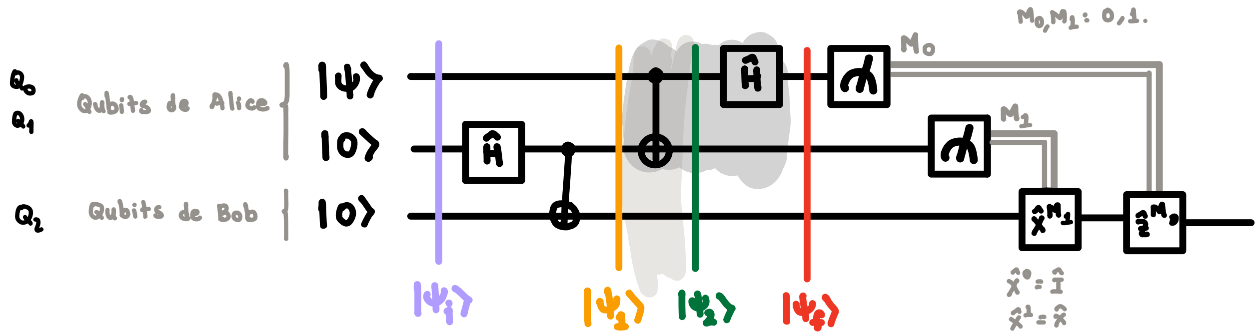
Día 3: Algoritmos cuánticos I

31/07/2024

Algoritmo de teleportación cuántica

Set-up del problema: Alice tiene un estado de qubit que desea transmitir a Bob. ¿Qué protocolo debe emplear ella para lograrlo?



$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$

Se hará el circuito con el estado inicial $|\psi_i\rangle = \underbrace{|\psi\rangle}_{Q0}\underbrace{|0\rangle}_{Q1}\underbrace{|0\rangle}_{Q2}$.

i) Paso 1:

$$|\psi_i\rangle = \underbrace{|\psi\rangle}_{Q1}\underbrace{|0\rangle}_{Q2}\underbrace{|0\rangle}_{Q3}$$

ii) Paso 2:

$$|\psi_1\rangle = (\hat{I}_0 \otimes \widehat{CNOT}_{12})(\hat{I}_0 \otimes \hat{H}_1 \otimes \hat{I}_2)|\psi_i\rangle$$

$$= |\psi\rangle|\beta_{00}\rangle$$

$$= (\alpha|0\rangle + \beta|1\rangle)\left(\frac{|00\rangle + |11\rangle}{\sqrt{2}}\right)$$

$$= \frac{\alpha|000\rangle + \alpha|011\rangle + \beta|100\rangle + \beta|111\rangle}{\sqrt{2}}$$

iii) Paso 3:

$$|\psi_2\rangle = \widehat{CNOT}_{01}|\psi_1\rangle = \widehat{CNOT}_{01}\left(\frac{\overbrace{\alpha|000\rangle + \alpha|011\rangle}^{\text{No son mod.}} + \overbrace{\beta|100\rangle + \beta|111\rangle}^{\text{Sí son mod.}}}{\sqrt{2}}\right)$$

$$= \frac{1}{\sqrt{2}}\left(\alpha|000\rangle + \alpha|011\rangle + \beta|110\rangle + \beta|101\rangle\right)$$

$$\hat{H}|0\rangle = |x;+\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}}$$

$$\hat{H}|1\rangle = |x;-\rangle = \frac{|0\rangle - |1\rangle}{\sqrt{2}}$$

iv) Paso 4:

$$|\psi_3\rangle = (\hat{H}_0 \otimes \hat{I} \otimes \hat{I})|\psi_2\rangle = (\hat{H}_0 \otimes \hat{I} \otimes \hat{I})\frac{1}{\sqrt{2}}\left(\alpha|000\rangle + \alpha|011\rangle + \beta|110\rangle + \beta|101\rangle\right)$$

$$= \frac{1}{\sqrt{2}}\left(\alpha\left(\frac{|0\rangle + |1\rangle}{\sqrt{2}}\right)|00\rangle + \alpha\left(\frac{|0\rangle + |1\rangle}{\sqrt{2}}\right)|11\rangle + \beta\left(\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right)|10\rangle + \beta\left(\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right)|01\rangle\right)$$

$$= \frac{1}{2}\left(\alpha|000\rangle + \alpha|100\rangle + \alpha|011\rangle + \alpha|111\rangle + \beta|010\rangle - \beta|110\rangle + \beta|001\rangle - \beta|101\rangle\right)$$

$$|\psi_f\rangle = \frac{1}{2}\left[|00\rangle\left(\alpha|0\rangle+\beta|1\rangle\right) + |01\rangle\left(\alpha|1\rangle+\beta|0\rangle\right) + |10\rangle\left(\alpha|0\rangle-\beta|1\rangle\right) + |11\rangle\left(\alpha|1\rangle-\beta|0\rangle\right)\right]$$

