SHEET 10: LIMITS and DERIVATIVES of FUNCTIONS

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The following lemma is very useful in proofs involving inequalities.

Lemma 10.0. *For any real numbers x and y, we have:*

$$\textit{The Triangle Inequality:} \quad |x+y| \leq |x| + |y|$$
$$\textit{The Reverse Triangle Inequality:} \quad ||x| - |y|| \leq |x-y|.$$

(Hint: the second inequality follows from the first.)

Proof. Note that $x \leq |x|$ and $y \leq |y|$. Then consider two cases:
Case 1: If $x + y > 0$, then we have that $x + y = |x + y|$. It follows then that $x + y \leq |x| + |y|$, so $|x + y| \leq |x| + |y|$.
Case 2: If $x + y < 0$, then we have $|x + y| = -x - y$. $-x \leq |-x|$, so we have that $|x+y| = -x-y \leq |-x|+|-y|$. Note that $|-x| = |x|$, so we have then that $|x+y| \leq |x|+|y|$.
To show the reverse triangle inequality, note that $|x+y| \leq |x|+|y|$ by the triangle inequality. It follows then that $|y| = |x + (y - x)| \leq |x| + |y - x|$ and similarly that $|x| = |y + (x - y)| \leq |y|+|x-y|$. It follows then that $|x-y| \geq |x|-|y|$ and that $|y-x| \geq |y|-|x|$. $|x-y| = |y-x|$, so we know that $|x - y| \geq |x| - |y|$ and that $|x - y| \geq |y| - |x|$. It follows directly then that $|x - y| \geq ||x| - |y||$. $\square$

Throughout this sheet, we let $f \colon A \longrightarrow \mathbb{R}$ be a real valued function with domain $A \subset \mathbb{R}$.

Definition 10.1. Let $a \in \mathbb{R}$ be such that there exists a region R containing a with $R \setminus \{a\} \subset A$. A *limit* of f at a point $a \in \mathbb{R}$ is a number $L \in \mathbb{R}$ satisfying the following condition: for every $\epsilon > 0$, there exists a $\delta > 0$ such that

$$\text{if } x \in A \text{ and } 0 < |x - a| < \delta, \quad \text{then } |f(x) - L| < \epsilon \, .$$

Lemma 10.2. *Limits are unique: if L and L' are both limits of f at a point a, then $L = L'$.*

Proof. Assume that L, L' are both limits of f at point a. We arbitrarily assume that $L' > L$, then we let $\epsilon = \frac{L'-L}{2}$. It follows then by Definition 10.1 that there exists δ, δ' such that for $x \in A$, if $0 < |x - a| < \delta$ then $|f(x) - L| < \epsilon$ and if $0 < |x - a| < \delta'$ then $|f(x) - L'| < \epsilon$. Without loss of generality we may note that $\delta' > \delta$. Then let $x \in A$ such that $0 < |x-a| < \delta$, then we have that $0 < |x - a| < \delta'$, so we know that $|f(x) - L| < \epsilon$ and that $|f(x) - L'| < \epsilon$. It follows then that $-\epsilon < f(x) - L < \epsilon$. Expanding, we get that $\frac{3L-L'}{2} < f(x) < \frac{L'+L}{2}$. Similarly, we have that $-\epsilon < f(x) - L' < \epsilon$. Expanding, we get that $\frac{L'+L}{2} < f(x) < \frac{3L'-L}{2}$, so we have a contradiction. Thus, we know that $L' = L$. $\square$

Definition 10.3. If the limit L of f at a exists, we write this as:

$$\lim_{x \to a} f(x) = L.$$

More generally, we can extend the above definition to any $a \in \mathbb{R}$ such that a is a limit point of A. In this case, the fact that a is a limit point guarantees that for all $\delta > 0$ there exists an x such that $x \in A$ and $0 < |x - a| < \delta$. So the "if" part of the statement is never vacuous.

Exercise 10.4. Give an example of a set $A \subset \mathbb{R}$, a function $f \colon A \longrightarrow \mathbb{R}$, and a point a (such that there exists a region R containing a with $R \setminus \{a\} \subset A$) such that $\lim_{x \to a} f(x)$ does not exist.

Proof. Let $f : A \to \mathbb{R}$ be defined as follows:

$$f(x) = \begin{cases} x \leq 0 & -1 \\ x > 0 & 1 \end{cases}$$

Then let $\epsilon = \frac{1}{2}$ and $a = 0 \in A$. It follows that there exists no $\delta > 0$ such that if $|x| < \delta$, then $|f(x)| < \frac{1}{2}$ because there does not exist $f(x)$ such that $|f(x)| < \frac{1}{2}$. $\square$

Theorem 10.5. *A function f is continuous at a if and only if*

$$\lim_{x \to a} f(x) = f(a).$$

Proof. Let f continuous at $a \in A$. Then by 9.21, we have that for all $\epsilon > 0$, there exists $\delta > 0$ such that for $a \in \mathbb{R}$, if $x \in A$ and $|x - a| < \delta$, then $|f(x) - f(a)| < \epsilon$. Let $L = f(a)$, then by Definition 10.1 we have that $L = f(a) = \lim_{x \to a} f(x)$.
Let $\lim_{x \to a} f(x) = f(a)$. Then we have by 10.1 that for all $\epsilon > 0$, there exists $\delta > 0$ such that if $x \in A$ and $0 < |x - a| < \delta$, then $|f(x) - f(a)| < \epsilon$. Then by 9.21 we have that f is continuous at a. $\square$

Exercise 10.6. (i) For every $c \in \mathbb{R}$, the constant function $f : \mathbb{R} \to \mathbb{R}$, defined by $f(x) = c$ for all $x \in \mathbb{R}$, is continuous.

