4. Computation of integration

The definition of $\int_M \theta$ is good/well-defined but hard to compute.

e.g. (It may be computable via definition in a rather low dimension)

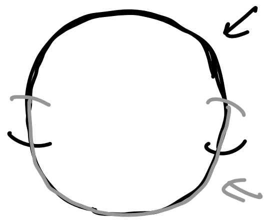
- $M = \{x_1, x_2, x_3\}$ three pts.

Consider open cover $\mathcal{U} = \{\{x_1\},\{x_2\}\{x_3\}\}$ and P.O.U. $\{\rho_i\}_{i=1}^{3}$ s.t.

$\rho_i(x_j) = \begin{cases} 1 & i=j \\ 0 & i\neq j \end{cases}$. Then for a $\underset{\Delta}{\overset{\dim M}{0}}$-form $\theta = F \in \Omega^0(M)$

$$\int_M \theta = \sum_{i=1}^{3} \int_M \rho_i \theta = \sum_{i=1}^{3} \int_M \rho_i F = F(x_1) + F(x_2) + F(x_3)$$

- $M = S^1$



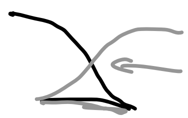
$\mathcal{U}_1$

$\mathcal{U}_2$

$\mathcal{U} = \{\mathcal{U}_1, \mathcal{U}_2\}$

P.O.U $= \{\rho_1, \rho_2\}$

ρ_1



ρ_2

$\frac{1}{2}$

intersection part

In order to write down a 1-form on S^1, let us parametrize by
(or embed into $\mathbb{R}^2$)

$$S^1 = \{ (x,y) \in \mathbb{R}^2 \mid x^2+y^2=1 \}$$

Then consider $\sigma := x\,dy - y\,dx$ (for $(x,y) \in S^1$).

Then $\int_{S^1} \sigma = \int_{S^1} \rho_1 \sigma + \int_{S^1} \rho_2 \sigma$ ← this σ is <u>globally</u> defined!

$$= \left(\int_{\cap} \rho_1 x\,dy - \rho_1 y\,dx \right) + \left(\int_{\cup} \rho_2 x\,dy - \rho_2 y\,dx \right)$$

$$= \left(\int_{\cap} x\,dy - y\,dx \right) + \left(\int_{\cup} x\,dy - y\,dx \right) +$$

b/c by our choice of p.o.u $\rho_1+\rho_2 \equiv 1$ over ( )

$$\int_{(\ )} (\rho_1+\rho_2) x\,dy - (\rho_1+\rho_2) y\,dx$$

$$= \int_{\bigcirc} x\,dy - \int_{\bigcirc} y\,dx$$

$x = \cos\theta$
$y = \sin\theta$

$$= \int_0^{2\pi} \cos^2\theta\, d\theta + \int_0^{2\pi} \sin^2\theta\, d\theta = 2\pi.$$

Something needs to think about (intrinsic vs. extrinsic)

Rmk The computation on S^1 can be written as follows:

Consider a parametrization $\underset{[0,2\pi)}{\overset{\cap}{\theta}} \xrightarrow{F} S^1$ by $F(\theta)=(\cos\theta, \sin\theta)$.

$$\int_{S^1} \overset{\overset{x\,dy-y\,dx}{\shortparallel}}{\sigma} = \int_{[0,2\pi)} F^*\sigma = \int_{[0,2\pi)} d\theta = 2\pi.$$

θ is viewed as the coordinate function

In fact, one can relax $[0,2\pi)$ to $(0,2\pi)$, no effect on integration result.

