

Theorem 1. If $\mathcal{G} = (G, \mathcal{L})$ is the minimizing right resolving presentation of an irreducible sofic shift X and X is an N -step shift of finite type, then $X_G \cong X_{\mathcal{G}}$.

Proof. Let x, y be walks in X_G . Suppose $\mathcal{L}_{\infty}(x) = \mathcal{L}_{\infty}(y)$. For any i , the paths $x_{[i-N, i-1]}$ and $y_{[i-N, i-1]}$ present the same word. Because that word is of length N , the word is synchronizing for \mathcal{G} (from 3.4.17), so those paths end at the same vertex. Since $\mathcal{L}(x_{[i]}) = \mathcal{L}(y_{[i]})$, \mathcal{G} is right resolving, and $x_{[i-N, i-1]}$ and $y_{[i-N, i-1]}$ end at the same vertex, then $x_{[i]} = y_{[i]}$ and hence $x = y$, so \mathcal{L}_{∞} is injective. By definition, \mathcal{L}_{∞} is surjective. Therefore, \mathcal{L}_{∞} is bijective and a conjugacy from X_G to $X_{\mathcal{G}}$. \square

Lemma 1. If X and Y are shift spaces, then $\mathcal{B}(X) \subseteq \mathcal{B}(Y)$ if and only if $X \subseteq Y$.

Proof. Let x be a point in X . Then every word that appears in x is in $\mathcal{B}(X)$. Since $\mathcal{B}(X) \subseteq \mathcal{B}(Y)$, then every word that appears in x is in $\mathcal{B}(Y)$, so $x \in Y$, hence $X \subseteq Y$.

Conversely, let w be a word in $\mathcal{B}(X)$. Then w occurs in some $x \in X$. Since $X \subseteq Y$, we have $x \in Y$, so w occurs in some $x \in Y$. Hence, $w \in \mathcal{B}(Y)$. \square

Let \mathcal{G} and \mathcal{H} be labeled graphs, I be a vertex from \mathcal{G} , and J be a vertex from \mathcal{H} . Define the *graph connecting \mathcal{G} to \mathcal{H} via I and J* as the disjoint union of the two graphs, adding an edge starting at I and ending at J , and adding a self loop on J . Label these two new edges with a symbol that does not appear in either graph. Since \mathcal{G} and \mathcal{H} are subgraphs of a graph connecting the two, it follows that the presentations of the individual graphs are subshifts of a presentation of a graph connecting the two - any bi-infinite walk in one of the graphs is a bi-infinite walk of the corresponding subgraph of the connected graphs. Additionally, observe that the graph is reducible, as any path starting in \mathcal{H} cannot end in \mathcal{G} .

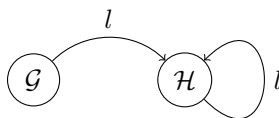


Figure 1: A graph connecting \mathcal{G} to \mathcal{H} .

Theorem 2. Let \mathcal{G} and \mathcal{H} be irreducible graphs, and \mathcal{K} be the graph connecting \mathcal{G} to \mathcal{H} via I and J . If $X_{\mathcal{K}}$ is irreducible, then $X_{\mathcal{G}} \subseteq X_{\mathcal{H}}$.

Proof. First, suppose that $X_{\mathcal{K}}$ is irreducible, and let $u \in \mathcal{B}(X_{\mathcal{G}})$. There is a path in \mathcal{G} that presents u , hence there is a path in the \mathcal{G} subgraph of \mathcal{K} that presents u . From the irreducibility of \mathcal{G} , there is a path from the terminating vertex of a path presenting u to I . Let v be the word such path presents and l be the label of the edge connecting \mathcal{G} to \mathcal{H} , so that we have $uvl \in \mathcal{B}(X_{\mathcal{K}})$ and $u \in \mathcal{B}(X_{\mathcal{K}})$. As $X_{\mathcal{K}}$ is irreducible, there exists a word $w \in \mathcal{B}(X_{\mathcal{K}})$ such that $uvlwu \in \mathcal{B}(X_{\mathcal{K}})$. A path presenting $uvlwu$ must have the subpath presenting wu visit vertices only from the \mathcal{H} subgraph

of \mathcal{K} . This implies that there is a path in \mathcal{H} presenting u , so we have $u \in \mathcal{B}(\mathcal{X}_{\mathcal{H}})$ and therefore $\mathcal{B}(\mathcal{X}_{\mathcal{G}}) \subseteq \mathcal{B}(\mathcal{X}_{\mathcal{H}})$, and $\mathcal{X}_{\mathcal{G}} \subseteq \mathcal{X}_{\mathcal{H}}$ via Lemma 1. \square

