1 Preliminaries

Definition 1. Let $\mathcal{A}$ be a finite set. The *full $\mathcal{A}$-shift* is the set $\mathcal{A}^{\mathbb{Z}}$ of all bi-infinite sequences over $\mathcal{A}$ (i.e. functions from $\mathbb{Z}$ to $\mathcal{A}$, hence the usual notation for the set of all functions from $\mathbb{Z}$ to $\mathcal{A}$).

A *block* (or *word*) is a finite sequence of letters over some alphabet $\mathcal{A}$. Let $x = (x_i)_{i\in\mathbb{Z}}$ be a bi-infinite sequence. For $i \le j$, the block from the ith coordinate to the jth coordinate is denoted

$$x_{[i,j]} \triangleq x_i x_{i+1} \dots x_j.$$

Definition 2. Let $\mathcal{F}$ be a set of words over some alphabet. A *subshift* is a subset $\mathsf{X}_{\mathcal{F}}$ of some full shift $\mathcal{A}^{\mathbb{Z}}$ such that no word in $\mathcal{F}$ appears in any point of the subshift, defined as

$$\mathsf{X}_{\mathcal{F}} \triangleq \left\{ (x_i)_{i\in\mathbb{Z}} \in \mathcal{A}^{\mathbb{Z}} \;:\; \forall i,j \in \mathbb{Z}, i \le j \quad x_{[i,j]} \notin \mathcal{F} \right\}.$$

Definition 3. A *graph* G is a 4-tuple $G = (\mathcal{V}, \mathcal{E}, i, t)$, where $\mathcal{V}$ is a finite set of *vertices*, $\mathcal{E}$ is a finite set of *edges*, and $i : \mathcal{E} \to \mathcal{V}$ and $t : \mathcal{E} \to \mathcal{V}$ are functions assigning an *initial* and *terminating* vertex for each edge, respectivley. For an arbitrary graph G, let $\mathcal{V}_G$, $\mathcal{E}_G$, i_G, and t_G denote the graph's vertices, edges, and intial and terminating vertex functions, respectivley. If the choice of G is understood, then the subscripts will be dropped for notational convenience.

For $I \in \mathcal{V}$, the *outgoing edges of* I is the set of edges starting at I, denoted

$$i^{-1}(I) = \{e \in \mathcal{E} : i(e) = I\}.$$

Similary, the *incoming edges of* I is the set of edges terminating at I, denoted

$$t^{-1}(I) = \{e \in \mathcal{E} : t(e) = I\}.$$

A graph is *essential* if all verticies have at least one incoming and outgoing edge; i.e. for all $I \in \mathcal{V}$, $i^{-1}(I) \neq \varnothing$ and $t^{-1}(I) \neq \varnothing$.

Definition 4. A *labeled graph* $\mathcal{G}$ is a pair $\mathcal{G} = (G, \mathcal{L})$, where G is a graph and $\mathcal{L} : \mathcal{E} \to \mathcal{A}$ is the *labeling function* from the edges of G onto some finite alphabet $\mathcal{A}$.

A labeled graph is *deterministic* if for each vertex, the labels of the outgoing edges at that vertex are all distinct (i.e. $\mathcal{L}|_{i^{-1}(I)}$ is injective for all $I \in \mathcal{V}$).

Definition 5. Let G be a graph. The *edge shift of* G is the set $\mathsf{X}_{\mathcal{G}}$ of all bi-infinite paths on G, defined as

$$\mathsf{X}_G \triangleq \left\{ (x_i)_{i\in\mathbb{Z}} \in \mathcal{E}^{\mathbb{Z}} \;:\; \forall i \in \mathbb{Z} \quad t(x_i) = i(x_{i+1}) \right\}.$$

As a consequence of this definintion, $\mathcal{B}(\mathsf{X}_G)$ is the set of all finite paths on G, so elements of $\mathcal{B}(\mathsf{X}_G)$ are called *paths on G*.

Definition 6. Let $\mathcal{G} = (G, \mathcal{L})$ be a labeled graph. The *presentation of* $\mathcal{G}$ is the set $\mathsf{X}_\mathcal{G}$ of the labels of all bi-infinite paths from X_G, defined as the image of $\mathcal{L}_\infty$ under X_G:

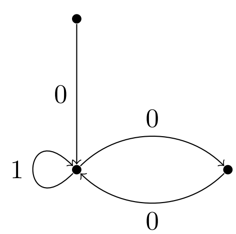
$$\mathsf{X}_\mathcal{G} \triangleq \mathcal{L}_\infty(\mathsf{X}_G).$$

We say a word $w \in \mathcal{B}(\mathsf{X}_\mathcal{G})$ is *presented by a path* $\pi \in \mathcal{B}(\mathsf{X}_G)$ if $\mathcal{L}(\pi) = w$.

