MATH 161, SHEET 3: THE TOPOLOGY OF THE CONTINUUM

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In this sheet we give the continuum C a topology. Roughly speaking, this is a way to describe how the points of C are 'glued together'.

Definition 3.1. A subset of the continuum is *closed* if it contains all of its limit points.

Theorem 3.2. *The sets $\emptyset$ and C are closed.*

Proof. We know that the set $\emptyset$ is a finite subset of C, so by Theorem 2.25 it has no limit points. The set $\emptyset$ then vacuously satisfies Definition 3.1, so it is closed. We know that by Definition 2.8, for all regions R, $R \subset C$. It follows then that if p is a limit point of C, then for all regions S such that $p \in S$, $p \in C$ as well, so C contains all of its limit points. $\square$

Theorem 3.3. *A subset of C containing a finite number of points is closed.*

Proof. We know that a finite subset $A \subset C$ has no limit points by Theorem 2.25. Therefore A vacuously satisfies Definition 3.1, as it has no limit points, so A is closed. $\square$

Definition 3.4. Let X be a subset of C. The *closure* of X is the subset $\overline{X}$ of C defined by:

$$\overline{X} = X \cup \{x \in C \mid x \text{ is a limit point of } X\}.$$

Theorem 3.5. *$X \subset C$ is closed if and only if $X = \overline{X}$.*

Proof. We let $X \subset C$. Assume that X is closed. Then we know that X contains all of its limit points, so $\{x \in C \mid x \text{ is a limit point of } X\} \subset X$, so $X \cup \{x \in C \mid x \text{ is a limit point of } X\} = X$. Then it follows by Definition 3.4 that $\overline{X} = X$. We now assume $X = \overline{X}$. Then it follows that $X = X \cup \{x \in C \mid x \text{ is a limit point of } X\}$. By properties of sets, we know then that $\{x \in C \mid x \text{ is a limit point of } X\} \subset X$, so X contains all of its limit points. Then by Definition 3.1, X is closed. We have now shown both directions, so we know that $X \subset C$ is closed if and only if $X = \overline{X}$. $\square$

Theorem 3.6. *The closure of $X \subset C$ satisfies $\overline{X} = \overline{\overline{X}}$.*

Proof. We know that $\overline{X} = X \cup \{x \in C \mid x \text{ is a limit point of } X\}$ and that if $x \in \overline{X}$, then $x \in \overline{\overline{X}}$ because by Definition 3.4 $\overline{X} \subset \overline{\overline{X}}$. We now assume that $x \in \overline{\overline{X}}$, so $\overline{\overline{X}} = \overline{X} \cup \{x \in C \mid x \text{ is a limit point of } \overline{X}\}$. If $x \in \overline{X}$, then it follows that $\overline{\overline{X}} \subset \overline{X}$. So we let x be a limit point of $\overline{X}$ such that $x \notin \overline{X}$. Then we know that for a region R, $\forall R, x \in R, R \cap \overline{X} \neq \emptyset$. So $\exists y$ such that $y \in R \cap \overline{X}$. It follows from the definition of $\overline{X}$ that for a region S, $\overline{X} = \{x \mid \forall S, x \in S, S \cap X \neq \emptyset\}$ because if $x \in X$ then $x \in S \cap X$, and if $x \notin X$ and x is a limit point of X then $S \cap X \neq \emptyset$. Because $y \in \overline{X}, \forall S, y \in S, S \cap X \neq \emptyset$. So we know that $\exists z$ such that $z \in S \cap X$. Because $y \in R$, we choose R as S, so we know that $z \in R \cap X$. So $\forall R, x \in R, R \cap X \neq \emptyset$. This means that x is a limit point of X, so it follows that $x \in \overline{X}$. Thus, we have shown that $\overline{X} \subset \overline{\overline{X}}$ and that $\overline{\overline{X}} \subset \overline{X}$, so $\overline{X} = \overline{\overline{X}}$. $\square$

Corollary 3.7. *Given any subset $X \subset C$, the closure $\overline{X}$ is closed.*

Proof. We know by Theorem 3.6 that for any subset $X \subset C$, $\overline{X} = \overline{\overline{X}}$. By Theorem 3.5 then it follows that $\overline{X}$ is closed. $\square$

Definition 3.8. A subset U of the continuum is *open* if its complement $C \setminus U$ is closed.

