$$\Phi_1 : \pi^{-1}(U_1) \longrightarrow U_1 \times \mathbb{R} \qquad [(s,t)] \longmapsto (e^{2\pi i s}, t)$$

$$\pi^{-1}(U_1) = \left\{ [(s,t)] \;\middle|\; \begin{array}{l} s \in (0,1) \\ t \in \mathbb{R} \end{array} \right\}$$

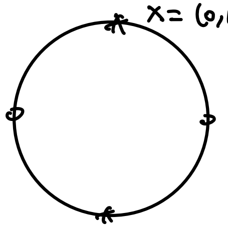
(identity on t)

$$\Phi_2 : \pi^{-1}(U_2) \longrightarrow U_2 \times \mathbb{R} \qquad [(s,t)] \longmapsto \begin{cases} (e^{2\pi i s}, t) & s \in [0, \frac{1}{2}) \\ (e^{2\pi i (s-1)}, -t) & s \in (\frac{1}{2}, 1] \end{cases}$$

$$\pi^{-1}(U_2) = \left\{ [(s,t)] \;\middle|\; \begin{array}{l} s \in [0, \frac{1}{2}) \cup (\frac{1}{2}, 1] \\ t \in \mathbb{R} \end{array} \right\}$$

Φ_2 is well-defined b/c $[(0,t)] \mapsto (1,t)$ and $[(1,-t)] \mapsto (1,t)$ ✓.

- About transition maps:

$U_1 \cap U_2 =$ 

$x = (0,1) \leftrightarrow s = \frac{1}{4}$

$y = (0,-1) \leftrightarrow s = \frac{3}{4}$

$$\pi^{-1}(x) = \{[(\tfrac{1}{4}, t)] \mid t \in \mathbb{R}\} \quad \xrightarrow{\Phi_1} (x,t), \quad \xrightarrow{\Phi_2} (x,t), \qquad (x,t) \downarrow (x,t) \qquad g_{12}(x) = \mathbb{1}$$

$$\pi^{-1}(y) = \{[(\tfrac{3}{4}, t)] \mid t \in \mathbb{R}\} \quad \xrightarrow{\Phi_1} (y,t), \quad \xrightarrow{\Phi_2} (y,-t), \qquad (y,t) \downarrow (y,-t) \qquad \underline{g_{12}(y) = -\mathbb{1}}.$$

$\det = -1$

<u>Rmk</u>. This $\begin{array}{c} E \\ \downarrow \\ S^1 \end{array}$ is called <u>Möbius bundle</u>

Prop Every real vector bundle over S^1 of rank 1 is either the trivial one $S^1 \times \mathbb{R}$ or the Möbius bundle.

- Observe that info of a vector bundle lies in the transition maps.
- In fact, one can construct a vector bundle via following data

Input: M^n open cover $\{U_\alpha\}_\alpha$ and $g_{\alpha\beta} : U_\alpha \cap U_\beta \to GL(n, k)$
GL(k, \R)
s.t. $g_{\alpha\alpha} = \mathbb{1}$ and $g_{\beta\gamma} \circ g_{\alpha\beta} = g_{\alpha\gamma}$.

\R^k

Output $E := \coprod_\alpha (U_\alpha \times \mathbb{R}^n) / \sim$ where $(x, v) \sim (y, w)$ iff $x = y$ and $w = g_{\alpha\beta}(v)$.

\R^k

e.g. If $g_{\alpha\beta} \equiv \mathbb{1}$, then $E = M \times \mathbb{R}^n$

$(U_\alpha, \varphi_\alpha : U_\alpha \simeq V_\alpha \subset \mathbb{R}^n)$
local chart

e.g. More interesting, take $g_{\alpha\beta}(x) \cdot v := d(\varphi_\beta \circ \varphi_\alpha^{-1})(x)(v)$

$=$ directional derivative of map $\varphi_\beta \circ \varphi_\alpha^{-1}$ at pt x along the direction v.

This construction contains info only from a mfd.

- Then $\begin{array}{c} E \\ \downarrow \pi \\ M \end{array}$ by $[(x,v)] \longmapsto x$ is a real vector bundle over M of rank n.