Prop. Given orientable, cpt mfd M^n and n-form θ, if

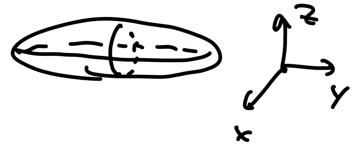
$$M \setminus Z = \bigcup_{i=1}^{k} F_i(U_i)$$

a union of some submfds of dim $< n$

where $\underset{\text{open}}{U_i} \subset \mathbb{R}^n$ and $F_i : U_i \to M$ orientation preserving, and $\theta \subset\subset \overline{M \setminus Z}$. Then

$$\int_M \theta = \sum_{i=1}^{k} \int_{U_i} F_i^* \theta.$$

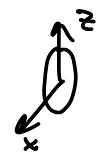
Exe. $M = \{(x,y,z) \in \mathbb{R}^3 \mid \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1\}$ ← ellipsoid.



Consider $F : (0, 2\pi) \times (0, \pi) \longrightarrow M$

$$\theta \qquad \varphi \longrightarrow (a\cos\theta\sin\varphi,\ b\sin\theta\sin\varphi,\ c\cos\varphi).$$

Compute $\displaystyle\int_M \frac{dx\wedge dy}{z} - \frac{dx\wedge dz}{y} + \frac{dy\wedge dz}{x}$ ← 2-form in $\Omega^2(\mathbb{R}^3\setminus\{0\})$

(Note that $Z = F(\underset{\theta=0}{\{0\}} \times (0,\pi)) \cup F((0,2\pi)\times \overset{\varphi=0}{\{0\}}) \subset$ 

).

5. Stokes' Thm

Thm Let M^n be an oriented cpt mfd with b/d, then for any $(n-1)$-form θ, we have
$$\int_{\underset{\text{also orientable}}{\partial M}} i^*\theta = \int_M d\theta,$$
where $i : \partial M \to M$ inclusion.

Rmk For non-cpt mfd, the Thm holds for any cpt supp $(n-1)$-form θ.

Pf By P.O.U. assume θ is cpt supp in a local chart U

Write $\theta = \sum_{i=1}^{n} f_i \, dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_n$. Then two cases.

If $U \cap \partial M = \emptyset$.

$$\int_M d\theta = \int_U d\theta$$

This should be dx_1 dx_2 … dx_n.

$$= \int_V \sum_{i=1}^{n} \pm \frac{\partial f_i}{\partial x_i} dx_1 \wedge \cdots \wedge dx_n$$

$$= \sum_{i=1}^{n} \pm \int_{\mathbb{R}^n} \frac{\partial f_i}{\partial x_i} dx_1 \wedge \cdots \wedge dx_n$$

$$= \sum_{i=1}^{n} \pm \int_{\mathbb{R}^{n-1}} \left(\int_{\mathbb{R}} \frac{\partial f_i}{\partial x_i} dx_i \right) dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_n$$

$$= \sum_{i=1}^{n} \pm \int_{\mathbb{R}^{n-1}} f_i \Big|_{x_i = -\infty}^{x_i = +\infty} dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_n$$

$$= 0 = \int_{\partial M} i^* \theta$$ ← b/c $\operatorname{supp}\theta \cap \partial M = \emptyset$.

If $U \cap \partial M \neq \emptyset$

$$\int_M d\theta = \int_U d\theta$$

This should be dx_1 dx_2 … dx_n.

$$= \int_V \sum_{i=1}^{n} \pm \frac{\partial f_i}{\partial x_i} dx_1 \wedge \cdots \wedge dx_n$$

$$= \sum_{i=1}^{n} \pm \int_{\mathbb{H}^n_+} \frac{\partial f_i}{\partial x_i} dx_1 \wedge \cdots \wedge dx_n$$

two parts

$$= \sum_{i=1}^{n-1} \pm \int_{\mathbb{R}^{n-1}} f_i \Big|_{x_i = -\infty}^{x_i = +\infty} dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_n$$

$$+ (-1)^{n-1} \int_{\mathbb{R}^{n-1}} f_n \Big|_{x_n = 0}^{x_n = +\infty} dx_1 \wedge \cdots \wedge dx_{n-1}$$

$$= (-1)^n \int_{\mathbb{R}^{n-1}} f_n(x_1, \ldots, x_{n-1}, 0) \, dx_1 \wedge \cdots \wedge dx_{n-1}$$

$\partial \mathbb{H}^n_+$

$$\overset{(*)}{=} \int_{\partial M} i^* \theta$$

Explanation of $(*)$: if $dx_1 \wedge \cdots \wedge dx_n$ is a top form on the interior of $\mathbb{H}^n_+$, then the "induced" top form on $\partial \mathbb{H}^n_+$ is

$(-1)^n dx_1 \wedge \cdots \wedge dx_{n-1}$.

