MATH 162, SHEET 7: THE RATIONAL NUMBERS

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In this script, we define the rational numbers $\mathbb{Q}$ and show that they form an ordered field. At the end of this script, we show that any ordered field contains a canonical copy of the rational numbers.

Definition 7.1. Let $X = \{(a, b) \mid a, b \in \mathbb{Z}, b \neq 0\}$. We define a relation $\sim$ on X as:

$$(a, b) \sim (c, d) \quad \text{if and only if} \quad ad = bc$$

Lemma 7.2.

1. *The relation $\sim$ is reflexive. That is, $(a, b) \sim (a, b)$ for every $(a, b) \in X$.*
2. *The relation $\sim$ is symmetric. That is, if $(a, b) \sim (c, d)$, then $(c, d) \sim (a, b)$.*
3. *The relation $\sim$ is transitive. That is, if $(a, b) \sim (c, d)$ and $(c, d) \sim (e, f)$, then $(a, b) \sim (e, f)$.*

Proof. Let $(a, b) \neq (a, b)$, then by Definition 7.1, $ab \neq ab$ for $a, b \in \mathbb{Z}$, which we know to be false. So $(a, b) \sim (a, b)$.
Let $(a, b) \sim (c, d)$, then $ad = bc$. We know that multiplication is commutative on $\mathbb{Z}$, so we have $da = cb$, which by Definition 7.1 gives that $(c, d) \sim (a, b)$.
Let $(a, b) \sim (c, d)$ and $(c, d) \sim (e, f)$, then we have that $ad = bc$ and $cf = de$. Multiplying both sides by ad and bc respectively, we get $acdf = bcde$. By the cancellation law on $\mathbb{Z}$, we get that $af = be$. Then by Definition 7.1 it follows that $(a, b) \sim (e, f)$. □

Remark 7.3. A relation on any set that is reflexive, symmetric, and transitive is called an *equivalence relation*. Thus, the preceding lemmas show that $\sim$ is an equivalence relation on X.

Remark 7.4. A *partition* of a set is a collection of non-empty disjoint subsets whose union is the original set. Any equivalence relation on a set creates a partition of that set by collecting into subsets all of the elements that are equivalent (related) to each other. When the partition of a set arises from an equivalence relation in this manner, the subsets are referred to as *equivalence classes*.

Remark 7.5. If we think of the set X as representing the collection of all fractions whose denominators are not zero, then the relation $\sim$ may be thought of as representing the equivalence of two fractions.

Definition 7.6. As a set, the *rational numbers*, denoted $\mathbb{Q}$, are the equivalence classes in the set X under the equivalence relation $\sim$. If $(a, b) \in X$, we denote the equivalence class of this element as $\dfrac{a}{b}$. So

$$\frac{a}{b} = \{(x_1, x_2) \in X \mid (x_1, x_2) \sim (a, b)\} = \{(x_1, x_2) \in X \mid x_1 b = x_2 a\}.$$

Then,

$$\mathbb{Q} = \left\{ \frac{a}{b} \mid (a,b) \in X \right\}.$$

Exercise 7.7. $\dfrac{a}{b} = \dfrac{a'}{b'} \iff (a,b) \sim (a',b')$.

Proof. Let $\frac{a}{b} = \frac{a'}{b'}$ for $a, b \in \mathbb{Z}$ and $\frac{a}{b}, \frac{a'}{b'} \in \mathbb{Q}$. We know that $(a', b') \sim (a', b')$ by Lemma 7.2, so $(a', b') \in \frac{a'}{b'}$ by Definition 7.6. Similarly, $(a, b) \in \frac{a}{b}$. $\frac{a}{b} = \frac{a'}{b'}$, so it follows that $(a', b') \in \frac{a}{b}$. $(a, b), (a', b') \in \frac{a}{b}$, so we have by 7.6 that $(a, b) \sim (a', b')$.
Let $(a, b) \sim (a', b')$ for $a, b \in \mathbb{Z}$ and $\frac{a}{b}, \frac{a'}{b'} \in \mathbb{Q}$. Let $(x, y) \in \mathbb{Q}$ be arbitrary such that $(x, y) \in \frac{a}{b}$. Then we know that $(x, y) \sim (a, b)$ by Definition 7.6, and that $(x, y) \sim (a', b')$ by Lemma 7.2 because $(x, y) \sim (a, b)$ and $(a, b) \sim (a', b')$. So by 7.6 again, we have that $(x, y) \in \frac{a'}{b'}$. It follows then that $\frac{a}{b} \subset \frac{a'}{b'}$. Applying the same logic in reverse, we get that $\frac{a'}{b'} \subset \frac{a}{b}$, so we know that $\frac{a}{b} = \frac{a'}{b'}$. $\square$

