

Ex 1.

(i). (ii).

For $GL(n, F)$, $F = \mathbb{R}$ or \mathbb{C} . $SL(n, F)$ is closed subgroup of $GL(n, F)$ so it is Lie subgroups of $GL(n, F)$.

Let $sl(n, F)$ be the subspace of $gl(n, F)$ s.t.

$$sl(n, F) := \{ A \in gl(n, F) \mid \text{trace}(A) = 0 \}.$$

For $B, C \in sl(n, F)$, $\text{Tr}([B, C]) = \text{Tr}(BC - CB) = 0$.

$$\Rightarrow [B, C]_{gl(n, F)} \in sl(n, F)$$

$\Rightarrow sl(n, F)$ is a Lie subalgebra.

The map $X \mapsto e^X$ gives a diffeomorphism

To end the proof. note that

$$\{ A \in gl(n, F) \mid \exp_{gl(n, F)}(tA) \in SL(n, F) \quad \forall t \in \mathbb{R} \}.$$

$$= sl(n, F).$$

Indeed.

$$\begin{aligned} e^{tA} \in SL(n, F) &\Leftrightarrow \det(e^{tA}) = 1 \quad \forall t \\ &\Leftrightarrow \text{Tr}(tA) = 0 \quad \forall t \\ &\Leftrightarrow \text{Tr}(A) = 0. \end{aligned}$$

(iii). (iv).

$O(n) / U(n)$ is closed subgroup of $GL(n, \mathbb{R}) / GL(n, \mathbb{C})$.

thus Lie subgroup. As Lie subalgebra :

$$g_{O(n)} = \{ A \in gl(n, \mathbb{R}) \mid \exp(tA) \in O(n) \quad \forall t \in \mathbb{R} \}.$$

$$\exp(tA) \in O(n) \Leftrightarrow e^{tA} (e^{tA})^T = I$$

$$\begin{aligned} \Leftrightarrow \forall t: \quad I &= (I + tA + \frac{1}{2}t^2A^2 + \dots)(I + tA^T + \frac{1}{2}t^2A^{T^2} + \dots) \\ &= I + t(A + A^T) + \frac{1}{2}t^2(A^2 + 2AA^T + A^{T^2}) + \dots \end{aligned}$$

\Rightarrow Each coefficient in this series must vanish except I .

$$\Leftrightarrow A + A^T = 0.$$

So

$$g_{O(n)} = O(n) = \{ A \in gl(n, \mathbb{R}) \mid A^T + A = 0 \}.$$

Similar case for

$$g_{\mathfrak{u}(n)} = u(n) = \{ A \in \mathfrak{gl}(n, \mathbb{C}) \mid A^* + A = 0 \}.$$

(V).

$$g_{\text{Sp}(2n)} = \{ A \in \mathfrak{gl}(2n, \mathbb{R}) \mid \exp(tA) \in \text{Sp}(2n) \quad \forall t \in \mathbb{R} \}.$$

$$e^{tA} J (e^{tA})^T = J.$$

$$\Leftrightarrow J e^{tA^T} = e^{-tA} J$$

$$\Leftrightarrow J(I + tA^T + \frac{1}{2}t^2 A^{T^2} + \dots) = (I - tA + \frac{1}{2}t^2 A^2) J.$$

$$\Leftrightarrow J \left(\frac{1}{n!} (A^T)^n \right) = (-1)^n \frac{1}{n!} A^n \cdot J. \quad (\#)$$

In particular : $J A^T + A J = 0$.

We prove it is equivalent to (*):

$$J(A^T)^n = - A J (A^T)^{n-1} = (-1) \cdot A (J \cdot (A^T)^{n-1}) \cdots = (-1)^n A^n \cdot J.$$

So

$$\begin{aligned} g_{\text{Sp}(2n)} &= \text{sp}(2n) = \{ A \in \mathfrak{gl}(2n, \mathbb{R}) \mid AJ + JA^T = 0 \} \\ &= \{ A \in \mathfrak{gl}(2n, \mathbb{R}) \mid A^T J + JA = 0 \}. \end{aligned}$$

Ex 2.

