MATH 161, SHEET 1: SETS, FUNCTIONS and CARDINALITY

Jeffrey Zhang IBL Script 1 Corrections (6 November 2013)

Exercise 1.16 Let $A = \{1, 2, 3\}$. Identify $\wp(A)$ by explicitly listing its elements. $\wp(A) =$ {Ø, {1},{2},{3},{1,2},{2,3},{1,3},{1,2,3}}

Lemma 1.25 Suppose that $f\colon A \to B$ is bijective. Then there exists a bijection $g\colon B \to A$.

Proof. We define $g(B) = \{a \in A \mid f(a) \in B\}$. We know that f is bijective, so f is injective and it follows by Definition 1.20 that for every $a \in A$ there exists a unique $b \in B$ such that $f(a) = b$. Thus, we know that g satisfies the definition of a function (Definition 1.17). Let $x, x' \in X$ such that $g(x) = g(x')$. Let $g(x) = a$ for some $a \in A$, then $f(a) = x$. $g(x) = g(x')$ so it follows that $g(x') = a$, $f(a) = x'$. Then $x = x'$, so g is injective (Definition 1.20). Let $a \in A$ and $f(a) = b$. We have shown that f is injective, so we know that there exists a unique $b \in B$ such that $f(a) = b$. It follows then that $\exists g(b) = a$ for some $b \in B$. So we can say that $\forall a \in A$ such that $f(a) = b$, $\exists b \in B$ such that $g(b) = a$. Then by definition 1.20, g is surjective. □

Lemma 1.29 Let A, B, and C be sets and suppose that there is a bijective correspondence between A and B and a bijective correspondence between B and C. Then there is a bijective correspondence between A and C.

Proof. Let $f : A \to B$, $g : B \to C$. We know f and g are bijections. Suppose $c \in C$, then $\exists b \in B$ such that $g(b) = c$ by Definition 1.20. By the same definition we also know that $\forall b \in B$ $\exists a \in A$ such that $f(a) = b$. It follows then that for $c \in C$, $\exists a \in A$ such that $g(f(a)) = c$. We define a function h such that $h = g \circ f$, so $h : A \to C$ and h is surjective. Suppose $a, a' \in A$ and $h(a) = h(a')$. It follows then that $g(f(a)) = g(f(a'))$. We know by Definition 1.20 that $\forall b, b' \in B$, if $g(b) = g(b')$ then $b = b'$. So we know that $f(a) = f(a')$. By the same definition, we know that $\forall a, a' \in A$, if $f(a) = f(a')$ then $a = a'$. Thus, if $h(a) = h(a')$ then $a = a'$, so h is injective. Because h is injective and surjective, h is bijective. □

Exercise 1.34 Let A and B be two finite sets. Then $|A \times B| = |A| \cdot |B|$.

Proof. Let $|A| = m, |B| = n$. We let the proposition $P(n) : |A \times B| = |A| \cdot |B|$. To prove the base case, we let $n = 1$. It follows then that $|A| = m$, $|B| = 1$, so $|A \times B| = m$, which is obviously true. Let the inductive hypothesis be that $|A \times B| = |A| \cdot |B|$ for two sets A, B. We define two sets A, B such that $|A| = m$ and $|B| = n + 1$. It follows then that $|A| \cdot |B| = m(n + 1)$. Let $B = C \cup D$ for two disjoint sets C, D such that $|C| = n$ and $|D| = 1$. We know that the set $|A \times B| = \{(a, b) | a \in A, b \in B\}$, and we know that $|A| + |(C \cup A)| + |C| = |A| \cdot |C| = m * n$ using the inductive hypothesis. It follows then that $|A \times B| = m * (n + 1)$. Thus, using induction we know that $|A \times B| = |A| \cdot |B|$ for two finite sets A, B. □