3. Symplectic embedding

Given two Liouville domain (U, ω_u) and (V, ω_v), a symp emb is an emb $\varphi: U \hookrightarrow V$ s.t. $\varphi^* \omega_v = \omega_u$.

Ex. $(\mathbb{R}^{2n}, \omega_{std})$ $(\simeq (\mathbb{C}^n, \omega_{std}))$

$$E(a_1, \cdots, a_n) = \left\{ (z_1, \cdots, z_n) \in \mathbb{C}^n \;\middle|\; \frac{\pi|z_1|^2}{a_1} + \cdots + \frac{\pi|z_n|^2}{a_n} \leq 1 \right\}$$ ← symplectic ellipsoid

$$B^{2n}(r) = E(r, \cdots, r) \quad , \quad Z^{2n}(R) = E(R, \underbrace{\infty, \cdots, \infty}_{\text{no constraint on } z_2, \cdots, z_n})$$

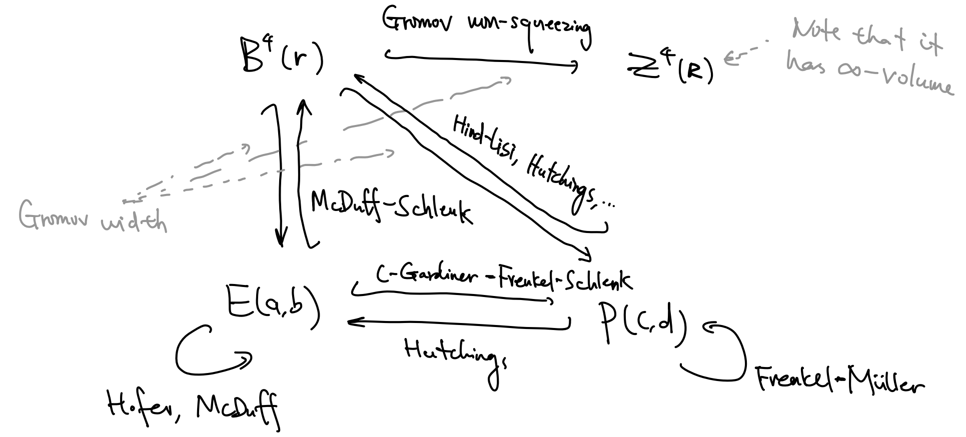
↑ symplectic ball ↑ symplectic cylinder.

$$P(a_1, \cdots, a_n) = \left\{ (z_1, \cdots, z_n) \in \mathbb{C}^n \;\middle|\; \pi|z_1|^2 \leq a_1, \cdots, \pi|z_n|^2 \leq a_n \right\}$$ ← polydisk

Rmk. $\partial P(a_1, \cdots, a_n)$ is not smooth.

Rmk All cases above admit $\mathbb{T}^2$-action by $(\theta_1, \cdots, \theta_n) \cdot (z_1, \cdots, z_n)$ defined by $(e^{i\theta_1} z_1, \cdots, e^{i\theta_n} z_n)$. $\Rightarrow$ toric domain

Finding obstructions of embedding is a central topic in symp. geo.



Thm (Gromov) $B^{2n}(r) \hookrightarrow Z^{2n}(R)$ iff $r \leq R$.

Thm $P(a,b) \hookrightarrow P(c,d)$ with $a \leq c$ but $b \geq d$, then $c \geq 2a$. //

For $U, V \ (\subset \mathbb{R}^{2n})$ Liouville domains (w.r.t Y radial v.f.).
$\Leftarrow$ also called star-shaped.

Assume $\varphi(U)$ is also star-shaped, then $V \setminus \mathrm{int}(\varphi(U)) =: W$ satisfies the following properties

- $\partial W = -\partial\varphi(U) \sqcup \partial V$
  with negative orientation

- Both $(\partial\varphi(U), \ker\lambda|_{\partial\varphi(U)})$ and $(\partial V, \ker\lambda|_{\partial V})$ are contact mfds
  where $\lambda = \iota_Y \omega_{std}$.

