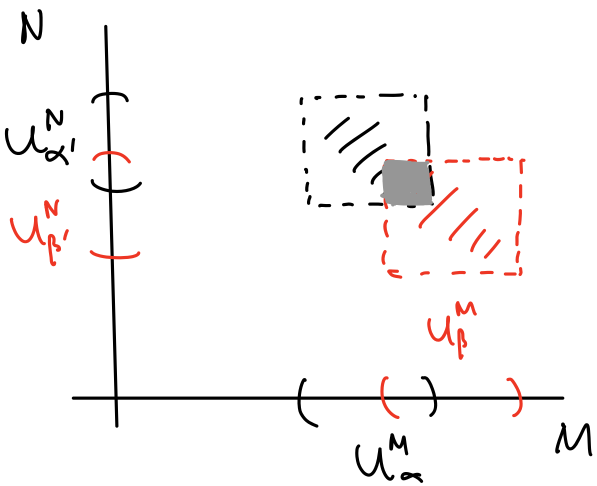
The first important task is to cook up manifolds.

① Known examples: $\mathbb{R}^n$, S^n, $SL(n,\mathbb{R})$.

② M^n, $N^{n'}$ are manifolds, then $M\times N$ is a manifold of dimension $n+n'$.



- $\{U^M_\alpha \times U^N_{\alpha'}\}_{\alpha\in I\ \alpha'\in J}$ is an open cover $M\times N$
- $\varphi^M_\alpha \times \varphi^N_{\alpha'} : U^M_\alpha \times U^N_{\alpha'} \xrightarrow{\sim} V^M_\alpha \times V^N_{\alpha'} \subset \mathbb{R}^{n+n'}$
- $(U^M_\alpha \times U^N_{\alpha'}) \cap (U^M_\beta \times U^N_{\beta'})$
  $= (U^M_\alpha \cap U^M_\beta) \times (U^N_{\alpha'} \cap U^N_{\beta'})$
- $(\varphi^M_\beta \times \varphi^N_{\beta'}) \circ (\varphi^M_\alpha \times \varphi^N_{\alpha'})^{-1}$
  $= (\varphi^M_\beta \circ (\varphi^M_\alpha)^{-1}) \times (\varphi^N_{\beta'} \circ (\varphi^N_{\alpha'})^{-1})$

e.g. $\underbrace{S^1 \times S^1 \times \cdots \times S^1}_{n \text{ times}} =: \mathbb{T}^n$ n-dim'l torus

(3) M^n manifold, then for any open subset $U \subset M$, it is also a manifold of dimension n.

$\{U_\alpha\}_\alpha$ open cover of M

$\varphi_\alpha : U_\alpha \xrightarrow{\sim} V_\alpha \subset \mathbb{R}^n$

$\varphi_\beta \circ \varphi_\alpha^{-1} : V_\alpha \cap V_\beta \to V_\alpha \cap V_\beta$ is a smooth diffeomorphism.

$\{U_\alpha \cap U\}_\alpha$ open cover of U

$\varphi_\alpha|_{U_\alpha \cap U} : U_\alpha \cap U \xrightarrow{\sim} V_\alpha' \subset V_\alpha$ $(\subset \mathbb{R}^n)$

$\varphi_\beta \circ \varphi_\alpha^{-1} : V_\alpha' \cap V_\beta' \to V_\alpha' \cap V_\beta'$ is a smooth diffeomorphism.

e.g. - $\mathbb{R}^n \setminus \{0\}$ is a manifold of dimension n

- $M_{n\times n}(\mathbb{R}) \setminus \underbrace{\{ A \in M_{n\times n}(\mathbb{R}) \mid \det(A) = 0 \}}_{F^{-1}(0) \text{ is a closed subset in } M_{n\times n}(\mathbb{R})} =: GL(n, \mathbb{R})$ is a manifold of dimension n^2.

④ Quotient manifold.

e.g. Take manifold $\mathbb{R}^3\setminus\{0\}$ and consider

equivalence relation

$$\mathbb{R}^3\setminus\{0\}/\sim \quad \text{where } x\sim y \text{ iff } \exists\lambda\neq 0 \text{ s.t } x=\lambda y.$$

quotient topology

We have natural projection $\pi: \mathbb{R}^3 \longrightarrow \mathbb{R}^3\setminus\{0\}/\sim$ by $x \rightarrow [x]$.

- $U_1 := \{[(x_1,x_2,x_3)] \mid x_1\neq 0\}$ is open in $\mathbb{R}^3\setminus\{0\}/\sim$.