<u>Cor</u> $M = [a,b]$ and $\theta = f \in \Omega^0(M)$, then

$$\int_{[a,b]} df = \int_M d\theta = \int_{\partial M} i^*\theta = \int_{\{a,b\}} i^* f$$

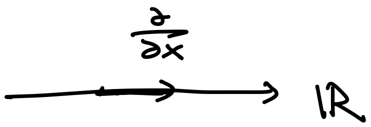
$$\text{LHS} = \int_{[a,b]} f'(x)\,dx = \int_a^b f'(x)\,dx$$

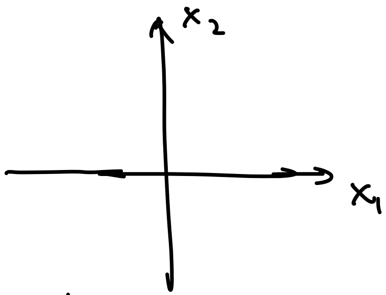
$$\boxed{\text{RHS}} = f(a) + f(b) \ ?$$

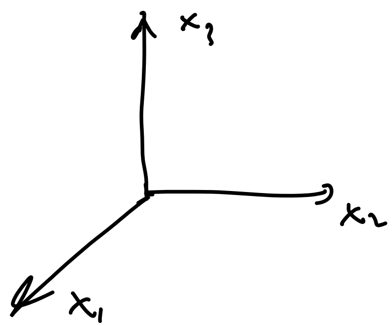
← something needs to be clarified – <u>orientation.</u>

<u>Def</u> For an orientable mfd M, <u>an orientation</u> is a choice of an <u>ordering</u> of the basis elements $\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}$ for an "orientable" local chart.

e.g.







$\frac{\partial}{\partial x}$ (positive orientation)

convention

so $-\frac{\partial}{\partial x}$ is a neg orientation

$\left\{\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}\right\}$ $+$

$\left\{\frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_3}\right\}$ $\overset{\text{def}}{=}$ $-$

$dx_2 \wedge dx_1 \wedge dx_3 = -dx_1 \wedge dx_2 \wedge dx_3$

Prop. The local orientations of M are not unique, but for an orientable mfd, the $\underbrace{+/-}_{\text{sign}}$ of local orientations can be chosen consistently.

- In the case of mfd with b/d (recall M is orientable $\Rightarrow \partial M$ is orientable).

Suppose $\left\{\frac{\partial}{\partial x_{i_1}}, \dots, \frac{\partial}{\partial x_{i_n}}\right\}$ is a pos. orientation of M, then define the sign of induced orientation $\left\{\frac{\partial}{\partial x_{i_1}}, \dots, \widehat{\frac{\partial}{\partial x_n}}, \dots, \frac{\partial}{\partial x_{i_n}}\right\}$ by the following number:

$$\operatorname{sign}\left\{\begin{matrix}\text{outward}\\ \text{normal vector}\\ \text{on } \partial M, n\end{matrix}, \frac{\partial}{\partial x_{i_1}}, \dots, \widehat{\frac{\partial}{\partial x_n}}, \dots, \frac{\partial}{\partial x_{i_n}}\right\}$$

有些同学提问说为什么微积分中并没有体现出orientation?

