Models for clustered data

Jon Zelner

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EPID 684
University of Michigan School of Public Health

jzelner@umich.edu
www.jonzelner.net

Agenda

- Re-intro to GLMs

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- Why?

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- Why?

- Radon!

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- Re-intro to GLMs

- Why?

- Radon!

- Self-assessments.

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- Model non-linear response using linear predictors

Advanced GLM tricks

- Represent hierarchical/multi-level structure

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- Model spatial autocorrelation

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- Model spatial autocorrelation

- Allow variance parameters to change with individual-level covariates to acommodate heteroskedasticity

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Working definition of a generalized linear model?

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- Linear relationship between X and Y quantified by regression coefficients β

- Outcome assumed to have distribution with mean determined by regression coefficients.

- Link function, $g()$, translates between linear predictor and the mean of the distribution function.

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- Conditional mean for individual i denoted by $\hat{y}_i = g(y_i^*)$

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And others:

- Gamma, Exponential, Negative Binomial...

GLM with Normally-distributed errors

What are assumptions of ordinary least squares regression?

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- $\epsilon_i \sim Normal(0, \sigma^2)$

Identity link function maps real numbers to real numbers
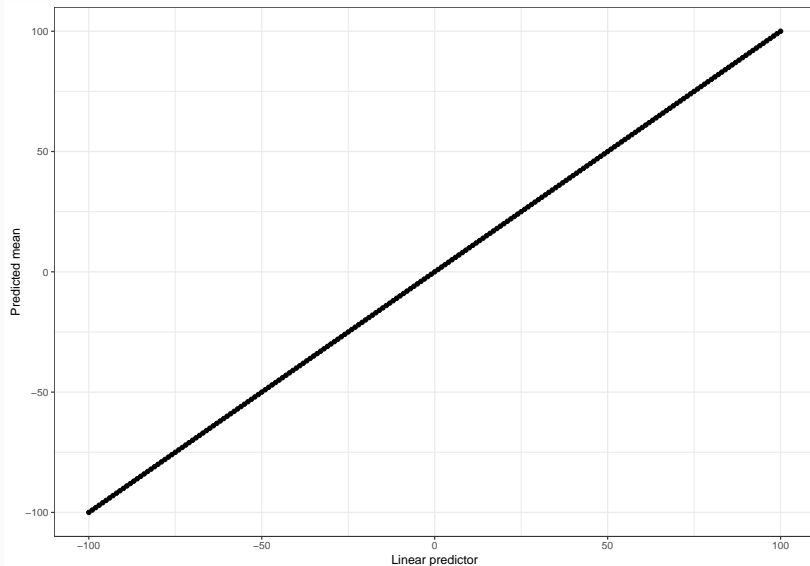
Interpreting results of Gaussian GLM

- α mean of y_i when $x_i = 0$

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- Just like in Merlo example, we also estimate the value of σ^2 that measures the amount of individual-level variability in outcomes

Generate some fake data in R

```r
## Predictors
x <- seq(from = 0.0, to = 10.0, by = 0.1)
## Regression coefficient
b <- 2.1
## Intercept
a <- 0.4
## Variance
sd <- 1.0
## The Data
y <- a + b * x + rnorm(length(x), 0, sd)
```

The Data
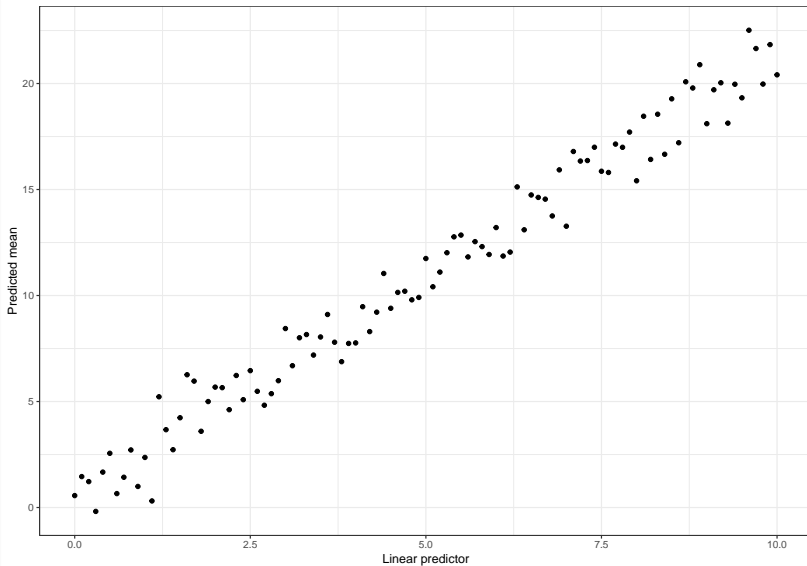
```
m <- glm(y ~ x, data = df)
```

