

Figure out correct theorem referencing.

1 Preliminaries

Definition 1. Let \mathcal{A} be a finite set. The *full \mathcal{A} -shift* is the set $\mathcal{A}^{\mathbb{Z}}$ of all bi-infinite sequences over \mathcal{A} (i.e. functions from \mathbb{Z} to \mathcal{A} , hence the usual notation for the set of all functions from \mathbb{Z} to \mathcal{A}).

A *block* (or *word*) is a finite sequence of letters over some alphabet \mathcal{A} . Let $x = (x_i)_{i \in \mathbb{Z}}$ be a bi-infinite sequence. For $i \leq j$, the block from the i th coordinate to the j th coordinate is denoted

$$x_{[i,j]} \triangleq x_i x_{i+1} \dots x_j.$$

Definition 2. Let \mathcal{F} be a set of words over some alphabet. A *subshift* is a subset $X_{\mathcal{F}}$ of some full shift $\mathcal{A}^{\mathbb{Z}}$ such that no word in \mathcal{F} appears in any point of the subshift, defined as

$$X_{\mathcal{F}} \triangleq \left\{ (x_i)_{i \in \mathbb{Z}} \in \mathcal{A}^{\mathbb{Z}} : \forall i, j \in \mathbb{Z}, i \leq j \quad x_{[i,j]} \notin \mathcal{F} \right\}.$$

Definition 3. A *graph* G is a 4-tuple $G = (\mathcal{V}, \mathcal{E}, i, t)$, where \mathcal{V} is a finite set of *vertices*, \mathcal{E} is a finite set of *edges*, and $i : \mathcal{E} \rightarrow \mathcal{V}$ and $t : \mathcal{E} \rightarrow \mathcal{V}$ are functions assigning an *initial* and *terminating* vertex for each edge, respectively. For an arbitrary graph G , let \mathcal{V}_G , \mathcal{E}_G , i_G , and t_G denote the graph's vertices, edges, and initial and terminating vertex functions, respectively. If the choice of G is understood, then the subscripts will be dropped for notational convenience.

For $I \in \mathcal{V}$, the *outgoing edges of I* is the set of edges starting at I , denoted

$$i^{-1}(I) \triangleq \{e \in \mathcal{E} : i(e) = I\}.$$

Similarly, the *incoming edges of I* is the set of edges terminating at I , denoted

$$t^{-1}(I) \triangleq \{e \in \mathcal{E} : t(e) = I\}.$$

A graph is *essential* if all vertices have at least one incoming and outgoing edge; i.e. for all $I \in \mathcal{V}$, $i^{-1}(I) \neq \emptyset$ and $t^{-1}(I) \neq \emptyset$.

Definition 4. A *labeled graph* \mathcal{G} is a pair $\mathcal{G} = (G, \mathcal{L})$, where G is a graph and $\mathcal{L} : \mathcal{E} \rightarrow \mathcal{A}$ is the *labeling function* from the edges of G onto some finite alphabet \mathcal{A} .

A labeled graph is *deterministic* if for each vertex, the labels of the outgoing edges at that vertex are all distinct (i.e. $\mathcal{L}|_{i^{-1}(I)}$ is injective for all $I \in \mathcal{V}$).

Definition 5. Let G be a graph. The *edge shift of G* is the set X_G of all bi-infinite paths on G , defined as

$$X_G \triangleq \left\{ (x_i)_{i \in \mathbb{Z}} \in \mathcal{E}^{\mathbb{Z}} : \forall i \in \mathbb{Z} \quad t(x_i) = i(x_{i+1}) \right\}.$$

As a consequence of this definition, $\mathcal{B}(X_G)$ is the set of all finite paths on G , so elements of $\mathcal{B}(X_G)$ are called *paths on G* .

