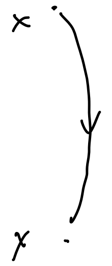
$\mathcal{M}(x,y) = \Big\{$  $\Big\}$ ← negative gradient flowline

new setting → 

Pick $\{J_t\}_{t\in[0,1] \text{ or } \mathbb{R}/\mathbb{Z}}$, define a metric on ΛM by

$$\langle \xi, \eta \rangle := \int_{S^1} \omega_{\gamma(t)}(\xi(t), \eta(t))\, dt$$

for any $\xi, \eta \in T_\gamma \Lambda M$ $\left(\simeq \Gamma(\gamma^* TM) = \{\text{vector fields along } \gamma(t)\}\right)$

Then by computation,

$$\underset{\text{w.r.t } \langle,\rangle}{\vec{\nabla}} \mathcal{A}_H(\gamma) = J_t(\dot\gamma - X_{H_t}(\gamma)) \Rightarrow \text{the negative flowline } u: \underset{s}{\mathbb{R}} \times \underset{t}{S^1} \to M \text{ is}$$

$$\partial_s u = -\nabla \mathcal{A}_H(u(s,t)) = J_t(u(s,t))\left(\partial_t u - X_{H_t}(u(s,t))\right) \quad (*)$$

which is a perturbed version of J-hol curve!

Again, one needs to study the moduli space

$$\mathcal{M}(\gamma, \gamma') = \{ u: \mathbb{R}\times S^1 \to M \mid (*) \}$$

☺: experience of J-hol curves can be borrowed

☹: the domain $\mathbb{R}\times S^1$ is non-cpt! ← study asymptotic behavior of such solutions

Assume things worked out, then

$$\partial\gamma = \sum_{\substack{\gamma' \\ \text{ind diff}=1}} \mathcal{M}(\gamma,\gamma')/_{\mathbb{R}} \cdot \gamma' \quad \text{is well-defined}$$

and $\partial\circ\partial = 0$

$\Rightarrow$ $CF_*(M,\omega,J,H) = \mathbb{Z}_2\langle \gamma \mid \deg(\gamma) = * \rangle \circlearrowleft \partial$

$\Rightarrow$ $HF_*(M,\omega,J,H)$ is well-defined.

Thm (M,ω) symp aspherical, then $HF_*(M,\omega,J,H) \simeq H_*(M;\mathbb{Z}_2)$

$\Rightarrow$ Arnold conj b/c

$$\#\mathrm{Fix}(\varphi_H^1) \geq \#\{\text{contractible closed orbits of } X_H\} \geq \text{total rank of } HF_*$$
$$= \sum b_i(M;\mathbb{Z}_2)$$

Rmk There are many other versions of Floer homologies, with generators admitting different alg/geo meanings.

Rmk Recall that in Morse theory, the key result is not the well-definedness of Morse homology $HM_*(X,f)$, instead it's the iso: $HM_*(X,f) \simeq H_*(X;\mathbb{Z}_2)$. //

Rmk Working over (M,ω,J) (instead of (M,J)) allows us to define/consider "metric" (on M) and "energy" of a J-hol curve.

2. Contact geometry

(X^{2n-1}, ξ) ξ a hyperplane field and called a (co-oriented) contact structure if $\exists\, \alpha \in \Omega^1(X)$ s.t.

$\operatorname{Ker}\alpha = \xi$ and $\alpha \wedge \underbrace{d\alpha \wedge \cdots d\alpha}_{n-1}$ is a volume form
α is called a contact form

Ex. $(\mathbb{R}^{2n-1}, \xi_{std})$ $\xi_{std} = \operatorname{Ker}\left(dz - \sum_{i=1}^{n-1} y_i dx_i\right)$
$x_1, y_1, \ldots, x_{n-1}, y_{n-1}, z$

Rmk All are encouraged to draw $(\mathbb{R}^3, \xi_{std})$ by hand.

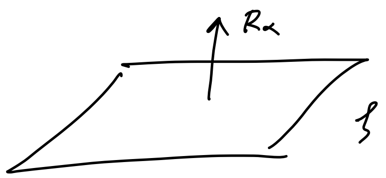
Rmk The data of contact forms is auxiliary. If α is a contact form of ξ, then $e^f \cdot \alpha$ is also a contact form of ξ.

