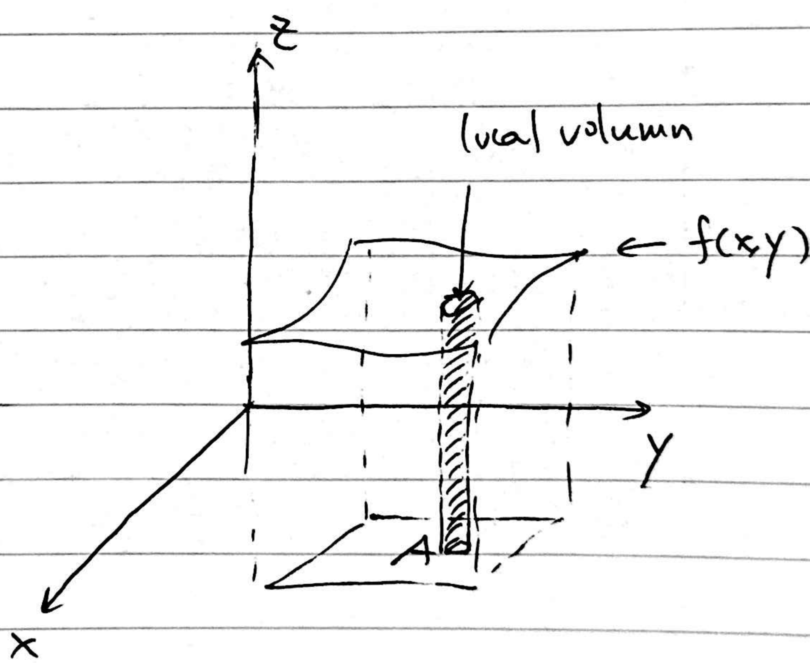
This notes was written on the plane, so it is in a different format from previous ones.

Lecture 5 Integration on manifolds

Motivation:

Recall in calculus, given a function $f(x,y)$: $A\ (\subset \mathbb{R}^n) \longrightarrow \mathbb{R}$,

$$\int_A f(x,y)\,dx\,dy := \text{sum of local volumn}$$



Two improvements on mfds:

① $A^{\subset \mathbb{R}^n} \rightsquigarrow M$ mfd (possibly with boundary)

② $f \in C^\infty(A;\mathbb{R}) \rightsquigarrow$ differential form $\theta \in \Omega^*(M)$

Naïve attempt to define integration:

define locally over a chart $U_\alpha \,(\simeq V_\alpha \subset \mathbb{R}^n)$ $\xrightarrow{\text{glue}}$ integration

Recall locally, θ can be written as

$$\theta = \sum_{1 \le i_1 < \cdots < i_k \le n} f^\alpha_{i_1 \cdots i_k}\, dx_{i_1} \wedge \cdots \wedge dx_{i_k} \qquad (\dim M = n)$$

Since integration always produces a number, we only consider $\theta \in \Omega^n(M)$.

$$\int_{V_\alpha (\subset \mathbb{R}^n)} f^\alpha\, dx_1 \wedge \cdots \wedge dx_n \quad (\in \mathbb{R}) \quad (= \text{local volumn})$$

Two issues:

- well-definedness? (U_α vs. U_β)

- how to glue?

1. <u>Partition of unity</u> (P.O.U)

<u>Question.</u> Given open domain $B^n(1) = \{ x \in \mathbb{R}^n \mid \|x\| < 1 \}$

how to construct a smooth function f that is compactly supported in $B^n(1)$?

