SHEET 11: UNIFORM CONTINUITY AND INTEGRATION

We will now consider a notion of continuity that is stronger than ordinary continuity.

Definition 11.1. Let $f\colon A \longrightarrow \mathbb{R}$ be a function. We say that f is *uniformly continuous* if for all $\epsilon > 0$, there exists a $\delta > 0$ such that for all $x, y \in A$

$$\text{if } |x-y| < \delta, \quad \text{then } |f(x) - f(y)| < \epsilon.$$

Theorem 11.2. *If f is uniformly continuous, then f is continuous.*

Proof. Let f be uniformly continuous, then we know that for $\epsilon > 0$, there exists $\delta > 0$ such that for all $x, y \in A$, if $|x-y| < \delta$, then $|f(x) - f(y)| < \epsilon$. This is a more general case of Theorem 9.21, so we have that f is clearly continuous by 9.21 as we can simply fix $a \in A$ to reach 9.21. □

Exercise 11.3. Determine with proof whether the following functions f are uniformly continuous on the given intervals A:

1. $f(x) = x^2$ on $A = \mathbb{R}$
2. $f(x) = x^2$ on $A = (-2, 2)$
3. $f(x) = \frac{1}{x}$ on $A = (0, +\infty)$
4. $f(x) = \frac{1}{x}$ on $A = [1, +\infty)$
5. $f(x) = \sqrt{x}$ on $A = [0, +\infty)$
6. $f(x) = \sqrt{x}$ on $A = [1, +\infty)$

Proof. 1. Let $f(x) = x^2$ for $f : \mathbb{R} \to \mathbb{R}$. Assume f is uniformly continuous, let $\epsilon = 1$, then we have that there exists $\delta > 0$ such that for all $x, y \in A$, if $|x-y| < \delta$, then $|f(x) - f(y)| < \epsilon$ by 11.1. Let $x = \frac{\epsilon}{\delta}$, $y = \frac{\epsilon}{\delta} + \frac{\delta}{2}$. Then simplifying $|x - y| < \delta$, we have that $\frac{\delta}{2} < \delta$ which is true for $\delta > 0$. It follows then by 11.1 that $|f(x) - f(x + \frac{\delta}{2}| < \epsilon$. Simplifying, we get that $|-\epsilon - \frac{\delta^2}{4}| < \epsilon$ for $\delta, \epsilon > 0$, so it follows that $\epsilon + \frac{\delta^2}{4} < \epsilon$ which is false becuase $\delta > 0$. Tus, this is a contradiction, and f is not uniformly continuous.
2. Let $f(x) = x^2$ for $f : (-2, 2) \to \mathbb{R}$. Let $\epsilon > 0$ be arbitrary, and let $\delta = \frac{\epsilon}{4}$. It follows then that for $x, y \in (-2, 2)$, $|x - y| < \frac{\epsilon}{4}$ implies that $|x^2 - y^2| < \epsilon$. Rearranging $|x^2 - y^2| < \epsilon$, we get that $|x-y| < \frac{\epsilon}{|x+y|}$. So we have that for $x, y \in (-2, 2)$, if $|x-y| < \frac{\epsilon}{4}$, then $|x-y| < \frac{\epsilon}{|x+y|}$. Note that for $x, y \in (-2, 2)$, we know that $0 \leq |x+y| < 4$. Thus we know that if $|x-y| < \frac{\epsilon}{4}$, then $|x-y| < \frac{\epsilon}{|x+y|}$. In the case that $|x+y| = 0$, then we have that $x = 0, y = 0$, so we know that $|x^2 - y^2| < \epsilon$ for $\epsilon > 0$. Thus we have shown that f is uniformly continuous by 11.1.
3. Let $f(x) = \frac{1}{x}$ for $f : (0, +\infty) \to \mathbb{R}$. Assume that f is uniformly continuous
4. Let $f(x) = \frac{1}{x}$ for $f : [1, +\infty) \to \mathbb{R}$. Let $\epsilon > 0$ be arbitrary and let $\delta = \frac{\epsilon}{2}$. Assume that

for $x, y \in [1, +\infty)$ that $|x - y| < \frac{\epsilon}{2}$. Then we have that $|x - y| < \epsilon$, because $\epsilon > 0$. Note that $x, y \geq 1$, so we know that $|xy| > 1$, so $\epsilon|xy| > \epsilon$. It follows then that $|x - y| < \epsilon|xy|$. Dividing by $|xy|$, we have that $\frac{|x-y|}{|xy|} < \epsilon$. Simplifying, we have that $|f(x) - f(y)| < \epsilon$, so by 11.1 we know that f is uniformly continuous.
5. Let $f(x) = \sqrt(x)$ for $f : [0, +\infty]$.
6. $\square$

Exercise 11.4. Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be defined by $f(x) = x^n$, for $n \in \mathbb{N}$. Show that f is uniformly continuous if, and only if, $n = 1$.

