Towards algorithms for reducible presentations of sofic shifts

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Abstract

Given an irreducible presentation of a sofic shift, it is well known that the shift it presents is irreducible and that there is a procedure that yields the unique vertex-minimal presentation for this shift in polynomial time. However, if one was given a reducible graph, there is not necessarily a unique vertex-minimal presentation, and a procedure for the construction of some minimal presentation in this case is unknown. It can be the case where the presentation still presents an irreducible shift, but the previous procedure will not produce the minimal presentation. In this thesis, we show complexity results around these problems.

1 Introduction

Symbolic dynamics studies discrete-time and discrete-state

2 Preliminaries

In this section we introduce basic concepts from symbolic dynamics. Definitions and notation follow [Lin+95] closely.

Definition 2.1. Let $\mathcal{A}$ be a finite set. The *full $\mathcal{A}$-shift* is the set $\mathcal{A}^{\mathbb{Z}}$ of all bi-infinite sequences over $\mathcal{A}$ (i.e. functions from $\mathbb{Z}$ to $\mathcal{A}$, hence the usual notation for the set of all functions from $\mathbb{Z}$ to $\mathcal{A}$).

A *word* is a finite sequence of letters over some alphabet $\mathcal{A}$. We use ϵ to denote the empty word. Let $x = (x_i)_{i\in\mathbb{Z}}$ be a bi-infinite sequence. For $i \leq j$, the block from the ith coordinate to the jth coordinate is denoted

$$x_{[i,j]} \triangleq x_i x_{i+1} \dots x_j.$$

Definition 2.2. Let $\mathcal{F}$ be a set of words over some alphabet, called the *forbidden words*. A *shift space* is a subset $\mathsf{X}_{\mathcal{F}}$ of some full shift $\mathcal{A}^{\mathbb{Z}}$ such that none of the forbidden words appear in any point of the subshift. That is,

$$\mathsf{X}_{\mathcal{F}} \triangleq \left\{ (x_i)_{i\in\mathbb{Z}} \in \mathcal{A}^{\mathbb{Z}} \ : \ \forall i,j \in \mathbb{Z}, i \leq j \quad x_{[i,j]} \notin \mathcal{F} \right\}.$$

Definition 2.3. Let X be a shift space. The *language of* X is the set

$$\mathcal{B}(X) = \{x_{[i,j]} : x \in X, i,j \in \mathbb{Z}, i < j\}$$

of non-empty words that appear in some point in X.

Instead of specifying what words are forbidden in a shift space, we can characterize shift spaces by their languages. If $L \subseteq \mathcal{A}^+$ is a set of non-empty words, then we say L is *factorial* if for every word $w \in L$, then every non-empty subword is in L. We say L is *prolongable* if there for every word $w \in L$, there are nonempty words $u, v \in L$ such that $uwv \in L$.

Theorem 2.4 ([Lin+95])**.** If $L \subset \mathcal{A}^+$ is a set of non-empty words, then $L = \mathcal{B}(X)$ for some shift space X if and only if L is factorial and prolongable. Furthermore, for any shift space, $X = \mathsf{X}_{\mathcal{A}^+ \setminus \mathcal{B}(X)}$, so two shift spaces are equal if and only if their languages are equal.

Theorem 2.5. If X and Y are shift spaces, then $X \subseteq Y$ if and only if $\mathcal{B}(X) \subseteq \mathcal{B}(Y)$.

Proof. Suppose $X \subseteq Y$. If $w \in \mathcal{B}(X)$, then w occurs in some point $x \in X$. But $x \in Y$ as $X \subseteq Y$, so w occurs in some point in Y. Therefore, $w \in \mathcal{B}(Y)$.

Conversely, suppose $\mathcal{B}(X) \subseteq \mathcal{B}(Y)$. If $x \in X$, then every word occuring in x is in $\mathcal{B}(X)$. But $\mathcal{B}(X) \subseteq \mathcal{B}(Y)$, so every word occuring in x is in $\mathcal{B}(Y)$, so $x \in Y$. $\square$

Definition 2.6. Let X be a shift space. If for all words $u, v \in \mathcal{B}(X)$, there is a word w such that $uwv \in \mathcal{B}(X)$, then we say X *is irreducible*.

Definition 2.7. A *graph* G is a 4-tuple $G = (\mathcal{V}, \mathcal{E}, i, t)$, where $\mathcal{V}$ is a finite set of *vertices*, $\mathcal{E}$ is a finite set of *edges*, and $i : \mathcal{E} \to \mathcal{V}$ and $t : \mathcal{E} \to \mathcal{V}$ are functions assigning an *initial* and *terminating* vertex for each edge, respectivley. For an arbitrary graph G, let $\mathcal{V}_G$, $\mathcal{E}_G$, i_G, and t_G denote the graph's vertices, edges, and intial and terminating vertex functions, respectivley.

Definition 2.8. Let G be a graph. A *path in* G is a non-empty finite sequence of edges $\pi = e_1 \dots e_n$ such that $t(e_i) = i(e_{i+1})$ for all $i < n$. If I is a vertex in G, we say a path starts at I if $i(e_1) = I$ and denote it $i(\pi) = I$. Similarly, we say a path ends at I if $t(e_n) = I$ and denote it $t(\pi) = I$.

Definition 2.9. Let G be a graph. A *bi-infinite walk in G* is a bi-infinite sequence x over the edges of G such that $t(x_i) = i(x_{i+1})$ for all $i \in \mathbb{Z}$. The set X_G of all bi-infinite walks on G is denoted

$$\mathsf{X}_G \triangleq \left\{ (x_i)_{i\in\mathbb{Z}} \in \mathcal{E}^{\mathbb{Z}} \ : \ \forall i \in \mathbb{Z} \quad t(x_i) = i(x_{i+1}) \right\}.$$

Theorem 2.10 ([Lin+95])**.** If G is a graph, the n X_G is a shift space.

Notice that every word in $\mathcal{B}(\mathsf{X}_G)$ is in G, but every path in G is not necessarily a word in $\mathcal{B}(\mathsf{X}_G)$. Call a vertex *stranded* if there is no edge starting at I or ending at I. No bi-infinite walk can go through a stranded vertex, because a bi-infinite path must always have a "next" or "previous" vertex, and stranded vertices are exactly the vertices which have none. If G has no stranded vertices, then we call G *essential*. Every non-essential graph can be made essential by removing the stranded vertices, and the resulting graph will have still have the same edge shift. Then, if G is essential, then every path in G is necessarily a word in $\mathcal{B}(\mathsf{X}_G)$. Thus, it simplifies our disucssion to work with essential graphs, as it allows us to paths and words in the language of an edge shift synonymously.

If G is a graph and I and J are vertices in G, then we say *J is reachable from I* if there is a path starting at I and ending at J. We say G is *irreducible* every pair of vertices reachable from each other. Otherwise, we say G is *reducible*. Every irreducible graph is necessarily essential. If G is irreducible, then it it not too hard to see that X_G is irreducible. However, as we will discuss in the next section X_G is not necessarily reducible if G is reducible.

Definition 2.11. A *labeled graph* $\mathcal{G}$ is a pair $(G, \mathcal{L})$, where G is a graph and $\mathcal{L} : \mathcal{E} \to \mathcal{A}$ is the *labeling function* from the edges of G onto some finite alphabet $\mathcal{A}$. We refer to G as the *underlying graph*. As we did with graphs, if $\mathcal{G}$ is an arbitrary labeled graph, then let $\mathcal{V}_{\mathcal{G}}, \mathcal{E}_{\mathcal{G}}, i_{\mathcal{G}}$, and $t_{\mathcal{G}}$ denote the vertices, edges, initial and terminating vertex functions of the underlying graph. Additionally, let $\mathcal{L}_{\mathcal{G}}$ denote the labeling function and $\mathcal{A}_{\mathcal{G}}$ denote the set of labels appearing in $\mathcal{G}$ (i.e. the image of $\mathcal{L}_{\mathcal{G}}$).

