1. Energy control 2

$\leftarrow$ Energy control 1 is the removal of singularities at the end of SFT-6. The following prop can be viewed as an application of removal of singularities.

Prop (Lemma 9.11 in [Wen])

$(M, (\omega,\lambda))$ stable Ham str. $J \in \mathcal{J}((\omega,\lambda))$. $u : (\dot{\Sigma}, j) \to (\mathbb{R}\times M, J)$, $\dot{\Sigma} = \mathbb{C} = S^2\setminus\{\infty\}$

J-hol and satisfies $E_\Sigma(u) < \infty$ and $\int_{\dot{\Sigma}} u^*(\pi_M^*\omega) = 0$, where $\pi_M : \mathbb{R}\times M \to M$ is the projection, then u is constant.

Pf. $T(\mathbb{R}\times M) = \langle \partial_r, \underset{R_{(\omega,\lambda)}}{\overset{\shortparallel}{R}}, \underset{\xi}{\overset{\shortparallel}{\ker(\lambda)}} \rangle$

- For $(s,t) \in \Sigma = \mathbb{C}$, we have

$$\begin{aligned}
u^*(\pi_M^*\omega)(\partial_s,\partial_t) &= \pi_M^*\omega(du(\partial_s), du(\partial_t)) \\
&= \pi_M^*\omega(du(\partial_s), du(j\partial_s)) \\
&= \pi_M^*\omega(du(\partial_s), J\,du(\partial_s)) \\
&= \omega(\underbrace{\quad}_{R\text{-part}} + \pi_\xi du(\partial_s), \underbrace{\quad}_{R\text{-part}} + \pi_\xi J du(\partial_s)) \\
&\overset{\omega \text{ vanishes in } R\text{-direction}}{=} \omega(\pi_\xi du(\partial_s), \pi_\xi J du(\partial_s)) \\
&= \omega|_\xi(\pi_\xi du(\partial_s), J_\xi \pi_\xi du(\partial_s))
\end{aligned}$$

By the defining condition of J, when restricted at ξ, J_ξ is $\omega|_\xi$ compatible, so $u^*(\pi_M^*\omega)(\partial_s,\partial_t) \geq 0$ and

$$u^*(\pi_M^*\omega)(\partial_s,\partial_t) = 0 \text{ iff } \operatorname{im}(du) \subset \operatorname{span}\langle \partial_r, R\rangle.$$

Observe (Exe) that there $\exists$ holomorphic map $\Phi : \mathbb{C} \to \mathbb{C}$ s.t.

not nec injective!

$$\begin{tikzcd}
(\dot{\Sigma}, j) = (\mathbb{C}, j) \arrow[rr, "u"] \arrow[dr, "\Phi"'] & & (\mathbb{R}\times M, J) \\
& (\mathbb{C}, \sqrt{-1}) \arrow[ur, "u_\gamma"'] &
\end{tikzcd}$$

"reparametrization"

$\mathbb{R}$-direction — no part in ξ — not nec a closed orbit

where $u_\gamma(s,t) = (s, \gamma(t))$ where $\gamma(t)$ is a flowline of R, that is, $u = u_\gamma \circ \Phi$. (Note that here we need to use $\pi_1(\mathbb{C}) = 0$.)

- For $\varphi \in \mathcal{T}_\varepsilon$, by the def of $E_\varepsilon(u)$, let's consider

$$(u^* \omega_\varphi) = u^*(\omega + d(\varphi(r)\lambda)) = \Phi^*\big(u_\gamma^*(\omega + d(\varphi(r)\lambda))\big)$$

Note that
$$\big(u_\gamma^*(\omega + d(\varphi(r)\lambda))\big)(\partial_s, \partial_t) = \big(\omega + \varphi(r)\, d\lambda + \varphi'(r)\, dr\wedge\lambda\big)(\partial_s, R)$$

from $(\mathbb{C}, \sqrt{-1})$

$$= \big(\varphi'(r)\, dr\wedge\lambda\big)(\partial_s, R).$$

$$\Rightarrow \quad u_\gamma^*\big(\omega + d(\varphi(r)\lambda)\big)(s,t) = \varphi'(s)\, ds\wedge dt$$

$$\Rightarrow \quad \int_{\dot{\Sigma}} u^*\omega_\varphi = \int_{\mathbb{C}} u^*\omega_\varphi = \int_{\mathbb{C}} \Phi^*(\varphi'(s)\, ds\wedge dt)$$

Since $\varphi'(s) > 0$, $\varphi'(s)\, ds\wedge dt$ is an area form on $(\mathbb{C}, \sqrt{-1})$.

