Def: (Global Kuranishi)

(1) G cpt lie gp

(2) $\mathcal{T}$ topological mfd, G action + finite stabilizers

(3) $E \to \mathcal{T}$ is a G-bundle

(4) $s: \mathcal{T} \to E$ is a G-section

Equivalently, this is an orbi-bundle with an orbi-section
$$\begin{tikzcd} E/G \arrow[r] & \mathcal{T}/G \arrow[l, bend left, "s"] \end{tikzcd}$$

A Kuranishi chart for M is $(G, \mathcal{T}, E, s)$ as above together with $M \to s^{-1}(0)/G$. $\mathrm{vdim}(G,\mathcal{T},E,s) = \dim(\mathcal{T}) - \mathrm{rank}(E) - \dim(G)$

Ex: $(0, S^2, T^*S^2, s=0)$ gives $M = S^2$ a global chart. vfc of this chart is $-2[pt] \in H_0(S^2)$.

① (Freeness) If G action is free (non-orbifold)

② (Smooth) If $(\mathcal{T}, E)$ smth and G-action smth. (not s)

③ (regular) s is transverse.

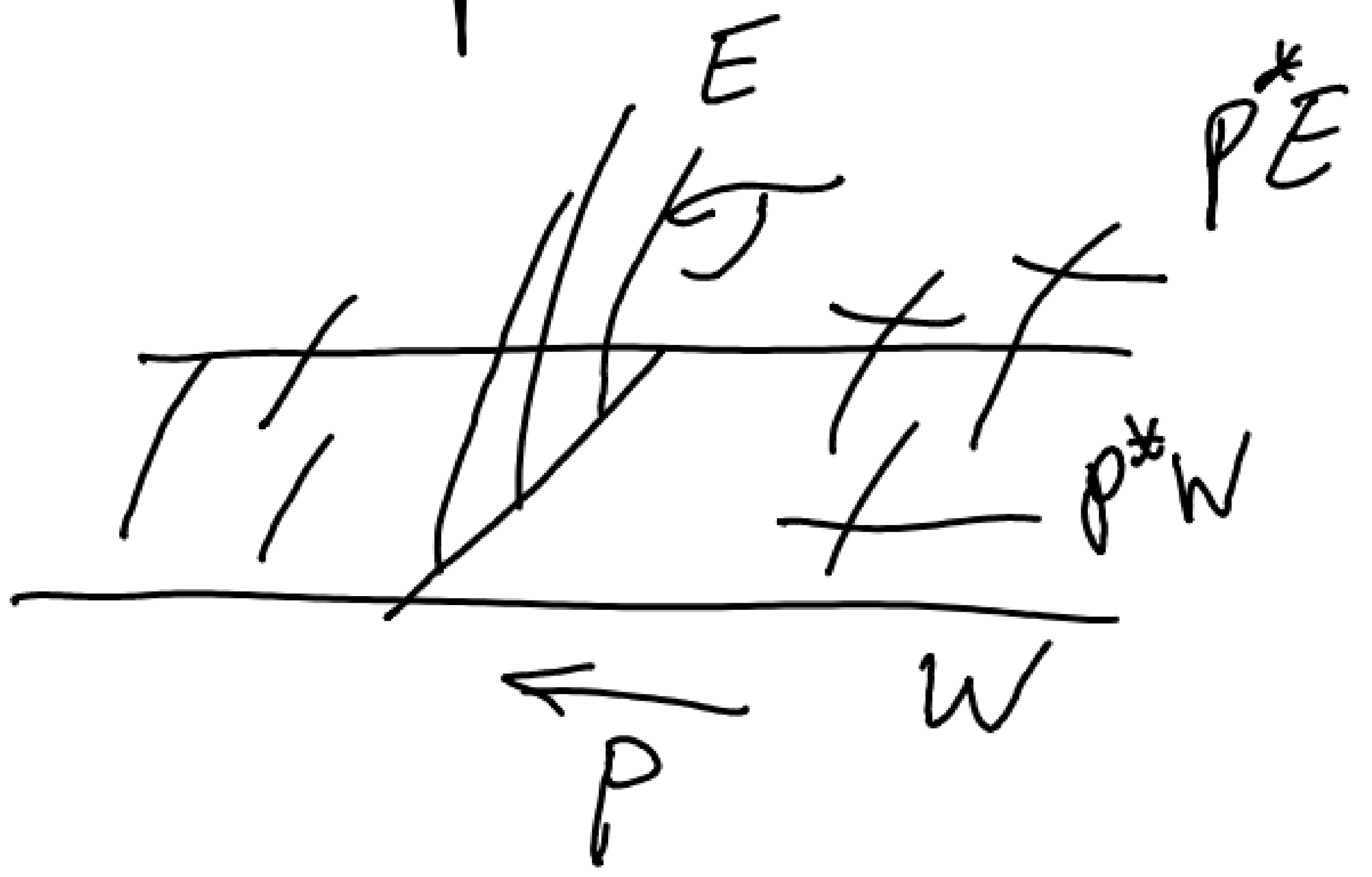
<u>Remark:</u> It is sufficient to have such properties near $s^{-1}(0)$.

