(\sigma)\}$ form a complete and orthonormal set.

orthogonality: $\displaystyle\int_a^b\int_c^d\int_f^g \psi_{nm\ell}^*(\xi,\eta,\sigma)\,\psi_{n'm'\ell'}(\xi,\eta,\sigma)\, d\sigma\, d\eta\, d\xi = \delta_{nn'}\,\delta_{mm'}\,\delta_{\ell\ell'}$

completeness: $\displaystyle\sum_{n,m,\ell=1}^{\infty} \psi_{nm\ell}^*(\xi,\eta,\sigma)\,\psi_{nm\ell}(\xi',\eta',\sigma') = \delta(\xi-\xi')\,\delta(\eta-\eta')\,\delta(\sigma-\sigma')$

From the completeness relation

$$f(\xi,\eta,\sigma) = \sum_{n,m,\ell=1}^{\infty} a_{nm\ell}\,\psi_{nm\ell}(\xi,\eta,\sigma)$$

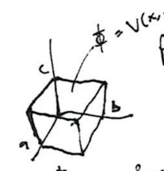
where $\displaystyle a_{nm\ell} = \int_a^b\int_c^d\int_f^g \psi_{nm\ell}^*(\xi,\eta,\sigma)\, f(\xi,\eta,\sigma)\, d\sigma\, d\eta\, d\xi$

A fxn that has a continuous index, say $f_k(\xi)$, where $-\infty < a < \infty$ can be orthogonal and complete

orthogonality: $\displaystyle\int_a^b f_k^*(\xi)\, f_{k'}(\xi)\, d\xi = \delta(k-k')$

completeness: $\displaystyle\int_{-\infty}^{\infty} f_k^*(\xi)\, f_k(\xi')\, dk = \delta(\xi-\xi')$

★ The Hollow Rectangular Box Problem



$\rho(\vec{x}) = 0$ inside the box

$\Phi = 0$ for all surfaces except the top

What is the potential inside the box?

Soln: Since $\rho(\vec{x})$ inside the box, we solve the Laplace eqn inside it wrt the given b.c.

Note that there is rectangular symmetry, so we use rectangular coord sys (cartesian):

$$\vec{\nabla}^2\Phi = 0 \Rightarrow \frac{\partial^2\Phi}{\partial x^2} + \frac{\partial^2\Phi}{\partial y^2} + \frac{\partial^2\Phi}{\partial z^2} = 0$$

Assume as a separable solution: $\Phi(x,y,z) = X(x)\, Y(y)\, Z(z)$

Substitute and divide

$$\frac{1}{X}\frac{d^2X}{dx^2} + \frac{1}{Y}\frac{d^2Y}{dy} + \frac{1}{Z}\frac{d^2Z}{dz} = 0$$

Each term depends on one variable only so that each term is equal to a constant

$$\frac{1}{X}\frac{d^2X}{dx} = -\alpha^2, \quad \frac{1}{Y}\frac{d^2Y}{dy^2} = -\beta^2, \Rightarrow -\alpha^2 - \beta^2 + \frac{1}{Z}\frac{d^2Z}{dz^2} = 0$$

$$\hookrightarrow \frac{1}{Z}\frac{d^2Z}{dz^2} = \alpha^2 + \beta^2 = \gamma^2$$

Thus, we need to solve for

$$\frac{d^2X}{dx} + \alpha^2 X = 0, \quad \frac{d^2Y}{dy} + \beta^2 Y = 0, \quad \frac{d^2Z}{dz^2} - \gamma^2 Z = 0$$

Assuming that $\alpha, \beta > 0$, we have

$$\frac{d^2X}{dx} + \alpha^2 X = 0 \Rightarrow X = A e^{i\alpha x} + B e^{-i\alpha x} = C \sin(\alpha x) + D\cos(\alpha x)$$

$$\frac{d^2Y}{dy} + \beta^2 Y = 0 \Rightarrow Y = C' \sin(\beta y) + D'\cos(\beta y)$$

$$\frac{d^2Z}{dz} - \gamma^2 Z = 0 \Rightarrow Z = A'' e^{\gamma z} + B'' e^{-\gamma z} = C'' \sinh(\gamma z) + D\cosh(\gamma z)$$