If $\mathcal{G}$ is a labeled graph, then we will let $\mathcal{G}$ inherit properties of its underlying graph. Specifically, we say $\mathcal{G}$ is essential if its underlying graph is essential, $\mathcal{G}$ is irreducible if its underlying graph is irreducible, and $\mathcal{G}$ is reducible if its underlying graph is reducible, and so on.

Definition 2.12. Let $\mathcal{G} = (G, \mathcal{L})$ be a labeled graph. If x is a bi-infinite walk in G, then the *label of x* is the bi-infinite sequence $(\xi_i)_{i\in\mathbb{Z}}$, where $\xi_i = \mathcal{L}(x_i)$ for all $i \in \mathbb{Z}$. The set of all labels of bi-infinite walks in G is denoted

$$\mathsf{X}_{\mathcal{G}} \triangleq \left\{ (\xi_i)_{i\in\mathbb{Z}} : \xi_i = \mathcal{L}(x_i), x \in \mathsf{X}_G \right\}.$$

Theorem 2.13 ([Lin+95])**.** If $\mathcal{G}$ is a labeled graph, then $\mathsf{X}_{\mathcal{G}}$ is a shift space.

If X is a shift space and $X = \mathsf{X}_{\mathcal{G}}$ for some labeled graph $\mathcal{G}$, then we say X is a *sofic shift*. We say $\mathcal{G}$ *presents* X, and will also refer to $\mathcal{G}$ as a *presentation of* X.

Let $\mathcal{G}$ be a labeled graph. If $\pi = e_1 \dots e_n$ is a path in $\mathcal{G}$, then we will abuse notation and define the *label of the path* $\mathcal{L}(\pi) \triangleq w$, where $w = \mathcal{L}(e_1) \dots \mathcal{L}(e_n)$, and say that π *presents* w. Similar to graphs, every word in $\mathcal{B}(\mathsf{X}_{\mathcal{G}})$ is presented by some path in $\mathcal{G}$ but every word presented by a path in $\mathcal{G}$ is not necessarily a word in $\mathcal{B}(\mathsf{X}_{\mathcal{G}})$, unless $\mathcal{G}$ is essential.

Definition 2.14. Let $\mathcal{G} = (G, L)$ be a labeled graph. For a vertex $I \in \mathcal{V}_{\mathcal{G}}$, the *follower set of* I is the set

$$F(I) \triangleq \left\{\mathcal{L}(\pi) : \pi \in \mathcal{B}(\mathsf{X}_G), i(\pi) = I\right\}.$$

For a word $w \in \mathcal{B}(\mathsf{X}_{\mathcal{G}})$, the *follower set of* w is the set

$$F(w) \triangleq \left\{u \in \mathcal{B}(\mathsf{X}_{\mathcal{G}}) : wu \in \mathcal{B}(\mathsf{X}_{\mathcal{G}})\right\}$$

Definition 2.15. Let $\mathcal{G} = (G, \mathcal{L})$ be a labeled graph, I be some vertex in $\mathcal{G}$, and $w \in B(\mathsf{X}_{\mathcal{G}})$. If every path π in $\mathcal{G}$ that presents w ends at I, then we say w *synchronizes to* I. We say that w is *synchronizing for* $\mathcal{G}$ if it synchronizes to some vertex in $\mathcal{G}$. If there is a word that synchronizes to I, then we say that the vertex I *is synchronizing*. Finally, we denote $S(\mathcal{G})$ as the set of all syncrhonizing words for $\mathcal{G}$.

Let $\mathcal{G}$ be a deterministic labeled graph. It follows that for every vertex I in $\mathcal{G}$ and every $w \in F(I)$, there is a unique path labeled w starting at I. We can define a partial *transition function* $\delta_{\mathcal{G}} : \mathcal{V}_{\mathcal{G}} \times \mathcal{A}_{\mathcal{G}}^* \to \mathcal{V}_{\mathcal{G}}$ with $\delta_{\mathcal{G}}(I, w) \triangleq t(\pi)$ if there is a path π labeled w starting at I. If $w = \mathcal{L}(\pi)$ and $\mathcal{L}(\tau)$ for paths π and τ, then by determinism, $\pi = \tau$ so $t(\pi) = t(\tau)$, so $\delta_{\mathcal{G}}$ is well defined. It will be useful to make $\delta_{\mathcal{G}}$ total by

defining $\delta_{\mathcal{G}}(I, w) \triangleq 0$, if $w \notin F(I)$ and $\delta_{\mathcal{G}}(0, w) = 0$, where 0 is a constant distinct from the vertices of $\mathcal{G}$. Finally, define the *subset transition function* $\Delta_{\mathcal{G}} : \mathcal{P}(\mathcal{V}) \times w \to \mathcal{P}(\mathcal{V})$ with $\Delta_{\mathcal{G}}(S, w) \triangleq \{\delta_{\mathcal{G}}(I, w) : I \in S \text{ and } \delta_{\mathcal{G}}(I, w) \neq 0\}$.

Theorem 2.16. Let $\mathcal{G}$ be a deterministic labeled graph, and I and J be verticies in $\mathcal{G}$. The following properties of the transition function are true:

(i) $w \in F(I)$ if and only if $\delta_{\mathcal{G}}(I, w) \neq 0$
(ii) $w \in \mathcal{B}(\mathsf{X}_{\mathcal{G}})$ if and only if $\Delta_{\mathcal{G}}(\mathcal{V}_{\mathcal{G}}, w) \neq \varnothing$
(iii) w synchronizes to I if and only if $\Delta_{\mathcal{G}}(\mathcal{V}_{\mathcal{G}}, w) = \{I\}$
(iv) If π is a path in $\mathcal{G}$ starting at I and ending at J, then $\delta_{\mathcal{G}}(I, \mathcal{L}(\pi)) = J$

Definition 2.17. Let $\mathcal{G}$ be an essential labeled graph. The $\mathcal{G}$ *kill state graph with alphabet* $\mathcal{A}$ is the labeled graph $\mathcal{G}^0$ taking the same vertices and edges as $\mathcal{G}$ with an additional vertex 0 and edges such that for each $a \in \mathcal{A}$ and vertex I in $\mathcal{G}$, if there is no edge starting at I labeled a, then there is an edge in $\mathcal{G}^0$ between I and 0 labeled a.

Definition 2.18. Let $\mathcal{G}$ and $\mathcal{H}$ be labeled graphs. The *label product of $\mathcal{G}$ and $\mathcal{H}$* is the labeled graph $\mathcal{G} * \mathcal{H}$ such that if there is an edge e_1 in $\mathcal{G}$ between I_1 and J_1 and an edge e_2 in $\mathcal{H}$ between I_2 and J_2 with $\mathcal{L}_{\mathcal{G}}(e_1) = \mathcal{L}_{\mathcal{H}}(e_2)$, then there is an edge in $\mathcal{G} * \mathcal{H}$ between (I_1, I_2) and (J_1, J_2) labeled $\mathcal{L}_{\mathcal{G}}(e_1) = \mathcal{L}_{\mathcal{H}}(e_2)$. That is,

$$
\begin{aligned}
\mathcal{V}_{\mathcal{G}*\mathcal{H}} &\triangleq \mathcal{V}_{\mathcal{G}} \times \mathcal{V}_{\mathcal{H}} \\
\mathcal{E}_{\mathcal{G}*\mathcal{H}} &\triangleq \{(e_1, e_2) \in \mathcal{E}_{\mathcal{G}} \times \mathcal{E}_{\mathcal{H}} \ : \mathcal{L}(e_1) = \mathcal{L}(e_2)\} \\
i_{\mathcal{G}*\mathcal{H}}(e_1, e_2) &\triangleq (i_{\mathcal{G}}(e_1), i_{\mathcal{G}}(e_2)) \\
t_{\mathcal{G}*\mathcal{H}}(e_1, e_2) &\triangleq (t_{\mathcal{H}}(e_1), t_{\mathcal{H}}(e_2)) \\
\mathcal{L}_{\mathcal{G}*\mathcal{H}}(e_1, e_2) &\triangleq \mathcal{L}_{\mathcal{G}}(e_1) \qquad (= \mathcal{L}_{\mathcal{H}}(e_2))
\end{aligned}
$$

It is not too hard to see that if π is a path in $\mathcal{G} * \mathcal{H}$ starting at (I_1, J_1) and ending at (I_2, J_2), then there is a path l

Theorem 2.19. If there is a path in $\mathcal{G} * \mathcal{H}$ labeled w starting at (I, J), then there is a path in $\mathcal{G}$ labeled w starting at I and a path in $\mathcal{H}$ labeled w starting at J.

