$$i : (x_1, \cdots, x_k) \longrightarrow (x_1, \cdots, x_k, 0, \cdots, 0)$$

So $\quad di : TN \longrightarrow TM \qquad p \to p \quad$ and $\quad v \mapsto \underset{m\times k}{Jac(i)} \cdot v$

$$= \begin{pmatrix} \boxed{\begin{matrix} 1 & & \\ & \ddots & \\ & & 1 \end{matrix}}_{k\times k} & 0 \\ 0 & 0 \end{pmatrix} v$$

2. Rank

An important observation:

$Jac^{\alpha}(F)$ and $Jac^{\beta}(F)$ in general are not the same.

$$Jac^{\beta}(F)_{m\times n} = \underbrace{\begin{pmatrix} & \\ & \end{pmatrix}_{m\times m}}_{\text{invertable}} \cdot Jac^{\alpha}(F)_{m\times n} \cdot \underbrace{\begin{pmatrix} & \\ & \end{pmatrix}_{n\times n}}_{\text{invertible}}$$

BUT

$\Longrightarrow$ rank $Jac^{\alpha}(F)$ = rank $Jac^{\beta}(F)$ $\quad \forall \alpha, \beta$

$\Longrightarrow$ rank of F is a globally defined value (over domain)

e.g. In examples above, rank $(F : M \to \mathbb{R}) = 1$ or 0. rank $(i : N \hookrightarrow M) = k$.

This e.g. right above induces the following two important definitions:

<u>Def</u> For $F: N^n \to M^m$, a pt $p \in N$ is called a <u>regular pt</u> of F if $\operatorname{rank}(F)(p) = m$ (= dim of target). ← mostly apply to the case $n \geq m$.

e.g. For those $p \in N$ s.t. $\operatorname{rank}(F)(p) < m$, they are called <u>critical pts</u> of F

<u>Rmk</u>. Crit value $:= \{ F(p) \in M \mid p \text{ is a critical pt of } F \}$

Regular value $:= M \setminus$ Crit value.

$\Rightarrow$ (i) Any pt in $M \setminus \operatorname{im}(F)$ is also called a regular value.

(ii) $p \in M$ is a regular value iff $\forall q \in F^{-1}(\{p\})$, it is a regular pt.

Recall (in the Introduction class), the following prop is handy to construct mfds.

Prop*: For $F: N^n \to M^m$ and any regular value $p \in M$, if $F^{-1}(\{p\})$ is not empty, then it is a (reg. or emb.) submfd in N^n with dim $n-m$.

also compact.

eg. Consider $F: \mathbb{R}^3 \to \mathbb{R}^2$ defined by $(x,y,z) \mapsto (x^3+y^3+z^3, x+y+z)$

Verify that $(1,0)$ is a regular value of F.

$$F = (F_1, F_2) \text{ where } F_1 = x^3+y^3+z^3,\ F_2 = x+y+z$$

$$\mathrm{Jac}(F) = \begin{pmatrix} 3x^2 & 3y^2 & 3z^2 \\ 1 & 1 & 1 \end{pmatrix}_{2\times 3} \Bigg|$$

evaluated at pts satisfying
$x^3+y^3+z^3=1$
$x+y+z=0$

which is always full rank $=2$.

Therefore, $F^{-1}((1,0)) = \left\{ (x,y,z) \in \mathbb{R}^3 \;\middle|\; \begin{array}{l} x^3+y^3+z^3=1 \\ x+y+z=0 \end{array} \right\}$

is a submfd of $\mathbb{R}^3$ of dim $= 3-2 = 1$

e.g. Consider $F: \mathbb{R}^3 \to \mathbb{R}$ defined by

$$(x,y,z) \longmapsto \Big(y^2 + \underbrace{x(x-1)^2(x-2)^2 \cdots (x-g+1)^2(x-g)}_{f_g(x)}\Big)^2 + z^2$$

for a fixed $g \in \mathbb{N}_{\geq 2}$.