1^o. From Ado's theorem:

Every finite-dimensional real Lie algebra is isomorphic to a subalgebra of $\mathfrak{gl}(n, \mathbb{R})$. Every finite-dimensional complex Lie algebra is isomorphic to a complex subalgebra of $\mathfrak{gl}(n, \mathbb{C})$.

So we can assume \mathfrak{g} is a matrix Lie algebra.

2^o. We can choose a neighborhood U_0 of $0 \in \mathfrak{g}$ on which \exp is a local diffeomorphism and \exp^{-1} is well defined on $\exp(U_0)$. Let U be an open ball in \mathfrak{g} s.t. $(\exp U)^2 (\exp U)^{-2} \subseteq \exp U_0$.

$\forall X, Y \in U$. $\gamma(t) := e^{tX} e^{tY}$. t can be choosed in a nghd of $[0, 1]$. $\gamma(t) \in \exp(U)$.

$\exists Z(t) \in U_0$ s.t. $\exp(Z(t)) = \exp(tX) \exp(tY)$.

We have:

$$\frac{d}{dt} \exp(Z(t)) = X \exp(Z(t)) + \exp(Z(t)) Y.$$

(a). If $X(t)$ is a smooth matrix-valued function, then

$$\begin{aligned} \frac{d}{dt} \exp(X(t)) &= \exp(X(t)) \left(\frac{I - \exp(-\text{ad}_{X(t)})}{\text{ad}_{X(t)}} \right) \frac{dX}{dt} \\ &= \left(\frac{\exp(\text{ad } X(t)) - I}{\text{ad}(X(t))} \right) \frac{dX}{dt} \exp(X(t)). \end{aligned}$$

From (a):

$$\left(\frac{\exp(\text{ad } Z(t)) - I}{\text{ad } Z(t)} \right) \frac{dZ(t)}{dt} \exp(Z(t)) = X \exp(Z(t)) + \exp(Z(t)) Y.$$

$Z(t) \in U_0$. \exp is a local diffeomorphism near $Z(t)$.

$\frac{\exp(\text{ad } Z(t)) - I}{\text{ad } Z(t)}$ is an invertible map on g .

$$\begin{aligned} \frac{dZ(t)}{dt} &= \left(\frac{\text{ad } Z(t)}{\exp(\text{ad } Z(t)) - I} \right) \cdot (X + \exp(Z(t)) Y \exp^{-1}(Z(t))) \\ &= \left(\frac{\text{ad } Z(t)}{\exp(\text{ad } Z(t)) - I} \right) \cdot (X + \text{Ad}(e^{Z(t)}) Y) \\ &= \left(\frac{\text{ad } Z(t)}{\exp(\text{ad } Z(t)) - I} \right) \cdot (X + \text{Ad}(e^{tX}) \text{Ad}(e^{tY}) Y) \\ &= \left(\frac{\text{ad } Z(t)}{\exp(\text{ad } Z(t)) - I} \right) \cdot (X + \exp(t\text{ad}(X)) \exp(t\text{ad}(Y)) Y) \\ &= \left(\frac{\text{ad } Z(t)}{\exp(\text{ad } Z(t)) - I} \right) \cdot (X + \exp(t\text{ad} X) Y). \end{aligned}$$

For $A = \text{ad } Z(t)$.