Definition 7.8. We define addition and multiplication in $\mathbb{Q}$ as follows. If $\dfrac{a}{b}$ and $\dfrac{c}{d}$ are in $\mathbb{Q}$, then:

$$\frac{a}{b} +_{\mathbb{Q}} \frac{c}{d} = \frac{ad + bc}{bd}$$
$$\frac{a}{b} \cdot_{\mathbb{Q}} \frac{c}{d} = \frac{ac}{bd}.$$

We use the notation $+_{\mathbb{Q}}$ and $\cdot_{\mathbb{Q}}$ to represent addition and multiplication in $\mathbb{Q}$ so as to distinguish these operations from the usual addition $(+)$ and multiplication $(\cdot)$ in $\mathbb{Z}$.

Theorem 7.9. *Addition in $\mathbb{Q}$ is well-defined. That is, if $(a, b) \sim (a', b')$ and $(c, d) \sim (c', d')$, then*

$$\frac{a}{b} +_{\mathbb{Q}} \frac{c}{d} = \frac{a'}{b'} +_{\mathbb{Q}} \frac{c'}{d'}.$$

Proof. Let $a, b, c, d, a', b', c', d' \in Z$ such that $(a, b) \sim (a', b')$ and $(c, d) \sim (c', d')$. We have that: $(ad + bc)(b'd') = ab'dd' + bb'cd'$ (Distributivity on $\mathbb{Z}$)

$= (ab')dd' + bb'(cd')$ (Associativity of multiplication on $\mathbb{Z}$)

$= (a'b)dd' + bb'(c'd)$ (Definition 7.1)

$= (a'd' + b'c')(bd)$ (Distributivity on $\mathbb{Z}$)

. So we have that $(ad + bc)(b'd') = (a'd' + b'c')(bd)$. Applying 7.1, we get that $(ad + bc, bd) \sim (a'd' + b'c', b'd')$. Then by Exercise 7.7 it follows that $\frac{ad+bc}{bd} = \frac{a'd'+b'c'}{b'd'}$. Then applying Definition 7.8, we get that $\frac{a}{b} +_{\mathbb{Q}} \frac{c}{d} = \frac{a'}{b'} +_{\mathbb{Q}} \frac{c'}{d'}$. $\square$

Theorem 7.10. *(Homework)*
Multiplication in $\mathbb{Q}$ is well-defined. That is, if $(a, b) \sim (a', b')$ and $(c, d) \sim (c', d')$, then

$$\frac{a}{b} \cdot_{\mathbb{Q}} \frac{c}{d} = \frac{a'}{b'} \cdot_{\mathbb{Q}} \frac{c'}{d'}.$$

Theorem 7.11. *The rational numbers $\mathbb{Q}$ with $+_{\mathbb{Q}}$ and $\cdot_{\mathbb{Q}}$ are a field.*