(ii) The identity function, defined by $g(x) = x$, is continuous.

Proof. (i) Let $c \in \mathbb{R}$ such that $f(x) = c$. Let $y \in \mathbb{R}$ be arbitrary, then we let $L = c$ and note that $|f(x) - c| < \epsilon$ is always true as $0 < \epsilon$. So we have then by Definition 10.1 that $\lim_{x \to a} f(x) = c = f(y)$, so by Theorem 10.5 we have that f is continuous at y. $y \in \mathbb{R}$ is arbitrary, so we have that f is continuous.
(ii) Let $y \in \mathbb{R}$ and let $g(x) = x$. Then we have that $g(y) = y$. Let $L = g(y) = y$, then for any $\epsilon > 0$, let $\delta = \epsilon$. It follows then that if $x \in A$ such that $0 < |x - y| < \delta$, then $0 < |f(x) - f(y)| = |x - L| < \delta = \epsilon$. We have then that $\lim_{x \to y} g(x) = L = y = g(y)$, so by 10.5 we know that f is continuous at y. y is arbitrary, so we know that f is continuous. $\square$

Exercise 10.7. Show that the absolute value function $f : \mathbb{R} \to \mathbb{R}$, $f(x) = |x|$ is continuous.

Proof. Let $f(x) = |x|$. Let $y \in \mathbb{R}$ be arbitrary, then for $\epsilon > 0$, let $\delta = \epsilon$. We have then that if $x \in \mathbb{R}$ such that $|x-y| < \delta$, then $||x|-|y|| < \epsilon$. By 10.0, we know that $||x|-|y|| \leq |x-y| < \epsilon$. $\delta = \epsilon$, so this is always true. Then it follows that $L = |y| = f(y)$, so $f(y) = \lim_{x\to y} f(x)$ and by 10.5 then we have that f is continuous at y. y is arbitrary, so we know then that f is continuous. $\square$

Given real-valued functions f and g, we define new functions $f+g$, fg and $\frac{1}{f}$ by

- $(f+g)(x) = f(x) + g(x)$
- $(fg)(x) = f(x) \cdot g(x)$
- $\frac{1}{f}(x) = \frac{1}{f(x)}$, provided that $f(x) \neq 0$.

We wish to understand the limits of $f+g$, fg and $\frac{1}{f}$ in terms of the limits of f and g.

Theorem 10.8. *Suppose that* $\lim\limits_{x\to a} f(x) = L$ *and* $\lim\limits_{x\to a} g(x) = M$. *Then*

$$\lim_{x\to a}(f+g)(x) = L + M.$$

Proof. Let $\epsilon > 0$ and define $\epsilon' = \frac{\epsilon}{2}$. By 10.1, we know that there exist δ_1, δ_2 such that if $0 < |x-a| \leq \delta_1$ then $|f(x) - L| < \epsilon'$ and if $0 < |x-a| \leq \delta_2$ then $|g(x) - M| < \epsilon'$. Let $\delta = \min(\delta_1, \delta_2)$. Then it follows that if $0 < |x-a| \leq \delta$, then $|f(x)-L| + |g(x)-M| < \epsilon' + \epsilon'$. Using the triangle inequality and simplifying, we get then that $|(f+g)(x) - (L+M)| < \epsilon$, so we have that $\lim_{x\to a}(f+g)(x) = L + M$. $\square$

Lemma 10.9. *If*

$$|x - x_0| < \min\Big(1, \frac{\epsilon}{2(|y_0|+1)}\Big) \quad \textit{and} \quad |y - y_0| < \frac{\epsilon}{2(|x_0|+1)},$$

then

$$|xy - x_0y_0| < \epsilon.$$

Proof. $|xy - x_0y_0| = |(y-y_0)x + y_0x - x_0y_0|$
$= |(y-y_0)x + y_0(x-x_0)|$
$\leq |(y-y_0)x| + |y_0(x-x_0)|$ by the triangle inequality
Note that $|x - x_0| \leq 1$ so we have that $x \leq x_0 + 1$.
Substituting, then we get that $|xy - x_0y_0| \leq |(y-y_0)(x_0+1)| + |y_0(x-x_0)|$
Note that $|y-y_0| < \frac{\epsilon}{2(|x_0|+1)}$ so substituting we have that $|(y-y_0)(x_0+1)| < \frac{\epsilon|x_0+1|}{2|x_0+1|} = \frac{\epsilon}{2}$.
Similarly we get that $|y_0(x-x_0)| < \frac{\epsilon(|y_0|)}{2(|y_0|+1)}$.
Note that $\frac{|y_0|}{|y_0|+1} < 1$, so we have that $|y_0(x-x_0)| < \frac{\epsilon}{2}$. Substituting into the original equation we get that $|xy - x_0y_0| \leq \frac{\epsilon}{2} + \frac{\epsilon}{2}$ so we have $|xy - x_0y_0| < \epsilon$. $\square$