$$A = \log(I + (\exp(A) - I)) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} (e^A - I)^n$$

$$\Rightarrow \frac{\text{ad } Z(t)}{\exp(\text{ad } Z(t)) - I} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} (\exp(t \text{ad } X) \exp(t \text{ad } Y) - I)^{n-1}$$

$$\frac{dZ(t)}{dt} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} (\exp(t \text{ad } X) \exp(t \text{ad } Y) - I)^{n-1} (X + \exp(t \text{ad } X) Y).$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \left[\sum_{(i,j) \neq (0,0)}^{\infty} \frac{t^{ij}}{i! j!} (\text{ad } X)^i (\text{ad } Y)^j \right]^{n-1} (X + \sum_{i=0}^{\infty} \frac{t^i}{i!} (\text{ad } X)^i Y).$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \left[\sum_{i_k+j_k \geq 1} \frac{t^{i_1+j_1+\dots+i_{n-1}+j_{n-1}}}{i_1! j_1! \dots i_{n-1}! j_{n-1}!} [(X)^{i_1}, (Y)^{j_1}, \dots, (X)^{i_{n-1}}, (Y)^{j_{n-1}}, X] \right]$$

$$+ \sum_{i_k+j_k \geq 1} \frac{t^{i_1+j_1+\dots+i_{n-1}+j_{n-1}+i_n}}{i_1! j_1! \dots i_{n-1}! j_{n-1}! i_n!} [(X)^{i_1}, (Y)^{j_1}, \dots, (X)^{i_{n-1}}, (Y)^{j_{n-1}}, (X)^{i_n}, Y]$$

using the notation that :

$$[X_n, \dots, X_3, X_2, X_1] = [X_n, \dots, [X_3, [X_2, X_1]] \dots]$$

$$[X_n^{(i_n)}, \dots, X_1^{(i_1)}] = [\underbrace{X_n, \dots, X_n}_{i_n}, \dots, \underbrace{X_1, \dots, X_1}_{i_1}]$$

Thus we have proved:

$$\exp(X) \exp(Y) = \exp(Z).$$

$$Z = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \sum_{i_k+j_k \geq 1} \frac{1}{(i_1+j_1+\dots+i_n+j_n)} \cdot \frac{1}{i_1! j_1! \dots i_n! j_n!} [X^{i_1}, Y^{j_1}, \dots, X^{i_n}, Y^{j_n}]$$

In particular,

$$\begin{aligned} \exp(X) \exp(Y) &= \exp(X+Y + \frac{1}{2}[X, Y] + \frac{1}{12}[X, [X, Y]] \\ &\quad - \frac{1}{12}[Y, [X, Y]] + \dots). \end{aligned}$$

3°. Now for t small enough

$$\exp(-tX) \exp(-tY) \exp(tX) \exp(tY)$$

$$= \exp(-tX - tY + tX + tY + \frac{1}{2}t^2[X, Y] - \frac{t^2}{2}[X, Y] - \frac{t^2}{2}[Y, X] + \frac{1}{2}t^2[X, Y] + O(t^3)).$$

$$= \exp(t^2[X, Y] + O(t^3)).$$

4°. Proof of (a):

$$\begin{aligned} \frac{d}{dt} \exp(X(t)) &= \exp(X(t)) \left(\frac{I - \exp(-\text{ad}_{X(t)})}{\text{ad}_{X(t)}} \right) \frac{dX}{dt} \\ &= \left(\frac{\exp(\text{ad}_{X(t)}) - I}{\text{ad}_{X(t)}} \right) \frac{dX}{dt} \exp(X(t)). \end{aligned}$$

Define

$$\Psi(s, t) = \exp(-sX(t)) \frac{\partial}{\partial t} \exp(sX(t))$$

$$\Psi(0, t) = 0. \quad \Psi(1, t) = \int_0^1 \frac{\partial}{\partial s} \Psi(s, t) ds$$

$$\frac{\partial}{\partial s} \Psi(s, t)$$

$$\begin{aligned} &= -X(t) \exp(-sX(t)) \frac{\partial}{\partial t} \exp(sX(t)) \\ &\quad + \exp(-sX(t)) \frac{\partial}{\partial t} [X(t) \exp(sX(t))] \end{aligned}$$

