can consider quotient space $\mathcal{J}(\Sigma_g)/\mathrm{Diff}(\Sigma_g)$ (usually denoted by M_g, the moduli space of cpx str on Σ_g)

This confirms what Grothendieck's statement above

$$\text{Quotient} \simeq \left\{ \begin{array}{ll} \text{orbit } 1 & j_1 \longrightarrow \varphi^* j_1 \longrightarrow \cdots \\ \text{orbit } 2 & j_2 \longrightarrow \varphi^* j_2 \longrightarrow \cdots \\ \cdots & \end{array} \right.$$

In the same way, one can add ordered pts Θ and consider

$$\big(\mathcal{J}(\Sigma_g) \times (\Sigma_g^{\ell} \setminus \Delta)\big)/\mathrm{Diff}(\Sigma_g) \quad \leftarrow \text{acting by diagonal.}$$

- It is $\simeq \mathcal{J}(\Sigma_g)/\mathrm{Diff}(\Sigma_g, \Theta)$ for any fixed Θ.

  where $\mathrm{Diff}(\Sigma_g, \Theta) \leq \mathrm{Diff}(\Sigma_g)$ that fixes Θ (stabilizer of $\mathrm{Diff}(\Sigma_g)$ w.r.t Θ).

- It can also be identified with $M_{g,\ell}$: $(\Sigma_g, j) \overset{\varphi}{\simeq} (\Sigma_g, j')$ iff

  $d\varphi \cdot j = j' \cdot d\varphi$ iff $j' = d\varphi \cdot j \cdot (d\varphi)^{-1}$.

In other words,

$$M_{g,\ell} \simeq \big(\mathcal{J}(\Sigma_g) \times (\Sigma_g^{\ell} \setminus \Delta)\big)/\mathrm{Diff}(\Sigma_g) \simeq \mathcal{J}(\Sigma_g)/\mathrm{Diff}(\Sigma_g, \Theta)$$

//

Introducing notation $\chi(\Sigma_g \setminus \Theta)$ = Euler char of punctured Σ_g
$= 2 - 2g - \ell$.

<u>FACT</u> (Prop 7.9 in [wen]) When $\chi(\Sigma_g \setminus \Theta) < 0$, for each $j \in \mathcal{J}(\Sigma_g)$

It is a Lie group as another fact $\to$ $\mathrm{Aut}(\Sigma, j, \Theta)$ = stabilizer of $\mathrm{Diff}(\Sigma_g, \Theta)$ w.r.t j is finite

Riemann-Roch.
$\Longrightarrow$ $\dim_{\mathbb{R}} M_{g,\ell} = 6g-6+2\ell$

(The intermediate step is in general, $\dim \mathrm{Aut}(\Sigma, j, \Theta) - \dim M_{g,\ell} = 3\chi(\Sigma) - 2\ell$.)

<u>Ex</u> $\chi(\Sigma_g|\Theta) \geq 0 \iff 2-2g-\ell \geq 0$ where both $g, \ell \geq 0$

$\iff (g, \ell) = (0,0),\ (1,0),\ (0,1),\ (0,2)$

$\chi(\Sigma_g|\Theta)=2 \quad \chi(\Sigma_g|\Theta)=0 \quad \chi(\Sigma_g|\Theta)=1 \quad \chi(\Sigma_g|\Theta)=0$

Usually, we call (Σ_g, j, Θ) <u>stable</u> if $\chi(\Sigma_g|\Theta) < 0$. Hence

for $g=0$ case, being stable $\iff$ $\ell \geq 3$

- being stable is common
- being stable $\Rightarrow$ $\mathrm{Aut}(\Sigma_g, j, \Theta)$ is a finite set.

<u>Rmk</u> (famous name/term)

$\mathrm{Diff}_0(\Sigma_g, \Theta) \leq \mathrm{Diff}(\Sigma_g, \Theta)$ identity component

$\Longrightarrow$ $\mathcal{M}(\Sigma_g, \Theta) := \mathrm{Diff}(\Sigma_g, \Theta) / \mathrm{Diff}_0(\Sigma_g, \Theta)$ mapping class group

$\Rightarrow$ $\mathcal{T}(\Sigma_g, \Theta) := \mathcal{J}(\Sigma_g) / \mathrm{Diff}_0(\Sigma_g, \Theta)$ Teichmüller space.