In-class Axioms 3,4,7,8,9

Homework Axioms 1,2,5,6,10

Proof. Axiom 3: Let the additive identity $e = \frac{0}{1}$. We know that $0, 1 \in \mathbb{Z}$ so $\frac{0}{1} \in \mathbb{Q}$ so $e \in \mathbb{Q}$. Let $\frac{x}{y} \in \mathbb{Q}$ be arbitrary for some $x, y \in \mathbb{Z}$. Then $\frac{x}{y} +_{\mathbb{Q}} \frac{0}{1} = \frac{x \cdot 1 + 0 \cdot y}{y \cdot 1}$ by the definition of addition on $\mathbb{Q}$. It follows then by the properties of the integers that $\frac{x \cdot 1 + 0 \cdot y}{y \cdot 1} = \frac{x \cdot 1}{y \cdot 1} = \frac{x}{y}$. So we have that e is the additive identity on $\mathbb{Q}$ because $\frac{x}{y} +_{\mathbb{Q}} e = \frac{x}{y}$.
Axiom 4: Let $\frac{x}{y} \in \mathbb{Q}$, so $x, y \in \mathbb{Z}$. Let $\frac{-x}{y}$ be the additive inverse. We know that $\frac{-x}{y} \in \mathbb{Q}$ because $x \in \mathbb{Z}$, so $-x \in \mathbb{Z}$. We have $\frac{x}{y} +_{\mathbb{Q}} \frac{-x}{y} = \frac{xy + -xy}{yy}$ by the definition of addition on $\mathbb{Q}$. It follows then by the properties of the integers that $\frac{xy + -xy}{yy} = \frac{0}{yy}$, and we know that $\frac{0}{yy}$ is in the same equivalence class as $\frac{0}{1}$.
Axiom 7: Let the multiplicative identity $e = \frac{1}{1}$. We know that $\frac{1}{1} \in \mathbb{Q}$ because $1 \in \mathbb{Z}$. Let $\frac{x}{y} \in \mathbb{Q}$ be arbitrary for some $x, y \in \mathbb{Z}$. Then $\frac{x}{y} \cdot_{\mathbb{Q}} \frac{1}{1} = \frac{x \cdot 1}{y \cdot 1}$. By the properties of the integers we have then that $\frac{x \cdot 1}{y \cdot 1} = \frac{x}{y}$, so we have that $e = \frac{1}{1}$ is the multiplicative identity on $\mathbb{Q}$.
Axiom 8: Let $\frac{x}{y} \in \mathbb{Q}$, with $x, y \in \mathbb{Z}$. Then the multiplicative inverse of $\frac{x}{y}$ is $\frac{y}{x}$. We know that $\frac{y}{x} \in \mathbb{Q}$ because $x, y \in \mathbb{Z}$. We verify that $\frac{y}{x}$ is the multiplicative inverse of $\frac{x}{y}$ by checking that $\frac{x}{y} \cdot_{\mathbb{Q}} \frac{y}{x} = \frac{xy}{yx}$. Using commutativity of multiplication on Z, we have that $\frac{xy}{yx} = \frac{xy}{xy}$, which is in the same equivalence class as $\frac{1}{1}$ in $\mathbb{Q}$, so $\frac{1}{1} \in \mathbb{Q}$, and thus $\frac{y}{x}$ is the multiplicative inverse of $\frac{x}{y}$ in $\mathbb{Q}$.
Axiom 9: Let $\frac{a}{b}, \frac{c}{d}, \frac{e}{f} \in \mathbb{Q}$, so $a, b, c, d, e, f \in \mathbb{Z}$. Consider
$\frac{a}{b} \cdot_{\mathbb{Q}} (\frac{c}{d} +_{\mathbb{Q}} \frac{e}{f}) = \frac{a}{b} \cdot_{\mathbb{Q}} (\frac{cf+de}{df})$ (Definition of Addition on $\mathbb{Q}$)
$= \frac{acf+ade}{bdf}$ (Definition of Addition on $\mathbb{Q}$).
Note that $(acf + ade)(bbdf) = (bdf)(bacf + bade)$ due to the properties of the integers, so we have that $(acf + ade, bdf) \sim (bacf + bade, bbdf)$. By Exercise 7.7 we have then that $\frac{acf+ade}{bdf} = \frac{bacf+bade}{bbdf}$. $\frac{bacf+bade}{bbdf} = \frac{acbf+aebd}{bdbf}$ by the properties of the integers, so it follows that $\frac{bacf+bade}{bbdf} = \frac{acbf+aebd}{bdbf} = \frac{ac}{bd} +_{\mathbb{Q}} \frac{ae}{bf}$. We have then that $\frac{ac}{bd} +_{\mathbb{Q}} \frac{ae}{bf} = \frac{a}{b} \cdot_{\mathbb{Q}} \frac{c}{d} +_{\mathbb{Q}} \frac{a}{b} \cdot_{\mathbb{Q}} \frac{e}{f}$ by the definition of multiplication on $\mathbb{Q}$. Thus we have that $\frac{a}{b} \cdot_{\mathbb{Q}} (\frac{c}{d} +_{\mathbb{Q}} \frac{e}{f}) = \frac{acf+ade}{bdf} = \frac{a}{b} \cdot_{\mathbb{Q}} \frac{c}{d} +_{\mathbb{Q}} \frac{a}{b} \cdot_{\mathbb{Q}} \frac{e}{f}$, so Axiom 9 (Distributivity) holds on $\mathbb{Q}$. □

Next, we define an ordering on $\mathbb{Q}$.