Si se hace una medición y se obtiene:

$(M_0, M_1) = (0,0)$:

$$\alpha|0\rangle+\beta|1\rangle = |\psi\rangle$$

$$|\psi_f\rangle = |00\rangle\left(\alpha|0\rangle+\beta|1\rangle\right)$$

$(M_0, M_1) = (0,1)$:

$$\hat{x}\left(\alpha|1\rangle+\beta|0\rangle\right) = \alpha|0\rangle+\beta|1\rangle = |\psi\rangle$$

$$|\psi_f\rangle = |01\rangle\left(\alpha|1\rangle+\beta|0\rangle\right)$$

$(M_0, M_1) = (1,0)$

$$\hat{z}\left(\alpha|0\rangle-\beta|1\rangle\right) = \alpha|0\rangle+\beta|1\rangle = |\psi\rangle$$

$(M_0, M_1) = (1,1)$

$$\hat{z}\hat{x}\left(\alpha|1\rangle-\beta|0\rangle\right) = \hat{z}\left(\alpha|0\rangle-\beta|1\rangle\right)$$
$$= \alpha|0\rangle+\beta|1\rangle = |\psi$$

Para el estado inicial $|\psi_i\rangle = |\psi\rangle|0\rangle|0\rangle$,

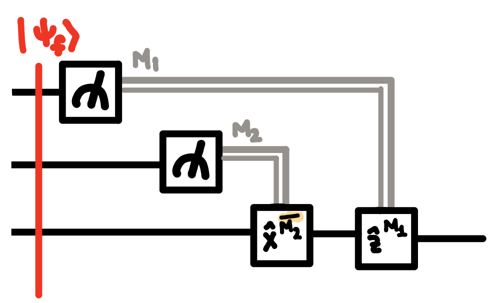
Alice mide		Bob obtiene	Para recuperar $|\psi\rangle$, Bob requiere aplicar
M_1	M_2		
0	0	$\alpha|0\rangle+\beta|1\rangle$	$\hat{I}$ $(\hat{z}^0\hat{x}^0)$
0	1	$\alpha|1\rangle+\beta|0\rangle$	$\hat{x}$ $(\hat{z}^0\hat{x}^1)$
1	0	$\alpha|0\rangle-\beta|1\rangle$	$\hat{z}$ $(\hat{z}^1\hat{x}^0)$
1	1	$\alpha|1\rangle-\beta|0\rangle$	$\hat{z}\hat{x}$ $(\hat{z}^1\hat{x}^1)$

$(\hat{z}^{M_1}\hat{x}^{M_2})$

Para el estado inicial $|\psi_i\rangle = |\psi\rangle|0\rangle|1\rangle$,

Alice mide		Bob obtiene	Para recuperar $|\psi\rangle$, Bob requiere aplicar
M_1	M_2		
0	0	$\alpha|1\rangle+\beta|0\rangle$	$\hat{x}$ $(\hat{z}^0\hat{x}^1)$
0	1	$\alpha|0\rangle+\beta|1\rangle$	$\hat{I}$ $(\hat{z}^0\hat{x}^0)$
1	0	$\alpha|1\rangle-\beta|0\rangle$	$\hat{z}\hat{x}$ $(\hat{z}^1\hat{x}^1)$
1	1	$\alpha|0\rangle-\beta|1\rangle$	$\hat{z}$ $(\hat{z}^1\hat{x}^0)$

$(\hat{z}^{M_1}\hat{x}^{\bar{M}_2})$



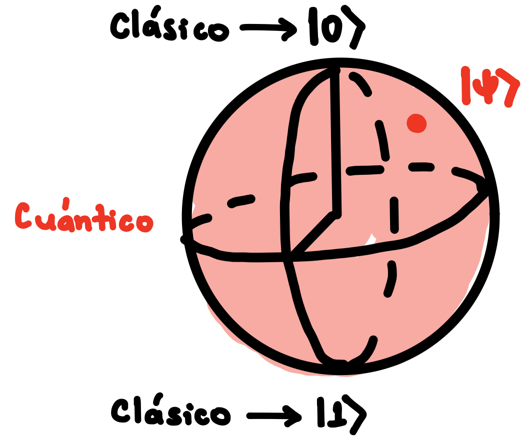
Tarea opcional: Resolver el protocolo de teleportación con los estados iniciales