$\int_{\mathbb{C}} \varphi'(s)\, ds\wedge dt = +\infty$

<u>Claim</u>. One can choose $\varphi = \varphi(s) \in \mathcal{T}_\varepsilon$ s.t. the area form $\varphi'(s)\, ds\wedge dt$ extends from $\mathbb{C}$ to $\mathbb{C}\cup\{\infty\}$ $(= S^2)$ (i.e. at ∞, this form won't degenerate too).

To avoid writing ∞ as a pt, let's change ∞ to 0 by

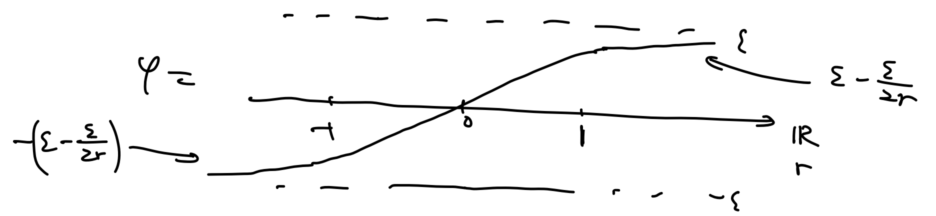
$$\mathbb{C}^* \longrightarrow \mathbb{C}^* \qquad z \to \frac{1}{z} \qquad (\infty \to 0).$$

Then $\varphi'(s)\,ds\wedge dt \xrightarrow[\bar z = s-\sqrt{-1}\,t]{z=s+\sqrt{-1}\,t} \varphi'\left(\frac{s}{|z|^2}\right)\cdot\frac{-1}{2\sqrt{-1}}\cdot\frac{1}{|z|^4}\,dz\wedge d\bar z$

$s+\sqrt{-1}\,t \to \frac{1}{s+\sqrt{-1}\,t} = \frac{s}{|z|^2}+\cdots$

$$= \underbrace{\varphi'\left(\tfrac{s}{|z|^2}\right)\Big/|z|^4\, ds\wedge dt}_{(*)} \quad \text{for } z = s+\sqrt{-1}\,t \in \mathbb{C}\setminus\{0\}.$$

We need to choose $\varphi \in T_\varepsilon$ s.t. $(*)$ won't degenerate to 0 at 0.



Then $\exists\, c>0$ (from cpt interval $[-1,1]$) s.t. $\forall\, r\in\mathbb{R}$

$$\varphi'(r) \geqslant \min\left\{c, \frac{\varepsilon}{2r^2}\right\}$$

$$\Rightarrow \varphi'\left(\frac{s}{|z|^2}\right)\Big/|z|^4 \underset{r=\frac{s}{|z|^2}}{\geqslant} \min\left\{\frac{c}{|z|^4}, \frac{\varepsilon|z|^4}{2s^2}\Big/|z|^4\right\} = \min\left\{\frac{c}{|z|^4}, \frac{\varepsilon}{2s^2}\right\}$$

and when $|z|\to 0$, both $\frac{c}{|z|^4}$ and $\frac{\varepsilon}{2s^2}$ blow up, so not degenerating to 0.

Then we have a setting: $\Phi : (\dot\Sigma, j) \xrightarrow{hol} (\underset{\mathbb{C}\cup\{\infty\}}{\underset{\shortparallel}{S^2}}, j_{std})$ and we have a well-defined area form $\varphi'(s)\,ds\wedge dt$ on (S^2, j_{std}) extended from $\varphi'(s)\,ds\wedge dt$.
(symplectic)

$\Rightarrow$ $\forall\, C>0$, $\exists$ a symplectic str Ω on (S^2, j_{std}) s.t.

$$\Omega \leq \varphi'(s)\, ds\wedge dt \quad \text{and} \quad \int_{S^2} \Omega > C.$$

Then

- If $\int_{\dot\Sigma} \Phi^*\Omega = \infty$, then

$$\int_{\dot\Sigma} u^*\omega_\varphi = \int_{\dot\Sigma} \Phi^*(\varphi'(s)\, ds\wedge dt) \geq \int_{\dot\Sigma} \Phi^*\Omega = \infty. \rightarrow\leftarrow$$

to $E_\varphi(u)<\infty$.