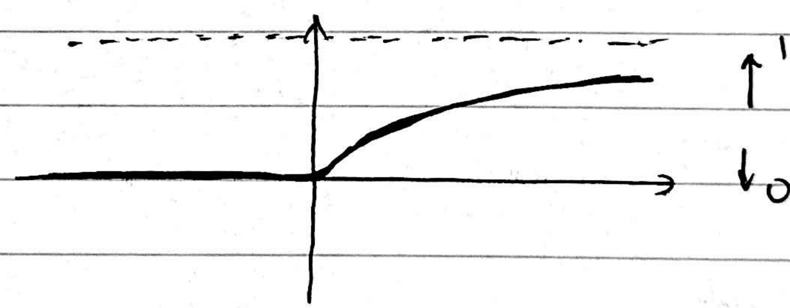
Better to write B^n(1+\ep), an open ball that is slightly bigger than B^n(1).

$$\operatorname{Supp}(f) = \{x \in \mathbb{R}^n \mid f(x) \neq 0\}$$

compactly supported: $\overline{\operatorname{Supp}(f)} \subset B^n(1)$

Notation: $\operatorname{supp}(f) \subset\subset B^n(1)$.

<u>Ans</u> (Explicit construction)

① $\varphi(x) =$ 

$\cap$

$C^\infty(\mathbb{R}; [0,1))$

$$\varphi(x) = \begin{cases} e^{-\frac{1}{x}} & x > 0 \\ 0 & x \leq 0 \end{cases}$$

$\varphi(x)$ is smooth at $x = 0$

② $h(x)$ constructed from $\varphi(x)$

smooth, even function on $\mathbb{R}$.

$$h(x) = \begin{cases} \dfrac{\varphi(-2x+2)}{\varphi(-2x+2)+\varphi(2x-1)} & x > 0 \\[2ex] \dfrac{\varphi(2x+2)}{\varphi(2x+2)+\varphi(-2x-1)} & x \leq 0 \end{cases}$$

Let's draw the picture of $h(x)$.

Note that denominator $\varphi(-2x+2)+\varphi(2x-1)$ or $\varphi(2x+2)+\varphi(-2x-1)$ never equals 0.

b/c $-2x+2 \leq 0$ & $2x-1 \leq 0$ ($\Leftrightarrow x \geq 1$ & $x \leq \frac{1}{2}$)

$\Rightarrow$ $h(x)$ is a smooth fcn over $\mathbb{R}$.

For $x \geq 1$, $-2x+2 \leq 0 \Rightarrow h(x) = 0$

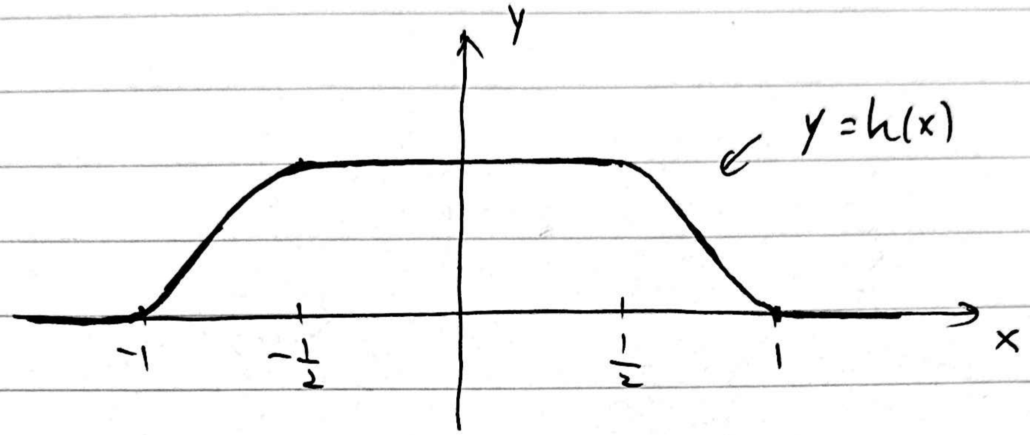
(b/c $\varphi(-2x+2) = 0$)

For $\frac{1}{2} \leq x < 1$, $0 \leq 2x-1 < 1 \Rightarrow h(x) \in (0,1]$

(where $\varphi(2x-1) \geq 0$)