Then
$$\frac{\partial F}{\partial x} = 2(y^2 + f_g(x))\, f_g'(x)$$
$$\frac{\partial F}{\partial y} = 2(y^2 + f_g(x)) \cdot 2y$$
$$\frac{\partial F}{\partial z} = 2z$$

For $p = (x,y,z) \in F^{-1}(\varepsilon)$ when $\varepsilon > 0$ and sufficiently small,

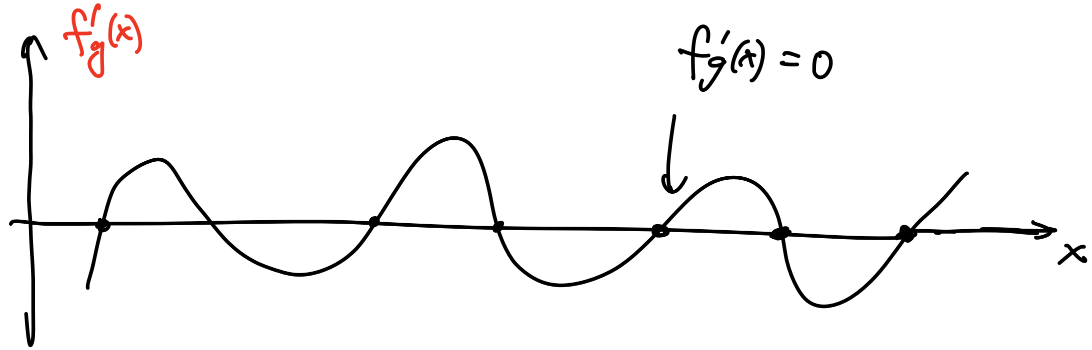
$y^2 + f_g(x) \geq 0$ and small or $z \geq 0$ and small.

- If $z \neq 0$, then $\left(\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z}\right)(p) \neq 0$. ✓

- If $z = 0$, then $y^2 + f_g(x) > 0$.

Suppose $\left(\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z}\right)(p) = 0$, then $y=0$ and $f_g'(x) = 0$

Note that $z = y = 0$ implies $(f_g(x))^2 = \varepsilon$



There are only <u>finitely many</u> zero's of $f_g'(x)$ (b/c $f_g(x)$ is a polynomial).

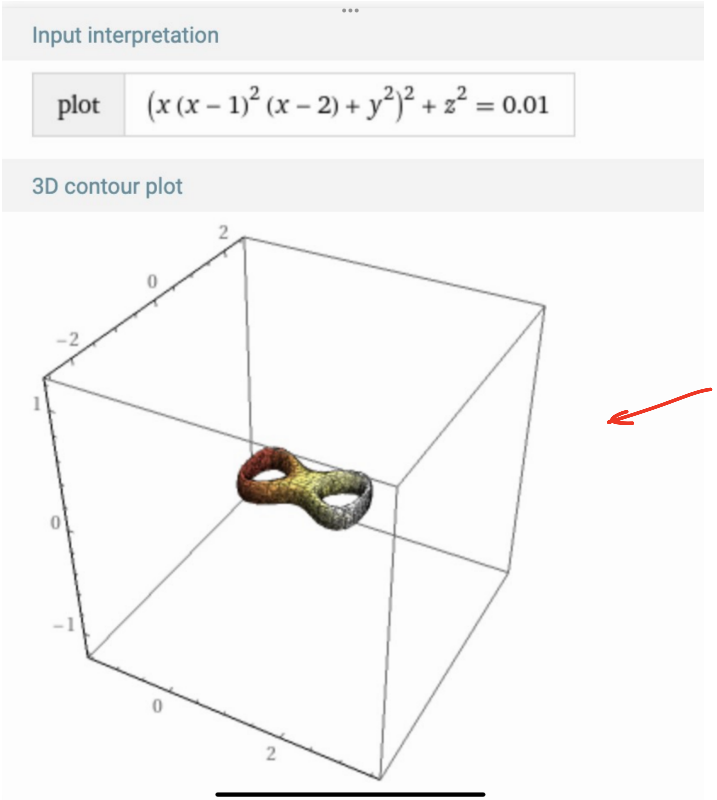
Set $\varepsilon < \min \{ f_g^2(x_0) \mid x_0 \text{ is a zero of } f_g'(x) \}$

Then $\{x \in \mathbb{R} \mid f_g'(x) = 0\} \cap \{x \in \mathbb{R} \mid f_g^2(x) = \varepsilon\} = \emptyset$.