Add theorem showing that $\mathsf{X}_{\mathcal{G}}$ and $\mathsf{X}_{\mathcal{G}}$ are shift spaces.

Add theorem showing that shift spaes are uniquely defined by a language.

3 Irreducibility

Consider the following reducible presentation:

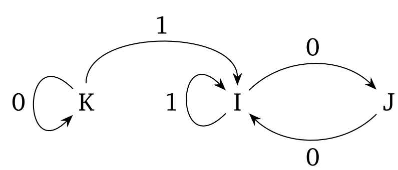

Figure 1: A reducible, follower-separated presentation of the even shift

The shift presented by subgraph induced by I and J is the even shift, and every word presented by a path starting from K is presented by some path starting at I and J. Hence, this graph presents the even shift. Additionly, it is also follower-separated as $01 \in F(K) \backslash F(I)$, $1 \in F(K) \backslash F(J)$, and $1 \in F(I) \backslash F(J)$. When do follower-separated reducible graphs present irreducible shifts? We will first look at a simple class of reducible graphs.

Let $\mathcal{G} \vec{\ } \mathcal{H}$ be an essential, deterministic, follower-separated graph with two irreducible components that induce two subgraphs, namely $\mathcal{G}$ and $\mathcal{H}$, such that there is exactly one edge starting in $\mathcal{G}$ and ending in $\mathcal{H}$. Some properties of $\mathcal{G}^{\to}\mathcal{H}$ are that the vertices of $\mathcal{G}$ and $\mathcal{H}$ partition the vertices of $\mathcal{G}^{\to}\mathcal{H}$, both $\mathcal{G}$ and $\mathcal{H}$ are essential, any vertex in $\mathcal{H}$ is reachable from any vertex in $\mathcal{G}^{\to}\mathcal{H}$, and no vertex in $\mathcal{G}$ is reachable from any vertex in $\mathcal{H}$.

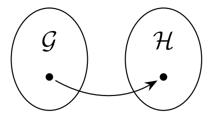

Figure 2: Representation of $\mathcal{G}^{\to}\mathcal{H}$.

Related to a comment below, but this is just a corollary of [Lind and Marcus 3.3.16] and the nature of $\mathcal{G}^{\to}\mathcal{H}$

Theorem 3.1. Every word $u \in \mathcal{B}(\mathsf{X}_{\mathcal{G}\to\mathcal{H}})$ can be extended on the right to a word $uw \in \mathcal{B}(\mathsf{X}_{\mathcal{G}\to\mathcal{H}})$ that syncrhonizes to a vertex I in $\mathcal{H}$.

Proof. Let $u \in \mathcal{B}(\mathsf{X}_{\mathcal{G}\to\mathcal{H}})$. As $\mathcal{G}^{\to}\mathcal{H}$ is deterministic and follower-separated, then by [Lin+95] Theorem 3.3.16, we can extend u on the right to $uw \in \mathcal{B}(\mathsf{X}_{\mathcal{G}\to\mathcal{H}})$ so that uw synchronizes to some vertex J. As I is a vertex in $\mathcal{H}$, I can be reached from J, so let v be the label of a path from I to J. Then, $uwv \in \mathcal{B}(\mathsf{X}_{\mathcal{G}\to\mathcal{H}})$ and any path presenting uwv must end at J, so uwv synchornizes to I. $\square$

Corollary 3.2. Every vertex in $\mathcal{H}$ is synchronizing for $\mathcal{G}^{\to}\mathcal{H}$.

Proof. For a vertex I in $\mathcal{H}$, take any word $u \in \mathcal{B}(\mathsf{X}_{\mathcal{G}\to\mathcal{H}})$ and use Theorem 3.1 to synchronize it to I. Hence, there is a word that synchronizes to I, so I is synchronizing. $\square$

Lemma 3.3. Let $u \in \mathcal{B}(\mathsf{X}_{\mathcal{G}\to\mathcal{H}})$. If $u \notin \mathcal{B}(\mathsf{X}_{\mathcal{H}})$, then $\mathsf{X}_{\mathcal{G}\to\mathcal{H}}$ is reducible.

Proof. By Theorem 3.1, we can extend u on the right to a word $uw \in \mathcal{B}(\mathsf{X}_{\mathcal{G}\to\mathcal{H}})$ so that any path presenting uw ends at some vertex I in $\mathcal{H}$. Hence, for any word $v \in F(uw)$, a path presenting uwv ends at some vertex in $\mathcal{H}$, so $F(uwv) \subseteq \mathcal{B}(\mathsf{X}_{\mathcal{H}})$. As $u \notin \mathcal{B}(\mathsf{X}_{\mathcal{H}})$, we have $u \notin F(uwv)$. Thus, there is no word in $\mathcal{B}(\mathsf{X}_{\mathcal{G}\to\mathcal{H}})$ joining the word uw and u, so $\mathsf{X}_{\mathcal{G}\to\mathcal{H}}$ is reducible. $\square$

Probably will expand this later. Argue that uwv is synchronizing, so $F(uwv) = F(J)$ for some vertex J in $\mathcal{H}$, so $F(uwv) = F(J) \subseteq \bigcup_{K\in\mathcal{V}_{\mathcal{H}}} F(K) = \mathcal{B}(\mathsf{X}_{\mathcal{H}})$

Theorem 3.4. $\mathsf{X}_{\mathcal{G}\to\mathcal{H}}$ is irreducible if and only if $\mathsf{X}_{\mathcal{G}\to\mathcal{H}} = \mathsf{X}_{\mathcal{H}}$.

Proof. Suppose $\mathsf{X}_{\mathcal{G}\to\mathcal{H}}$ is irreducible, and let $w \in \mathcal{B}(\mathsf{X}_{\mathcal{G}\to\mathcal{H}})$. By Lemma 3.3, as $\mathsf{X}_{\mathcal{G}\to\mathcal{H}}$ is irreducible, then $w \in \mathcal{B}(\mathsf{X}_{\mathcal{H}})$, so $\mathcal{B}(\mathsf{X}_{\mathcal{G}\to\mathcal{H}}) \subseteq \mathcal{B}(\mathsf{X}_{\mathcal{H}})$. By construction of $\mathcal{G}^{\to}\mathcal{H}$, we have $\mathcal{B}(\mathsf{X}_{\mathcal{H}}) \subseteq \mathcal{B}(\mathsf{X}_{\mathcal{G}\to\mathcal{H}})$. Therefore, $\mathcal{B}(\mathsf{X}_{\mathcal{G}\to\mathcal{H}}) = \mathcal{B}(\mathsf{X}_{\mathcal{H}})$ so $\mathsf{X}_{\mathcal{G}\to\mathcal{H}} = \mathsf{X}_{\mathcal{H}}$. Conversely, suppose $\mathsf{X}_{\mathcal{G}\to\mathcal{H}} = \mathsf{X}_{\mathcal{H}}$. As $\mathsf{X}_{\mathcal{H}}$ is irreducible, then so is $\mathsf{X}_{\mathcal{G}\to\mathcal{H}}$. $\square$

Corollary 3.5. If $\mathsf{X}_{\mathcal{G}\to\mathcal{H}}$ is irreducible, then $\mathsf{X}_{\mathcal{G}} \subseteq \mathsf{X}_{\mathcal{H}}$.

