

4. Computation of integration

The definition of $\int_M \theta$ is good/well-defined but hard to compute.

e.g. (It may be computable via definition in a rather low dimension)

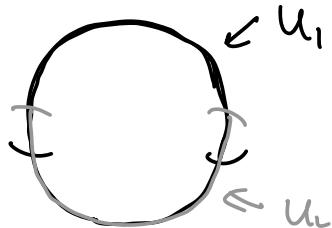
- $M = \{x_1, x_2, x_3\}$ three pts.

Consider open cover $\mathcal{U} = \{\{x_1\}, \{x_2\}, \{x_3\}\}$ and P.O.U. $\{P_i\}_{i=1}^3$ s.t.

$P_i(x_j) = \begin{cases} 1 & i=j \\ 0 & i \neq j \end{cases}$. Then for a $\overset{\dim M}{\underset{\Delta}{\circ}}\text{-form } \theta = F \in \Sigma^\circ(M)$

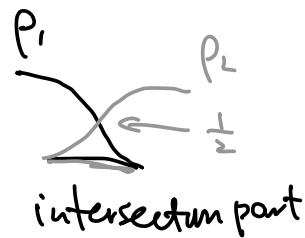
$$\int_M \theta = \sum_{i=1}^3 \int_M P_i \theta = \sum_{i=1}^3 \int_M P_i F = F(x_1) + F(x_2) + F(x_3)$$

- $M = S^1$



$$\mathcal{U} = \{U_1, U_2\}$$

$$\text{P.O.U} = \{P_1, P_2\}$$



In order to write down a 1-form on S^1 , let us parametrize by
(or embed into \mathbb{R}^2)

$$S^1 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$$

Then consider $\sigma := x dy - y dx$ (for $(x, y) \in S^1$).

Then $\int_{S^1} \sigma = \int_{S^1} \rho_1 \sigma + \int_{S^1} \rho_2 \sigma \quad \leftarrow \text{This } \sigma \text{ is } \underline{\text{globally defined!}}$

$$= \left(\int_{\cap} \rho_1 x dy - \rho_1 y dx \right) + \left(\int_{\cup} \rho_2 x dy - \rho_2 y dx \right)$$

$$= \left(\int_{\cap} x dy - y dx \right) + \left(\int_{\cup} x dy - y dx \right) +$$

b/c by our
choice of ρ -fun

$$\int_{\cap} (\rho_1 + \rho_2) x dy - (\rho_1 + \rho_2) y dx$$

$\rho_1 + \rho_2 \equiv 1$
over $()$

$$= \int_{\cap} x dy - \int_{\cup} y dx$$

$$x = \cos \theta$$

$$y = \sin \theta \quad \Rightarrow \quad = \int_0^{2\pi} \cos^2 \theta d\theta + \int_0^{2\pi} \sin^2 \theta d\theta = 2\pi.$$

Something needs
to think about
(intrinsic vs.
extrinsic)

Rank The computation on S' can be written as follows.

Consider a parametrization $\begin{array}{ccc} \theta & \xrightarrow{F} & S' \\ \cap \\ [0, 2\pi) & & \end{array}$ by $F(\theta) = (\cos \theta, \sin \theta)$.

$$\int_{S'} \sigma = \int_{[0, 2\pi)} F^* \sigma = \int_{[0, 2\pi)} d\theta \quad \text{Q is viewed as the coordinate function} = 2\pi.$$

In fact, one can relax $[0, 2\pi]$ to $(0, 2\pi]$, no effect on integration result.

Prop. Given M^n and n -form θ , if

$$M/\mathcal{Z} = \bigcup_{i=1}^k F_i(U_i)$$

a union of some submfds
of dim $< n$

where $U_i \subset \mathbb{R}^n$ and $F_i: U_i \rightarrow M$, and $\theta \in \overline{M/\mathcal{Z}}$. Then

$$\int_M \theta = \sum_{i=1}^k \int_{U_i} F_i^* \theta.$$

Exe. $M = \{(x, y, z) \in \mathbb{R}^3 \mid \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1\}$ \leftarrow ellipsoid.



Consider $F: (0, 2\pi) \times (0, \pi) \rightarrow M$

$$\theta \quad \varphi \quad \rightarrow (a \cos \theta \sin \varphi, b \sin \theta \sin \varphi, c \cos \varphi).$$

Compute $\int_M \frac{dx \wedge dy}{z} - \frac{dx \wedge dz}{y} + \frac{dy \wedge dz}{x} \leftarrow$ 2-form
in $S^2(\mathbb{R}^3 \setminus \{0\})$

(Note that $\mathcal{E} = F(\overset{\varphi=0}{\{(0)\}} \times (0, \pi)) \cup F(\overset{\theta=0}{(0, 2\pi)} \times \{0\}) \subset \overset{\varphi=0}{\text{a circle}}$).

