e.g. $\alpha \in \Lambda^1 V^*$ and $\beta \in \Lambda^2 V^*$, then

$$2(\alpha\wedge\beta)(u,v,w) = \underset{\sigma=(1)}{(\alpha\otimes\beta)(u,v,w)} - \underset{\sigma=(1,2)}{(\alpha\otimes\beta)(v,u,w)} - \underset{\sigma=(1,3)}{(\alpha\otimes\beta)(w,v,u)}$$

$$- \underset{\sigma=(2,3)}{(\alpha\otimes\beta)(u,w,v)} + \underset{\sigma=(1,2)(1,3)}{(\alpha\otimes\beta)(v,w,u)} + \underset{\sigma=(1,3)(2,3)}{(\alpha\otimes\beta)(w,u,v)}$$

<u>Exe</u> Suppose $\{e^1,\cdots,e^n\}$ is a basis of V^*, then

$$\{e^{i_1}\wedge\cdots\wedge e^{i_k} \mid 1\le i_1<\cdots<i_k\le n\}$$

form a basis of $V^{*,\otimes k}$. Therefore $\dim V^{*,\otimes k} = \binom{n}{k} = \dfrac{n!}{k!(n-k)!}$. //

$\Rightarrow$ ① $\underbrace{V^*\wedge\cdots\wedge V^*}_{>n} = 0$

$\Rightarrow$ ② $\underbrace{V^*\wedge\cdots\wedge V^*}_{=n}$ has $\dim = 1$ $(=\mathrm{span}\{e^1\wedge\cdots\wedge e^n\})$

where $(e^1\wedge\cdots\wedge e^n)(\underbrace{v_1,\cdots,v_n}_{\text{vectors}}) = \det\begin{pmatrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{pmatrix}$

$\Rightarrow$ ① $\dim \bigoplus_k \Lambda^k V^* = \sum_{k=0}^{n} \binom{n}{k} = 2^n$

A general formula (proved by induction)

$\alpha_i \in \Lambda^{r_i} V^*$

$$\alpha_1 \wedge \cdots \wedge \alpha_k = \frac{(r_1 + \cdots + r_k)!}{r_1! \cdots r_k!} \mathrm{Alt}(\alpha_1 \otimes \cdots \otimes \alpha_k)$$

<u>Rmk</u> (about ② above) $(e^1 \wedge \cdots \wedge e^n)(e_1, \cdots, e_n) = 1$

$$\Rightarrow (e^1 \wedge \cdots \wedge e^n)(v_1, \cdots, v_n) \underset{v_i = \sum_{j=1}^n v_{ij} e_j}{=} \sum_{\sigma \in S_n} \mathrm{sgn}(\sigma)\, v_{1\sigma(1)} \cdots v_{n\sigma(n)} = \det \begin{pmatrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{pmatrix}.$$

<u>Rmk</u> For $\sigma \in S_n$ and $\sigma \neq \mathbb{1}$, then $e^{\sigma(1)} \wedge \cdots \wedge e^{\sigma(n)}$ is not a basis of $\Lambda^n V^*$ (by def). However, since $\dim \Lambda^n V^* = 1$, $\exists \lambda \in \mathbb{R}$ s.t.

$$e^{\sigma(1)} \wedge \cdots \wedge e^{\sigma(n)} = \lambda \cdot e^1 \wedge \cdots \wedge e^n$$

Apply this relation to $(e_1, \cdots, e_n) (= e_1 \otimes \cdots \otimes e_n)$, we know $\lambda = \mathrm{sgn}(\sigma) = \pm 1$

- $\bigoplus_k \Lambda^k V$ admits the (Hodge) duality, via operation $*$:

$$*: \Lambda^k V^* \longrightarrow \Lambda^{n-k} V^* \quad \text{for any } k \in \{0, \cdots, n\}.$$

Explicitly on the level of basis element.

$$*(e^{i_1}\wedge\cdots\wedge e^{i_k}) := \pm (e^{i_{k+1}}\wedge\cdots\wedge e^{i_n}) \quad \text{where } i_{k+1}<\cdots<i_n.$$

How to determine $+1$ or -1 ?