Challenge: Let $p : \mathbb{R} \to \mathbb{R}$ be a polynomial with real coefficients. Show that p is uniformly continuous on $\mathbb{R}$ if and only if $\deg(p) \leq 1$.

There is no deadline for the Challenge problem. Once a student has a solution he/she should present it to Sarah or Laurie and we will then schedule a class presentation of it.

Proof. Let f be uniformly continuous. Assume that $n \geq 2$ and let $\epsilon = 1$. Then by the definition of uniform continuity, we know there exists $\delta > 0$ such that we can choose x and $y = x + \frac{\delta}{2}$. $|x - y| = \frac{\delta}{2} < \delta$, so we have by uniform continuity that $|f(x) - f(y)| < \epsilon$. It follows then that $|f(x) - f(x + \frac{\delta}{2})| < 1$ for any $x \in \mathbb{R}$. So we have that $|x^n - (x + \frac{\delta}{2})^n| < 1$. Expanding, we get that $x^{n-1}(\frac{\delta}{2}) + \cdots + x(\frac{\delta}{2})^{n-1} + \frac{delta}{2}^n < 1$. Note that all terms are positive, so if we let $x = \frac{1}{\frac{\delta}{2}}^{n-1}$, then we get that $1 + p < 1$ where p is some positive value. This is a contradiction, so we know that if f is uniformly continuous, then $n = 1$.
Let $f(x) = x$, then we have that for $\epsilon > 0$, choose $\delta = \frac{\epsilon}{2}$. Then we know that for any $x, y \in \mathbb{R}$, if $|x - y| < \frac{\epsilon}{2}$, then clearly $|x - y| < \frac{\epsilon}{2} < \epsilon$, so by definition of uniform continuity we have that f is uniformly continuous. $\square$

Exercise 11.5. Let f and g be uniformly continuous on $A \subset \mathbb{R}$. Show that:

1. The function $f + g$ is uniformly continuous on A.
2. For any constant $c \in \mathbb{R}$, the function $c \cdot f$ is uniformly continuous on A.

We will now prove that continuous functions with compact domain are automatically uniformly continuous. To this end, first consider:

Lemma 11.6. *Let $f\colon A \longrightarrow \mathbb{R}$ be continuous. Fix $\epsilon > 0$. By the definition of continuity, for each $p \in A$ there exists $\delta(p) > 0$ such that for all $x \in A$*

$$\text{if } |x - p| < \delta(p), \quad \text{then } |f(x) - f(p)| < \frac{\epsilon}{2}.$$

For each $p \in A$, define $U(p) = \{x \in \mathbb{R} \mid |x - p| < \frac{1}{2}\delta(p)\}$. Then the collection $\{U(p) \mid p \in A\}$ is an open cover of A.

Theorem 11.7. *Suppose that $X \subset \mathbb{R}$ is compact and $f\colon X \longrightarrow \mathbb{R}$ is continuous. Then f is uniformly continuous.*

Corollary 11.8. *Suppose that $f\colon [a, b] \longrightarrow \mathbb{R}$ is continuous. Then f is uniformly continuous.*

Definition 11.9. We say that a function $f : A \to \mathbb{R}$ is *bounded* if $f(A)$ is a bounded subset of $\mathbb{R}$.

Theorem 11.10. *Suppose that $X \subset \mathbb{R}$ is compact and $f \colon X \longrightarrow \mathbb{R}$ is continuous. Then f is bounded.*

Exercise 11.11. Show that if f and g are bounded on A and uniformly continuous on A, then fg is uniformly continuous on A.

We are now ready to turn to integration.

Definition 11.12. A *partition* of the interval $[a, b]$ is a finite set of points in $[a, b]$ that includes a and b:

$$a = t_0 < t_1 < \cdots < t_{n-1} < t_n = b.$$

If P and Q are partitions of the interval $[a, b]$ and $P \subset Q$, we refer to Q as a *refinement* of P.

We usually write partitions as ordered lists $P = \{t_0, t_1, \dots, t_n\}$ with $t_{i-1} < t_i$ for each $i = 1, \dots, n$.