5. Stokes' Thm

Thm Let M^n be an oriented, cpt mfd with b/d, then for any $(n-1)$ -form Ω ,

we have $\int_{\partial M} i^* \Omega = \int_M d\Omega$, where $i: \partial M \hookrightarrow M$ inclusion.

also
orientable

Rmk For non-cpt mfd, the Thm holds for any cpt supp $(n-1)$ -form Ω .

pf By P.Q.U. assume θ is cpt supp in a local chart U

Write $\theta = \sum_{i=1}^n f_i dx_1 \wedge \dots \wedge dx_i \wedge \dots \wedge dx_n$. Then two cases.

If $U \cap \partial M = \emptyset$.

$$\int_M d\theta = \int_U d\theta$$

$$= \int_V \sum_{i=1}^n \pm \frac{\partial f_i}{\partial x_i} dx_1 \wedge \dots \wedge dx_n$$

This should be
 $dx_1 dx_2 \dots dx_n$.

$$= \sum_{i=1}^n \pm \int_{\mathbb{R}^n} \frac{\partial f_i}{\partial x_i} dx_1 \wedge \dots \wedge dx_n$$

$$= \sum_{i=1}^n \pm \int_{\mathbb{R}^{n-1}} \left(\int_{\mathbb{R}} \frac{\partial f_i}{\partial x_i} dx_i \right) dx_1 \wedge \dots \wedge \widehat{dx_i} \wedge \dots \wedge dx_n$$

$$= \sum_{i=1}^n \pm \int_{\mathbb{R}^{n-1}} f_i \Big|_{x_i=-\infty}^{x_i=+\infty} dx_1 \wedge \dots \wedge \widehat{dx_i} \wedge \dots \wedge dx_n$$

$$= 0 = \int_{\partial M} i^* \theta$$

\hookrightarrow b/c $\text{supp } \theta \cap \partial M = \emptyset$.

If $U \cap \partial M \neq \emptyset$

$$\int_M d\theta = \int_U d\theta$$

$$= \int_V \sum_{i=1}^n \pm \frac{\partial f_i}{\partial x_i} dx_1 \wedge \dots \wedge dx_n$$

This should be
 $dx_1 dx_2 \dots dx_n$.

$$= \sum_{i=1}^n \pm \int_{H_f^n} \frac{\partial f_i}{\partial x_i} dx_1 \wedge \dots \wedge dx_n$$

$$\Rightarrow \sum_{i=1}^{n-1} \pm \int_{\mathbb{R}^{n-1}} f_i \Big|_{\substack{x_i=+\infty \\ x_i=-\infty}} dx_1 \wedge \dots \wedge \widehat{dx_i} \wedge \dots \wedge dx_n$$

$$\text{two parts} \quad + (-1)^{n-1} \int_{\mathbb{R}^{n-1}} f_n \Big|_{x_n=0} dx_1 \wedge \dots \wedge dx_{n-1}$$

$$\partial H_f^n = (-1)^n \int_{\mathbb{R}^{n-1}} f_n(x_1, \dots, x_{n-1}, 0) dx_1 \wedge \dots \wedge dx_{n-1}$$

$$(*) \quad \int_{\partial M} i^* \theta$$

Explanation of (*): if $dx_1 \wedge \cdots \wedge dx_n$ is a top form on the interior of Int_+ , then the "induced" top form on ∂M^+ is $(-1)^n dx_1 \wedge \cdots \wedge dx_{n-1}$.

Car $M = [a, b]$ and $\theta = f \in \mathcal{D}^\circ(M)$, then

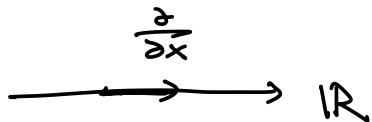
$$\int_{[a,b]} df = \int_M d\theta = \int_{\partial M} i^* \theta = \int_{\{a,b\}} i^* f$$

$$\text{LHS} = \int_{[a,b]} f'(x) dx = \int_a^b f'(x) dx$$

$$\boxed{\text{RHS}} = f(a) + f(b) ? \quad \leftarrow \begin{matrix} \text{something needs to be} \\ \text{clarified - orientation.} \end{matrix}$$

Def For an orientable mfd M , an orientation is a choice of an ordering of the basis elements $\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}$ for an "orientable" local chart.

e.g.