Definition 7. Let $\mathcal{G} = (G, \mathcal{L})$ be a labeled graph.

2 Irreducibility

Consider the graph $\mathcal{G}$:



Theorem 1. If $v \in \mathcal{V}_\mathcal{H}$, then v is synchronizing for $\mathcal{G}\mathcal{H}$.

Proof. For any word $w \in \mathcal{B}(\mathsf{X}_\mathcal{H})$, Let □

Theorem 2. If $\mathsf{X}_{\mathcal{G}\mathcal{H}}$ is irreducible, then $\mathsf{X}_\mathcal{G} \subseteq \mathsf{X}_\mathcal{H}$.

Proof. Let $\mathcal{L}(\pi) \in \mathcal{B}(\mathsf{X}_\mathcal{G})$. As $\mathcal{G}$ is strongly connected, there is a path $\tau \in \mathcal{B}(\mathsf{X}_G)$ with $t(\tau) = J$ and $\pi\tau \in \mathcal{B}(\mathsf{X}_G)$. Hence, the word $\mathcal{L}(\pi\tau e) = w$ □

Theorem 3. If $\mathcal{G} = (G, \mathcal{L})$ is the minimizing right resolving presentation of an irreducible sofic shift X and X is and N-step shift of finite type, then $\mathsf{X}_G \cong \mathsf{X}_\mathcal{G}$.

Proof. Let x, y be walks in X_G. Suppose $\mathcal{L}_\infty(x) = \mathcal{L}_\infty(y)$, For any i, the paths $x_{[i-N,i-1]}$ and $y_{[i-N,i-1]}$ present the same word. Because that word is of length N, the word is synchronizing for $\mathcal{G}$ (from 3.4.17), so those paths end at the same vertex. Since $\mathcal{L}(x_{[i]}) = \mathcal{L}(y_{[i]})$, $\mathcal{G}$ is right resolving, and $x_{[i-N,i-1]}$ and $y_{[i-N,i-1]}$ end at the same vertex, then $x_{[i]} = y_{[i]}$ and hence $x = y$, so $\mathcal{L}_\infty$ is injective. By definition, $\mathcal{L}_\infty$ is surjective. Therefore, $\mathcal{L}_\infty$ is bijective and a conjugacy from X_G to $\mathsf{X}_\mathcal{G}$. □

Lemma 1. If X and Y are shift spaces, then $\mathcal{B}(X) \subseteq \mathcal{B}(Y)$ if and only if $X \subseteq Y$.

Proof. Let x be a point in X. Then every word that appears in x is in $\mathcal{B}(X)$. Since $\mathcal{B}(X) \subseteq \mathcal{B}(Y)$, then every word that appears in x is in $\mathcal{B}(Y)$, so $x \in Y$, hence $X \subseteq Y$.

Conversely, let w be a word in $\mathcal{B}(X)$. Then w occurs in some $x \in X$. Since $X \subseteq Y$, we have $x \in Y$, so w occurs in some $x \in Y$. Hence, $w \in \mathcal{B}(Y)$. $\square$

Let $\mathcal{G}$ and $\mathcal{H}$ be labeled graphs, I be a vertex from $\mathcal{G}$, and J be a vertex from $\mathcal{H}$. Define the *graph connecting* $\mathcal{G}$ *to* $\mathcal{H}$ *via* I *and* J as the disjoint union of the two graphs, adding an edge starting at I and ending at J, and adding a self loop on J. Label these two new edges with a symbol that does not appear in either graph. Since $\mathcal{G}$ and $\mathcal{H}$ are subgraphs of a graph connecting the two, it follows that the presentations of the individual graphs are subshifts of a presentation of a graph connecting the two - any bi-infinite walk in one of the graphs is a x bi-infinite walk of the corresponding subgraph of the connected graphs. Additionally, observe that the graph is reducible, as any path starting in $\mathcal{H}$ cannot end in $\mathcal{G}$.

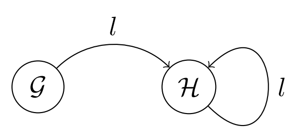


Figure 1: A graph connecting $\mathcal{G}$ to $\mathcal{H}$.

Theorem 4. Let $\mathcal{G}$ and $\mathcal{H}$ be irreducible graphs, and $\mathcal{K}$ be the graph connecting $\mathcal{G}$ to $\mathcal{H}$ via I and J. If $\mathsf{X}_{\mathcal{K}}$ is irreducible, then $\mathsf{X}_{\mathcal{G}} \subseteq \mathsf{X}_{\mathcal{H}}$.