How does $F^{-1}(\{\varepsilon\})$ look like?

e.g. $g=2$ $\quad F^{-1}(\{\varepsilon\}) = \{ (x(x-1)^2(x-2)+y^2)^2 + z^2 = \varepsilon \}$

$\varepsilon = 0.01$



A closed surface (as a submfd in $\mathbb{R}^3$) with genus $g=2$.

In general, $F^{-1}(\{\varepsilon\})$ is a closed surface of genus g.

Rmk: There are many examples showing F^{-1}(crit value) is not a submfd.

Def. $F: N^n \to M^m$ is called an <u>immersion</u> (浸入) iff

$\mathrm{rank}(F)(p) = n$ for <u>any</u> $p \in N$. ← the same way to define "immersion at pt p"

(A dual def: F is called <u>submersion</u> (淹没) iff

$\mathrm{rank}(F)(p) = m$ for <u>any</u> $p \in N$ $\Longrightarrow$ any $p \in N$ is a regular pt
↑ the same way to define "submersion at pt p" (regular pt)
any $q \in M$ is a regular value.)

e.g. $F: \mathbb{R} \to \mathbb{R}$ by $x \to x^3$ is (surprisely) <u>not</u> immersion.

$dF(x) = 3x^2$ and $\mathrm{rank}(F)(0) = 0$ $(\neq 1)$.

(but F is an injective map)

Rmk. This example also shows F is <u>not</u> submersion, but F is a surjective map (point $x=0$ is a crit pt of F).

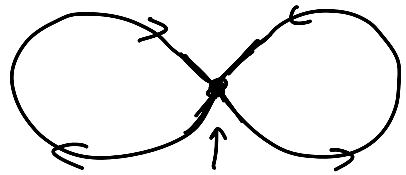
<u>Question</u>: relation between F and dF

F inj $\not\Rightarrow$ F is immersion

F surj $\not\Rightarrow$ F is submersion

} example above

F immersion $\not\Rightarrow$ F is injective



double pt

F submersion $\not\Rightarrow$ F is surjective

$F: \mathbb{R} \to \mathbb{R} \quad x \mapsto \arctan(x)$

$\left(dF(x) = \frac{1}{1+x^2} \neq 0\right)$

This shows that in general F and dF are relatively <u>independent</u> to each other.

BUT: some magic things happen . . .

3. Local behavior (F vs. dF)

特别牛B的定理！

Thm (Constant Rank Thm)

Let $F: N^n \to M^m$ be a smooth map s.t. near $p \in N$, $\operatorname{rank}(F) = r$ (constant). Then $\exists$ local charts $(U, \varphi: U \to V \subset \mathbb{R}^n)$ near p and $(W, \psi: W \to Z \subset \mathbb{R}^m)$ near $F(p)$ s.t.

$$\psi \circ F \circ \varphi^{-1}: \begin{array}{ccc} V & \to & Z \\ \cap & & \cap \\ \mathbb{R}^n & & \mathbb{R}^m \end{array} \quad \text{in the form of} \quad \underbrace{(x_1, \cdots, x_n) \longmapsto (x_1, \cdots, x_r, 0, \cdots, 0)}_{\text{projection to the first } r \text{ coordinates.}}$$

Cor: If $F: N^n \to M^m$ is an immersion, then locally F is

$$(x_1, \cdots, x_n) \longmapsto (x_1, \cdots, x_n, 0, \cdots, 0)$$

Cor. If $F: N^n \to M^m$ is an submersion, then locally F is

$$(x_1, \cdots, x_m, x_{m+1}, \cdots, x_n) \longmapsto (x_1, \cdots, x_m)$$

$\Rightarrow$ Any submersion is also an open map (open $\mapsto$ open)

$\Rightarrow$ If N is cpt and connected, M is connected, then any submersion $F: N \to M$ is surjective (i.e., $F(N) = M$).