Any contact mfd (X, ξ) admits a dyn system itself: fix α s.t. $\xi = \ker\alpha$, one can solve a vector field R_α from

$$d\alpha(R_\alpha, -) = 0 \quad \text{and} \quad \alpha(R_\alpha) = 1.$$

$d\alpha|_\xi$ is non-deg.

in a unique way. It is called the Reeb vector field of α.



$\varphi^t_{R_\alpha}$ the Reeb flow (w.r.t. α)

Again, one should be interested in closed orbits of $\varphi_{R_\alpha}^t$.

Conjecture (Weinstein) For any closed contact mfd (X,ξ), for any fixed α s.t $\xi = \ker\alpha$, there $\exists$ at least 1 closed orbit of $\varphi_{R_\alpha}^t$.

Rmk. Different from Arnold conj, we do not restrict to time-one.

Rmk. Conclusion is wrong if (X,ξ) is non-cpt: R_α in $(\mathbb{R}^3, \xi_{std})$ is ∂_z, so $\varphi_{R_\alpha}^t(z) = z+t$.

Rmk In dim 3, this has been proved by Taubes.
Other dims have individual results, but in general it is contraversial.

//

Ex. (M^4,ω) symp mfd. $H: M \to \mathbb{R}$ autonomous Ham fcn.

Consider the level set $H^{-1}(c) \subset M$, where c is regular.

- $H^{-1}(c)$ is a mfd of dim 3.
- flow lines of X_H stay inside $H^{-1}(c)$ $\leftarrow$ $dH(X_H) = -\omega(X_H, X_H) = 0$

Suppose near $H^{-1}(c)$, $\exists$ a vector field $Y \pitchfork H^{-1}(c)$ and $\underbrace{\mathcal{L}_Y\omega = \omega}_{(*)}$

$(*) \Longleftrightarrow d\underbrace{\iota_Y\omega}_{=:\lambda} + \cancel{\iota_Y d\omega}^{0} = \omega$

$\Longleftrightarrow \exists$ 1-form λ (near $H^{-1}(c)$) s.t. $\omega = d\lambda$.
called a Liouville form

Then one can check that $\left(H^{-1}(c), \underbrace{\xi = \ker(\lambda|_{H^{-1}(c)})}_{\text{dim 2 hyperplane field}}\right)$ is a contact mfd. Moreover,

$$R_\lambda = \frac{X_H}{\underbrace{dH(Y)}_{\text{non-zero b/c } Y \pitchfork H^{-1}(c)}}$$

Verify:
$$\begin{aligned}
\lambda(R_\lambda) &= (\iota_Y \omega)\left(\frac{X_H}{-dH(Y)}\right) \\
&= \omega\left(Y, \frac{X_H}{dH(Y)}\right) \\
&= -\frac{1}{dH(Y)}\,\omega(X_H, Y) = 1.
\end{aligned}$$

$$\iota_{X_H} \underbrace{d\lambda}_{\lambda|_{H^{-1}(c)}} = \iota_{X_H}\, \omega|_{H^{-1}(c)} = \omega|_{H^{-1}(c)}(X_H, -) = -dH(\underbrace{-}_{\text{input from } H^{-1}(c)}) = 0$$

Then Taubes's result implies that $\exists$ a closed characteristic (closed orbit) of the Ham flow X_H (not nec at $t=1$) on $H^{-1}(c)$!

<u>Rmk</u> Weinstein, Rabinowitz proved this $\exists$-result in 1978. for any dim. (Of course, not every cpt contact str can be viewed as the level set of a Ham system).

Example above gives a hint that symp geo is related with contact geo in a natural way.

Def A Liouville domain is a symp mfd with boundary (W, ω) s.t. $\exists\, Y$ a v.f. on W, $\pitchfork \partial W$, and $\underline{\mathcal{L}_Y \omega = \omega}$.

ω exponentially grows or shrinks.

