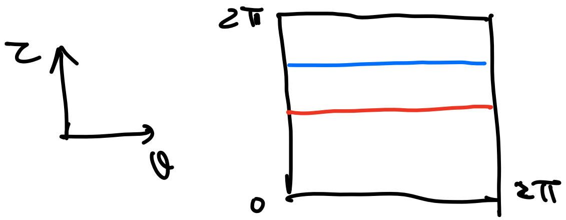
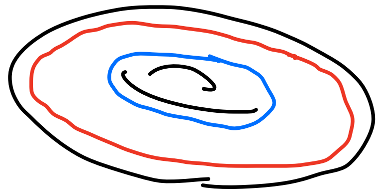
One can of course add stronger condition on embeddings

e.g. Recall $F : (N, g_N) \longrightarrow (M, g_M)$ is called an <u>isometry</u>
if $F^* g_M = g_N$. In particular, F preserves lengths.



$\xrightarrow[\text{embedding}]{F}$



$\subset \mathbb{R}^3$

$S^1 \times S^1 = T^2$ (top 2-torus)

$$(\theta, z) \longmapsto \begin{cases} x = (3 + \cos(z)) \cos\theta \\ y = (3 + \cos(z)) \sin\theta \\ z = \sin z \end{cases}$$

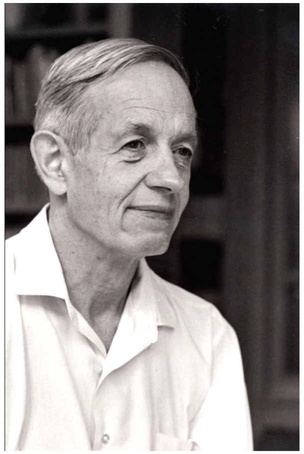
Note that F is an embedding but <u>not</u> isometric.

<u>Question</u>: Is it possible find a(n) (smooth) embedding from T^2 to $\mathbb{R}^3$ that is also an isometry?

Ans: ① There always exists an isometric emb $\mathbb{T}^2 \hookrightarrow \mathbb{R}^k$ where $k \leq 10$. (Nash, C^∞-embedding Thm 1956)

② there exists a isometric C^1- emb $\mathbb{T}^2 \hookrightarrow \mathbb{R}^3$ (Nash, C^1-embedding Thm 1954)

$\frac{m(3m+11)}{2}$



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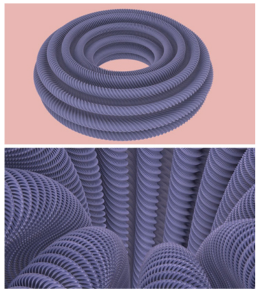
Differential Geometry

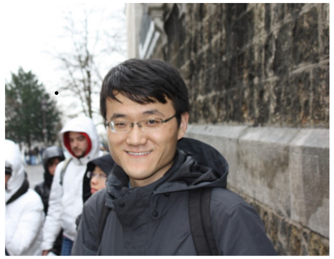
(Fall 2017)



抽象曲面：平坦环面

在空间中的C^1嵌入
CNRS website

V. Borrelli, S. Jabrane, F. Lazarus and B. Thibert,
Flat tori in three dimensional space and convex integration, PNAS, April 2012



$n+1$ (Kuiper)

Rmk: why are Nash's emb Thms useful?

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Usefulness of Nash embedding theorem

Asked 4 years, 11 months ago Modified 1 year, 10 months ago Viewed 9k times

Nash embedding theorem states that every smooth Riemannian manifold can be smoothly isometrically

The Overflo

Ans. $\Longleftarrow$

allowed one to avoid using this scheme. (For Nash embedding itself, a somewhat similar trick of Gunther in

Günther, Matthias, Isometric embeddings of Riemannian manifolds, Proc. Int. Congr. Math., Kyoto/Japan 1990, Vol. II, 1137-1143 (1991). ZBL0745.53031.

can be used to also avoid applying Nash-Moser iteration.)

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edited Dec 1, 2022 at 10:52
Martin Sleziak

answered Oct 6, 2019 at 21:12
Terry Tao

5. Relation between submfd & mfd ← ambient mfd

Exe For a mfd M^m, its tangent bundle TM and cotangent bundle T^*M are dim-$2m$ mfd.