Theorem 3. Let \mathcal{G} and \mathcal{H} be irreducible, minimal, right-resolving presentations. If $\mathcal{X}_{\mathcal{G}} = \mathcal{X}_{\mathcal{H}}$, then there exists a pair of vertices $(I, J) \in (\mathcal{V}_{\mathcal{G}} \times \mathcal{V}_{\mathcal{H}})$ such that both the graph connecting \mathcal{G} to \mathcal{H} via I and J and the graph connecting \mathcal{H} to \mathcal{G} via J and I both present irreducible shifts.

Equivalently, if for every pair of vertices $(I, J) \in (\mathcal{V}_{\mathcal{G}}, \mathcal{V}_{\mathcal{H}})$ the graph connecting \mathcal{G} to \mathcal{H} via I and J and the graph connecting \mathcal{H} to \mathcal{G} via J and I do not both present irreducible shifts.

Proof. Suppose $\mathcal{X}_{\mathcal{G}} = \mathcal{X}_{\mathcal{H}}$. Since \mathcal{G} and \mathcal{H} are irreducible, minimal, right-resolving presentations of the same shift, they must be isomorphic. Let $(\partial\Phi, \Phi)$ be a graph isomorphism between them. Choose an arbitrary vertex I from \mathcal{G} , and then let \mathcal{K} be the graph connecting \mathcal{G} to \mathcal{H} via I and $\partial\Phi(I)$. Let f be the self loop on $\partial\Phi(I)$ added in the construction of \mathcal{K} , and \mathcal{H}^+ be the \mathcal{H} of \mathcal{K} subgraph plus f . As \mathcal{H} is irreducible, then \mathcal{H}^+ is irreducible. It suffices to show that $\mathcal{X}_{\mathcal{K}} = \mathcal{X}_{\mathcal{H}^+}$ to show $\mathcal{X}_{\mathcal{K}}$ is irreducible.

Let u be a word from $\mathcal{B}(\mathcal{X}_{\mathcal{K}})$, and π be a path that presents it. Without loss of generality, assume u and π are nonempty. If π starts in \mathcal{H}^+ , then $u \in \mathcal{B}(\mathcal{X}_{\mathcal{H}^+})$. Otherwise, it starts in \mathcal{G} and either ends in \mathcal{G} or ends in \mathcal{H}^+ . For the case of π ending in \mathcal{G} , then $\Phi(\pi)$ is a path in \mathcal{H} presenting u , as $(\partial\Phi, \Phi)$ is a graph isomorphism, so $u \in \mathcal{B}(\mathcal{X}_{\mathcal{H}^+})$. For the case of π ending in \mathcal{H}^+ , we can split π into $\pi = \pi_1 e \pi_2$, where π_1 and π_2 are (possibly empty) paths from \mathcal{G} and \mathcal{H}^+ , respectively, and e is the edge connecting \mathcal{G} and \mathcal{H} . Note that $\Phi(\pi_0)$ is a path in \mathcal{H} and that it terminates at $\partial\Phi(I)$ as π_0 terminates at I , assuming π_0 is nonempty. From this, we have that $\Phi(\pi_0)f\pi_1$ is path in \mathcal{H}^+ that presents u , so $u \in \mathcal{B}(\mathcal{X}_{\mathcal{H}^+})$.