  $U_2 := \{[(x_1,x_2,x_3)] \mid x_2\neq 0\}$ is open in $\mathbb{R}^3\setminus\{0\}/\sim$.

  $U_3 := \{[(x_1,x_2,x_3)] \mid x_3\neq 0\}$ is open in $\mathbb{R}^3\setminus\{0\}/\sim$.

$\{U_1, U_2, U_3\}$ is an open cover of $\mathbb{R}^3\setminus\{0\}/\sim$.

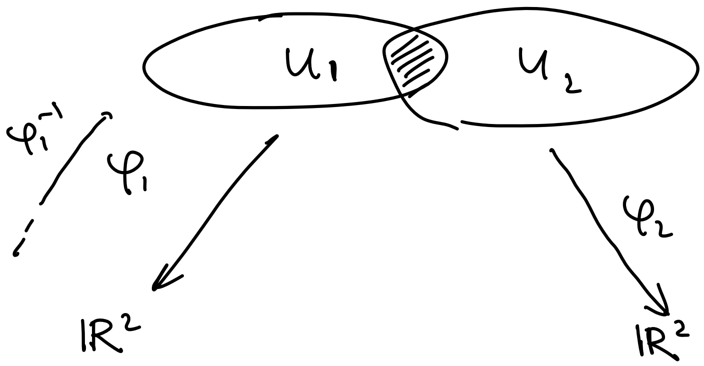
- $\varphi_1 : U_1 \xrightarrow{\sim} V_1 := \mathbb{R}^2$ by $[(x_1,x_2,x_3)] \longrightarrow \left(\frac{x_2}{x_1}, \frac{x_3}{x_1}\right)$

$$[(x_1,x_2,x_3)] \longrightarrow \left(\frac{x_2}{x_1}, \frac{x_3}{x_1}\right)$$
$$= [(1, \tfrac{x_2}{x_1}, \tfrac{x_3}{x_1})]$$

(inverse: $[(1,x,y)] \longleftarrow (x,y)$)

Similarly, define homeomorphisms $\varphi_2 : U_2 \simeq V_2$ and $\varphi_3 : U_3 \simeq V_3$.

- Over intersection, say $U_1 \cap U_2$, then



$$(x,y) \xrightarrow{\varphi_1^{-1}} [(1,x,y)] \text{ where } x \neq 0 \xrightarrow{\varphi_2} \left(\frac{1}{x}, \frac{y}{x}\right)$$

$$\left.\begin{aligned} \varphi_2 \circ \varphi_1^{-1}(x,y) &= \left(\frac{1}{x}, \frac{y}{x}\right) \\ \varphi_3 \circ \varphi_1^{-1}(x,y) &= \left(\frac{1}{y}, \frac{x}{y}\right) \\ \varphi_3 \circ \varphi_2^{-1}(x,y) &= \left(\frac{x}{y}, \frac{1}{y}\right) \\ &\cdots \end{aligned}\right\} \text{smooth diffeomorphisms} \Longrightarrow \begin{array}{l} \mathbb{R}^3 \setminus \{0\} / \sim \; =: \mathbb{RP}^2 \\ \text{is a manifold.} \\ \text{(real projective space)} \end{array}$$

$*$ $\dim \mathbb{R}^3 = 3$ but $\dim \mathbb{RP}^2 = 2$.

<u>Rmk</u> Let's consider "Jacobian matrices" of the transition maps

$$J_{\varphi_2\circ\varphi_1^{-1}} = \begin{pmatrix} -\frac{1}{x^2} & -\frac{y}{x^2} \\ 0 & \frac{1}{x} \end{pmatrix} \xrightarrow{\det} -\frac{1}{x^3}$$

could be negative

$$J_{\varphi_3\circ\varphi_1^{-1}} = \begin{pmatrix} 0 & \frac{1}{y} \\ -\frac{1}{y^2} & -\frac{x}{y^2} \end{pmatrix} \xrightarrow{\det} \frac{1}{y^3}$$

$$J_{\varphi_3\circ\varphi_2^{-1}} = \begin{pmatrix} \frac{1}{y} & 0 \\ -\frac{x}{y^2} & -\frac{1}{y^2} \end{pmatrix} \xrightarrow{\det} -\frac{1}{y^3}$$

...