(One can define the $\mathcal{T}(\Sigma_g)$ in the same way without Θ.)

Then $\mathcal{T}(\Sigma_g, \Theta) / \mathcal{M}(\Sigma_g, \Theta) \overset{\text{check the group action descents}}{=} \mathcal{J}(\Sigma_g) / \mathrm{Diff}(\Sigma_g, \Theta) = M_{g,\ell}$.

a smooth mfd as a classical result

usually a discrete group

usually not a mfd but an orbifold

In general, $M_{g,\ell}$ is a finite-dim'l orbifold.
(locally modelled by $\mathbb{R}^{\dim}/\text{group}$)

One way to compactify $M_{g,\ell}$ is called the Deligne-Mumford compactification. $\overline{M_{g,\ell}}$ ($= M_{g,\ell} \cup$ "singular curves").

Def A nodal Riem surface with $\ell \geq 0$ marked pts and $N \geq 0$ nodes is a tuple (S, j, Θ, Δ):

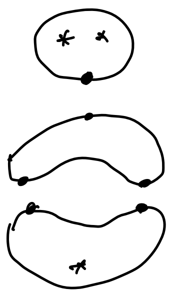
Here we use S instead of Σ to indicate S is not nec connected.

- A closed but not nec connected Riem surface (S, j)
- An ordered set of ℓ pts $\Theta \subset S$
- An unordered set of $2N$ pts $\Delta \subset S \setminus \Theta$ equipped with involution $\sigma: \Delta \to \Delta$. Each pair $\{z, \sigma(z)\}$ for $z \in \Delta$ is (often referred as z^+, z^-) referred to as a node

$\Longrightarrow$ $\hat{S}$ = closed surface by performing connected sums on S at each ALL node $\{z^+, z^-\} \subset \Delta$.

(Then $z^+ \sim z^-$ in $\hat{S}$ as a "double pt".) the easiest singular pt

e.g.

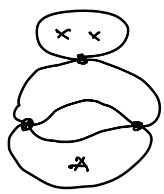
$S =$ 

3 connected components

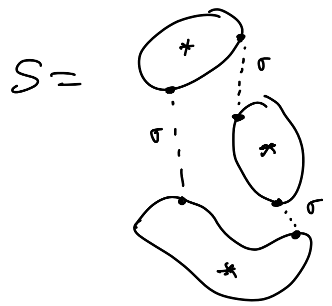
$\ell = 3$

$N = 3$ (3 pairs)

(Each connected component has genus 0)

$\Rightarrow \hat{S} =$ 

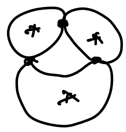
$g(\hat{S}) = 1$



3 connected components

$\ell = 3$

$N = 3$ (3 pairs)

$\Rightarrow \hat{S} =$ 

Terms

(1) (S, j, Θ, Δ) is connected if $\hat{S}$ is connected.

(2) In the connected case, $g(\hat{S})$ is called the arithmetic genus of (S, j, Θ, Δ)

(3) (S, j, Θ, Δ) is stable if each connected component of $S \setminus (\Theta \cup \Delta)$ has negative Euler char.

(4) $(S, j, \Theta, \Delta) \sim (S', j', \Theta', \Delta')$ iff $\exists$ bihol $\varphi : (S, j) \to (S', j')$ taking Θ to Θ' w.r.t order and taking Δ to Δ s.t. nodes mapped to nodes.

e.g. Every element in $M_{g,\ell}$ is a nodal Riem surface that has $N = 0$, connected, and arithmetic genus g.
(so $\Delta = \emptyset$)

Note that so far this is just a notation, no properties proved yet.

Def $\overline{M_{g,\ell}} = \{$ equ classes of stable connected nodal Riem surface with ℓ marked pts and arithmetic genus $g \}$

assume $2g + \ell \geq 3$
(stable condition)

(so $\mathcal{M}_{g,\ell} \subset \overline{\mathcal{M}_{g,\ell}}$).

<u>Example</u> $g=0,\ \ell=3$: $\quad \mathcal{M}_{g,\ell} = \mathcal{M}_{0,3} \simeq \{pt\}$

since for $(S^2, j, (p_1,p_2,p_3)) \underset{\sim}{\overset{\exists !}{\longrightarrow}} (S^2, j, (0,1,\infty))$

$\Rightarrow \quad \overline{\mathcal{M}_{0,3}} = \mathcal{M}_{0,3}$.