Theorem 3.9. *The sets $\emptyset$ and C are open.*

Proof. By Theorem 3.2 we know that $\emptyset$ and C are both closed. $\emptyset$ is the complement of C, so C is open by Definition 3.8. C is the complement of $\emptyset$, so $\emptyset$ is similarly open. $\square$

The following is a very useful criterion to determine whether a set of points is open.

Theorem 3.10. *Let $U \subset C$. Then U is open if and only if for all $x \in U$, there exists a region R such that $x \in R \subset U$.*

Proof. Let $U^c = C \setminus U$. We first assume that U is open, and thus we know that U^c is closed. Because U^c is closed, it follows that $\forall x \in U, x$ is not a limit point of U^c. So we know that for a region R, $\forall x \in U, \exists R$ such that $x \in R, R \cap (U^c \setminus \{x\}) = \emptyset$. But we know that U and U^c are disjoint, so $x \notin U^c$. So $R \cap U^c = \emptyset$, and thus $R \subset U$. So if U is open, $\forall x$, there exists a region R such that $x \in R \subset U$. We now assume that $\forall x \in U$, there exists a region R such that $x \in R \subset U$ and thus $R \cap U^c = \emptyset$. This means that $\forall x \in U$, x is not a limit point of U^c, so U^c is closed. It follows then that U must be open. $\square$

Corollary 3.11. *Every region R is open. Every complement of a region $C \setminus R$ is closed.*

Proof. Let R be a region. Then it is clear that $\forall x \in R$, $x \in R \subset R$. Thus by Theorem 3.10 we know that R is open. $\square$

Corollary 3.12. *Let $a \in C$. Then the sets $\{x \mid x < a\}$ and $\{x \mid a < x\}$ are open.*

Proof. We first consider the set $\{x \mid x < a\}$. We know that $\forall x$ such that $x < a$, there exists b such that $b < x$ by Axiom 3. Thus we know that $x \in \underline{ba}$, so $\forall x \in \{x \mid x < a\}$, there exists a region $\underline{ba}$ such that $x \in \underline{ba}$. So by Theorem 3.10, $\{x \mid x < a\}$ is open. Similarly, for the set $\{x \mid a < x\}$, we know that $\forall x$ such that $a < x$ there exists c such that $x < c$ by Axiom 3. So it follows that $x \in \underline{ac}$, so $\forall x \in \{x \mid a < x\}$, there exists a region $\underline{ac}$ such that $x \in \underline{ac}$. So by Theorem 3.10, $\{x \mid a < x\}$ is open. $\square$

Theorem 3.13. *Let U be a nonempty open set. Then U is the union of a collection of regions.*

Proof. We know by Theorem 3.10 that for all x such that $x \in U$, there exists a region R_x such that $x \in R_x \subset U$. If $x \in U$, then $x \in \bigcup_{x \in U} R_x$ because there exists an R_x such that $x \in R_x \subset U$ for all x. If $x \in \bigcup_{x \in U} R_x$, then we know that $x \in U$ because for all R_x, $R_x \subset U$. So it follows that $U = \bigcup_{x \in U} R_x$, so U is the union of a collection of regions. $\square$

Exercise 3.14. Do there exist subsets $X \subset C$ that are neither open nor closed?