Definition 6. Let $\mathcal{G} = (G, \mathcal{L})$ be a labeled graph. The *presentation of \mathcal{G}* is the set $X_{\mathcal{G}}$ of the labels of all bi-infinite paths from X_G , defined as the image of X_G under \mathcal{L}_{∞} :

$$X_{\mathcal{G}} \triangleq \mathcal{L}_{\infty}(X_G).$$

We say a word $w \in \mathcal{B}(X_{\mathcal{G}})$ is *presented by a path $\pi \in \mathcal{B}(X_G)$* if $\mathcal{L}(\pi) = w$.

Synchronizing definition.

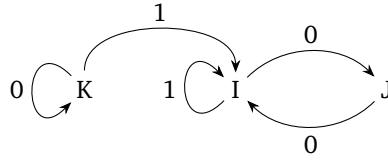
Definition 7. Let $\mathcal{G} = (G, \mathcal{L})$ be a labeled graph.

2 Basic properties of sofic shifts

Irreducible presentations, synchronizing presentations

3 Irreducibility

Consider the reducible graph \mathcal{G} :



The shift that subgraph induced by I and J is the even shift, and every word presented by a path starting from K is presented by some path starting at I and J . Hence, this graph presents the even shift. Additionally, it is also follower-separated as $01 \in F(K) \setminus F(I)$, $1 \in F(K) \setminus F(J)$, and $1 \in F(I) \setminus F(J)$.

When do follower-separated reducible graphs present irreducible shifts? We will first look at a simple class of reducible graphs. Let $\mathcal{G} = (G, \mathcal{L}_{\mathcal{G}})$ and $\mathcal{H} = (H, \mathcal{L}_{\mathcal{H}})$ be irreducible, follower-separated graphs. Fix two vertices, $I \in \mathcal{V}_{\mathcal{G}}$ and $J \in \mathcal{H}$, and some label ℓ not in the outgoing labels of I (i.e. $\ell \notin \mathcal{L}_{\mathcal{G}}(i^{-1}(I))$). Define the graph $\mathcal{G} \rightarrow \mathcal{H} = (G \rightarrow H, \mathcal{L}_{\mathcal{G} \rightarrow \mathcal{H}})$ be the disjoint union of \mathcal{G} and \mathcal{H} plus one distinct edge e with $i(e) = I$, $t(e) = J$, and $\mathcal{L}(e) = \ell$.

Some immediate properties of $\mathcal{G} \rightarrow \mathcal{H}$ are that the graph is deterministic $X_{\mathcal{G}} \subseteq X_{\mathcal{G} \rightarrow \mathcal{H}}$, $X_{\mathcal{H}} \subseteq X_{\mathcal{G} \rightarrow \mathcal{H}}$, and that there exists a path from any vertex in $\mathcal{G} \rightarrow \mathcal{H}$ to any vertex in \mathcal{H} .

Theorem 1. Every word $u \in \mathcal{B}(X_{\mathcal{G} \rightarrow \mathcal{H}})$ can be extended on the right to a word $us_K \in \mathcal{B}(X_{\mathcal{G} \rightarrow \mathcal{H}})$ that synchronizes to a vertex $K \in \mathcal{V}_{\mathcal{H}}$.

Related to a comment below, but this is just a corollary of [Lind and Marcus 3.3.16] and the nature of $\mathcal{G} \rightarrow \mathcal{H}$

Proof. Let $u \in \mathcal{B}(X_{\mathcal{G} \rightarrow \mathcal{H}})$. As $\mathcal{G} \rightarrow \mathcal{H}$ is deterministic and follower-separated, then by [Lind and Marcus 3.3.16], we can extend u on the right to a synchronizing word $uw \in \mathcal{B}(X_{\mathcal{G} \rightarrow \mathcal{H}})$. As $K \in \mathcal{V}_{\mathcal{H}}$, K can be reached from any vertex in $\mathcal{G} \rightarrow \mathcal{H}$, so let v be the label of some path connecting the vertex uw synchronizes to and some vertex in $I \in \mathcal{V}_{\mathcal{H}}$. Then, uvw must end at I since $\mathcal{G} \rightarrow \mathcal{H}$ is deterministic and in addition, $uvw \in \mathcal{B}(X_{\mathcal{G} \rightarrow \mathcal{H}})$. \square