Proof. By Theorem 3.4, if $\mathsf{X}_{\mathcal{G}\to\mathcal{H}}$ is irreducible, then $\mathsf{X}_{\mathcal{G}\to\mathcal{H}} = \mathsf{X}_{\mathcal{H}}$. As $\mathsf{X}_{\mathcal{G}} \subseteq \mathsf{X}_{\mathcal{G}\to\mathcal{H}}$, then $\mathsf{X}_{\mathcal{G}} \subseteq \mathsf{X}_{\mathcal{H}}$. □

Theorem 3.6. $\mathsf{X}_{\mathcal{G}} \subseteq \mathsf{X}_{\mathcal{H}}$ if and only if no vertex in $\mathcal{G}$ is synchronizing for $\mathcal{G} \vec{\ } \mathcal{H}$.

Proof. Suppose $\mathsf{X}_{\mathcal{G}} \nsubseteq \mathsf{X}_{\mathcal{H}}$. Then, there is a word $w \in \mathcal{B}(\mathsf{X}_{\mathcal{G}})$ but $w \notin \mathcal{B}(\mathsf{X}_{\mathcal{H}})$. We can extend w to a word $wu \in \mathcal{B}(\mathsf{X}_{\mathcal{G}})$ such that wu is synchronizing for $\mathcal{G}$ □

Corollary 3.7.

Theorem 3.8. If $\mathsf{X}_{\mathcal{G}\to\mathcal{H}}$ is irreducible, then no vertex in $\mathcal{G}$ is synchronizing for $\mathcal{G} \vec{\ } \mathcal{H}$.

Proof. Suppose there were a vertex I in $\mathcal{G}$ that is synchronizing for $\mathcal{G} \vec{\ } \mathcal{H}$. Let v be a word that synchronizes to I in $\mathcal{G} \vec{\ } \mathcal{H}$. By Corollary 3.2, every vertex in $\mathcal{H}$ is synchronizing for $\mathcal{G} \vec{\ } \mathcal{H}$, so let J be some vertex in $\mathcal{H}$ and u be a word that synchronizes to J in $\mathcal{G} \vec{\ } \mathcal{H}$. As $\mathsf{X}_{\mathcal{G}\to\mathcal{H}}$ is irreducible, there is a word $w \in \mathcal{B}(\mathsf{X}_{\mathcal{G}\to\mathcal{H}})$ such that $uwv \in \mathcal{B}(\mathsf{X}_{\mathcal{G}\to\mathcal{H}})$. However, any path presenting uwv must first visit J (as u is synchronizing) and then visit I, implying that I is reachable from J. But this contradicts the structure of $\mathcal{G} \vec{\ } \mathcal{H}$, so no vertex in $\mathcal{G}$ can be synchronizing for $\mathcal{G} \vec{\ } \mathcal{H}$. □

Theorem 3.9. $\mathsf{X}_{\mathcal{G}\to\mathcal{H}} = \mathsf{X}_{\mathcal{H}}$ if and only if $S(\mathcal{G} \vec{\ } \mathcal{H}) \subseteq S(\mathcal{H})$.

Proof. Suppose $\mathsf{X}_{\mathcal{G}\to\mathcal{H}} = \mathsf{X}_{\mathcal{H}}$. Let $w \in B(\mathsf{X}_{\mathcal{G}\to\mathcal{H}})$ synchronize to some vertex I in $\mathcal{G} \vec{\ } \mathcal{H}$. If π is some path in $\mathcal{H}$ presenting w, then it must end at I. By Theorem 3.8, I cannot be a vertex in $\mathcal{G}$, so it must be a vertex in $\mathcal{H}$. Therefore, every path in $\mathcal{H}$ presenting w ends at I, so $S(\mathcal{G} \vec{\ } \mathcal{H}) \subseteq S(\mathcal{H})$.

Conversely, suppose $S(\mathcal{G} \vec{\ } \mathcal{H}) \subseteq S(\mathcal{H})$. Let $u \in \mathcal{B}(\mathsf{X}_{\mathcal{G}\to\mathcal{H}})$. By Theorem 3.1, we can extend u to a word $uw \in \mathcal{B}(\mathsf{X}_{\mathcal{G}\to\mathcal{H}})$ that synchronizes to some vertex I in $\mathcal{G} \vec{\ } \mathcal{H}$. Hence, uw is synchronizing for $\mathcal{G} \vec{\ } \mathcal{H}$, so it is also synchronizing for $\mathcal{H}$. But if uw is synchronizing for $\mathcal{H}$, then by definition, $uw \in \mathcal{B}(\mathsf{X}_{\mathcal{H}})$. Therefore, $\mathcal{B}(\mathsf{X}_{\mathcal{G}\to\mathcal{H}}) \subseteq \mathcal{B}(\mathsf{X}_{\mathcal{H}})$, so $\mathsf{X}_{\mathcal{G}\to\mathcal{H}} = \mathsf{X}_{\mathcal{H}}$. □

Theorem 3.10. If $w \in S(\mathcal{G} \vec{\ } \mathcal{H})$ but $w \notin S(\mathcal{H})$, then $w \notin \mathcal{B}(\mathsf{X}_{\mathcal{H}})$.

Proof. As $w \in S(\mathcal{G} \vec{\ } \mathcal{H})$, there is a vertex I in $\mathcal{G} \vec{\ } \mathcal{H}$ such that every path in $\mathcal{G} \vec{\ } \mathcal{H}$ presenting w ends at I. If $w \notin S(\mathcal{H})$, then either $w \notin \mathcal{B}(\mathsf{X}_{\mathcal{H}})$ or for every vertex J in $\mathcal{H}$ there is a path in $\mathcal{H}$ that presents w but does not end at J. Hence if the latter condition were true, then this implies there is a path in $\mathcal{H}$ that presents w and does not end at I. But this path is also in $\mathcal{G} \vec{\ } \mathcal{H}$ and presents w, so it should end at I, which is a contradiction. Therefore, $w \notin \mathcal{B}(\mathsf{X}_{\mathcal{H}})$. □

Theorem 3.11. $\mathsf{X}_{\mathcal{G}\to\mathcal{H}}$ is irreducible if and only if $\mathsf{X}_{\mathcal{G}} \subseteq \mathsf{X}_{\mathcal{H}}$ and for all vertices I in $\mathcal{G}$ there is a word w and vertex J in $\mathcal{H}$ such that $\delta_{\mathcal{G}\to\mathcal{H}}(I, w) = \delta_{\mathcal{G}\to\mathcal{H}}(J, w)$.

