The three paths of hierarchical modeling

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Agenda

- Hierarchical GLM Notation

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- Likelihood and model fit activity

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- Likelihood and model fit activity
- Radon!

A re-introduction to Generalized Linear Models (GLMs) for *clustered* data

Notation for a classic un-clustered GLM

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- $x_i \in [0, 1]$ is an indicator of exposure.
- α is expected outcome when x_i = 0
- ϵ_i are independently and identically distributed (i.i.d.) errors

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Another way of writing it:

- $y_i \sim N(\alpha + \beta x_i, \sigma^2)$

Which door will you choose?

Door #1: Ignore clustering and fit a normal GLM

- *Pool* data across all units, i.e. ignore clustering.
- i.e. fit model $y_{ij} = \alpha + \beta x_i + \epsilon_i$

Is this a good idea? Why or why not?

NO!

Complete pooling ignores potential sources of *observed* and *unobserved*. unit-level confounding.

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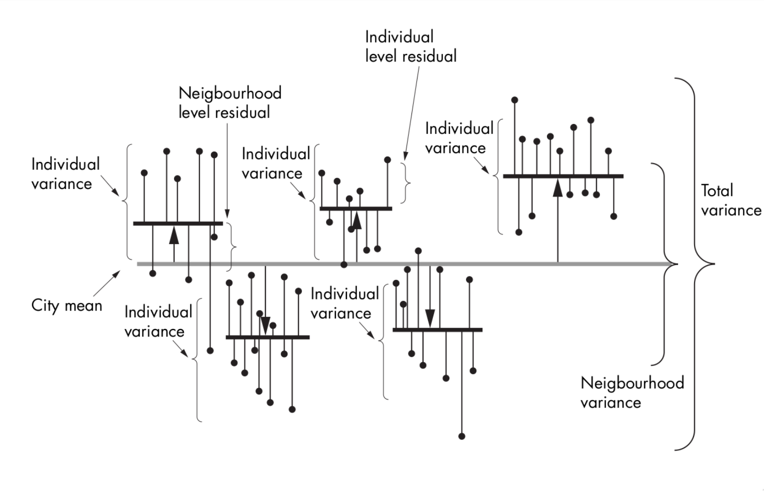
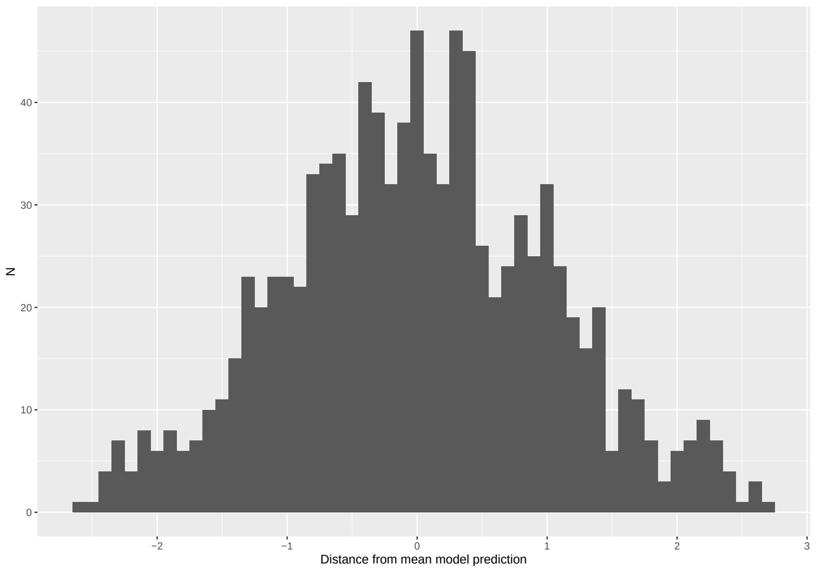
- y_i is a combination of systematic variation ($\alpha + \beta x$) and *uncorrelated* random noise (ϵ_i) where:

$$i.i.d.\ \epsilon \sim Normal(0, \sigma^2) \tag{2}$$

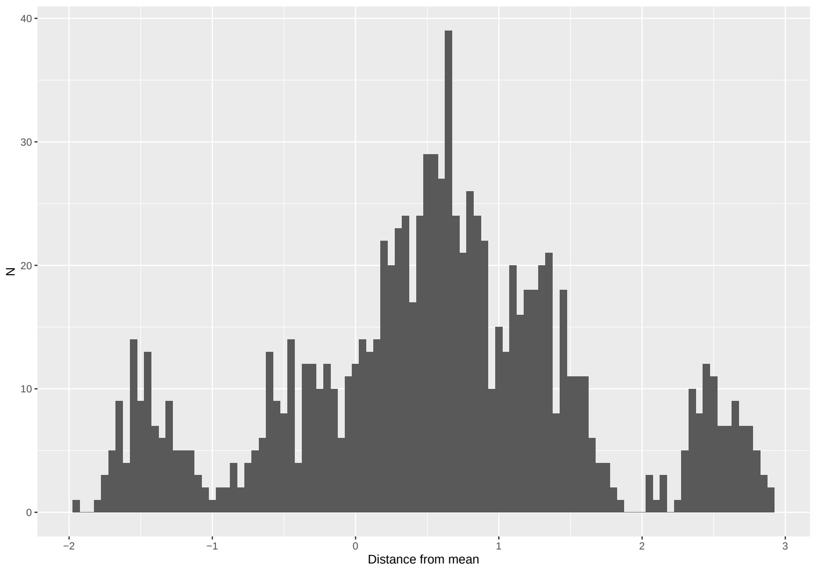
Clustering may result in correlation between average differences from mean



Your residuals should look like this



When you ignore clustering you may see something like:



Door #2: Fit a different model to each cluster

Fit *unpooled* model to each unit (j), assuming outcomes in each unit are independent:

- $y_{ij} = \alpha_j + \beta_j x_i + \epsilon_{ij}$
- $\epsilon_{ij} \sim N(0, \sigma_j^2)$

More danger!

Totally unpooled models run the risk of overfitting the data, particularly in small samples.

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- Some units (e.g. counties) may have few observations, making *unpooled* models impractical
- We may want to allow some effect of exposure (e.g. having a basement) to be consistent across counties.

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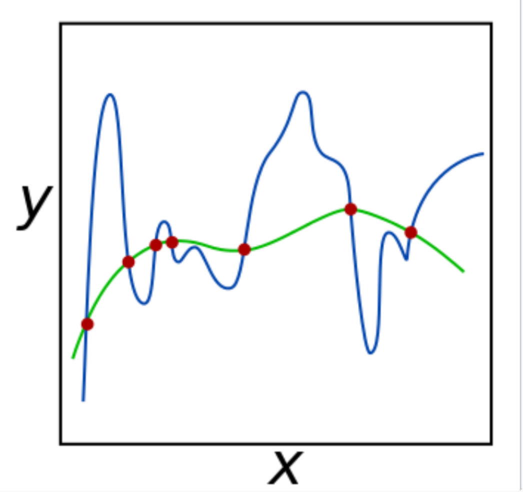
- This approach accommodates variation across units without assuming they have no similarity.
- More likely to make accurate out-of-sample predictions than the fully-pooled or unpooled examples.

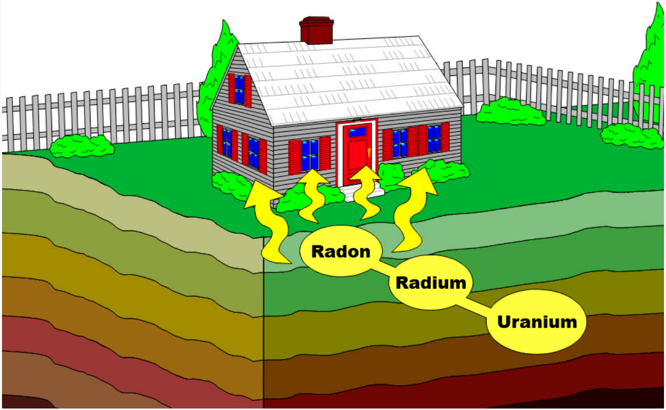
Partial pooling = Regularization



Both functions fit the data perfectly...which one should you prefer?