Explicitly,

$$\pi^{-1}(U_\alpha) \left(= \{ [(x,v)] \in E \mid x \in U_\alpha \} \right) \xrightarrow{\Phi_\alpha} U_\alpha \times \mathbb{R}^n$$

$$[(x,v)] \longmapsto (x, v) \quad \text{fiberwise } \checkmark$$

$$\Phi_\beta \circ \Phi_\alpha^{-1} : (U_\alpha \cap U_\beta) \times \mathbb{R}^n \longrightarrow (U_\alpha \cap U_\beta) \times \mathbb{R}^n$$

$$(x,v) \longrightarrow [(x,v)] \longrightarrow (x, g_{\alpha\beta}(x) \cdot v)$$

$$[(x,v)] = [(x, g_{\alpha\beta}(x) \cdot v)]$$

where $g_{\alpha\beta}(x)$ is a linear isomorphism

- This $\begin{array}{c} E \\ \downarrow \\ M \end{array}$ is called the tangent bundle of M, denoted by TM.

Rmk Following the same construction as the e.g. above, replace $\mathbb{R}^n$ by its dual space $(\mathbb{R}^*)^n = \{(f_1, \cdots, f_n) \mid \underset{\text{linear map}}{f_i : \mathbb{R} \to \mathbb{R}}\}$, then the resulting bundle is called <u>cotangent bundle</u> of M, denoted by T^*M. The transition maps are $(g_{\alpha\beta}(x)^T)^{-1}$.

余切从

② **Def** For a vector bundle $\begin{matrix} E \\ \downarrow \pi \\ M \end{matrix}$, a section $S: M \to E$ is a smooth map s.t. $(\pi \circ S)(x) = x$ for any x. The set of all sections of $\begin{matrix} E \\ \downarrow \pi \\ M \end{matrix}$ is denoted by $\Gamma(M, E)$ (or simply $\Gamma(E)$).

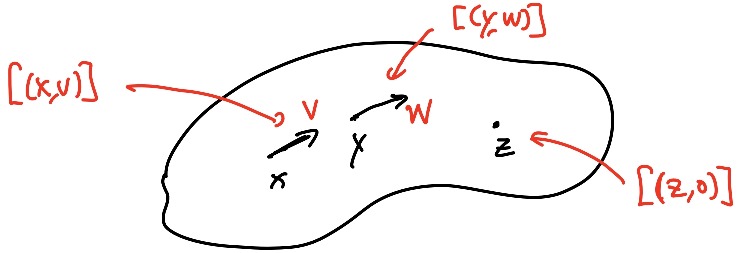
e.g. $E = M \times \mathbb{R}$, then a section $S: M \to M \times \mathbb{R}$ can be identified with smooth fcns on M. Moreover

$$\Gamma(M, M \times \mathbb{R}) \simeq C^\infty(M; \mathbb{R})$$

Rmk. For any $\begin{matrix} E \\ \downarrow \pi \\ M \end{matrix}$, $\Gamma(M, E)$ is a $C^\infty(M; \mathbb{R})$-module.

if $S \in \Gamma(M, E)$, then $f \cdot S \in \Gamma(M, E)$ for any $f \in C^\infty(M; \mathbb{R})$

e.g $E = TM$, then a section $s: M \to TM$ is called a <u>vector field</u> on M (向量場)

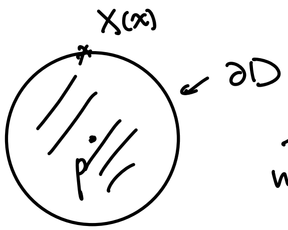


$$\Gamma(M, TM) = \{ \text{(smooth) vector field } X \text{ on } M \}$$

- One of the most interesting questions in diff top is asking for a given $X \in \Gamma(M, TM)$, how many 0's does X possess ?