Proof. Suppose $\mathsf{X}_{\mathcal{G}} \subseteq \mathsf{X}_{\mathcal{H}}$ and for all vertices I in $\mathcal{G}$ there is a word w and vertex J in $\mathcal{H}$ such that $\delta_{\mathcal{G}\to\mathcal{H}}(I,w) = \delta_{\mathcal{G}\to\mathcal{H}}(J,w)$. Let $w \in \mathcal{B}(\mathsf{X}_{\mathcal{G}\to\mathcal{H}})$ and π be a path in $\mathcal{G}^{\to}\mathcal{H}$ presenting w. If π starts in $\mathcal{H}$, then $w \in \mathcal{B}(\mathsf{X}_{\mathcal{H}})$. Otherwise, $\square$

4 Subshift testing

Algorithm 4.1 Subshift testing

Require: $\mathcal{G}$ and $\mathcal{H}$ are deterministic presentations

1: **procedure** IS_SUBSHIFT($\mathcal{G}, \mathcal{H}$)
2: $\quad I \leftarrow$ any element in $\mathcal{V}_{\mathcal{G}}$
3: $\quad X \leftarrow \mathcal{V}_{\mathcal{H}}$
4: $\quad w \leftarrow \epsilon$
5: $\quad$ **repeat**
6: $\quad\quad J \leftarrow$ any element in X
7: $\quad\quad$ find a word u such that $\delta_{\mathcal{G}}(I, u) \neq 0$ and $\delta_{\mathcal{H}}(J, u) = 0$
8: $\quad\quad$ **if** u exists **then**
9: $\quad\quad\quad w \leftarrow wu$
10: $\quad\quad\quad I \leftarrow \delta_{\mathcal{G}}(I, u)$
11: $\quad\quad\quad X \leftarrow \Delta_{\mathcal{H}}(X, u)$
12: $\quad\quad\quad$ **if** $X = \varnothing$ **then**
13: $\quad\quad\quad\quad$ **return** false
14: $\quad\quad\quad$ **end if**
15: $\quad\quad$ **end if**
16: $\quad$ **until** u does not exist
17: $\quad$ **return** true
18: **end procedure**

Lemma 4.1. If I_0 is the value of I before entering the loop on lines 5-16, the invariants

(i) $\delta_{\mathcal{G}}(I_0, w) = I$

(ii) $I \neq 0$

(iii) $\Delta_{\mathcal{H}}(\mathcal{V}_{\mathcal{H}}, w) = X$

hold before the beginning of each iteration the loop. Additionally, the loop always terminates.

Proof. Clearly, $I = I_0$ and $w = \epsilon$ before entering the loop, so $\Delta_{\mathcal{G}}(I_0, w) = \delta_{\mathcal{G}}(I, \epsilon) = I$. As $I_0 \in \mathcal{V}_{\mathcal{G}}$, we have that $I \neq 0$. Finally, as $X = \mathcal{V}_{\mathcal{G}}$ before entering the loop, $\Delta_{\mathcal{H}}(\mathcal{V}_{\mathcal{H}}, w) = \Delta_{\mathcal{H}}(X, \epsilon) = X)$, so the invariants hold before entering the loop.

Suppose the invariants were true before the current iteration of the loop and I, J, w, and X had the values I_n, J_n, w_n and X_n, respectively. Furthermore, suppose the loop does not exit after the current iteration. Because of this, then the algorithm must enter the if statement on line 8 (and not enter the if statement on line 12), so there is a word

u such that $\delta_{\mathcal{G}}(I_n, u) \neq 0$ and $\delta_{\mathcal{H}}(J_n, u) = 0$. We can see that $w = w_n u$, $I = \delta_{\mathcal{G}}(I_n, u)$, and $X = \Delta_{\mathcal{H}}(X_n, u)$ at the end of the current iteration, so from this, we can see

$$\delta_{\mathcal{G}}(I_0, w) = \delta_{\mathcal{G}}(I_0, w_n u) = \delta_{\mathcal{G}}(\delta_{\mathcal{G}}(I_0, w_n), u) = \delta_{\mathcal{G}}(I_n, u) = I$$

and similarly,

$$\Delta_{\mathcal{H}}(\mathcal{V}_{\mathcal{H}}, w) = \Delta_{\mathcal{H}}(\mathcal{V}_{\mathcal{H}}, w_n u) = \Delta_{\mathcal{H}}(\Delta_{\mathcal{H}}(\mathcal{V}_{\mathcal{H}}, w_n), u) = \Delta_{\mathcal{H}}(X_n, u) = X.$$

Therefore, all the invariants hold before the next iteration of the loop. Finally, note that as $\delta_{\mathcal{H}}(J_n, u) = 0$ and $J_n \in X$, $|X|$ is strictly less than $|X_n|$. This guarantees the loop terminates as if the exit condition is never true, then $|X| = 0$ will be true for some iteration so $X = \varnothing$ and the algorithm exits on line 13. □

Theorem 4.2. If Algorithm 4.1 returns false, then $\mathsf{X}_{\mathcal{G}} \nsubseteq \mathsf{X}_{\mathcal{H}}$.

Proof. If the algorithm returned false, then it exited at line 13, so we know $X = \varnothing$. By Lemma 4.1, $\delta_{\mathcal{G}}(I_0, w) = I \neq 0$ so $w \in \mathcal{B}(\mathsf{X}_{\mathcal{G}})$ and $\Delta_{\mathcal{H}}(\mathcal{V}_{\mathcal{H}}, w) = \varnothing$ so $w \notin \mathcal{B}(\mathsf{X}_{\mathcal{H}})$. Therefore, $\mathsf{X}_{\mathcal{G}} \nsubseteq \mathsf{X}_{\mathcal{H}}$. □

Theorem 4.3. If $\mathcal{G}$ is irreducible and $\mathsf{X}_{\mathcal{G}} \nsubseteq \mathsf{X}_{\mathcal{H}}$, then Algorithm 4.1 returns false.

Proof. As $\mathsf{X}_{\mathcal{G}} \nsubseteq \mathsf{X}_{\mathcal{H}}$, there is a word $w \in \mathcal{B}(\mathsf{X}_{\mathcal{G}})$ but $w \notin \mathcal{B}(\mathsf{X}_{\mathcal{H}})$. As $w \in \mathcal{B}(\mathsf{X}_{\mathcal{G}})$, then $\delta_{\mathcal{G}}(I^*, w) \neq 0$ for some vertex I^* in $\mathcal{G}$, and as $w \notin \mathcal{B}(\mathsf{X}_{\mathcal{H}})$, then $\delta_{\mathcal{H}}(J^*, w) = 0$ for all vertices J^* in $\mathcal{H}$. Therefore, for any value of I, as $\mathcal{G}$ is irreducible, there is a path π starting at I and ending at I^*, so

$$\delta(I, \mathcal{L}(\pi)w) = \delta_{\mathcal{G}}(\delta_{\mathcal{G}}(I, \mathcal{L}(\pi)), w) = \delta_{\mathcal{G}}(I^*, w) \neq 0$$

and for any value of J, $\delta_{\mathcal{H}}(J, \mathcal{L}(\pi)w) = \delta_{\mathcal{H}}(\delta_{\mathcal{H}}(J, \mathcal{L}(\pi)), w) = \varnothing$. Hence, the algorithm can always find a word u on line 7. If the algorithm picks u to be $\mathcal{L}(\pi)w$, then the algorithm terminates, as $\Delta_{\mathcal{H}}(\mathcal{V}_{\mathcal{H}}, \mathcal{L}(\pi)w) = \varnothing$. If not, then as discussed in Lemma 4.1, the the current value of $|X|$ is strictly less than the previous value of $|X|$. Therefore, as the algorithm can always find a word u on line 7, then $|X| = 0$ eventually and thus will return false eventually. □

Theorem 4.4. Algorithm 4.1 can be computed in $O(|\mathcal{V}_{\mathcal{G}}| \cdot |\mathcal{V}_{\mathcal{H}}|^3 + |\mathcal{E}_{\mathcal{G}}| \cdot |\mathcal{E}_{\mathcal{H}}| \cdot |\mathcal{V}_{\mathcal{H}}|)$ time.

