HOMEWORK FOR LECTURE 8

This homework problem set can be accomplished with the help of references. DO NOT LEAVE ANY PROBLEM BLANK! It is due to 19:30 pm on January 6, 2025 (sharp).

Exercise 1 [3 points]. Complete the following question on Hodge-Laplace operator.

(1) [1 points] Let M be a connected closed manifold and $f : M \to \mathbb{R}$ be a smooth function. Fix a volume form Ω on M. Prove that $\Delta f = 0$ or $\Delta(f\Omega) = 0$ if and only if f is a constant function.

(2) [2 points] Under the same hypothesis of (1) above. Prove that $\int_M f\Omega = 0$ if and only if there exists a smooth function $g : M \to \mathbb{R}$ such that $\Delta g = f$.

Exercise 2 [4 points]. A *contact 1-form* on M^3 is a 1-form $\alpha \in \Omega^1(M)$ such that $d\alpha \wedge \alpha$ is nowhere vanishing (i.e., a volume form). Complete the following questions.

(1) [1 points] Prove that the hyperplane field $\mathcal{D}^2$ defined by

$$\mathcal{D}^2(p) := \ker \alpha(p) = \{v \in T_pM \mid \alpha_p(v) = 0\}$$

for any $p \in M$ is *not* integrable anywhere (called *completely non-integrable*). Such a completely non-integrable $\mathcal{D}^2$ is called a contact structure on M^3.

(2) [2 points] Following the terminology in (1) right above, for $\mathbb{T}^3 = (\mathbb{R}/\mathbb{Z})^3$ in coordinate (x, y, z), prove that $\mathcal{D}^2$ defined as follows,

$$\mathcal{D}^2 = \mathrm{span}_{\mathbb{R}} \left\langle \frac{\partial}{\partial z},\ \cos(2\pi z)\frac{\partial}{\partial x} - \sin(2\pi z)\frac{\partial}{\partial y} \right\rangle$$

is a contact structure on $\mathbb{T}^3$. (Hint: find a 1-form $\alpha \in \Omega^1(\mathbb{T}^3)$ with kernel equal to $\mathcal{D}^2$ and check that α is a contact 1-form.)

(3) [1 points] Draw a closed curve γ in $\mathbb{T}^3$ such that everywhere its tangent vector lies in $\mathcal{D}^2$. Note that this does *not* contradict the Frobenius integrability theorem!

Exercise 3 [3 points]. Use Sard's theorem and stereographic projection to prove that the n-sphere S^n (for $n \geq 2$) is simply connected. (Recall that a smooth manifold X is *simply connected* if any smooth map $S^1 \to X$ can be continuously deformed to a constant map.)