- If $\int_{\dot\Sigma} \Phi^*\Omega < \infty$, then by removal of singularities, Φ extends to a (j_{std}-) hol map $\Phi : (S^2, j) \to (S^2, j_{std})$, and

$$\int_{\dot\Sigma} u^*\omega_\varphi = \int_{\dot\Sigma} \Phi^*(\varphi'(s)\, ds\wedge dt) \geq \int_{\dot\Sigma} \Phi^*\Omega = \int_{\Sigma} \Phi^*\Omega$$
$$= \deg(\Phi) \int_{S^2} \Omega$$

Φ is hol $\Rightarrow$ $\deg(\Phi) \geq 0$ and then $\deg(\Phi)\int_{S^2}\Omega \geq \deg(\Phi)\cdot C \to \infty$ as $C\to\infty$. Therefore, $\int_{\dot\Sigma} u^*\omega_\varphi = \infty$ $\rightarrow\leftarrow$. $\square$

<u>Rmk</u> One can also consider $\dot\Sigma = S^2\setminus\{p,q\} \simeq \mathbb{R}\times S^1$. The observation above changes to $u_\gamma : \mathbb{R}\times S^1 \longrightarrow \mathbb{R}\times M$ by

$$u_\gamma(s,t) = (Ts, \gamma(Tt))$$

where T is the minimal period of the flow of R.

and $\Phi : (\mathbb{R}\times S^1, j) \to (\mathbb{R}\times S^1, j_{std})$

By a similar argument, one can show either u is constant or

$$u(s,t) = (kTs, \gamma(kTt))$$

that is, a k-fold covering of the "trivial cylinder".

2. Convergence to critical pts

Let's discuss in Morse case first. $(M^{\times \text{closed}}, F)$ also such a metric g fixed. F is a Morse fcn.

Prop. Suppose that $u(s): \mathbb{R} \to M$ is a gradient flow line

$$\frac{du(s)}{dt} = \nabla F(u(s)).$$

then there exist $p^{\pm} \in \mathrm{Crit}(F)$ s.t. $\lim_{s \to \pm\infty} u(s) = p^{\pm}$.

Pf. By def of Morse fcn, each critical pt is "non-deg" (i.e. near crit pt p, $F(x_1, \dots, x_n) = \sum_{i=1}^{n} \pm x_i^2$), so since M is closed, $\exists$ finitely many critical pts

$$\mathrm{Crit}(F) = \{p_1, \dots, p_N\}.$$

Choose $\{U_i\}_{i=1}^{N}$, NBHs of p_i, disjoint.

- $\exists\, \varepsilon_0 > 0$ s.t. $\|\nabla F(x)\| \geq \varepsilon_0$ for any $x \in M \setminus \bigcup_{i=1}^{N} U_i$.

Suppose not, $\exists\, x_n \in \underbrace{M \setminus \bigcup_{i=1}^{N} U_i}_{\text{closed} \Rightarrow \text{cpt}}$ s.t. $\|\nabla F(x_n)\| < \frac{1}{n}$ $\forall n$.

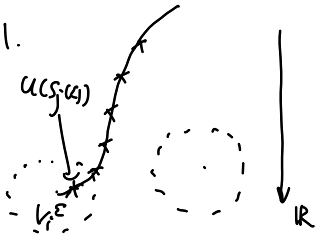
Then $x_n \to x_*$ and $\|\nabla F(x_*)\| = 0$. So, $x \in \mathrm{Crit}(F) \to \Leftarrow$

For every $\varepsilon \in (0, \varepsilon_0)$, denote the following NBH of $p_i \in \mathrm{Crit}(F)$,

$$V_i^{\varepsilon} := \{x \in U_i \mid \|\nabla F(x)\| < \varepsilon\} \subset U_i$$

- $\forall \varepsilon > 0$, $\exists$ a seq $s_j(\varepsilon) \in \mathbb{R}$ and $s_j(\varepsilon) \to +\infty$ as $j \to \infty$ s.t.

$$u(s_j(s)) \in \bigcup_{i=1}^{N} V_i^{\varepsilon} \quad \text{when } j \gg 1.$$



Suppose not, $\exists\, s_*$ s.t. $\forall s \geq s_*$, we have

$$u(s) \notin \bigcup_{i=1}^{N} V_i^{\varepsilon} \underset{\substack{\text{then} \\ \text{by def of } V_i^{\varepsilon}}}{\Longrightarrow} \|\nabla F(u(s))\| \geq \varepsilon \quad (\forall s \geq s_*).$$

Recall $E(u) := \int_{-\infty}^{\infty} \left\|\frac{\partial u}{\partial s}\right\|_g^2 ds \quad \left(= \lim_{s\to\infty} F(u(s)) - \lim_{s\to-\infty} F(u(s)) \right)$

This shows that in a closed/cpt mfd, the energy of a gradient flow is also finite!