$$\begin{aligned} &= -\exp(-sX(t)) X(t) \frac{\partial}{\partial t} \exp(sX(t)) + \exp(-sX(t)) \frac{\partial}{\partial t} X(t) \exp(sX(t)) \\ &\quad + \exp(-sX(t)) X(t) \frac{\partial}{\partial t} \exp(sX(t)) \end{aligned}$$

$$= \exp(-sX(t)) \frac{\partial}{\partial t} X(t) \exp(sX(t))$$

$$= \text{Ad}(\exp(-sX(t))) \frac{\partial}{\partial t} X(t) = \exp(-s\text{ad}X(t)) \frac{\partial}{\partial t} X(t).$$

$$\begin{aligned}
 \Phi(1, t) &= \int_0^1 \exp(-s \operatorname{ad} X(t)) \frac{\partial}{\partial t} X(t) \cdot ds \\
 &= \int_0^1 \sum_{n=0}^{\infty} \frac{(-s)^n}{n!} (\operatorname{ad} X(t))^n \frac{\partial}{\partial t} X(t) ds \\
 &= \sum_{n=0}^{\infty} \frac{(-1)^n s^{n+1}}{(n+1)!} (\operatorname{ad} X(t))^n \frac{\partial}{\partial t} X(t) \Big|_{s=0}^{s=1} \\
 &= \left(\frac{I - \exp(-\operatorname{ad} X(t))}{\operatorname{ad} X(t)} \right) \frac{dX}{dt}
 \end{aligned}$$

To show the second part, note that

$$\begin{aligned}
 L_{\exp(X(t))} &= R_{\exp(X(t))} \circ \operatorname{Ad}(\exp(X(t))) \\
 &= R_{\exp(X(t))} \circ \exp(\operatorname{ad} X(t))
 \end{aligned}$$

Ex 3.

(1).

(i). If $A \in M_n(\mathbb{R})$ is diagonalizable, that is.

$$P^{-1}AP = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix} = A_0 \quad P \in GL(n, \mathbb{R}).$$

$$\begin{aligned}
 \Rightarrow e^A &= Id + A + \frac{1}{2}A^2 + \dots \\
 &= Id + PA_0P^{-1} + \frac{1}{2}(PA_0P^{-1})^2 + \dots \\
 &= P(Id + A_0 + \frac{1}{2}A_0^2 + \dots)P^{-1} \\
 &= P e^{A_0} P^{-1}
 \end{aligned}$$

$$\det(e^A) = \det(e^{A_0}) = \prod_{i=1}^n e^{\lambda_i} = e^{\sum_{i=1}^n \lambda_i} = e^{\operatorname{tr}(A_0)} = e^{\operatorname{tr}(A)}$$

(ii). If $A \in M_n(\mathbb{R})$ is nilpotent, that is.

$$P^{-1}AP = \begin{pmatrix} 0 & * \\ 0 & \ddots \\ 0 & 0 \end{pmatrix} = A_0.$$

$$e^{A_0} = Id + A_0 + \frac{1}{2}A_0^2 + \dots = \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$$

$$e^A = P e^{A_0} P^{-1}.$$

$$\det(e^A) = 1 = e^{\text{tr}(A)} = e^{\text{tr}(A_0)}.$$

(iii). $\forall A \in M_n(\mathbb{R})$. $\exists N$ nilpotent & S diagonalizable

$$\text{s.t. } A = N + S \quad \& \quad NS = SN.$$

$$\Rightarrow e^A = e^N e^S$$

$$\begin{aligned} \det(e^A) &= \det(e^N) \det(e^S) = e^{\text{tr}(N)} \cdot e^{\text{tr}(S)} \\ &= e^{\text{tr}(N+S)} = e^{\text{tr}(A)}. \end{aligned}$$

