Lecture 5 Integration on mfds

Motivation ...

1. Partition of unity
2. Definition of integration

} see hand-written notes in a different format.

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3. Manifold with boundary

Recall a submfd is defined via the local model

$$\underbrace{\{x_{k+1} = \cdots = x_n\}}_{\text{a linear subspace}} \subseteq \mathbb{R}^n$$

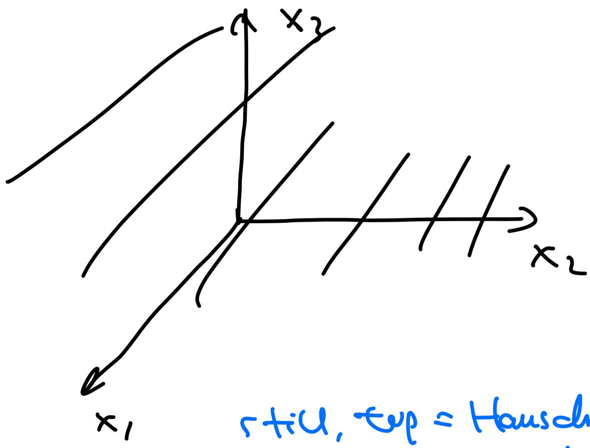
Similarly, mfd with b/d is model by the following "half plane"

$$\mathbb{H}^n_+ = \{(x_1, \cdots, x_n) \in \mathbb{R}^n \mid x_n \geq 0\}$$

$\mathbb{H}^2_+$

2-dim

$\partial\mathbb{H} := \{(x_1, \cdots, x_{n-1}, 0)\}$



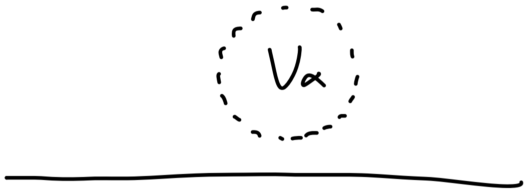
$\mathbb{H}^3_+$

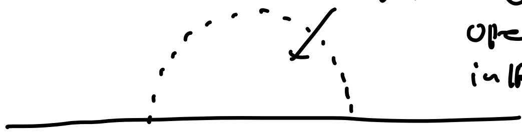
3-dim

still, top = Hausdorff
second countable

<u>Def</u> A mfd with b/d M^n is a mfd that admits a local chart $(U_\alpha, \varphi_\alpha : U_\alpha \longrightarrow V_\alpha)$ s.t. V_α is an open subset of $\mathbb{H}^n_+$

Open subset of $\mathbb{H}$ has two types





$V_\alpha = U \cap \mathbb{H}$

open in $\mathbb{R}^n$

Notation: $\partial M = \{ p \in M \mid \varphi_\alpha(p) \in \partial\mathbb{H} \}$.

Prop ∂M^n is an $(n-1)$-dim'l mfd without boundary.

Moreover, if M is orientable, then ∂M is also orientable.

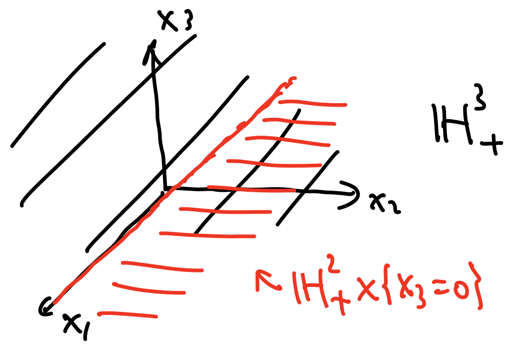
pf. ① A local chart $\{U_\alpha, \varphi_\alpha : U_\alpha \to V_\alpha \subset \mathbb{H}^n_+\}$ of M^n

$\Rightarrow \{U_\alpha \cap \partial M, \varphi_\alpha|_{U_\alpha \cap \partial M} : U_\alpha \cap \partial M \to \underbrace{V_\alpha \cap \partial \mathbb{H}^n_+}_{\text{an open subset of } \mathbb{R}^{n-1}}\}$

is a local chart of ∂M.

(Moreover, since the local model is from $\mathbb{R}^{n-1}$, so ∂M has no boundary.)