Lemma 1. Let $u \in \mathcal{B}(X_{\mathcal{G} \rightarrow \mathcal{H}})$. If $u \notin \mathcal{B}(X_{\mathcal{H}})$, then $X_{\mathcal{G} \rightarrow \mathcal{H}}$ is reducible.

Proof. By Theorem 1, we can extend u on the right to a word $us_J \in \mathcal{B}(X_{\mathcal{G} \rightarrow \mathcal{H}})$ so that any path presenting us_J ends at J . Hence, for any word $w \in F(us_J)$, a path presenting us_Jw ends at some vertex in $\mathcal{V}_{\mathcal{H}}$, so $F(us_Jw) \subseteq \mathcal{B}(X_{\mathcal{H}})$. As $u \notin \mathcal{B}(X_{\mathcal{H}})$, we have $u \notin F(us_Jw)$. Therefore, there is no word in $\mathcal{B}(X_{\mathcal{G} \rightarrow \mathcal{H}})$ joining the word us_J and u , so $X_{\mathcal{G} \rightarrow \mathcal{H}}$ is reducible. \square

Theorem 2. If $X_{\mathcal{G} \rightarrow \mathcal{H}}$ is irreducible, then $X_{\mathcal{G}} \subseteq X_{\mathcal{H}}$.

Proof. Suppose $X_{\mathcal{G} \rightarrow \mathcal{H}}$ is irreducible, and let $u \in \mathcal{B}(X_{\mathcal{G}})$. By Theorem 1, we can extend u on the right to $us_J \in \mathcal{B}(X_{\mathcal{G} \rightarrow \mathcal{H}})$, so that any path presenting us_J ends at J . As $X_{\mathcal{G} \rightarrow \mathcal{H}}$ is irreducible, there exists a word $w \in \mathcal{B}(X_{\mathcal{G} \rightarrow \mathcal{H}})$ such that $us_Jwu \in \mathcal{B}(X_{\mathcal{G} \rightarrow \mathcal{H}})$. The wu subword must be presented by some path in \mathcal{H} as us_J synchronizes to $J \in \mathcal{V}_{\mathcal{H}}$. Hence $u \in \mathcal{B}(X_{\mathcal{H}})$, so $X_{\mathcal{G}} \subseteq X_{\mathcal{H}}$. \square

Theorem 3. If $X_{\mathcal{G} \rightarrow \mathcal{H}}$ is irreducible, then $\ell \in \mathcal{L}(t_H^{-1}(J))$.

Proof. Suppose $\ell \notin \mathcal{L}(t_H^{-1}(J))$. Considering just \mathcal{H} , if $\ell \in \mathcal{B}(X_{\mathcal{H}})$, then a path presenting ℓ does not end at J . As \mathcal{H} is follower-separated, there is a word $w \in F(J)$ and $w \notin F(K)$ for all $K \in \mathcal{V}_{\mathcal{H}} \setminus \{J\}$. Hence, we have $\ell w \in \mathcal{B}(X_{\mathcal{G} \rightarrow \mathcal{H}})$ but $\ell w \notin \mathcal{B}(X_{\mathcal{H}})$, so by Lemma 1, $X_{\mathcal{G} \rightarrow \mathcal{H}}$ is reducible. \square

This assumption is incorrect.

Theorem 4. The following are equivalent:

- (i) $X_{\mathcal{G} \rightarrow \mathcal{H}}$ is irreducible
- (ii) for every path π in $\mathcal{G} \rightarrow \mathcal{H}$ that ends at J there is another path τ in \mathcal{H} that ends at J and presents the same word as π .