<u>3 equivalences:</u>

(1) (Germ equivalence): take $U \subset \mathcal{T}$ open near $s^{-1}(0)$

(2) (Stabilization): $p: W \to \mathcal{T}$ a G-bundle, replace the chart by $(G, W, p^*E \oplus p^*W, p^*s \oplus \Delta)$, where Δ = tautological diagonal section of

$$p^*W \to W$$



<u>Example:</u> $W = \mathcal{T} \times \mathbb{R}^n$, then we take $E \times \mathbb{R}^n \times \mathbb{R}^n$, and $p^*s(x,v) = (s, v, v)$, whose zero is still $s^{-1}(0)$.

(3) (Group enlargement) $G' = $ cpt Lie, $q: P \to \mathcal{T}$ be G-equiv. principal G'-bdle, and replace by $(G \times G', P, q^*E, q^*s)$

<u>Lemma</u>: (Smoothing the evaluation map)

$\mathcal{K} = (G, \mathcal{T}, E, s)$ <u>smth</u> GKC for M.

$ev: \mathcal{T} \to X^{(\text{smth})}$ continuous. (not nec. <u>smooth</u>)

Then $\exists$ $\begin{cases} \text{cont. } G\text{-equiv. } \widetilde{ev}: ev^*TX \to X, \\ \left(\begin{array}{c} C^0\text{-small fiber-preserving } G\text{-equiv. homeo} \\ h: ev^*TX \to ev^*TX \end{array}\right) \end{cases}$

<u>s.t.</u> (i) $\widetilde{ev}|_{\mathcal{T}} = ev$

(ii) ev^*TX admits a smth str. s.t. $\widetilde{ev} \circ h$ smth submersion near $\mathcal{T}$.

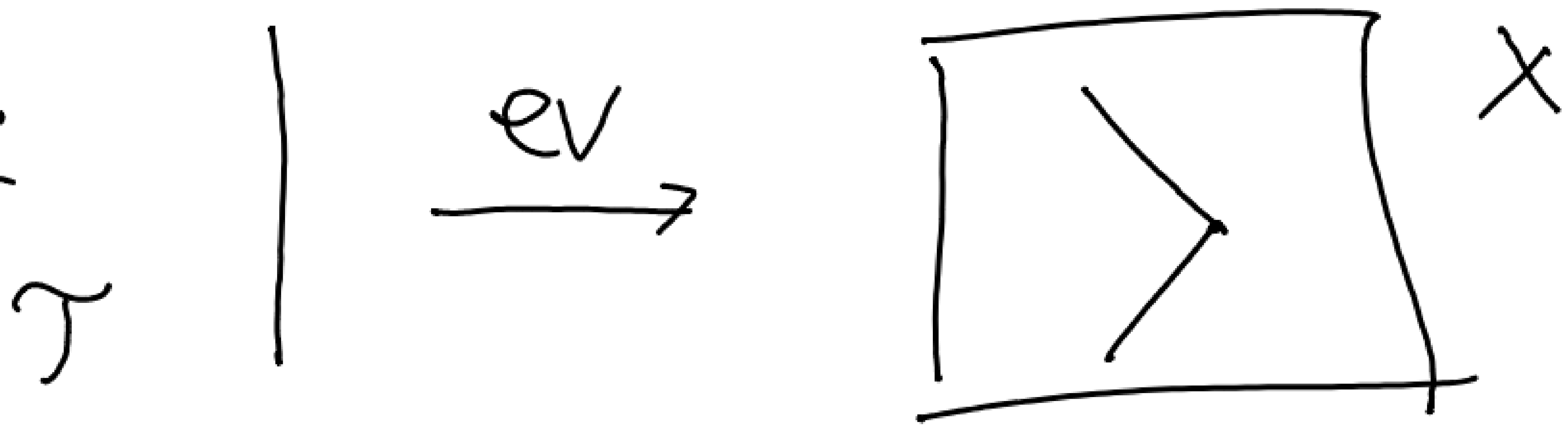
(iii) $h = C^0$-small, isotopic through G-equ. homeo to id.

