(so as a ring $H^*_{dR}(\mathbb{C}P^n;\mathbb{R}) \simeq \mathbb{R}[t]/(t^{n+1}=0)$ polynomial ring up to degree n.

3. cptly supp deRham cohomology

This is designed to deal with non-cpt mfd M.

$\Omega^k_c(M;\mathbb{R}) = \{$ cpt supp k-form on $M\}$ $\leftarrow$ This is closed under exterior derivative

KEY observation: it could be $d: \underset{\Omega^k(M)}{\alpha} \longrightarrow \underset{\Omega^k_c(M)}{d\alpha}$

BUT α is not cptly supp. e.g. $f \equiv 1 \in \Omega^0(\mathbb{R}^n)$.

Define $$H^k_c(M;\mathbb{R}) := \frac{\ker\{d_k : \Omega^k_c(M;\mathbb{R}) \to \Omega^{k+1}_c(M;\mathbb{R})\}}{\operatorname{Im}\{d_{k-1} : \Omega^{k-1}_c(M;\mathbb{R}) \to \Omega^k_c(M;\mathbb{R})\}}$$

e.g. compute $H^*_c(\mathbb{R}^1;\mathbb{R})$

$$H^0_c(\mathbb{R}^1;\mathbb{R}) = \frac{\ker\{d_0 : \Omega^0_c(\mathbb{R}^1;\mathbb{R}) \to \Omega^1_c(\mathbb{R};\mathbb{R})\}}{0}$$

b/c $\Omega^0_c(\mathbb{R}^1;\mathbb{R}) \ni 0$ (df = 0 ⇒ f is constant, and it has to be 0 in order to be cptly supp)

$$= 0/0 = 0.$$

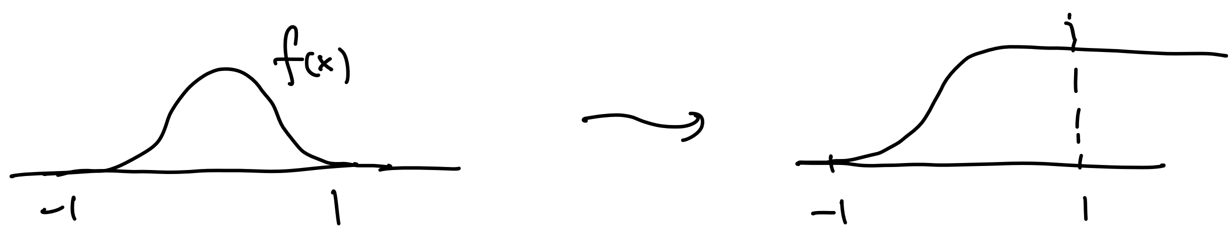
This is in a sharp contrast to $H^0_{dR}(\mathbb{R}^1;\mathbb{R})$ ($\cong \mathbb{R}$). In general, when M is non-cpt, $H^0_c(M;\mathbb{R}) = 0$.

$$H^1_c(\mathbb{R}^1;\mathbb{R}) = \frac{\ker(d_1 : \Omega^1_c(\mathbb{R}^1;\mathbb{R}) \to 0)}{\operatorname{Im}(d_0 : \Omega^0_c(\mathbb{R}^1;\mathbb{R}) \to \Omega^1_c(\mathbb{R}^1;\mathbb{R}))}$$

$$= \Omega^1_c(\mathbb{R}^1;\mathbb{R}) \big/ \operatorname{Im}(d_0) = \frac{\{f(x)dx \mid f \subset\subset \mathbb{R}\}}{\{dg \mid g \subset\subset \mathbb{R}\}}$$

Naive: For any $f(x)dx$, consider $g(x) = \int_{-\infty}^{x} f(t)\,dt$ (then $dg = f(x)dx$)

but g is <u>not</u> nec inside $\Omega^0_c(\mathbb{R}^1;\mathbb{R})$.



(This infact indicates that $H^1(\mathbb{R}^1;\mathbb{R})=0$.)

Consider $S: \Omega_c^1(\mathbb{R}^1;\mathbb{R}) \twoheadrightarrow \mathbb{R}$ by $f(x)dx \overset{S}{\longmapsto} \int_{\mathbb{R}} f(x)dx < +\infty$.

