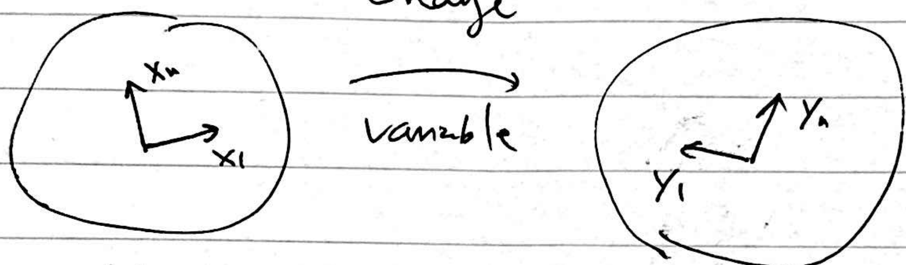
2. Definition of integration

Model: $\displaystyle\int_{\underset{\mathbb{R}^n}{\overset{\cap}{V_\alpha}}} f^\alpha\, dx_1\wedge\cdots\wedge dx_n$

Recall change of variable formula of integration in calculus:



$V_x \subset \mathbb{R}^n$
domain for coordinate
$(x_1, \cdots, x_n)$

$V_y \subset \mathbb{R}^n$
domain for coordinate
$(y_1, \cdots, y_n)$

$$y_i = y_i(x_1, \cdots, x_n) \xRightarrow{\text{denote}} y = \varphi(x)$$

n variable
n output

$$\int_{V_y} f(y)\, dy_1 \cdots dy_n = \int_{V_x} f(\varphi(x))\, |\det(J(\varphi))|\, dx_1 \cdots dx_n$$

absolute value !

As a comparison, view $dy_1 \cdots dy_n = dy_1 \wedge \cdots \wedge dy_n$

then both $dx_1 \wedge \cdots \wedge dx_n,\ dy_1 \wedge \cdots \wedge dy_n \in \Omega^n(\mathbb{R}^n)$

and $\dim \Omega^n(\mathbb{R}^n) = 1$, so

$$dy_1 \wedge \cdots \wedge dy_n = \det(J(\varphi))\, dx_1 \wedge \cdots \wedge dx_n$$

no absolute value

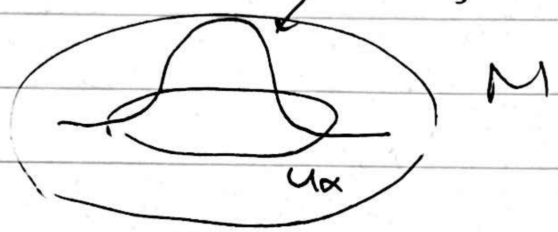
To resolve the issue of taking 1-1, we will always assume M is <u>orientable</u> when taking / computing integration.

(Recall M is orientable if $\exists$ open cover $\{U_\alpha\}_\alpha$ and local chart $U_\alpha \xrightarrow{\varphi_\alpha} V_\alpha \subset \mathbb{R}^n$ s.t. $\det(J(\varphi_{\alpha\beta})) > 0$. for any α, β)

The definition of an integration of n-form θ on an n-dim'l mfd is produced by the following three steps:

① Assume θ is cpt supp in a local chart U_α.

$\theta = f^\alpha dx_1 \wedge \cdots \wedge dx_n$ over $\varphi_\alpha(U_\alpha) = V_\alpha \subset \mathbb{R}^n$

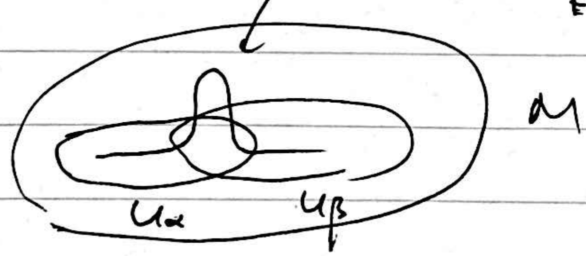


better to write $dx_1 \cdots dx_n$

<u>Defini</u> $\int_M \theta := \int_{V_\alpha} f^\alpha \, dx_1 \wedge \cdots \wedge dx_n$

Verify that this is well-defined.