$\leftarrow \dim_{\mathbb{R}} \mathcal{M}_{0,4} = 6\cdot 0 - 6 + 2\cdot 4 = 2$,

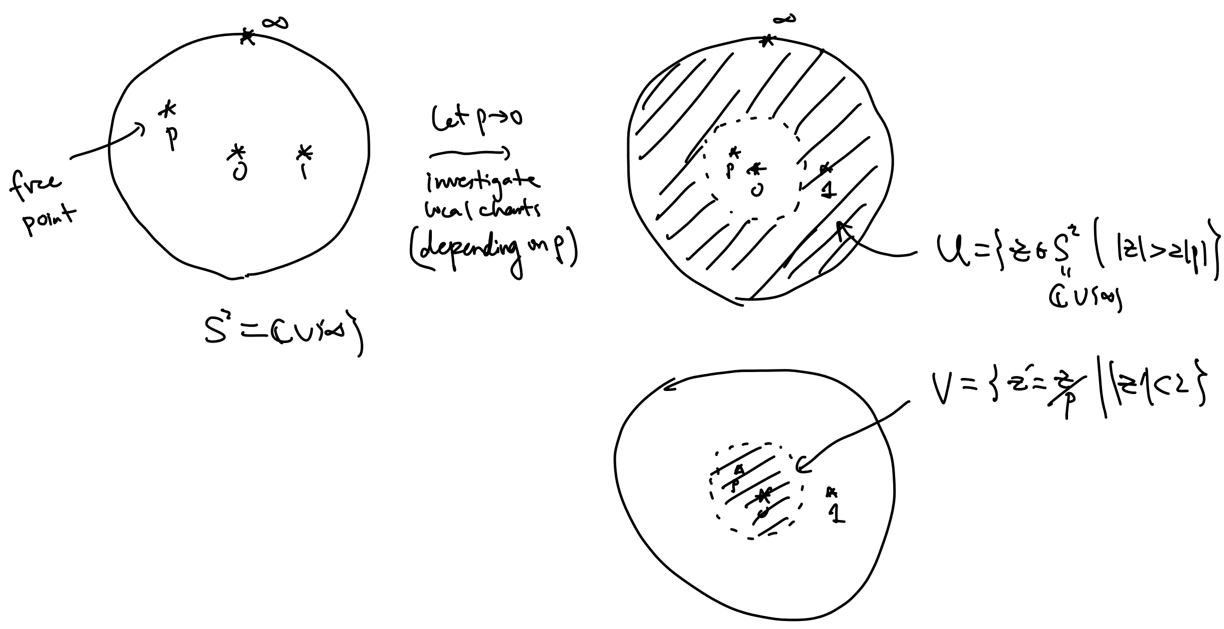
<u>Example</u> $g=0,\ \ell=4$: $\quad \mathcal{M}_{g,\ell} = \mathcal{M}_{0,4} \simeq S^2 \setminus \{0,1,\infty\}$.

$$\mathcal{M}_{0,4} = \{[(S^2, j, (p_1,p_2,p_3,p_4))]\} = \{[(S^2, j, (0,1,\infty,p))]\}$$

when $p \neq 0,1,\infty$.

* Note that $S^2 \setminus \{0,1,\infty\}$ is <u>not</u> cpt and a naive way to compactify it is by adding "3 pts" back and get S^2.

A standard argument to find out these extra pts is by a limit argument:



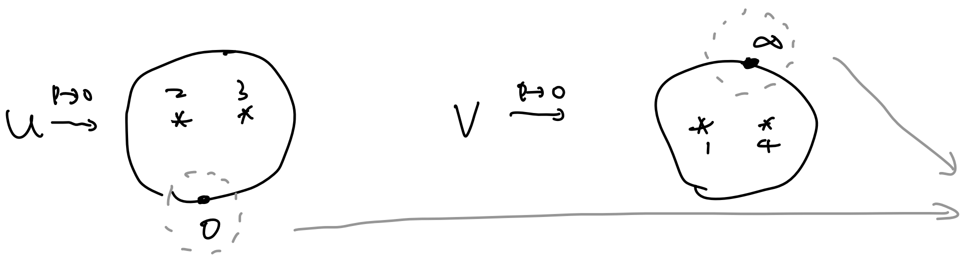
- $U \cap V = \emptyset$.