Parameter estimates

Table 1: Fitting generalized (gaussian/identity) linear model: y ~ x

	Estimate	Std. Error	t value	Pr(>|t|)
(Intercept)	0.55	0.2	2.7	0.0071
x	2.1	0.035	60	1.6e-79

Model Residuals

Histogram of resid(m)
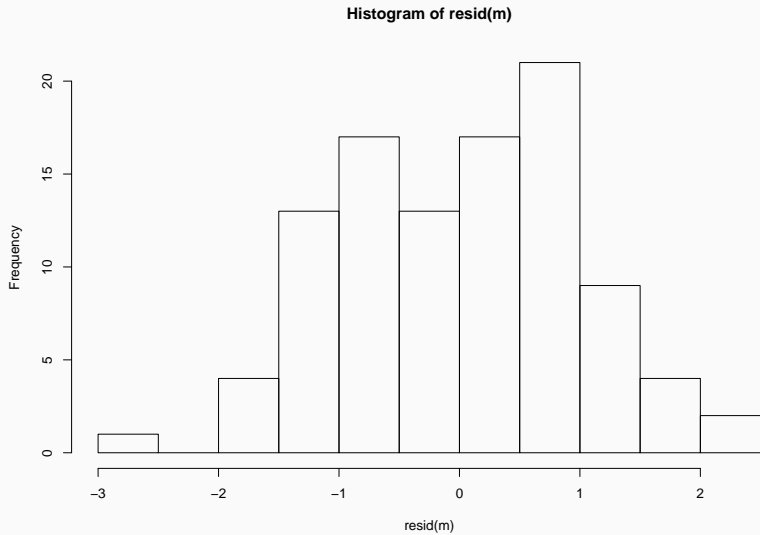
Table 2: Analysis of Variance Model

	Df	Sum Sq	Mean Sq	F value	Pr(>F)
x	1	3721	3721	3579	1.6e-79
Residuals	99	103	1	NA	NA

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- Both of these necessitate calculating the likelihood of the data.

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For Gaussian GLM, f = Normal distribution, $\theta = \{\alpha, \beta, \sigma^2\}$

Motivations for Multilevel Modeling

- What is the distinction Gelman and Hill make between *hierarchical* and *multilevel* models?

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- When might you want model *slopes* to vary and *intercepts* to be fixed?

- When might you want them to both vary?

A re-introduction to Generalized Linear Models (GLMs) for clustered data

Notation for a classic un-clustered GLM

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- $x_i \in [0, 1]$ is an indicator of exposure.
- α is expected outcome when x_i = 0
- ϵ_i are independently and identically distributed (i.i.d.) errors

Classic assumption is that:

- $\epsilon_i \sim N(0, \sigma^2)$

Independent errors

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In plain-ish English:

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In plain-ish English:

- Observation y_{ij} of individual i is a function of $\alpha + \beta x_i$ and normally distributed errors (ϵ_i) with mean zero and variance σ^2.

Another way of writing it:

- $y_i \sim N(\alpha + \beta x_i, \sigma^2)$

Which door will you choose?

- *Pool* data across all units, i.e. ignore clustering.
- i.e. fit model $y_{ij} = \alpha + \beta x_i + \epsilon_i$

Is this a good idea? Why or why not?

Complete pooling ignores potential sources of *observed* and *unobserved*. unit-level confounding.