Proof. As $|X|$ goes down by at minimum once every iteration of the loop, and $|X| = O(|\mathcal{V}_{\mathcal{H}}|)$, the loop repeats $O(|\mathcal{V}_{\mathcal{H}}|)$ times. To compute line 7, one can construct the graph $\mathcal{G} * \mathcal{H}^0$ and and determine if there is a path from (I, J) to any vertex in the set $\{(K, 0) : K \in \mathcal{V}_{\mathcal{G}}\}$ using, say, a depth-first search. As $|\mathcal{V}_{\mathcal{G}*\mathcal{H}^0}| = O(|\mathcal{V}_{\mathcal{G}}| \cdot |\mathcal{V}_{\mathcal{H}}|)$ and $|\mathcal{E}_{\mathcal{G}*\mathcal{H}^0}| = O(|\mathcal{E}_{\mathcal{G}}| \cdot |\mathcal{E}_{\mathcal{H}}|)$, then with depth-first search, this step can be computed in $O(|\mathcal{V}_{\mathcal{G}}| \cdot |\mathcal{V}_{\mathcal{H}}| + |\mathcal{E}_{\mathcal{G}}| \cdot |\mathcal{E}_{\mathcal{H}}|)$ time.

As $|u| = O(|\mathcal{V}_\mathcal{G}| \cdot |\mathcal{V}_\mathcal{H}|)$, computing $\delta_\mathcal{G}(I, u)$ takes $O(|\mathcal{V}_\mathcal{G}| \cdot |\mathcal{V}_\mathcal{H}|)$ time. Similarly, computing $\Delta_\mathcal{H}(X, u)$ is reduces to computing $\delta_\mathcal{H}$ at most $|X|$ times, so this can be computed in $O(|\mathcal{V}_\mathcal{G}| \cdot |\mathcal{V}_\mathcal{H}|) \cdot O(|\mathcal{V}_\mathcal{H}|) = O(|\mathcal{V}_\mathcal{G}| \cdot |\mathcal{V}_\mathcal{H}|^2)$ time. In total, this is

$$\begin{aligned}
&O(|\mathcal{V}_\mathcal{H}|) \cdot \big(O(|\mathcal{V}_\mathcal{G}| \cdot |\mathcal{V}_\mathcal{H}| + |\mathcal{E}_\mathcal{G}| \cdot |\mathcal{E}_\mathcal{H}|) + O(|\mathcal{V}_\mathcal{G}| \cdot |\mathcal{V}_\mathcal{H}|) + O(|\mathcal{V}_\mathcal{G}| \cdot |\mathcal{V}_\mathcal{H}|^2)\big) \\
&= O(|\mathcal{V}_\mathcal{H}|) \cdot O(|\mathcal{V}_\mathcal{G}| \cdot |\mathcal{V}_\mathcal{H}|^2 + |\mathcal{E}_\mathcal{G}| \cdot |\mathcal{E}_\mathcal{H}|) \\
&= O(|\mathcal{V}_\mathcal{G}| \cdot |\mathcal{V}_\mathcal{H}|^3 + |\mathcal{E}_\mathcal{G}| \cdot |\mathcal{E}_\mathcal{H}| \cdot |\mathcal{V}_\mathcal{H}|).
\end{aligned}$$

□

5 Discussion and future work

6 Minimality

This section will probably hardness of sofic shift problems. As the one of the earlier sections that was written, this could be rewritten and streamlined a bit with the other sections. Other things to do: define "irreducible shift" decision problem. Give reduction from GI to irreducible shift, conclude irreducible shift is GI-hard. Give reduction from irreducible shift to minimality, conclude minimality is GI-hard.

Move this to an appropriate section.

Theorem 6.1. If $\mathcal{G} = (G, \mathcal{L})$ is the minimizing right resolving presentation of an irreducible sofic shift X and X is and N-step shift of finite type, then $\mathsf{X}_G \cong \mathsf{X}_{\mathcal{G}}$.

Proof. Let x, y be walks in X_G. Suppose $\mathcal{L}_\infty(x) = \mathcal{L}_\infty(y)$, For any i, the paths $x_{[i-N,i-1]}$ and $y_{[i-N,i-1]}$ present the same word. Because that word is of length N, the word is synchronizing for $\mathcal{G}$ (from 3.4.17), so those paths end at the same vertex. Since $\mathcal{L}(x_{[i]}) = \mathcal{L}(y_{[i]})$, $\mathcal{G}$ is right resolving, and $x_{[i-N,i-1]}$ and $y_{[i-N,i-1]}$ end at the same vertex, then $x_{[i]} = y_{[i]}$ and hence $x = y$, so $\mathcal{L}_\infty$ is injective. By definition, $\mathcal{L}_\infty$ is surjective. Therefore, $\mathcal{L}_\infty$ is bijective and a conjugacy from X_G to $\mathsf{X}_{\mathcal{G}}$. □

Move this to an appropriate section.

Lemma 6.2. If X and Y are shift spaces, then $\mathcal{B}(X) \subseteq \mathcal{B}(Y)$ if and only if $X \subseteq Y$.

Proof. Let x be a point in X. Then every word that appears in x is in $\mathcal{B}(X)$. Since $\mathcal{B}(X) \subseteq \mathcal{B}(Y)$, then every word that appears in x is in $\mathcal{B}(Y)$, so $x \in Y$, hence $X \subseteq Y$.

Conversely, let w be a word in $\mathcal{B}(X)$. Then w occurs in some $x \in X$. Since $X \subseteq Y$, we have $x \in Y$, so w occurs in some $x \in Y$. Hence, $w \in \mathcal{B}(Y)$. □

Let $\mathcal{G}$ and $\mathcal{H}$ be labeled graphs, I be a vertex from $\mathcal{G}$, and J be a vertex from $\mathcal{H}$. Define the *graph connecting $\mathcal{G}$ to $\mathcal{H}$ via I and J* as the disjoint union of the two graphs, adding an edge starting at I and ending at J, and adding a self loop on J. Label these two new edges with a symbol that does not appear in either graph. Since $\mathcal{G}$ and $\mathcal{H}$ are subgraphs of a graph connecting the two, it follows that the presentations of the individual graphs are subshifts of a presentation of a graph connecting the two - any bi-infinite walk in one of the graphs is a x bi-infinite walk of the corresponding subgraph of the connected graphs. Additionally, observe that the graph is reducible, as any path starting in $\mathcal{H}$ cannot end in $\mathcal{G}$.

Theorem 6.3. Let $\mathcal{G}$ and $\mathcal{H}$ be irreducible graphs, and $\mathcal{K}$ be the graph connecting $\mathcal{G}$ to $\mathcal{H}$ via I and J. If $\mathsf{X}_{\mathcal{K}}$ is irreducible, then $\mathsf{X}_{\mathcal{G}} \subseteq \mathsf{X}_{\mathcal{H}}$.

Proof. First, suppose that $\mathsf{X}_{\mathcal{K}}$ is irreducible, and let $u \in \mathcal{B}(\mathsf{X}_{\mathcal{G}})$. There is a path in $\mathcal{G}$ that presents u, hence there is a path in the $\mathcal{G}$ subgraph of $\mathcal{K}$ that presents u. From the irreducibility of $\mathcal{G}$, there is a path from the terminating vertex of a path presenting u to

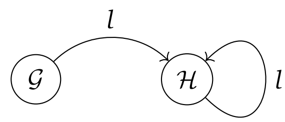

Figure 3: A graph connecting $\mathcal{G}$ to $\mathcal{H}$.