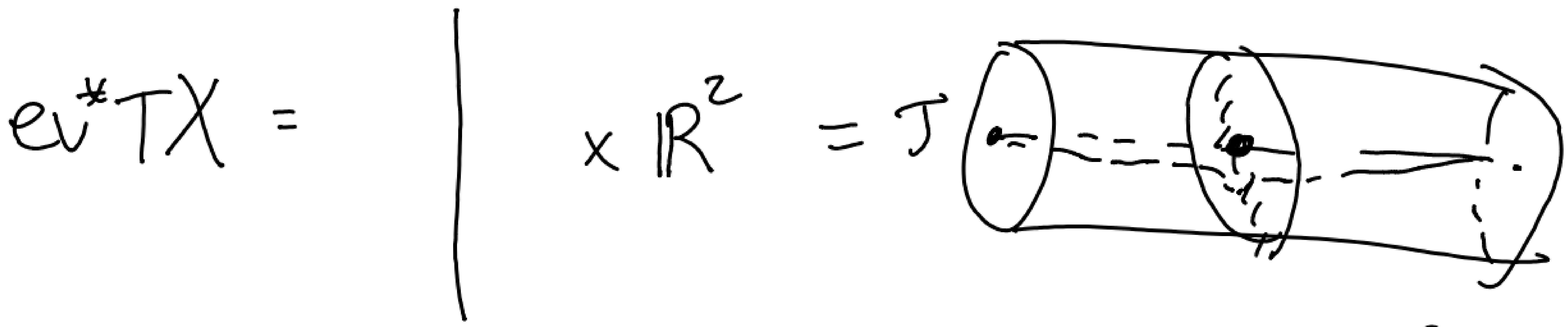
(iv) if ev smth near $K \subset \mathcal{T}$, $h \equiv \mathrm{id}$ near K.

In particular, stabilization of $\mathcal{K}$ by ev^*TX is smooth + smooth submersion to X extending ev.

(example-proof next page)

Ex:





to make $\widetilde{ev}$ smth, construct a map in $J \times \mathbb{R}^2$ by a cut-off exp. map, so that $\widetilde{ev} \circ h$ smth submersion.