We chose the eqn for Z to be equated to a positive constant because of b.c.
Construct the appropriate relation by imposing the b.c.:

- At the origin: $(x, y, z) = (0, 0, 0) \to \Phi = 0$

  The solns that satisfy this condition are

  $$X(x) = C\sin(\alpha x) \qquad Y(y) = C'\sin(\beta y) \qquad Z = C''\sinh\left(\sqrt{\alpha^2+\beta^2}\, z\right)$$

- At $x = a$; $y = b$: $\Phi = 0$

  $$X(a) = C\sin(\alpha a) = 0 \quad \to \quad \alpha a = n\pi \to \alpha_n = \frac{n\pi}{a}$$

  $$Y(b) = C'\sin(\beta b) = 0 \quad \to \quad \beta b = m\pi \to \beta_m = \frac{m\pi}{b}$$

  $$Z(z) = C''\sinh\left(\pi\sqrt{\frac{n^2}{a^2} + \frac{m^2}{b^2}}\, z\right) \quad \text{where } \gamma_{nm} = \pi\sqrt{\frac{n^2}{a^2} + \frac{m^2}{b^2}}$$

The product: $X(x)\,Y(y)\,Z(z) = \sin\left(\frac{n\pi}{a}x\right)\sin\left(\frac{m\pi}{b}y\right)\sinh(\gamma_{nm} z)$
is a soln to the Laplace eqn and satisfies the b.c. at $x = y = z = 0$ and at $x = a$, $y = b$. BUT it does NOT SATISFY the condition at $z = c$ b/c at $z = c$:
$\Phi(x,y,z) = V(x,y) \neq XYZ$ as $V(x,y)$ is arbitrary.

What do we do now? We appeal to the fact $\sin\left(\frac{n\pi}{a}x\right)$ and $\sin\left(\frac{m\pi}{b}y\right)$ are orthogonal and complete at the interval $(0,a)$ and $(0,b)$, respectively.

$$\int_0^a \sin\left(\frac{n\pi}{a}x\right)\sin\left(\frac{m\pi}{a}x\right)dx = \frac{a}{2}\delta_{nm}$$

By normalizing: $\sin\left(\frac{n\pi}{a}x\right) \to \sqrt{\frac{2}{a}}\sin\left(\frac{n\pi}{a}x\right)$

$$\sum_{n=1}^{\infty}\sqrt{\frac{2}{a}}\sin\left(\frac{n\pi}{a}x\right)\sqrt{\frac{2}{a}}\cdot\sin\left(\frac{n\pi}{a}x'\right) = \delta(x-x')$$

We do the same for y:

$$\int_0^b \sin\left(\frac{n\pi}{b} y\right) \sin\left(\frac{m\pi}{b} y\right) dy = \frac{b}{2} \delta_{nm}$$

$$\sum_{m=1}^{\infty} \frac{2}{b} \sin\left(\frac{m\pi}{b} y\right) \sin\left(\frac{m\pi}{b} y'\right) = \delta(y-y')$$

Construct a soln by using superposition

$$\Phi(x,y,z) = \sum_{nm=1}^{\infty} A_{nm} \sin(\alpha_n x) \sin(\beta_m y) \sin(\gamma_{nm} z)$$

Since the Laplace eqn is linear, their linear superposition is a soln

$$\Phi(x,y,c) = V(x,y)$$

$$= \sum_{n,m}^{\infty} A_{nm} \sin(\alpha_n x) \sin(\beta_m y) \sinh(\gamma_{nm} c)$$

Using the orthogonality of solns

$$A_{nm} = \frac{4}{ab \sinh(\gamma_{nm} c)} \int_0^a \int_0^b V(x,y) \sin(\alpha_n x) \sin(\beta_m y)\, dy\, dx$$