(assume X only has "isolated" 0's and $\dim_{\mathbb{R}} X = 2$)

Pick a disk around zero pt



$X(p) = 0$

normalization

$$\partial D \xrightarrow{\text{Gauss map}} S^1 = \mathbb{R}/\mathbb{Z} \quad \text{unit circle in } \mathbb{R}^2$$

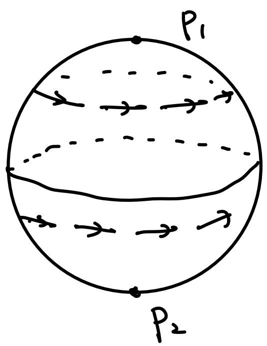
$$x \in \partial D \longmapsto \frac{X(x)}{\|X(x)\|}$$

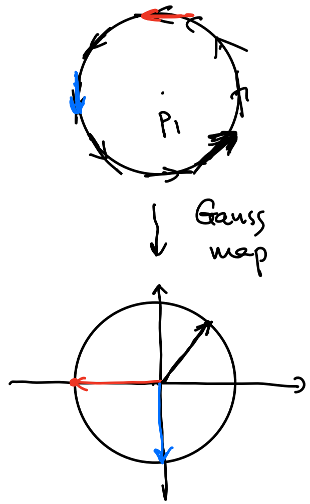
Counting rotation number gives index(p). ← could be negative
well-defined on higher dim'l

<u>Fact</u> (Poincaré–Hopf) For a closed mfd M, for $X \in \Gamma(TM)$ with isolated 0's, we have

analysis
local →
$$\sum_{\substack{p \in X \\ \text{zero}}} \text{index}(p) = \text{top invariant of } M$$
(Euler char. of M) ← topology
global

e.g.





index$(p_1) = 1$

Similarly, index$(p_2) = 1$

so by Poincaré–Hopf:

$$\text{index}(p_1) + \text{index}(p_2)$$
$$= 1 + 1$$
$$= 2 \quad (= \text{Euler char. of } S^2).$$

$\Rightarrow$ If M has Euler char non-zero, then any $X \in \Gamma(TM)$ has a zero pt.

(and then TM is <u>not</u> a trivial bundle).

e.g. TS^{even} is not trivial.

<u>Fact</u>. TS^n is trivial only for $n = 1, 3, 7$.

e.g $E = T^*M$, then a section $s: M \to T^*M$ is called a <u>1-form</u> on M

$$\Gamma(M, T^*M) =: \Omega^1(M)$$

Note that we have a natural pairing for $\alpha \in \Omega^1(M)$ and $X \in \Gamma(TM)$

$$\langle \alpha, X \rangle \text{ or } \alpha(X) \in C^\infty(M; \mathbb{R}).$$

<u>Rmk</u> Forms works better than vector fields.

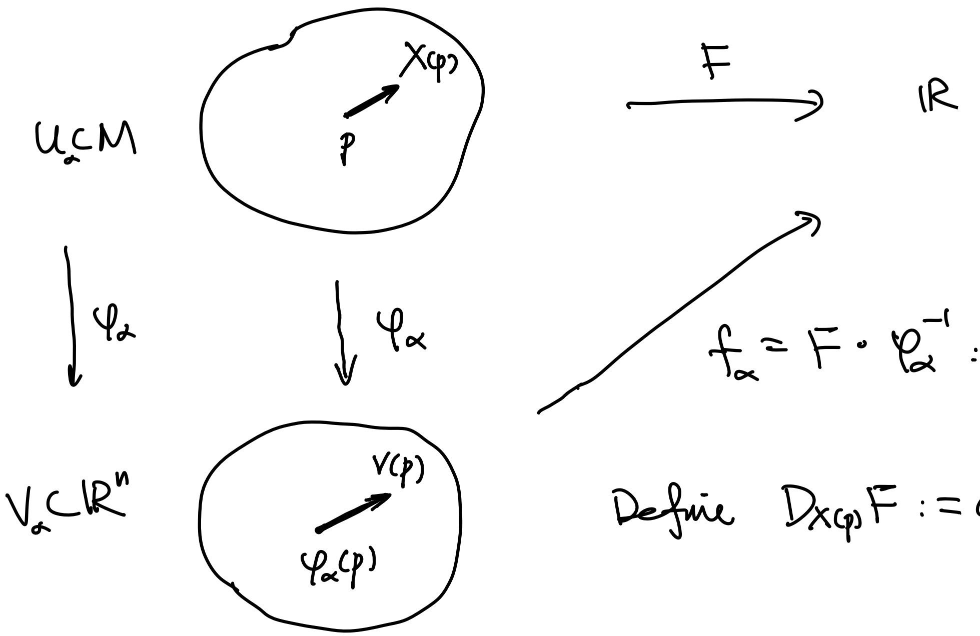
<u>Rmk</u> For any pt $x \in M^n$, the fiber $\pi^{-1}(\{x\})$ of $\begin{array}{c} TM \\ \downarrow \pi \\ M \end{array}$ is denoted by T_xM $(\simeq \mathbb{R}^n)$. similarly T_x^*M $(\simeq (\mathbb{R}^*)^n)$.