In general, for $\mathbb{RP}^n = \mathbb{R}^{n+1}\setminus\{0\}/\sim$, $U_i \overset{\varphi_i}{\xrightarrow{\simeq}} \mathbb{R}^n$ for $i=1,\dots,n+1$

and $$\det\left(J_{\varphi_j\circ\varphi_i^{-1}}(x_1,\dots,x_n)\right) = \frac{(-1)^{(n+1)j}}{x_j^{n+1}} \qquad (*)$$

Def A manifold M^n is orientable if $\exists$ open cover $\{U_\alpha\}_{\alpha\in I}$ and homeomorphisms $\varphi_\alpha : U_\alpha \to V_\alpha \subset \mathbb{R}^n$ s.t for any $\alpha, \beta \in I$, the transition $\varphi_\beta \circ \varphi_\alpha^{-1}$ satisfies

multi-dim'l gradient $\rightarrow$ $$\det\left(J_{\varphi_\beta\circ\varphi_\alpha^{-1}}\right) > 0 \quad \text{(for any pt in } V_\alpha \cap V_\beta).$$

Otherwise, M^n is called non-orientable.

(*) : n is odd $\Longrightarrow$ $\mathbb{RP}^n$ is orientable.

~~Prop~~ : n is odd $\Longleftrightarrow$ $\mathbb{RP}^n$ is orientable.

"$\Leftarrow$" is equivalent :

n is even $\Longrightarrow$ $\mathbb{RP}^n$ is non-orientable.

Standard approach : derive a necessary condition (obstruction (cf. char. classes)) for M being orientable, then prove $\mathbb{RP}^{\text{even}}$ violate this condition.

Rmk There are numerous ways to define "orientable".

e.g. $\mathbb{T}^n = \mathbb{R}^n / \mathbb{Z}^n$ ← discrete.

"$\sim$" → action by a group (later)

$\mathbb{Z}^n$ is generated by the relations $(x_1, \cdots, x_i, \cdots, x_n) \sim (x_1, \cdots, x_i + 1, \cdots, x_n)$ for $i = 1, \cdots, n$

e.g. $M := \mathbb{R}^4 / \Gamma$ ← discrete

Γ is generated by the relations

$$(x_1, x_2, x_3, x_4) \sim (x_1 + 1, x_2, x_3, x_4)$$
$$(x_1, x_2, x_3, x_4) \sim (x_1, x_2 + 1, x_3, x_4)$$
$$(x_1, x_2, x_3, x_4) \sim (x_1, x_2 + x_4, x_3 + 1, x_4)$$
$$(x_1, x_2, x_3, x_4) \sim (x_1, x_2, x_3, x_4 + 1)$$

Fact (Kodaira-Thurston): M is cpx but <u>not</u> Kähler.

Rmk In this notation,

$$\mathbb{R}P^n = \mathbb{R}^{n+1} \setminus \{0\} / \mathbb{R}^* \quad \text{where } \mathbb{R}^* = \mathbb{R} \setminus \{0\}$$

← continuous, not connected

$$\mathbb{C}P^n = \mathbb{C}^{n+1} \setminus \{0\} / \mathbb{C}^* \quad \text{where } \mathbb{C}^* = \mathbb{C} \setminus \{0\}$$

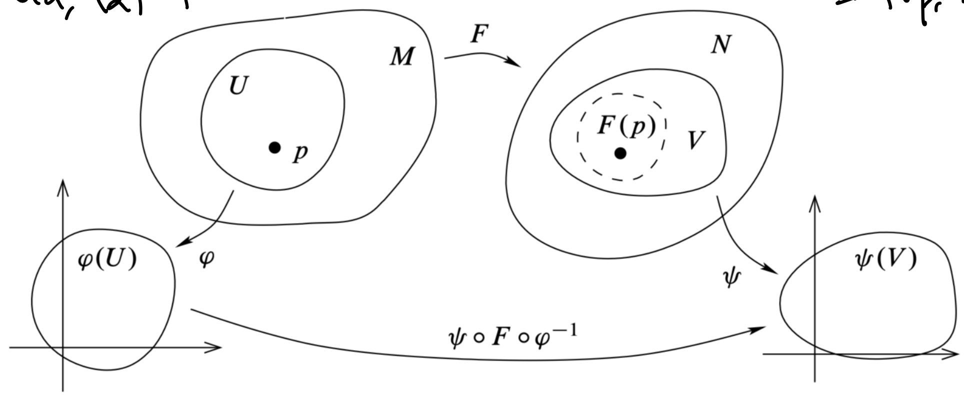
← continuous and connected

⑤ Lie group

- preparation: smooth maps $F: M^n \to N^{n'}$ (defined via the following picture)

$\exists \{U_\alpha, \varphi_\alpha\}$ of M $\qquad\qquad$ $\exists \{V_\beta, \psi_\beta\}$ of N

$\mathbb{R}^n \supset$



$\subset \mathbb{R}^{n'}$

(smooth maps between subsets in Euclidean spaces are well-defined.)