Definition 7.12. Note that, given an element in $\mathbb{Q}$ is is always possible to choose a representative (a, b) with $b > 0$. If $\dfrac{a}{b}, \dfrac{c}{d} \in \mathbb{Q}$ and $b, d > 0$ in $\mathbb{Z}$, we say that $\dfrac{a}{b} <_{\mathbb{Q}} \dfrac{c}{d}$ if and only if $ad < bc$. Note that we use the symbol $<_{\mathbb{Q}}$ to denote our ordering relation in $\mathbb{Q}$ so as to distinguish it from the usual ordering $<$ in $\mathbb{Z}$.

Theorem 7.13. *Ordering in $\mathbb{Q}$ is well-defined. That is, if $(a, b) \sim (a', b')$ and $(c, d) \sim (c', d')$, then*

$$\frac{a}{b} <_{\mathbb{Q}} \frac{c}{d} \quad \textit{if and only if} \quad \frac{a'}{b'} <_{\mathbb{Q}} \frac{c'}{d'}.$$

Proof. Let $a, b, c, d, a', b', c', d' \in \mathbb{Z}$ such that $(a, b) \sim (a', b')$ and $(c, d) \sim (c', d')$.
We know by Definition that 7.12 that b, b', d, d' can always be represented such that they are positive. Note also that $(a, b) \sim (a', b')$ so $ab' = a'b$. b, b' are positive, so we have that a and a' have the same sign. Similarly, c and c' have the same sign. We then consider two cases: where $a \cdot c, a' \cdot c'$ are negative and where $a \cdot c, a' \cdot c'$ are positive.
Assume $\frac{a}{b} <_{\mathbb{Q}} \frac{c}{d}$. Then we have that $ad < bc$ by Definition 7.12.
Let $a \cdot c$, $a' \cdot c'$ be positive. Because $a', b', c', d' \in \mathbb{Z}$, we know that $ad(a'b'c'd') < bc(a'b'c'd')$. So it follows by the properties of $\mathbb{Z}$ that $a'd'(ab')(c'd) < b'c'(a'b)(cd')$. We have that $(a, b) \sim (a', b')$ and $(c, d) \sim (c', d')$, so $ab' = a'b$ and $cd' = c'd$ by Definition 7.1. It follows then by substituting that $a'd'(ab')(cd') < b'c'(ab')(cd')$. Rearranging using the properties of the integers, we get that $a'd'(ac)(b'd') < b'c'(ac)(b'd')$. We know that ac, b', d' are positive, so by cancelling we get $a'd' < b'c'$. It follows then by Definition 7.12 that $\frac{a'}{b'} <_{\mathbb{Q}} \frac{c'}{d'}$.
Let $a \cdot c$, $a' \cdot c'$ be negative. We have then that $ad(a'b'c'd') > bc(a'b'c'd')$ because we are multiplying both sides by $a' \cdot c'$, which is negative. Similar to the positive case, we substitute and rearrange using the properties of the integers to get $a'd'(ac)(b'd') > b'c'(ac)(b'd')$. Here we have that $a \cdot c$ is negative, so when we cancel we get $a'd' < b'c'$. It follows then by Definition 7.12 that $\frac{a'}{b'} <_{\mathbb{Q}} \frac{c'}{d'}$. Assume $\frac{a'}{b'} <_{\mathbb{Q}} \frac{c'}{d'}$. Then we have that $a'd' < b'c'$ by Definition 7.12.
Let $a \cdot c$, $a' \cdot c'$ be positive. Because $a, b, c, d \in \mathbb{Z}$, we know that $a'd'(abcd) < b'c'(abcd)$. Similarly, substituting and rearranging using the properties of $\mathbb{Z}$, we get that $ad(ac)(b'd') < bc(ac)(b'd')$. It follows then by cancelling that $ad < bc$ because ac, b', d' are all positive.Then by applying Definition 7.12, we get that $\frac{a}{b} <_{\mathbb{Q}} \frac{c}{d}$.
Let $a \cdot c$, $a' \cdot c'$ be negative. Because $a, b, c, d \in \mathbb{Z}$, we know that $a'd'(abcd) > b'c'(abcd)$ because $a \cdot c$ is negative, and b, d, are positive. Similarly substituting and rearranging using the properties of $\mathbb{Z}$, we get that $ad(ac)(b'd') > bc(ac)(b'd')$. It follows then by cancelling that $ad < bc$ because $a \cdot c$ is negative and b', d' are positive. Then by applying Definition 7.12, we get that $\frac{a}{b} <_{\mathbb{Q}} \frac{c}{d}$. □

Theorem 7.14. *(Homework)*
The relation $<_{\mathbb{Q}}$ is an ordering on the rational numbers $\mathbb{Q}$.