$\theta \subset\subset U_\alpha \cap U_\beta$



$U_\alpha \xrightarrow{\varphi_\alpha} V_\alpha \quad (x_1, \cdots, x_n)$

$U_\beta \xrightarrow{\varphi_\beta} V_\beta \quad (y_1, \cdots, y_n)$

$\theta = f^\beta dy_1 \wedge \cdots \wedge dy_n$

Observe that $f^\beta dy_1 \wedge \cdots \wedge dy_n \; (= f^\alpha dx_1 \wedge \cdots \wedge dx_n)$

$$= f^\beta \det(J(\varphi_{\alpha\beta})) \, dx_1 \wedge \cdots \wedge dx_n$$

$$\Rightarrow f^\alpha \left(\det(J(\varphi_{\alpha\beta}))^{-1}\right) = f^\beta.$$

This is a number pointwise

Now, viewed from U_β, by def.

$$\int_M \theta = \int_{V_\beta} f^\beta \, dy_1 \wedge \cdots \wedge dy_n$$

change of variable formula

$$= \int_{V_\alpha} f^\alpha \left(\det J(\varphi_{\alpha\beta})^{-1}\right) \det(J(\varphi_{\alpha\beta}))\, dx_1 \wedge \cdots \wedge dx_n$$

$$= \int_{V_\alpha} f^\alpha \, dx_1 \wedge \cdots \wedge dx_n$$

✓

② Assume $\theta = \sum_{i=1}^{k} \theta_i$ where all θ_i are cpt supp in a local chart U_α.

Define $\int_M \theta := \sum_{i=1}^{k} \int_M \theta_i$

This is natural since $\theta = \sum f_i^\alpha \, dx_1 \wedge \cdots \wedge dx_n$

③ Assume θ is cptly supp in M

(if M is already cpt, then any θ is cpt supp automatically).

Take a finite open cover $\{U_\alpha\}_{\alpha=1}^{k}$ of $\mathrm{supp}(\theta)$.

Consider any p.o.u for $\{U_\alpha\}_{\alpha=1}^{k}$, denoted by $\{\rho_\alpha\}_{\alpha=1}^{k}$. Then

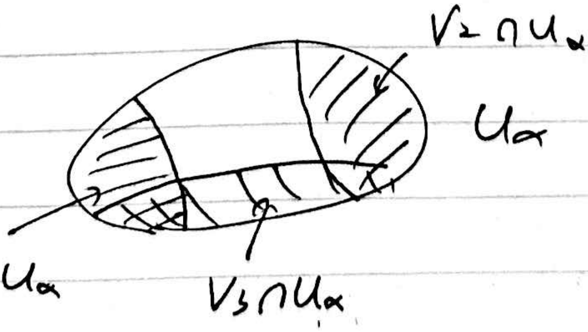
$$\rho_\alpha \cdot \Theta \subset\subset U_\alpha$$

<u>Define</u> $$\int_M \Theta := \sum_{\alpha=1}^{k} \underbrace{\int_M}_{\text{This is well-defined by Step ①.}} \rho_\alpha \cdot \Theta \qquad (*)$$

Verify that $(*)$ is independent of the choice of open cover of $\mathrm{Supp}(\Theta)$ and p.o.u.

