- From calculus, we know in $\mathbb{R}^3$,

$$\text{curl} \circ \text{grad} = 0 \qquad \text{and} \qquad \text{div} \circ \text{curl} = 0$$

These are in fact concrete expressions of $d \circ d = 0$

e.g. $f(x,y,z) \xrightarrow{d} \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz$

$$\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right) =: \text{grad}(f)$$

$$\alpha = P\,dx + Q\,dy + R\,dz \xrightarrow{d} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx \wedge dy + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} \right) dz \wedge dx + \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \right) dy \wedge dz$$

$$\left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}, \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}, \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) =: \text{curl}\,(\overbrace{P, Q, R}^{\text{vector field}}) = \nabla \times (P, Q, R)$$

$$\beta = A\,dx \wedge dy + B\,dz \wedge dx + C\,dy \wedge dz \xrightarrow{d} \underbrace{\left(\frac{\partial C}{\partial x} + \frac{\partial B}{\partial y} + \frac{\partial A}{\partial z} \right)}_{} dx \wedge dy \wedge dz$$

$$=: \text{div}(C, B, A) = \nabla \cdot (C, B, A)$$

vector field

- Recall the celebrated Maxwell's equations.

Denote by E electricity field, B magnetic field.

E, B are vector fields on $\mathbb{R}^3$ AND time-dependent!

$E = E(t, x, y, z)$

$B = B(t, x, y, z)$

$$\begin{cases} \nabla \cdot B = 0 \\ \nabla \cdot E = 4\pi\rho \leftarrow \text{constant} \\ \dfrac{\partial B}{\partial t} + \nabla \times E = 0 \\ \dfrac{\partial E}{\partial t} - \nabla \times B = -4\pi J \leftarrow \text{constant} \end{cases}$$

The standard spacetime is the Minkowski space $(\underset{t,x,y,z}{\mathbb{R}^4}, \overset{\text{still non-deg}}{\langle , \rangle})$ where

$$\langle (t_1, x_1, y_1, z_1), (t_2, x_2, y_2, z_2) \rangle := t_1 t_2 - x_1 x_2 - y_1 y_2 - z_1 z_2$$

Using what we have learned, equations above simplifies

$$d\mathbf{F} = 0 \quad \text{and} \quad d * \mathbf{F} = 4\pi * \mathbf{J}$$

Hodge star operator

where $\mathbf{F} \in \Omega^2(\mathbb{R}^4)$ and $\mathbf{J} \in \Omega^1(\mathbb{R}^4)$.

Here,

$E_x = E_x(t,x,y,z)$

$$\mathbf{F} = E_x\, dx\wedge dt + E_y\, dy\wedge dt + E_z\, dz\wedge dt$$
$$+ B_x\, dy\wedge dz + B_y\, dz\wedge dx + B_z\, dx\wedge dy$$

Faraday 2-form (Minkowski)

$$\mathbf{J} = \rho\, dt + J_x\, dx + J_y\, dy + J_z\, dz$$

$\frac{\partial B}{\partial t}$

- $$d\mathbf{F} = 0 \iff \left(\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} + \frac{\partial B_z}{\partial t}\right) dx\wedge dy\wedge dt$$

$\nabla \times E =$

$$+ \left(\frac{\partial E_z}{\partial x} - \frac{\partial E_x}{\partial z} - \frac{\partial B_y}{\partial t}\right) dx\wedge dz\wedge dt$$
$$+ \left(\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} + \frac{\partial B_x}{\partial t}\right) dy\wedge dz\wedge dt$$
$$+ \left(\frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z}\right) dx\wedge dy\wedge dz = 0$$

$= \nabla \cdot B$

$$\iff \nabla\cdot B = 0 \quad \text{and} \quad \frac{\partial B}{\partial t} + \nabla \times E = 0$$

- $*: \Lambda^2((\mathbb{R}^4)^*) \longrightarrow \Lambda^2((\mathbb{R}^4)^*)$

$\underline{\underline{** = -1}}$

$$dx \wedge dt \longrightarrow dy \wedge dz \qquad dx\ dt\ dy\ dz$$

$\text{sgn}(\sigma) = -1$ (cancelled with the signature of $\langle , \rangle$ in spacetime)