Then $\ker(S) = \{ f(x)dx \mid \int_{\mathbb{R}} f(x)dx = 0 \}$.

By construction above, consider $g(x) = \int_{-\infty}^{x} f(t)dt$ for any $f(x)dx$ in $\ker(S)$ and $g \in \operatorname{Im} d_0$ ($\Rightarrow \ker(S) = \operatorname{im}(d_0)$).

$\Rightarrow H_c^1(\mathbb{R}^1;\mathbb{R}) \simeq \mathbb{R}$ so $\dim H_c^1(\mathbb{R}^1;\mathbb{R}) = 1$.

Recall $H^1_{dR}(\mathbb{R}^1;\mathbb{R})=0$.

- From $H_c^0(\mathbb{R}^1;\mathbb{R})$, we obtain a general result:

$$H_c^0(M;\mathbb{R}) = \mathbb{R}^{\#\text{ cpt connect component of } M}.$$

- From $H_c^0(\mathbb{R}^1;\mathbb{R})$, we know that $H_c^*(M;\mathbb{R})$ is <u>not</u> an invariant up to homotopy equivalence (b/c $\mathbb{R}^1 \simeq \{pt\}$ but $H_c^0(\{pt\},\mathbb{R}) = \mathbb{R}$).

- $f: \mathbb{R} \to \mathbb{R}$ by $f(x) \equiv 0$, then f^*: $\Omega^1(\mathbb{R}^1;\mathbb{R}) \to$
for the pullback

$\Omega^1(\mathbb{R}^1;\mathbb{R})$, $f^*(0) = \Omega^1(\mathbb{R}^1;\mathbb{R})$ (much bigger than $\Omega^1_c(\mathbb{R}^1;\mathbb{R})$).

so f^* does not pullback well in terms of cpt supp forms.

To fix this, one usually consider two variants

① assume f is proper (preimage of cpt under f is cpt).

② instead of f^*, consider "pushforward f_*".

For ①, $f: N \to M$ and proper then similarly to the standard case, we have $f^*: H^*_c(M;\mathbb{R}) \to H^*_c(N;\mathbb{R})$.

For ②, only works for special cases:

e.g. If $N \subset M$ and $f: N \to M$ is the inclusion, then define

$f_*(\underset{\Omega^k_c(V;\mathbb{R})}{\theta}) =$ extension by zero of $\theta \in \Omega^k_c(M;\mathbb{R})$.

e.g. If $\begin{matrix} E \\ \downarrow \pi \\ M \end{matrix}$ a vector bundle. $\theta \in \Omega^k_c(E)$, then define

$$\left(\pi_*(\theta)\right)(p) := \int_{\pi^{-1}(p)} \theta \in \Omega^{k - \text{rank of } \begin{matrix} E \\ \downarrow \\ M \end{matrix}}$$ ← called integration along fiber → reference, Thom class

(DIY) For $H^*_c(M;\mathbb{R})$, we also have a MV-seq (but with opposite direction):

$$\cdots \to H^k_c(U\cap V;\mathbb{R}) \to H^k_c(U;\mathbb{R}) \oplus H^k_c(V;\mathbb{R}) \to H^k_c(M;\mathbb{R}) \xrightarrow{+1} \cdots$$

Thm (Poincaré duality) Let M be an oriented mfld (without b/d). Then

$$H^*_{dR}(M;\mathbb{R}) \simeq \left(H_c^{\dim M - *}(M;\mathbb{R})\right)^*$$ ← dual of a vector space

This isomorphism is explicitly given by $[\theta] \mapsto$ a linear map $PD([\theta])$

defined by $PD([\theta])([\sigma]) := \int_M \theta \wedge \sigma \in \mathbb{R}$. ← this is in top degree.