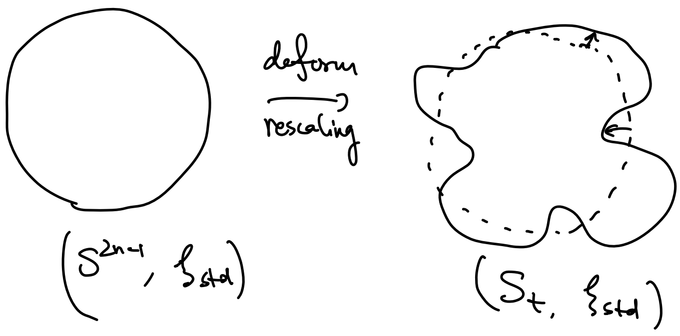
By discussion above, $(\partial W,\ \xi = \ker(\iota_Y \omega)|_{\partial W})$ is a contact mfd.

Ex. $(W, \omega) = (\{\sum x_i^2 + y_i^2 \leq 1\},\ \omega_{std}|_{\{\dots\}})$ is a Liouville domain
$\cap$
$(\mathbb{R}^{2n}, \omega_{std})$

and $Y = \frac{1}{2} \sum_{i=1}^{n} (x_i \partial x_i + y_i \partial y_i)$ radial vector field.

$\Rightarrow (\partial W, \xi) = (S^{2n-1}, \xi_{std})$ contact mfd with the standard contact str.

observation: $\lambda = \iota_Y \omega$ is fixed but $\lambda|_{\partial W}$ varies, depending on ∂W.



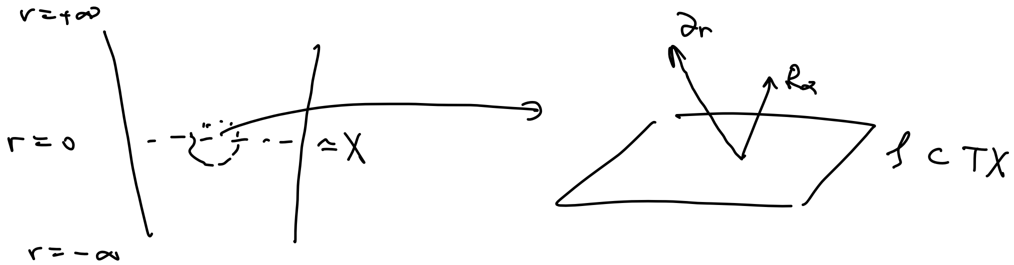
One can also fix S^{2n-1} but deform ξ_{std} to ξ_t (through contact str), via deforming Y, then Gray's stability thm (Thm1.6 in [Wen]) shows that $\exists\, \varphi_t : S^{2n-1} \circlearrowleft$ s.t. $(\varphi_t)_* \xi_t = \xi_{std}$.

contactomorphism

//

From contact mfd (X, ξ), one can build a symp mfd via "symplectization": fix α s.t $\xi = \ker \alpha$, then

$$M := \underset{r}{\mathbb{R}} \times X \quad \text{and} \quad \omega = d(e^r \alpha) \qquad e^r dr \wedge \alpha$$



<u>Rmk</u> Sometimes papers use another convention $M = [0, \infty) \times X$ and $\omega = dr \wedge \alpha$ and $X \simeq \{1\} \times X$.

How does a J-hol curve look like in a symplectization?

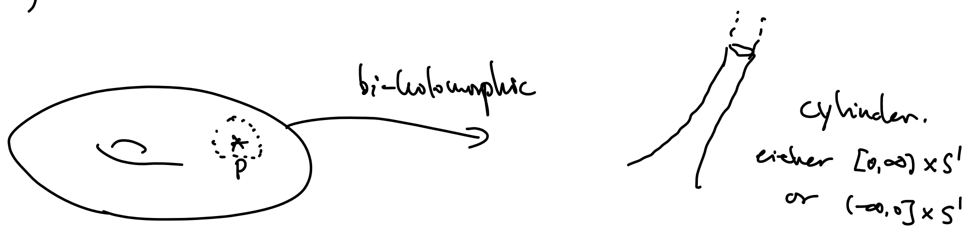
Pick J on $\mathbb{R} \times X$ s.t.