$$dy \wedge dt \longrightarrow dz \wedge dx \qquad dy\ dt\ dz\ dx$$

$$dz \wedge dt \longrightarrow dx \wedge dy \qquad dz\ dt\ dx\ dy$$

$*: \Lambda^1((\mathbb{R}^4)^*) \longrightarrow \Lambda^3((\mathbb{R}^4)^*)$

$\underline{\underline{*\cdot * = 1}}$

$$dt \longrightarrow dx \wedge dy \wedge dz \qquad dt\ dx\ dy\ dz$$

$$dx \longrightarrow -dy \wedge dz \wedge dt \qquad dx\ dy\ dz\ dt$$

$$dy \longrightarrow dx \wedge dz \wedge dt \qquad dy\ dx\ dz\ dt$$

$$dz \longrightarrow -dx \wedge dy \wedge dt \qquad dz\ dx\ dy\ dt$$

$$*F = -B_x\, dx\wedge dt - B_y\, dy \wedge dt - B_z\, dz \wedge dt + E_x\, dy \wedge dz + E_y\, dz \wedge dx + E_z\, dx \wedge dy$$

switch E_* and B_* (with a minus sign before B_*)

$$*\mathbf{J} = \rho\, dx\wedge dy\wedge dz - J_x\, dy\wedge dz\wedge dt + J_y\, dx\wedge dz\wedge dt - J_z\, dx\wedge dy\wedge dt$$

$$d*\mathbf{F} = 4\pi *\mathbf{J} \iff \underbrace{\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z}}_{\nabla\cdot E} = 4\pi\rho$$

$$\begin{aligned}
\frac{\partial B_y}{\partial z} - \frac{\partial B_z}{\partial y} + \frac{\partial E_x}{\partial t} &= -4\pi J_x \\
\frac{\partial B_x}{\partial z} - \frac{\partial B_z}{\partial x} - \frac{\partial E_y}{\partial t} &= 4\pi J_y \\
\frac{\partial B_x}{\partial y} - \frac{\partial B_z}{\partial x} + \frac{\partial E_z}{\partial t} &= -4\pi J_z
\end{aligned}$$

$$-\nabla\times B \qquad + \frac{\partial E}{\partial t} \qquad -4\pi J$$

Therefore, Maxwell's equations are

$$d\mathbf{F} = 0 \quad \text{and} \quad d*\mathbf{F} = 4\pi *\mathbf{J}$$

Rmk. More concise, more meaningful!

② Flows

Def Given a vector field $X \in \Gamma(TM)$, an <u>integral curve</u> (or a <u>flow line</u>) is a map $\gamma : \overset{\text{open interval}}{I} \longrightarrow M$ s.t. $\dot{\gamma}(t) = X(\gamma(t))$.

(usually with some fixed initial condition)

e.g. $M = \mathbb{R}^2 \setminus \{0\}$, in polar coordinate (ρ, θ)

- $X(\rho, \theta) = (0, \overset{\text{constant}}{1}) \quad (= 0 \cdot \partial_\rho + 1 \cdot \partial_\theta)$

Then an integral curve $\gamma(t) = (\rho(t), \theta(t))$ satisfies

$$\dot{\gamma}(t) = (\dot{\rho}(t), \dot{\theta}(t)) = (0, 1)$$

$$\Rightarrow \begin{cases} \rho(t) = \underline{\rho(0)} & (\text{constant}) \\ \underbrace{\theta(t) = t + \underline{\theta(0)}}_{\text{rotation in a constant speed.}} \end{cases}$$

where $(\rho(0), \theta(0))$ is the initial cond.

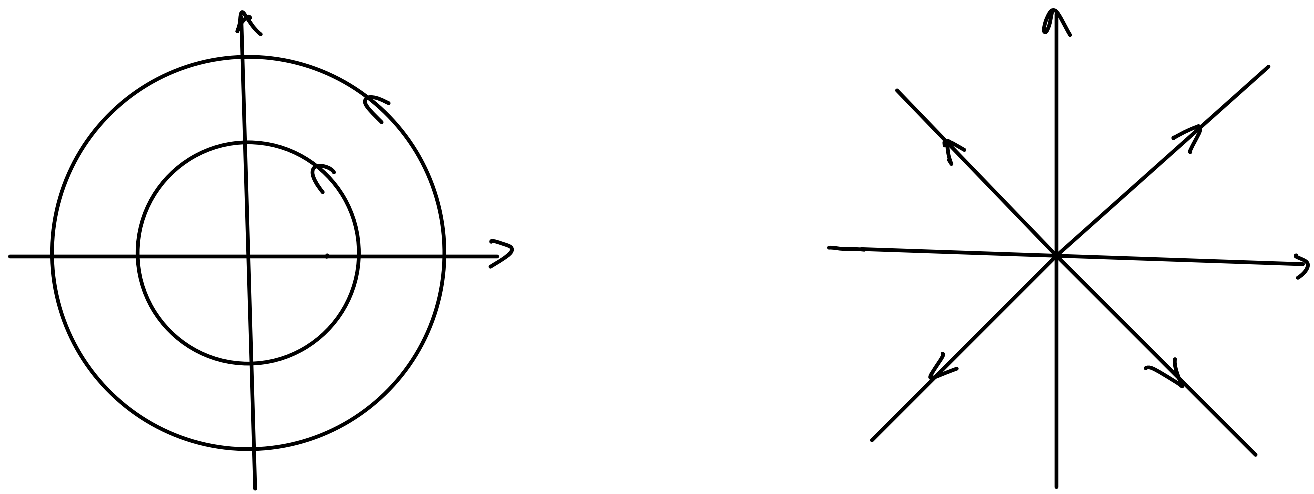
- $X(\rho, \theta) = (\rho, 0)$ ← non-constant $(= \rho \partial_\rho + 0 \cdot \partial_\theta)$