Questions about smooth maps $F: M \to N$:

① Is the image $Im(F)$ a manifold ?

② Recall for linear map $F: \mathbb{R}^n \to \mathbb{R}^{n'}$, one often consider $Ker(F)$.

For a smooth map $F: M \to N$, is it possible to define $\mathrm{Ker}(F)$?

③ What is the derivative of F?

...

We will discuss these questions later.

Rmk "smoothness" of $F: M \to N$ is independent of the choices of $\{U_\alpha, \varphi_\alpha\}$ and $\{V_\beta, \psi_\beta\}$.

- An abstract perspective:

$\mathcal{C}$ category $\begin{cases} ob(\mathcal{C}) = \{\text{smooth mflds } M\} \\ \mathrm{Hom}_{\mathcal{C}}(M,N) = \{\text{smooth maps from } M \text{ to } N\} \end{cases}$

* Notation: when $N = \mathbb{R}$ or $\mathbb{C}$, $\mathrm{Hom}_{\mathcal{C}}(M,N) =: C^\infty(M,\mathbb{R})$ or $C^\infty(M,\mathbb{C})$

Prop $\mathcal{C}$ is a well-defined category.

In particular, for any M,

$$\mathrm{Hom}_c(M,M) \supset \underbrace{\left\{ F: M \to M \;\middle|\; \begin{array}{l} F \text{ is bijective} \\ \text{and } F^{-1} \text{ is also smooth} \end{array} \right\}}_{\text{group of smooth diffeomorphisms } =: \mathrm{Diff}(M)}$$

a nec. condition

(光滑微分同胚)

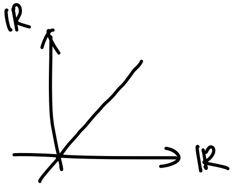
Studying algebraic properties of $\mathrm{Diff}(M)$ is a central problem in diff top.

e.g. Take $M = S^1 = \mathbb{R}/\mathbb{Z}$ and to get an element in $\mathrm{Diff}(S^1)$, consider

$$\text{smooth } f: \mathbb{R} \to \mathbb{R} \quad \text{s.t.} \quad f' > 0 \quad \text{and} \quad f(x+1) = f(x) + 1 .$$

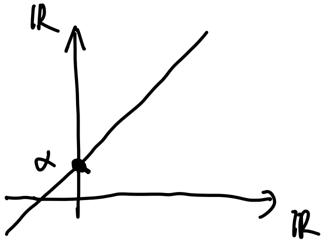
viewed as the universal cover of S^1

Then f descents to a well-defined smooth diffeo φ_f on S^1

f: 

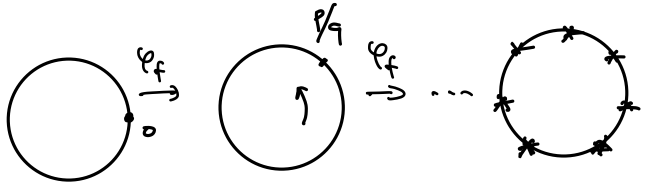
$f(x) = x$, then $\varphi_f : S^1 \to S^1$ by $e^{2\pi i \theta} \mapsto e^{2\pi i \theta}$
θ

(identity map)

$f:$ 

$f(x) = x + \alpha$ then $\varphi_f : S^1 \to S^1$ by $e^{2\pi i \theta} \to e^{2\pi i (\theta + \alpha)}$

(shift by α)

- If α is rational $= \frac{p}{q}$, then 

  and $\varphi_f^q = \mathbb{1}$

- If α is irrational, then 

covers a dense subset of S^1

b/c $\{\{n\alpha\} \in [0,1)\}_{n\in\mathbb{Z}}$ is a dense subset of $[0,1)$ (if α is irrational).

Although dynamics of these two cases are different, they share a common features: both are shifted/rotated by α.

It is tempting to use this rotation to study $\mathrm{Diff}(S^1)$.

Prop. ① $\mathrm{Diff}(S^1) = \mathrm{Diff}^-(S^1) \cup \mathrm{Diff}^+(S^1)$ where $\mathrm{Diff}^+(S^1)$ can be identified with $\{$ smooth $f : \mathbb{R} \to \mathbb{R}$ s.t $f' > 0$ and $f(x+1) = f(x)+1 \}$

② The value $\mathrm{rot}(\varphi_f) := \lim_{n\to\infty} \frac{f^n(0)}{n}$ is a well-defined map on $\mathrm{Diff}^+(S^1)$ satisfying a "quasi-morphism" property;

$\exists\, C > 0$ s.t $\forall\, \varphi_f, \varphi_g \in \mathrm{Diff}^+(S^1)$, we have

$$\left|\mathrm{rot}(\varphi_f \cdot \varphi_g) - \mathrm{rot}(\varphi_f) - \mathrm{rot}(\varphi_g)\right| < C$$

This value $\mathrm{rot}(\varphi_f)$ is called <u>Poincaré rotation number</u>.