Proof. Suppose (ii) does not hold. Then, there exists a path π in $\mathcal{G} \rightarrow \mathcal{H}$ that ends at J such that for all other paths τ in \mathcal{H} , if $\mathcal{L}(\tau) = \mathcal{L}(\pi)$, then τ does not end at J . If there no path τ in \mathcal{H} that presents $\mathcal{L}(\pi)$, then $\mathcal{L}(\pi) \notin \mathcal{B}(X_{\mathcal{H}})$, and so by Lemma 1, $X_{\mathcal{G} \rightarrow \mathcal{H}}$ is reducible.

The rest of $\neg(ii) \implies \neg(i)$ requires the same incorrect assumption above.

Suppose (ii) does hold. We need to show that for $u, v \in \mathcal{B}(X_{\mathcal{G} \rightarrow \mathcal{H}})$, there is a word $w \in \mathcal{B}(X_{\mathcal{G} \rightarrow \mathcal{H}})$ such that $uwv \in \mathcal{B}(X_{\mathcal{G} \rightarrow \mathcal{H}})$. Let π, τ be paths in $\mathcal{G} \rightarrow \mathcal{H}$ that present u and v , respectively. If τ starts in \mathcal{H} , then by the construction of $\mathcal{G} \rightarrow \mathcal{H}$, there is a path ρ in

Instead of using (ii) and instead assuming there is a path τ' in \mathcal{H} presenting $\mathcal{L}(\tau)$, the proof is still valid. The fact that this path ends at J is superfluous.

Redux of previous.

$\mathcal{G} \rightarrow \mathcal{H}$ that starts at $t(\pi)$ and ends at $i(\tau)$. Hence, $u\mathcal{L}(\rho)v \in \mathcal{B}(X_{\mathcal{G} \rightarrow \mathcal{H}})$. Otherwise, if τ does not start in \mathcal{H} , then it must start in \mathcal{G} . Similarly, by the construction of $\mathcal{G} \rightarrow \mathcal{H}$, there is a path ρ_1 in $\mathcal{G} \rightarrow \mathcal{H}$ that starts at $t(\tau)$ and ends at J . Hence, the path $\tau\rho_1$ ends at J . By (ii), there is a path τ' in \mathcal{H} that ends at J and presents $\mathcal{L}(\tau\rho_1)$. By the construction of $\mathcal{G} \rightarrow \mathcal{H}$, there is a path ρ_2 that starts at $t(\pi)$ and ends at $i(\tau')$. Hence,

$$\mathcal{L}(\pi\rho_2\tau') = \mathcal{L}(\pi)\mathcal{L}(\rho_2)\mathcal{L}(\tau') = u\mathcal{L}(\rho_2)\mathcal{L}(\tau\rho_1) = u\mathcal{L}(\rho_2)\mathcal{L}(\tau)\mathcal{L}(\rho_1) = u\mathcal{L}(\rho_2)v\mathcal{L}(\rho_1).$$

As $\mathcal{L}(\pi\rho_2\tau') \in \mathcal{B}(X_{\mathcal{G} \rightarrow \mathcal{H}})$, then $u\mathcal{L}(\rho_2)v\mathcal{L}(\rho_1) \in \mathcal{B}(X_{\mathcal{G} \rightarrow \mathcal{H}})$ and thus $u\mathcal{L}(\rho_2)v \in \mathcal{B}(X_{\mathcal{G} \rightarrow \mathcal{H}})$. \square

Theorem 5. The following are equivalent:

- (i) $X_{\mathcal{G} \rightarrow \mathcal{H}}$ is irreducible
- (ii) for every path π in $\mathcal{G} \rightarrow \mathcal{H}$ that starts in \mathcal{G} there is another path τ in \mathcal{H} presents the same word as π .

