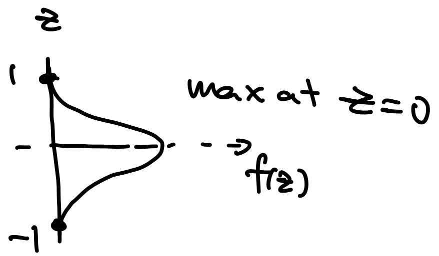
So in case ① $D_X F \equiv 0$

in case ② $D_Y F = f(z)$ where



Reflection: in case ① $\overset{=z}{F}$ is constant along each level set $F^{-1}(\{z\})$.
纬线 (Longitude)

in case ② $\overset{=z}{F}$ is increasing along each latitude.
经线

- Def A connection on vector bundle $\begin{matrix} E \\ \downarrow \pi \\ M \end{matrix}$ is a map

$$\nabla : \Gamma(TM) \times \Gamma(E) \longrightarrow \Gamma(E)$$

satisfying, for $f, g \in C^\infty(M)$,

① $\nabla_{fX+gY} s = f \nabla_X s + g \nabla_Y s$

② $\nabla_X (s_1 + s_2) = \nabla_X s_1 + \nabla_X s_2$

③ $\nabla_X (fs) = \underbrace{X(f)}_{\in C^\infty(M)} \cdot s + f \nabla_X s$

e.g. $E = M \times \mathbb{R}^k$ then recall $s \in \Gamma(E)$ identifies with fcns
$\downarrow$
M
$s : M \to \mathbb{R}^k$.

Then define $\nabla_X s$ by $D_X s$.

①, ② are satisfied automatically.

③: $\nabla_X(fs) = D_X(fs) \underset{\substack{\uparrow \\ \text{Lebniz rule}}}{=} (D_X f)\cdot s + f(D_X s) = X(f)\cdot s + f\nabla_X s$.

Exe $\Longrightarrow$ For any $\begin{matrix} E \\ \downarrow \pi \\ M \end{matrix}$, there always exists a connection.

eg. $E = TM$, then it might be the first time you see the following
$\downarrow$
M

structure: $\nabla^{a} : \underbrace{\Gamma(TM) \times \Gamma(TM)}_{\text{two vector fields as inputs}} \longrightarrow \underbrace{\Gamma(TM)}_{\text{one vector field as output.}}$ affine $(2,0) \to (1,0)$

By def, $\nabla_X Y$ means "directional derivative of Y along X".

e.g. $\begin{array}{c} E = T^*M \\ \downarrow \\ M \end{array}$, then $\nabla : \Gamma(TM) \times \Gamma(T^*M) \to \Gamma(T^*M)$ $\quad (1,1) \to (0,1)$

$$X \qquad \alpha \quad \to \quad \nabla_X \alpha$$

e.g. Fix ∇.

$$\begin{array}{ccc} \Gamma(TM) \times \Gamma(E) & (X, s) & (Y, s) \\ \downarrow & \downarrow & \downarrow \\ \Gamma(TM) \times \Gamma(E) & (Y, \nabla_X s) & (X, \nabla_Y s) \\ \downarrow & \downarrow & \downarrow \\ \Gamma(E) & \nabla_Y \nabla_X s & \nabla_X \nabla_Y s \end{array}$$

Are $\nabla_Y \nabla_X s$ and $\nabla_X \nabla_Y s$ the same?

Consider $\begin{array}{c} E = \mathbb{R}^2 \times \mathbb{R} \\ \downarrow \\ M = \mathbb{R}^2 \end{array}$ $\quad X((x,y)) = (1,0) \quad Y((x,y)) = (0,x)$

$s = f : \mathbb{R}^2 \to \mathbb{R}$ by $f((x,y)) = y$

$$\nabla_Y \nabla_X s = D_Y(D_X f) = 0$$

$$\nabla_X \nabla_Y s = D_X(D_Y f) = D_X(x) = 1$$

So in general they are <u>NOT</u> the same!

Rmk Recall in calculus, $\forall f: \mathbb{R}^n \to \mathbb{R}$, one has the following identity

$$\frac{\partial^2 f}{\partial x_i \partial x_j} = \frac{\partial^2 f}{\partial x_j \partial x_i} \quad \forall\, i, j$$

This is explained by

i-th position

$$X(x_1, \cdots, x_n) = (0, \cdots, 1, \cdots 0)$$

$$Y(x_1, \cdots, x_n) = (0, \cdots \cdots, 1, \cdots 0)$$

j-th position

$$\frac{\partial^2 f}{\partial x_i \partial x_j} = D_Y(D_X f) \qquad \frac{\partial^2 f}{\partial x_j \partial x_i} = D_X(D_Y f).$$

For these two vector fields X, Y, $D_Y D_X = D_X D_Y$. ← X, Y "commute"

Here is a beautiful bi-operator (two inputs) on $\Gamma(TM)$ that detects the "commutativity":

$$X, Y \in \Gamma(TM) \longmapsto [X, Y] \in \Gamma(TM)$$

s.t.

$$D_{[X,Y]} f = D_X D_Y f - D_Y D_X f \quad \text{for any } f: M \to \mathbb{R}.$$

defined implicitly

(反向定义)