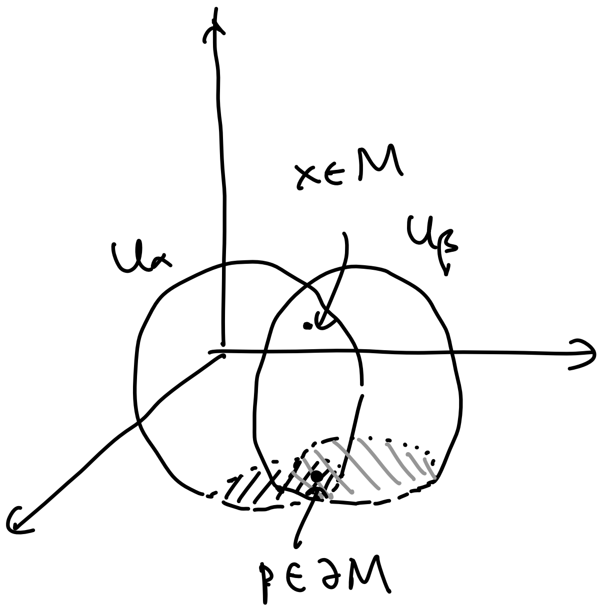
Rmk. What happens if ∂M also has boundary?



$\mathbb{H}^3_+ \rightsquigarrow C^3 := \{(x_1,x_2,x_3) \in \mathbb{R}^3 \mid x_3 > 0\}$
$\cup \underbrace{\{(x_1,x_2,0) \in \mathbb{R}^3 \mid x_2 \geq 0\}}_{\mathbb{H}^2_+ \times \{x_3=0\}}$

By our definition, C^3 is not a mfd (or mfd with b/d).

② About orientation:



$\varphi_{\alpha\beta}(x)$ transition map of M.

orientable : $\det\left(J(\varphi_{\alpha\beta})(x)\right) > 0$ $\forall x \in V_\alpha \subset \mathbb{H}^n_+$

Important :

$\varphi_{\alpha\beta}((x_1, \cdots, x_{n-1}, 0)) \in \partial \mathbb{H}^n_+$

$\Longrightarrow$ if $\varphi_{\alpha\beta} = (\varphi^1_{\alpha\beta}, \cdots, \varphi^n_{\alpha\beta})$,

then $\varphi^n_{\alpha\beta}((x_1, \cdots, x_{n-1}, 0)) = 0$.

Compute the Jacobian $J(\varphi_{\alpha\beta})$ restricted on $\partial \mathbb{H}^n_+$:

$$J(\varphi_{\alpha\beta})(\underbrace{(x_1, \cdots, x_{n-1}, 0)}_{\substack{= x \\ \cap \\ \partial \mathbb{H}^n_+}}) = \begin{pmatrix} \left(\dfrac{\partial \varphi^i_{\alpha\beta}}{\partial x_j}(x)\right)_{(n-1)\times(n-1)} & \begin{matrix} \dfrac{\partial \varphi^1_{\alpha\beta}}{\partial x_n}(x) \\ \vdots \end{matrix} \\ \underbrace{\dfrac{\partial \varphi^n_{\alpha\beta}}{\partial x_1}(x)}_{=0} \cdots \underbrace{\dfrac{\partial \varphi^n_{\alpha\beta}}{\partial x_{n-1}}(x)}_{=0} & \dfrac{\partial \varphi^n_{\alpha\beta}}{\partial x_n}(x) \end{pmatrix}$$

$$\det\left(J(\varphi_{\alpha\beta})(x)\right) = \det\left(J(\varphi_{\alpha\beta}|_{\partial \mathbb{H}^n_+})(x)\right) \cdot \frac{\partial \varphi^n_{\alpha\beta}}{\partial x_n}(x) > 0$$

- what's the sign of $\frac{\partial \varphi^n_{\alpha\beta}}{\partial x_n}(x)$?

By definition $$\frac{\partial \varphi^n_{\alpha\beta}}{\partial x_n}(x) = \lim_{h \to 0^+} \frac{\overbrace{\varphi^n_{\alpha\beta}(x+h)}^{\in \mathbb{H}^n_+} - \varphi^n_{\alpha\beta}(x)}{h} = \lim_{h \to 0} \frac{\varphi^n_{\alpha\beta}(x+h)}{h} > 0$$

b/c transition map $\varphi_{\alpha\beta}$ maps $\mathbb{H}^n_+$ to $\mathbb{H}^n_+$, in particular $\varphi^n_{\alpha\beta}(x+h) \geq 0$.

Therefore, $\det\left(J(\underbrace{\varphi_{\alpha\beta}|_{\partial \mathbb{H}^n_+}}_{\text{transition map on } \partial \mathbb{H}^n_+})(x)\right) > 0$ and ∂M is orientable $\square$

<u>Rmk</u>: If one defines mfd with b/d via local model $\mathbb{H}^n_- = \{x_n \leq 0\}$ the proof above goes through as well.

<u>Exe</u> If M, N are mfd with b/d, then $\partial(M \times N) \overset{\text{as subsets}}{=} (\partial M \times N) \cup (M \times \partial N)$.