Proof. Suppose (ii) does not hold. Then, there exists a path π in $\mathcal{G} \rightarrow \mathcal{H}$ that starts in \mathcal{G} such that for every other path τ in \mathcal{H} , $\mathcal{L}(\tau) \neq \mathcal{L}(\pi)$. The latter condition is equivalent to $\mathcal{L}(\pi) \notin \mathcal{B}(X_{\mathcal{G} \rightarrow \mathcal{H}})$, and we have $\mathcal{L}(\pi) \in \mathcal{B}(X_{\mathcal{G} \rightarrow \mathcal{H}})$, so by Lemma 1, $X_{\mathcal{G} \rightarrow \mathcal{H}}$ is reducible, so (ii) does not hold.

Suppose (ii) does hold. We need to show that for $u, v \in \mathcal{B}(X_{\mathcal{G} \rightarrow \mathcal{H}})$, there is a word $w \in \mathcal{B}(X_{\mathcal{G} \rightarrow \mathcal{H}})$ such that $uwv \in \mathcal{B}(X_{\mathcal{G} \rightarrow \mathcal{H}})$. Let π, τ be paths in $\mathcal{G} \rightarrow \mathcal{H}$ that present u and v , respectively. If τ starts in \mathcal{H} , then by the construction of $\mathcal{G} \rightarrow \mathcal{H}$, there is a path ρ in $\mathcal{G} \rightarrow \mathcal{H}$ that starts at $t(\pi)$ and ends at $i(\tau)$. Hence, $\pi\rho\tau$ is a path in $\mathcal{G} \rightarrow \mathcal{H}$ and

$$\mathcal{L}(\pi\rho\tau) = \mathcal{L}(\pi)\mathcal{L}(\rho)\mathcal{L}(\tau) = u\mathcal{L}(\rho)v,$$

so $u\mathcal{L}(\rho)v \in \mathcal{B}(X_{\mathcal{G} \rightarrow \mathcal{H}})$. Otherwise, if τ does not start in \mathcal{H} , then it must start in \mathcal{G} . By (ii), there is a path τ' in \mathcal{H} with that presents $\mathcal{L}(\tau)$. By the construction of $\mathcal{G} \rightarrow \mathcal{H}$, there is a path ρ in $\mathcal{G} \rightarrow \mathcal{H}$ that starts at $t(\pi)$ and ends at $i(\tau')$. Hence, $\pi\rho\tau'$ is a path in $\mathcal{G} \rightarrow \mathcal{H}$ and

$$\mathcal{L}(\pi\rho\tau') = \mathcal{L}(\pi)\mathcal{L}(\rho)\mathcal{L}(\tau') = u\mathcal{L}(\rho)\mathcal{L}(\tau) = u\mathcal{L}(\rho)v,$$

so $u\mathcal{L}(\rho)v \in \mathcal{B}(X_{\mathcal{G} \rightarrow \mathcal{H}})$. In either case, we found a word $w \in \mathcal{B}(X_{\mathcal{G} \rightarrow \mathcal{H}})$ such that $uwv \in \mathcal{B}(X_{\mathcal{G} \rightarrow \mathcal{H}})$. Therefore, $\mathcal{G} \rightarrow \mathcal{H}$ is irreducible, so (i) holds. \square

I think this should be a lemma or something for clarity. All this argument is that any vertex in \mathcal{H} is reachable from any other vertex in $\mathcal{G} \rightarrow \mathcal{H}$, by the nature of $\mathcal{G} \rightarrow \mathcal{H}$.

4 Minimality

This section will probably hardness of sofic shift problems. As the one of the earlier sections that was written, this could be rewritten and streamlined a bit with the other sections. Other things to do: define "irreducible shift" decision problem. Give reduction from GI to irreducible shift, conclude irreducible shift is GI-hard. Give reduction from irreducible shift to minimality, conclude minimality is GI-hard.

Move this to an appropriate section.