(2). If $e^A = \begin{pmatrix} -2 & 0 \\ 0 & -1 \end{pmatrix} =: B$

$$\Rightarrow B = Id + A + \frac{1}{2}A^2 + \dots$$

$$\Rightarrow BA = AB. \quad A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} -2a_{11} & -a_{12} \\ -2a_{21} & -a_{22} \end{pmatrix} = \begin{pmatrix} -2a_{11} & -2a_{12} \\ -a_{21} & -a_{22} \end{pmatrix} \Rightarrow a_{12} = a_{21} = 0$$

$$A = \begin{pmatrix} a_{11} & 0 \\ 0 & a_{22} \end{pmatrix} \Rightarrow e^A = \begin{pmatrix} e^{a_{11}} & 0 \\ 0 & e^{a_{22}} \end{pmatrix} = \begin{pmatrix} -2 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\Rightarrow a_{11}, a_{22} \notin \mathbb{R}.$$

Ex 4.

$$R(X, Y, Z, W) = g(R(X, Y)Z, W).$$

$$(i). R(X, Y, Z, W) + R(Y, Z, X, W) + R(Z, X, Y, W) = 0.$$

Let S be the operator :

$$ST(X, Y, Z) := T(X, Y, Z) + T(Y, Z, X) + T(Z, X, Y).$$

$$\begin{aligned} SR(X, Y)Z &= S(\nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X, Y]} Z) \\ &= S(\nabla_X \nabla_Y Z) - S(\nabla_Y \nabla_X Z) - S(\nabla_{[X, Y]} Z) \\ &= S(\nabla_Z \nabla_X Y) - S(\nabla_Z \nabla_Y X) - S(\nabla_{[X, Y]} Z) \\ &= S(\nabla_Z [X, Y]) - S[\nabla_{[X, Y]} Z] \\ &= S([Z, [X, Y]]) = 0. \end{aligned}$$

$$(ii). R(X, Y)Z = -R(Y, X)Z$$

$$\Rightarrow R(X, Y, Z, W) = -R(Y, X, Z, W).$$

For the second part: Note that

$$R(X, Y)f = X(Yf) - Y(Xf) - [X, Y]f = 0.$$

$$\begin{aligned} \Rightarrow 0 &= R(X, Y)(\frac{1}{2}g(Z, Z)) \\ &= \frac{1}{2} X(Y<Z, Z>) - \frac{1}{2} Y(X<Z, Z>) - \frac{1}{2} [X, Y]<Z, Z> \\ &= D_X <\nabla_Y Z, Z> - D_Y <\nabla_X Z, Z> - <\nabla_{[X, Y]} Z, Z> \\ &= <\nabla_X \nabla_Y Z, Z> - <\nabla_Y \nabla_X Z, Z> - <\nabla_{[X, Y]} Z, Z> \\ &\quad + <\nabla_Y Z, \nabla_X Z> - <\nabla_X Z, \nabla_Y Z> \\ &= <R(X, Y)Z, Z> = R(X, Y, Z, Z). \end{aligned}$$

$$\text{So } R(X, Y, Z+W, Z+W) = 0$$

$$\Rightarrow R(X, Y, Z, W) = -R(X, Y, W, Z).$$

(iii).

$$R(X, Y, Z, W) = -R(Y, Z, X, W) - R(Z, X, Y, W).$$

$$= R(Y, Z, W, X) + R(Z, X, W, Y)$$

$$= -R(Z, W, Y, X) - R(W, Y, Z, X) - R(X, W, Z, Y) - R(W, Z, X, Y).$$

$$\begin{aligned}
 &= 2R(W, Z, Y, X) + R(W, Y, X, Z) + R(X, W, Y, Z). \\
 &= 2R(W, Z, Y, X) - R(Y, X, W, Z). \\
 \Rightarrow 2R(X, Y, Z, W) &= 2R(Z, W, X, Y). \\
 R(X, Y, Z, W) &= R(Z, W, X, Y).
 \end{aligned}$$

Ex 5.

