SHEET 9: SEQUENCES and LIMITS

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We will now work with the real numbers $\mathbb{R}$ instead of an arbitrary continuum C. Accordingly, let us now use the standard notation (a, b) for the region $\underline{ab} = \{x \in \mathbb{R} : a < x < b\}$. Even though the notation is the same, this is *not* the same object as the ordered pair (a, b).

Definition 9.1. A *sequence* (of real numbers) is a function $a \colon \mathbb{N} \longrightarrow \mathbb{R}$.

By setting $a_n = a(n)$, we can think of a sequence a as a list $a_1, a_2, a_3, \ldots$ of real numbers. We use the notation $(a_n)_{n=1}^{\infty}$ for such a sequence, or if there is no possibility of confusion, we sometimes abbreviate this and write simply (a_n).

Definition 9.2. We say that a sequence (a_n) *converges* to a point $p \in \mathbb{R}$ if, for every region R containing p, there exists $N \in \mathbb{N}$ such that if $n \geq N$, then $a_n \in R$. If (a_n) does not converge to any point, we say that the sequence *diverges.*

Exercise 9.3. Show that if a sequence (a_n) converges to p, then any region containing p contains all but finitely many terms in the sequence.

Proof. Let (a_n) converge to p, and R be an arbitrary region such that $p \in R$. Then we know by 9.2 that there exists $N \in \mathbb{N}$ such that if $n \geq N$, $a_n \in R$. R contains all $a_n \in R$ such that $n \geq N$, so R contains at all but at most $N - 1$ terms in the sequence. $N - 1$ is finite, so R contains all but finitely many terms in the sequence. □

Exercise 9.4. Which of the following sequences converge? Which diverge? If one converges, what does it converge to? If one diverges, what can you say about the nature of its divergence?

(a) $a_n = 5$	(c) $a_n = 1/n$	(e) $a_n = (-1)^n \cdot n$
(b) $a_n = n$	(d) $a_n = (-1)^n$	(f) $a_n = (-1)^n \cdot \frac{1}{n}$

Proof. (a) $a_n = 5$ converges to 5.
(b) $a_n = n$ diverges to infinity.
(c) $a_n = \frac{1}{n}$ converges to 0.
(d) $a_n = (-1)^n$ diverges, alternating between -1 and 1.
(e) $a_n = (-1)^n \cdot n$ diverges.
(f) $a_n = (-1)^n \cdot \frac{1}{n}$ converges to 0. □

Theorem 9.5. *Suppose that (a_n) converges both to p and to p'. Then $p = p'$.*

Proof. Let (a_n) converge to p and p' such that $p \neq p'$. Then we know that we can choose two regions R, R' around p, p' respectively such that $R \cap R' = \emptyset$. Then we know by Exercise 9.3 that R, R' each contain all but finitely many terms of the sequence. So it follows then that because R and R' are disjoint, R and R' can each contain only finitely many terms in the sequence. This is a contradiction, because it means that a_n has finitely many terms, and by Definition 9.1 we have that a_n has infinitely many terms. So we have a contradiction, and p must be equal to p'. □

Definition 9.6. If a sequence (a_n) converges to $p \in \mathbb{R}$, we call p the *limit* of (a_n) and write

$$\lim_{n\to\infty} a_n = p.$$

Definition 9.7. Let A and B be ordered sets. A function $f : A \to B$ is said to be *increasing* if $a < a'$ implies $f(a) < f(a')$, and *decreasing* if $a < a'$ implies $f(a) > f(a')$.

Definition 9.8. Let (a_n) be a sequence. A *subsequence* of (a_n) is a sequence $b \colon \mathbb{N} \to \mathbb{R}$ defined by the composition $b = a \circ i$, where $i \colon \mathbb{N} \to \mathbb{N}$ is an increasing function. If (a_n) has a subsequence with limit p, we call p a *subsequential limit* of (a_n).

If we let $n_k = i(k) \in \mathbb{N}$, we can write $b_k = a_{n_k}$, so that (b_k) is the sequence $b_1, b_2, b_3, \ldots,$ which is equal to the sequence $a_{n_1}, a_{n_2}, a_{n_3}, \ldots,$ where $n_1 < n_2 < n_3 < \cdots$.

Theorem 9.9. *If (a_n) converges to p, then so do all of its subsequences.*

Proof. Let b_n be a subsequence of a_n such that $b_k = a_{i(k)}$ for i an increasing function. Suppose a_n converges to $p \in \mathbb{R}$, then by 9.2 we know that for any region R such that $p \in R$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, $a_n \in R$. Then choose $m > N$ such that $m = i(k_0)$ for $k_0 \in \mathbb{N}$. Then for all $k \geq k_0$, we know that $i(k) \geq i(k_0)$ as i is increasing. It follows then that $k \geq k_0$, so we know that $i(k) \geq m > N$. $a_{i(k)} = b_k$ and $b_k \in R$ for all $k \geq k_0$, so we have by 9.2 that b_n converges to p . □

Exercise 9.10. Construct a sequence with two subsequential limits. Construct a sequence with infinitely many subsequential limits.