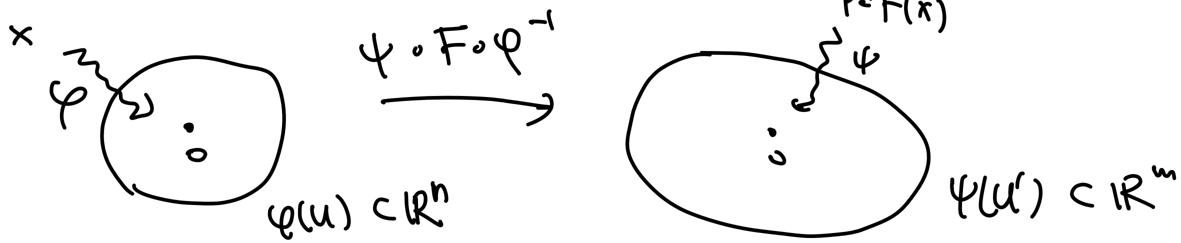
Rmk Cor's above says locally immersion $\Rightarrow$ injective.
submersion $\Rightarrow$ surjective

Cor ($=$ Prop*) $F: N^n \to M^m$, $p \in M$ regular value, then $F^{-1}(\{p\})$ is a submfd of M of dim $= n-m$.

~~pf~~. By def, $x \in N$, $\operatorname{rank}(F)(x) = m$. full rank

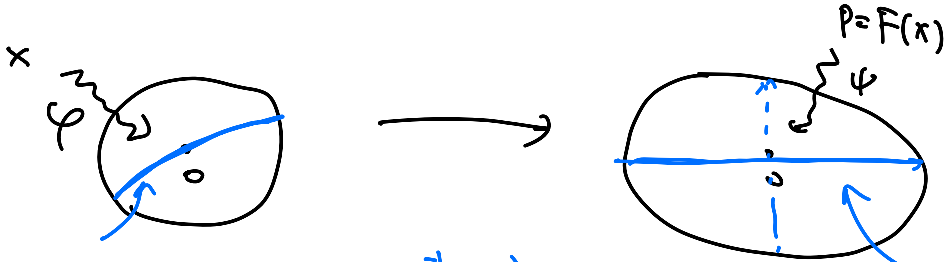
Then $\exists$ NBH U of x in N s.t. $\operatorname{rank}(F) = m$ over U.

Then locally F is like



$$(\psi\circ F\circ\varphi^{-1})((x_1,\cdots,x_n)) = (x_1,\cdots,x_m)$$

$$\Rightarrow \varphi\circ F^{-1}\circ\psi^{-1}(\underbrace{0,\cdots,0}_{m}) = \varphi(F^{-1}(\{p\})) = (\underbrace{0,\cdots,0}_{m}, x_{m+1},\cdots,x_n)$$



$\varphi(F^{-1}(\{p\}))\cap\varphi(U) = \varphi(U\cap F^{-1}(\{p\}))$

$\psi(U')\cap\{(0,\cdots,0,\underbrace{x_{m+1},\cdots,x_n}_{n-m})\}$

This verifies $F^{-1}(\{p\})$ is a submfd of dim $= n-m$. D.

<u>Cor</u> (Global constant rank thm) (see Thm 4.14 in Lee's fat book)

If $F: M\to N$ has constant rank (globally), then

(i) F surjective $\Rightarrow$ submersion

(ii) F injective $\Rightarrow$ immersion

(iii) F bijective $\Rightarrow$ diffeomorphism

everywhere local diffeo
$+\ \exists$ inverse
$=$ global diffeo.

In reality, where do we see global constant rank?

eg If $\varphi: G \to H$ is a Lie group homomorphism, then it is global constant rank.

~~pf~~. Note that for any $g \in G$, $d(\underbrace{g\cdot}_{\text{(left)-multiplication by } g})(e)$ is an isomorphism.

(b/c $d(g\cdot)\circ d(g^{-1}\cdot) = d(g^{-1}\cdot)\, d(g\cdot) = d(\underbrace{e\cdot}_{\text{identity}}) = \text{identity}$)

Then

$$\begin{array}{ccc} G & \longrightarrow & H \\ \downarrow & & \downarrow \\ G & \longrightarrow & H \end{array} \qquad \begin{array}{ccc} e & \xrightarrow{\varphi} & e \\ {\scriptstyle g\cdot}\downarrow & & \downarrow{\scriptstyle \varphi(g)\cdot} \\ g & \xrightarrow[\varphi]{} & \varphi(g) \end{array}$$

$$\Longrightarrow \quad d\varphi(g)\cdot \underbrace{d(g\cdot)(e)}_{\text{iso}} = \underbrace{d(\varphi(g)\cdot)(e)}_{\text{iso}} \cdot d\varphi(e) \qquad \square$$

Rmk (i) $\ker(\varphi)$ is a (reg. or emb.) submfd of G of dim equal to $\dim G - \operatorname{rank}(\varphi)(e)$.

algebra

(ii) If Lie group homomorphism $\varphi: G \to H$ is an iso, then φ is a diffeomorphism. In particular, φ^{-1} is also a diffeomorphism (smooth). ← geometry

(In general, a smooth map $F: M \to N$, with inverse F^{-1}, does <u>not</u> implies that F^{-1} is also smooth! Example?)