原因是在求（多重）积分的时候我们可以自由交换积分变量的顺序而不改变最后得到的积分的值。直接的原因是多重积分的“变量替换”公式中总是被强行的对Jacobian的行列式取了绝对值。

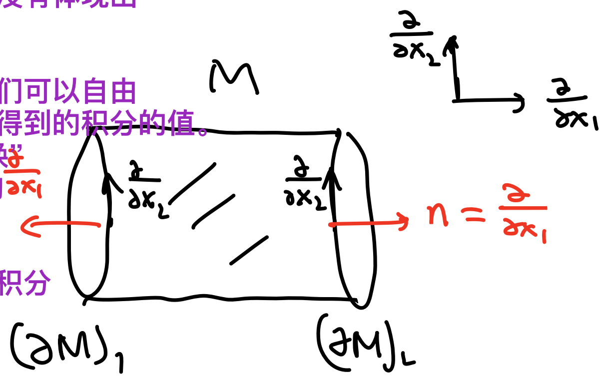
一个更具体的例子是如果在求二重积分的时候，变量替换为

(x,y) \to (y,x)

虽然Jacobian的行列式是-1， 但是取了绝对值后得到的和原来的积分值是完全一样的，而这个变量替换正好就是上面提到的交换变量的顺序。

在谈论流形积分的时候，我们被积分的对象是forms，不能随意交换变量的顺序（会产生符号差），因此在比较流形积分和微积分中的多重积分的时候，符号差就来源于对于（局部坐标）x_1, ..., x_n的一个顺序的选取（而这个正是orientation的原始定义）。

e.g.

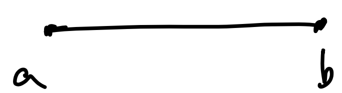


- On $(\partial M)_1$, we have induced orientation $\{\frac{\partial}{\partial x_2}\}$ with sign

$$\operatorname{sign}\left\{-\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}\right\} = -1$$

- On $(\partial M)_2$ $\{\frac{\partial}{\partial x_2}\} \rightsquigarrow \operatorname{sign}\{\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}\} = 1$

e.g. On a (special) case M



$\{a\} = (\partial M)_1$ $\{b\} = (\partial M)_2$

- On $(\partial M)_1$, we assign the orientation sign at a by -1
- On $(\partial M)_2$, we assign the orientation sign at b by $+1$.

Back to Cor above,

extra sign from orientation

$$\boxed{\text{RHS}} = -f(a) + f(b) = f(b) - f(a)$$

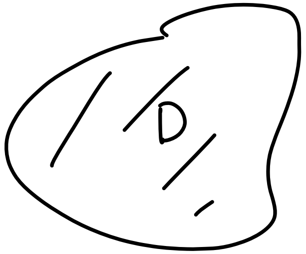
$\Longrightarrow$ Newton-Leibniz $\int_a^b f'(x)dx = f(b) - f(a)$

A more global way to define an orientation is the follwg:

Def For a mfd M^n, an orientation is a non-vanishing $\Omega \in \Omega^n(M)$

Prop: If M is orientable, then M has an orientation.

e.g.

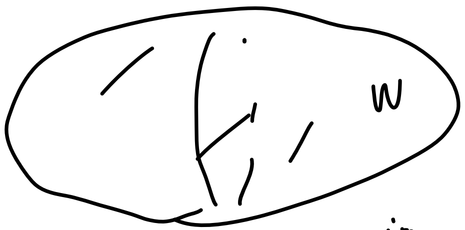


$\subset \mathbb{R}^2(x,y)$ $\quad \theta = P(x,y)\,dx + Q(x,y)\,dy$
$\quad\quad \rotatebox{90}{$\in$}\ \Omega^1(\mathbb{R}^2)$

in calculus $\rightsquigarrow$ $$\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy = \int_{\partial D} P(x,y)\,dx + Q(x,y)\,dy$$

in manifold $\rightsquigarrow$ $$\int_D d\theta = \int_{\partial D} \theta$$

e.g.



$\subset \mathbb{R}^3(x,y,z)$ $\quad \theta = P\,dy\wedge dz + Q\,dz\wedge dx + R\,dx\wedge dy$
$\quad\quad \rotatebox{90}{$\in$}\ \Omega^2(\mathbb{R}^3)$

in calculus $\to$ $$\iiint_W \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}\right) dx\,dy\,dz = \iint_{\partial W} \cdots$$

in mfd $\longrightarrow$ $$\int_W d\theta = \int_{\partial W} \theta.$$

<u>Rmk</u> $\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$ is the <u>divergence</u> of vector field (P, Q, R).