Theorem 6. If $\mathcal{G} = (G, \mathcal{L})$ is the minimizing right resolving presentation of an irreducible sofic shift X and X is an N -step shift of finite type, then $X_G \cong X_{\mathcal{G}}$.

Proof. Let x, y be walks in X_G . Suppose $\mathcal{L}_{\infty}(x) = \mathcal{L}_{\infty}(y)$. For any i , the paths $x_{[i-N, i-1]}$ and $y_{[i-N, i-1]}$ present the same word. Because that word is of length N , the word is synchronizing for \mathcal{G} (from 3.4.17), so those paths end at the same vertex. Since $\mathcal{L}(x_{[i]}) = \mathcal{L}(y_{[i]})$, \mathcal{G} is right resolving, and $x_{[i-N, i-1]}$ and $y_{[i-N, i-1]}$ end at the same vertex, then $x_{[i]} = y_{[i]}$ and hence $x = y$, so \mathcal{L}_{∞} is injective. By definition, \mathcal{L}_{∞} is surjective. Therefore, \mathcal{L}_{∞} is bijective and a conjugacy from X_G to $X_{\mathcal{G}}$. \square

Move this to an appropriate section.

Lemma 2. If X and Y are shift spaces, then $\mathcal{B}(X) \subseteq \mathcal{B}(Y)$ if and only if $X \subseteq Y$.

Proof. Let x be a point in X . Then every word that appears in x is in $\mathcal{B}(X)$. Since $\mathcal{B}(X) \subseteq \mathcal{B}(Y)$, then every word that appears in x is in $\mathcal{B}(Y)$, so $x \in Y$, hence $X \subseteq Y$.

Conversely, let w be a word in $\mathcal{B}(X)$. Then w occurs in some $x \in X$. Since $X \subseteq Y$, we have $x \in Y$, so w occurs in some $x \in Y$. Hence, $w \in \mathcal{B}(Y)$. \square

Let \mathcal{G} and \mathcal{H} be labeled graphs, I be a vertex from \mathcal{G} , and J be a vertex from \mathcal{H} . Define the graph connecting \mathcal{G} to \mathcal{H} via I and J as the disjoint union of the two graphs, adding an edge starting at I and ending at J , and adding a self loop on J . Label these two new edges with a symbol that does not appear in either graph. Since \mathcal{G} and \mathcal{H} are subgraphs of a graph connecting the two, it follows that the presentations of the individual graphs are subshifts of a presentation of a graph connecting the two - any bi-infinite walk in one of the graphs is a bi-infinite walk of the corresponding subgraph of the connected graphs. Additionally, observe that the graph is reducible, as any path starting in \mathcal{H} cannot end in \mathcal{G} .

Theorem 7. Let \mathcal{G} and \mathcal{H} be irreducible graphs, and \mathcal{K} be the graph connecting \mathcal{G} to \mathcal{H} via I and J . If $X_{\mathcal{K}}$ is irreducible, then $X_{\mathcal{G}} \subseteq X_{\mathcal{H}}$.

Proof. First, suppose that $X_{\mathcal{K}}$ is irreducible, and let $u \in \mathcal{B}(X_{\mathcal{G}})$. There is a path in \mathcal{G} that presents u , hence there is a path in the \mathcal{G} subgraph of \mathcal{K} that presents u . From the irreducibility of \mathcal{G} , there is a path from the terminating vertex of a path presenting u to

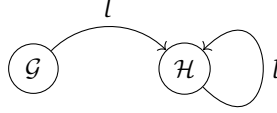


Figure 1: A graph connecting \mathcal{G} to \mathcal{H} .