Definition 11.13. Suppose that $f \colon [a, b] \longrightarrow \mathbb{R}$ is bounded and that $P = \{t_0, t_1, \dots, t_n\}$ is a partition of $[a, b]$. Define:

$$m_i = \inf\{f(x) \mid t_{i-1} \le x \le t_i\}$$
$$M_i = \sup\{f(x) \mid t_{i-1} \le x \le t_i\}.$$

The *lower sum* of f for the partition P is the number:

$$L(f, P) = \sum_{i=1}^{n} m_i(t_i - t_{i-1}).$$

The *upper sum* of f for the partition P is the number:

$$U(f, P) = \sum_{i=1}^{n} M_i(t_i - t_{i-1}).$$

Notice that it is always the case that $L(f, P) \le U(f, P)$.

Lemma 11.14. *Suppose that P and Q are partitions of $[a, b]$ and that Q is a refinement of P. Then:*

$$L(f, P) \le L(f, Q) \quad \textit{and} \quad U(f, P) \ge U(f, Q).$$

Theorem 11.15. *Let P_1 and P_2 be partitions of $[a, b]$ and suppose that $f \colon [a, b] \longrightarrow \mathbb{R}$ is bounded. Then:*

$$L(f, P_1) \le U(f, P_2).$$

Definition 11.16. Let $f : [a, b] \to \mathbb{R}$ be bounded. We define:

$$L(f) = \sup\{L(f, P) \mid P \text{ is a partition of } [a, b]\}$$
$$U(f) = \inf\{U(f, P) \mid P \text{ is a partition of } [a, b]\}$$

to be, respectively, the *lower integral* and *upper integral* of f from a to b.

Exercise 11.17. Why do $L(f)$ and $U(f)$ exist? Find a function f for which $L(f) = U(f)$. Find a function f for which $L(f) \neq U(f)$. Is there a relationship between $L(f)$ and $U(f)$ in general?

Definition 11.18. Let $f : [a, b] \longrightarrow \mathbb{R}$ be bounded. We say that f is *integrable* on $[a, b]$ if $L(f) = U(f)$. In this case, the common value $L(f) = U(f)$ is called the *integral* of f from a to b and we write it as:

$$\int_a^b f.$$

When we want to display the variable of integration, we write the integral as follows, including the symbol dx to indicate that variable of integration:

$$\int_a^b f(x)\, dx.$$

For example, if $f(x) = x^2$, we would write $\displaystyle\int_a^b x^2\, dx$ but not $\displaystyle\int_a^b x^2$.

Theorem 11.19. *Let $f : [a, b] \longrightarrow \mathbb{R}$ be bounded. Then f is integrable if and only if for every $\epsilon > 0$ there exists a partition P of $[a, b]$ such that*

$$U(f, P) - L(f, P) < \epsilon.$$

Theorem 11.20. *If $f : [a, b] \longrightarrow \mathbb{R}$ is continuous, then f is integrable.*

(Hint: Use theorem 11.19 and uniform continuity.)

Exercise 11.21. Fix $c \in \mathbb{R}$ and let $f : [a, b] \to \mathbb{R}$ be defined by $f(x) = c$, for each $x \in [a, b]$. Show that f is integrable on $[a, b]$ and that $\int_a^b f = c(b - a)$.

Exercise 11.22. Define $f : [0, b] \to \mathbb{R}$ by the formula $f(x) = x$. Show that f is integrable on $[0, b]$ and that $\int_0^b f = \frac{b^2}{2}$.

Exercise 11.23. Show that the converse of theorem 11.20 is false in general.

Exercise 11.24. Let $f : [0, 1] \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q}. \end{cases}$$

Show that f is not integrable on $[0, 1]$. Compute the upper and lower integrals of f on $[0, 1]$.

Theorem 11.25. *Let $a < b < c$. A function $f\colon [a,c] \longrightarrow \mathbb{R}$ is integrable on $[a,c]$ if and only if f is integrable on $[a,b]$ and $[b,c]$. When f is integrable on $[a,c]$, we have*

$$\int_a^c f = \int_a^b f + \int_b^c f.$$

If $b < a$, we define

$$\int_a^b f = -\int_b^a f,$$

whenever the latter integral exists. With this notational convention, it follows that the equation

$$\int_a^c f = \int_a^b f + \int_b^c f$$

always holds, regardless of the ordering of a, b and c, whenever f is integrable on the largest of the three intervals.