In fact, eg. above is a special case of the following prop.

Given an orientable mfd M, fix an n-form Ω on M + nowhere vanishing. For any vector field X on M, define $\mathrm{div}(X)$ (divergence of X) by

$$\underbrace{\mathcal{L}_X \Omega}_{\text{which is again an } n\text{-form}} = \mathrm{div}(X) \cdot \Omega .$$

Rmk $\mathrm{div}(X)$ depends on Ω. Suppose $\Omega' = f \cdot \Omega$, then

$$\begin{aligned}
\mathcal{L}_X \Omega' &= d\, \iota_X (f\Omega) + \iota_X d(f\Omega) \\
&= d(f(\iota_X \Omega)) + \iota_X (df \wedge \Omega + f \overset{0}{d\Omega}) \\
&= \cancel{df \wedge \iota_X \Omega} + f \cdot d\iota_X \Omega + (\iota_X df) \wedge \Omega - \cancel{df \wedge \iota_X \Omega} \\
&= f \cdot \mathcal{L}_X \Omega + (\iota_X df) \wedge \Omega \\
&= f \cdot \mathrm{div}(X) \cdot \Omega + df(X) \cdot \Omega \\
&= \mathrm{div}(X)\, \Omega' + \underbrace{df(X) \cdot \Omega}_{\text{extra term}}
\end{aligned}$$

This proposition is usually
called **Divergence Theorem**.

<u>Prop.</u> Let M^n be an orientable mfd with b/d. Fixing $\Omega \in \Omega^n(M)$ that is nowhere vanishing, then

$$\int_M \mathrm{div}(X)\,\Omega = \int_{\partial M} i^*(\iota_X \Omega)$$

Pf

$$\int_M \mathrm{div}(X)\,\Omega = \int_M L_X \Omega = \int_M d\,\iota_X \Omega = \int_{\partial M} i^*(\iota_X \Omega) \quad \square$$

<u>Prop (again)</u> Let M^n be an orientable cpt mfd with b/d. Then $\nexists$ smooth map $F: M \to \partial M$ s.t. $F|_{\partial M}: \partial M \to \partial M$ is the identity.

<u>Pf</u> Suppose $\exists$ such F. Pick a nowhere vanishing Ω on $\Omega^{n-1}(\partial M)$ s.t. $\int_{\partial M} \Omega > 0$. Then

$$0 < \int_{\partial M} \Omega = \int_{\partial M} F^* \Omega = \int_M d(F^*\Omega) = \int_M F^*(\cancel{d\Omega}^{\,0}) = 0 \quad \rightarrow\leftarrow.$$

<u>Def</u> ① For a mfd M^n, a <u>volume form</u> is a nowhere vanishing $\Omega \in \Omega^n(M)$.

- M is orientable $\Longrightarrow$ $\exists$ a volume form.

① For an orientable mfd M^n, the volume w.r.t. a volume form $\Omega \in \Omega^n(M)$ is defined by $\int_M \Omega$.

- If furthermore M is cpt, then $\int_M \Omega < +\infty$.

Rmk If $\Omega \in \Omega^n(M)$ is a volume form, then $f \cdot \Omega$ is also a volume form if f is a nowhere vanishing function.

Rmk To some extent, $\operatorname{div}(X)$ detects how the volume w.r.t Ω changes. More precisely,

$$\int_{\substack{D \\ \cap \\ M}} \operatorname{div}(X)\,\Omega = \int_D \mathcal{L}_X \Omega \overset{\text{by def}}{=} \int_D \left.\frac{d}{dt}\right|_{t=0} (\varphi_X^t)^* \Omega$$

$$= \left.\frac{d}{dt}\right|_{t=0} \int_D (\varphi_X^t)^* \Omega = \left.\frac{d}{dt}\right|_{t=0} \int_{(\varphi_X^t)(D)} \Omega .$$