1. Local properties of J-hol curve

Slogan: J-hol curve $\overset{\text{locally}}{\simeq}$ holomorphic curve

Recall for a J-hol curve u, locally $u: D \to \mathbb{R}^{2n}$ satisfies

open subset containing 0 in $\mathbb{C}$

$$\frac{\partial u}{\partial s} + J(u(z)) \cdot \frac{\partial u}{\partial t} = 0$$

Let's consider a more general equ (for $u: D \to \mathbb{R}^{2n}$)

$$\frac{\partial u}{\partial s} + J(z) \frac{\partial u}{\partial t} + C(z)\, u(z) = 0 \qquad (*)$$

where $J(z): D \to \mathrm{End}(\mathbb{R}^{2n})$ (s.t. $J^2(z) = -\mathbb{1}$) and $C(z): D \to \mathrm{End}(\mathbb{R}^{2n})$, both varying smoothly for $z \in D$.

__Rmk__ A J-hol curve is a special case of $(*)$ by choosing $C \equiv 0$ and $J(z) = J \circ u(z)$.

Here we pre-assuming u satisfies certain regularity so that u is in fact smooth (see the end of SFT-3).

__Rmk__ A holomorphic curve is usually specifying $J(z) = J_0$ constant and the standard one.

To relate $J(z)$ with J_0, let's start from the following lemma.

__Lemma__ Given $J(z): D \to \mathrm{End}(\mathbb{R}^{2n})$, a family of a.c.s on $\mathbb{R}^{2n}$, there exists $\Phi: D'(\subset D) \to GL(2n, \mathbb{R})$ s.t. $\Phi(z)^{-1} \cdot J(z) \cdot \Phi(z) = J_0$.

__Pf.__ Consider map $GL(2n, \mathbb{R}) \xrightarrow{f} \{ J \in GL(2n, \mathbb{R}) \mid J^2 = -\mathbb{1} \}$

by $\Phi \mapsto \Phi \cdot J_0 \cdot \Phi^{-1}$. (then $(\Phi \cdot J_0 \cdot \Phi^{-1})^2 = -\mathbb{1}$)

Then one can check that $J(0)$ is a regular point of f, ♡ so the implicit function thm solve $f(\Phi(z)) = J(z)$ when z is sufficiently close to 0 (so $J(z)$ is sufficiently close to $J(0)$).

In particular, we know $\exists\, D' \subset D$ and $\Phi : D' \to GL(2n, \mathbb{R})$ s.t

$$\Phi(z) \cdot J_0 \cdot \Phi(z)^{-1} = J(z) \iff \Phi(z)^{-1} \cdot J(z) \cdot \Phi(z) = J_0$$

To check ♡ above, for $A \in M_{2n \times 2n}(\mathbb{R})$ $(\cong T_\Phi GL(2n, \mathbb{R})$ where $\Phi J_0 \Phi^{-1} = J(0))$

when t is small, it is invertible.

$$\begin{aligned}
df\big|_\Phi (A) &= \lim_{t \to 0} \frac{f(\Phi + tA) - f(\Phi)}{t} \\
&= \lim_{t \to 0} \frac{(\Phi + tA) \cdot J_0 (\Phi + tA)^{-1} - \Phi \cdot J_0 \cdot \Phi^{-1}}{t} \\
&= \lim_{t \to 0} \frac{(\Phi + tA) \cdot J_0 \left((\mathbb{1} + t\Phi^{-1}A)^{-1} \Phi^{-1} \right) - \Phi J_0 \Phi^{-1}}{t} \\
&= \lim_{t \to 0} \frac{(\Phi + tA) J_0 \left(\Phi^{-1} - t\Phi^{-1} A \Phi^{-1} + o(t) \right) - \Phi J_0 \Phi^{-1}}{t} \\
&= \lim_{t \to 0} \frac{t\left(A J_0 \Phi^{-1} - \Phi J_0 \Phi^{-1} A \Phi^{-1} \right) + o(t)}{t}
\end{aligned}$$

b/c $\Phi J_0 \Phi^{-1} = J(0)$

$$= A J_0 \Phi^{-1} - J(0) A \Phi^{-1}$$

b/c $J_0 \Phi^{-1} = \Phi^{-1} J(0)$

$$= A \Phi^{-1} J(0) - J(0) A \Phi^{-1}$$

$(J(0) + tB)^2 = -\mathbb{1} + t(J(0)B + BJ(0)) + \ldots$

On the other hand, at $J(0)$, the tangent space $T_{J(0)} \{ J^2 = -\mathbb{1} \}$ consists of all matrices $B \in M_{2n \times 2n}(\mathbb{R})$ s.t. $BJ(0) + J(0)B = 0$.