③ Connection (聯絡)

- Consider a vector field $X \in \Gamma(TM)$ and a smooth function $F: M \to \mathbb{R}$. define "directional derivative of F along X", denoted by

$$X(F) \quad \text{or} \quad D_X F$$

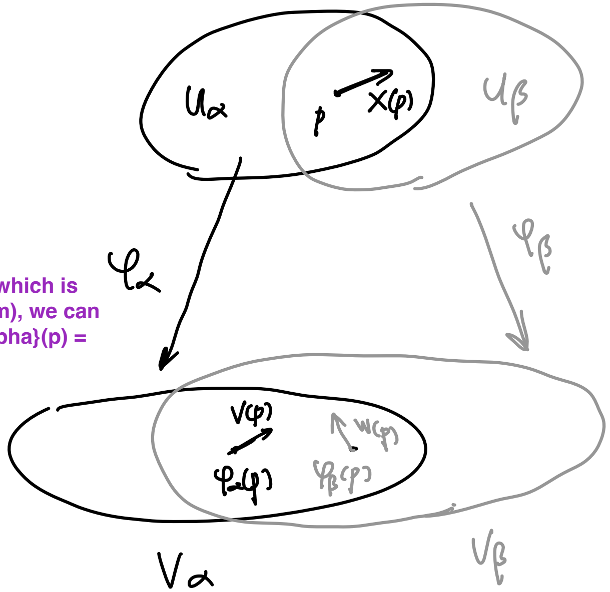
pointwise by directional derivative of F at $\underset{\in T_pM}{X(p)}$ for any $p \in M$. $\leftarrow D_{X(p)}F$



$$f_\alpha = F \circ \varphi_\alpha^{-1} : V \subset \mathbb{R}^n \to \mathbb{R}$$

Define $D_{X(p)}F := df_\alpha(\varphi_\alpha(p))(V(p))$ $\leftarrow$ directional derivative in calculus

Well-definedness:



up to a shift in \R^n (which is also a diffeomorphism), we can assume that \phi_{\alpha}(p) = \phi_{\beta}(p).

Similarly, one define directional derivative of $F: M \to \mathbb{R}^k$ along a vector field $X \in \Gamma(TM)$.

$$df_\alpha(\varphi_\alpha(p))(V(p))$$

$$= d(F\circ\varphi_\alpha^{-1})(\varphi_\alpha(p))(V(p))$$

$$= d\big((F\circ\varphi_\beta^{-1})\circ(\varphi_\beta\circ\varphi_\alpha^{-1})\big)(\varphi_\alpha(p))(V(p))$$

$$= d\big(f_\beta\circ(\varphi_\beta\circ\varphi_\alpha^{-1})\big)(\varphi_\alpha(p))(V(p))$$

recall

$$W(p) = d(\varphi_\beta\circ\varphi_\alpha^{-1})\, V(p)$$

$$\Big(\text{so}\quad V(p) = d(\varphi_\alpha\circ\varphi_\beta^{-1})(W(p))\Big)$$

$$= \Big(df_\beta(\varphi_\beta(p))\circ d(\varphi_\beta\circ\varphi_\alpha^{-1})(\varphi_\alpha(p))\Big)\Big(d(\varphi_\alpha\circ\varphi_\beta^{-1})(W(p))\Big)$$

$$= df_\beta(\varphi_\beta(p))(W(p)) \qquad \checkmark$$

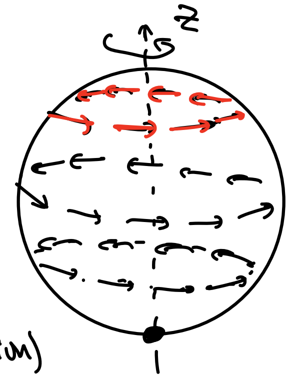
Rmk For $F: M \to \mathbb{R}^k$ and $X \in \Gamma(TM)$, the directional derivative $X(F)$ (or $D_X F$) is also a (smooth) function from M to $\mathbb{R}^k$

e.g.