We will see more surprising results when studying Lie group systematically later.

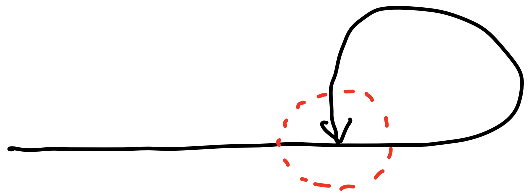
4. Embedding

<u>Def</u>. $F: N^n \to M^m$ is called an <u>embedding</u> if F is an immersion (i.e. $\operatorname{rank}(F) = n$) and $F: N \to F(N)$ is a homeomorphism.

Rmk (Important). Topology of $F(N)$ is the subspace top of $F(N)$ in M.

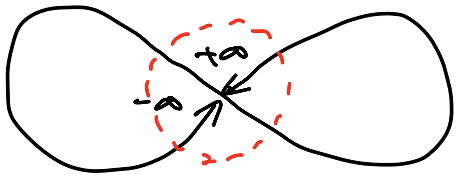
e.g. (immersion but not embedding)

$F: \mathbb{R} \to \mathbb{R}$



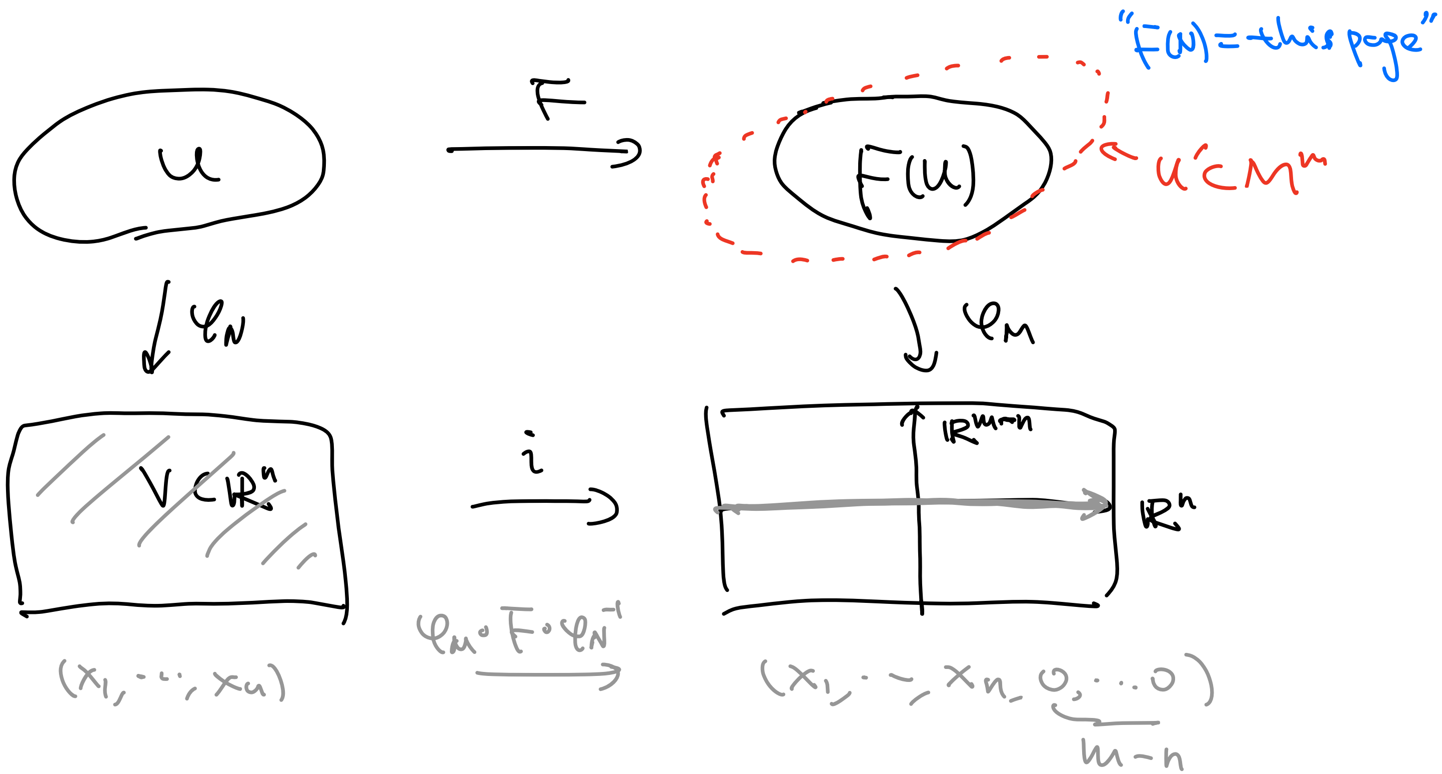
or

$F: \mathbb{R} \to \mathbb{R}$



Prop. The image $F(N)$ of an embedding $F: N^n \to M^m$ is a dim-n (reg. or emb.) submfd of M.

Pf. F is an immersion $\Rightarrow$ locally F is the "standard inclusion"



F is a homeomorphism (in particular, open map), so $F(U)$ is an open subset of $F(N)$ $\Longrightarrow$ $\exists$ open subset $U' \subset M$ s.t.

$$F(U) = U' \cap F(N).$$

$$\varphi_M(U' \cap F(N)) = \varphi_M(F(U)) = \underset{\text{inclusion"}}{\text{"standard}}\,(\varphi_N(U)) \subset \mathbb{R}^n \times \{0\}.$$

Hence, $F(N)$ is an n-dim'l submfd of M. $\square$

Rmk Any (reg.-emb.) submfd $N \subset M$ is the image of an embedding: consider inclusion $i : N \hookrightarrow M$.

$$\operatorname{rank}(i) = \dim N \quad \text{and} \quad N \underset{\text{homeo}}{\simeq} i(N)$$

