Background on symplectic and contact geometry

1. Symplectic geometry

(M^{2n}, ω) $\quad \omega \in \Omega^2(M)$ non-deg and closed

$$\underbrace{\omega \wedge \cdots \wedge \omega = \text{a volume form}}_{\text{local condition}} \qquad \underbrace{d\omega = 0}_{\text{global condition}}$$

Ex. $(M, \omega) = (\mathbb{R}^{2n}, \omega_{std} = \sum_{i=1}^{n} dx_i \wedge dy_i)$

$(M, \omega) = (\mathbb{C}^n \quad \omega_{std} = \frac{\sqrt{-1}}{2} \sum_{i=1}^{n} dz_i \wedge d\bar{z}_i)$

$(M, \omega) = (T^*Q, \omega_{can} = \sum_{i=1}^{n} dq_i \wedge dp_i$ ← p_i coord of fiber $T^*_q Q$.)

coord. q_i $\quad$ in local coordinate

All cases above are exact: $\omega = d\lambda$ for some $\lambda \in \Omega^1(M)$.

Ex $(M, \omega) = (\Sigma_{g \geq 0}, \omega_{area})$

$(M, \omega) = (\mathbb{CP}^n, \omega_{FS})$ ← Fubini-Study form obtained by gluing a symp str on local charts.

Ex $(M, \omega_M), (N, \omega_N) \Longrightarrow (M \times N, \omega_{prod} = \pi_M^* \omega_M + \pi_N^* \omega_N)$ //

(M, J) almost cpx str. where $J_p : T_pM \circlearrowleft$ a linear map s.t. $J_p^2 = -\mathbb{1}_{T_pM}$

<u>Ex</u> Any complex mfd (M, i) is an almost cpx str b/c locally

$i \cdot z = \sqrt{-1} \cdot z$ identified $J(x_1, y_1, \cdots; x_n, y_n) = (y_1, -x_1, \cdots; y_n, -x_n)$

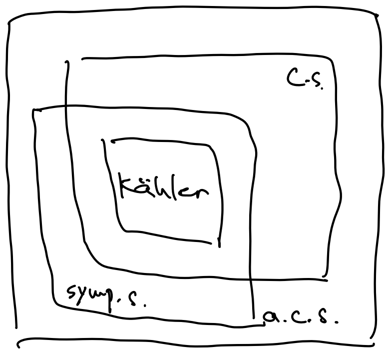
$\uparrow$ easy matrix

In general, J could be much more complicated.

<u>Rmk</u> In general, to check if J is induced by a cpx str, we apply Newlander-Nirenberg criterion. $\Longrightarrow$ On Σ_g, any almost cpx is a cpx str.

Notation: (Σ_g, j) a Riem surface.

Relations between different geometries.



① Every subset-containing is strict.

② For any (M, ω), $\exists J$ that is ω-compatible:

- $\omega(v, Jv) > 0$
- $\omega(v, w) = \omega(Jv, Jw)$ (*)

Thm (Gromov) $\mathcal{J}(M,\omega) = \{ \omega\text{-compatible a.c.s } J \text{ on } M \}$
is contractible (and non-empty).

Rmk The same conclusion holds for

$$\mathcal{J}_\tau(M,\omega) := \left\{ \begin{array}{l} \text{a.c. } J \text{ on } M \text{ that satisfies the} \\ \text{second condition of } (*) \end{array} \right\}$$

$\Longleftarrow$ J is ω-tamed.

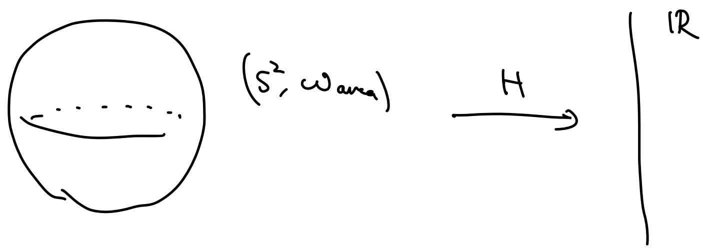
Rmk The $\{\text{symp. s}\} \subset \{\text{a.c.s}\}$ is the key to bring symp. geo to a.c. geometry, where techniques in complex geo can apply.

(M,ω), $H: [0,1]\times M \to \mathbb{R}$, non-deg of ω implies

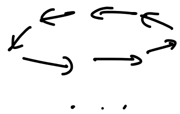
$$\exists\, X_H \quad \text{s.t.} \quad \omega(X_H, -) = -dH$$

$\longrightarrow$ Hamiltonian vector field.

Ex



$H: S^2 \to \mathbb{R}$
autonomous, height fcn.