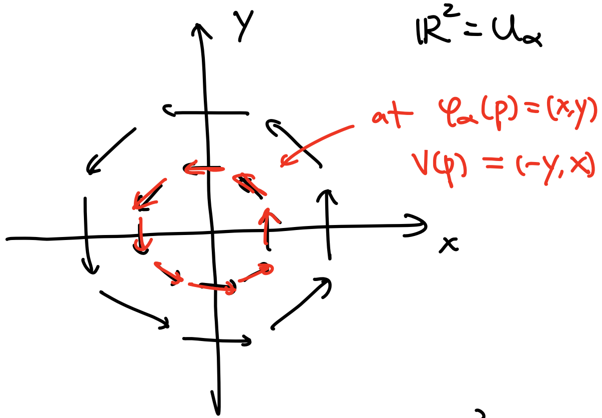
Case 1

vector field

X (rotation)



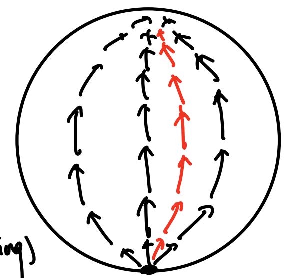
stereographic projection (from south pole)



Case 2

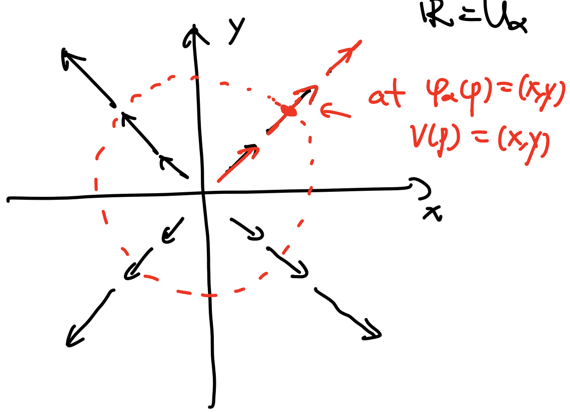
vector field

Y (ascending)



stereographic projection (from south pole)

north pole

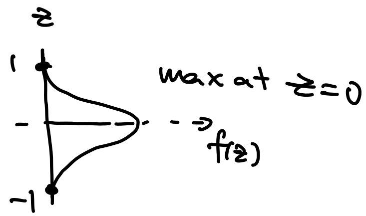


$F: S^2 \to \mathbb{R}$ by height function ($=$ z-coordinate) $\longleftrightarrow$ $f_\alpha(x,y) = \frac{1}{2}(x^2+y^2)$

$\Rightarrow df_\alpha(x,y) = (x,y)$

So in case ① $D_X F \equiv 0$

in case ② $D_Y F = f(z)$ where



Reflection: in case ① $\overset{=z}{F}$ is constant along each level set $F^{-1}(\{z\})$.
纬线 (Longitude)

in case ② $\overset{=z}{F}$ is increasing along each latitude.
经线

- Def A connection on vector bundle $\begin{array}{c} E \\ \downarrow \pi \\ M \end{array}$ is a map

$$\nabla : \Gamma(TM) \times \Gamma(E) \longrightarrow \Gamma(E)$$

satisfying, for $f, g \in C^\infty(M)$,

① $\nabla_{fX+gY}\, s = f \nabla_X s + g \nabla_Y s$

② $\nabla_X (s_1 + s_2) = \nabla_X s_1 + \nabla_X s_2$

③ $\nabla_X (fs) = \underbrace{X(f)}_{\in C^\infty(M)} \cdot s + f \nabla_X s$