Statistical Thinking for Forensic Practitioners

Quiz on Part 2: The Language of Probability

1 Reversing Conditional Probabilities

We will practice using Bayes Theorem to reverse conditional probabilities.

Correctional Offender Management Profiling for Alternative Sanctions (COMPAS) is a risk assessment tool that predicts how likely a defendant is to recitivate (reoffend). In particular, a numerical score quantifying the "risk of recidivism" is assigned to a defendant. It has been used by some jurisdictions to inform parole decisions. A 2016 ProPublica article analyzed how accurate the risk assessment tool was at predicting recidivism rates among 7,214 individuals. In particular, they compared how the tool classified defendants as "low" or "high" risk to whether these individuals actually went on to recidivate after being released from prison. They found that the tool incorrectly labeled someone who did not recidivate post-release as "high risk" 32.35% of the time. The tool also incorrectly labeled someone who did go on to recidivate post-release as "low risk" 37.4% of the time. We are interested in calculating the probability that an individual goes on to recidivate given that the COMPAS tool classifies them as high risk.

1. Let H represent the event "the tool labeled a defendant as high risk" and R the event "the defendant went on to recidivate post-release." Using mathematical notation similar to that on slides 55-60 of the lecture slides, express the information given in the antepenultimate and penultimate sentences of the problem preamble above in terms of these events.

   The event *the tool incorrectly labeled someone who did not recidivate post-release as "high risk"* is equivalent to the event *an individual was classified as "high risk" given that they did not recidivate.* Expressing the event in this way makes it easier to see how we should express the probability: $P(H|R^c) = .3235$ remembering that R^c represents the complementary (i.e., opposite) event to R (this is an example of a false positive). Similarly, *the tool incorrectly labeled someone who did go on to recidivate post-release as "low risk"* can be expressed as the event $H^c|R$, meaning the probability is $P(H^c|R) = .374$ (this is an example of a false negative).

2. Use Bayes' Theorem to express the probability than an individual goes on to recidivate given the algorithm classified them as high risk. (Note: you do not need to evaluate the formula.)

   By Bayes' Theorem,

$$P(R|H) = \frac{P(H|R)P(R)}{P(H)}.$$

3. The event $H|R$ is complementary to one of the events identified in question 1. Use this fact to calculate $P(H|R)$. (Hint: it may be helpful to translate $H|R$ back in to English and consider its opposite.)

   $H|R$ is the event *the tool labeled a defendant as high risk given they recidivated.* Said another way, *the tools correctly labeled someone who did go on to recidivate post-release as "high risk."* This is an example of a "true positive." The opposite of this event is *the tool incorrectly classified someone as "low risk" who went on to recidivate.* You will recognize this as the false negative event $H^c|R$ (i.e., the "opposite" of a true positive is a false negative). Thus, $P(H|R) = 1 - P(H^c|R) = .626$.

4. The Law of Total Probability states that for two events A and B:

$$P(A) = P(A|B)P(B) + P(A|B^c)P(B^c).$$

Use this to re-express $P(H)$ in terms of conditional probabilities. (Note: you do not need to evaluate the formula.)

By the Law of Total Probability, $P(H) = P(H|R)P(R) + P(H|R^c)P(R^c)$.

5. There is still one piece of information required to evaluate $P(R|H)$ using Bayes' Rule (i.e., we still don't know one of the values in the Bayes' Rule expression you gave in question 2). Identify what this quantity represents in the context of the problem.

We still do not know $P(R)$ (or, equivalently, $P(R^c)$), which is the probability that an individual will recidivate post-release (or, equivalently will not recidivate post-release).

6. A 2018 report from Bureau of Justice Statistics concluded that 83% of released prisoners were arrested within 9 years of their release (Source). Use this to calculate $P(R|H)$.

Using the statistic of .83 as an estimate for $P(R)$, we have that

$$\begin{aligned} P(R|H) &= \frac{P(H|R)P(R)}{P(H)} \\ &= \frac{P(H|R)P(R)}{P(H|R)P(R) + P(H|R^c)P(R^c)} \\ &= \frac{(.626)(.83)}{(.626)(.83) + (.3325)(1 - .83)} \\ &= .902 \end{aligned}$$

rounding to 3 decimal places.

7. Do you find your answer to question 6 surprising given the information in the problem preamble? Why? (Hint: reference slides 59 and 60)

One might be initially surprised that the probability that someone actually recidivates given the tool labels them as "high risk" is so high (90.2%) based on the relatively large false positive and false negative rates of the tool. However, when we note that the prevalence of recidivism is large in general (we used 83%), the results are perhaps not that surprising. This is another illustration of why the "prior" information matters a great deal when interpreting results. For example, instead of using 83%, we could have used the statistic that 68% of released prisoners are arrested within 3 years of their release in which case $P(R|H) = .80$. We will discuss how one could estimate a reasonable prevalence rate such as $P(R)$ in Part 8 of the course.

2 Probability and betting

Let us reconsider the example discussed on slide 15 of the Part 2 lecture slides. You repeatedly play a game in which the only outcomes are winning or losing (e.g., winning if a coin flip is heads). Whether you win or lose in one round does not affect the probability of winning in the next round (i.e., the outcomes are independent). The game requires you to bet a certain amount of money each time you play. If you lose, then you lose the money you bet. If you win, then you gain some amount of money.

