Models for clustered data

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Agenda

- Re-intro to GLMs

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- Why?

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- Why?
- Radon!

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- Why?
- Radon!
- Self-assessments.

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- Gives flexibility in the types of outcomes and inputs we can model.
- Model non-linear response using linear predictors

Advanced GLM tricks

- Represent hierarchical/multi-level structure

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- Model spatial autocorrelation

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- Model spatial autocorrelation
- Allow variance parameters to change with individual-level covariates to acommodate heteroskedasticity

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- Linear relationship between X and Y quantified by regression coefficients β
- Outcome assumed to have distribution with mean determined by regression coefficients.
- Link function, $g()$, translates between linear predictor and the mean of the distribution function.

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- Conditional mean for individual i denoted by $\widehat{y}_i = g(y_i^*)$

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And others:

- Gamma, Exponential, Negative Binomial...

GLM with Normally-distributed errors

GLM with normally distributed errors $\approx$ OLS regression

What are assumptions of ordinary least squares regression?

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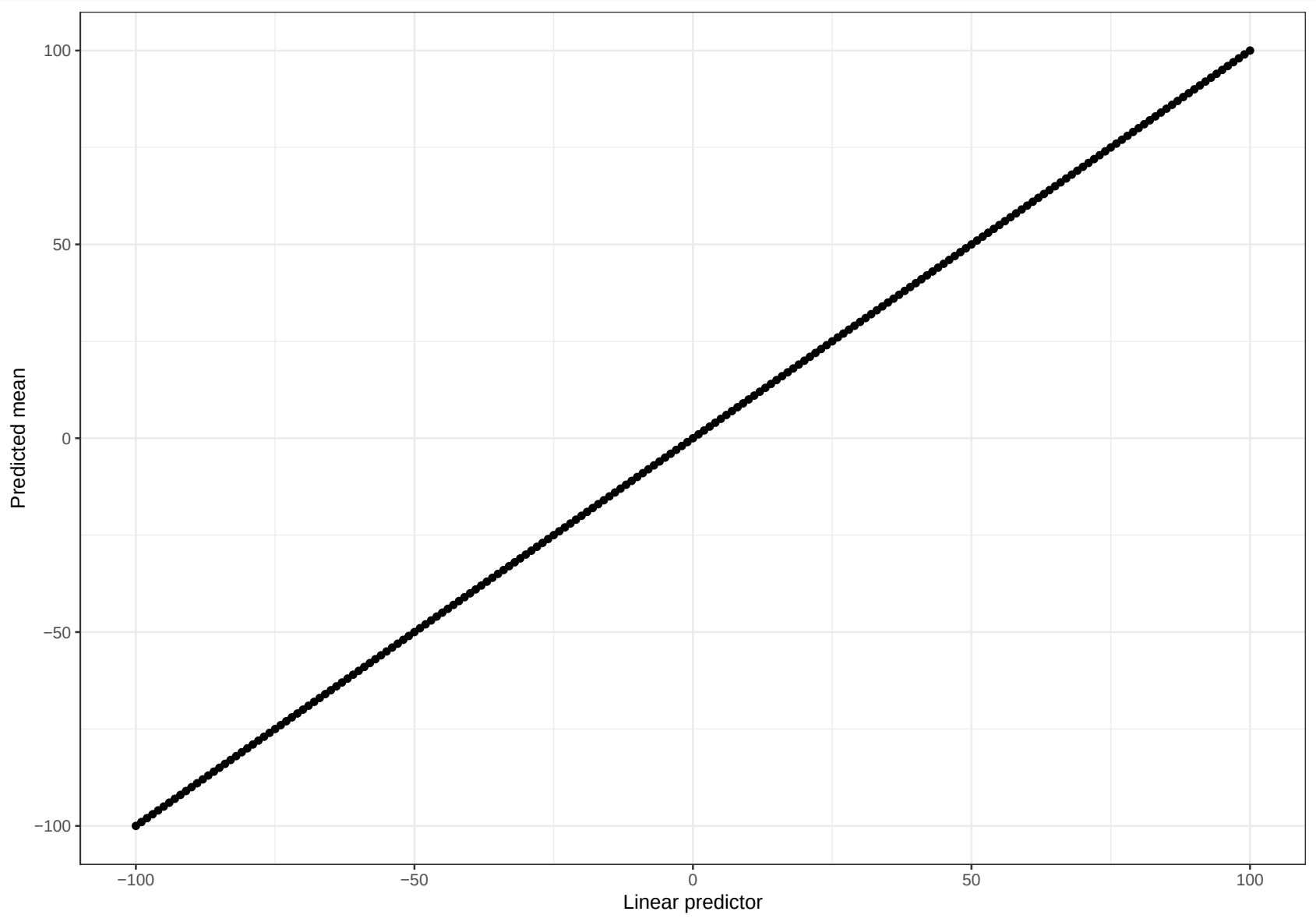
- $\epsilon_i \sim Normal(0, \sigma^2)$