Hence, we have $\mathcal{B}(\mathcal{X}_{\mathcal{K}}) \subseteq \mathcal{B}(\mathcal{X}_{\mathcal{H}^+})$, and from the construction of \mathcal{K} , we also have $\mathcal{B}(\mathcal{X}_{\mathcal{H}^+}) \subseteq \mathcal{B}(\mathcal{X}_{\mathcal{K}})$, so $\mathcal{B}(\mathcal{X}_{\mathcal{K}}) = \mathcal{B}(\mathcal{X}_{\mathcal{H}^+})$ and $\mathcal{X}_{\mathcal{K}} = \mathcal{X}_{\mathcal{H}^+}$. Thus, we have shown the graph connecting \mathcal{G} to \mathcal{H} via I and $\partial\Phi(I)$ presents an irreducible shift. A similar argument can be made showing that the graph connecting \mathcal{H} to \mathcal{G} via $\partial\Phi(I)$ and I presents an irreducible shift (start from “... and then let \mathcal{K} ”, and replace $\mathcal{G} \mapsto \mathcal{H}, \mathcal{H} \mapsto \mathcal{G}, I \mapsto \partial\Phi(I), \partial\Phi(I) \mapsto I, \mathcal{H}^+ \mapsto \mathcal{G}^+, \Phi(\pi_0) \mapsto \Phi^{-1}(\pi_0)$). \square

Theorem 4. Given an oracle for minimizing a reducible presentation, deciding if two irreducible, minimal, right-resolving labeled graphs are isomorphic can be determined in polynomial time.

Proof. Let the decision procedure be as follows: for every pair of vertices $(I, J) \in (\mathcal{V}_{\mathcal{G}} \times \mathcal{V}_{\mathcal{H}})$ between the graphs, construct and set $\mathcal{G}_I \rightarrow \mathcal{H}_J$ as the graph connecting \mathcal{G} to \mathcal{H} via I and J , and similarly, construct and set $\mathcal{H}_J \rightarrow \mathcal{G}_I$ as the graph connecting \mathcal{H} to \mathcal{G} via J and I . Then, using the oracle, minimize $\mathcal{G}_I \rightarrow \mathcal{H}_J$ and check if it is strongly connected (that is, irreducible), and if so, then from Theorem 2, we know we can set $\mathcal{X}_{\mathcal{G}} \subseteq \mathcal{X}_{\mathcal{H}}$ to be true. Similary, minimize $\mathcal{H}_J \rightarrow \mathcal{G}_I$ and check if it is strongly connected, and if so, set $\mathcal{X}_{\mathcal{H}} \subseteq \mathcal{X}_{\mathcal{G}}$ to be true. If at any point both $\mathcal{X}_{\mathcal{G}} \subseteq \mathcal{X}_{\mathcal{H}}$ and $\mathcal{X}_{\mathcal{H}} \subseteq \mathcal{X}_{\mathcal{G}}$ are set to true, then $\mathcal{X}_{\mathcal{G}} = \mathcal{X}_{\mathcal{H}}$ and thus can conclude $\mathcal{G} \cong \mathcal{H}$ (as \mathcal{G} and \mathcal{H} are unique, minimal, and irreducible presentations). If you find that after every pair of vertices that one or both of $\mathcal{X}_{\mathcal{G}} \subseteq \mathcal{X}_{\mathcal{H}}$ and $\mathcal{X}_{\mathcal{H}} \subseteq \mathcal{X}_{\mathcal{G}}$ were not set to true, then we have that for all pairs of vertices, the presentations of the pair of graphs constructed were not both irreducible, as if they were, then both $\mathcal{X}_{\mathcal{G}} \subseteq \mathcal{X}_{\mathcal{H}}$ and $\mathcal{X}_{\mathcal{H}} \subseteq \mathcal{X}_{\mathcal{G}}$ would be true, so via Theorem 3, we have that $\mathcal{X}_{\mathcal{G}} \neq \mathcal{X}_{\mathcal{H}}$ and can conclude $\mathcal{G} \not\cong \mathcal{H}$ (as again, \mathcal{G} and \mathcal{H} are irreducible, minimal, right-resolving graphs).

The worst case runtime of the decision procedure is if $\mathcal{G} \not\cong \mathcal{H}$, as we minimize and check strongly connected-ness twice (of a graph that is potentially the size of both \mathcal{G} and \mathcal{H}) for each pair of vertices, so the runtime is $O((V + E) \cdot V^2)$, where V is the number of vertices of one graph and E is the number of edges of one graph. \square