$\boxed{+ \text{ page } 9}$

Micro-bundles

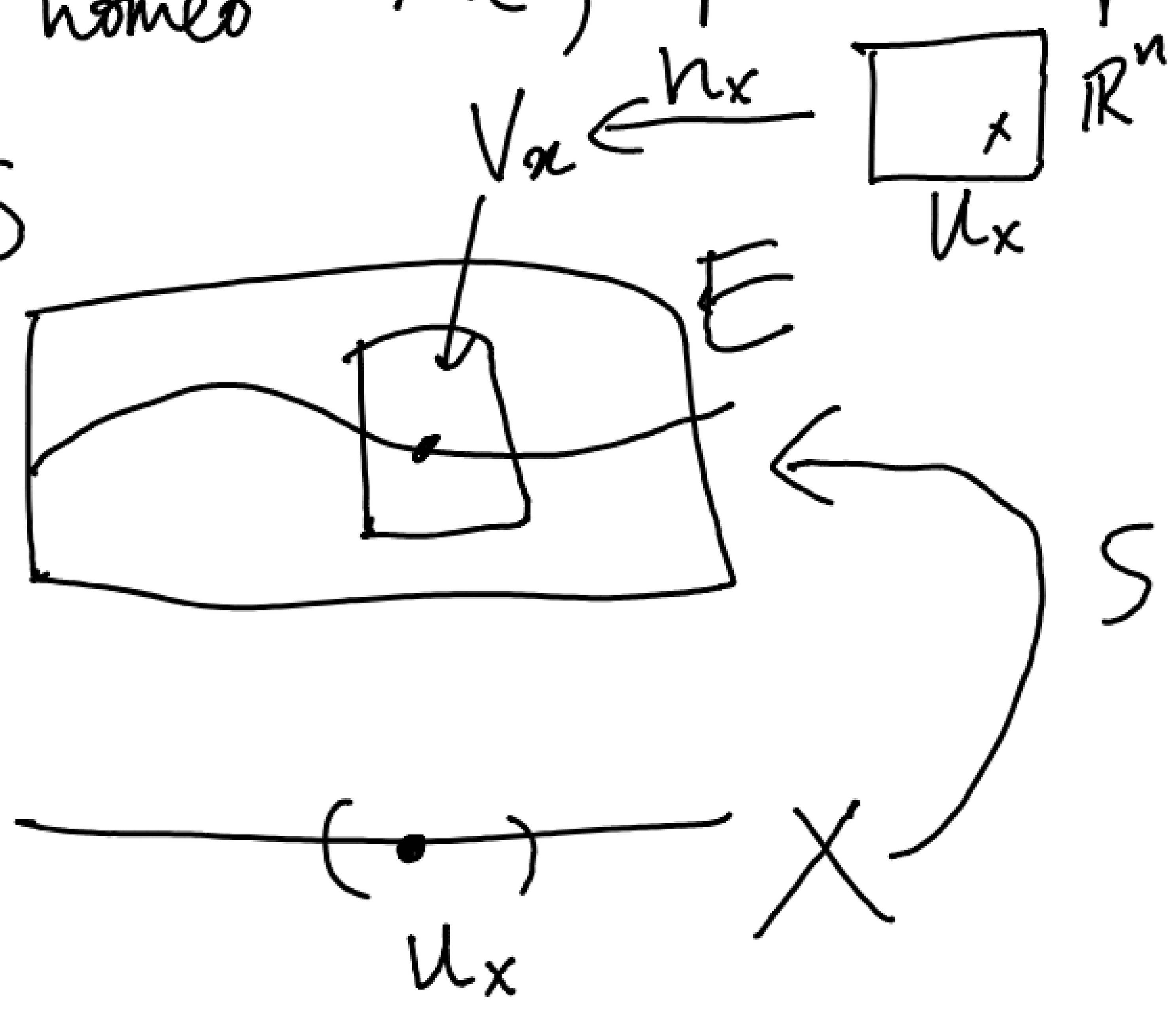
Def: $\{ X \xrightarrow{s} E \xrightarrow{p} X \}$ is a microbundle

(1) $p \circ s = id_X$

(2) $\forall x \in X$, $\exists\, U_x \subset X$, $s(x) \in V_x \subset E$,

$h_x : U_x \times \mathbb{R}^n \xrightarrow[\text{homeo}]{\sim} V_x$, $p \circ h_x = pr_{U_x}$,

$h_x|_{U_x \times \{0\}} = s$



Morphism/Isom between microbundles

$$
\begin{tikzcd}
 & E_1 \arrow[dr, "p_1"] & \\
X \arrow[r, "s_1"] \arrow[d, equal] & U_1 \arrow[u, hook] \arrow[d, "\phi"] & X \arrow[d, equal] \\
X \arrow[r, "s_2"] & E_2 \arrow[r, "p_1"] & X
\end{tikzcd}
$$

If ϕ is a homeomorphism, E_1/E_2 are Isom.

Def: (Tangent microbundle) $T_\mu X$

$$\{ X \xrightarrow{\Delta} X \times X \xrightarrow{p_1} X \}$$

Rmk: For smooth X, one observes $\exists$ Isom. b/w N_{Δ_X} and T_X by

$$T_X \ni (x, v) \longmapsto ((x,x), (v, -v)).$$

Def: (Vector bundle lift of E) $\exists$ vector bundle V with isom from V_μ to E.