Identity link function maps real numbers to real numbers



Interpreting results of Gaussian GLM

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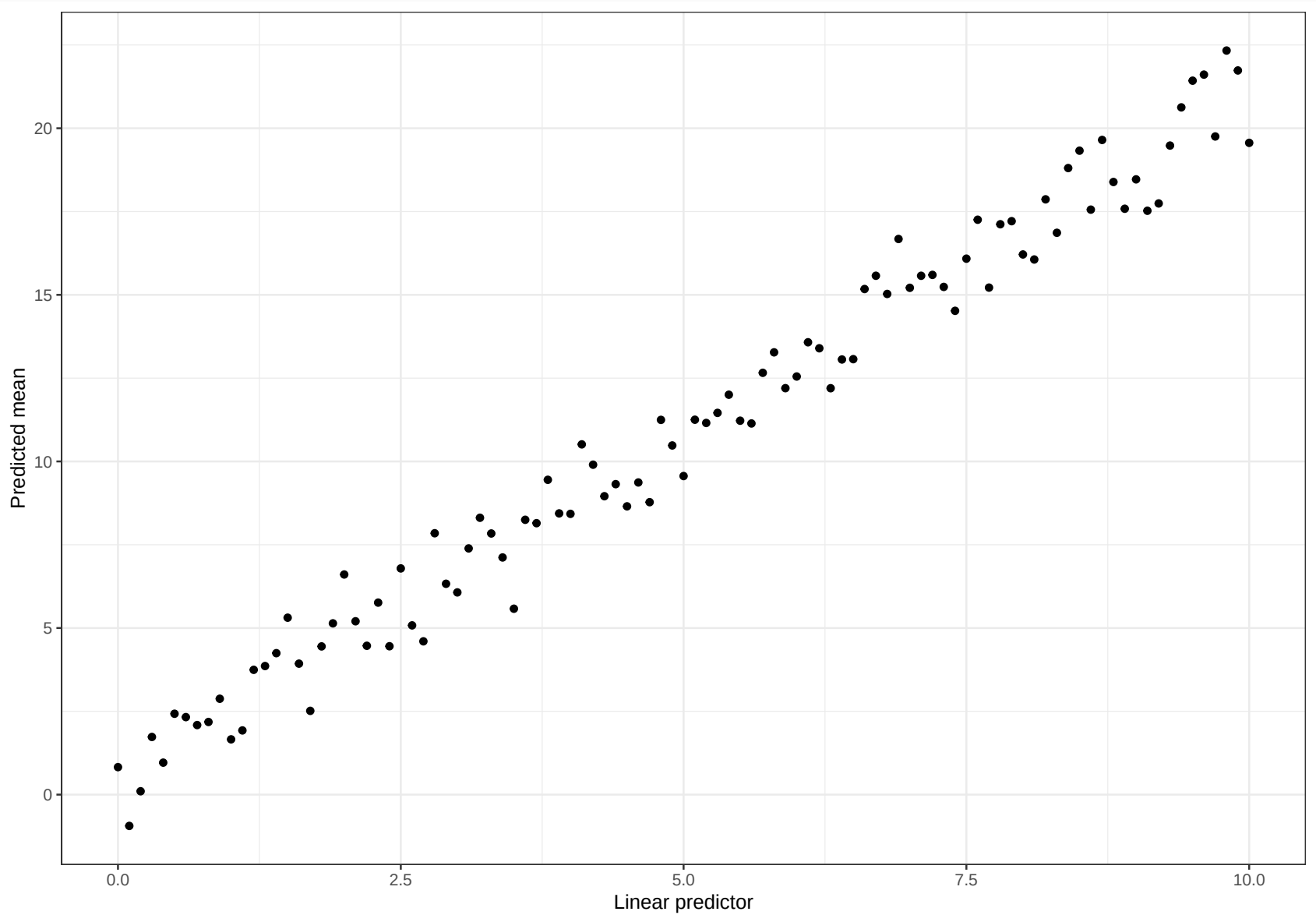
- α mean of y_i when $x_i = 0$
- β is change in mean of y_i for each one-unit change in x_i
- Just like in Merlo example, we also estimate the value of σ^2 that measures the amount of individual-level variability in outcomes

Generate some fake data in R

```r
## Predictors
x <- seq(from = 0.0, to = 10.0, by = 0.1)
## Regression coefficient
b <- 2.1
## Intercept
a <- 0.4
## Variance
sd <- 1.0
## The Data
y <- a + b * x + rnorm(length(x), 0, sd)
```

The Data



Recover the parameters

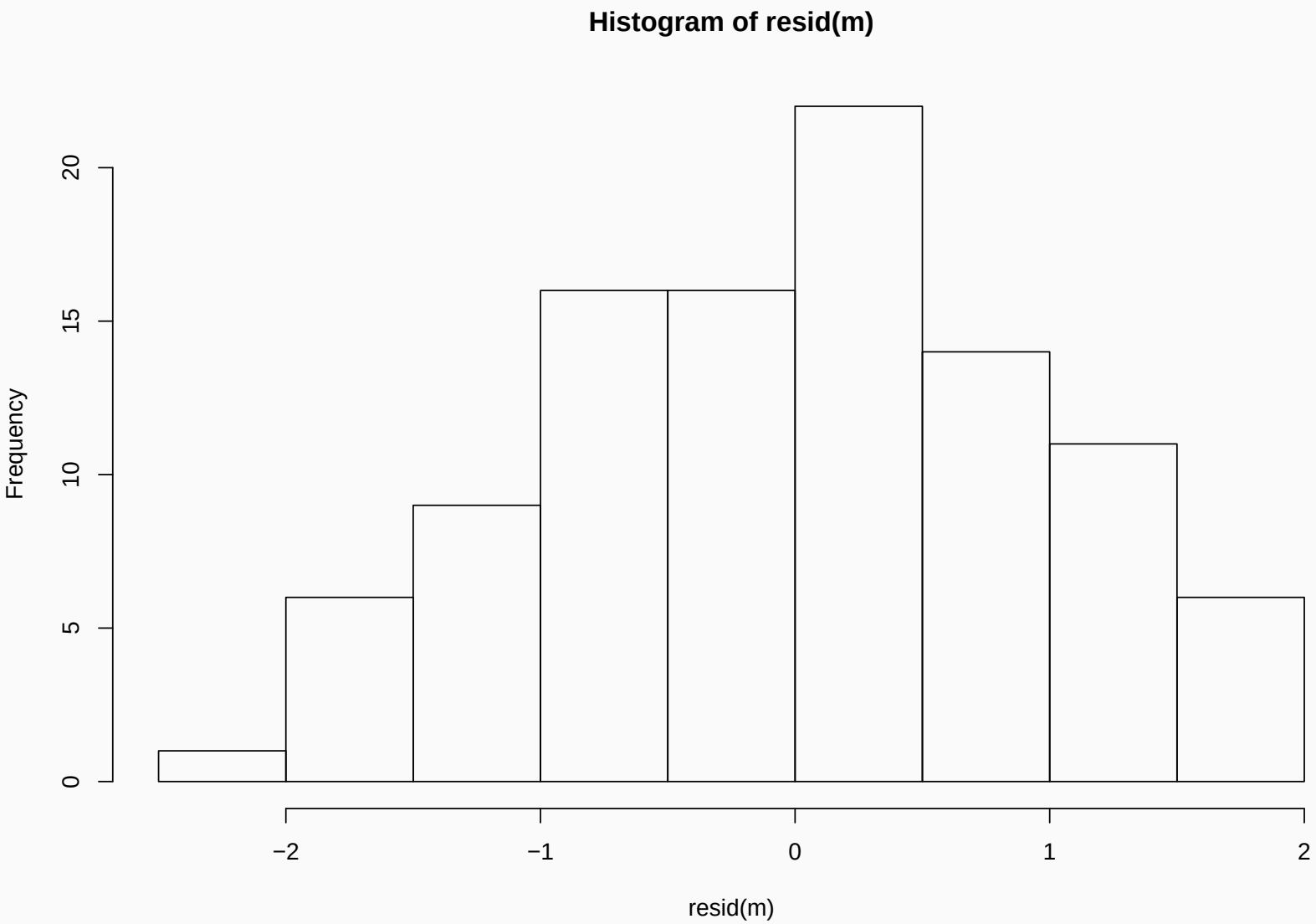
```
m <- glm(y ~ x, data = df)
```

Parameter estimates

Table 1: Fitting generalized (gaussian/identity) linear model: y ~ x

	Estimate	Std. Error	t value	Pr(>|t|)
(Intercept)	0.59	0.19	3.2	0.0019
x	2	0.032	64	2.5e-82