(i). If $\gamma(t)$ is a path from $e = I \in G$. $\gamma(0) = I$.

$$G \ni \gamma(t) = \begin{pmatrix} x(t) & y(t) \\ 0 & 1 \end{pmatrix}$$

$$\frac{d}{dt} \gamma(t) = \begin{pmatrix} x'(t) & y'(t) \\ 0 & 0 \end{pmatrix} \in g_G.$$

$$\text{If } \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \in g_G \quad \gamma(t) = \begin{pmatrix} e^{at} & bt \\ 0 & 1 \end{pmatrix} \subset G \text{ for } t \in (-\varepsilon, \varepsilon).$$

$$\left. \frac{d}{dt} \right|_{t=0} \gamma(t) = \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \in T_e G.$$

$$\Rightarrow g_G = T_e G = \left\{ \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \mid a, b \in \mathbb{R} \right\}$$

(ii). For $p = \begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix} \in G$. the corresponding left-invariant vector fields are

$$\begin{aligned}
 X_1(p) &= \begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix} & X_2(p) &= \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix} \\
 &= (x, 0) & &= (0, x). \\
 &\equiv x \frac{\partial}{\partial x} & &\equiv x \frac{\partial}{\partial y}
 \end{aligned}$$

$g_{ij} = x^{-2} \delta_{ij}$. is the left invariant metric g on G :

$$g = x^{-2} (dx \otimes dx + dy \otimes dy)$$

(iii). Note that $[X_1, X_2] = X_2$ $[X_2, X_1] = -X_2$:

$$\langle \nabla_{X_1} X_1, X_1 \rangle = \frac{1}{2} X_1 \langle X_1, X_1 \rangle = 0.$$

$$\begin{aligned}\langle \nabla_{X_1} X_1, X_2 \rangle &= \frac{1}{2} \{ X_1 \langle X_1, X_2 \rangle + X_1 \langle X_2, X_1 \rangle - X_2 \langle X_1, X_1 \rangle \\ &\quad + \langle X_2, [X_1, X_1] \rangle \}.\end{aligned}$$

$$= 0.$$

$$\Rightarrow \nabla_{X_1} X_1 = 0.$$

$$\langle \nabla_{X_2} X_2, X_1 \rangle = X_1 \langle X_2, X_1 \rangle - \langle X_2, \nabla_{X_1} X_1 \rangle = 0$$

$$\langle \nabla_{X_2} X_2, X_2 \rangle = \frac{1}{2} X_1 \langle X_2, X_2 \rangle = 0.$$

$$\Rightarrow \nabla_{X_2} X_2 = 0.$$

$$\langle \nabla_{X_2} X_1, X_1 \rangle = \frac{1}{2} X_2 \langle X_1, X_1 \rangle = 0.$$

$$\begin{aligned}\langle \nabla_{X_2} X_1, X_2 \rangle &= \frac{1}{2} \{ - \langle X_2, [X_1, X_2] \rangle + \langle X_1, [\nabla_{X_2} X_2] \rangle + \langle X_2, [X_2, X_1] \rangle \} \\ &= \frac{1}{2} \cdot (-1 - 1) = -1\end{aligned}$$

$$\Rightarrow \nabla_{X_2} X_1 = -X_2.$$

$$\langle \nabla_{X_2} X_2, X_1 \rangle = X_2 \langle X_2, X_1 \rangle - \langle X_2, \nabla_{X_2} X_1 \rangle = 1$$

$$\langle \nabla_{X_2} X_2, X_2 \rangle = 0.$$

$$\Rightarrow \nabla_{X_2} X_2 = X_1.$$

We use the Koszul formula. (Proven in HW2 EX9).

(iv). $K(p)(X_1, X_2) = \langle R(X_1 \cdot X_2) X_2, X_1 \rangle$

$$= \langle \nabla_{X_1} \nabla_{X_2} X_2 - \nabla_{X_2} \nabla_{X_1} X_2 - \nabla_{[X_1, X_2]} X_2, X_1 \rangle$$

$$= -1$$

$$\Rightarrow K(p) = -1.$$