Def: $X \xrightarrow{s} E \xrightarrow{p} X$ microbundle and $f : \check{X} \longrightarrow X$ cont map then pullback is $\check{X} \xrightarrow{f^*s} f^*E \xrightarrow{f^*p} \check{X}$ where

$$f^*E = \{ (e, x) \in E \times \check{X} \mid p(e) = f(x) \} \subset E \times \check{X}$$

$f^*s : x \mapsto (sf(x), x)$, $f^*p : (e, x) \mapsto x$. (If $\check{X}$ is a subset of X and f the corresponding inclusion then $E|_{\check{X}}$ to be the corresponding pullback f^*E.)

4.3 Smoothing theory

Prop (Lashof) M top manifold with a cont. action of a compact Lie group G. Assume

(1) There are finite orbit type

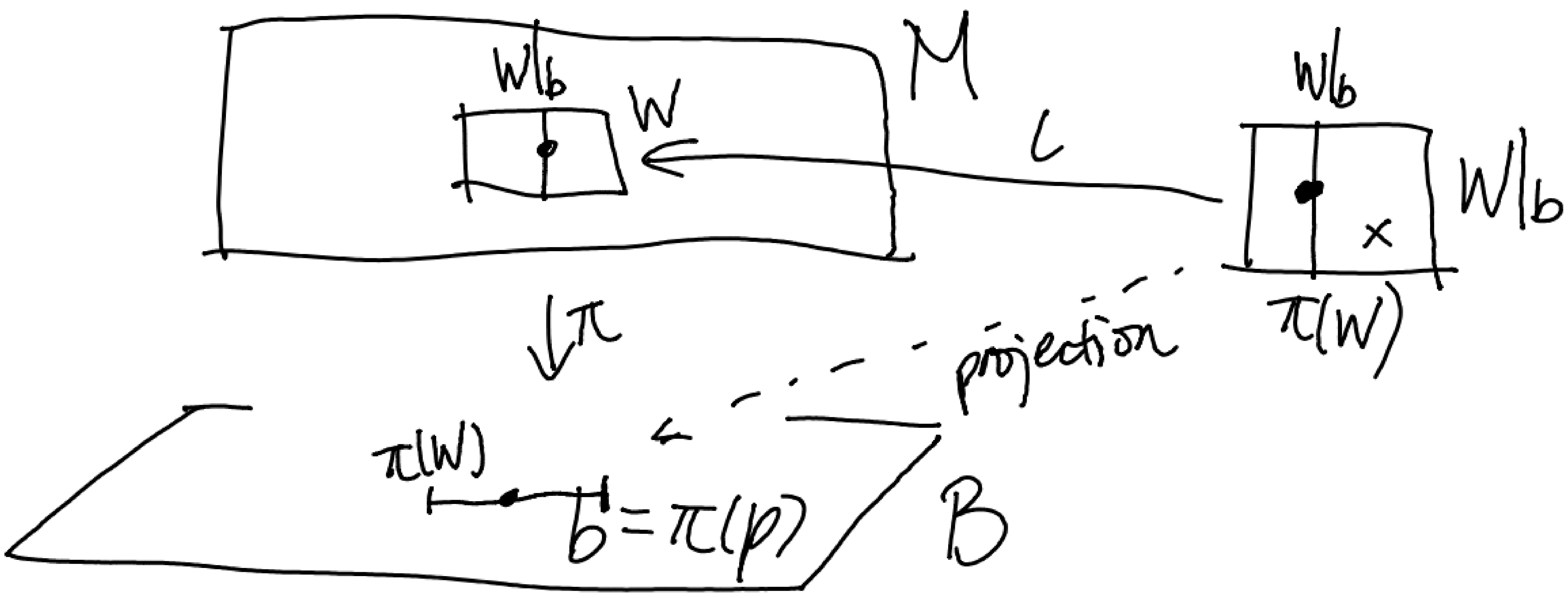
(2) Microbundle $T_\mu M$ admits a G-vect. bdle lift $l: E \to T_\mu M$.