(Recall) For categories $\mathcal{C}$ and $\mathcal{D}$, a covariant functor $\Phi : \mathcal{C} \to \mathcal{D}$ is a map

$$\begin{cases} \text{on the level of objects} & \underset{\mathrm{Ob}(\mathcal{C})}{\overset{X}{\cap}} \longmapsto \underset{\mathrm{Ob}(\mathcal{D})}{\overset{\Phi(X)}{\cap}} \\ \text{on the level of hom-set:} & \varphi \longmapsto \Phi(\varphi) \end{cases}$$

$$\begin{tikzcd} X \arrow[r, mapsto] \arrow[d, "\varphi"'] \arrow[dr, phantom, "\circlearrowleft"] & \Phi(X) \arrow[d, "\Phi(\varphi)"] \\ Y \arrow[r, mapsto] & \Phi(Y) \end{tikzcd}$$

and satisfies $\Phi(\mathbb{1}_X) = \mathbb{1}_{\Phi(X)}$ and $\Phi(\psi \circ \varphi) = \Phi(\psi) \circ \Phi(\varphi)$

Rmk. A contravariant functor $\Phi : \mathcal{C} \to \mathcal{D}$ is defined in a similar way, but with two exceptions:

$$\varphi : X \to Y \text{ in } \mathcal{C} \quad \longmapsto \quad \Phi(\varphi) : \Phi(Y) \longrightarrow \Phi(X).$$

(so, $\Phi(\psi \circ \varphi) = \Phi(\varphi) \cdot \Phi(\psi)$).

e.g. "Taking tangent bundle and differential" is a covariant functor.

$$\Phi : \underset{\substack{\text{cat of } m\text{-dim} \\ \text{smooth mfd}}}{\mathcal{M}^m} \longrightarrow \underset{\substack{\text{cat of } m\text{-dim} \\ \text{smooth mfd}}}{\mathcal{M}^{2m}} \qquad \begin{cases} M \longmapsto \Phi(M) := TM \\ F : N \to M \longmapsto \Phi(F) := dF : TN \to TM \end{cases}$$

(b/c $\Phi(G \circ F) = d(G \circ F) = dG \circ dF$)

e.g. "Taking cotangent bundle and pullback" is a contravariant functor.

In particular, $\Phi(F : N \to M) := F^* : T^*M \to T^*N$.

Using (vector) bundle language, $dF: TN \to TM$ is called a <u>bundle</u> map b/c the following diagram commutes.

$dF(F(x))(v)$

$$\begin{tikzcd} TN \arrow[r, "dF"] \arrow[d, "\pi_N"'] \arrow[dr, phantom, "\circlearrowleft"] & TM \arrow[d, "\pi_M"] \\ N \arrow[r, "F"] & M \end{tikzcd} \qquad \begin{tikzcd} (x,v) \arrow[r] \arrow[d, "\pi_N"'] \arrow[dr, phantom, "\circlearrowleft"] & (F(x), dF(v)) \arrow[d, "\pi_M"] \\ x \arrow[r] & F(x) \end{tikzcd}$$

The key in this def is $dF(v) \in \pi^{-1}(\{F(x)\})$.

<u>Def</u>. For two vector bundles $\begin{matrix} E_1 \\ \downarrow \\ N \end{matrix}$ and $\begin{matrix} E_2 \\ \downarrow \\ M \end{matrix}$, a bundle map from $\begin{matrix} E_1 \\ \downarrow \\ N \end{matrix}$ to $\begin{matrix} E_2 \\ \downarrow \\ M \end{matrix}$ is a pair of smooth mass

$$\Phi: E_1 \to E_2 \quad \text{and} \quad f: N \to M$$

s.t. the following diagram commutes:

$$\begin{tikzcd} E_1 \arrow[r, "\Phi"] \arrow[d, "\pi_1"'] & E_2 \arrow[d, "\pi_2"] \\ N \arrow[r, "f"] & M \end{tikzcd} \quad + \quad \Phi\big|_{\pi_1^{-1}(\{x\})} : \pi_1^{-1}(\{x\}) \to \pi_2^{-1}(\{f(x)\})$$

is a linear map.

Rmk. Often $f = \mathbb{1} : N \to N$.