Pooling clustered data violates assumption of independent errors

A pooled model:

$$y_i = \alpha + \beta x + \epsilon_i \tag{1}$$

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A pooled model:

$$y_i = \alpha + \beta x + \epsilon_i \qquad (1)$$

- y_i is a combination of systematic variation ($\alpha + \beta x$) and *uncorrelated* random noise (ϵ_i) where:

$$i.i.d. \ \epsilon \sim Normal(0, \sigma^2) \qquad (2)$$
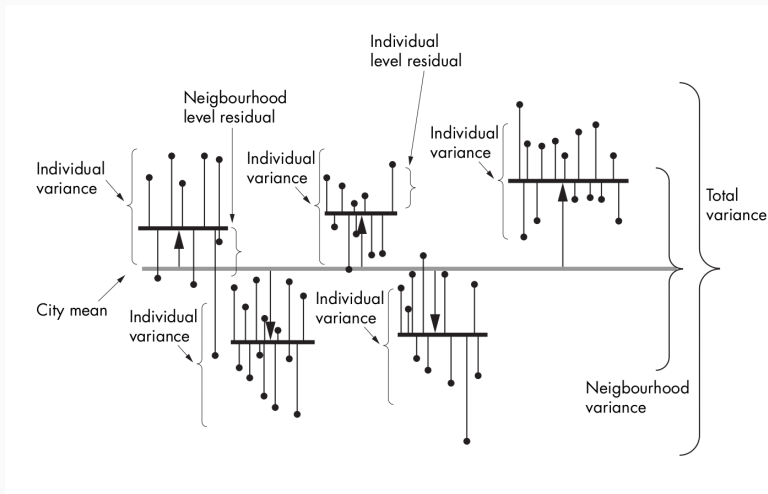
Your residuals should look like this
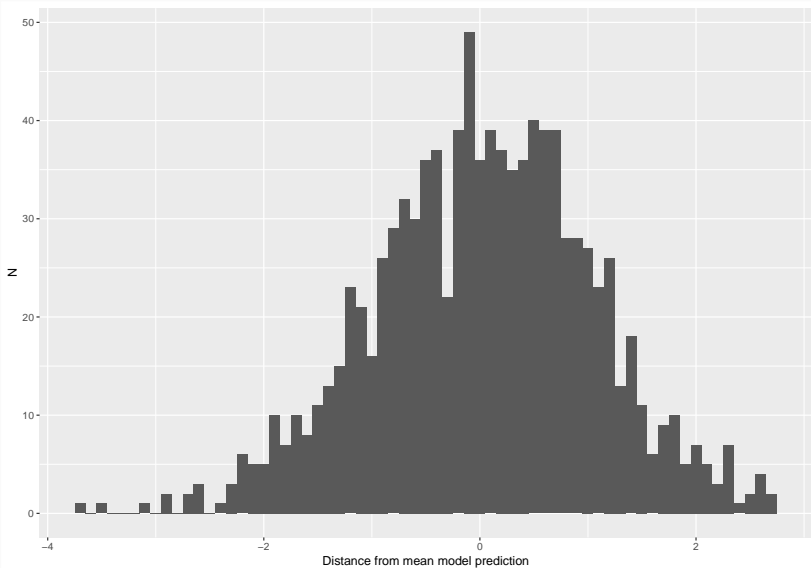
When you ignore clustering you may see something like:
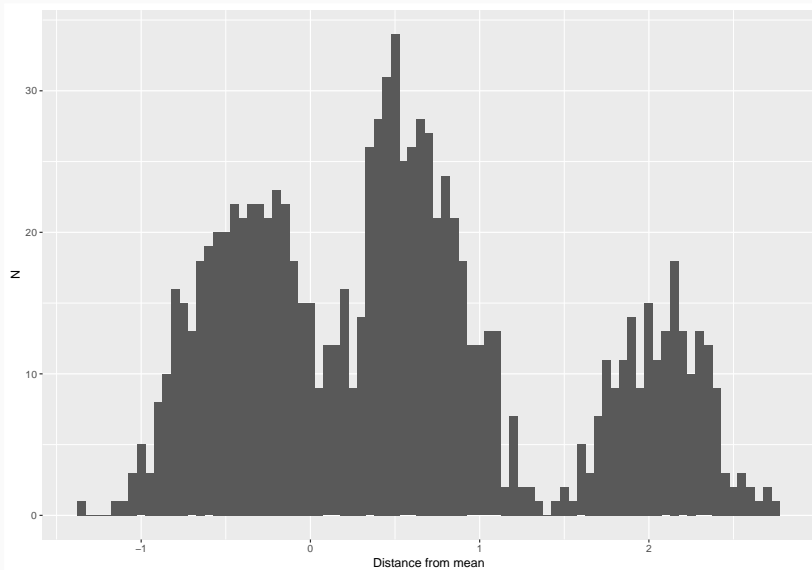
Fit *unpooled* model to each unit (j), assuming outcomes in each unit are independent:

- $y_{ij} = \alpha_j + \beta_j x_i + \epsilon_{ij}$
- $\epsilon_{ij} \sim N(0, \sigma_j^2)$

Totally unpooled models run the risk of overfitting the data, particularly in small samples.