Then $\exists$ G-rep'n V for which $V \times M$ admits G-equ. smth str. (actually, if $\dim M \neq 4$, M itself admits G-smth structure)

Corollary: Let $\mathcal{K} = (G, T, E, S)$ be a GKS so that G-action on T admits only finite orbit types, and so that $T_\mu T$ admits G-vect. bundle lift $l: V \to T_\mu T$. Then $\exists$ G-rep'n V, and a stabilization $\mathcal{K}'$ of (G, T, E, S) with thickening $T \times V$, which is equivalent to a smth GKC.

<u>Topological submersion</u>:

- $p \in M$, $b = \pi(p) \in B$.
- Product nbh W of p is a homeo



- π = top. submersion $\Leftrightarrow$ $\forall p \in M$ admit product nbh.
- When π = top. subm., $T^{vt}_{\mu}(\pi)$ of π is the microbdle

$$M \xrightarrow{\Delta} M \times_B M \xrightarrow{\pi \times \pi} M,$$

where $M \times_B M$ is the fiber product of π with itself, $(\pi \times \pi)(p,q) = \pi(p) = \pi(q)$.

<u>Example</u>: In the GW-problem, we will construct

$\mathcal{F}$:= moduli space of deg=d <u>stable</u> curves in $\mathbb{C}P^d$

$\mathcal{T}$:= moduli space of curves s.t. $\bar{\partial} u = \xi$ for some perturbation ξ.

And we will have a forgetful map

$$\Pi : \mathcal{T} \to \mathcal{F}$$

being a topological submersion. Here, $\mathcal{F}$ has a natural smooth structure.

To obtain a smooth structure of $\mathcal{T}$, we need to understand additional structures of Π, called "fiberwise smooth".

<u>Corollary 4.26</u>: B = loc. linear, $\pi : M \to B$ top smth, + <u>fiberwise linear</u>, then

$$T_\mu M \cong T^{vt}_\mu(M) \oplus \pi^*(T_\mu B)$$

<u>Lemma 4.29</u>: $\pi : M \to B$ is a <u>fiberwise smth C^1_{loc} bundle</u> (smooth fiber + C^1_{loc} dependence).

Then $T^{vt}_\mu M$ has a lift to $T^{vt} M$.

Therefore $T_\mu M$ can be lifted to a vector bundle and obtain a smth GKC when implemented to $\Pi : \mathcal{T} \to \mathcal{F}$.

Locally linear action:

Def: $\forall p$, $\exists$ chart U_p, s.t. G_p acts linearly on U_p.

Def: $\pi: M \to B$ top. submersion, π = fiberwise loc linear if G_b acts loc. linear on $\pi^{-1}(b)$, $\forall b \in B$.

Lemma: $\pi: M \to B$ G-equivariant top. submersion, fiberwise linear, then $\forall p \in M$, $\exists$ chart $(x_1, \cdots, x_n): U_p \to \mathbb{R}^n$.

- π = projection to last k coordinate
- $g \cdot U_p = U_p$
- $g(x_1, \cdots, x_n) = (A_g(x_1, \cdots, x_{n-k}), B_g(x_{n-k+1}, \cdots, x_n))$

block + linear.

概念

$\rightsquigarrow T_\mu M \simeq T_\mu^{vt} M \oplus \pi^* T_\mu B$

C^1_{loc} structures in smoothing thy. ($\pi : M \to B$, $\pi^{-1}(b)$ has smooth str.)

(1) $\iota_i : W_i \to W_i|_{b_i} \times \pi(W_i)$, $i=1,2$, product nbh's.