Theorem 7.15. *The rational numbers $\mathbb{Q}$ form an ordered field.*

Proof. We have by Theorem 7.11 that $\mathbb{Q}$ is a field. To show that $\mathbb{Q}$ is ordered, we show that addition on $\mathbb{Q}$ respects the ordering and that multiplication on $\mathbb{Q}$ respects the ordering.
Let $a, b, c, d, e, f \in \mathbb{Z}$ and $\frac{a}{b}, \frac{c}{d}, \frac{e}{f} \in \mathbb{Q}$. Let $\frac{a}{b} <_{\mathbb{Q}} \frac{c}{d}$, then we have by 7.12 that $ad < bc$. We know that $f \in \mathbb{Z}$ such that f is positive by 7.12, so we know that $adf < bcf$. We know that addition respects the ordering on $\mathbb{Z}$, so $adf + bde < bcf + bde$. We also know that b, d, f are positive, so $\frac{adf+bde}{bdf} <_{\mathbb{Q}} \frac{bcf+bde}{bdf}$. Cancelling d on the left and b on the right, we get that $\frac{af+be}{bf} <_{\mathbb{Q}} \frac{cf+de}{df}$. Applying the definition of addition on $\mathbb{Q}$, we get that $\frac{a}{b} + \frac{e}{f} <_{\mathbb{Q}} \frac{c}{d} + \frac{e}{f}$, so it follows then that addition respects the ordering on $\mathbb{Q}$ by 6.19.
Let $a, b, c, d \in \mathbb{Z}$ such that $\frac{a}{b}, \frac{c}{d} \in \mathbb{Q}$ and $\frac{0}{1} <_{\mathbb{Q}} \frac{a}{b}$, $\frac{0}{1} <_{\mathbb{Q}} \frac{c}{d}$. It follows then that $0 \cdot b < a \cdot 1$ and $0 \cdot d < c \cdot 1$. Using the properties of the integers, we then get that $0 < c$, $0 < a$. We have

that $0 < 1, 0 < a, 0 < c$, so because multiplication respects the ordering on $\mathbb{Z}$, we have that $0 < 1 \cdot a \cdot c$. We also know that $0 \cdot b \cdot d = 0$ for $b, d \in \mathbb{Z}$, so using the associative property of multiplication on $\mathbb{Z}$, we have that $0 \cdot (b \cdot d) < 1 \cdot (a \cdot c)$. Applying 7.12 then, we get that $\frac{0}{1} <_{\mathbb{Q}} \frac{ac}{bd}$. Using the definition of multiplication on $\mathbb{Q}$ then, it follows that $\frac{0}{1} <_{\mathbb{Q}} (\frac{a}{b} \cdot_{\mathbb{Q}} \frac{c}{d})$. So then by 6.19 we know that multiplication respects the ordering on $\mathbb{Q}$. □

There is an important sense in which every ordered field contains a "copy" of the rational numbers. Let F be an ordered field with additive identity 0_F and multiplicative identity 1_F. To prevent ambiguity, denote the additive identity in the rational numbers by $0_{\mathbb{Q}}$ and the multiplicative identity in the rational numbers by $1_{\mathbb{Q}}$.

Theorem 7.16. *Any ordered field F contains a copy of the rational numbers $\mathbb{Q}$ in the following sense. There exists an injective map $i : \mathbb{Q} \longrightarrow F$ that respects all of the axioms for an ordered field. In particular:*

- $i(0_{\mathbb{Q}}) = 0_F$
- $i(1_{\mathbb{Q}}) = 1_F$
- *If $a, b \in \mathbb{Q}$, then $i(a + b) = i(a) + i(b)$.*
- *If $a, b \in \mathbb{Q}$, then $i(a \cdot b) = i(a) \cdot i(b)$.*
- *If $a, b \in \mathbb{Q}$ and $a < b$, then $i(a) < i(b)$.*

Since we know that $\mathbb{Q}$ is an ordered field, this result says that $\mathbb{Q}$ is the "minimal" ordered field.

Proof. We define a_F for $a > 0$ such that $a_F = 1_{F_0} + \cdots + 1_{F_a}$. Then for $a < 0$ we define a_F such that a_F is the additive inverse of $(-a)_F$. So we can then define $i : \mathbb{Q} \to F$ such that $i(\frac{a}{b}) = a_F b_F^{-1}$. Note that $(ad)_F = a_F d_F$ because $(ad)_F = i(ad)$, and we have that $i(a \cdot d) = 1_{F_0} + \cdots + 1_{F_{a \cdot d}}$ which is equal to $a_F \cdot d_F$.