For $0 \leq x < \frac{1}{2}$, $-1 \leq 2x-1 < 0 \Rightarrow h(x) = 1$

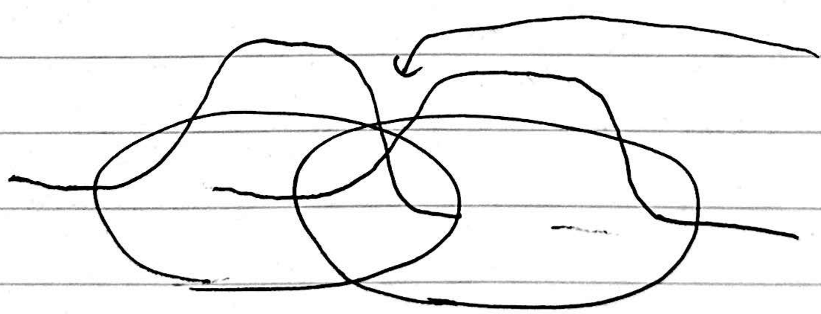
(b/c $\varphi(2x-1) = 0$)



③ Define $f(x) : \mathbb{R}^n \longrightarrow \mathbb{R}$ by $f(x) = h(\|x\|)$.

Prop. For any cpt subset $A \subset M$ and an open subset $U \supset A$, there exists a $f \in C^\infty(M; [0,1])$ s.t. $f|_A \equiv 1$ and $\text{supp}(f) \subset\subset U$.

Pf. Do this in local charts



do linear combination of two "bump" functions

Caution: the parts where $\equiv 1$ can not be two small! $\square$.

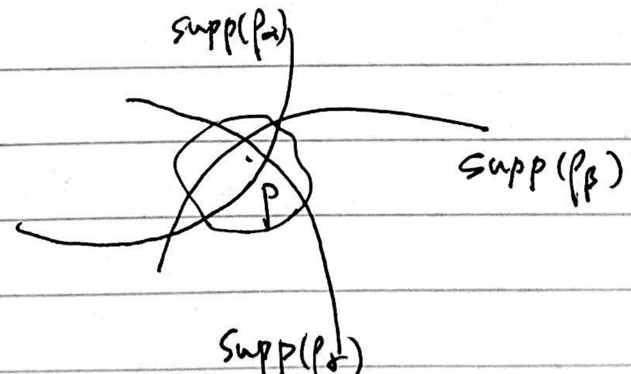
Here is the definition of P.O.U..

<u>Def</u> M mfd (not nec. compact),
$\mathcal{U} = \{U_\alpha\}_\alpha$ (not nec. finite) an open cover of M.
A P.O.U. subordinated to $\mathcal{U}$ is a collection
(从属)
of smooth fcns $\{\rho_\alpha\}_\alpha$ satisfying

① $0 \leq \rho_\alpha \leq 1 \quad \forall \alpha$
② $\mathrm{supp}(\rho_\alpha) \Subset U_\alpha$
③ $\forall p \in M$, $\exists$ a NBH that intersects only finitely many $\mathrm{supp}(\rho_\alpha)$
④ $\sum_\alpha \rho_\alpha(p) = 1. \ \forall p \in M.$

③ $\Rightarrow$ ④ is well-defined (finite sum)

③:



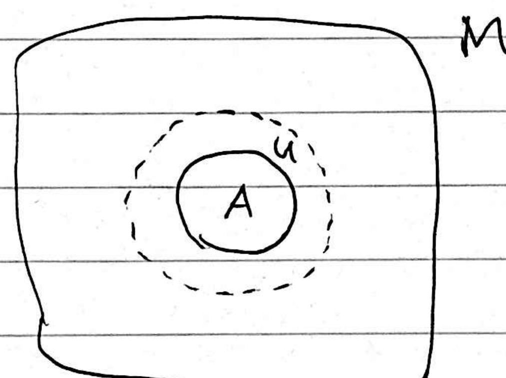
$\rho_\alpha(p) = \frac{1}{2} \quad \rho_\beta(p) = \frac{1}{3} \quad \rho_\gamma(p) = \frac{1}{6}$

<u>Thm</u> Any mfd admits a P.O.U (in the sense that for any open cover $\mathcal{U} = \{U_\alpha\}_\alpha$, $\exists$ P.O.U subordinated to $\mathcal{U}$).