Then

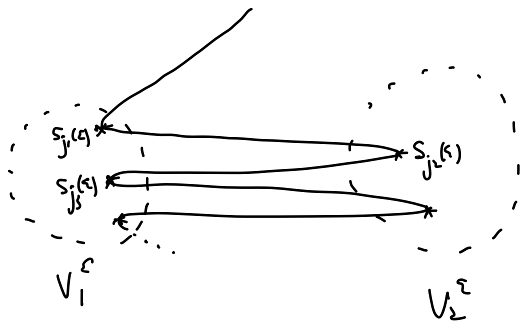
$$\max_M F - \min_M F \geq E(u) = \int_{-\infty}^{\infty} \left\|\frac{\partial u}{\partial s}\right\|^2 ds$$

$$= \int_{-\infty}^{\infty} \|\nabla F(u(s))\|^2 ds$$

$$\geq \int_{\alpha}^{\infty} \|\nabla F(u(s))\|^2 ds \geq \int_{\alpha}^{\infty} \varepsilon^2 ds = \infty.$$

$\rightarrow\leftarrow$.

Note that this won't directly finish the proof since we could have the following "jumping" picture



Since $E(u) < \infty$, choose $\sigma_\varepsilon \in \mathbb{R}$ s.t.

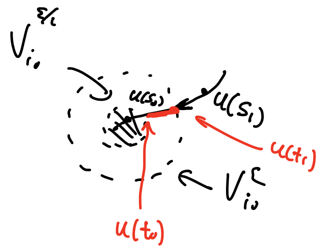
$$\int_{\sigma_\varepsilon}^{\infty} \left\|\frac{\partial u}{\partial s}\right\|^2 ds \leq \frac{\varepsilon}{4} \cdot \min_{i \in \{1,\cdots,N\}} d\Big(\underbrace{\overline{V_i^{\varepsilon/2}},\ M \setminus V_i^{\varepsilon}}_{>0}\Big)$$

- $\exists\, i \in \{1, \cdots, N\}$ s.t. $u(s) \in V_i^{\varepsilon}$ for $s \geq \sigma_{\varepsilon}$.

Since there are only finitely many V_i^{ε}'s that $u(s)$ can "jump," so

$\exists\, S_0 \geq \sigma_{\varepsilon}$ and $i \in \{1, \cdots, N\}$ s.t. $u(S_0) \in V_i^{\varepsilon/2}$ $\leftarrow$ apply the argument above to $V_i^{\varepsilon/2}$ and take $s_0 = s_j(\varepsilon/2)$ for sufficiently large $j \gg 1$.

Now, suppose $\forall\, i \in \{1, \cdots, N\}$, in particular i_0, there exist $S_1 \geq \sigma_{\varepsilon}$ s.t. $u(S_1) \notin V_{i_0}^{\varepsilon}$



$\Rightarrow$ (if nec, one can take S_0 sufficiently large s.t. $S_0 > S_1$)

$\exists$ time interval $[t_1, t_0]$ s.t. $u(t_0) \in \partial V_{i_0}^{\varepsilon/2}$, $u(t_1) \in \partial V_{i_0}^{\varepsilon}$, and $u(t) \in V_{i_0}^{\varepsilon} \setminus V_{i_0}^{\varepsilon/2}$.

By def of $V_{i_0}^{\varepsilon/2}$, we know for $t \in [t_1, t_0]$,

$$\left\| \frac{\partial u}{\partial s}(t) \right\| = \| \nabla F(u(t)) \| \geq \varepsilon/2 \quad \leftarrow \quad \left\| \frac{\partial u}{\partial s}(t) \right\|^{-1} \leq \frac{2}{\varepsilon}.$$

$$\Rightarrow \min_{i \in \{1, \cdots, N\}} d(\overline{V_i^{\varepsilon/2}}, M \setminus V_i^{\varepsilon}) \leq d(u(t_1), u(t_0)) \leq \int_{t_1}^{t_0} \left\| \frac{\partial u}{\partial s} \right\| ds$$

$$= \int_{t_1}^{t_0} \left\| \frac{\partial u}{\partial s} \right\|^2 \cdot \left\| \frac{\partial u}{\partial s} \right\|^{-1} ds$$

$$\leq \frac{2}{\varepsilon} \int_{\sigma_{\varepsilon}}^{\infty} \left\| \frac{\partial u}{\partial s} \right\|^2 ds$$

$$\leq \frac{2}{\varepsilon} \cdot \frac{\varepsilon}{4} \cdot \min_{i \in \{1, \cdots, N\}} d(\overline{V_i^{\varepsilon/2}}, M \setminus V_i^{\varepsilon})$$

$$= \frac{1}{2} \min_{i \in \{1, \cdots, N\}} d(\overline{V_i^{\varepsilon/2}}, M \setminus V_i^{\varepsilon}). \quad \rightarrow\leftarrow$$

Finally, let $\varepsilon \to 0$, we know $u(s) \to p_{i_0} \in \mathrm{Crit}(F)$. $\square$