$X_H =$  which is $\perp$ to the gradient vector field ∇H.

$\Rightarrow$ dyn or geo meaning of X_H is more mysterious.

For dyn, one aims to study closed orbits of X_H: $\gamma: S^1 = \mathbb{R}/\mathbb{Z}$

$\to M$ s.t. $\dot{\gamma}(t) = X_H(\gamma(t))$.

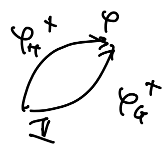
$$\{\text{closed orbits of } X_H\} \xleftrightarrow{1:1} \left\{\begin{array}{l}\text{fixed pts of the} \\ \text{time-1 map of the} \\ \text{flow of } X_H\end{array}\right\}$$

φ_H^t flow (i.e. $\frac{d\varphi_H^t}{dt} = X_H(\varphi_H^t)$) Hamiltonian flow

$\varphi = \varphi_H^1 \in \mathrm{Diff}(M)$ Hamiltonian diffeo.

Observations:

- φ preserves ω $\varphi^*\omega = \omega$ ($\Rightarrow$ φ preserves volume of M)
- There could be other fcn G s.t. $\varphi_G^1 = \varphi$.



- $\overbrace{\{\varphi = \varphi_H^1 \text{ for some } H\}}^{\mathrm{Ham}(M,\omega)} \subseteq \mathrm{Diff}_{vol}(M)$.
  subgroup
- $\exists$ $\varphi \in \mathrm{Diff}_{vol}(M) \setminus \mathrm{Ham}(M,\omega)$ Hint: $(\underset{=\Sigma_{g=1}}{\mathbb{T}^2}, \omega_{area})$

$\Rightarrow$ $\varphi \in \mathrm{Ham}(M,\omega)$ is more special and it reveals some rigidities in symp geo.

<u>Thm</u> (Arnold conj) For any $\varphi \in \mathrm{Ham}(M,\omega)$, under some non-deg,

$$\#\mathrm{Fix}(\varphi) \geq \sum b_i(M;\mathbb{Z}_2)$$

betti number.

Rmk Lefschetz fixed pt thm gives an alternating sum. //

Def Given (Σ, j), (M, J), a J-hol curve is $u : (\Sigma, j) \to (M, J)$ satisfying

no symp str needed!

$$J \circ u_* = u_* \circ j$$

One can also write $\quad u_* + J \cdot u_* \circ j = 0 \Rightarrow$ introduce a notation $\bar\partial_J$ by

$$\bar\partial_J u := \frac{1}{2}\left(u_* + J \cdot u_* \circ j\right)$$

Then u is J-hol iff $u \in \bar\partial_J^{-1}(0)$. ← We will make this more rigorous later.

One can study J-hol from various perspectives, mostly split into the following two:

- local properties of J-hol curves

Ex u, v are J-hol curve $(\Sigma, j) \to (M^4, J)$ closed. Denote

$$A = [\mathrm{Im}(u)] \qquad B = [\mathrm{Im}(v)] \qquad \text{and} \qquad A \neq B \text{ in } H_2(M; \mathbb{Z})$$

If they are simple (not multiple cover), then

① $A \cdot B \geq \# \{ (z_0, z_1) \in \Sigma \times \Sigma \mid u(z_0) = v(z_1) \}$

intersection number

②
$$A \cdot A \geq \# \{ (z_0, z_1) \in \Sigma \times \Sigma \setminus \Delta \mid u(z_0) = u(z_1) \} + c_1(A) - \chi(\Sigma)$$

diagonal

① is called positivity of intersection (cf. the same conclusion for hol curves in a cpx mfd).

② is called the adjunction formula.

In a local coordinate of Σ, (s,t) s.t. $j\partial_s = \partial_t$, $j\partial_t = -\partial_s$,

$$\bar\partial_J(u) = \frac{1}{2}(\partial_s u + J(u)\partial_t u)\,ds + \frac{1}{2}(\partial_t u - J(u)\partial_s u)\,dt$$

$$\Rightarrow \bar\partial_J(u) = 0 \text{ iff } \partial_s u + J(u)\partial_t u = 0$$

More concretely, if (Σ, j) and (M, J) are locally modelled by $(\mathbb{C}, i)$, then $\bar\partial_J(u) = 0$ iff u satisfies Cauchy-Riem eqn.

- global properties

$$\mathcal{M} := \{ J\text{-hol curve } u : (\Sigma, j) \to (M, J) \}$$

moduli space + "decorations" $\left\{ \begin{array}{l} \text{top constraints } [u] = A \in H_2(M;\mathbb{Z}) \\ \text{simplicity or multiple covering} \\ \text{marked points} \\ \vdots \end{array} \right.$

mfd str of $\mathcal{M}$; compactness of $\mathcal{M}$. limiting curves in $\overline{\mathcal{M}}$ ← compactification

...