W_i are C^1_{loc}-compatible if for $\forall p \in W_1 \cap W_2$,

$\exists$ product nbh of p;

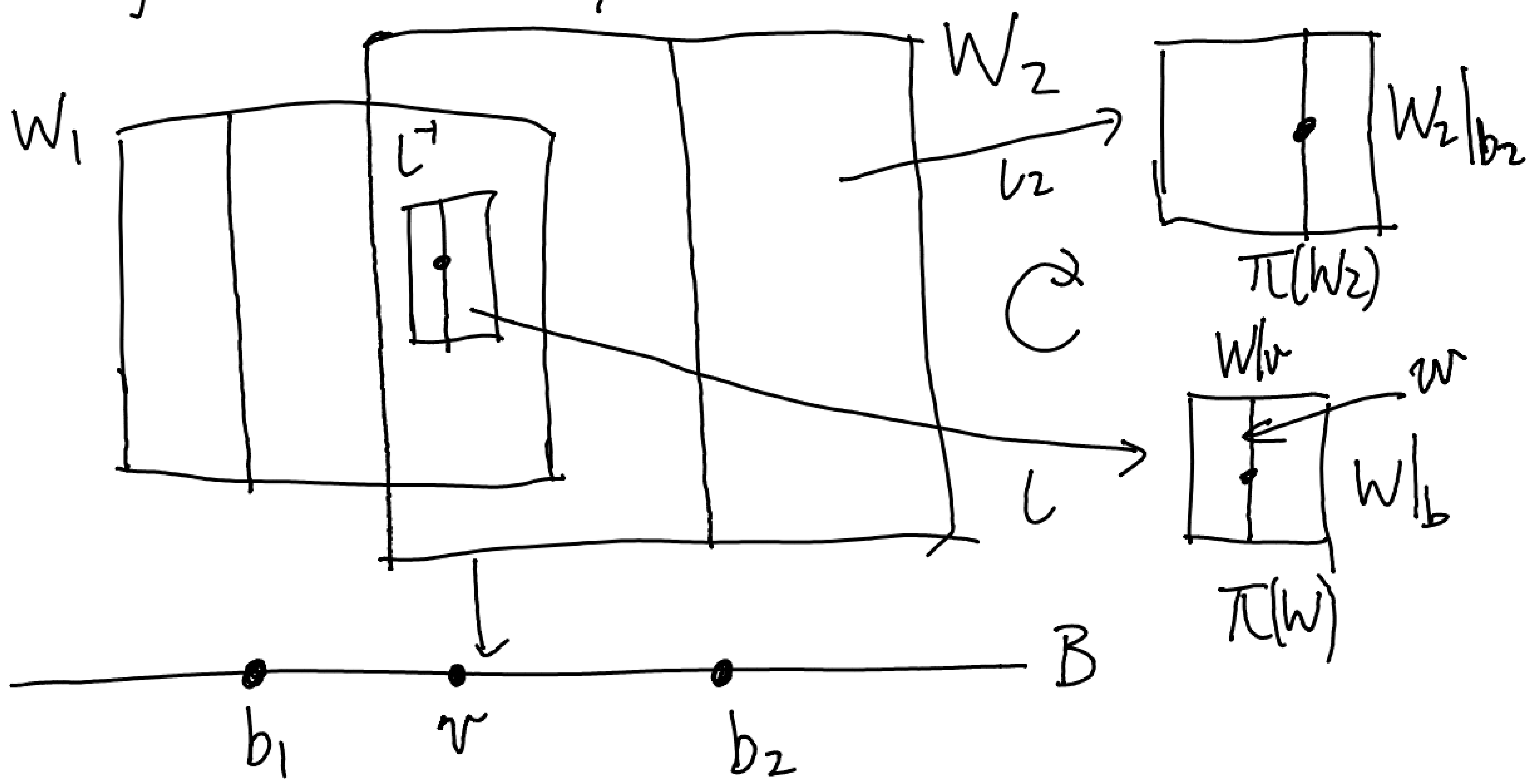
$$\iota : W \to W|_b \times \pi(W), \quad b = \pi(p)$$

so that we consider family

$$\eta_v : W|_v \to W_i|_{b_i}, \quad v \in \pi(W)$$

$$w \mapsto \Pi_i(\iota_i(\iota|_{W|_v})^{-1}(w)))$$ are smth,

and vary continuously w.r.t. C^1_{loc}-top. (Π_i to $W_i|_{b_i}$)



Def: $\pi: M \to B$ is a G-equiv. It is a fiberwise smooth C^1_{loc} G-bundle if

- $\exists$ collection of product nbh $\left(\iota_i : W_i \to W_i|_{b_i} \times \pi(W_i)\right)_{i\in I}$ around p_i which covers M.
- Each pair of product nbh are C^1_{loc}-compatible.