Prop (Exe) Locally, if $X = (X^1, \cdots, X^n)$, $Y = (Y^1, \cdots, Y^n)$, then

$$[X, Y] = (D_X Y^1 - D_Y X^1, \cdots, D_X Y^n - D_Y X^n)$$

Prop $[\cdot,\cdot]$ satisfies Lie bracket properties on $\Gamma(TM)$. i.e,

① bi-linear on inputs

② anti-symmetric $\quad [X, Y] = -[Y, X]$

expand out

③ Jacobi identity $\quad [X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0.$

Summary :

$$E = M \times \mathbb{R} \\ \downarrow \\ M$$

$\nabla : \Gamma(TM) \times C^\infty(M) \to C^\infty(M)$

$$\nabla = D$$

dir-derivative

- $[\cdot,\cdot]$ defines a product on $\Gamma(TM)$ s.t. $\nabla_{[X,Y]} = \nabla_X \nabla_Y - \nabla_Y \nabla_X$
- $(\Gamma(TM), [\cdot,\cdot])$ is a Lie algebra

$C^\infty(M)$-mod $\longrightarrow$ not compatible!

Question (discussed later)

For any $\begin{array}{c} E \\ \pi \downarrow \\ M \end{array}$ and connection $\nabla : \Gamma(TM) \times \Gamma(E) \longrightarrow \Gamma(E)$, does it always hold $\nabla_{[X,Y]} = \nabla_X \nabla_Y - \nabla_Y \nabla_X$?

curvature

④ Tensor algebra (张量代数)

Assume* everyone knows what "$\otimes$" is.

Test: What is $\mathbb{Z}/p\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/q\mathbb{Z} = ?$ where p, q are coprime.

- Take V, W are two vector spaces,

formal linear combination of ordered pair (v_i, w_i)

$$V \otimes W := \frac{\{ \sum r_i (v_i, w_i) \mid (v_i, w_i) \in V \times W \}}{\sim}$$

where eqn relation $\sim$ is defined by "linear at each input".

Denote $v \otimes w := [(v, w)]$.

By def, $(\lambda v+v', w) \sim \lambda(v,w) + (v', w)$, so

$$(\lambda v+v')\otimes w = [(\lambda v+v', w)] = [(\lambda v, w)+(v', w)] = \lambda v\otimes w + v'\otimes w.$$

$\Rightarrow$ $-\otimes-$ is a bilinear operator

ordering ? $(V_1\otimes V_2)\otimes V_3$
$V_1\otimes(V_2\otimes V_3)$

Similarly, one defines $V_1\otimes \cdots \otimes V_k$. and it is "multi-linear".

<u>Rmk</u>. $V_1\otimes\cdots\otimes V_k$ is the "largest" vector space that carries "multi-linear" structure – universal property (Optional Exe)

given multi-linear map

$$\begin{tikzcd}
V_1\times\cdots\times V_k \arrow[r, "\widetilde{F}"] \arrow[d, "\pi \text{ quotient map}"'] & X \\
V_1\otimes\cdots\otimes V_k \arrow[ur, dotted, "\exists F"']
\end{tikzcd}$$

exist a unique multi-linear map

e.g. $V = \mathrm{span}\langle e_1,\cdots,e_n\rangle$
$W = \mathrm{span}\langle e_1,\cdots, e_m\rangle$ $\Rightarrow$ $V\otimes W = \mathrm{span}\left\langle e_i\otimes e_j \;\middle|\; \begin{matrix}1\le i\le n\\ 1\le j\le m\end{matrix}\right\rangle$

e.g. $\mathbb{R} \otimes \mathbb{R} = \mathbb{R}$

$\quad e \quad e \qquad e \otimes e$

e.g. $\dim(V \otimes W) = \dim(V) \times \dim(W)$

<u>Rmk</u> The eqn relation "$\sim$" to define $\otimes$ does <u>not</u> include "switch".

In general $V \otimes W$ and $W \otimes V$ are different. ← otherwise the dim will drop.

e.g. $V = \mathbb{R}^2 = \mathrm{span}\langle e_1, e_2 \rangle$, then $e_1 \otimes e_1 + e_2 \otimes e_2 \in V \otimes V$ is not "decomposable"

in the sense that $e_1 \otimes e_1 + e_2 \otimes e_2 = v_1 \otimes v_2$.

e.g. $V_1 \otimes \cdots \otimes V_k$ is well-defined.

e.g. For a vector space V and its dual space V^*, there is a

linear maps on V.

well-defined map $V \otimes V^* \longrightarrow \mathbb{R}$ $\quad (v, f) \mapsto f(v)$.

In this way, V, V^* "cancel" each other.

- Consider

$$\underbrace{T}_{\text{tensor}}{}^{\overbrace{(k,l)}^{\text{index}}} V := \underbrace{V \otimes \cdots \otimes V}_{k \text{ copies}} \otimes \underbrace{V^* \otimes \cdots \otimes V^*}_{l \text{ copies}}$$

(By notation, $T^{(0,0)} = \mathbb{R}$)

$T^{(k,l)} V$ is a vector space of $\dim = \dim(V)^{k+l}$.

e.g. $V = \mathbb{R}^n$ and consider $\det \in T^{(0,n)} \mathbb{R}^n$ by

$$\det(\underset{\mathbb{R}^n}{\underset{\cap}{v_1}}, \cdots, v_n) := \det \underbrace{\begin{pmatrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{pmatrix}}_{n \times n \text{ matrix}}$$

Important observation on ordering: switch v_i, v_j. det changes sign. This property is called "alternating" or "antisymmetric". In other words, some elements in $T^{(0,l)} V^*$ are more special (elaborated next Lecture).