$\underline{\text{Rmk}}$ well-definedness of this def. $\theta + d\tau$, then

$$\int_M (\theta + d\tau) \wedge \sigma = \int_M \theta \wedge \sigma + \underbrace{\int_M d(\tau \wedge \sigma)}_{=0 \text{ by Stokes' Thm.}}$$

↖ b/c σ is closed

$$= \int_M \theta \wedge \sigma.$$

e.g. Let $M = \mathbb{R}^n$, then

$$H^k_c(\mathbb{R}^n; \mathbb{R}) = \left(H^{n-k}_{dR}(\mathbb{R}^n; \mathbb{R})\right)^* = \begin{cases} \mathbb{R} & k = n \\ 0 & k \neq n \end{cases}$$

e.g. If M^n is connected, $\underline{\text{non-cpt}}$, orientable

$$H^n_c(M; \mathbb{R}) \simeq \left(H^0_{dR}(M; \mathbb{R})\right)^* = \mathbb{R}$$

(NEW) $H^n_{dR}(M; \mathbb{R}) \simeq \left(H^0_c(M; \mathbb{R})\right)^* = 0$

Recall in Lecture 5, we defined and calculated $H^n_{dR}(M; \mathbb{R})$ when M is cpt e.g. M cpt, non-orientable $\Longrightarrow$ $H^n_c(M;\mathbb{R}) = H^n_{dR}(M;\mathbb{R}) = 0$

Rmk (NEW) M non-cpt non-orientable $\Longrightarrow$ $H^n_c(M;\mathbb{R}) = 0$ (see proof in Lee's book Thm 17.34)

we can't apply Poincaré duality as stated above (which works for orientable cases).

e.g. If M is closed, then $H^*_c(M;\mathbb{R}) \simeq H^*_{dR}(M;\mathbb{R})$.

$\Longrightarrow$ M^3, then $H^0_{dR}(M;\mathbb{R}) \simeq H^3_{dR}(M;\mathbb{R})$ (*)

$H^1_{dR}(M;\mathbb{R}) \simeq H^2_{dR}(M;\mathbb{R})$.

$$\Rightarrow \chi(M) = \sum_{k=0}^{3} (-1)^k b_k(M;\mathbb{R}) = \cancel{\dim H^0} - \dim H^1 + \dim \cancel{H^2 - \dim H^3}$$
$$= 0.$$

(In general, any M^{odd} has $\chi(M) = 0$).

Rmk Every odd-dim'l closed mfd is orientable. (by (*))

Rmk We will prove Poincaré duality in next section.

To end this section, let us demonstrate an application of $H^*_c(M;\mathbb{R})$.

For $f: N^n \longrightarrow M^n$ proper, consider $f^*: H^n_c(M;\mathbb{R}) \longrightarrow H^n_c(N;\mathbb{R})$

orientable, connected without b/d. $\qquad$ $H^n_c(M;\mathbb{R}) \simeq \mathbb{R}$, $\quad H^n_c(N;\mathbb{R}) \simeq \mathbb{R}$

Then fix any generators $\alpha \in H^n_c(M;\mathbb{R})$, we have

defined by $\int_N \theta$ for any rep θ of class α

$$\int_N f^*\alpha = \lambda \cdot \int_M \alpha \quad \rightsquigarrow \quad \text{define } \deg(f) = \lambda.$$

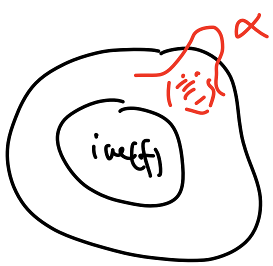
degree of f

The degree $\deg(f)$ is independent of the choice of the generator.

<u>FACT</u> (proved in next lecture) $\deg(f) \in \mathbb{Z}$.

Here are trivial observations directly from def.

- If $f: N \to M$ is not surjective, then $\deg(f) = 0$.



$M \implies f^*\alpha = 0 \implies \deg(f) = 0$

- If $f: M \circlearrowleft$ is the identity map, then $\deg(f) = 1$.
- $L \xrightarrow{f} N \xrightarrow{g} M \Rightarrow \deg(g \circ f) = \deg(f)\deg(g)$.

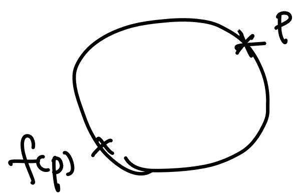
$f, g: N \to M$ proper homotopic then $\deg(f) = \deg(g)$.