We begin by showing that i respects the axioms of an ordered field. First we consider $i(\frac{0}{1})$. By definition of i, we have that $i(\frac{0}{1}) = 0_F \cdot 1_F^{-1} = 0_F \cdot 1_F = 0_F$. So we have that $i(0_{\mathbb{Q}}) = 0_F$.

We then consider $i(\frac{1}{1})$. We have that $i(\frac{1}{1}) = 1_F \cdot 1_F^{-1} = 1_F \cdot 1_F = 1_F$. So we have that $i(1_{\mathbb{Q}}) = 1_F$.

We then consider $i(\frac{a}{b} +_{\mathbb{Q}} \frac{c}{d})$ for $\frac{a}{b}, \frac{c}{d} \in \mathbb{Q}$. We have then that this equals $i(\frac{ad+bc}{bd})$ by the definition of addition on $\mathbb{Q}$.
$= (ad + bc)_F (bd)_F^{-1}$
$= (ad)_F (bd)_F^{-1} + (bc)_F (bd)_F^{-1}$ Distributivity
$= (a_F d_F b_F^{-1} d_F^{-1}) + (b_F c_F b_F^{-1} d_F^{-1})$ Associativity
$= (a_F b_F^{-1})(d_F d_F^{-1}) + c_F d_F^{-1}(b_F b_F^{-1})$ Commutativity and Associativity

$= a_F b_F^{-1}(1) + c_F d_F^{-1}(1)$ Multiplicative Inverse
$= a_F b_F^{-1} + c_F d_F^{-1}$ Multiplicative Identity
Then we use the definition of i to get that $a_F b_F^{-1} + c_F d_F^{-1} = i(\frac{a}{b}) + i(\frac{c}{d})$. So we have that $i(\frac{a}{b} +_{\mathbb{Q}} \frac{c}{d}) = i(\frac{a}{b}) + i(\frac{c}{d})$.

Consider $i(\frac{a}{b} \cdot_{\mathbb{Q}} \frac{c}{d})$. We have that this is equal to $i(\frac{ac}{bd})$ by the definition of multiplication on $\mathbb{Q}$. $i(\frac{ac}{bd}) = (ac)_F(bd)_F^{-1} = a_F c_F b_F^{-1} d_F^{-1}$. We have that $i(\frac{a}{b}) \cdot i(\frac{c}{d}) = a_F b_F^{-1} c_F d_F^{-1}$, so by commutativity we have that $i(\frac{a}{b}) \cdot i(\frac{c}{d}) = a_F c_F b_F^{-1} d_F^{-1}$. It follows then that $i(\frac{a}{b} \cdot_{\mathbb{Q}} \frac{c}{d}) = i(\frac{a}{b}) \cdot i(\frac{c}{d})$.

Let $\frac{a}{b} < \frac{c}{d}$, so we have that $ad < bc$. It follows then that $a_F d_F < b_F c_F$. So
$a_F d_F(1_F) < b_F c_F(1_F)$ Multiplicative Identity
$a_F d_F(b_F^{-1} b_F) < c_F b_F(d_F^{-1} d_F)$ Multiplicative Inverse
$a_F b_F^{-1} b_F d_F < c_F d_F^{-1} b_F d_F$ Associativity and Commutativity
$a_F b_F^{-1} < c_F d_F^{-1}$ (Note that $b_F, d_F > 0$ because $b, d > 0$)
$i(\frac{a}{b}) < i(\frac{c}{d})$
. So we have shown that if $\frac{a}{b} < \frac{c}{d}$, then $i(\frac{a}{b}) < i(\frac{c}{d})$.

To show that i is injective, we let $i(\frac{a}{b}) = i(\frac{c}{d})$. Then we have that
$a_F b_F^{-1} = c_F d_F^{-1}$
$a_F b_F^{-1}(b_F d_F) = c_F d_F^{-1}(b_F d_F)$
$a_F d_F(b_F^{-1} b_F) = c_F b_F(d_F d_F^{-1})$ Associative and Commutative laws of multiplication
$a_F d_F(1_F) = c_F b_F(1_F)$ Multiplicative inverse
$a_F d_F = c_F b_F$ Multiplicative identity.
$a_F d_F = c_F b_F$, so we have that $ad = bc$. Then by 7.1 we know that $(a, b) \sim (c, d)$. Finally by 7.7 we have that $\frac{a}{b} = \frac{c}{d}$. $\square$