Theorem 10.10. *Suppose that* $\lim_{x\to a} f(x) = L$ *and* $\lim_{x\to a} g(x) = M$. *Then*

$$\lim_{x\to a}(fg)(x) = L \cdot M.$$

Proof. Let $\epsilon > 0$ be arbitary. Then let $p = \min\Big(1, \frac{\epsilon}{2(|y_0|+1)}\Big)$ and $q = \frac{\epsilon}{2(|x_0|+1)}$. $p, q > 0$, so we know there exist δ_1, δ_2 such that if $0 < |x - a| < \delta_1$ then $|f(x) - L| < p$ and if $|x - a| < \delta_2$ then $|g(x) - M| < q$. Let $\delta = \min \delta_1, \delta_2$. Then we have that for any $\epsilon > 0$, there exists $\delta > 0$ such that if $0 < |x - a| < \delta$, then $|f(x) - L| < p$ and $|g(x) - M| < q$, and thus by Theorem 10.9 then $|(fg)(x) - LM| < \epsilon$. Then by Definition 10.1 we have that $\lim_{x\to a}(fg)(x) = L \cdot M$. □

Theorem 10.11. *Suppose that* $\lim_{x\to a} f(x) = L \neq 0$. *Then*

$$\lim_{x\to a} \frac{1}{f}(x) = \frac{1}{L}.$$

Proof. Let $\epsilon > 0$ and let $|f(x) - L| < \sigma$. We know that $|f(x)L||\frac{1}{f(x)} - \frac{1}{L}| < \sigma$ implies that $|\frac{1}{f(x)} - \frac{1}{L}| < \frac{\sigma}{|f(x)L|}$. We have that $f(x) \in (L - \sigma, L + \sigma)$, so it follows that $0 < \sigma < |L|$. Suppose that $L + \sigma < 0$, then we have that $|f(x)| > |L + \sigma|$. Suppose that $L - \sigma > 0$, then we have that $|f(x)| > |L - a|$. We have then that $\sigma < \frac{|L|}{2}$ and $|f(x)| > |L - \sigma|$. It follows then that $|f(x)| \geq |L| - |L - f(x)|$. Substituting we get that $|L - f(x)| < |L| - \frac{|L|}{2} = \frac{|L|}{2}$. So we have then that $|f(x)| > \frac{|L|}{2}$. It follows then that $\frac{\sigma}{|f(x)L|} < \frac{2\sigma}{L^2}$.
So we have $\sigma = \min(\frac{|L|}{2}, \frac{\epsilon|L|^2}{2})$ and because $\sigma' > 0$, we know that there exists $\delta' > 0$ such that if $0 < |x - a| < \delta'$, then $|f(x) - L| < \sigma$. It follows then by substituting that $|\frac{1}{f(x)} - \frac{1}{L}| < \epsilon$, so we have that $\lim_{x\to a} \frac{1}{f}(x) = \frac{1}{L}$. □

Corollary 10.12. *If* f *and* g *are continuous at* a, *then* $f + g$ *and* fg *are continuous at* a. *Also,* $\frac{1}{f}$ *is continuous at* a, *provided that* $f(a) \neq 0$.

Proof. f, g continuous at a, so we have that $\lim_{x\to a} f(x) = f(a)$ and $\lim_{x\to a} g(x) = g(a)$.
To show that $f + g$ is continuous at a, we have by definition that $\lim_{x\to a}(f + g)(x) = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$, so $\lim_{x\to a}(f + g)(x) = f(a) + g(a) = (f + g)(a)$. So by 10.5 $f + g$ is continuous at a.
To show that fg is continuous at a, we similarly have that $\lim_{x\to a}(fg)(x) = f(a) \cdot g(a) = (fg)(a)$ and thus by 10.5 we know that fg is continuous at a.
To show that $\frac{1}{f}$ is continuous at a, we similarly have that $\lim_{x\to a} \frac{1}{f}(x) = \frac{1}{f(a)}$, and thus we know by 10.5 that $\frac{1}{f}$ is continuous at a. □

Definition 10.13. A *polynomial in one variable with real coefficients* is a function f of the form

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

for some $n \in \mathbb{N} \cup \{0\}$, where $a_i \in \mathbb{R}$ for $0 \leq i \leq n$. A *rational function in one variable with real coefficients* is a function of the form $h(x) = \frac{f(x)}{g(x)}$ where f and g are polynomials in one variable with real coefficients.

Corollary 10.14. *Polynomials in one variable with real coefficients are continuous. A rational function in one variable with real coefficients $h(x) = \frac{f(x)}{g(x)}$ is continuous at all $a \in \mathbb{R}$ where $g(a) \neq 0$.*