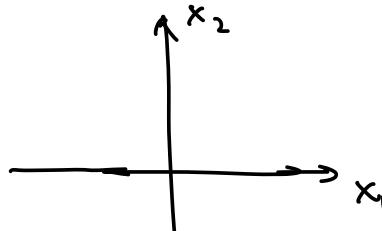


$$\frac{\partial}{\partial x}$$

(positive orientation)

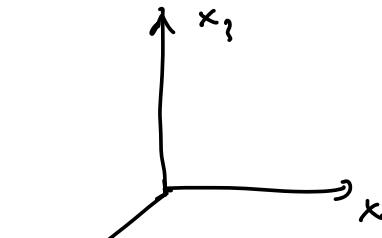
convention

$\text{so } -\frac{\partial}{\partial x}$ is a neg
orientation



$$\left\{ \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2} \right\}$$

+



$$\left\{ \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_3} \right\}$$

def —

$$dx_2 \wedge dx_1 \wedge dx_3 = -dx_1 \wedge dx_2 \wedge dx_3$$

Prop. The local orientations of M are not unique, but for an orientable mfld, the $\frac{+/-}{\text{sign}}$ of local orientations can be chosen consistently.

- In the case of mfld with b/c (recall M is orientable $\Rightarrow \partial M$ is orientable).

Suppose $\left\{ \frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n} \right\}$ is a ^{pos.} orientation of M , then define the sign of induced orientation $\left\{ \frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}, \dots, \frac{\partial}{\partial x_n} \right\}$ by the following number:

Sign $\left\{ \begin{array}{l} \text{outward} \\ \text{normal vector, } \frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}, \dots, \frac{\partial}{\partial x_n} \end{array} \right\}$
on $\partial M, n$

有些同学提问说为什么微积分中并没有体现出 orientation?

原因是在求(多重)积分的时候我们可以自由交换积分变量的顺序而不改变最后得到的积分的值。

直接的原因是多重积分的“变量替换”公式中总是被强行的对 Jacobian 的行列式取了绝对值。

一个更具体的例子是如果在求二重积分的时候，变量替换为

$(x, y) \mapsto (y, x)$

虽然 Jacobian 的行列式是 -1，但是取了绝对值后得到的和原来的积分值是完全一样的，而这个变量替换正好就是上面提到的交换变量的顺序。

e.g. On a (special) case

在谈论流形积分的时候，我们被积分的对象是 forms，不能随意交换变量的顺序（会产生符号差），因此在比较流形积分和微积分中的多重积分的时候，符号差就来源于对于（局部坐标） x_1, \dots, x_n 的一个顺序的选取（而这个正是 orientation 的原始定义）。

$$\begin{array}{ccc} & \xrightarrow{\quad} & \\ a & & b \\ \{a\} = (\partial M)_1 & & \{b\} = (\partial M)_2 \end{array}$$

Back to Cor above,
extra sign from orientation

$$\boxed{\text{RHS}} = -f(a) + f(b) = f(b) - f(a)$$

$$\Rightarrow \text{Newton-Leibniz} \quad \int_a^b f'(x) dx = f(b) - f(a)$$

- On $(\partial M)_1$, we have induced orientation $\{\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}\}$ with sign

$$\text{Sign} \left\{ -\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2} \right\} = -1$$

- On $(\partial M)_2$, $\{\frac{\partial}{\partial x_2}\} \rightarrow \text{Sign} \left\{ \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2} \right\} = 1$

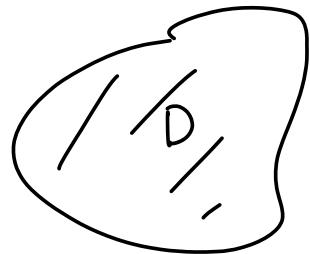
- On $(\partial M)_1$, we assign the orientation sign at a by -1
- On $(\partial M)_2$, we assign the orientation sign at b by $+1$.

A more global way to define an orientation is the following:

Def For a manifold M^n , an orientation is a non-vanishing $\Omega \in \Omega^n(M)$

Imp: If M is orientable, then M has an orientation.

e.g.