$$\Rightarrow \quad \left\{ \begin{array}{c} \text{embed} \\ \text{on the level of maps} \end{array} \right\} \Longleftrightarrow \left\{ \begin{array}{c} \text{embed} \\ \text{on the level submfds.} \end{array} \right\}.$$

Question: Given an embedded submfd $N \subset M$, will we study the embeddings $F : \underset{\substack{\uparrow \\ \text{domain}}}{X} \to F(X) = N$ or study the relation between N and M (inside M)?

Ans. Both!

Rmk: The first one (on maps) is harder.

e.g. <u>Thm</u> (Whitney embedding Thm)

Every k-dim'l mfd N can be embedded into (via some embedding) $\mathbb{R}^{2k+1}$.

<u>Rmk</u>. If we only use immersions, then $N \xrightarrow{\text{immersed}} \mathbb{R}^{2k}$

<u>Rmk</u> (Strong Whitney embedding) $N^{k\geq 1} \xhookrightarrow{\text{embed}} \mathbb{R}^{2k}$ ⟸ sharp

more difficult proof

e.g. $\mathbb{RP}^2 \xhookrightarrow{\text{emb.}}\!\!\!\!\!\!\!\!\times\;\;\; \mathbb{R}^3$ (later)

<u>Rmk</u> $N \hookrightarrow \mathbb{R}^m$: stronger condition on N, less m will be.

e.g. (Fuquan Fang 方复全) : N cpt, orientable, 4-dim'l, then $N \xhookrightarrow{\text{emb.}} \mathbb{R}^7$.

(Haefliger-Hirsch) : N cpt, orientable, $2^{k\geq 3}$-dim'l, then $N \xhookrightarrow{\text{emb.}} \mathbb{R}^{2^{k+1}-1}$

e.g. $S^n \xhookrightarrow{\text{emb}} \mathbb{R}^{\underline{n+1}}$ ✓ ↙sharp but $S^n \xhookrightarrow{\text{emb}}\!\!\!\!\!\!\!\times\;\;\; \mathbb{R}^n$ (WHY?)