Proof. We know by 10.6 that $f(x) = a$ is continuous for all $x \in \mathbb{R}$. Additionally, by 10.12 we know the function $f_2(x) = (f + f)(x) = 2a$ is also continuous. Using induction, assume $f_n(x) = n \cdot a$ is continuous, then we have that $f_{n+1}(x) = na + a$ is also continuous. We also know that a function $g(x) = x$ is continuous, so by 10.12 the function $g_2(x) = (g \cdot g)(x) = x^2$ is continuous as well. Applying induction, we assume $g_n(x) = x^n$ is continuous, then it follows closely that $g_{n+1}(x) = x^n \cdot x$ is also continuous. So through induction we know that $g_n(x) = x^n$ is continuous for $n \in \mathbb{N}$ and $x \in \mathbb{R}$.
We know that any $f_n(x) = na$ and any $g_n(x) = x^n$ are continuous, so any $(f_n \cdot g_n)(x) = a_n x^n$ is also continuous. It follows then that the sum of these $(f_n \cdot g_n)(x)$ is also continuous, so we know that any polynomial in one variable is continuous.
For a rational function $h(x) = \frac{f(x)}{g(x)}$, we have just shown that f, g are continuous, so $h(x) = \frac{f(x)}{g(x)} = f(x) * \frac{1}{g(x)}$ is continuous at all $a \in \mathbb{R}$ such that $g(a) \neq 0$ as we have that $\frac{1}{g(x)}$ is continuous at all $a \in \mathbb{R}$ such that $g(a) \neq 0$. $\square$

Now we want to look at limits of the composition of functions. It is not quite true in general that if $\lim_{x \to a} g(x) = M$ and $\lim_{y \to M} f(y) = L$, then $\lim_{x \to a} f(g(x)) = L$, but it is true in some cases.

Proposition 10.15. *Let $a \in \mathbb{R}$. Then*

$$\lim_{x \to a} f(x) = \lim_{h \to 0} f(a + h),$$

assuming that the limit on the left exists. (Hint: You can think of the right hand side as the composition of f with the function $g(h) = a + h$.)

Proof. Let $a \in \mathbb{R}$ and let $\lim_{x \to a} f(x) = L$. Let $\epsilon > 0$ be arbitary, then the limit on the LHS exists so we know that there exists $\delta > 0$ such that if $0 < |x - a| < \delta$ then $|f(x) - L| < \epsilon$ by Definition 10.1. Choose $h = x - a$, so it follows that $0 < |h| < \delta$. We have then that $|(h + a) - L| < \epsilon$, so $L = \lim_{h \to 0} f(a + h)$. So we get $\lim_{x \to a} f(x) = L = \lim_{h \to 0} f(a + h)$. $\square$

Theorem 10.16. *If $\lim_{x \to a} g(x) = M$ and f is continuous at M, then $\lim_{x \to a} f(g(x)) = f(M)$.*

Proof. Let $\lim_{x \to a} g(x) = M$, then for $\epsilon, \epsilon' > 0$, there exists $\delta, \delta' > 0$ such that if $0 < |x - a| < \delta$ then $|f(x) - f(M)| < \epsilon$ and if $0 < |x - a| < \delta'$ then $|g(x) - M| < \epsilon'$. Let $\epsilon' = \delta$, then we have that for $\epsilon > 0$, there exists δ such that if $|g(x) - M| < \delta$, then $|f(g(x)) - f(M)| < \epsilon$. It follows then by 10.1 that $\lim_{x \to a} f(g(x)) = f(M)$. $\square$

Remark 10.17. This theorem can also be rewritten as

$$\lim_{x \to a} f(g(x)) = f\left(\lim_{x \to a} g(x)\right),$$

which can be remembered as "limits pass through continuous functions."

Corollary 10.18. *If g is continuous at a and f is continuous at $g(a)$, then $f \circ g$ is continuous at a.*

Proof. f is continuous at a, so by Remark 10.17 we can express $\lim_{x\to a} f(g(x)) = f(\lim_{x\to a} g(x))$. $g(x)$ is continuous at a, so we know $\lim_{x\to a} g(x) = g(a)$. It follows then that $\lim_{x\to a} f(g(x)) = f(\lim_{x\to a} g(x)) = f(g(a))$. So by 10.5, we have that $f \circ g$ is continuous at a. □

We now assume the domain $A \subset \mathbb{R}$ is open.

Definition 10.19. The *derivative* of f at a point $a \in A$ is the number $f'(a)$ defined by the following limit:

$$f'(a) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h},$$

provided the limit on the right hand side exists. If $f'(a)$ exists, we say that f is *differentiable at a*. If f is differentiable at all points of its domain, we say that f is *differentiable*. In this case, the values $f'(a)$ define a new function $f' \colon A \longrightarrow \mathbb{R}$ called the *derivative* of f.

Here is a useful reformulation of the definition of the derivative.

Theorem 10.20. *If f is differentiable at a, the derivative of f at a is given by the limit:*

$$f'(a) = \lim_{x\to a} \frac{f(x) - f(a)}{x - a}.$$

Proof. Let $x = a + h$, then $f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$
Substituting we get $f'(a) = \lim_{x\to a} \frac{f(x)-f(a)}{x-a}$ □

Theorem 10.21. *If f is differentiable at a, then f is continuous at a.*

Proof. Let f differentiable at a. Note that $\lim_{x\to a}(f(x)-f(a)) = \lim_{x\to a} \frac{f(x)-f(a)}{x-a} \cdot \lim_{x\to a}(x-a) = f'(a) \cdot 0 = 0$. So we have that $\lim_{x\to a}(f(x) - f(a)) = 0$ and $\lim_{x\to a} f(a) = f(a)$, so it follows then that $\lim_{x\to a}(f(x) - f(a)) + \lim_{x\to a} f(a) = f(a) = 0 + f(a)$. So $\lim_{x\to a}(f(x) - f(a) + f(a)) = \lim_{x\to a}(f(x)) = f(a)$ by 10.8. So we have then by 10.5 that $f(x)$ is continuous at a. □

Exercise 10.22. Show that the converse of Theorem 10.21 is not true in general.