Ex $(M,\omega) = (S^2 \times S^2, \omega_{prod})$, $J = J_{std} \times J_{std}$, $A \in [S^2 \times \{pt\}]$, $B = [\{pt\} \times S^2]$

$S^2 = \Sigma_{g=0}$

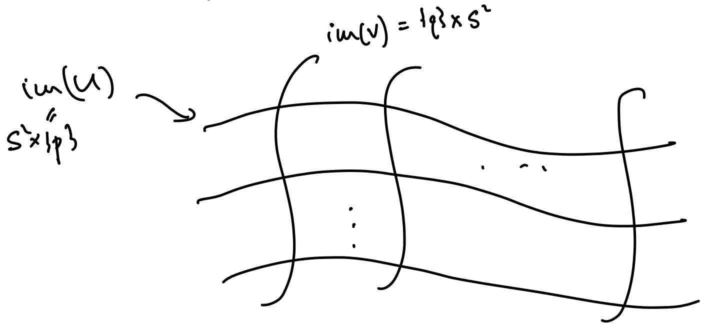
identify S^2 with $\mathbb{C}P^1$

$$\mathcal{M}_{A,J} = \{ u : (S^2, j) \to (M, J) \mid [u] = A \}$$

$$\mathcal{M}_{B,J} = \{ \quad \cdots \quad \mid [u] = B \}$$

One can show both moduli spaces are cpt mfd of (real) dim = 2.

Positivity + adjunction $\Longrightarrow$ $\mathcal{M}_{A,J}$ and $\mathcal{M}_{B,J}$ foliate $S^2 \times S^2$



Therefore $\mathcal{M}_{A,J} \times \mathcal{M}_{B,J} \simeq S^2 \times S^2$: $S^2 \times \{p\} \cap \{q\} \times S^2 \longleftrightarrow (p,q)$

Any $S^2 \times \{p\}$ or $\{q\} \times S^2$ is a symplectic submfd $\leftarrow$ $\omega|_{\cdots}$ is non-deg.

Thm (Thm 1.4 in [Wen]) (M^4, ω) satisfies

① $\nexists$ symp submfd $S \subset M^4$ diffeo to S^2 and $[S] \cdot [S] = -1$

helps to prove $\mathcal{M}$ is a mfd (no bubbling)

② $\exists$ symp mfd $S_1, S_2 \subset M^4$, diffeo to S^2 and

$$[S_1] \cdot [S_1] = [S_2] \cdot [S_2] = 0$$

and have exact 1 intersection pt with each other, $\pitchfork$ and positive.

construct foliation

Then $(M, \omega) \simeq (S^2 \times S^2, \omega_{prod})$

complete rigidity.

//

SFT is based on a version of Floer theory.

Floer theory (mostly Floer homology) is an ∞-dim'l Morse theory but on path or loop space of (M,ω).

Ham Floer homology: Assume (M,ω) satisfies $\omega|_{\pi_2(M)} = 0$. Given H,

$$\underset{\substack{\uparrow \\ \text{actional fcn'l} \\ \text{(as a Morse fcn)}}}{\mathcal{A}_H} : \overset{\text{loop space}}{\Lambda M} \longrightarrow \mathbb{R} \qquad \gamma \to \mathcal{A}_H(\gamma)$$

where

ind of the "capping" w due to our assumption.

$$\mathcal{A}_H(\gamma) = \mathcal{A}_H((\gamma, w)) = -\int_{\mathbb{D}^2} w^*\omega + \int_{S^1} H_t(\gamma(t))\,dt$$

$w : \mathbb{D}^2 \to M$ s.t. $w|_{\partial\mathbb{D}^2} = \gamma$

By computation,

$$\begin{aligned} \mathrm{Crit}(\mathcal{A}_H) &= \{ \gamma \subset \overset{\text{contractible component}}{\Lambda M} \mid \dot{\gamma} = X_H(\gamma) \} \\ &= \{ \text{closed orbits of } X_H \} \\ &\simeq \{ \text{fixed pts of } \varphi_H^1 \} \end{aligned}$$

Following Morse theory, we need to construct a homology theory (chain cpx) with generators in $\mathrm{Crit}(\mathcal{A}_H)$

Recall in Morse homology $\partial_{\text{Morse}} x = \sum_{\substack{y \\ \text{ind diff}=1}} \# \mathcal{M}(x,y)/\mathbb{R} \cdot y$