I . Let v be the word such path presents and l be the label of the edge connecting \mathcal{G} to \mathcal{H} , so that we have $uvl \in \mathcal{B}(X_{\mathcal{K}})$ and $u \in \mathcal{B}(X_{\mathcal{K}})$. As $X_{\mathcal{K}}$ is irreducible, there exists a word $w \in \mathcal{B}(X_{\mathcal{K}})$ such that $uvlwu \in \mathcal{B}(X_{\mathcal{K}})$. A path presenting $uvlwu$ must have the subpath presenting wu visit vertices only from the \mathcal{H} subgraph of \mathcal{K} . This implies that there is a path in \mathcal{H} presenting u , so we have $u \in \mathcal{B}(X_{\mathcal{H}})$ and therefore $\mathcal{B}(X_{\mathcal{G}}) \subseteq \mathcal{B}(X_{\mathcal{H}})$, and $X_{\mathcal{G}} \subseteq X_{\mathcal{H}}$ via Lemma 1. \square

Theorem 8. Let \mathcal{G} and \mathcal{H} be irreducible, minimal, right-resolving presentations. If $X_{\mathcal{G}} = X_{\mathcal{H}}$, then there exists a pair of vertices $(I, J) \in (\mathcal{V}_{\mathcal{G}} \times \mathcal{V}_{\mathcal{H}})$ such that both the graph connecting \mathcal{G} to \mathcal{H} via I and J and the graph connecting \mathcal{H} to \mathcal{G} via J and I both present irreducible shifts.

Equivalently, if for every pair of vertices $(I, J) \in (\mathcal{V}_{\mathcal{G}}, \mathcal{V}_{\mathcal{H}})$ the graph connecting \mathcal{G} to \mathcal{H} via I and J and the graph connecting \mathcal{H} to \mathcal{G} via J and I do not both present irreducible shifts.

Proof. Suppose $X_{\mathcal{G}} = X_{\mathcal{H}}$. Since \mathcal{G} and \mathcal{H} are irreducible, minimal, right-resolving presentations of the same shift, they must be isomorphic. Let $(\partial\Phi, \Phi)$ be a graph isomorphism between them. Choose an arbitrary vertex I from \mathcal{G} , and then let \mathcal{K} be the graph connecting \mathcal{G} to \mathcal{H} via I and $\partial\Phi(I)$. Let f be the self loop on $\partial\Phi(I)$ added in the construction of \mathcal{K} , and \mathcal{H}^+ be the \mathcal{H} of \mathcal{K} subgraph plus f . As \mathcal{H} is irreducible, then \mathcal{H}^+ is irreducible. It suffices to show that $X_{\mathcal{K}} = X_{\mathcal{H}^+}$ to show $X_{\mathcal{K}}$ is irreducible.

Let u be a word from $\mathcal{B}(X_{\mathcal{K}})$, and π be a path that presents it. Without loss of generality, assume u and π are nonempty. If π starts in \mathcal{H}^+ , then $u \in \mathcal{B}(X_{\mathcal{H}^+})$. Otherwise, it starts in \mathcal{G} and either ends in \mathcal{G} or ends in \mathcal{H}^+ . For the case of π ending in \mathcal{G} , then $\Phi(\pi)$ is a path in \mathcal{H} presenting u , as $(\partial\Phi, \Phi)$ is a graph isomorphism, so $u \in \mathcal{B}(X_{\mathcal{H}^+})$. For the case of π ending in \mathcal{H}^+ , we can split π into $\pi = \pi_1 e \pi_2$, where π_1 and π_2 are (possibly empty) paths from \mathcal{G} and \mathcal{H}^+ , respectively, and e is the edge connecting \mathcal{G} and \mathcal{H} . Note that $\Phi(\pi_0)$ is a path in \mathcal{H} and that it terminates at $\partial\Phi(I)$ as π_0 terminates at I , assuming π_0 is nonempty. From this, we have that $\Phi(\pi_0) f \pi_1$ is path in \mathcal{H}^+ that presents u , so $u \in \mathcal{B}(X_{\mathcal{H}^+})$.