Proof. First, suppose that $\mathsf{X}_{\mathcal{K}}$ is irreducible, and let $u \in \mathcal{B}(\mathsf{X}_{\mathcal{G}})$. There is a path in $\mathcal{G}$ that presents u, hence there is a path in the $\mathcal{G}$ subgraph of $\mathcal{K}$ that presents u. From the irreducibility of $\mathcal{G}$, there is a path from the terminating vertex of a path presenting u to I. Let v be the word such path presents and l be the label of the edge connecting $\mathcal{G}$ to $\mathcal{H}$, so that we have $uvl \in \mathcal{B}(\mathsf{X}_{\mathcal{K}})$ and $u \in \mathcal{B}(\mathsf{X}_{\mathcal{K}})$. As $\mathsf{X}_{\mathcal{K}}$ is irreducible, there exists a word $w \in \mathcal{B}(\mathsf{X}_{\mathcal{K}})$ such that $uvlwu \in \mathcal{B}(\mathsf{X}_{\mathcal{K}})$. A path presenting $uvlwu$ must have the subpath presenting wu visit verticies only from the $\mathcal{H}$ subgraph of $\mathcal{K}$. This implies that there is a path in $\mathcal{H}$ presenting u, so we have $u \in \mathcal{B}(\mathsf{X}_{\mathcal{H}})$ and therefore $\mathcal{B}(\mathsf{X}_{\mathcal{G}}) \subseteq \mathcal{B}(\mathsf{X}_{\mathcal{H}})$, and $\mathsf{X}_{\mathcal{G}} \subseteq \mathsf{X}_{\mathcal{H}}$ via Lemma 1. $\square$

Theorem 5. Let $\mathcal{G}$ and $\mathcal{H}$ be irreducible, minimal, right-resolving presentations. If $\mathsf{X}_{\mathcal{G}} = \mathsf{X}_{\mathcal{H}}$, then there exists a pair of verticies $(I, J) \in (\mathcal{V}_{\mathcal{G}} \times \mathcal{V}_{\mathcal{H}})$ such that both the graph connecting $\mathcal{G}$ to $\mathcal{H}$ via I and J and the graph connecting $\mathcal{H}$ to $\mathcal{G}$ via J and I both present irreducible shifts.

Equivalently, if for every pair of verticies $(I, J) \in (\mathcal{V}_{\mathcal{G}}, \mathcal{V}_{\mathcal{H}})$ the graph connecting $\mathcal{G}$ to $\mathcal{H}$ via I and J and the graph connecting $\mathcal{H}$ to $\mathcal{G}$ via J and I do not both present irreducible shifts.

Proof. Suppose $\mathsf{X}_{\mathcal{G}} = \mathsf{X}_{\mathcal{H}}$. Since $\mathcal{G}$ and $\mathcal{H}$ are irreducible, minimal, right-resolving presentations of the same shift, they must be isomorphic. Let $(\partial\Phi, \Phi)$ be a graph isomorphism between them. Choose an arbitrary vertex I from $\mathcal{G}$, and then let $\mathcal{K}$ be the graph connecting $\mathcal{G}$ to $\mathcal{H}$ via I and $\partial\Phi(I)$. Let f be the self loop on $\partial\Phi(I)$ added in the construction of $\mathcal{K}$, and $\mathcal{H}^+$ be the $\mathcal{H}$ of $\mathcal{K}$ subgraph plus f. As $\mathcal{H}$ is irreducible, then $\mathcal{H}^+$ is irreducible. It suffices to show that $\mathsf{X}_{\mathcal{K}} = \mathsf{X}_{\mathcal{H}^+}$ to show $\mathsf{X}_{\mathcal{K}}$ is irreducible.

Let u be a word from $\mathcal{B}(\mathsf{X}_{\mathcal{K}})$, and π be a path that presents it. Without loss of generality, assume u and π are nonempty. If π starts in $\mathcal{H}^+$, then $u \in \mathcal{B}(\mathsf{X}_{\mathcal{H}^+})$. Otherwise, it starts in $\mathcal{G}$ and either ends in $\mathcal{G}$ or ends in $\mathcal{H}^+$. For the case of π ending in $\mathcal{G}$, then $\Phi(\pi)$ is a path in $\mathcal{H}$ presenting u, as $(\partial\Phi, \Phi)$ is a graph isomorphism, so $u \in \mathcal{B}(\mathsf{X}_{\mathcal{H}^+})$. For the case of π ending in $\mathcal{H}^+$, we can split π into $\pi = \pi_1 e \pi_2$, where π_1 and π_2 are (possibly empty) paths from $\mathcal{G}$ and $\mathcal{H}^+$, respectivley, and e is the edge connecting $\mathcal{G}$ and $\mathcal{H}$. Note that $\Phi(\pi_0)$ is a path in $\mathcal{H}$ and that it terminates at $\partial\Phi(I)$ as π_0 terminates at I, assuming π_0 is nonempty. From this, we have that $\Phi(\pi_0) f \pi_1$ is path in $\mathcal{H}^+$ that presents u, so $u \in \mathcal{B}(\mathsf{X}_{\mathcal{H}^+})$.