Def: $\pi: M \to B$ is fiberwise smth C^1_{loc} G-bundle. Then there is a G-vector bundle over M, denoted $T^{vt}M$,

Over a point $p \in M$, $(T^{vt}M)_p := T_p(M|_{\pi(p)})$

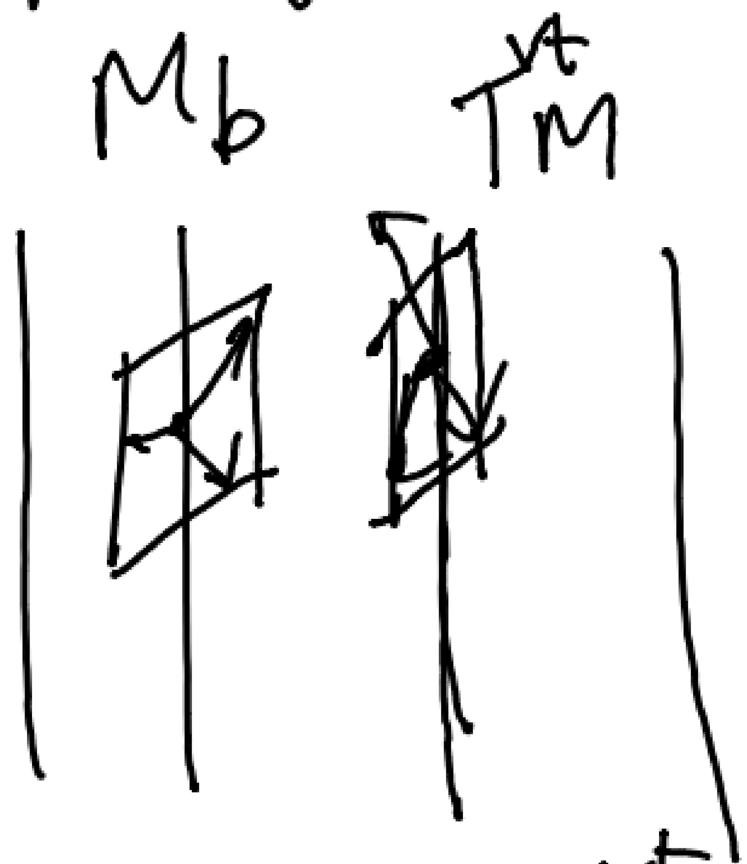
(C^1_{loc}-condition ensures that the vertical tangent space varies cont.)

Lemma: $\pi: M \to B$ fiberwise smth C^1_{loc} G-bundle, B = smth. Then $\exists$ natural G-equ. lift from $T_\mu M$ to $T^{vt}M$.

Pf: By a fiberwise PoU, $T^{vt}M$ has a metric, s.t. $M|_b = \pi^{-1}(b)$ has a smooth Riem. metric. Let

$$\exp : T^{vt}M \to M.$$

The restriction to each fiber $T^{vt}M|_{(M|_b)}$ is the exponential map of $M|_b$.



$$\text{Take} \quad T^{vt}M \xrightarrow{\exp} T^{vt}_{\mu}M \subset M \times_B M \quad (\text{maps to same fiber})$$
$$v \longmapsto (p, \exp(v)), \; p \in M, \; v \in T_p^{vt}M.$$

This defines a lift tautologically. $\boxtimes$

<u>Proof of main Result</u>: $\mathcal{K} := (G, \mathcal{T}, E, s)$ GKC,

$\mathcal{T} \longrightarrow B$ is a C^1_{loc} G-bdle over B = smth G-mfd.
Also, G-action on $\mathcal{T}$ has finite orbits (for smoothing)
Then $\exists$ stabilization $\mathcal{K}'$ of $\mathcal{K}$ which admits a smth str.

$$\mathcal{K}' = (G, \mathcal{T} \times V, E \times V \times V, s' = s \oplus \Delta_V)$$

pf: this follows from that $T_{\mu}\mathcal{T} \simeq T^{vt}_{\mu}\mathcal{T} \oplus \pi^* TB$ where both have a smooth lift.