Proof. Let $f(x) = |x|$. Then by 10.7, we know that f is continuous. Let $x < 0$, then we have that $\lim_{x\to 0} \frac{f(x)-f(0)}{x-0} = \lim_{x\to 0} \frac{|x|}{x}$. $x < 0$, so we have that $L = \lim_{x\to 0} \frac{|x|}{x} < 0$. Let $x > 0$, then we have that $L = \lim_{x\to 0} \frac{|x|}{x} > 0$. So we have a contradiction, as $L > 0$ and $L < 0$, and we know by 10.2 that limits are unique. Therefore $f(x) = |x|$ is not differentiable at 0. □

Exercise 10.23. (i) Show that for all $n \in \mathbb{N}$,

$$x^n - a^n = (x-a)(x^{n-1} + ax^{n-2} + a^2x^{n-3} + \cdots + a^{n-2}x + a^{n-1}),$$

or more formally,

$$x^n - a^n = (x-a)\left(\sum_{i=0}^{n-1} x^{n-1-i}a^i\right).$$

(ii) Use this to prove that if $f(x) = x^n$ for some $n \in \mathbb{N}$, then $f'(a) = na^{n-1}$.

Proof. (i) We know by expanding that $(x-a)(x^{n-1} + ax^{n-2} + ... + a^{n-2}x + a^{n-1}) = (x^n + ax^{n-1} + ... + a^{n-2}x^2 + a^{n-1}x) - (ax^{n-1} + a^2x^{n-2} + ... + a^{n-1}x + a^n)$. Notice that all terms except the first and last cancel, so we are left with $(x-a)(x^{n-1} + ax^{n-2} + ... + a^{n-2}x + a^{n-1}) = (x^n + ax^{n-1} + ... + a^{n-2}x^2 + a^{n-1}x) - (ax^{n-1} + a^2x^{n-2} + ... + a^{n-1}x + a^n) = x^n - a^n$.
(ii) $f'(a) = \lim_{x\to a} \frac{f(x)-f(a)}{x-a}$. $\lim_{x\to a} \frac{x^n - a^n}{x-a} = \lim_{x\to a}(x^{n-1} + ax^{n-2} + ... + a^{n-1})$. We know that the limit exists because polynomials are continuous, and note that each term inside the sum $(x^{n-1} + ax^{n-2} + ... + a^{n-1})$ approaches a^{n-1} as x approaches a. There are n terms, so we have that $f'(a) = \lim_{x\to a} \frac{x^n-a^n}{x-a} = na^{n-1}$. □

Lemma 10.24. *If $f : (a,b) \to \mathbb{R}$ is continuous and $f(c) > 0$ for some $c \in (a,b)$, then there exists a region $R \subset (a,b)$ such that $c \in R$ and $f(x) > 0$ for all $x \in R$. The analogous statement is true if $f(c) < 0$.*

Proof. Let $f : (a,b) \to \mathbb{R}$ be continuous such that $f(c) > 0$ for some $c \in (a,b)$. Let $\epsilon = f(c)$. f is continuous, so we know there exists $\delta > 0$ such that if $x \in (a,b)$, then if $0 < |x - c| < \delta$, then $|f(x) - f(c)| < \epsilon = f(c)$. If $f(x) - f(c) < f(c)$, then we have that $f(x) < 2f(c)$. If $f(c) - f(x) < f(c)$, then we have that $0 < f(x)$. Then we have that $0 < f(x) < 2f(c)$, so we know that $x \in R$ where R is the region $R = (c - \delta, c + \delta)$. □

Exercise 10.25. Suppose that f and g are differentiable at a.

(i) Compute $(f+g)'(a)$ in terms of $f'(a)$ and $g'(a)$.

(ii) Compute $(fg)'(a)$ in terms of $f(a)$, $g(a)$, $f'(a)$ and $g'(a)$.

(iii) Compute $\left(\frac{1}{f}\right)'(a)$ in terms of $f'(a)$ and $f(a)$. What assumption do you need to make?

These results are known as the Sum Rule, Product Rule, and (a special case of the) Quotient Rule for Derivatives, respectively.

Proof. (i) We know the limit of $\frac{(f+g)(x)-(f+g)(a)}{x-a}$ exists because f is differentiable at a so f is continuous at a. We compute the limit $\lim_{x\to a} \frac{(f+g)(x)-(f+g)(a)}{x-a} = \lim_{x\to a} \frac{f(x)-f(a)}{x-a} + \lim_{x\to a} \frac{g(x)-g(a)}{x-a} = f'(a) + g'(a)$. Thus, we have that $(f+g)'(a) = f'(a) + g'(a)$.
(ii) We know the limit of $\frac{(fg)(x)-(fg)(a)}{x-a}$ exists becuase f is differentiable at a so f is continuous at a. We compute the limit $\lim_{x\to a} \frac{(fg)(x)-(fg)(a)}{x-a} = \lim_{x\to a} \frac{f(x)(g(x)-g(a))+g(a)(f(x)-f(a))}{x-a} = f(a)g'(a) + g(a)f'(a)$. So we have that $(fg)'(a) = f(a)g'(a) + g(a)f'(a)$.
(iii) Assuming $f(a) \neq 0$, we know that the limit of $\frac{\frac{1}{f(x)} - \frac{1}{f(a)}}{x-a}$ exists because f is differentiable at a so f is continuous at a. We compute the limit $\lim_{x\to a} \frac{\frac{1}{f(x)} - \frac{1}{f(a)}}{x-a} = \lim_{x\to a} \frac{f(a)-f(x)}{f(x)f(a)(x-a)} = \frac{-f'(a)}{f(a)^2}$. So we have that $\frac{1}{f'(a)} = \frac{-f'(a)}{f(a)^2}$. □

Exercise 10.26. Suppose f and g are differentiable at a and $g(a) \neq 0$.