Suppose $\begin{tikzcd} E_1 \arrow[rr, "\Phi"] \arrow[dr, "\pi_1"'] & & E_2 \arrow[dl, "\pi_2"] \\ & N & \end{tikzcd}$, then Φ is injective if $\Phi\big|_{\pi_1^{-1}(\{x\})}$ is injective for any $x \in N$.

Suppose $\begin{tikzcd} E_1 \arrow[rr, "\Phi"] \arrow[dr, "\pi_1"'] & & E_2 \arrow[dl, "\pi_2"] \\ & N & \end{tikzcd}$, then Φ is surjective if $\Phi\big|_{\pi_1^{-1}(\{x\})}$ is surjective for any $x \in N$.

e.g. Given vector bundles (over N) E_1, E_2, one can construct a vector bundle E_3 (over N) s.t. the following short exact sequence holds:

$$0 \to E_1 \to E_3 \longrightarrow E_2 \longrightarrow 0.$$

tran. map of E_1

The transition map of E_3 is given by $\begin{pmatrix} \Phi^1_{\alpha\beta} & \\ & \Phi^2_{\alpha\beta} \end{pmatrix}$

tran map of E_2

This E_3 is called the direct (or Whitney) sum of E_1, E_2.

e.g. Given injective $\Phi: E_1 \to E_2$, one can construct a vector bundle E_3 s.t. the following short-exact seq. holds

$$0 \to E_1 \xrightarrow{\Phi} E_2 \to E_3 \to 0$$

where $\pi_3^{-1}(\{x\}) = \pi_2^{-1}(\{x\}) \big/ \underset{\Phi(\pi_1^{-1}(\{x\}))}{\pi_1^{-1}(\{x\})}$ (and the trans. map is the induced map on E_3 from E_2).

This E_3 is called the quotient bundle of $\Phi: E_1 \to E_2$.

e.g. $N \subset M$ embedded submfd and $i: N \hookrightarrow M$ inclusion.

Then $$0 \to TN \xrightarrow{di} \underbrace{TM\big|_N}_{:= TM \cap \pi^{-1}(N)} \longrightarrow \underbrace{TM|_N / TN}_{:= \text{normal bundle of } N \text{ in } M.} \to 0 \qquad (*)$$

Rmk Normal bundle of N in M is often denoted by $\nu_N M$.

Similarly, taking dual of $(*)$, one defines <u>conormal bundle</u>. denoted by $\nu_N^* M$.

$$\nu_N^* M \underset{\substack{\uparrow \\ \text{locally}}}{=} \Big\{ (\underset{\substack{\cap \\ T^*M}}{p, f}) \ \Big| \ \begin{array}{l} p \in N, \ f: T_pM \to \mathbb{R} \\ \text{s.t. } f(v) = 0 \text{ for any } v \in T_pN \end{array} \Big\}$$

$$\dim \nu_N^* M = \dim N + (\dim M - \dim N) = \dim M = \tfrac{1}{2} \dim T^*M$$

e.g. Recall that $S^2 = \{ (x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1 \}$ is an embedded submfd in $\mathbb{R}^3$. (WHY?) $\Leftarrow$ $S^2 = F^{-1}(\{1\})$ where $F: \mathbb{R}^3 \to \mathbb{R}$, $(x, y, z) \mapsto x^2 + y^2 + z^2$

Let us work out $\nu_{S^2} \mathbb{R}^3$.

$\Uparrow$ by def

Let us work out TS^2.

Observe $i: S^2 \overset{\text{inclusion}}{\hookrightarrow} \mathbb{R}^3$ composed with $F: \mathbb{R}^3 \to \mathbb{R}$ (above)

satisfies $\underset{S^2}{F \circ i\,(p)} = 1$ (constant map), so

$$\begin{tikzcd} TS^2 \arrow[r, "di"] \arrow[rr, bend right, "d(F\circ i) = 0"'] & T\mathbb{R}^3 \arrow[r, "dF"] & T\mathbb{R} \end{tikzcd}$$

(i.e. $dF \circ di = 0$)

injective

and equal to identity on TS^2

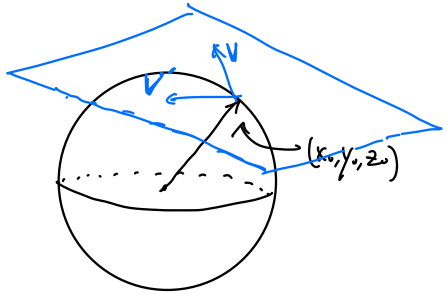
$\Rightarrow$ For every pt $p \in S^2$, $T_pS^2 \subset \ker dF(p)$.

