SHEET 12: THE FUNDAMENTAL THEOREM OF CALCULUS AND INVERSE FUNCTIONS

Lemma 12.1. *Let $f\colon [a,b] \longrightarrow \mathbb{R}$ be continuous at $p \in (a,b)$. Define functions m and M by:*

$$m(h) = \begin{cases} \inf\{f(x) \mid p \le x \le p+h\} & \text{if } h \ge 0, \\ \inf\{f(x) \mid p+h \le x \le p\} & \text{if } h < 0 \end{cases}$$

$$M(h) = \begin{cases} \sup\{f(x) \mid p \le x \le p+h\} & \text{if } h \ge 0, \\ \sup\{f(x) \mid p+h \le x \le p\} & \text{if } h < 0 \end{cases}$$

Then $\lim_{h\to 0} m(h) = f(p)$ and $\lim_{h\to 0} M(h) = f(p)$.

Theorem 12.2 (The First Fundamental Theorem of Calculus)**.** *Suppose that f is integrable on $[a,b]$. Define $F\colon [a,b] \longrightarrow \mathbb{R}$ by*

$$F(x) = \int_a^x f.$$

If f is continuous at $p \in (a,b)$, then F is differentiable at p and

$$F'(p) = f(p).$$

Lemma 12.3. *Suppose that $f\colon [a,b] \longrightarrow \mathbb{R}$ is integrable and that I is a number satisfying*

$$L(f,P) \le I \le U(f,P) \qquad \text{for every partition } P \text{ of } [a,b].$$

Then

$$\int_a^b f = I.$$

Theorem 12.4 (The Second Fundamental Theorem of Calculus)**.** *Suppose that f is integrable on $[a,b]$ and that $f = F'$ on (a,b) for some function F that is continuous on $[a,b]$. Then*

$$\int_a^b f = F(b) - F(a).$$

Corollary 12.5 (Integration by Parts)**.** *Let f, g be functions defined on some open interval containing $[a,b]$ such that f' and g' exist and are continuous on $[a,b]$. Then*

$$\int_a^b fg' = [f(b)g(b) - f(a)g(a)] - \int_a^b f'g.$$

Corollary 12.6 (Change of Variables). *Let g be a function defined on some open interval containing $[a, b]$ such that g' is continuous on $[a, b]$. Suppose that $g([a, b]) \subset [c, d]$ and $f : [c, d] \longrightarrow \mathbb{R}$ is continuous. Define $F : [c, d] \longrightarrow \mathbb{R}$ by $F(x) = \int_c^x f$. Then*

$$\int_a^b f(g(x)) \cdot g'(x)dx = F(g(b)) - F(g(a)).$$

Now, we prove another very important theorem that tells us about inverse functions and their derivatives. To get there we will need a few lemmas.

Definition 12.7. Let $A \subset \mathbb{R}$, and let $f : A \to \mathbb{R}$ be a function. We say that f is *strictly increasing* on A if, for $x, y \in A$, if $x < y$, then $f(x) < f(y)$. Similarly, we say that f is *strictly decreasing* on A if, for $x, y \in A$, if $x < y$, then $f(x) > f(y)$.

Exercise 12.8. Show that if f is strictly increasing or strictly decreasing on an interval, then f is injective.

Lemma 12.9. *If $f : (a, b) \to \mathbb{R}$ is continuous and injective, then f is either strictly increasing or strictly decreasing on (a, b).*

Theorem 12.10. *If $f : (a, b) \to \mathbb{R}$ is continuous and injective, then there exists a function $g : f(a, b) \to (a, b)$ such that $g(f(x)) = x$ for all $x \in (a, b)$ and g is continuous.*

We call the function g from Theorem **??** the *inverse function* of f and we denote it by f^{-1}.

Lemma 12.11. *If $f : (a, b) \to \mathbb{R}$ is differentiable and $f'(x) > 0$ for all $x \in (a, b)$, then f is strictly increasing on (a, b). Similarly, if f is differentiable on (a, b) and $f'(x) < 0$ for all $x \in (a, b)$, then f is strictly decreasing on (a, b).*

Lemma 12.12. *If $f : (a, b) \to \mathbb{R}$ is continuous and $f(c) > 0$ for some $c \in (a, b)$, then there exists a region $R \subset (a, b)$ such that $c \in R$ and $f(x) > 0$ for all $x \in R$. The analogous statement is true if $f(c) < 0$.*

Theorem 12.13. *Suppose that $f : (a, b) \to \mathbb{R}$ is differentiable and that the derivative $f' : (a, b) \to \mathbb{R}$ is continuous. Also suppose that there is a point $p \in (a, b)$ such that $f'(p) \neq 0$. Then there exists a region $R \subset (a, b)$ such that $p \in R$ and f with domain restricted to R is injective. Furthermore, $f^{-1} : f(R) \to R$ is differentiable at the point $f(p)$ and*

$$(f^{-1})'(f(p)) = \frac{1}{f'(p)}.$$

Exercise 12.14. Consider the function $f(x) = x^n$ for a fixed $n \in \mathbb{N}$. If n is even, then f is strictly increasing on the set of non-negative real numbers. If n is odd, then f is strictly increasing on all of $\mathbb{R}$. For a given n, let A be the aforementioned set on which f is strictly increasing. Define the inverse function $f^{-1} : f(A) \to A$ by $f^{-1}(x) = \sqrt[n]{x}$, which we sometimes also denote $f^{-1}(x) = x^{1/n}$. Use Theorem **??** to find the points $y \in f(A)$ at which f^{-1} is differentiable, and determine $(f^{-1})'(y)$ at these points.