Proof. Let f be integrable on $[a,c]$. Then by Theorem 11.19, we know that for any $\epsilon > 0$, $U(f,P_0) - L(f,P_0) < \epsilon$ for some partition $P_0 = \{a, t_1, t_2, \cdots, t_{n-1}, c\}$ of $[a,c]$. Let $P = \{a, t_1, t_2, \cdots, t_{n-1}, c\} \cup \{b\}$ be a partition of $[a,c]$. Then we define partitions $Q = \{a, t_1, t_2, \cdots, b\}$ and $R = \{b, \cdots, t_{n-1}, c\}$. By Definition 11.13 and properties of sums, we know that $L(f,P) = L(f,Q) + L(f,R)$ and that $U(f,P) = U(f,Q) + U(f,R)$. We also have that $U(f,P) - L(f,P) < \epsilon$ so substuting we have that $[U(f,Q) + U(f,R)] - [L(f,Q) + L(f,R)] < \epsilon$. Rearranging, we get $[U(f,Q) - L(f,Q)] + [U(f,R) - L(f,R)] < \epsilon$. Note that both terms are positive, as $U(f,Q) > L(f,Q)$ and $U(f,R) > L(f,R)$. Thus we have that $U(f,Q) - L(f,Q) < \epsilon$ and similarly that $U(f,R) - L(f,R) < \epsilon$. So we have by 11.19 that f is integrable on $[a,b]$ and $[b,c]$.
Let f be integrable on $[a,b]$ and $[b,c]$ and let $\epsilon > 0$ be arbitrary. Choose $\epsilon' = \frac{\epsilon}{2}$. Then by 11.19, we get that $U(f,Q) - L(f,Q) < \epsilon'$ and that $U(f,R) - L(f,R) < \epsilon'$ for some partitions Q and R. Combining, we have that $U(f,Q) - L(f,Q) + U(f,R) - L(f,R) < 2\epsilon' = \epsilon$. Let $P = Q \cup R$, then it follows that $U(f,P) - L(f,P) < \epsilon$, so we have by 11.19 that f is integrable on $[a,c]$. □

Theorem 11.26. *Suppose that f and g are integrable functions on $[a,b]$ and that $c \in \mathbb{R}$ is a constant. Then $f+g$ and cf are integrable on $[a,b]$ and*

(i) $\displaystyle\int_a^b (f+g) = \int_a^b f + \int_a^b g,$

(ii) $\displaystyle\int_a^b c \cdot f = c\int_a^b f.$

Theorem 11.27. *Suppose that f is integrable on $[a,b]$ and that there exists numbers m and M such that:*

$$m \le f(x) \le M \qquad \textit{for all } x \in [a,b].$$

Then:

$$m(b-a) \leq \int_a^b f \leq M(b-a).$$

Proof. Let P be any partition of $[a, b]$. We write $P = \{t_0, t_1, \cdots, t_n\}$ for $n \in \mathbb{N}$. Then we know that $L(f, P) \leq L(f) \leq \sum_{i=1}^{n} M_i(t_i, t_{i-1}) \leq M(b-a)$. f is integrable, so $L(f) = \int_a^b$. It follows then that $\int_a^b f \leq M(b-a)$. Without loss of generality, the same argument can be used to show that $m(b-a) \leq \int_a^b f$ using $U(f)$ and $U(f, P)$. □

Theorem 11.28. *Suppose that f is integrable on $[a, b]$. Define $F\colon [a, b] \longrightarrow \mathbb{R}$ by*

$$F(x) = \int_a^x f$$

Then F is continuous.

Proof. Let $F(x) = \int_a^x f$ and suppose that f is integrable on $[a, b]$. Then choose $\epsilon > 0$. We take two cases, where $x > c$ and $x < c$ (note that if $x = c$, then $|x - c| < \delta$ and $|F(x) - F(c)| < \epsilon$ will always be true and so f will be continuous).
Case 1: Let $x > c$, then we let $\delta = \frac{\epsilon}{M}$ for M such that $f(x) \leq M$. Now let $|x - c| < \frac{\epsilon}{M}$. It follows then that $x > c$ so $M(x - c) < \epsilon$. We know by 11.26 and 11.27 that $|F(x) - F(c)| = |\int_c^x f| < M(x - c) < \epsilon$. So we have that f is continuous by definition.
Case 2: Let $x < c$, then we let $\delta = \frac{\epsilon}{m}$ for m such that $m \leq f(x)$. Now let $|x - c| < \frac{\epsilon}{m}$. It follows then that $x < c$ so $m(c - x) < \epsilon$. $x < c$, so we know by 11.25, 11.26, and 11.27 that $|F(x) - F(c)| = \int_x^c f \leq m(c - x) < \epsilon$. So we have that f is continuous by definition. □