$\subset \mathbb{R}^2(x,y)$

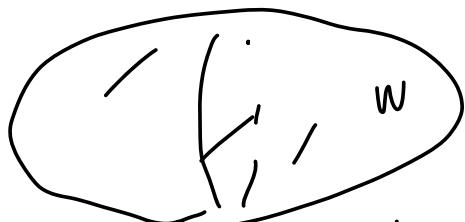
$$\begin{array}{c} \text{Q} = P(x,y) dx + Q(x,y) dy \\ \uparrow \\ \Sigma^1(\mathbb{R}^2) \end{array}$$

in calculus $\rightarrow \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx dy = \int_{\partial D} P(x,y) dx + Q(x,y) dy$

in manifold \rightarrow

$$\int_D da = \int_{\partial D} 0$$

e.g.



$\subset \mathbb{R}^3(x,y,z)$

$$\begin{array}{c} \theta = P dy \wedge dz + Q dz \wedge dx + R dx \wedge dy \\ \uparrow \\ \Sigma^2(\mathbb{R}^3) \end{array}$$

in calculus $\rightarrow \iint_W \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} \right) dx dy dz = \iint_{\partial W} \dots$

in manifold \rightarrow

$$\int_W da = \int_{\partial W} \theta.$$

Rank $\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$ is the divergence of vector field (P, Q, R) .

In fact, e.g. above is a special case of the following prop.

Given an orientable mfld M , fix an n -form Ω on M . For any vector field X on M , define $\text{div}(X)$ (divergence of X) by

$$\underbrace{L_X \Omega}_{\text{which is again an } n\text{-form}} = \text{div}(X) \cdot \Omega.$$

Rank $\text{div}(X)$ depends on Ω . Suppose $\Omega' = f \cdot \Omega$, then

$$\begin{aligned} L_X \Omega' &= dL_X(f\Omega) + L_X d(f\Omega) \\ &= d(f(L_X \Omega)) + L_X (df \wedge \Omega + f \cdot d\overset{\circ}{\Omega}) \\ &= \cancel{df \wedge L_X \Omega} + f \cdot dL_X \Omega + (L_X df) \wedge \Omega - \cancel{df \wedge L_X \Omega} \\ &= f \cdot L_X \Omega + (L_X df) \wedge \Omega \\ &= f \cdot \text{div}(X) \cdot \Omega + df(X) \cdot \Omega \\ &= \text{div}(X) \Omega' + \underbrace{df(X) \cdot \Omega}_{\text{extra term}} \end{aligned}$$

+ nowhere vanishing

Prop. Let M^n be an orientable mfd with b/d. Fixing $\omega \in \mathcal{D}^n(M)$ that is nowhere vanishing, then

$$\int_M \operatorname{div}(x) \omega = \int_{\partial M} i^*(\iota_x \omega)$$

If $\int_M \operatorname{div}(x) \omega = \int_M L_x \omega = \int_M d\iota_x \omega = \int_{\partial M} i^*(\iota_x \omega) \quad \square$

Prop (again) Let M^n be an orientable cpt mfd with b/d. Then

\exists smooth map $F: M \rightarrow \partial M$ s.t. $F|_{\partial M}: \partial M \rightarrow \partial M$ is the identity.

If Suppose \exists such F . Pick a nowhere vanishing ω on $\mathcal{D}^{n-1}(\partial M)$

s.t. $\int_{\partial M} \omega > 0$. Then

$$0 < \int_{\partial M} \omega = \int_{\partial M} F^* \omega = \int_M d(F^* \omega) = \int_M F^*(d\omega) = 0 \Rightarrow \leftarrow .$$

Def ^① For a mfd M^n , a volume form is a nowhere vanishing $\omega \in \mathcal{D}^n(M)$.

- M is orientable $\Rightarrow \exists$ a volume form.
- ② For an orientable wfd M^n , the volume w.r.t. a volume form $\Omega \in \mathcal{D}^n(M)$ is defined by $\int_M \Omega$.
- If furthermore M is cpt, then $\int_M \Omega < \infty$.

Rank If $\Omega \in \mathcal{D}^n(M)$ is a volume form, then $f \cdot \Omega$ is also a volume form if f is a nowhere vanishing function.

Rank To some extent, $\text{div}(X)$ detects how the volume w.r.t Ω changes.

More precisely,

$$\begin{aligned} \int_D \text{div}(X) \Omega &= \int_D L_X \Omega = \int_D \frac{d}{dt} \Big|_{t=0} (\varphi_t^+)^* \Omega \\ &= \frac{d}{dt} \Big|_{t=0} \int_D (\varphi_t^+)^* \Omega = \frac{d}{dt} \Big|_{t=0} \int_{(\varphi_t^+(D))} \Omega. \end{aligned}$$