What else could go wrong here?

Specific dangers of unpooled models

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- Some units (e.g. counties) may have few observations, making *unpooled* models impractical

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- Some units (e.g. counties) may have few observations, making *unpooled* models impractical

- We may want to allow some effect of exposure (e.g. having a basement) to be consistent across counties.

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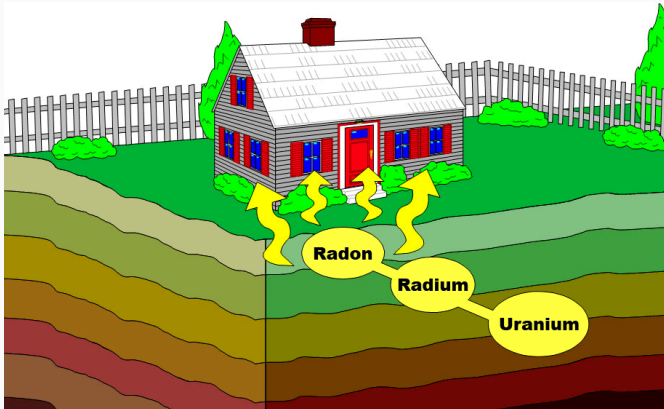- More likely to make accurate out-of-sample predictions than the fully-pooled or unpooled examples.
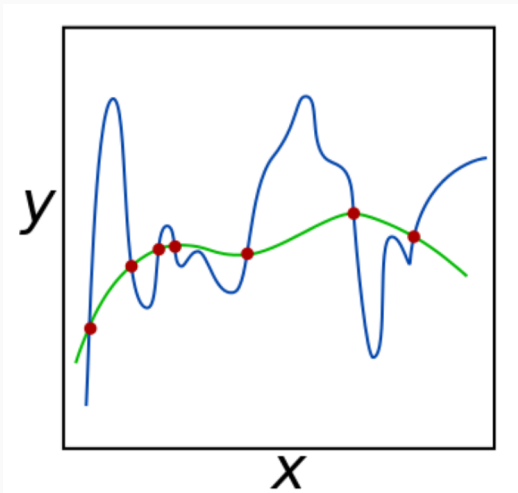
Both functions fit the data perfectly...which one should you prefer?

Radon Example

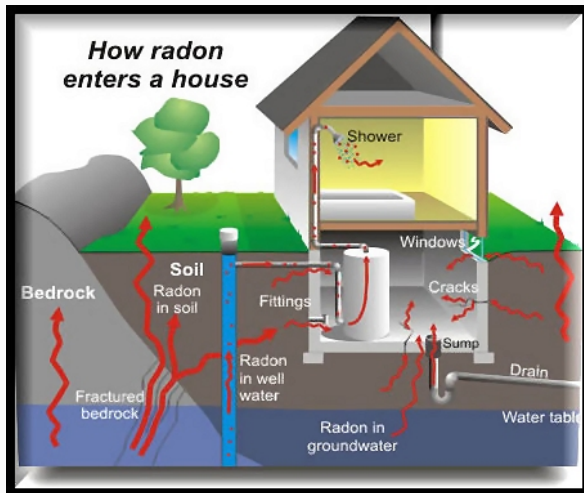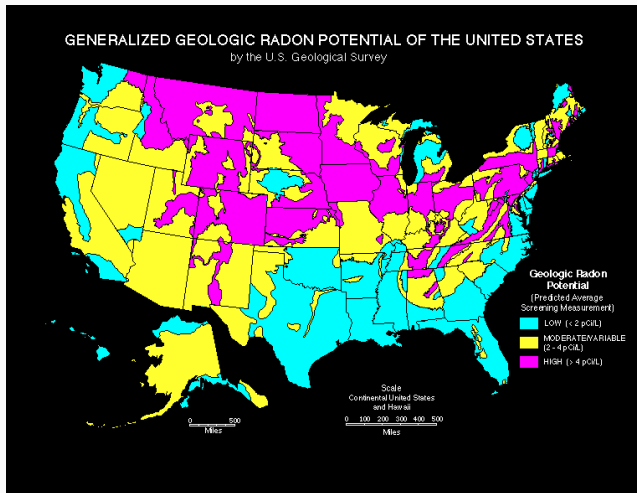# Radon is a carcinogenic gas