I. Let v be the word such path presents and l be the label of the edge connecting $\mathcal{G}$ to $\mathcal{H}$, so that we have $uvl \in \mathcal{B}(\mathsf{X}_{\mathcal{K}})$ and $u \in \mathcal{B}(\mathsf{X}_{\mathcal{K}})$. As $\mathsf{X}_{\mathcal{K}}$ is irreducible, there exists a word $w \in \mathcal{B}(\mathsf{X}_{\mathcal{K}})$ such that $uvlwu \in \mathcal{B}(\mathsf{X}_{\mathcal{K}})$. A path presenting $uvlwu$ must have the subpath presenting wu visit vertices only from the $\mathcal{H}$ subgraph of $\mathcal{K}$. This implies that there is a path in $\mathcal{H}$ presenting u, so we have $u \in \mathcal{B}(\mathsf{X}_{\mathcal{H}})$ and therefore $\mathcal{B}(\mathsf{X}_{\mathcal{G}}) \subseteq \mathcal{B}(\mathsf{X}_{\mathcal{H}})$, and $\mathsf{X}_{\mathcal{G}} \subseteq \mathsf{X}_{\mathcal{H}}$ via Lemma 1. □

Theorem 6.4. Let $\mathcal{G}$ and $\mathcal{H}$ be irreducible, minimal, right-resolving presentations. If $\mathsf{X}_{\mathcal{G}} = \mathsf{X}_{\mathcal{H}}$, then there exists a pair of vertices $(I, J) \in (\mathcal{V}_{\mathcal{G}} \times \mathcal{V}_{\mathcal{H}})$ such that both the graph connecting $\mathcal{G}$ to $\mathcal{H}$ via I and J and the graph connecting $\mathcal{H}$ to $\mathcal{G}$ via J and I both present irreducible shifts.

Equivalently, if for every pair of vertices $(I, J) \in (\mathcal{V}_{\mathcal{G}}, \mathcal{V}_{\mathcal{H}})$ the graph connecting $\mathcal{G}$ to $\mathcal{H}$ via I and J and the graph connecting $\mathcal{H}$ to $\mathcal{G}$ via J and I do not both present irreducible shifts.

Proof. Suppose $\mathsf{X}_{\mathcal{G}} = \mathsf{X}_{\mathcal{H}}$. Since $\mathcal{G}$ and $\mathcal{H}$ are irreducible, minimal, right-resolving presentations of the same shift, they must be isomorphic. Let $(\partial\Phi, \Phi)$ be a graph isomorphism between them. Choose an arbitrary vertex I from $\mathcal{G}$, and then let $\mathcal{K}$ be the graph connecting $\mathcal{G}$ to $\mathcal{H}$ via I and $\partial\Phi(I)$. Let f be the self loop on $\partial\Phi(I)$ added in the construction of $\mathcal{K}$, and $\mathcal{H}^+$ be the $\mathcal{H}$ of $\mathcal{K}$ subgraph plus f. As $\mathcal{H}$ is irreducible, then $\mathcal{H}^+$ is irreducible. It suffices to show that $\mathsf{X}_{\mathcal{K}} = \mathsf{X}_{\mathcal{H}^+}$ to show $\mathsf{X}_{\mathcal{K}}$ is irreducible.

Let u be a word from $\mathcal{B}(\mathsf{X}_{\mathcal{K}})$, and π be a path that presents it. Without loss of generality, assume u and π are nonempty. If π starts in $\mathcal{H}^+$, then $u \in \mathcal{B}(\mathsf{X}_{\mathcal{H}^+})$. Otherwise, it starts in $\mathcal{G}$ and either ends in $\mathcal{G}$ or ends in $\mathcal{H}^+$. For the case of π ending in $\mathcal{G}$, then $\Phi(\pi)$ is a path in $\mathcal{H}$ presenting u, as $(\partial\Phi, \Phi)$ is a graph isomorphism, so $u \in \mathcal{B}(\mathsf{X}_{\mathcal{H}^+})$. For the case of π ending in $\mathcal{H}^+$, we can split π into $\pi = \pi_1 e \pi_2$, where π_1 and π_2 are (possibly empty) paths from $\mathcal{G}$ and $\mathcal{H}^+$, respectivley, and e is the edge connecting $\mathcal{G}$ and $\mathcal{H}$. Note that $\Phi(\pi_0)$ is a path in $\mathcal{H}$ and that it terminates at $\partial\Phi(I)$ as π_0 terminates at I, assuming π_0 is nonempty. From this, we have that $\Phi(\pi_0) f \pi_1$ is path in $\mathcal{H}^+$ that presents u, so $u \in \mathcal{B}(\mathsf{X}_{\mathcal{H}^+})$.

Hence, we have $\mathcal{B}(\mathsf{X}_{\mathcal{K}}) \subseteq \mathcal{B}(\mathsf{X}_{\mathcal{H}^+})$, and from the construction of $\mathcal{K}$, we also have

$\mathcal{B}(\mathsf{X}_{\mathcal{H}^+}) \subseteq \mathcal{B}(\mathsf{X}_{\mathcal{K}})$, so $\mathcal{B}(\mathsf{X}_{\mathcal{K}}) = \mathcal{B}(\mathsf{X}_{\mathcal{H}^+})$ and $\mathsf{X}_{\mathcal{K}} = \mathsf{X}_{\mathcal{H}^+}$. Thus, we have shown the graph connecting $\mathcal{G}$ to $\mathcal{H}$ via I and $\partial\Phi(I)$ presents an irreducible shift. A similar argument can be made showing that the graph connecting $\mathcal{H}$ to $\mathcal{G}$ via $\partial\Phi(I)$ and I presents an irreducible shift (start from "...and then let $\mathcal{K}$", and replace $\mathcal{G} \mapsto \mathcal{H}, \mathcal{H} \mapsto \mathcal{G}, I \mapsto \partial\Phi(I), \partial\Phi(I) \mapsto I, \mathcal{H}^+ \mapsto \mathcal{G}^+, \Phi(\pi_0) \mapsto \Phi^{-1}(\pi_0)$). □

Theorem 6.5. Given an oracle for minimizing a reducible presentation, deciding if two irreducible, minimal, right-resolving labeled graphs are isomorphic can be determined in polynomial time.

Proof. Let the decision procedure be as follows: for every pair of vertices $(I, J) \in (\mathcal{V}_{\mathcal{G}} \times \mathcal{V}_{\mathcal{H}})$ between the graphs, construct and set $\mathcal{G}_I \to \mathcal{H}_J$ as the graph connecting $\mathcal{G}$ to $\mathcal{H}$ via I and J, and similarly, construct and set $\mathcal{H}_J \to \mathcal{G}_I$ as the graph connecting $\mathcal{H}$ to $\mathcal{G}$ via J and I. Then, using the oracle, minimize $\mathcal{G}_I \to \mathcal{H}_J$ and check if it is strongly connected (that is, irreducible), and if so, then from Theorem 2, we know we can set $\mathsf{X}_{\mathcal{G}} \subseteq \mathsf{X}_{\mathcal{H}}$ to be true. Similary, minimize $\mathcal{H}_J \to \mathcal{G}_I$ and check if it is strongly connected, and if so, set $\mathsf{X}_{\mathcal{H}} \subseteq \mathsf{X}_{\mathcal{G}}$ to be true. If at any point both $\mathsf{X}_{\mathcal{G}} \subseteq \mathsf{X}_{\mathcal{H}}$ and $\mathsf{X}_{\mathcal{H}} \subseteq \mathsf{X}_{\mathcal{G}}$ are set to true, then $\mathsf{X}_{\mathcal{G}} = \mathsf{X}_{\mathcal{H}}$ and thus can conclude $\mathcal{G} \cong \mathcal{H}$ (as $\mathcal{G}$ and $\mathcal{H}$ are unique, minimal, and irreducible presentations). If you find that after every pair of verticies that one or both of $\mathsf{X}_{\mathcal{G}} \subseteq \mathsf{X}_{\mathcal{H}}$ and $\mathsf{X}_{\mathcal{H}} \subseteq \mathsf{X}_{\mathcal{G}}$ were not set to true, then we have that for all pairs of verticies, the presentations of the pair of graphs constructed were not both irreducible, as if they were, then both $\mathsf{X}_{\mathcal{G}} \subseteq \mathsf{X}_{\mathcal{H}}$ and $\mathsf{X}_{\mathcal{H}} \subseteq \mathsf{X}_{\mathcal{G}}$ would be true, so via Theorem 3, we have that $\mathsf{X}_{\mathcal{G}} \neq \mathsf{X}_{\mathcal{H}}$ and can conclude $\mathcal{G} \ncong \mathcal{H}$ (as again, $\mathcal{G}$ and $\mathcal{H}$ are irreducible, minimal, right-resolving graphs).