Then an integral curve $\gamma(t) = (\rho(t), \theta(t))$ satisfies

$$\dot{\gamma}(t) = (\dot{\rho}(t), \dot{\theta}(t)) = (\rho(t), 0)$$

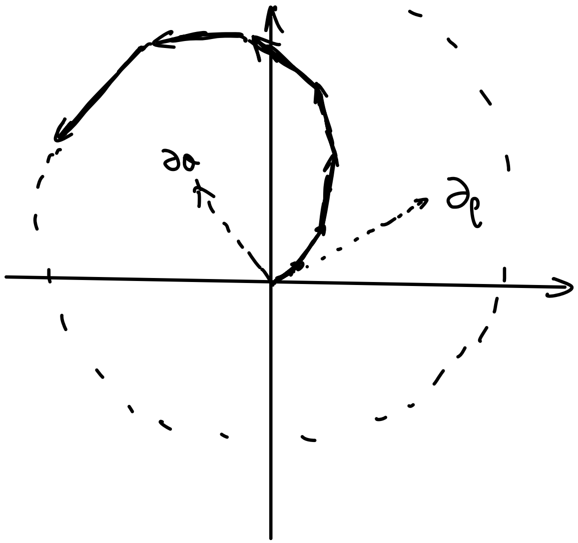
$$\Rightarrow \begin{cases} \rho(t) = e^t \cdot \underline{\rho(0)} \\ \theta(t) = \underline{\theta(0)} \end{cases}$$ where $(\rho(0), \theta(0))$ is the initial cond.

diverging exponentially.



- $X(\rho, \theta) = (\rho, 1) \; (= \rho \partial_\rho + \partial_\theta)$

Then one can roughly draw the picture of integral curves



- This will be curves "spiralling out"
- There are no "periodic" integral curve (except $\vec{0}$).

e.g. (M, g) Riem manifold. Recall that at every pt $p \in M$, $g(p)$ is a <u>non-deg</u> bi-linear form, which induces an iso:

$$\Omega^1(M) \; (= \Gamma(T^*M)) \xrightarrow{\sim} \Gamma(TM)$$

Given a smooth fcn $F: M \longrightarrow \mathbb{R}$, $dF \longrightarrow \nabla F$ via

gradient vector field of F w.r.t g. $\quad g(\nabla F, -) = dF.$

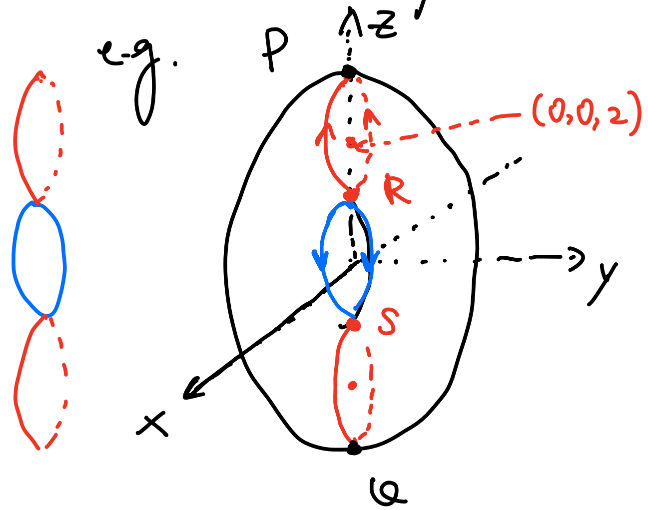
Then for any integral curve of negative gradient v.f. $-\nabla F$,

i.e. $\gamma: I \longrightarrow M$ s.t. $\dot{\gamma}(t) = -\nabla F(\gamma(t))$.