Hence, we have $\mathcal{B}(\mathsf{X}_{\mathcal{K}}) \subseteq \mathcal{B}(\mathsf{X}_{\mathcal{H}^+})$, and from the construction of $\mathcal{K}$, we also have $\mathcal{B}(\mathsf{X}_{\mathcal{H}^+}) \subseteq \mathcal{B}(\mathsf{X}_{\mathcal{K}})$, so $\mathcal{B}(\mathsf{X}_{\mathcal{K}}) = \mathcal{B}(\mathsf{X}_{\mathcal{H}^+})$ and $\mathsf{X}_{\mathcal{K}} = \mathsf{X}_{\mathcal{H}^+}$. Thus, we have shown the graph connecting $\mathcal{G}$ to $\mathcal{H}$ via I and $\partial\Phi(I)$ presents an irreducible shift. A similar argument can be made showing that the graph connecting $\mathcal{H}$ to $\mathcal{G}$ via $\partial\Phi(I)$ and I presents an irreducible shift (start from "...and then let $\mathcal{K}$", and replace $\mathcal{G} \mapsto \mathcal{H}, \mathcal{H} \mapsto \mathcal{G}, I \mapsto \partial\Phi(I), \partial\Phi(I) \mapsto I, \mathcal{H}^+ \mapsto \mathcal{G}^+, \Phi(\pi_0) \mapsto \Phi^{-1}(\pi_0)$). $\square$

Theorem 6. Given an oracle for minimizing a reducible presentation, deciding if two irreducible, minimal, right-resolving labeled graphs are isomorphic can be determined in polynomial time.

Proof. Let the decision procedure be as follows: for every pair of vertices $(I, J) \in (\mathcal{V}_{\mathcal{G}} \times \mathcal{V}_{\mathcal{H}})$ between the graphs, construct and set $\mathcal{G}_I \to \mathcal{H}_J$ as the graph connecting $\mathcal{G}$ to $\mathcal{H}$ via I and J, and similarly, construct and set $\mathcal{H}_J \to \mathcal{G}_I$ as the graph connecting $\mathcal{H}$ to $\mathcal{G}$ via J and I. Then, using the oracle, minimize

$\mathcal{G}_I \to \mathcal{H}_J$ and check if it is strongly connected (that is, irreducible), and if so, then from Theorem 2, we know we can set $\mathsf{X}_\mathcal{G} \subseteq \mathsf{X}_\mathcal{H}$ to be true. Similary, minimize $\mathcal{H}_J \to \mathcal{G}_I$ and check if it is strongly connected, and if so, set $\mathsf{X}_\mathcal{H} \subseteq \mathsf{X}_\mathcal{G}$ to be true. If at any point both $\mathsf{X}_\mathcal{G} \subseteq \mathsf{X}_\mathcal{H}$ and $\mathsf{X}_\mathcal{H} \subseteq \mathsf{X}_\mathcal{G}$ are set to true, then $\mathsf{X}_\mathcal{G} = \mathsf{X}_\mathcal{H}$ and thus can conclude $\mathcal{G} \cong \mathcal{H}$ (as $\mathcal{G}$ and $\mathcal{H}$ are unique, minimal, and irreducible presentations). If you find that after every pair of verticies that one or both of $\mathsf{X}_\mathcal{G} \subseteq \mathsf{X}_\mathcal{H}$ and $\mathsf{X}_\mathcal{H} \subseteq \mathsf{X}_\mathcal{G}$ were not set to true, then we have that for all pairs of verticies, the presentations of the pair of graphs constructed were not both irreducible, as if they were, then both $\mathsf{X}_\mathcal{G} \subseteq \mathsf{X}_\mathcal{H}$ and $\mathsf{X}_\mathcal{H} \subseteq \mathsf{X}_\mathcal{G}$ would be true, so via Theorem 3, we have that $\mathsf{X}_\mathcal{G} \neq \mathsf{X}_\mathcal{H}$ and can conclude $\mathcal{G} \ncong \mathcal{H}$ (as again, $\mathcal{G}$ and $\mathcal{H}$ are irreducible, minimal, right-resolving graphs).

The worst case runtime of the decision procedure is if $\mathcal{G} \ncong \mathcal{H}$, as we minimize and check strongly connected-ness twice (of a graph that is potentially the size of both $\mathcal{G}$ and $\mathcal{H}$) for each pair of verticies, so the runtime is $O((V+E) \cdot V^2)$, where V is the number of verticies of one graph and E is the number of edges of one graph. $\square$