Proof. Exe

Rmk: not every closed subset is cpt!

<u>Cor</u> For any <u>closed</u> subset $A \subset M$ and an open subset $U \supset A$, there exists a smooth function $f \in C^\infty(M; [0,1])$ s.t. $f|_A \equiv 1$ and $\mathrm{supp}(f) \subset\subset U$.



Pf. Consider open cover $\{U, M \setminus A\}$ of M.

By Thm, $\exists$ P.O.U. $\rho_1, \rho_2 \in C^\infty(M; [0,1])$ s.t.

$\mathrm{supp}(\rho_1) \subset\subset U$ and $\mathrm{supp}(\rho_2) \subset M \setminus A$.

Then for $x \in A$,

$$\sum_\alpha \rho_\alpha(x) = \rho_1(x) + \underbrace{\rho_2(x)}_{=0} = \rho_1(x) = 1$$

by def of P.O.U.

Take $f = \rho_1$ $\square$

<u>Cor</u> (smooth extension)

$A \subset M$ submfd. $F : A \to \mathbb{R}$ s.t. $\forall x \in A$

$\exists$ NBH U_x and $F_x : U_x \to \mathbb{R}$ s.t. $F_x|_{U_x \cap A} = F|_{U_x \cap A}$,

then $\exists$ NBH V of A and $\tilde{F} \in C^\infty(V; \mathbb{R})$ st.

$\tilde{F}|_A = F$.

Pf. $\mathcal{U} = \{U_x\}_{x \in A}$ is an open cover of $\cancel{A}\, V := \bigcup_{x \in A} U_x$

By Thm, $\exists \{\rho_x\}_{x \in A}$. Consider

$$\rho_x \cdot F_x \quad \text{which is cpt supp in } U_x.$$

Then extend by zero to be defined on V. Then define $\tilde{F} = \sum_{x \in A} \rho_x F_x$.

Verify: $\forall p \in A$, $\tilde{F}(p) = \sum_{x \in A} \rho_x(p) \underbrace{F(p)}_{\text{ind of } x} = F(p) \underbrace{\sum_{x \in A} \rho_x(p)}_{=1}$

$= F(p)$ ✓ □.

<u>Cor</u> (<u>Soft</u> Whitney embedding) "Compact"

For any cpt mfd M^n, there exists an embedding $M^n \hookrightarrow \mathbb{R}^N$ for a sufficiently large N.

There is no need to enlarge the "\equiv 1" part of the P.O.U. (which is in fact incorrect) as what I explained in class. See the correct treatment on this page. THANKS for those students who pointed this out right after class!

Pf. Take any open cover $\mathcal{U} = \{U_\alpha\}_\alpha$ with

$$\varphi_\alpha : U_\alpha \xrightarrow{\cong} V_\alpha \subset \mathbb{R}^n$$

Here since M is cpt, $\mathcal{U}$ is a finite open cover, set $\alpha = 1, \dots, k$.

Consider map $F : M \longrightarrow \mathbb{R}^{nk+k}$ defined by

$$x \longmapsto (\underbrace{\rho_1(x) \cdot \varphi_1(x)}_{\in \mathbb{R}^n}, \dots, \rho_k(x) \cdot \varphi_k(x), \rho_1(x), \dots, \rho_k(x))$$

a number   a vector in $\mathbb{R}^n$

where $\{\rho_\alpha\}_{\alpha=1}^k$ is a P.O.U. wrt $\mathcal{U}$.

(Rmk. $\rho_i(x) \cdot \varphi_i(x)$ extends by zero outside U_i)

Verify that F is an embedding.

- e.g. F is injective: if $F(x) = F(y)$, then for some i, $\rho_i(x) = \rho_i(y) \neq 0$ (WHY?)

$\Rightarrow x, y \in \operatorname{supp}(\rho_i) \subset U_i$

$\Rightarrow \varphi_i(x) = \varphi_i(y) \Rightarrow x = y$ b/c φ_i is a homeomorphism.

The rest verification is left as an exercise.

Strongly recommend readers to finish the rest of the proof!