(i) Show that $\frac{f(x)}{g(x)}$ is continuous at a.

(ii) Show that $\frac{f(x)}{g(x)}$ is differentiable at a, and compute $\left(\frac{f}{g}\right)'(a)$ in terms of $f(a)$, $g(a)$, $f'(a)$ and $g'(a)$.

This last result is known as the Quotient Rule for Derivatives.

Proof. (i) We know that $\frac{f(x)}{g(x)} = \frac{1}{g(x)}f(x)$, and we have that $\frac{1}{g(x)}$ is continuous, so we know that $\frac{f(x)}{g(x)}$ is also continuous.
(ii) $\frac{f'(a)}{g'(a)} = \lim_{x\to a} \frac{\frac{f'(x)}{g'(x)} - \frac{f'(a)}{g'(a)}}{x-a}$
$= \lim_{x\to a} \frac{f(x)g(a)-g(x)f(a)}{(x-a)g(a)g(x)} = \lim_{x\to a}(\frac{f(x)-f(a)}{x-a}g(a) - \frac{g(x)-g(a)}{x-a}f(a))\frac{1}{g(a)g(x)}$.
$= \frac{f'(a)g(a)-g'(a)f(a)}{g(a)^2}$ provided that $g(a) \neq 0$. $\square$

One of the most important results concerning the differentiation of functions is the Chain Rule for the derivative of a composition of functions. Let $f : B \to \mathbb{R}$, $g : A \to \mathbb{R}$ be functions such that $g(A) \subset B$. Thus, the composition $(f \circ g)(x) = f(g(x))$ is defined for all $x \in A$.

Lemma 10.27. *Given f and g as above, with f differentiable at $g(a)$, define a new function $\varphi : B \to \mathbb{R}$ (note that the domain is the same as the domain of f) by*

$$\varphi(y) = \begin{cases} \dfrac{f(y) - f(g(a))}{y - g(a)} & \text{if } y \neq g(a), \\ f'(g(a)) & \text{if } y = g(a). \end{cases}$$

Then

a) φ is continuous at the point $g(a)$

b) For all $x \neq a$,

$$\varphi(g(x))\frac{g(x) - g(a)}{x - a} = \frac{f(g(x)) - f(g(a))}{x - a}.$$

Proof. a) We have that $\varphi(y) = \frac{f(g)-f(g(a)}{y-g(a)}$ for $y \neq g(a)$. It follows then that $f'(g(a)) = \lim_{y\to g(a)} \varphi(y)$. We also know that $\varphi(g(a)) = f'(g(a))$ so it follows then that $\lim_{y\to g(a)} \varphi(y) = \varphi(g(a))$, so by 10.5 φ is continuous.
b) Let $x \neq a$, then we have that $\frac{g(x)-g(a)}{x-a}$ exists. It follows then that $\varphi(g(x))\frac{g(x)-g(a)}{x-a} = \frac{f(g(x))-f(g(a))}{g(x)-g(a)}(\frac{g(x)-g(a)}{x-a})$. Cancelling then, we get that $\varphi(g(x))\frac{g(x)-g(a)}{x-a} = \frac{f(g(x))-f(g(a))}{x-a}$. In the case that $g(x) = g(a)$, we have that $\varphi(g(x)) * 0 = 0$, which is self-evident. $\square$

Theorem 10.28 (Chain Rule)**.** *Let $a \in A$, suppose that g is differentiable at a and f is differentiable at $g(a)$. Then $f \circ g$ is differentiable at a and:*

$$(f \circ g)'(a) = f'(g(a)) \cdot g'(a).$$

Proof. $(f \circ g)'(a) = \lim_{x\to a} \frac{f(g(x))-f(g(a))}{x-a}$
$= \lim_{x\to a} \varphi(g(x)) \frac{g(x)-g(a)}{x-a}$ by Lemma 10.27
$= \lim_{x\to a} \varphi(g(x)) \lim_{x\to a} \frac{g(x)-g(a)}{x-a}$
$= \varphi(g(a))g'(a) = f'(g(a))g'(a)$. □

We finish this sheet with a discussion of maxima and minima of real-valued functions and the most important theorem in differential calculus, the Mean Value Theorem.

Definition 10.29. Let $f\colon A \longrightarrow \mathbb{R}$ be a function. If $f(a)$ is the last point of $f(A)$, then $f(a)$ is called the *maximum value* of f. If $f(a)$ is the first point of $f(A)$, then $f(a)$ is the *minimum value* of f. We say that $f(a)$ is a *local maximum value* of f if there exists a region R containing a such that $f(a)$ is the last point of $f(A \cap R)$. We say that $f(a)$ is a *local minimum value* of f if there exists a region R containing a such that $f(a)$ is the first point of $f(A \cap R)$.