Rule: consider

$$e^{i_1}\wedge\cdots\wedge e^{i_k}\wedge e^{i_{k+1}}\wedge\cdots\wedge e^{i_n} = \mathrm{sgn}(\sigma)\, e^1\wedge\cdots\wedge e^n$$

where $\sigma = \begin{pmatrix} 1 & \cdots & & \cdots & n \\ i_1 & \cdots & i_k, i_{k+1} & \cdots & i_n \end{pmatrix}$.

If $\mathrm{sgn}(\sigma)=+1$, then choose $+1$ in def of $*$ above

If $\mathrm{sgn}(\sigma)=-1$, then choose -1 in def of $*$ above.

e.g. $*(1) = e^1\wedge\cdots\wedge e^n$, $\quad *(e^1\wedge\cdots\wedge e^n)=1$

e.g. $*(e^1\wedge\cdots\wedge e^k) = e^{k+1}\wedge\cdots\wedge e^n \Rightarrow *(*(e^1\wedge\cdots\wedge e^k)) = *(e^{k+1}\wedge\cdots\wedge e^n) = \pm e^1\wedge\cdots\wedge e^n$

where sign $\pm$ is precisely given by $(-1)^{k(n-k)}$.

<u>Exe</u> $\quad * \circ * = (-1)^{k(n-k)}\cdot \mathbb{1}_{\wedge^k V}$ $\quad$ ($\Rightarrow$ $*$ is an isomorphism)

Rmk Later we will see operator "$*$" enables us to carry out analysis on the space of $(0,k)$-tensors over a manifold, which eventually leads to a powerful homological theory — Hodge theory.

① Exterior derivative

- Recall that over a manifold M^n, the local model $\{U_\alpha \times (\mathbb{R}^n)^{*,\otimes k}\}_\alpha$ glue together to get $(T^*M)^{\otimes k}$, where elements inside are called $(0,k)$-tensors.

In the same way,

$$\begin{array}{c}\text{local model}\\ \{U_\alpha \times \Lambda^k(\mathbb{R}^n)^*\}_\alpha\end{array} \xrightarrow[\text{together}]{\text{glue}} \Lambda^k T^*M \quad \left(\begin{array}{c}\text{a bundle over } M\\ \text{of rank } \binom{n}{k}\end{array}\right)$$

Denote by
$$\Omega^k(M) := \{\underset{\substack{\uparrow\\ \text{a } \underline{k\text{-form}} \text{ on } M.}}{\text{sections}} : M \to \Lambda^k T^*M\}$$

- $\Omega(M) := \bigoplus_{k\geq 0} \Omega^k(M)$ is an associative super-commutative algebra over the $C^\infty(M; \mathbb{R} \text{ or } \mathbb{C})$.

$\alpha, \beta \to \alpha\wedge\beta$ wedge product.

$\alpha \longrightarrow *\alpha$ Hodge star operator.

Rmk By def, $\Omega^0(M) = C^\infty(M; \mathbb{R} \text{ or } \mathbb{C})$.

There is a famous $(\mathbb{R} \text{ or } \mathbb{C})$-linear operator on $\Omega(M)$, defined as follows:

$\alpha \in \Omega^k(M) \rightsquigarrow d\alpha \in \Omega^{k+1}(M)$ by

For computation

$$d\alpha(X_1, \dots, X_{k+1}) := \sum_{1\leq i\leq k+1} (-1)^{i-1} X_i \big(\overbrace{\alpha(X_1, \dots, \hat{X_i}, \dots, X_{k+1})}^{\text{a function}}\big)$$

vector fields in $\Gamma(TM)$

$$+ \sum_{1\leq i<j\leq k+1} (-1)^{i+j} \alpha([X_i, X_j], X_1, \dots, \hat{X_i}, \dots, \hat{X_j}, \dots, X_{k+1}).$$

bracket of vector fields

e.g. $\alpha = f \in \Omega^0(M)$, then $df(X) = D_X f$

(Therefore, the correct way to explain the "differentiation df" (全微分) in calculus is that it is a 1-form on M.)

e.g. $\alpha \in \Omega^1(M)$, then $d\alpha(X_1, X_2) = X_1\alpha(X_2) - X_2\alpha(X_1) - \alpha([X_1, X_2])$

e.g. $\alpha \in \Omega^2(M)$, then
$$d\alpha(X_1, X_2, X_3) = X_1\alpha(X_2, X_3) - X_2\alpha(X_1, X_3) + X_3\alpha(X_1, X_2) - \alpha([X_1, X_2], X_3) + \alpha([X_1, X_3], X_2) - \alpha([X_2, X_3], X_1).$$

How were this formula of $d\alpha$ derived? ← asking for the origin of $d\alpha$

(optional to explain the materials below)

//• Since $\alpha = \sum \underbrace{\alpha_1 \wedge \cdots \wedge \alpha_k}_{\alpha \text{ are 1-forms}}$. by requiring d to be $\mathbb{K}$-linear, it suffices to figure out how d and $\wedge$ are related.