- $Y \pitchfork \partial V$ pointing outward, $Y \pitchfork \partial\varphi(U)$ pointing inward.

Such $(W, \omega_{std}|_W)$ is a Liouville cobordism from $\partial\varphi(U)$ to ∂V.
or exact

Rmk More general, if Y is not globally defined, i.e. when $\partial W = -\partial W_- \sqcup \partial W_+$
$\exists\, Y_-$ near ∂W_- and Y_+ near ∂W_+.

Ex $(X, \xi = \ker\alpha) \rightsquigarrow (M = \mathbb{R}\times X,\ \omega = d(e^r\alpha))$

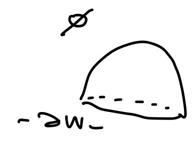
Given any two functions $f_-, f_+ : X \to \mathbb{R}$ s.t. $f_- < f_+$ ptwise,

$\{(r,x)\in\mathbb{R}\times X \mid r \le f_+(x)\}$ $\quad$ $\{(r,x)\in\mathbb{R}\times X \mid r \ge f_-(x)\}$ $\quad$ $W = \{f_-(x) \le r \le f_+(x)\}$

Then W is a Liouville cobordism from $\{r = \underline{f}(x)\}$ to $\{r = f_+(x)\}$

- In particular, a Liouville domain $\mathcal{U}$ is a Liouville cobordism W with $\partial W_- = \emptyset$.
- For a contact mfd (X,ξ), a Liouville filling (or an exact filling) is a Liouville domain $(\mathcal{U}, \underset{d\lambda}{\overset{\shortparallel}{\omega}})$ s.t. $(\partial\mathcal{U}, \ker\lambda|_{\partial\mathcal{U}}) = (X,\xi)$.

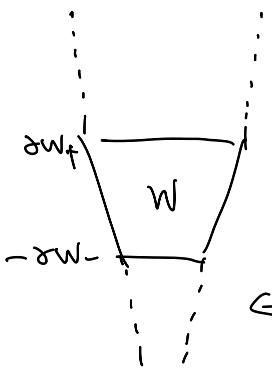
Rmk $\nexists$ Liouville cobordism $(W, \omega = d\lambda)$ s.t. $\partial W_+ = \emptyset$.



$$\underbrace{\int_W \omega\wedge\cdots\wedge\omega}_{>0} = \int_W d(\lambda\wedge d\lambda\wedge\cdots\wedge d\lambda)$$
$$= \int_{\partial W_+} \lambda\wedge d\lambda\wedge\cdots\wedge d\lambda - \int_{\partial W_-} \lambda\wedge d\lambda\wedge\cdots\wedge d\lambda$$
$$= 0 - (\text{positive}) < 0 \qquad \rightarrow\leftarrow$$

$\Rightarrow$ symplectic cobordism is NOT an equivalence relation.
(not symmetric). //

Given a symplectic cobordism (W, ω), one can "complete" it by gluing <u>positive cylindrical end</u> and <u>negative cylindrical end</u>



$\leftarrow$ $[0,\infty) \times \partial W_+ \subset$ symplectization of $(\partial W_+, \ker \lambda_+|_{\partial W_+})$

$\leftarrow$ $(-\infty, 0] \times (-\partial W_-) \subset$ symplectization of $(\partial W_-, \ker \lambda_-|_{\partial W_-})$

<u>Ex</u> For a Liouville domain $(U, \omega = d\lambda)$, its completion is a non-cpt exact symp mfd $(\hat{U}, \omega = d\lambda)$

$$E(a_1, \cdots, a_n) \xrightarrow{\text{completion}} (\mathbb{C}^n, \omega_{std})$$

$$(Q, g) \to U_g^* Q = \{(q,p) \mid \|p\|_{g^*} \leq 1\} \quad \text{unit codisk bundle}$$

$$\xrightarrow{\text{completion}} (T^*Q, \omega_{can})$$

<u>Rmk</u> Completion allows us to work with a non-cpt (exact) symp mfd instead of a mfd with b/d (which is always harder). //