Model Residuals



Residual variance

Table 2: Analysis of Variance Model

	Df	Sum Sq	Mean Sq	F value	Pr(>F)
x	1	3604	3604	4091	2.5e-82
Residuals	99	87	0.88	NA	NA

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- We'll talk about these soon enough, but first...
- Both of these necessitate calculating the likelihood of the data.

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For Gaussian GLM, f = Normal distribution, $\theta = \{\alpha, \beta, \sigma^2\}$

Motivations for Multilevel Modeling

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- When might you want them to both vary?

A re-introduction to Generalized Linear Models (GLMs) for clustered data

Notation for a classic un-clustered GLM

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- ϵ_i are independently and identically distributed (i.i.d.) errors

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In plain-ish English:

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Another way of writing it:

- $y_i \sim N(\alpha + \beta x_i, \sigma^2)$

Three Approaches to Modeling Clustered Data

Which door will you choose?

Door #1: Ignore clustering and fit a normal GLM

- *Pool* data across all units, i.e. ignore clustering.
- i.e. fit model $y_{ij} = \alpha + \beta x_i + \epsilon_i$

Is this a good idea? Why or why not?

NO!

Complete pooling ignores potential sources of *observed* and *unobserved*. unit-level confounding.

Pooling clustered data violates assumption of independent errors

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$$y_i = \alpha + \beta x + \epsilon_i \tag{1}$$

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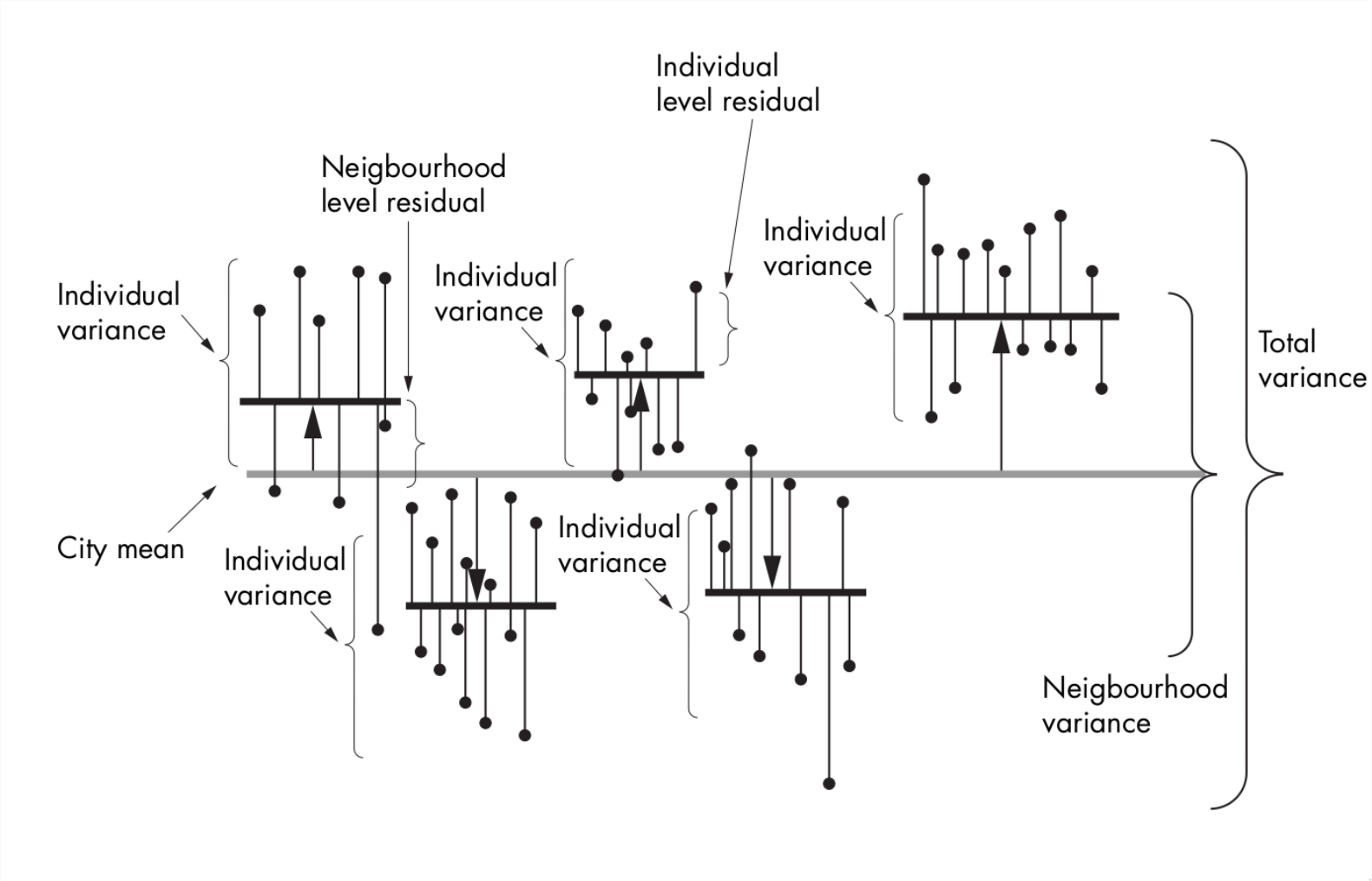
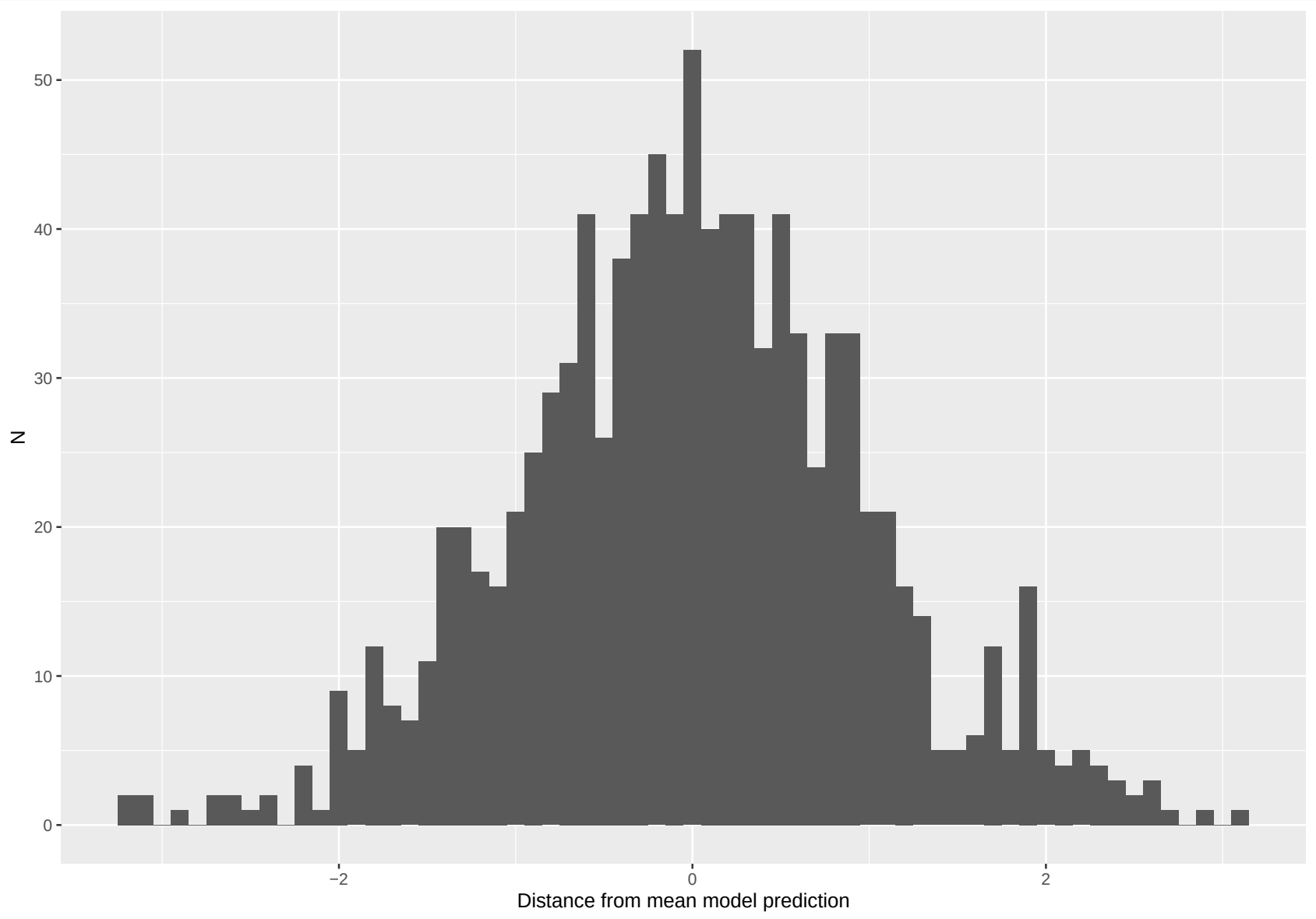
- y_i is a combination of systematic variation ($\alpha + \beta x$) and *uncorrelated* random noise (ϵ_i) where:

$$i.i.d.\ \epsilon \sim Normal(0, \sigma^2) \tag{2}$$

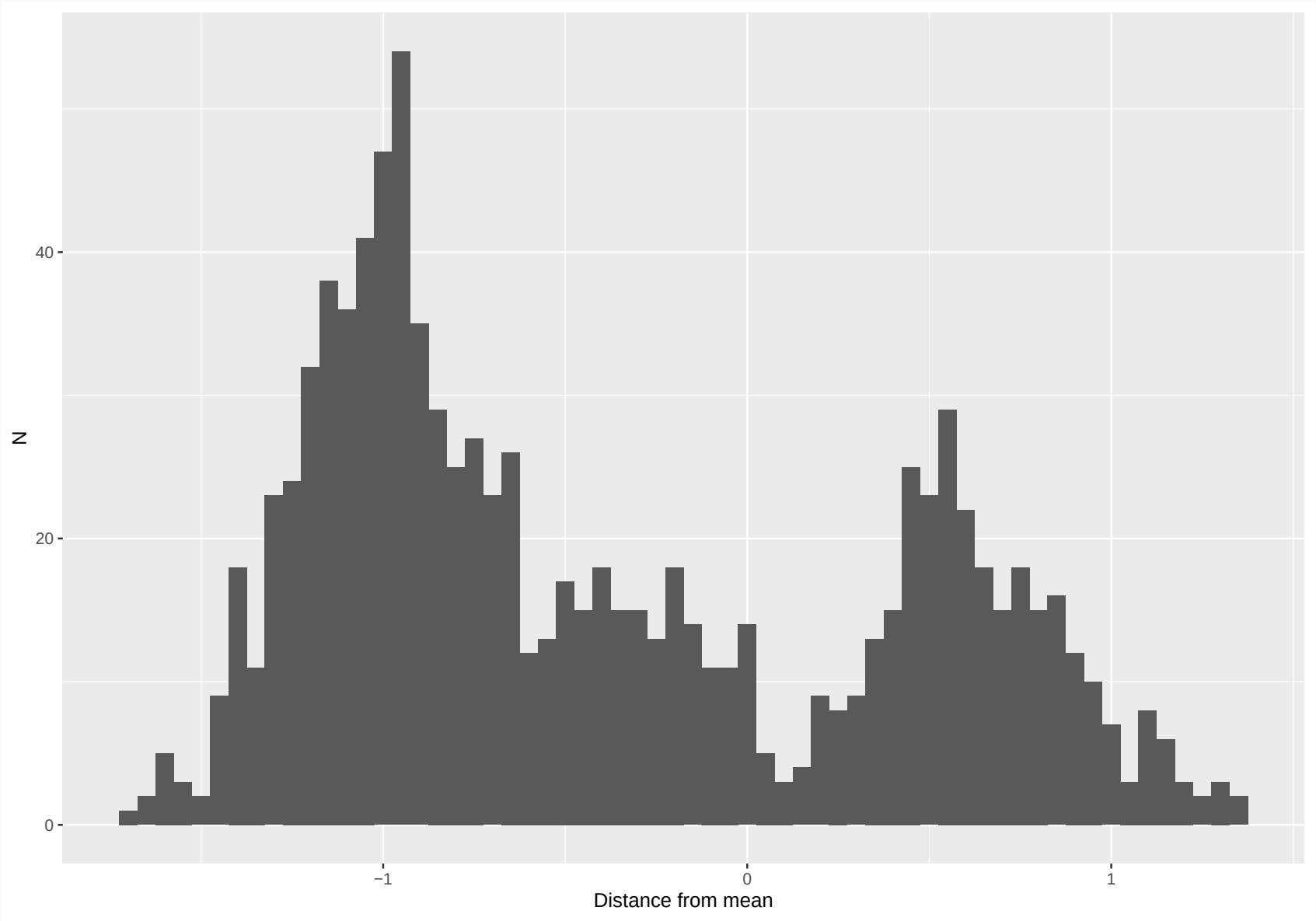
Clustering may result in correlation between average differences from mean