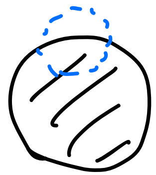
Note that the RHS may <u>not</u> be a smooth mfd (see e.g later).

In particular, when N is a mfd without b/d, then

$\partial(M \times N) = \partial M \times N$ (which is a mfd).

- Example of mfd with b/d

e.g. $\mathbb{H}^n_+$, closed ball $\bar{B}^n(1) = \{ x \in \mathbb{R}^n \mid \|x\| \le 1 \}$



We can view $\bar{B}^n(1) = \{ F(x) \le 1 \}$ (cf. $\underset{\substack{\cap \\ \mathbb{R}^n}}{S^{n-1}} = \{ F(x) = 1 \}$)

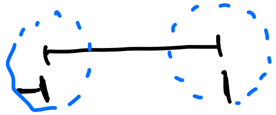
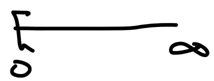
where $F(x) = \|x\| = \sqrt{x_1^2 + \cdots + x_n^2}$.

In general, consider $F: M \to \mathbb{R}$, for any regular value $r \in \mathbb{R}$, the "sublevel set"

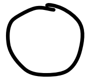
$$M = \{ F \le r \}$$

is a mfd with b/d and its b/d is $\partial M = \{ F = r \}$.

e.g. Boundary ∂M may not be connected!


locally modelled by $\mathbb{H}^1_+ =$


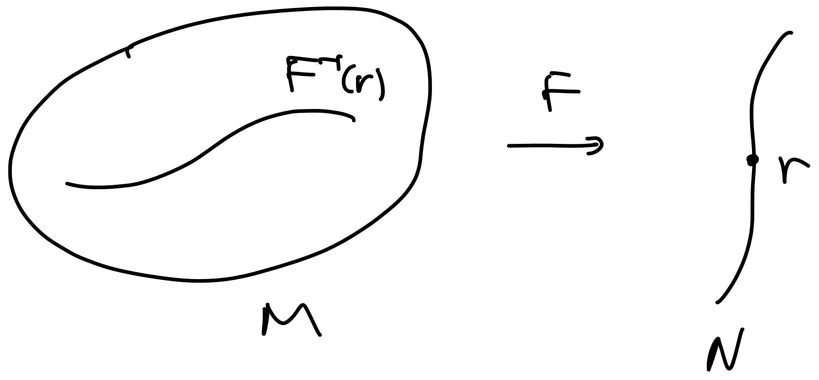
$\partial([-1,1]) = \{-1\} \cup \{1\}$

FACT: Any (smooth) cpt connected 1-dim mfd, up to diffeo, is either  or .

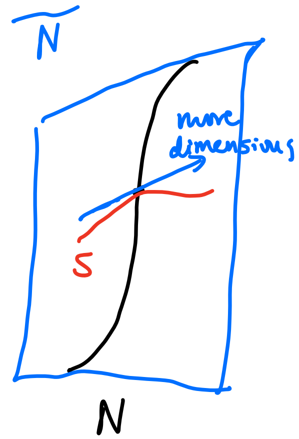
$\Rightarrow$ If M' is cpt, then $\#(\partial M')$ is always even.
(not nec. connected) 0-dim'l mfd

This simple observation can lead to something deep.

Recall one of our favorite prop: $F: M \to N$ and if $r \in N$ is a regular value of F, then $F^{-1}(r)$ is an embedded mfd of $\dim = \dim M - \dim N$.



one can $\rightsquigarrow$ easily break the regular condition by "enlarging" N



⌈ skip ? YES, skip (staring from here).

However, it's possible that there exists some submfd $S \subset \tilde{N}$

s.t. $r \in S$ and $dF(p)(T_pM) + T_rS = T_r\tilde{N}$ (for any $p \in F^{-1}(r)$)

@ $F^{-1}(r)$

(This is sometimes called $F \pitchfork S$ at r).
F ← map, S ← submfd

– If so, then $F^{-1}(r)$ is also an embedded submfd of M with dim

$$\dim M - (\dim \tilde{N} - \dim S)$$

(In old prop, $S = \emptyset$ or $\{r\}$, $\tilde{N} = N$).

- If $F \pitchfork S$ for any pt in S, then $F^{-1}(S)$ is an embedded submfd of M with dim

$$\dim M - (\dim \tilde{N} - \dim S). \quad (*)$$

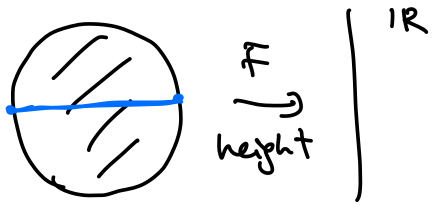
- For the case of mfd with b/d, we have a similar conclusion:
$F: M \to \tilde{N}$ where M is a mfd with b/d. If $\exists$ submfd $S \subset \tilde{N}$ s.t. $F \pitchfork S$ (for every pt in S), then $F^{-1}(S)$ is a mfd (as a submfd of M) with dim $(*)$ and (new)

$$\partial(F^{-1}(S)) = F^{-1}(S) \cap \partial M.$$

skip ?