- observe: $dF(-\nabla F) = g(\nabla F, -\nabla F) = -\|\nabla F\|^2 \leq 0$,

so F is always non-increasing along the integral curve $\gamma(t)$.

e.g.



2-torus $\mathbb{T}^2$ in $\mathbb{R}^3$ defined by

$f^{-1}(1)$ where $f: \mathbb{R}^3 \longrightarrow \mathbb{R}$ by

$$f(x,y,z) = \left(\sqrt{y^2+z^2} - 2\right)^2 + x^2$$

Consider $F: \mathbb{T}^2 \to \mathbb{R}$ by projecting to the z-axis.

Then starting from pt R, blue curves can be integral curves, red curves can not be!

- For a fixed $\tau \in \mathbb{R}$, define "reparametrization" $\tilde{\gamma}(t) = \gamma(t+\tau)$.

Then $\dot{\tilde{\gamma}}(t) = \dot{\gamma}(t+\tau) = -\nabla F(\gamma(t+\tau)) = -\nabla F(\tilde{\gamma}(t))$

Therefore $\tilde{\gamma}(t)$ is also an integral curve. ← "unparametrized" integral curve $\gamma(t)/\sim$ translation in t

<u>Rmk</u> $\tilde{\gamma}(t) = \gamma(2t)$, then

$$\dot{\tilde{\gamma}}(t) = 2\dot{\gamma}(2t) = -2\nabla F(\gamma(2t)) = -\underset{\substack{\uparrow \\ \text{extra factor}}}{2}\nabla F(\tilde{\gamma}(t))$$

This reparametrized $\tilde{\gamma}(t)$ is <u>not</u> an integral curve of $-\nabla F$.

- Those pts $p \in M$ s.t. $dF(p) \equiv 0$ are called <u>critical pts</u> of F. Integral curves "stop" at critical pts.

e.g. Back to the 2-torus above, $R = (0,0,1)$ is a critical pt of the projection $F(x,y,z) := z$.

<u>Prop</u> If $r \in \mathbb{R}$ is a regular value of f, then $f^{-1}(r)$ is a mfd and at any $p \in f^{-1}(r)$, $T_p f^{-1}(r) = \ker df(p)$.

In this case:

$$df((x,y,z)) = 2x\,dx + 2\left(\sqrt{y^2+z^2} - 2\right)\cdot\left(\frac{y}{\sqrt{y^2+z^2}}\,dy + \frac{z}{\sqrt{y^2+z^2}}\,dz\right)$$

At $R = (0,0,1)$, $df(R) = -2\,dz \Rightarrow T_R T^2 = \operatorname{span}\langle \partial_x, \partial_y \rangle$

Therefore $\underset{\substack{\parallel \\ dz}}{dF(R)} : T_R T^2 \to \mathbb{R}$ vanishes identically.

e.g. Denote $J = \begin{pmatrix} 0 & I_{n\times n} \\ -I_{n\times n} & 0 \end{pmatrix} \in M_{2n\times 2n}(\mathbb{R})$ $\quad$ $x_i \to y_i$, $y_i \to -x_i$

Then $J^2 = -\mathbb{1}$ and $J^T = -J$.

left multiplication

Consider a vector field in $M_{2n\times 2n}(\mathbb{R})$ defined by $X(A) = JS\cdot A$

where S is a fixed symmetric matrix in $M_{2n\times 2n}(\mathbb{R})$

<u>Claim</u>: For any integral curve $A(t)$ in $M_{2n\times 2n}(\mathbb{R})$ of v.f. X, starting from $\mathbb{1}$, we have $A(t)^T J A(t) = J$.

Indeed,
$$\dot{A}(t)^T J A(t) + A(t)^T J \dot{A}(t)$$
$$= A(t)^T S \underbrace{J^T \cdot J}_{\mathbb{1}} A(t) + A(t)^T \underbrace{J\cdot J}_{-\mathbb{1}} S\cdot A(t)$$
$$= A(t)^T S A(t) - A(t)^T S A(t) = 0$$

$\dot{A}(t) = X(A(t))$
$= JS\cdot A(t)$

Denote $Sp(2n) = \{ A \in M_{2n\times 2n}(\mathbb{R}) \mid \underbrace{A(t)^T J A(t) = J}_{\text{symplectic matrix (辛矩阵)}} \}$.

It is a Lie group of $\dim_{\mathbb{R}} = 2n^2 + n$.