Your residuals should look like this



When you ignore clustering you may see something like:



Door #2: Fit a different model to each cluster

Fit *unpooled* model to each unit (j), assuming outcomes in each unit are independent:

- $y_{ij} = \alpha_j + \beta_j x_i + \epsilon_{ij}$
- $\epsilon_{ij} \sim N(0, \sigma_j^2)$

More danger!

Totally unpooled models run the risk of overfitting the data, particularly in small samples.

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- Some units (e.g. counties) may have few observations, making *unpooled* models impractical
- We may want to allow some effect of exposure (e.g. having a basement) to be consistent across counties.

Door #3: Partial Pooling!

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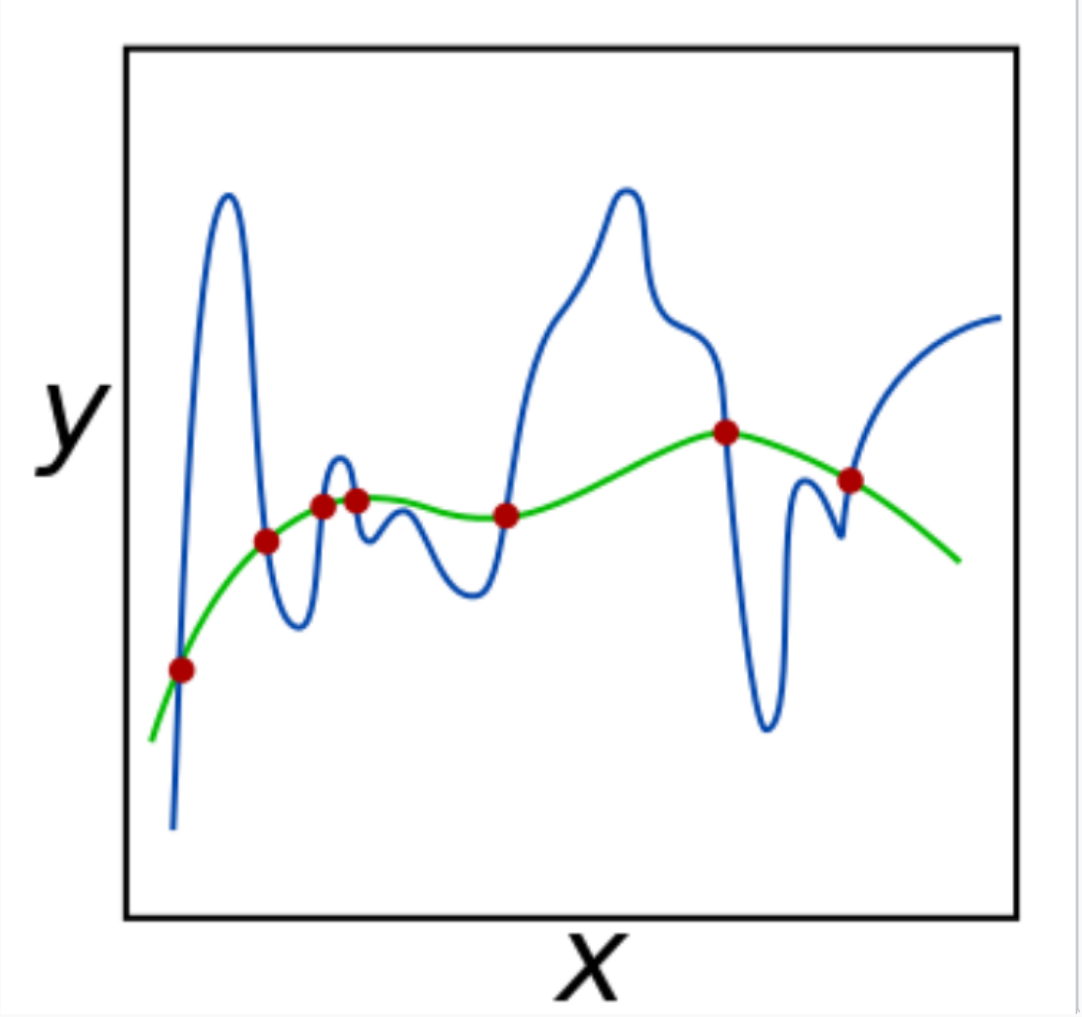
- This approach accommodates variation across units without assuming they have no similarity.
- More likely to make accurate out-of-sample predictions than the fully-pooled or unpooled examples.