- J is invariant under translation in r, i.e. $J_{(a,x)} = J_{(a+r,x)}$
- $J(\partial r) = R_\alpha$ and $J(R_\alpha) = -\partial r$
- Restricted at ξ, $\omega(\cdot, J\cdot)$ is a metric (on bundle $\begin{smallmatrix}\xi\\ \downarrow\\ X\end{smallmatrix}$)

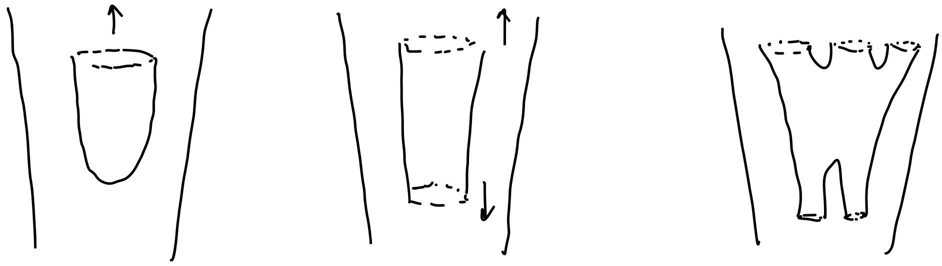
<u>Prop</u> $u: (\Sigma, j) \to (\mathbb{R} \times X, \omega)$ by $(u_{\mathbb{R}}, u_X)$. if u is J-hol, then $u_{\mathbb{R}}$ is a subharmonic function $(\Delta u_{\mathbb{R}} \geq 0)$

$\Rightarrow$ Σ can not be compact.

Modify Σ to be punctured $\dot{\Sigma} = \Sigma \setminus \{p_1, \cdots, p_n\}$.



<u>Ex</u>



$\Sigma \setminus \{p\} \simeq \mathbb{D}^2$ $\quad$ $\Sigma \setminus \{p,q\} \simeq$ cylinder $\quad$ $\Sigma \setminus \{p_1, \cdots, p_5\}$

$\lim_{s \to \infty} u(s,t) = \gamma(t)$ positive (asymptotic) end

$\lim_{s \to -\infty} u(s,t) = \gamma(t)$ negative (asymptotic) end

<u>Thm</u>. Under a finite energy condition, $\gamma(t)$ is a closed Reeb orbit.
(later)

<u>Rmk</u> Recall that in (Ham) Floer homology, ∂ is also involving the study of asymptotic behavior when $s \to \pm\infty$.

<u>Rmk</u> Similarly to symp geo situation, studying the moduli space of punctured $u: (\dot{\Sigma}, j) \to (\mathbb{R} \times M, J)$ is also crucial.

need to add asymp. cond.

3. Symplectic embedding

Given two Liouville domain (U, ω_u) and (V, ω_v), a symp emb is an emb $\varphi : U \hookrightarrow V$ s.t. $\varphi^* \omega_v = \omega_u$.

Ex. $(\mathbb{R}^{2n}, \omega_{std})$ $(\simeq (\mathbb{C}^n, \omega_{std}))$

$$E(a_1, \dots, a_n) = \left\{ (z_1, \dots, z_n) \in \mathbb{C}^n \;\middle|\; \frac{\pi |z_1|^2}{a_1} + \dots + \frac{\pi |z_n|^2}{a_n} \leq 1 \right\}$$ ← symplectic ellipsoid

$$B^{2n}(r) = E(r, \dots, r) , \quad Z^{2n}(R) = E(R, \underbrace{\infty, \dots, \infty}_{\text{no constraint on } z_2, \dots, z_n})$$

$B^{2n}(r)$: symplectic ball; $Z^{2n}(R)$: symplectic cylinder.

$$P(a_1, \dots, a_n) = \left\{ (z_1, \dots, z_n) \in \mathbb{C}^n \;\middle|\; \pi |z_1|^2 \leq a_1, \dots, \pi |z_n|^2 \leq a_n \right\}$$ ← polydisk

Rmk. $\partial P(a_1, \dots, a_n)$ is not smooth.

Rmk All cases above admit $\mathbb{T}^n$-action by $(\theta_1, \dots, \theta_n) \cdot (z_1, \dots, z_n)$ defined by $(e^{i\theta_1} z_1, \dots, e^{i\theta_n} z_n)$. $\Longrightarrow$ toric domain

Finding obstructions of embedding is a central topic in symp. geo.