Then by the dim-counting ($\dim T_pS^2 = \dim \ker dF(p) = 2$), we know $TS^2 = \ker(dF)$.

$$\Rightarrow \nu_{S^2}\mathbb{R}^3 = T\mathbb{R}^3|_{S^2} \big/ TS^2 = T\mathbb{R}^3|_{S^2} \big/ \ker(dF)$$

At pt $p=(x_0, y_0, z_0) \in S^2$,

$$\ker(dF)(p) = \{ v \in \underset{\substack{\parallel \\ T_p\mathbb{R}^3}}{\mathbb{R}^3} \mid (x_0, y_0, z_0)\cdot(v_1, v_2, v_3) = 0 \}$$



Then $\nu_{S^2}\mathbb{R}^3$ is a rank-1 vector bundle over S^2, where each fiber is represented by any a vector <u>not</u> lying in the blue plane.

<u>Rmk</u> In order to specify a vector in $\mathbb{R}^3$, representing the fiber of $\nu_{S^2}\mathbb{R}^3$, one needs an inner product (use its non-deg).

<u>Rmk</u> Argument in e.g. above works in general, so we proved the following prop (seen earlier):

conclusion 1 conclusion 2

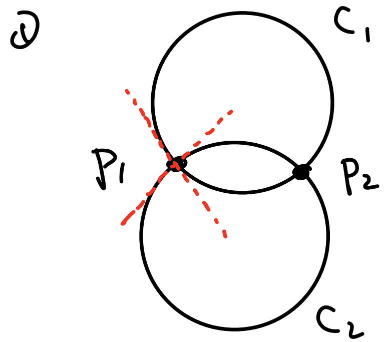
$$F^{-1}(\{\text{reg. value}\}) = \underset{N \subset M}{\text{emb submfd}} \quad + \quad TN = \ker(dF).$$

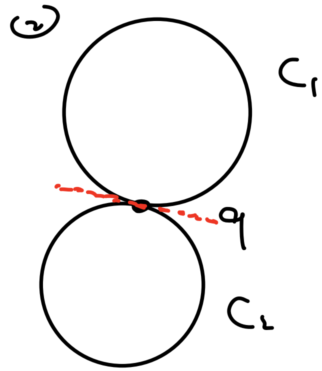
Exe. Prove that $T_e O(n) = \{ A \in M_{n\times n} \mid A + A^T = 0 \}$.

identity

6. Relation between submfds

In $\mathbb{R}^2$, two circles may interact in the following two ways:





Key difference:

① At either p_1 or p_2, $\quad T_{p_i}C_1 + T_{p_i}C_2 = T_{p_i}\mathbb{R}^2$

② At q $\quad T_qC_1 + T_qC_2$ is 1-dim $\neq T_q\mathbb{R}^2$

Rmk. Case ① seems appearing more often than case ②.

Def Given two emb. submfds $N_1^{n_1}$, $N_2^{n_2}$ in M^m, we call them <u>intersect transversally</u> if for any $p \in N_1 \cap N_2$,

$$T_pN_1 + T_pN_2 = T_pM \qquad (*)$$

not nec a direct sum. (i.e. $T_pN_1 \cap T_pN_2$ may be non-trivial)

Notation: $N_1 \pitchfork N_2$.

Rmk. If $N_1 \cap N_2 = \phi$, then they do <u>not</u> call intersect trans..

e.g. Case ① above, $C_1 \pitchfork C_2$; Case ②, they do <u>not</u>.

Intersection in a transverse manner is the "best" one can hope.
(intersection theory in general) is a <u>hard</u> subject in math :c).

Rmk To satisfy condition $(*)$ above, $n_1 + n_2 \geq m$.

Why do we like transverse intersection?