This provides an algebraic way to study $\mathrm{Diff}(S^1)$.

Back to the Lie group.

<u>Def</u> A <u>Lie group</u> G is a group with a smooth mfd structure s.t. $\cdot : G \times G \to G$ is a smooth map. ⇐ compatibility of two structures

group product

Rmk (prop) Later we will prove that in a Lie group G, the inverse map $G \to G$ $g \mapsto g^{-1}$ is also a smooth map.

e.g. $(\mathbb{R}^n, +, \underset{\substack{\uparrow \\ \text{unit}}}{0})$ abelian group

$$+ : \underbrace{\mathbb{R}^n \times \mathbb{R}^n}_{M} \longrightarrow \underbrace{\mathbb{R}^n}_{N}$$

$$(x_1, \cdots, x_n) \ (y_1, \cdots, y_n) \longrightarrow (x_1+y_1, \cdots, x_n+y_n)$$

Take $U_\alpha = \mathbb{R}^{2n}$ and $V_\alpha = \mathbb{R}^n$, then $F = (f_1, \cdots, f_n)$ where

$$f_i(x_1, \cdots, y_n) = x_i + y_i \quad \text{(linear map, so smooth).}$$

e.g. $(\mathbb{R}^* = \mathbb{R} \setminus \{0\}, \times, \underset{\uparrow \text{unit}}{1})$ abelian group

$$\times : \mathbb{R}^* \times \mathbb{R}^* \longrightarrow \mathbb{R}^* \qquad (a, b) \longrightarrow ab \qquad \text{smooth.}$$

e.g. S^1 smooth structure. define $\bullet : S^1 \times S^1 \to S^1$ by

$$e^{2\pi i\theta}\cdot e^{2\pi i\tau}=e^{2\pi i(\theta+\tau)}.$$

This is a group structure.

$\Rightarrow$ $\mathbb{T}^n=S^1\times\cdots\times S^1$ is also a Lie group.

<u>Fact</u> Any connected cpt abelian Lie group is a torus.

e.g S^3 smooth structure. define a group structure on S^3.

- Motivation: $S^1=\{x_1^2+x_2^2=1\}\subset\mathbb{R}^2$

$\mathbb{R}^2\simeq\mathbb{C}$

Put another group str. on $\mathbb{R}^2$ by identifying it with $\underset{\text{real part}}{x_1}+\underbrace{x_2}_{\text{imaginary part}}\cdot i$

where $i^2=-1$.

$$(x_1,x_2)\cdot(y_1,y_2) := (x_1y_1-x_2y_2,\ x_1y_2+x_2y_1)$$

$$\begin{array}{ccc} \wr\!\wr & & \wr\!\wr \\ (x_1+x_2\cdot i)\cdot(y_1+y_2\cdot i) & = & (x_1y_1-x_2y_2)+(x_1y_2+x_2y_1)\,i \end{array}$$

Importantly, $x_1^2+x_2^2=1$, $y_1^2+y_2^2=1 \Rightarrow$

$$(x_1y_1-x_2y_2)^2+(x_1y_2+x_2y_1)^2 = x_1^2\underbrace{(y_1^2+y_2^2)}_{=1}+x_2^2\underbrace{(y_1^2+y_2^2)}_{=1}=1$$

This group str on $\mathbb{R}^2$ restricts to S^1.

- For $S^3=\{x_1^2+x_2^2+x_3^2+x_4^2=1\}\subset\mathbb{R}^4$

Consider i, j, k satisfying

$$i^2=j^2=k^2=-1 \quad \text{and} \quad \begin{array}{l} ij=k,\ jk=i,\ ki=j. \\ ji=-k,\ kj=-i,\ ik=-j \end{array}$$

quaternion 四元数 $\mathbb{H}$

then identify $\mathbb{R}^4$ with $\underset{\text{real part}}{x_1} + \underbrace{x_2 i + x_3 j + x_4 k}_{\text{imaginary part}}$ $(=: \mathbf{x})$

Define a group str. on $\mathbb{R}^4$ by $\mathbf{x}\cdot\mathbf{y}$, and it restricts to S^3.

<u>Fact</u>. Only $\underset{\text{two pts}}{S^0}$, S^1, S^3 admit Lie group str.