Remark 10.30. Equivalently, $f(a)$ is a *local maximum (resp. minimum) value* of f if there exists U open *in* A such that $f(a)$ is the last (resp. first) point of $f(U)$.

Theorem 10.31. *Let $f : A \longrightarrow \mathbb{R}$ be differentiable at a. Suppose that $f(a)$ is the maximum value or minimum value of f. Then $f'(a) = 0$.*

Proof. Let $f(a)$ be the maximum value of f. f is differentiable at a, so we know that there exists $x, y \in A$ such that $x < a$ and $a < y$. It follows then that $\frac{f(x)-f(a)}{x-a} \geq 0$ because $f(x) \leq f(a)$ and $x < a$. Similarly, we have that $\frac{f(y)-f(a)}{y-a} \leq 0$ because $f(y) \leq f(a)$ and $y > a$.
Note that we may take the limit of both sides of the inequality without contradicting the inequality. This can be proven through contradiction by considering a function $F(x)$ such that $F(x) \leq 0$. Suppose $\lim_{x\to a} F(x) > 0$, then by the definition of a limit (10.1) we can choose $\epsilon = \lim_{x\to a} F(x)$ and we know there exists $\delta > 0$ such that if $0 < |x-a| < \delta$, then $|F(x) - \epsilon| < \epsilon$. It follows then that $F(x) > 0$, which contradicts $F(x) \leq 0$. Thus we know that $\lim_{x\to a} F(x) \leq 0$.
We have shown then that $\frac{f(y)-f(a)}{y-a} \leq 0$ implies that $\lim_{y\to a} \frac{f(y)-f(a)}{y-a} \leq 0$ and similarly that $\lim_{x\to a} \frac{f(x)-f(a)}{x-a} \geq 0$. It follows then that $f'(a) \leq 0$ and $f'(a) \geq 0$ by Theorem 10.20. We have then that $f'(a) = 0$.
Without loss of generality we may let $f(a)$ be the minimum value of f and similarly prove that $f'(a) = 0$. □

Corollary 10.32. *Let $f : A \longrightarrow \mathbb{R}$ be differentiable at a. Suppose that $f(a)$ is a local maximum or local minimum value of f. Then $f'(a) = 0$.*

Proof. Let $f(a)$ be a local maximum value of f. By Definition 10.29 we know that there exists a region R containing a such that $f(a)$ is the last point of $f(A \cap R)$. It follows then that $f(a)$ is the maximum value of $f|_R : A \to \mathbb{R}$. Since $\mathbb{R}$ is a region containing a, we know that $f|_R : A \to \mathbb{R}$ is differentiable at a. Applying Theorem 10.31 then, we have that

$f'(a) = 0$.
Without loss of generality we may let $f(a)$ be a local minimum value of f and similarly prove that $f'(a) = 0$. $\square$

Theorem 10.33 (Rolle's Theorem)**.** *Suppose that $f\colon [a,b] \longrightarrow \mathbb{R}$ is continuous, differentiable on (a,b) and that $f(a) = f(b) = 0$. Then there exists a point $c \in (a,b)$ such that $f'(c) = 0$.*

Proof. We know that $[a,b]$ is non-empty, closed, and bounded, and that f is continuous, so by the Extreme Value Theorem (5.18) we know that $f([a,b]$ has a first and last point. We are given $f(a) = f(b) = 0$, so we have that $f((a,b)) = f([a,b]) \setminus \{0\}$. We now have two cases:
Case 1: Let 0 be both the first and last point of $f([a,b])$. Then we know that for all $c \in (a,b)$, $f(c) = 0$ and so $f'(c) = 0$.
Case 2: Let 0 be not both the first and last point of $f([a,b])$. Then we know that at least one of the first and last point lies in (a,b). We call this point c, so we have that $c \in (a,b)$ and c is the first or last point of $f((a,b))$. It follows then by Corollary 10.31 that $f'(c) = 0$. $\square$

Corollary 10.34 (The Mean Value Theorem)**.** *Suppose that $f\colon [a,b] \longrightarrow \mathbb{R}$ is continuous on $[a,b]$ and differentiable on (a,b). Then there exists a point $c \in (a,b)$ such that:*

$$f(b) - f(a) = f'(c)(b-a).$$

Proof. Let $g : [a,b] \to \mathbb{R}$ such that $g(x) = f(x)(b-a) - x(f(b) - f(a)) + f(b)a - f(a)b$. By Exercise 10.25, we know that g is continuous and differentiable on (a,b), and we can easily verify that $g(a) = g(b) = 0$. Thus we know by Rolle's Theorem (10.33) that there exists a point $c \in (a,b)$ such that $g'(c) = 0$. We know that $g'(c) = f'(c)(b-a) - f(b) + f(a)$ by computing the derivative, so we have that $f'(c)(b-a) - f(b) + f(a) = g'(c) = 0$. It follows then that $f'(c)(b-a) = f(b) - f(a)$ for some $c \in (a,b)$. $\square$

Corollary 10.35. *Suppose that $f : [a,b] \to \mathbb{R}$ and $g : [a,b] \to \mathbb{R}$ are continuous on $[a,b]$, differentiable on (a,b), and $f'(x) = g'(x)$, for all $x \in (a,b)$. Then there is some $c \in \mathbb{R}$ such that $f(x) = g(x) + c, \forall x \in [a,b]$.*