Proof. It depends on the realization of C, as all subsets on $\mathbb{Z}$ are both open and closed, so on $\mathbb{Z}$ there exist no $X \subset \mathbb{Z}$ such that X is neither open nor closed. $\square$

Theorem 3.15. *Let $\{X_\lambda\}$ be an arbitrary collection of closed subsets of the continuum. Then the intersection $\bigcap_\lambda X_\lambda$ is closed.*

Proof. We know by DeMorgan's Laws that $C \setminus \bigcap_\lambda X_\lambda = \bigcup_\lambda (C \setminus X_\lambda)$. By Definition 3.8, we know that $C \setminus X_\lambda$ is open because X_λ is closed. We let $A_\lambda = (C \setminus X_\lambda)$. So for $\{A_\lambda\}$, if $x \in \bigcup_\lambda A_\lambda$, $x \in A_\lambda$ for some λ. Since A_λ is open, $x \in A_\lambda$, by Theorem 3.10 there exists a region R such that $x \in R \subset A_\lambda$. So it follows that $R \subset \bigcup_\lambda A_\lambda$, so $\bigcup_\lambda A_\lambda$ is open. This means that $\bigcup_\lambda (C \setminus X_\lambda)$ is open, so $C \setminus \bigcap_\lambda X_\lambda$ is open, so by Definition 3.8, $\bigcap_\lambda X_\lambda$ is closed. $\square$

Theorem 3.16. *Let $U_1, \ldots, U_n$ be a finite collection of open subsets of the continuum. Then the intersection $U_1 \cap \cdots \cap U_n$ is open.*

Proof. If $U_1 \cap \cdots \cap U_n = \emptyset$ then we know that $U_1 \cap \cdots \cap U_n$ is open by Theorem 3.9. Assume $U_1 \cap \cdots \cap U_n \neq \emptyset$. Then we choose an arbitrary $x \in U_1 \cap \cdots \cap U_n$. It follows then by Theorem 3.10 that there exist regions $R_1, \ldots, R_n$ such that $x \in R_1, \ldots, R_n$. So we know that $x \in R_1 \cap \cdots \cap R_n$. We let $S = R_1 \cap \cdots \cap R_n$. By Corollary 2.18, we know that S is a region, and we know that $x \in S$. We also know that $S \subset U_1 \cap \cdots \cap U_n$ because $R_1 \subset U_1, \cdots, R_n \subset U_n$. Thus for an arbitrary $x \in U_1 \cap \cdots \cap U_n$ we know that $x \in S \subset U_1 \cap \cdots \cap U_n$, so by Theorem 3.10 we have shown that $U_1 \cap \cdots \cap U_n$ is open. $\square$

Exercise 3.17. Is it necessarily the case that the intersection of an infinite number of open sets is open?

Proof. It is not necessarily the case. We let R_n be defined as the region $\overline{\frac{-1}{n}\frac{1}{n}}$ over $\mathbb{R}$ for $n \in \mathbb{N}$. We know that R_n is open because it is a region (Corollary 3.11). It follows that $\bigcap R_n = \{0\}$. We know that $\{0\}$ is a closed set on $\mathbb{R}$ because we cannot construct a region S around $\{0\}$ such that $S \subset \{0\}$ in $\mathbb{R}$. $\square$

Corollary 3.18. *Let $\{U_\lambda\}$ be an arbitrary collection of open subsets of the continuum. Then the union $\bigcup_\lambda U_\lambda$ is open. Let $X_1, \ldots, X_n$ be a finite collection of closed subsets of the continuum. Then the union $X_1 \cup \cdots \cup X_n$ is closed.*

Proof. Let $x \in \bigcup_\lambda U_\lambda$, then $x \in U_\lambda$ for some U_λ. We know that U_λ is open, so by Theorem 3.10 there exists a region R such that $x \in R \subset U_\lambda$. So it follows that $R \subset \bigcup_\lambda U_\lambda$. Thus $\forall x \in \bigcup_\lambda U_\lambda$, there exists a region R such that $x \in R \subset \bigcup_\lambda U_\lambda$ so by Theorem 3.10 $\bigcup_\lambda U_\lambda$ is open. We let $X_1, ..., X_n$ be finite closed subsets of C. For each X_i, we define $U_i = C \setminus X_i$ is open by Definition 3.8. We know by DeMorgan's Laws that $C \setminus \bigcap_i U_i = \bigcup_i X_i$. By Theorem 3.16 we have that the intersection of a finite number of open sets is open, so $\bigcap_i U_i$ is open. By Definition 3.8, we know $C \setminus \bigcap_i U_i$ is closed so $\bigcup_i X_i$ is closed. $\square$

