- Consider

$$\underset{\text{tensor}}{T}^{\overset{\text{index}}{(k,l)}} V := \underbrace{V \otimes \cdots \otimes V}_{k \text{ copies}} \otimes \underbrace{V^* \otimes \cdots \otimes V^*}_{l \text{ copies}}$$

(By notation, $T^{(0,0)} = \mathbb{R}$)

$T^{(k,l)} V$ is a vector space of $\dim = \dim(V)^{k+l}$.

e.g. $V = \mathbb{R}^n$ and consider $\det \in T^{(0,n)} \mathbb{R}^n$ by

$$\det(\underset{\mathbb{R}^n}{\underset{\cap}{v_1}}, \cdots, v_n) := \det \underbrace{\begin{pmatrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{pmatrix}}_{n \times n \text{ matrix}}$$

Important observation on ordering: switch v_i, v_j. det changes sign

This property is called "alternating" or "antisymmetric". In other words, some elements in $T^{(0,l)} V^*$ are more special (elaborated next Lecture).

Back to the section-name "tensor algebra":

$$\mathcal{T}(V) := \bigoplus_{k,l} T^{(k,l)}V = \mathbb{R} \oplus (V \oplus V^*) \oplus \left(V^{\otimes 2} \oplus V \otimes V^* \oplus (V^*)^{\otimes 2}\right) \oplus \cdots$$

Then $\mathcal{T}(V)$ is an algebra under "$\otimes$" b/c

$$a \in T^{(k_1,l_1)}(V),\quad b \in T^{(k_2,l_2)}(V) \xmapsto{\otimes} a \otimes b \in T^{(k_1+k_2,\, l_1+l_2)}(V)$$

Recall that tangent bundle $TM = \bigsqcup_\alpha (U_\alpha \times \mathbb{R}^n) / \sim$ via $d(\varphi_\beta \circ \varphi_\alpha^{-1})(-)$

Similarly, one can construct <u>(k,l)-type tensor bundle</u>

$$T^{(k,l)}M = \bigsqcup_\alpha \left(U_\alpha \times T^{(k,l)}\mathbb{R}^n\right) / \sim$$

where $(x, (v_1, \cdots, v_k, w_1, \cdots, w_l)) \sim (x, (d(\varphi_\beta \circ \varphi_\alpha^{-1})(x) v_1, \cdots, d(\varphi_\beta \circ \varphi_\alpha^{-1})(x) v_k,$

$$\left(d(\varphi_\beta \circ \varphi_\alpha^{-1})^T\right)^{-1}(x) w_1, \cdots, \left(d(\varphi_\beta \circ \varphi_\alpha^{-1})^T\right)^{-1}(x) w_l)$$

e.g. $TM = T^{(1,0)}M$, $T^*M = T^{(0,1)}M$

e.g.
$$\begin{tikzcd} T^{(0,2)}M \arrow[d, "\pi"'] \\ M \arrow[u, bend right, "s"'] \end{tikzcd}$$
A section s is called a $(0,2)$-tensor field.

A Riemannian metric g is a special $(0,2)$-tensor field satisfying

non-degeneracy

(1) $g(x)(X,X) \geq 0$ and $\underline{g(x)(X,X) = 0 \text{ iff } X = 0}$ $\forall x \in M$ and $X \in T_xM$

(2) $g(x)(X,Y) = g(x)(Y,X)$ for any $x \in M$, $X, Y \in T_xM$.

Locally under a preferred basis, $g(x)$ is a matrix $(g_{ij})^{(x)}_{1\le i,j\le n}$, symmetric and positive definite (so its signature type is $(n, 0)$).

\# positive eigenvalue

\# negative eigenvalue.

Rmk As a comparison, Lorentzian metric is of signature type $(n-1, 1)$.

- Recall that a connection $\nabla: \Gamma(TM) \times \Gamma(E) \to \Gamma(E)$ is an
  $\qquad E = TM$, $E = TM$
  operator that "eats" 2 vector fields and "spits out" 1 vector field.
  One may wonder if ∇ is a $(1,2)$-tensor field.