The worst case runtime of the decision procedure is if $\mathcal{G} \ncong \mathcal{H}$, as we minimize and check strongly connected-ness twice (of a graph that is potentially the size of both $\mathcal{G}$ and $\mathcal{H}$) for each pair of verticies, so the runtime is $O((V + E) \cdot V^2)$, where V is the number of verticies of one graph and E is the number of edges of one graph. □

7 Graph isomorphisms

Definition 7.1. An *undirected graph* is a pair $G = (V, E)$, where V is a finite set of vertices and $E \subseteq V \times V$ is a symmetric and irreflexive relation on V.

Definition 7.2. Given undirected graphs $G = (V_G, E_G)$ and $H = (V_H, E_H)$, an isomorphism from G to H is a bijection $\varphi : V_G \to V_H$ such that for all $I, J \in V_G$, $(I, J) \in E_G$ if and only if $(\varphi(I), \varphi(J)) \in E_H$. If there is and isomorphism from G to H, then we say G and H are isomorphic.

Given labeled graphs $\mathcal{G}$ and $\mathcal{H}$, an isomorphism from $\mathcal{G}$ to $\mathcal{H}$ is a pair of bijections $\partial\Phi : \mathcal{V}_{\mathcal{G}} \to \mathcal{V}_{\mathcal{H}}$ and $\Phi : \mathcal{E}_{\mathcal{G}} \to \mathcal{E}_{\mathcal{H}}$ such that for all $e \in \mathcal{E}_{\mathcal{G}}$, $i_{\mathcal{H}}(\Phi(e)) = \partial\Phi(i_{\mathcal{G}}(e))$, $t_{\mathcal{H}}(\Phi(e)) = \partial\Phi(t_{\mathcal{G}}(e))$, and $\mathcal{L}_{\mathcal{H}}(\Phi(e)) = \mathcal{L}_{\mathcal{G}}(e)$.

Definition 7.3. Define the decision problem GI to be the set of all pairs of isomorphic undirected graphs:

$$\mathsf{GI} \triangleq \{\langle G, H\rangle : G \text{ and } H \text{ are isomorphic undirected graphs}\}$$

Similary, define the decision problem LGI to be the set of all pairs of isomorphic labeled graphs:

$$\mathsf{LGI} \triangleq \{\langle \mathcal{G}, \mathcal{H}\rangle : \mathcal{G} \text{ and } \mathcal{H} \text{ are isomorphic labeled graphs}\}$$

Definition 7.4. Let A be some decision problem. We say A is *GI-hard* if GI is polynomial-time Turing reducible to A. We say A is *GI-complete* if A is GI-hard and A is polynomial-time Turing reducible to GI.

Theorem 7.5. LGI is GI-hard.

Proof. First, we will show that LGI is GI-hard. Let G and H be undirected graphs. Construct the labeled graph $\mathcal{G}$ with $\mathcal{V}_{\mathcal{G}} \triangleq V_G$, $\mathcal{E}_{\mathcal{G}} \triangleq E_G$ and for all $I, J \in \mathcal{E}_{\mathcal{G}}$, $i_{\mathcal{G}}(I, J) \triangleq I$, $t_{\mathcal{G}}(I, J) \triangleq J$, and $\mathcal{L}_{\mathcal{G}}(I, J) \triangleq \ell$. Construct the labeled graph $\mathcal{H}$ in a similar fashion. Obviously, this construction can be done in polynomial-time with respect to the size of G and H.

Suppose φ is an isomorphism from G to H. Define the map $\partial\Phi : \mathcal{V}_{\mathcal{G}} \to \mathcal{V}_{\mathcal{H}}$ with $\partial\Phi(I) \triangleq \varphi(I)$ for $I \in \mathcal{V}_{\mathcal{G}}$ and $\Phi : \mathcal{E}_{\mathcal{G}} \to \mathcal{E}_{\mathcal{H}}$ with $\Phi(I, J) \triangleq (\varphi(I), \varphi(J))$ for $I, J \in \mathcal{E}_{\mathcal{G}}$. Let $(I, J) \in \mathcal{E}_{\mathcal{G}}$. Then,

$$\begin{aligned}
\partial\Phi(i_{\mathcal{G}}(I, J)) &= \partial\Phi(I) \\
&= \varphi(I) \\
&= i_{\mathcal{H}}(\varphi(I), \varphi(J)) \qquad (\star) \\
&= i_{\mathcal{H}}(\Phi(I, J)),
\end{aligned}$$

and $(\star)$ is well-defined as $(I,J) \in \mathcal{E}_{\mathcal{G}} = E_G$ if and only if $(\varphi(I), \varphi(J)) \in E_H = \mathcal{E}_{\mathcal{H}}$. A similar deduction can be used to show $\partial\Phi(t_{\mathcal{G}}(I,J)) = t_{\mathcal{H}}(\Phi(I,J))$. As φ is a bijection from V_G to V_H, then $\partial\Phi$ is a bijection from $\mathcal{V}_{\mathcal{G}}$ to $\mathcal{V}_{\mathcal{H}}$. One can check that for $\Phi^{-1}(I,J) \triangleq (\varphi^{-1}(I), \varphi^{-1}(J))$, $\Phi^{-1} \circ \Phi = id_{\mathcal{E}_{\mathcal{G}}}$ and $\Phi \circ \Phi^{-1} = id_{\mathcal{E}_{\mathcal{H}}}$ so Φ is a bijection from $\mathcal{G}$ to $\mathcal{H}$. Hence, $(\partial\Phi, \Phi)$ is an isomorphism from $\mathcal{G}$ to $\mathcal{H}$.

Conversely, suppose $(\partial\Phi, \Phi)$ is an isomorphism from $\mathcal{G}$ to $\mathcal{H}$. Define $\varphi : V_G \to V_H$ with $\varphi(I) \triangleq \partial\Phi(I)$ for $I \in V_G$. Note that for $(I,J) \in \mathcal{E}_{\mathcal{H}}$, $(I,J) = (i_{\mathcal{H}}(I,J), t_{\mathcal{H}}(I,J))$. If $(I,J) \in E_G$, then

$$\begin{aligned} i_{\mathcal{H}}(\Phi(I,J)) &= \partial\Phi(i_{\mathcal{G}}(I,J)) \\ &= \partial\Phi(I) \\ t_{\mathcal{H}}(\Phi(I,J)) &= \partial\Phi(t_{\mathcal{G}}(I,J)) \\ &= \partial\Phi(J), \end{aligned}$$

so $\Phi(I,J) = (i_{\mathcal{H}}(\Phi(I,J)), t_{\mathcal{H}}(\Phi(I,J))) = (\partial\Phi(I), \partial\Phi(J)) = (\varphi(I), \varphi(J))$. Therefore, as $\Phi(I,J) \in \mathcal{E}_{\mathcal{H}} = E_H$, then $(\varphi(I), \varphi(J)) \in E_H$. Simlarly, note that for $I,J \in E_{\mathcal{G}}$, $(I,J) = (i_{\mathcal{G}}(I,J), t_{\mathcal{G}}(I,J))$. If $(\varphi(I), \varphi(J)) \in E_H$, then

$$\begin{aligned} i_{\mathcal{H}}(\Phi(I,J)) &= \partial\Phi(i_{\mathcal{G}}(I,J)) \\ &= \partial\Phi(I) \\ t_{\mathcal{H}}(\Phi(I,J)) &= \partial\Phi(t_{\mathcal{G}}(I,J)) \\ &= \partial\Phi(J), \end{aligned}$$

□

References

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