Note that $\left(A\Phi^{-1}J(0) - J(0)A\Phi^{-1}\right)J(0) + J(0)\left(A\Phi^{-1}J(0) - J(0)A\Phi^{-1}\right)$

$$= -A\Phi^{-1} - J(0)A\Phi^{-1}J(0) + J(0)A\Phi^{-1}J(0) + A\Phi^{-1} = 0 \checkmark$$

and for any such B (satisfying $BJ(0)+J(0)B=0$), set

$$A := \frac{J(0)}{2} B \Phi .$$

$\left(\Rightarrow A\Phi^{-1}J(0) - J(0)A\Phi^{-1} = \frac{1}{2}\left(J(0)BJ(0) + B\right) = \frac{1}{2}(2B) = B \checkmark\right)$. $\square$

Therefore, if u satisfies $(*)$ above, then set $V(z) := \Phi^{-1}(z) \cdot u(z)$ for $\Phi(z) : D' \to GL(2n, \mathbb{R})$ from Lemma above,

$$0 = \frac{\partial u}{\partial s} + J(z)\frac{\partial u}{\partial t} + C(z)u(z)$$

$$= \frac{\partial(\Phi \cdot V)}{\partial s} + J(z)\frac{\partial(\Phi V)}{\partial t} + C(z)(\Phi \cdot V)(z)$$

$$= \Phi \cdot \frac{\partial V}{\partial s} + J(z)\Phi \cdot \frac{\partial V}{\partial t} + \frac{\partial \Phi}{\partial s} \cdot V + J(z)\frac{\partial \Phi}{\partial t} \cdot V + C(z) \cdot \Phi(z) \cdot V(z)$$

$$= \Phi\left(\frac{\partial V}{\partial s} + \underbrace{\Phi^{-1}J\Phi}_{\text{by lemma } = J_0} \cdot \frac{\partial V}{\partial t}\right) + \left(\frac{\partial \Phi}{\partial s} + J(z)\frac{\partial \Phi}{\partial t} + C(z) \cdot \Phi(z)\right)V(z)$$

$$= \Phi\left(\frac{\partial V}{\partial s} + J_0 \frac{\partial V}{\partial t}\right) + (\cdots)V(z)$$

Set $B(z) := \Phi^{-1}\underset{\left(\frac{\partial \Phi}{\partial s} + J(z)\frac{\partial \Phi}{\partial t} + C(z)\Phi(z)\right)V(z)}{(\cdots)}$, then we get the following pre-Carleman Similarity principle:

Prop If $u : D \xrightarrow{C^\infty} \mathbb{R}^{2n}$ satisfies $(*)$ (or u is J-hol), then $\exists\, 0 \in D' \subset D$ and $\Phi : D' \to GL(2n, \mathbb{R})$ s.t. for $V = \Phi^{-1} \cdot u$, we have

$$\frac{\partial V}{\partial s} + \underset{\text{constant}}{J_0} \frac{\partial V}{\partial t} + \underset{\text{related with derivatives of } \Phi}{B(z)} \cdot V = 0 \qquad (**)$$

We actually want more: could v be even holomorphic?

with a fresh start from (**), D' relabelled by D, v relabelled by u (i.e. no extra term about B).

<u>Goal</u>: $\exists_{0 \in} D' \subset D$ and $\Phi : D' \to \underbrace{GL^{hol}}_{\text{hol part}}(2n, \mathbb{R})$ s.t.

hol part $= \{ B \in GL(2n,\mathbb{R}) \mid BJ_0 = J_0 B \} =: \mathbb{C}^{n\times n}$

$v := \Phi \cdot u$ and v is holomorphic.