- If $f: M \circlearrowleft$ diffeo, then $\forall\, \alpha \in H^n_c(M; \mathbb{R})$, we have

$$\underbrace{\int_M f^*\alpha = \pm \int_M \alpha}_{\text{change variable}} \Rightarrow \deg(f) = \begin{cases} 1 & \text{orientation preserving} \\ -1 & \text{orientation reversing.} \end{cases}$$

b/c $\deg(f) \cdot \deg(f^{-1}) = \deg(\mathbb{1}) = 1$
$\Rightarrow \deg(f) = \pm 1$

e.g. $f: S^1 \to S^1$



$p \to -p$. $(x, y) \mapsto (-x, -y)$

then take a closed 1-form $\theta = -y\,dx + x\,dy$ for $(x, y) \in S^1$ (i.e. $x^2 + y^2 = 1$).

$$\int_{S^1} f^*\theta = \int_{S^1} \theta \quad \text{b/c both } x, y \text{ change sign} \Rightarrow \deg(f) = 1$$

One can image, for S^2, the "antipotal" map $p \to -p$ will have $\deg(f) = -1$.

In general, $\deg\left(\underset{\text{antipotal}}{f: S^n \to S^n}\right) = \begin{cases} +1 & \text{if } n \text{ is odd} \\ -1 & \text{if } n \text{ is even} \end{cases}$

$\Longrightarrow$ On S^{2n}, there does not exist any nowhere vanishing vector fields.

general hairball thm

Pf. Set $S^{2n} \subset \mathbb{R}^{2n+1}$ as the standard sphere, then vector field X at pt $p \in S^{2n}$ lies in the tangent plane $T_p S^{2n}$, which is orthogonal to $p \subset \mathbb{R}^{2n+1}$. In particular, $X(p)$, p are always linearly indep in $\mathbb{R}^{2n+1}$.

since $X(p) \neq 0$ $\forall p$.

a continuous map par by t.

Suppose $X(p) \neq 0$ $\forall p$, then consider a htp

$$\cos(\pi t)\, p + \sin(\pi t)\, X(p) \quad \text{for } t \in [0,1]$$

Then maps $f_t : p \longmapsto \cos(\pi t) p + \sin(\pi t) X(p)$ is a htp from $f_0 = \mathbb{1}$ to $f_1 =$ antipodal map $\Rightarrow f_0^* \alpha = f_1^* \alpha \Rightarrow \deg(f_0) = \deg(f_1)$

$\to\leftarrow$.

<u>Prop</u> M^n cpt mfd with b/d ∂M

X^{n-1} cpt orientable mfd

If $f : \partial M \to X^{n-1}$ can be extended to $g : M \to X^{n-1}$,

then $\deg(f) = 0$

(How to apply: suppose $X^{n-1} = \partial M$ for some M, say $M = B^n$ n-dim'l ball, then $X^{n-1} = \partial M = S^{n-1}$. Under the hypothesis above, such f can not be homotopic to either $\mathbb{1}$ or antipotal map!)

~~Pf of prop~~. Fix a volume form Ω on X^{n-1} s.t. $\int_X \Omega = 1$

Then $\deg(f) = \deg(f)\int_X \Omega = \int_{\partial M} f^*\Omega$.

$$\begin{tikzcd} \partial M \arrow[r, "i"] \arrow[rr, bend right, "f = g \circ i"'] & M \arrow[r, "g"] & X^{n-1} \end{tikzcd} \Longrightarrow \int_{\partial M} f^*\Omega = \int_{\partial M} i^*(g^*\Omega)$$

$$= \int_M d(g^*\Omega) = \int_M g^* \overset{0}{\cancel{d\Omega}} = 0$$ ▯.

Extension reading topics

- de Rham coh groups on a Lie group ← Chevalley - Eilenberg's original paper 1948
- Thom class (integration over fiber) ← Bott-Tu's book. Chapter 1
- Poincaré duality of submfds (intersection theory; basic).

notes from Nicolaescu (Notre Dame).