Algoritmo de codificación superdensa

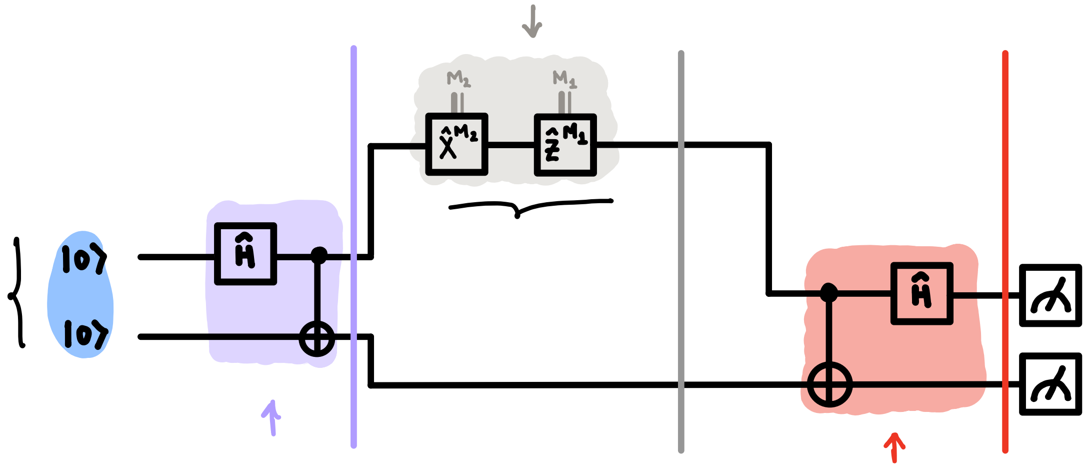
- ¿Cuánta información puede portar un qubit?

- ¿Cuántos bits clásicos pueden extraerse de un qubit?

  R: A lo más, un bit clásico de información por qubit (Teorema de Holevo).



Set-up del problema: Alice quiere hacerle llegar 2 bits clásicos a Bob. Para ello, necesita 2 qubits.



Paso 1: Generación del estado de Bell, $|\beta_{00}\rangle$.

$$|\psi_{00}\rangle = |\beta_{00}\rangle$$

Paso 2: La decisión de Alice.

(M_0, M_1)	Compuerta aplicada
$(0,0)$	$\hat{Z}^0\hat{X}^0 \otimes \hat{I} = \hat{I}\otimes\hat{I}$
$(0,1)$	$\hat{Z}^0\hat{X}^1 \otimes \hat{I} = \hat{X}\otimes\hat{I}$
$(1,0)$	$\hat{Z}^1\hat{X}^0 \otimes \hat{I} = \hat{Z}\otimes\hat{I}$
$(1,1)$	$\hat{Z}^1\hat{X}^1 \otimes \hat{I} = \hat{Z}\hat{X}\otimes\hat{I}$

Caso 1: $(M_1, M_2) = (0,0)$

$$|\psi_2\rangle = (\hat{I} \otimes \hat{I}) |\beta_{00}\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}} = |\beta_{00}\rangle$$

Caso 2: $(M_1, M_2) = (0,1)$

$$|\psi_2\rangle = (\hat{X} \otimes \hat{I}) |\beta_{00}\rangle = \frac{|10\rangle + |01\rangle}{\sqrt{2}} = |\beta_{10}\rangle$$

Caso 3: $(M_1, M_2) = (1,0)$

$$|\psi_2\rangle = (\hat{Z} \otimes \hat{I}) |\beta_{00}\rangle = \frac{|00\rangle - |11\rangle}{\sqrt{2}} = |\beta_{01}\rangle$$

Caso 4: $(M_1, M_2) = (1,1)$

$$|\psi_2\rangle = (\hat{Z}\hat{X} \otimes \hat{I}) |\beta_{00}\rangle = \frac{-|10\rangle + |01\rangle}{\sqrt{2}} = |\beta_{11}\rangle$$

Paso 3: Retorno a la base computacional.

Caso 1: $(M_1, M_2) = (0,0)$

$$\begin{aligned}
|\psi_2\rangle &= (\hat{H} \otimes \hat{I})\, \widehat{CNOT}\, |\beta_{00}\rangle \\
&= (\hat{H} \otimes \hat{I})\, \widehat{CNOT}\, \frac{|00\rangle + |11\rangle}{\sqrt{2}} \\
&= (\hat{H} \otimes \hat{I}) \left(\frac{|00\rangle + |10\rangle}{\sqrt{2}} \right) \\
&= \frac{1}{2} \Big((|0\rangle + |1\rangle)|0\rangle + (|0\rangle - |1\rangle)|0\rangle \Big) \\
&= |00\rangle
\end{aligned}$$

Caso 2: $(M_1, M_2) = (0,1)$

Se obtiene $|\psi_2\rangle = |01\rangle$.

$Pr(01) = 1$

$Pr(00) = Pr(10) = Pr(11) = 0$

Caso 3: $(M_1, M_2) = (1,0)$

Se obtiene $|\psi_2\rangle = |10\rangle$.

$Pr(10) = 1$

$Pr(00) = Pr(01) = Pr(11) = 0$

Caso 4: $(M_1, M_2) = (1,1)$

Se obtiene $|\psi_2\rangle = |11\rangle$.

$Pr(11) = 1$

$Pr(00) = Pr(01) = Pr(10) = 0$

Tarea opcional