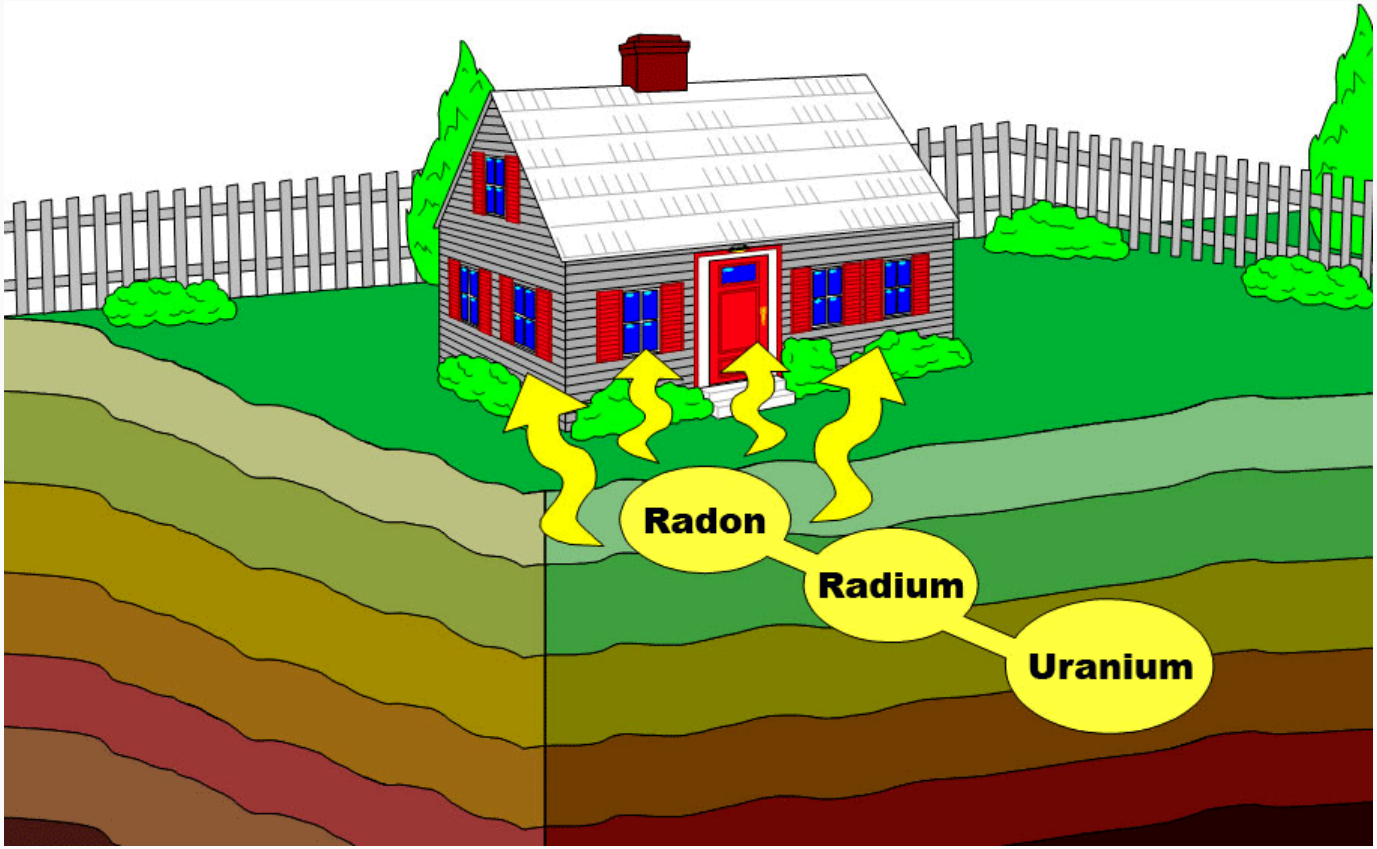
Partial pooling = Regularization



Both functions fit the data perfectly...which one should you prefer?