- $\{\overset{1}{0}, \overset{4}{p}\} \notin U$ and $\{\overset{2}{1}, \overset{3}{\infty}\} \in U$

  when $p \to 0$, $\overset{1}{0}, \overset{4}{p}$ coincide (at 0)

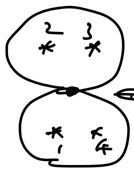
- coordinate in V is $\left(\frac{0}{p}, \frac{1}{p}, \frac{\infty}{p}, \frac{p}{p}\right) = \left(\overset{1}{0}, \overset{2}{\frac{1}{p}}, \overset{3}{\infty}, \overset{4}{1}\right)$

  In V, $\underbrace{\overset{1}{0}, \overset{4}{1} \text{ never coincide}}_{\text{when } p \to 0}$, so $\overset{2}{\frac{1}{p}}, \overset{3}{\infty}$ coincide (at ∞)



These two are the same pt (I simply change the coordinate).

$\Longrightarrow$ $p \to 0$ the limit is



$\leftarrow$ nodal pt $\in \overline{M}_{0,4} \setminus M_{0,4}$.

Note that there are three ways to have such "2-spheres breaking"

so $\overline{M}_{0,4} = M_{0,4} \cup \{3\text{pts}\}$ $(\simeq S^2)$.

<u>Example</u> $g = 0$, $\ell = 5$

$$M_{0,5} = \left\{ [(S^2, j, (0,1,\infty,p,q))] \;\middle|\; \begin{array}{l} p \neq 0,1,\infty,\ q \neq 0,1,\infty \\ p \neq q \end{array} \right\}$$

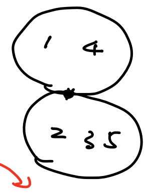
$\dim_{\mathbb{R}} M_{0,5}$
$= 6\cdot 0 - 6 + 2\cdot 5 = 4$

$$\simeq \left(S^2 \setminus \{0,1,\infty\}\right) \times \left(S^2 \setminus \{0,1,\infty\}\right) - \underbrace{\Delta \text{ of the previous product}}_{\simeq S^2 \setminus \{0,1,\infty\}}$$

$$\overline{M_{0,5}} = M_{0,5} \cup \begin{pmatrix} \text{codim}_{\mathbb{R}} 2 \\ \text{statum} \end{pmatrix} \cup \begin{pmatrix} \text{codim}_{\mathbb{R}} 4 \\ \text{statum} \end{pmatrix}$$

$\updownarrow$ 1 pairs of marked pts coincide

$\updownarrow$ 2 pairs of marked pts coincide

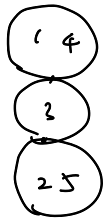


$\in M_{0,3}$

$\in M_{0,4}$

$\dim_{\mathbb{R}} = 2$

$$\underset{\binom{5}{3}}{10} \times M_{0,3} \times M_{0,4}$$



$\in M_{0,3}$

$\in M_{0,3}$

$\in M_{0,3}$

$$15 \times M_{0,3} \times M_{0,3} \times M_{0,3}$$

$\dim_{\mathbb{R}} = 0$

One can check that $\overline{M_{0,5}}$ is <u>not</u> a mfd.

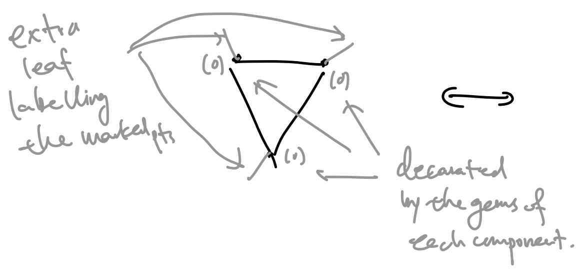
(In fact, $\overline{M_{0,5}} = S^2 \times S^2 - 7 \times (S^3 \setminus \{0,1,\infty\}) - 9$ pts.

$+ 10 \times (S^3 \setminus \{0,1,\infty\}) + 15$ pts.)

<u>Rmk</u> One can use graph to describe elements in $\overline{M_{g,\ell}} \setminus M_{g,\ell}$:

vertices $\longleftrightarrow$ component in $(\Sigma, j, \cdot)$

edges $\longleftrightarrow$ nodal pts in Σ.



$\longleftrightarrow$