Proof. The sequence $a_n = (-1)^n$ has two subsequential limits, as we have the subsequence defined by $b_n = 1$ and the subsequence defined by $c_n = -1$. b_n converges to 1 and c_n converges to -1.
The sequence $a_n = 1, 1, 2, 1, 2, 3, 1, 2, 3, 4, 1, 2, 3, 4, 5, \cdots, n$. Then we can find a subsequence b_n such that $b_n = 1, b_n = 2, b_n = 3, \cdots, b_n = n$. These subsequences then converge to $1, 2, 3, \cdots, n$ respectively, so we have infinitely many subsequential limits. □

Let $p \in \mathbb{R}$ and for each natural number $k \geq 1$, define R_k to be the region $(p - \frac{1}{k}, p + \frac{1}{k})$.

Lemma 9.11. *The R_k form a descending collection of regions $R_1 \supset R_2 \supset R_3 \supset \cdots$ whose intersection is the point p:*

$$\bigcap_{k\geq 1} R_k = \{p\}.$$

Proof. We know that $p - \frac{1}{k} < p - \frac{1}{k+1}$ and that $p + \frac{1}{k+1} < p + \frac{1}{k}$ so it folllows that $(p - \frac{1}{k}, p + \frac{1}{k}) \supset (p - \frac{1}{k+1}, p + \frac{1}{k+1})$. It follows then that $R_1 \supset R_2 \supset R_3 \cdots$.
We know that $p \in \bigcap_{k\geq 1} R_k$, so we now show that $\bigcap_{k\geq 1} R_k = \{p\}$. Let $q \in \bigcap_{k\geq 1} R_k$ such that $q \neq p$. Then we know that $q > p$ or $q < p$. If $q > p$, then we have that $q - p \in \mathbb{R}$ and $q - p > 0$ so we have by 8.25 that $q - p > \frac{1}{k} > 0$ for some $k \in \mathbb{N}$. Then it follows that $q > p + \frac{1}{k}$ for $k \in \mathbb{N}$, so $q \notin (p - \frac{1}{k}, p + \frac{1}{k})$. This is a contradiction as we now have that $q \notin \bigcap_{k\geq 1} R_k$. The same result can be shown for the case of $q < p$ without loss of generality. So we have that $\bigcap_{k\geq 1} R_k = \{p\}$. □

Lemma 9.12. *Let R be a region containing p. Then there exists a natural number N such that $R_k \subset R$ for all $k \geq N$.*

Proof. Let $S = (a, b)$ be a region containing p for $a, b \in \mathbb{R}$. Then we have that $a < p$, so $p - a > 0$ and we now that $p - a \in \mathbb{R}$. It follows then by 8.25 that there exists M such that $p - a > \frac{1}{M}$ for $M \in \mathbb{N}$. Note then that we have $a < p - \frac{1}{M}$ for $M \in \mathbb{N}$. We now consider $p + \frac{1}{M}$ and take two cases, where $p + \frac{1}{M} \geq b$ and where $p + \frac{1}{M} < b$.
Consider $p + \frac{1}{M} \geq b$. $b > p$, so we know there exists $N \in \mathbb{N}$ such that $b - p > \frac{1}{N}$ by 8.25. So it follows then that $b > p + \frac{1}{N}$. Then we have $p + \frac{1}{M} \geq b > p + \frac{1}{N}$. It follows then that $p + \frac{1}{M} > p + \frac{1}{N}$, so $p - \frac{1}{M} < p - \frac{1}{N}$. Recall $p - \frac{1}{M} > a$, so then we have $a < p - \frac{1}{N}$ and that $p + \frac{1}{N} < b$, so we know that $R_N \subset S$ and by 9.11 we know that the collection of regions is descending, so we have that $R_k \subset S$ for all $k \geq N$.
Consider $p + \frac{1}{M} < b$, then we have that $a < p - \frac{1}{M}$ and that $p + \frac{1}{M} < b$, so we have that $R_M \subset S$ and similarly then that $R_k \subset S$ for all $k \geq M$. $\square$

Theorem 9.13. *Let $A \subset \mathbb{R}$. Then $p \in \overline{A}$ if and only if there exists a sequence (a_n), with each $a_n \in A$, that converges to p.*

Proof. Let $A \subset \mathbb{R}$ and suppose $p \in \overline{A}$. Then we know that for all regions R such that $p \in R$, $R \cap A \neq \emptyset$. We also know that for a region R such that $p \in R$, by 9.12 there exists $N \in \mathbb{N}$ such that $R_k \subset R$ for all $k \geq N$. So we choose $a_k \in R_k$, then we have that $a_k \in R_k \subset R$, so we know then by 9.2 that a_k converges to p.
Suppose there exists a sequence a_n such that for $a_n \in A$, a_n converges to p. By 9.3, we know that for any $p \in R$ for a region R, R contains all but finitely many terms in the sequence. So we have that for any R such that $p \in R$, $R \cap A \neq \emptyset$. It follows then that $p \in \overline{A}$. $\square$

Definition 9.14. A sequence (a_n) is *bounded* if its range $\{a_n \mid n \in \mathbb{N}\}$ is bounded.