Skipping ends here.

Bring back to the special case where $S = \{r\}$ and $\tilde{N} = N$, then if r is regular value of $F: M \to N$, we have $\dim F^{-1}(r) = \dim M - \dim N$ and $\partial(F^{-1}(r)) = F^{-1}(r) \cap \partial M$.



Pointed by 李华滨, this conclusion holds only when assuming F^{-1}(r) indeed admits a non-empty boundary. Otherwise, it is possible that F^{-1}(r) itself is a boundary (of some other manifold) - then the left-hand side will be empty.

Apply this to the following situation. $F: M \xrightarrow{\text{cpt}} \partial M$ (i.e. $N = \partial M$).

Pick a regular value $x \in \partial M$ of F ← Its existence is promised by Sard's Thm (later).

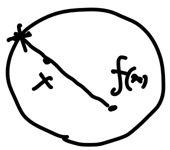
Then $F^{-1}(x)$ is a cpt 1-dim mfd with b/d.

Observe that $F|_{\partial M}: \partial M \to \partial M$ can not be the identity map! (b/c otherwise $\partial F^{-1}(x) = \underbrace{F^{-1}(x) \cap \partial M}_{\substack{\text{all pts on } \partial M \\ \text{that maps to } x \text{ by } F}} = \{x\}$. $\to\leftarrow$).

e.g. $M = \bar{B}^n(1)$ and $\partial M = S^{n-1}(1)$. No (smooth) map $F: B^n(1) \to S^{n-1}(1)$ that restricts to the identity on $S^{n-1}(1)$.

<u>Thm</u> (smooth version of Brouwer fixed pt Thm) Any smooth map f from $\bar{B}^n(1)$ to itself must have a fixed pt.

<u>pf</u>.

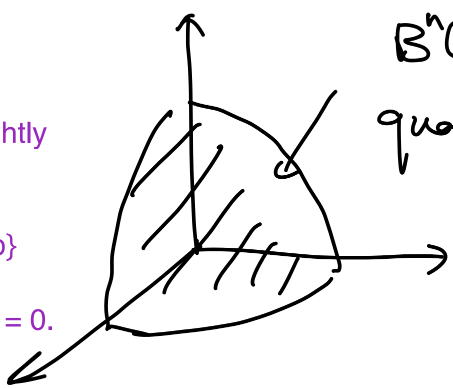


$F: \bar{B}^n(1) \to S^{n-1}(1)$ by projecting from $f(x)$ to x until hitting $S^{n-1}(1)$.

Rmk Brouwer fixed pt thm holds for continuous map.

Cor $A \in M_{n\times n}(\mathbb{R})$ where all entries are positive, then it admits a positive (real) eigenvalue.

pf:

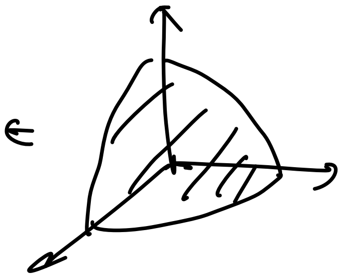


In fact, one needs to take a slightly smaller region:

\bar{B}^n(1) cap \R^n_{\geq \ep}

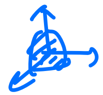
then there will be no issue A(0) = 0.

$B^n(1)$ intersects 1st quadrant of $\mathbb{R}^n \ni x \overset{F}{\longmapsto} \dfrac{A(x)}{\|A(x)\|} \in$



More rigorously

$\varphi \circ F \circ \varphi^{-1}$ where $\varphi : B^n(1) \to$



Then up to homeomorphism (lower regularity), $F : B^n(1) \longrightarrow B^n(1)$ continuous map

$\underset{\text{Brouwer}}{\Longrightarrow} \exists x_* \text{ s.t. } F(x_*) = x_*$

$\Rightarrow \dfrac{A(x_*)}{\|A(x_*)\|} = x_* \Longleftrightarrow A(x_*) = \underbrace{\|A(x_*)\|}_{\text{positive eigenvalue}} \cdot x_*$ $\square$.

This corner aims to show that there exists a manifold with boundary that is non-orientable, but its boundary is orientable. This example is the Mobi\"us strip (cut out from a Mobi\"us bundle over S^1).

add example