8. Suppose the probability of winning is .2 (20% chances) and you are required to bet \$5. If you win, you gain \$4. You want to determine if you should play the game 100 times based on whether you are expected to make money on average. Using similar calculations as shown in slide 15, determine whether you would make money on average.

You would be expected to make $20(4) - 80(5) = -320$ dollars. You would not make money on average and thus shouldn't play the game.

9. Now suppose you play the game only 10 times. How much money would you win (or lose) on average? (Hint: this should require a small a change to your calculations in part (a))

   If you play 10 times with a 20% chance of winning, then you're expected to win 2 times and lose 8 times. Then you would make $2(4) - 8(5) = -32$ dollars.

10. Now suppose you play the game only once. How much money would you win (or lose) on average? (Hint: continue the pattern used to solve part (b))

    While it doesn't make sense from a physical perspective, simply following the pattern from (a) and (b) means that you would be expected to win .2 times and lose .8 times if only playing once. Then you would make $.2(4) - .8(5) = -3.2$ dollars.

11. We might refer to the amount of money you would make from the game on average as the *expected winnings*. Let p is the probability of winning the game, w the amount of money you could win (in dollars), and b the amount you must bet (in dollars). How might we express the expected winnings after playing the game once? (Hint: refer to your answer from part (c))

    expected winnings $= p \cdot w - (1-p) \cdot b$

12. Suppose that you are allowed to choose the amount that you could win. If $p = .2$ and $b = 5$, use your formula from part (d) to calculate w such that you can expect to break even (win 0 dollars) on average.

    $0 = .2w - (1 - .2) \cdot 5 \implies w = 20$ after solving the equation for w.

3 Probabilistic evidence of a double-sided coin

A surprising number of historically important decisions were made by a coin toss. We could wonder how many such *seemingly* random choices have been nefariously decided by a double-sided coin (i.e., both sides have the same face). Is there some way that we could distinguish between a fair coin and a double-sided by simply observing the outcome of tossing them? We will investigate this question here.

Suppose you and some friends are testing out this hypothesis. You place a fair coin and a coin with heads on both sides in a bag. The two coins are indistinguishable other than by their fairness. Your most trustworthy friend takes the bag into another room (where you can't observe the tosses) and randomly selects one of the two coins from the bag (Source).

13. Prior to observing the outcome of the first flip, what is the probability that your friend selected the double-sided coin?

    If F corresponds to the event that the coin pulled is fair and U that the coin pulled is unfair, then

$$P(F) = \frac{1}{2} = P(U)$$

14. Your friend tosses the coin once and calls out "Heads!" What is the probability that the coin is fair given this outcome? (Hint: 3 out of the 4 possible outcomes are heads. Bayes' Theorem will be useful.)

$$P(F|H) = \frac{P(H|F)P(F)}{P(H)} = \frac{(\frac{1}{2})(\frac{1}{2})}{\frac{3}{4}} = \frac{1}{3}$$

15. What is the probability that the coin is fair had your friend instead flipped a tails?

$$P(F|T) = \frac{P(T|F)P(F)}{P(T)} = \frac{(\frac{1}{2})(\frac{1}{2})}{\frac{1}{4}} = 1$$

16. Your friend tosses the coin a second time and again calls out "Heads!" What is the probability the coin is fair given that these first two flips were heads? (Hint: 5 out of the 8 possible outcomes result in two heads)

$$P(F|HH) = \frac{P(HH|F)P(F)}{P(HH)} = \frac{\left(\frac{1}{2} * \frac{1}{2}\right)\left(\frac{1}{2}\right)}{\frac{5}{8}}$$

17. Your friend tosses the coin a third time and a third time calls out "Heads!" What is the probability the coin is fair given these first three flips were heads? (Hint: 9 out of 16 possible outcomes result in three heads.)

$$P(F|HHH) = \frac{P(HHH|F)P(F)}{P(HHH)} = \frac{\left(\frac{1}{2} * \frac{1}{2} * \frac{1}{2}\right)\left(\frac{1}{2}\right)}{\frac{9}{16}}$$

18. What pattern do you see between these probabilities? Suppose your friend flips the coin n times and gets heads all n times. Express the probability that the coin is fair given the first n flips are heads. (Hint: powers of 2 can be expressed as 2^k for some whole number k.)

$$P(F|\underbrace{HH...H}_{n \text{ times}}) = \frac{1}{2^n + 1}$$

19. After the 3rd heads, one of your prosecutorial friends makes the claim: "After seeing these probabilities, I'm convinced that the coin is double-sided! The chances of seeing so many heads in a row if the coin *were* fair is minuscule." How do you respond to your friend? Either explain why they are correct or provide a probability contradicting the claim.

The claim seems to be contradicted by the fact that the probability of seeing 3 heads in a row given the coin is fair is not "minuscule," but is actually equal to .125 (1/8).

$$P(HHH|F) = \frac{P(F|HHH)P(HHH)}{P(F)} = \frac{\left(\frac{1}{9}\right)\left(\frac{9}{16}\right)}{\frac{1}{2}} = \frac{1}{8}.$$