Theorem 9.15. *Every convergent sequence is bounded.*

Proof. Let a_n be a convergent sequence such that a_n converges to p for $p \in \mathbb{R}$. Let $R = (a, b)$ for $a, b \in \mathbb{R}$ be a region, then we know there exists $N \in \mathbb{N}$ such that for all $n \geq N$, $a_n \in (a, b)$. Thus, we have that for all $n \geq N$, $a_n < b$. We have for $\{a_1, a_2, \cdots, a_(n-1)\}$ that there exists a greatest element a_k. If $b > a_k$, then b is an upper bound of a_n and we have that a_n is bounded. If $a_k > b$, then $a_k \geq a_n$ and we have that $\max(a_k, b)$ is an upper bound of a_n. So we know that a_n is bounded above. Without loss of generality an analogous proof may be used to show that convergent sequences are bounded below. $\square$

The converse is not true, but there are two important partial converses.

Theorem 9.16 (Monotone Bounded Sequence Theorem)**.** *Every bounded increasing sequence converges to its range's supremum. Every bounded decreasing sequence converges to its range's infimum.*

Proof. Let a_n be a bounded increasing sequence. Then by 8.22, we know that $s = \sup(a_n)$ exists. Note that by definition of $\sup(a_n)$, we know that for $\epsilon > 0$, $s - \epsilon < a_M$ for some $a_M \in a_n$. We know that a_n is increasing, so for $m > M$, $a_m > a_M$. It follows then that $s - a_m < s - a_M$, so $s - a_m < s - a_M < \epsilon$ (recall that $s - \epsilon < a_M$) and so we have that for all $n \geq M$, $a_n \in R$. Thus we know that for a region S such that $s = \sup(a_n) \in S$, we have that some $a_n \in S$ so a_n converges to $s = \sup(a_n)$. Similarly, it can be shown that a bounded decreasing sequence converges to its range's infimum. □

Theorem 9.17. *Every bounded sequence has a convergent subsequence.* Hint: sometimes this is called the Bolzano–Weierstrass theorem.

Proof. Let a_n be bounded, then we know that the range of a_n is bounded. Let A be the range of a_n. Then we know that A is bounded and we consider two cases, where A is finite and where A is infinite.

We consider the case where A is finite, then we know that for any $n \in bbN$, a_n has finitely many possible values. Let t be the finite number of possible values. Then the set $\{a_n, \cdots, a_{n+t}\}$ has $t+1$ values in it so there must be at least one repeated value. We know that the sequence is infinite, so the cycles of $\{a_n, \cdots, a_{n+t}\}$ are repeated infinitely. Thus, some number p must be repeated infinitely many times, so we take the subsequence $\{p, \cdots, p\}$ which we know is convergent.

Now consider the case where A is infinite, then we know by Bolzano-Weierstrauss that a limit point p of A exists. So for all regions R containing p, we know that $R \cap (A \setminus \{p\}) \neq \emptyset$. Then we know that for a region $R_k = (p - \frac{1}{k}, p + \frac{1}{k})$, there exists some $a_{n_k} \in R_k$. Let R be an arbitrary region containing p, then by 9.12, we know that there exists $N \in \mathbb{N}$ such that $R_k \subset R$ for all $k \geq \mathbb{N}$. Thus we know that there exists some $k \geq N$ such that $a_{n_k} \in R_k \subset R$, so we have that there exists some $a_{n_k} \in R$. So by 9.2, we know there exists a convergent subsequence. □

Mathematicians often use the letters δ and ϵ to denote small positive numbers.