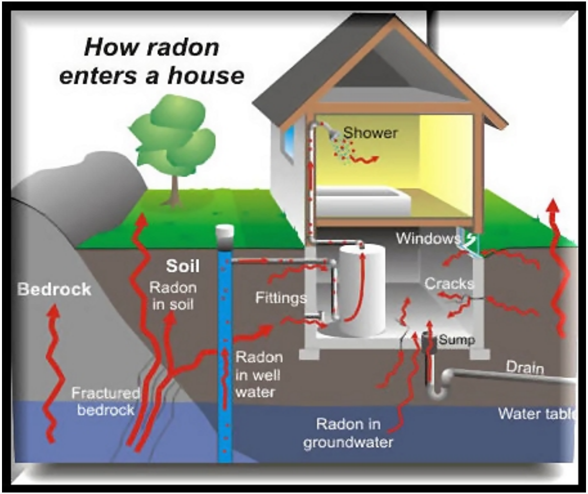
Radon Example

Radon is a carcinogenic gas



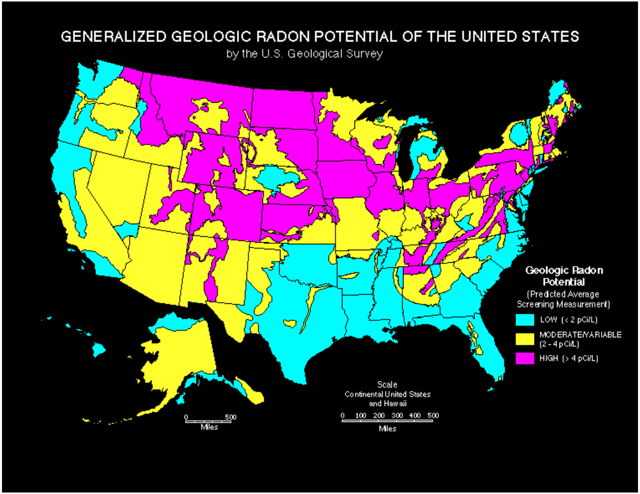
Radon is a byproduct of decaying soil uranium.

Radon enters a house more easily when it is built into the ground



Ann Arbor is a radon hotspot!

Considerable geographic variation in radon potential



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Trust me on this one...

My very own radon mitigation system.

What should a model that accounts for important sources of variation in household radon potential include?

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- County-level variation in soil uranium.
- Whether or not the radon measurement was taken in a basement.

Random intercepts account for county-level variation

Gelman [@Gelman2006] proposes a multi-level model to measure household radon in household i in county j, y_{ij}:

- $y_{ij} \sim N(\alpha_j + \beta x_{ij}, \sigma_y^2)$, for $i = 1, \ldots, n_j$, $j = 1, \ldots, J$

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Where:

- α_j is average, non-basement radon measure at county level
- β is fixed effect measuring average change in radon level in houses with a basement.
- σ_y^2 represents within-county variation in risk

Include predictors of county-level variation in second level

County-level random intercept is a function of county soil uranium measure, u_j:

- $\alpha_j \sim N(\gamma_0 + \gamma_1 u_j, \sigma_\alpha^2)$, for $j = 1, \ldots, J$

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Where:

- γ_0 is expected household radon measure when $u_j = 0$
- γ_1 scales expected county-level uranium with u_j
- σ_α^2 is between-county variation in radon risk not measured by u_j.

Putting it all together

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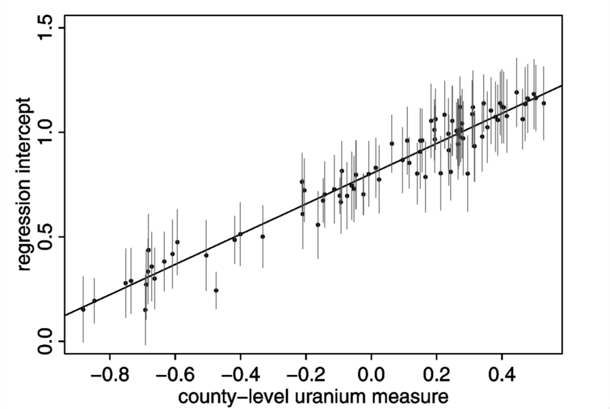
Household-level radon measure is a function of having a basement and county-level intercept:

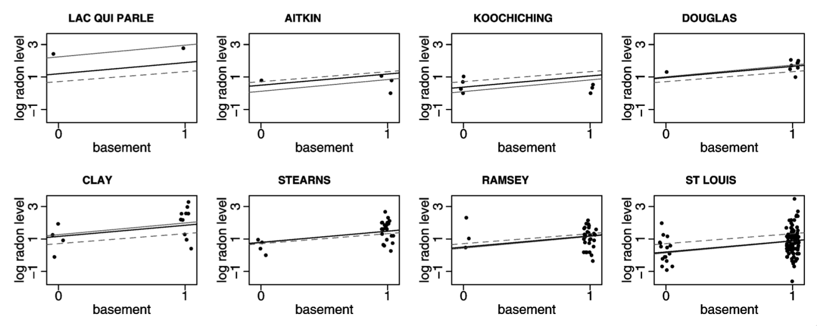
- $y_{ij} \sim N(\alpha_j + \beta x_{ij}, \sigma_y^2)$

County-level radon levels vary with soil uranium measures



County-level intercept, α_j, (± 1 standard error) as a function of county-level uranium.

Model predictions vs. radon measures by county



Multi-level regression line, $y = \alpha_j + \beta x$, from 8 Minnesota counties. Unpooled estimates = light grey line; Totally pooled estimates = dashed grey line.

Next Time

- Hands-on with the Radon example

References

References i