Proof. Let $f : [a,b] \to \mathbb{R}$, $g : [a,b] \to \mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b) such that $f'(x) = g'(x)$ for all $x \in (a,b)$. We define $h : [a,b] \to \mathbb{R}$ such that $h(x) = f(x) - g(x)$ and it follows that $h(x)$ is continuous on $[a,b]$ and differentiable on (a,b).
Let $x \in [a,b]$ be arbitrary. We know $h'(x) = f'(x) - g'(x)$ by Exercise 10.25 and that $f'(x) = g'(x)$ so $h'(x) = 0$. h is continuous on $[a,b]$ and differentiable on (a,b), so we know that h is continuous on $[x,b]$ and differentiable on (x,b) because $x \in [a,b]$. Then it follows by the Mean Value Theorem (Corollary 10.34) that $h(x) - h(b) = h'(c)(x-b)$. We know $h'(c) = 0$ because $h'(x) = 0$ for all $x \in [a,b]$ and $c \in (a,b)$. So we have that $h(x) - h(b) = 0$, so $h(x) = h(b)$. It follows then that $f(x) - g(x) = f(b) - g(b) = c$ for some $c \in \mathbb{R}$. Then we have that $f(x) - g(x) = c$, so $f(x) = g(x) + c$ for $c \in \mathbb{R}$. $\square$

Corollary 10.36. *Suppose that $f\colon [a,b] \longrightarrow \mathbb{R}$ is continuous on $[a,b]$ and differentiable on (a,b). Then*

1. *If $f'(x) > 0$ for all $x \in (a,b)$, then f is increasing on $[a,b]$.*
2. *If $f'(x) < 0$ for all $x \in (a,b)$, then f is decreasing on $[a,b]$.*
3. *If $f'(x) = 0$ for all $x \in (a,b)$, then f is constant on $[a,b]$.*

Proof. Suppose that $f : [a,b] \to \mathbb{R}$ is continuous on $[a,b]$ and differentiable on (a,b).
1. Let $f'(x) > 0$ for all $x \in (a,b)$. Let $x_1, x_2 \in (a,b)$ be arbitrarily chosen such that $a \leq x_1 < x_2 \leq b$. Then we know that there exists x_3 by Corollary 10.34 such that $a \leq x_1 < x_3 < x_2 \leq b$ and $f'(x_3) = \frac{f(x_2)-f(x_1)}{x_2-x_1}$. We know that $f'(x) > 0$ for all $x \in (a,b)$, so $f(x_3) > 0$. Also, we have that $x_2 > x_1$, so it follows that $f(x_2) > f(x_1)$. Thus we know that f is increasing on $[a,b]$.
2. Without loss of generality, this follows from a similar proof to the proof for 1.
3. Let $f'(x) = 0$ for all $x \in (a,b)$. Let $x_1, x_2 \in (a,b)$ be arbitrarily chosen such that $a \leq x_1 < x_2 \leq b$. Then we know that there exists x_3 by Corollary 10.34 such that $a \leq x_1 < x_3 < x_2 \leq b$ and $f'(x_3) = \frac{f(x_2)-f(x_1)}{x_2-x_1}$. We know that $f'(x) = 0$ for all $x \in (a,b)$, so $f'(x_3) = 0$ and thus $f(x_2) = f(x_1)$. It follows then that f is constant on $[a,b]$. $\square$

Corollary 10.37 (The Cauchy Mean Value Theorem)**.** *Suppose that $f\colon [a,b] \longrightarrow \mathbb{R}$ and $g\colon [a,b] \longrightarrow \mathbb{R}$ are continuous on $[a,b]$ and differentiable on (a,b). Then there is a point $c \in (a,b)$ such that:*

$$(f(b) - f(a))g'(c) = (g(b) - g(a))f'(c).$$

Proof. Let $f : [a,b] \to \mathbb{R}$ and $g : [a,b] \to \mathbb{R}$ continuous on $[a,b]$ and differentiable on (a,b). Define $h : [a,b] \to \mathbb{R}$ such that $h(x) = k_1 f(x) + k_2 g(x)$. Then it follows that:
$k_1 f(a) + k_2 g(a) = k_1 f(b) + k_2 g(b)$
$k_2(g(a) - g(b)) = k_1(f(b) - f(a))$
$\frac{k_2}{k_1} = \frac{f(b)-f(a)}{g(a)-g(b)}$
Let $k_2 = f(b) - f(a)$ and $k_1 = g(a) - g(b)$, then define $i[a,b] \to \mathbb{R}$ such that $i(x) = h(x) - h(a)$. It follows then that $i(a) = i(b) = 0$, then applying Rolle's Theorem we get that there exists a point $c \in (a,b)$ such that $f'(c) = 0$. Since $i'(c) = h'(c)$, we have then that there exists c such that $h'(c) = 0$. So it follows for c that $k_1 f'(c) + k_2 g'(c) = 0$. So we have for c that $\frac{k_2}{k_1} = \frac{-f'(c)}{g'(c)}$. Substituting we get that $\frac{f'(a)}{g'(a)} = \frac{f(b)-f(a)}{g(b)-g(a)}$. Expanding and simplifying we get that there exists $c \in (a,b)$ such that $f'(c)(g(b) - g(a)) = g'(c)(f(b) - f(a))$. $\square$