Lemma 9.18. *Let R be a region containing the point a. Then there exists a number $\delta > 0$ such that $(a - \delta, a + \delta) \subset R$.*

Proof. Let R be a region containing a. We write $R = (m, n)$ for $m, n \in \mathbb{R}$. Then we let $\delta = \min(|a - m|, |a - n|)$. It follows then that $\delta \leq |a - m|$ and $\delta \leq |a - n|$, so we know that $a - \delta \geq m$ and that $a + \delta \leq n$. Then we have that $(a - \delta, a + \delta) \subset R$. □

Definition 9.19. The *absolute value* of a real number x is the non-negative number $|x|$ defined by:

$$|x| = \begin{cases} x & \text{if } x \geq 0, \\ -x & \text{if } x < 0. \end{cases}$$

Exercise 9.20. Prove that:

$$(a - \delta, a + \delta) = \{x \in \mathbb{R} : |x - a| < \delta\}.$$

Proof. Let $p \in (a-\delta, a+\delta)$. Then we have $a-\delta < p < a+\delta$, and so $|p-a| < \delta$, so $p \in \{x \in \mathbb{R} : |x-a| < \delta\}$.
Let $p \in \{x \in \mathbb{R} : |x-a| < \delta\}$, then we know $a-\delta < p < a+\delta$ so $p \in (a-\delta, a+\delta)$.
So we have $(a-\delta, a+\delta) = \{x \in \mathbb{R} : |x-a| < \delta\}$. □

We deduce a more concrete characterization of continuity at a point (Definition 5.11).

Theorem 9.21. *Let $A \subset \mathbb{R}$, $f : A \to \mathbb{R}$ and $a \in A$. Then f is continuous at a if and only if the following condition holds: for every $\epsilon > 0$, there exists $\delta > 0$ such that*

$$\text{if} \quad x \in A \quad \text{and} \quad |x-a| < \delta, \qquad \text{then} \quad |f(x) - f(a)| < \epsilon.$$

Proof. We first show the forward direction.
Let $\epsilon > 0$, then let a region $R = (f(a)-\epsilon, f(a)+\epsilon)$. We are given f is continuous at a, so we know by 5.11 that there exists a region S such that for $a \in S$, $f(S \cap A) \subset R$. $a \in S$, so it follows then by 9.18 that there exists $\delta > 0$ such that $(a-\delta, a+\delta) \subset S$. Let $x \in A$ and $x \in (a-\delta, a+\delta)$, then it follows that $|x-a| < \delta$ by 9.20. $x \in (S \cap A)$, so $f(x) \in f(S \cap A)$. Then we have that $f(x) \in f(S \cap A) \subset (f(a)-\epsilon, f(a)+\epsilon)$, so it follows again by 9.20 that $|f(x) - f(a)| < \epsilon$. So we have that the forward direction holds.
We now show the reverse direction.
Let R be a region such that $f(a) \in R$. We express $R = (m, n)$ for $m, n \in \mathbb{R}$. Then we define $\epsilon = \min(|f(a)-m|, |f(a)-n|)$. Note that $(f(a)-\epsilon, f(a)+\epsilon) \subset R$. $\epsilon > 0$, so we are given that there exists $\delta > 0$ such that $|x-a| < \delta$ for all $x \in A$. Then we let the region $S = (a-\delta, a+\delta)$. Let $x \in S \cap A$, then we know that $|x-a| < \delta$ and $x \in A$ so $|f(x) - f(a)| < \epsilon$. It follows then by 9.20 that $f(x) \in (f(a)-\epsilon, f(a)+\epsilon)$, so $f(x) \in R$ because $(f(a)-\epsilon, f(a)+\epsilon) \subset R$. So we have that $f(S \cap A) \subset R$, and thus f is continuous at a by 5.11. So we have that the reverse direction holds.

□

Exercise 9.22.
1. Let $a, b \in \mathbb{R}$ and let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be given by $f(x) = ax + b$. Show that f is continuous at every $x \in \mathbb{R}$.
2. Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be given by $f(x) = \begin{cases} 1 & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$ Show that f is not continuous at 0.

Proof. 1. Let $\epsilon > 0$ be arbitrary. Let $x \in \mathbb{R}$ be arbitrary. Then we have $f(x) = ax + b$ for $a, b \in \mathbb{R}$. Let $p \in \mathbb{R}$, and let $\delta = \frac{\epsilon}{|a|}$ such that $|x-p| < \delta$. It follows then that $|a(x-p)| < \epsilon$. Note that $|f(x) - f(a)| = |ax + b - ap - b| = |ax - ap| = |a(x-p)|$, so we have that $|f(x) - f(a)| < \epsilon$. So we know then that f is continuous at p for every $p \in \mathbb{R}$.

2. Assume that f is continuous at 0. Let $\epsilon = 0.5$ and $a = 0$. Then we have that there exists δ such that $|x-a| < \delta$. So we have $|x-0| < \delta$. We choose $x \in \mathbb{R}$ such that $x \neq 0$. We assume f is continuous at 0, so we know that $|x-0| < \delta$ implies that $|f(x) - f(0)| < \epsilon$. However, we have $x \neq 0$ so $f(x) = 1$ and thus $|f(x) - f(0)| = |1-0| = 1 \not< 0.5$, so we have a contradiction. Thus, f must not be continuous at 0. □