An easy observation: for u satisfying (**)

$$
\begin{aligned}
\frac{\partial v}{\partial s} + J_0 \frac{\partial v}{\partial t} &= \frac{\partial(\Phi \cdot u)}{\partial s} + J_0 \frac{\partial(\Phi\cdot u)}{\partial t} \\
\Phi J_0 = J_0 \Phi \longrightarrow &= \left(\frac{\partial \Phi}{\partial s} + J_0 \frac{\partial \Phi}{\partial t}\right) u + \Phi\left(\frac{\partial u}{\partial s} + J_0 \frac{\partial u}{\partial t}\right) \\
&= (\partial_{\bar z} \Phi) u - \Phi(z) \cdot B(z) \cdot u
\end{aligned}
$$

recall $\partial_{\bar z} = \frac{1}{2}\left(\frac{\partial}{\partial s} + J_0 \frac{\partial}{\partial t}\right)$

Therefore, if one can solve $\partial_{\bar z}\Phi = \Phi(z) \cdot B(z)$ (for Φ, given B), then v is holomorphic. In fact, we have the following result

<u>Thm</u> $B \in L^\infty(D; \mathbb{C}^{n\times n})$, then $\exists_{0\in} D' \subset D$ and $\Phi : D' \to \mathbb{C}^{n\times n}$ s.t.

$$\partial_{\bar z}\Phi = \Phi \cdot B$$ ← in a similar way solved via the operator T introduced in SFT-3.

Moreover, $\forall\, p < \infty$, $\Phi \in W^{1,p}(D', \mathbb{C}^{n\times n})$ and $\Phi(z)$ is invertible for every $z \in D'$.

<u>Rmk</u> (Exe) Here is an implicit step, when reaching (**) in Prop above, one can furthermore upgrade it to replace $B : D' \to GL(2n,\mathbb{R})$ to $B \in L^\infty(D; \mathbb{C}^{n\times n})$.

Carleman Similarity principle $\Longrightarrow$ If u satisfies (*), then $\exists\ \Phi : \overset{(0\in)}{D'} \subset D \to GL(2n,\mathbb{R})$ (of class $W^{1,p}$, $p<\infty$) s.t. Φu is holomorphic.

Here is a useful corollary of Carleman similarity principle.

Recall any holomorphic map $u: D \to \mathbb{C}^n$ admit local Taylor expansion near pt $0 \in D$.

$$u(z) = a_0 + a_1 z + \frac{a_2^2}{2!} z^2 + \cdots \qquad \text{(only involving power of } z\text{)}$$

Then if $\lim_{|z| \to 0} \frac{|u(z)|}{|z|^k} = 0$ for every $k \in \mathbb{N}_{\geq 0}$ then $u \equiv 0$ near 0.

$\Rightarrow$ for two holomorphic fcns $u_0, u_1 : D \to \mathbb{C}^n$,

$$\lim_{|z| \to 0} \frac{|u_0(z) - u_1(z)|}{|z - z_0|^k} = 0 \text{ for every } k \in \mathbb{N}_{\geq 0} \Rightarrow u_0 = u_1 \text{ near } z_0.$$

at z_0, u_0 and u_1 "agree to infinite order".

Prop (unique continuation) $u_0, u_1 : (\Sigma, j) \to (M, J)$ J-hol curve that agree to infinite order at some pt $z_0 \in \Sigma$, then $u_0 \equiv u_1$.

connected

← Another way to express this, either u has isolated zero or vanishes identically.

Pf $S = \{ z \in \Sigma \mid u_0 \equiv u_1 \text{ to infinite order} \}$ is closed obviously and non-empty. $(z_0 \in S)$

Near z_0, in local chart D we have for $i = 0, 1$,

$$\frac{\partial u_i}{\partial s} + J(u_i(z)) \frac{\partial u_i}{\partial t} + \underline{\underline{0}} = 0$$

Then consider $w = u_1 - u_0$, then

$$\frac{\partial w}{\partial s} + J(u_1(z)) \frac{\partial u_1}{\partial t} - J(u_0(z)) \frac{\partial u_0}{\partial t} = 0$$

$$\Leftrightarrow \left(\frac{\partial w}{\partial s} + J(u_1(z)) \frac{\partial w}{\partial t} \right) + \underbrace{\left(J(u_1(z)) - J(u_0(z)) \right) \frac{\partial u_0}{\partial t}}_{(*)} = 0$$