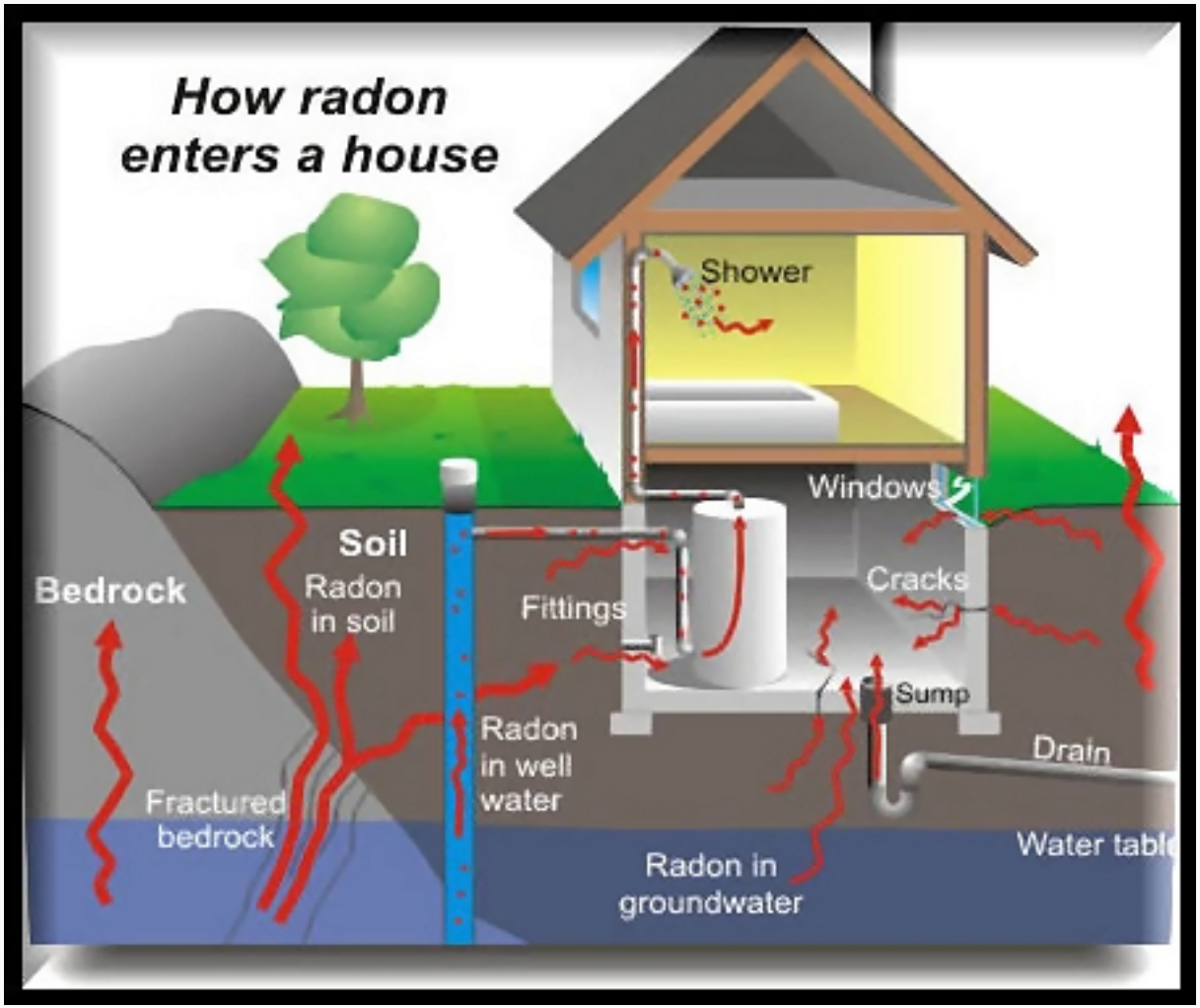
Radon Example

Radon is a carcinogenic gas



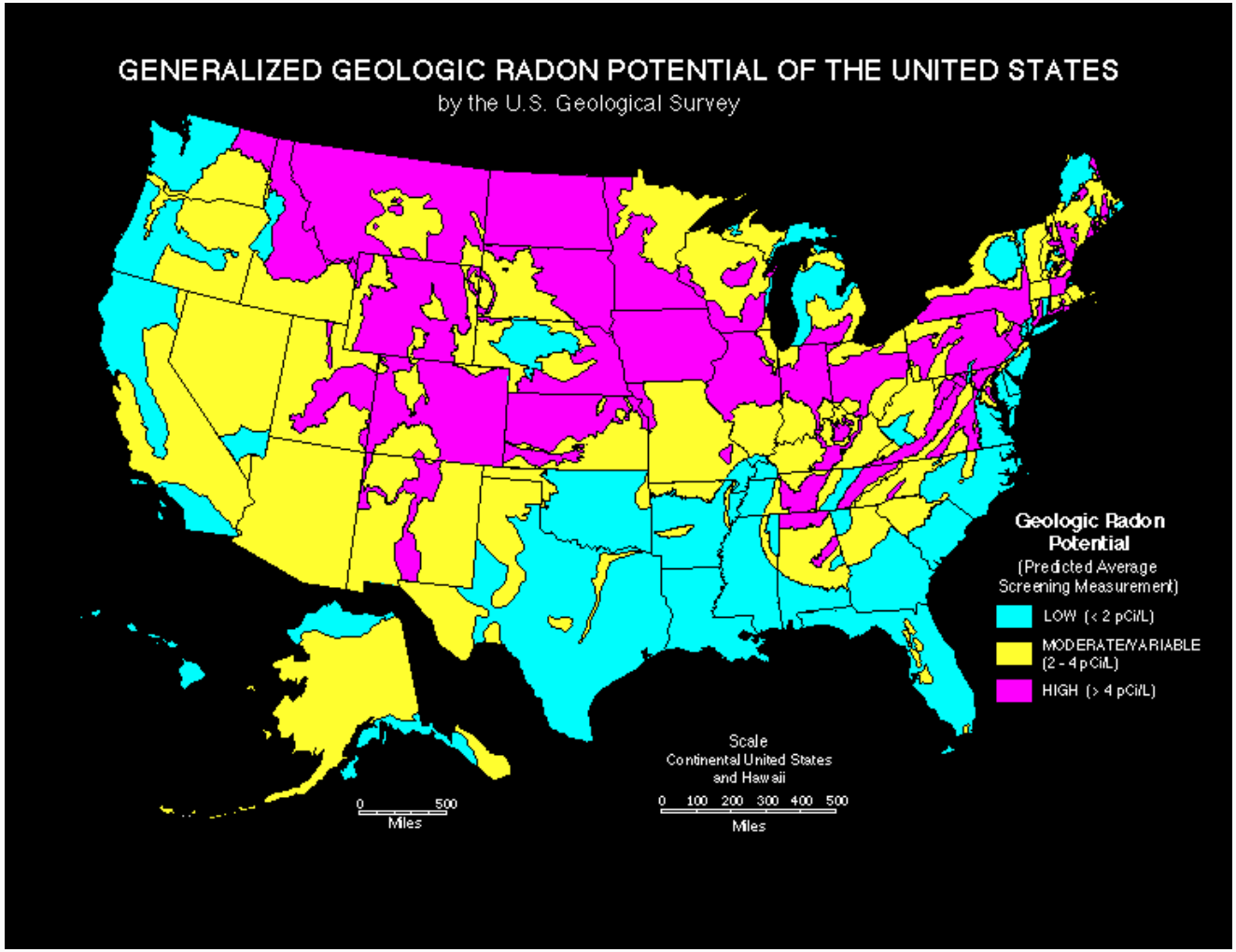
Radon is a byproduct of decaying soil uranium.

Radon enters a house more easily when it is built into the ground



Ann Arbor is a radon hotspot!

Considerable geographic variation in radon potential



Ann Arbor is a radon hotspot!

Trust me on this one...

My very own radon mitigation system.

What should a model that accounts for important sources of variation in household radon potential include?

What should a model that accounts for important sources of variation in household radon potential include?

- County-level variation in soil uranium.
- Whether or not the radon measurement was taken in a basement.

Random intercepts account for county-level variation

Gelman [@Gelman2006] proposes a multi-level model to measure household radon in household i in county j, y_{ij}:

- $y_{ij} \sim N(\alpha_j + \beta x_{ij}, \sigma_y^2)$, for $i = 1, \dots, n_j$, $j = 1, \dots, J$

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Where:

- α_j is average, non-basement radon measure at county level
- β is fixed effect measuring average change in radon level in houses with a basement.
- σ_y^2 represents within-county variation in risk

Include predictors of county-level variation in second level

County-level random intercept is a function of county soil uranium measure, u_j:

- $\alpha_j \sim N(\gamma_0 + \gamma_1 u_j, \sigma^2_\alpha)$, for $j = 1, \ldots, J$

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Where:

- γ_0 is expected household radon measure when $u_j = 0$
- γ_1 scales expected county-level uranium with u_j
- σ^2_α is between-county variation in radon risk not measured by u_j.