Suppose $\{V_\beta\}_{\beta=1}^{k'}$ is another open cover of $\mathrm{Supp}(\Theta)$ and $\{\rho_\beta\}_{\beta=1}^{k'}$ is a p.o.u for $\{V_\beta\}_{\beta=1}^{k'}$. Then

- $\{U_\alpha \cap V_\beta\}_{\substack{1\le\alpha\le k \\ 1\le\beta\le k'}}$ is an open cover of $\mathrm{Supp}(\Theta)$



In particular, by Step ②, for each α,

$$\rho_\alpha \cdot \theta = \sum_{\beta=1}^{k'} \rho_\beta \cdot \rho_\alpha \cdot \theta \Rightarrow \int_M \rho_\alpha \theta = \sum_{\beta=1}^{k'} \int_M \rho_\beta \rho_\alpha \cdot \theta$$

Also $\{\rho_\beta \cdot \rho_\alpha\}_{\alpha,\beta}$ is a P.O.U of $\{U_\alpha \cap V_\beta\}_{\alpha,\beta}$.

Then

$$\sum_{\alpha=1}^{k} \int_M \rho_\alpha \cdot \theta = \sum_{\alpha=1}^{k} \sum_{\beta=1}^{k'} \int_M \rho_\beta \cdot \rho_\alpha \cdot \theta$$

$$= \sum_{\beta=1}^{k'} \left(\sum_{\alpha=1}^{k} \int_M \rho_\beta \cdot \rho_\alpha \cdot \theta \right)$$

$$= \sum_{\beta=1}^{k'} \int_M \rho_\beta \cdot \theta \qquad \checkmark$$

Basic properties

- $\lambda \int_M \theta + \eta \int_M \sigma = \int_M \lambda \theta + \eta \sigma$ for $\lambda, \eta \in \mathbb{R}$, $\theta, \sigma \in \Omega^n(M)$ cpt supp.

- If $\varphi: M \to N$ is an orientable preserving diffeo between two orientable mfds, then

$$\int_M \varphi^* \theta = \int_N \theta \quad \text{for any } \theta \in \Omega^n(N)$$
cpt supp
(and dim $N = n$)

<u>Prop</u> If M^n is a cpt orientable mfd <u>without</u> boundary, then for any $(n-1)$-degree form θ, we have

$$\int_M d\theta = 0$$

<u>Rmk</u> This is a special case of Stokes' Thm.

<u>Pf</u> By a P.O.U, we only focus on a $(n-1)$-form θ locally and cpt supp in a local chart,

$$\theta = \sum_{i=1}^n f^i \, dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_n$$

Then

$$d\theta = \left(\sum_{i=1}^n \pm \frac{\partial f^i}{\partial x_i} \right) dx_1 \wedge \cdots \wedge dx_n .$$

One can extend (by zero) θ and $d\theta$ to be over $\mathbb{R}^n$

Then

$$\int_M d\alpha = \int_{\mathbb{R}^n} \sum_{i=1}^{n} \pm \frac{\partial f^i}{\partial x_i} dx_1 \cdots dx_n$$

$$= \sum_{i=1}^{n} \pm \int_{\mathbb{R}^n} \frac{\partial f^i}{\partial x_i} dx_1 \cdots dx_n$$

A student pointed out to me that in the expression of multi-variable integration, switching the order of dx_i's won't affect the sign. So there is still a \pm sign at the third equation (and also the fourth equation). However, since f^i vanished at infinity, there is no effect to the final answer in this verification.

$$= \sum_{i=1}^{n} \pm \int_{\mathbb{R}^{n-1}} \left(\int_{\mathbb{R}} \frac{\partial f^i}{\partial x_i} dx_i \right) dx_1 \cdots \widehat{dx_i} \cdots dx_n$$

$$= \sum_{i=1}^{n} \pm \int_{\mathbb{R}^{n-1}} \left(f^i \Big|_{x_i = -\infty}^{x_i = +\infty} \right) dx_1 \cdots \widehat{dx_i} \cdots dx_n$$

$$= 0$$

<u>Rmk</u> It could be that for some mfd, x_i can not reach to $\pm\infty$, so f^i may not be zero.

This leads to a new concept — mfd with boundary.

<u>Rmk</u> Definition of $\int_M \alpha$ is <u>not</u> computable!

We will see a different expression of $\int_M \alpha$ for the purpose of computation.