<u>Prop</u> An $\mathbb{R}$-multi-linear operator A on $(T^*M)^{\otimes k} \otimes (TM)^{\otimes l}$ is tensor field iff A is $C^\infty(M)$-multi-linear.

e.g. ∇^a is <u>not</u> a tensor field b/c $\nabla^a_X(fY) = X(f)\cdot Y + f\nabla^a_X Y$

However, if ∇'^a is another connection, then $\nabla^a - \nabla'^a$ is a tensor field!

e.g. Recall the bracket $[\cdot,\cdot]: \Gamma(TM)\otimes\Gamma(TM) \longrightarrow \Gamma(TM)$ $X, Y \mapsto [X,Y]$

It is <u>not</u> a tensor field b/c

$$D_{[fX,Y]}F \overset{\text{def}}{=} D_{fX}D_Y F - D_Y D_{fX} F$$

$$= f(D_X D_Y F) - D_Y(f\cdot D_X F)$$

$$= f(D_X D_Y F) - Y(f) D_X F - f D_Y D_X F$$

$$= -Y(f) D_X F + D_{f[X,Y]} F$$

- On tangent bundle $TM = T^{(1,0)}M$, one can associate two structures.

One is $\nabla^a : \Gamma(TM) \times \Gamma(TM) \longrightarrow \Gamma(TM)$

The other is Riem metric $g : \Gamma(TM) \times \Gamma(TM) \longrightarrow \mathbb{R}$

the following compatibility condition, $\forall X, Y, Z \in \Gamma(TM)$

$$Z\, g(X,Y) = g(\nabla^a_Z X, Y) + g(X, \nabla^a_Z Y) \quad (*)$$

has a clear geometric meaning.

e.g. On $\mathbb{R}^3 = \mathbb{R}^3(x,y,z)$, consider vector fields

$$X((x,y,z)) = (1,0,0) \quad Y((x,y,z)) = (0,1,0), \quad Z((x,y,z)) = (0,0,1)$$

Define a connection ∇^a by

$$\nabla^a_X Y = Z \qquad \nabla^a_X Z = -Y \qquad \nabla^a_Y Z = X$$

$$\nabla^a_Y X = -Z \qquad \nabla^a_Z X = Y \qquad \nabla^a_Z Y = -X$$

and extend it over $C^\infty(\mathbb{R}^3)$.

Then ∇^a and g (= standard inner product) are compatible.

$$\underset{\overset{\shortparallel}{0}}{\nabla^a_Z g(X,X)} = g(\underbrace{\nabla^a_Z X, X}_{\overset{\shortparallel}{0}}) + g(\underbrace{X, \nabla^a_Z X}_{\overset{\shortparallel}{0}})$$

Observe that $\nabla^a_X Y - \nabla^a_Y X = Z - (-Z) = 2Z \ (\neq 0)$

$[X, Y] = 0$ (by Lie bracket formula)

(so $\nabla^a_X Y - \nabla^a_Y X \neq [X,Y]$).

<u>Thm (Exe)</u> On (M, g), there <u>exists</u> a <u>unique</u> connection ∇^a that is (i) compatible with g by $(*)$; (ii) $\underbrace{\nabla^a_X Y - \nabla^a_Y X = [X,Y]}_{\text{torsion free}}$ $\forall X, Y$.

<u>Rmk.</u> In e.g. above, one computes $\nabla_{[X,Y]} = \nabla_X \nabla_Y - \nabla_Y \nabla_X$. $\forall X, Y$.

So curvature of this ∇^a is 0 (flat). curvature $\neq$ torsion !

Pointed by 何玟辛 (and other students), this last Remark is not correct - \nabla is not flat... One can check this by apply it to X.
Still, this example above (with carefully desiged \nabla) shows an example that \nabla is **not** torsion-free.
Question: can one give an example of \nabla that is **flat** but not torsion-free?