Hence, we have $\mathcal{B}(X_{\mathcal{K}}) \subseteq \mathcal{B}(X_{\mathcal{H}^+})$, and from the construction of \mathcal{K} , we also have

$\mathcal{B}(X_{\mathcal{H}^+}) \subseteq \mathcal{B}(X_{\mathcal{K}})$, so $\mathcal{B}(X_{\mathcal{K}}) = \mathcal{B}(X_{\mathcal{H}^+})$ and $X_{\mathcal{K}} = X_{\mathcal{H}^+}$. Thus, we have shown the graph connecting \mathcal{G} to \mathcal{H} via I and $\partial\Phi(I)$ presents an irreducible shift. A similar argument can be made showing that the graph connecting \mathcal{H} to \mathcal{G} via $\partial\Phi(I)$ and I presents an irreducible shift (start from “...and then let \mathcal{K} ”, and replace $\mathcal{G} \mapsto \mathcal{H}, \mathcal{H} \mapsto \mathcal{G}, I \mapsto \partial\Phi(I), \partial\Phi(I) \mapsto I, \mathcal{H}^+ \mapsto \mathcal{G}^+, \Phi(\pi_0) \mapsto \Phi^{-1}(\pi_0)$). \square

Theorem 9. Given an oracle for minimizing a reducible presentation, deciding if two irreducible, minimal, right-resolving labeled graphs are isomorphic can be determined in polynomial time.

Proof. Let the decision procedure be as follows: for every pair of vertices $(I, J) \in (\mathcal{V}_{\mathcal{G}} \times \mathcal{V}_{\mathcal{H}})$ between the graphs, construct and set $\mathcal{G}_I \rightarrow \mathcal{H}_J$ as the graph connecting \mathcal{G} to \mathcal{H} via I and J , and similarly, construct and set $\mathcal{H}_J \rightarrow \mathcal{G}_I$ as the graph connecting \mathcal{H} to \mathcal{G} via J and I . Then, using the oracle, minimize $\mathcal{G}_I \rightarrow \mathcal{H}_J$ and check if it is strongly connected (that is, irreducible), and if so, then from Theorem 2, we know we can set $X_{\mathcal{G}} \subseteq X_{\mathcal{H}}$ to be true. Similarly, minimize $\mathcal{H}_J \rightarrow \mathcal{G}_I$ and check if it is strongly connected, and if so, set $X_{\mathcal{H}} \subseteq X_{\mathcal{G}}$ to be true. If at any point both $X_{\mathcal{G}} \subseteq X_{\mathcal{H}}$ and $X_{\mathcal{H}} \subseteq X_{\mathcal{G}}$ are set to true, then $X_{\mathcal{G}} = X_{\mathcal{H}}$ and thus can conclude $\mathcal{G} \cong \mathcal{H}$ (as \mathcal{G} and \mathcal{H} are unique, minimal, and irreducible presentations). If you find that after every pair of vertices that one or both of $X_{\mathcal{G}} \subseteq X_{\mathcal{H}}$ and $X_{\mathcal{H}} \subseteq X_{\mathcal{G}}$ were not set to true, then we have that for all pairs of vertices, the presentations of the pair of graphs constructed were not both irreducible, as if they were, then both $X_{\mathcal{G}} \subseteq X_{\mathcal{H}}$ and $X_{\mathcal{H}} \subseteq X_{\mathcal{G}}$ would be true, so via Theorem 3, we have that $X_{\mathcal{G}} \neq X_{\mathcal{H}}$ and can conclude $\mathcal{G} \not\cong \mathcal{H}$ (as again, \mathcal{G} and \mathcal{H} are irreducible, minimal, right-resolving graphs).

The worst case runtime of the decision procedure is if $\mathcal{G} \not\cong \mathcal{H}$, as we minimize and check strongly connected-ness twice (of a graph that is potentially the size of both \mathcal{G} and \mathcal{H}) for each pair of vertices, so the runtime is $O((V + E) \cdot V^2)$, where V is the number of vertices of one graph and E is the number of edges of one graph. \square