• Working locally, d is defined first in the Euclidean space $\mathbb{R}^n$

Suppose coordinates of $\mathbb{R}^n$ are $x_1, \cdots, x_n$, then

① 1-forms are $C^\infty(\mathbb{R}^n)$-linear combinations of $dx_1, \cdots, dx_n$. $\leftarrow$ basis of $\Omega^1(\mathbb{R}^n)$

② Define for $f \in C^\infty(\mathbb{R}^n)$, $df := \frac{\partial f}{\partial x_1} dx_1 + \cdots + \frac{\partial f}{\partial x_n} dx_n$

③ For $\alpha \in \Omega^1(\mathbb{R}^n)$ as $\alpha = \sum_{i=1}^n g_i dx_i$, define

$$d\alpha := \sum_{i=1}^n dg_i \wedge dx_i = \sum_{i=1}^n \left(\sum_{j=1}^n \frac{\partial g_i}{\partial x_j} dx_j \right) \wedge dx_i$$

$$= \sum_{1 \le i < j \le n} \left(\frac{\partial g_j}{\partial x_i} - \frac{\partial g_i}{\partial x_j} \right) dx_i \wedge dx_j$$

④ Define $d(\alpha \wedge \overset{\in \Omega^1(\mathbb{R}^n)}{\beta}) := d\alpha \wedge \beta - \alpha \wedge d\beta$.

$\left(\Rightarrow \alpha \in \Omega^k(\mathbb{R}^n), \beta \in \Omega^l(\mathbb{R}^n), \ d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^k \alpha \wedge d\beta \right)$.

⑤ Verify that d is defined globally ("compatible" with transition map in the overlap $U_\alpha \cap U_\beta$).

Therefore, one can view d as the differentiation with respect to multi-index. //

Rmk $$d(df) = d\left(\sum_{i=1}^{n} \frac{\partial f}{\partial x_i} dx_i\right) = \sum_{1\le i<j\le n} \left(\frac{\partial^2 f}{\partial x_i \partial x_j} - \frac{\partial^2 f}{\partial x_j \partial x_i}\right) dx_i \wedge dx_j = 0.$$

In a similar way, $d(d\alpha) = 0$ for $d\alpha \in \Omega^1(\mathbb{R}^n)$. Then, for $\alpha, \beta \in \Omega^1(\mathbb{R}^n)$,

$$d(d(\alpha\wedge\beta)) = d(d\alpha\wedge\beta - \alpha\wedge d\beta) = d\alpha\wedge d\beta - d\alpha\wedge d\beta = 0$$

$\Rightarrow \quad d\circ d = 0$

added: Fix a vector field X on M, define $\iota_X : \Omega^k(M) \to \Omega^{k-1}(M)$ by $(\iota_X \alpha)(X_1, \cdots, X_{k-1}) := \alpha(X, X_1, \cdots, X_{k-1})$

always the 1st position

Rmk

$$\Omega^k(M) \xrightarrow{\iota_X} \Omega^{k-1}(M) \xrightarrow{d} \Omega^k(M)$$

$$\Omega^k(M) \xrightarrow{d} \Omega^{k+1}(M) \xrightarrow{\iota_X} \Omega^k(M)$$

are they the same?

e.g. $M = \mathbb{R}^2 \qquad X(x_1, x_2) = (0, x_2)$

$$dx_1\wedge dx_2 \xrightarrow{\iota_X} -x_2 dx_1 \xrightarrow{d} dx_1\wedge dx_2$$

$$dx_1\wedge dx_2 \xrightarrow{d} 0 \xrightarrow{\iota_X} 0$$

in general, $d\circ\iota_X$ and $\iota_X\circ d$ are <u>not</u> the same.

Later we will see that $d\circ\iota_X + \iota_X\circ d$ has a geometric meaning.