HOMEWORK TWO

This homework problem set can be accomplished with the help of references. Every problem worths 2 point and DO NOT LEAVE ANY PROBLEM BLANK! It is due to 11:59 pm on November 27 (sharp).

Exercise 1. Recall that we have defined an operator $T : C_0^\infty(\mathbb{C};\mathbb{C}) \to C_0^\infty(\mathbb{C};\mathbb{C})$ by

$$T(f) := \partial_z(\varphi * f)$$

where $\varphi(w) = \frac{1}{\pi w}$ defined on $\mathbb{C}\backslash\{0\}$. Prove that $\|Tf\|_{L^2} = \|f\|_{L^2}$ for any $f \in C_0^\infty(\mathbb{C};\mathbb{C})$. In particular, T extends to an isometry of $L^2(\mathbb{C},\mathbb{C})$. (In class, we state a theorem: for any $f \in C_0^\infty(\mathbb{C};\mathbb{C})$ and $1 < p < \infty$, we have $\|Tf\|_{L^p} \leq C_p\|f\|_{L^p}$. This exercise asks to manually verify this conclusion for $p = 2$, with equality in the conclusion and $C_2 = 1$.

Exercise 2. Let $D : X \to Y$ be a Fredholm operator. Then, for a sufficiently small bounded linear operator $P : X \to Y$ such that $D + P$ is also a Fredholm operator, prove that $\mathrm{ind}(D + P) = \mathrm{ind}(D)$.

Exercise 3. Complete, with as many details as possible, the proof that the linearization of the J-holomorphic operator $\bar{\partial}_J : \mathcal{B} \to \mathcal{E}$, denoted by

$$D_u : W^{k,p}(u^*TM) \to W^{k-1,p}(\Omega^{0,1}(\Sigma, u^*TM)),$$

is a Fredholm operator for any $u \in W^{k,p}(\Sigma, M) \cap C^1(\Sigma, M)$ where $k \geq 1$ and $p > 2$. Recall that in class, we sketched the proof that D_u is *semi-Fredholm* in the sense that $\ker(D_u)$ has finite dimension and its image is closed, therefore, it suffices to consider dual or adjoint operator of D_u.

Exercise 4. Consider the following operator discussed in the class:

$$A = -J\frac{\partial}{\partial t} - S : W^{1,2}(S^1, \mathbb{R}^2) \to L^2(S^1, \mathbb{R}^2)$$

where J is the standard almost complex structure on $\mathbb{R}^2$ and $S : S^1 \to \mathrm{Sym}(2)$ is a *constant* symmetric matrix with negative determinant. Prove that $\ker(A) = \{0\}$ (in other words, the value 0 is not in the set of the eigenvalues of A).

Exercise 5. Show that function $u : \mathbb{R}^n \to \mathbb{R}$ defined by $u(x) = \log\log\left(1 + \frac{1}{|x|}\right)$ for $x \neq 0$ and $u(0) = 0$ satisfies $\int_{B_1(0)} |\nabla u|^n d\mathrm{vol} < \infty$ for all $n \geq 2$, where $B_1(0)$ is the ball centered at $0 \in \mathbb{R}^n$ with radius 1. Therefore, $\chi u \in W^{1,n}(\mathbb{R}^n)$ for every $\chi \in C_0^\infty(\mathbb{R}^n)$. In particular, Sobolev embedding theorem for $p > n$ case fails to extend to the *"borderline"* case where $p = n$.