Radon is a byproduct of decaying soil uranium.

Radon enters a house more easily when it is built into the ground

Ann Arbor is a radon hotspot!

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My very own radon mitigation system.

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- County-level variation in soil uranium.
- Whether or not the radon measurement was taken in a basement.

Gelman [@Gelman2006] proposes a multi-level model to measure household radon in household i in county j, y_{ij}:

- $y_{ij} \sim N(\alpha_j + \beta x_{ij}, \sigma_y^2)$, for $i = 1, ..., n_j$, $j = 1, ..., J$

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Where:

- α_j is average, non-basement radon measure at county level
- β is fixed effect measuring average change in radon level in houses with a basement.
- σ_y^2 represents within-county variation in risk

County-level random intercept is a function of county soil uranium measure, u_j:

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Where:

- γ_0 is expected household radon measure when $u_j = 0$
- γ_1 scales expected county-level uranium with u_j
- σ_α^2 is between-county variation in radon risk not measured by u_j.

County-level intercept is a function of county soil uranium measure, u_j:

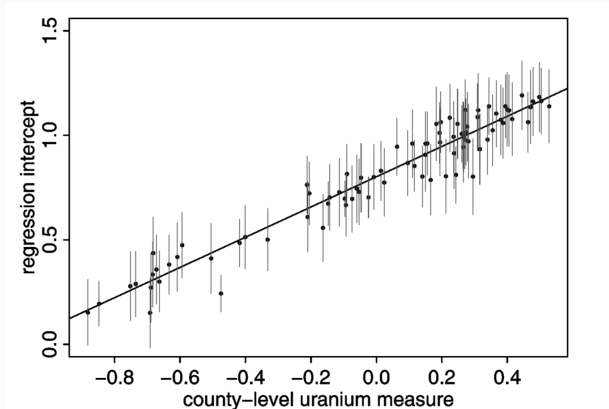- $\alpha_j \sim N(\gamma_0 + \gamma_1 u_j, \sigma_\alpha^2)$

Putting it all together

County-level intercept is a function of county soil uranium measure, u_j:

- $\alpha_j \sim N(\gamma_0 + \gamma_1 u_j, \sigma_\alpha^2)$

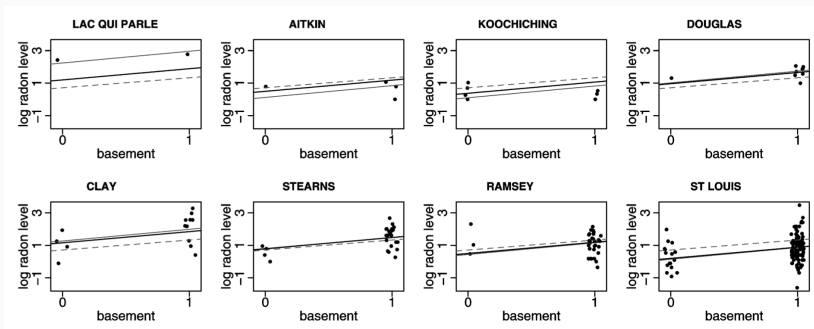Household-level radon measure is a function of having a basement and county-level intercept:

- $y_{ij} \sim N(\alpha_j + \beta x_{ij}, \sigma_y^2)$

County-level radon levels vary with soil uranium measures

County-level intercept, α_j, (± 1 standard error) as a function of county-level uranium.

Model predictions vs. radon measures by county

Multi-level regression line, $y = \alpha_j + \beta x$, from 8 Minnesota counties. Unpooled estimates = light grey line; Totally pooled estimates = dashed grey line.

- Hands-on with the Radon example

References