Theorem 3.9 and Corollary 3.18 say that the collection $\mathscr{T}$ of open subsets of the continuum is a topology on C, in the following sense:

Definition 3.19. Let X be any set. A *topology* on X is a collection $\mathscr{T}$ of subsets of X that satisfy the following properties:

1. X and $\emptyset$ are elements of $\mathscr{T}$.
2. The union of an arbitrary collection of sets in $\mathscr{T}$ is also in $\mathscr{T}$.
3. The intersection of a finite number of sets in $\mathscr{T}$ is also in $\mathscr{T}$.

The elements of $\mathscr{T}$ are called the *open sets* of X. The set X with the structure of the topology $\mathscr{T}$ is called a *topological space*[1].

Theorem 3.13 says that every nonempty open set is the union of a collection of regions. This necessary condition for open sets is also sufficient:

Theorem 3.20. *Let $U \subset C$ be nonempty. Then U is open if and only if U is the union of a collection of regions.*

Proof. We first assume that $U \subset C$ is nonempty and open. We know then by Theorem 3.13 that U is the union of a collection of regions. We now assume that $U \subset C$ is the union of a collection of regions. By Corollary 3.11 we know that all regions are open, so by applying Theorem 3.18 we get that U is open. □

Definition 3.21. A topological space X is *discrete* if every subset of X is open.

Exercise 3.22. Find a realization of the continuum that is discrete. Must every realization be discrete?

Proof. We know that every subset of $\mathbb{Z}$ is both open and closed, so $\mathbb{Z}$ is a discrete realization of the continuum. It is not a necessary condition that a realization of the continuum be discrete, as $\mathbb{R}$ is a non-discrete realization of the continuum (the set $\{0\}$ on $\mathbb{R}$ is not open because no region can be constructed such that the region is contained within the set $\{0\}$). □

Definition 3.23. Let A and B be nonempty disjoint subsets of a topological space X. We say that A and B are *separated* if each contains no point of the closure of the other, i.e. $A \cap \overline{B} = \emptyset$ and $\overline{A} \cap B = \emptyset$.

Theorem 3.24. *Let $\underline{ab}$ be a region in C. Then the sets $\underline{ab}$ and $\operatorname{ext} ab$ are separated.*

Proof. We know that no limit points of $\underline{ab}$ are in $\operatorname{ext} ab$ and that no limit points of $\operatorname{ext} ab$ are in $\underline{ab}$ by Lemma 2.16. We also know that $\underline{ab}$ and $\operatorname{ext} ab$ are disjoint, so it follows that $\underline{ab}$ and $\operatorname{ext} ab$ are separated. □

Definition 3.25. Let X be a topological space. X is *disconnected* if it may be written as the union $X = A \cup B$ of two separated sets. X is *connected* if it is not disconnected.

Exercise 3.26. Is the continuum connected?

[1]The word *topology* comes from the Greek word *topos* ($\tau \acute{o} \pi o \zeta$), which means "place".

Proof. As defined thus far, the continuum is not necessarily connected. We know that $\mathbb{Z}$ is a realization of the continuum as defined thus far, and $\mathbb{Z}$ can be written as $\mathbb{Z} = \{x \in \mathbb{Z} \mid \text{x is even}\} \cup \{x \in \mathbb{Z} \mid \text{x is odd}\}$. Any subset of $\mathbb{Z}$ has no limit points, so we know that the set of even integers and the set of odd integers are separated. $\square$