Putting it all together

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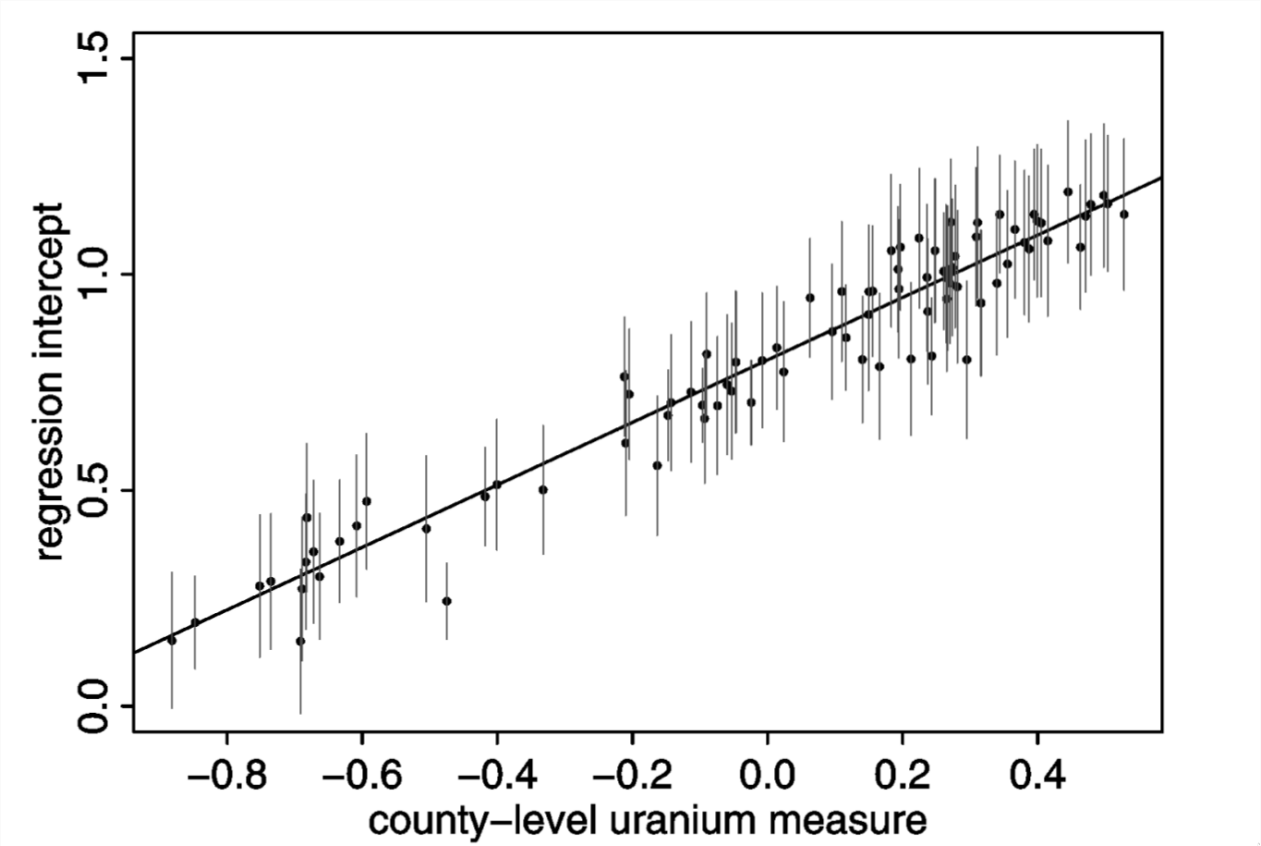
Household-level radon measure is a function of having a basement and county-level intercept:

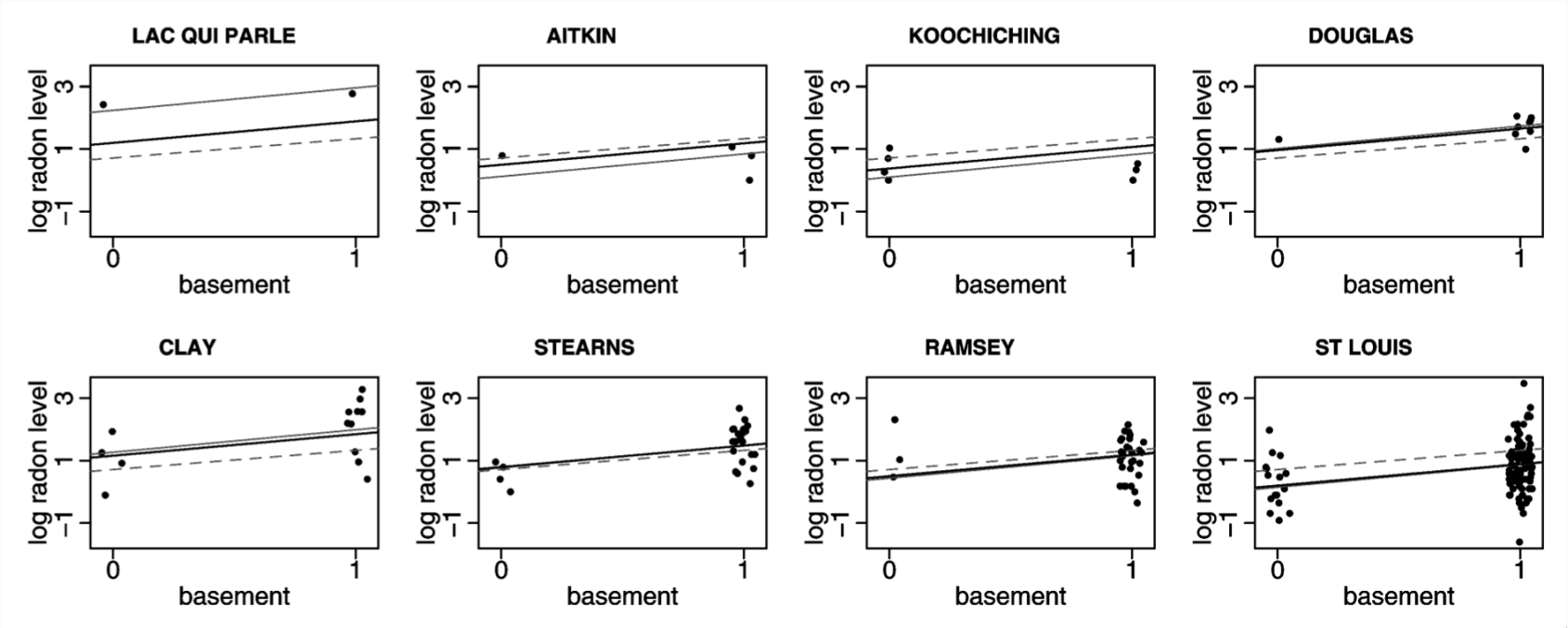
- $y_{ij} \sim N(\alpha_j + \beta x_{ij}, \sigma_y^2)$

County-level radon levels vary with soil uranium measures



County-level intercept, α_j, (± 1 standard error) as a function of county-level uranium.

Model predictions vs. radon measures by county



Multi-level regression line, $y = \alpha_j + \beta x$, from 8 Minnesota counties. Unpooled estimates = light grey line; Totally pooled estimates = dashed grey line.

Next Time

- Hands-on with the Radon example

References

References i