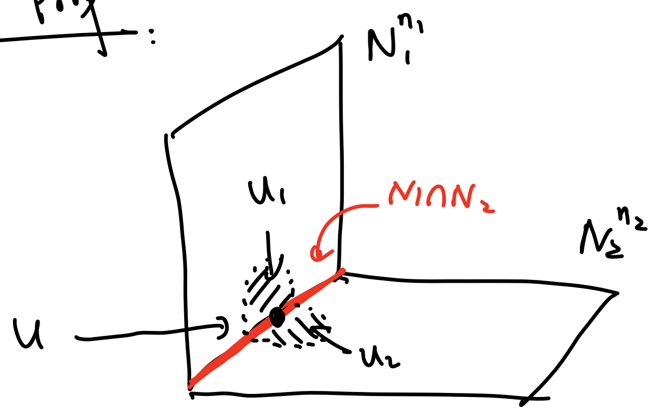
Prop Suppose $N_1^{n_1} \pitchfork N_2^{n_2}$ in M^m, then $N_1 \cap N_2$ is an embedded submfd in M. Moreover, $\dim(N_1 \cap N_2) = n_1 + n_2 - m$.

Rmk If $n_1 + n_2 = m$, then $N_1 \cap N_2$ = collection of pts.

Rmk A better way to memorize the dim formula in Prop above is the "additivity of codim": if $N_1 \pitchfork N_2$ in M, then

$$\operatorname{codim}(N_1 \cap N_2) = \operatorname{codim} N_1 + \operatorname{codim} N_2$$

pf of prop:



$$U_1 = U \cap N_1 \simeq \{\overbrace{x_{n_1+1} = \cdots = x_m}^{m-n_1} = 0\}$$

$$U_2 = U \cap N_2 \simeq \{\overbrace{y_{n_2+1} = \cdots = y_m}^{m-n_2} = 0\}$$

$$U \cap (N_1 \cap N_2) \simeq \{\underbrace{z_m = \cdots = z_{n_1+n_2-m+1}}_{\text{locally constant rank}} = 0\}$$

✓

e.g. In $\mathbb{C}^5\setminus\{0\}$, $N_1 = \{ z_1^2 + z_2^2 + z_3^2 + z_4^3 + z_5^{6k-1} = 0 \}$ ← it has only 1 singular pt at $0 \in \mathbb{C}^5$

$$N_2 = \{ |z_1|^2 + |z_2|^2 + |z_3|^2 + |z_4|^2 + |z_5|^2 = 1 \}$$

Then one can check $N_1 \pitchfork N_2$ and it is homeomorphic to S^7.

For $k = 1, \cdots, 28$, there are all 28 different exotic differentiable str.
not equ. to the standard one.

$N_1 \pitchfork N_2$ is an example of Brieskorn spheres.

By discussion above, $N_1 = F(X)$ and $N_2 = G(Y)$ for some embeddings $F: X \to M$ and $G: Y \to M$. Then $N_1 \pitchfork N_2$ can be translated to transversality of maps F, G.

Def Two embeddings F and G are transverse if

$$\operatorname{Im} dF(x) + \operatorname{Im} dG(y) = T_p M$$

for any $x \in X$, $y \in Y$, $p \in M$ satisfying $F(x) = G(y) = p$.

Notation: $F \pitchfork G$

e.g. $\underset{\substack{\uparrow \\ \text{submersion}}}{F} : X \to M$ and $i : \underset{\substack{\uparrow \\ \text{emb. submfd}}}{N} \xhookrightarrow{\text{inclusion}} M$, then $F \pitchfork i$.

e.g. The condition $F \pitchfork G$ implies $F \times G : X \times Y \longrightarrow M \times M$

intersect transversely with diagonal map $\Delta : M \longrightarrow M \times M$.
$p \longmapsto (p, p)$

pf. - $(F \times G)(x, y) = \Delta(p) \iff F(x) = G(y) = p$

- $\operatorname{Im} d(F \times G)(x, y) \simeq \operatorname{Im} dF(x) + \operatorname{Im} dG(y)$ this "sum" should be product

$\operatorname{Im} d\Delta(p) \simeq T_p M$. ✓

This implies that $\operatorname{im}(F \times G) \cap \operatorname{im} \Delta$ is a submfd of

under the condition that both F and G are embeddings

$\dim(X \times Y) + \dim M - \dim(M \times M) = \dim X + \dim(Y) - \dim M$.

This submfd is called the <u>fiber product</u> of $F : X \to M$ and $G : Y \to M$. (of $M \times M$)