MATH 161, SHEET 5: CONTINUOUS FUNCTIONS

Jeffrey Zhang, IBL Script 5 Journal

Definition 5.1. Let $A \subset X \subset C$. We say that A is *open in* X if it is the intersection of X with an open set, and *closed in* X if it is the intersection of X with a closed set. (This is called the subspace topology on X.)

Exercise 5.2. Let $A \subset X \subset C$. Show that $X \setminus A$ is closed in X if, and only if, A is open in X.

Proof. We let $X \setminus A$ be closed in X. Then we know that $X \setminus A = X \cap S$, where S is some closed set. S is closed, so $C \setminus S$ is open. It follows then that $X \cap (C \setminus S) = X \setminus (X \setminus A)$. $X \setminus (X \setminus A) = A$ so $X \cap (C \setminus S) = A$. $C \setminus S$ is open, so by Definition 5.1 A is open in X.
Now, let A be open in X. Then $A = X \cap R$, where R is some open set. $C \setminus R$ is closed then, so $X \cap (C \setminus R) = X \setminus A$. So $X \setminus A$ is closed in X by Definition 5.1. $\square$

Exercise 5.3. a. Let $X = [a, b] \subset C$. Give an example of a set $A \subset X$ such that A is open in X but not in C.

b. Give an example of sets $A \subset X \subset C$ such that A is closed in X but not in C.

Proof. a. Let $A = X$. Then $A = X \cap \emptyset$, and $\emptyset$ is open, so A is open in X. A is closed in C, and $A \neq \emptyset$ and $A \neq C$, so we know that A is not open.
b. Let $A = \underline{ab}$ where $a, b \in C$ and $A \neq \emptyset$, $A \neq C$. Then it follows that A is open and not closed. Let $X = A$, then $A = X \cap C$. C is closed, so A is closed in X, but not closed in C. $\square$

Definition 5.4. Let $X \subset C$. A function $f \colon X \to C$ is *continuous* if for every open set $U \subset C$, the preimage $f^{-1}(U) = \{x \in X \mid f(x) \in U\}$ is open in X.

Exercise 5.5. Let $X \subset C$. A function $f \colon X \to C$ is continuous if, and only if, for every closed set $F \subset C$, the preimage $f^{-1}(F)$ is closed in X.

Proof. Let $X \subset C$. Let a function $f : X \to C$ be continuous. Then we have by Definition 5.4 that the preimage $f^{-1}(U)$ is open in X where U is some open set. So we know $U = C \setminus F$, where F is some closed set. We have then that $f^{-1}(F) = \{x \in X \mid f(x) \in F\}$. It follows that $f^{-1}(C \setminus F) = \{x \in X \mid f(x) \notin F\}$. $f^{-1}(C \setminus F)$ is open in X because $f^{-1}(U)$ is open in X and $U = C \setminus F$. This means then that $X \setminus \{x \in X \mid f(x) \notin F\}$ is closed in X by Exercise 5.2. $X \setminus \{x \in X \mid f(x) \notin F\} = f^{-1}(F)$, so it follows that $f^{-1}(F)$ is closed in X.
Let $f : X \to C$ and $f^{-1}(F)$ be closed in X. Then for every closed set $F \subset C$, $\{x \in X \mid f(x) \in F\}$ is closed in X. It follows then that $X \setminus \{x \in X \mid f(x) \in F\}$ is open in X. $X \setminus \{x \in X \mid f(x) \in F\} = \{x \in X \mid f(x) \notin F\} = f^{-1}(C \setminus F)$. It follows then that $f^{-1}(C \setminus F)$ is open in X. F is an arbitrary closed set, so for every open set U, U can be expressed as $C \setminus F$ for some closed set F. Thus, for every open set U, we have that $f^{-1}(U)$ is open in X. So by Definition 5.4, $f : X \to C$ is continuous. $\square$

Proposition 5.6. *Let $X \subset Y \subset C$. If $f\colon Y \to C$ is continuous, then the restriction of f to X (denoted $f|_X\colon X \to C$) is continuous.*

Proof. Let U be an open set in C and $f : Y \to C$ be continuous. We know then that $f^{-1}(U) = \{x \in X \mid f(x) \in U\}$. Also, by Definition 5.4, we have that $f^{-1}(U) = Y \cap V$ such that V is some open set. It follows then that $f^{-1}(U) \cap X = (Y \cap V) \cap X$. We can express this as $f^{-1}(U) \cap X = (Y \cap X) \cap V$. $X \subset Y$, so $Y \cap X = X$. We have then that $f^{-1}(U) \cap X = X \cap V$. $f^{-1}(U) \cap X = f^{-1}|_X(U)$, so $f^{-1}|_X(U) = X \cap V$. So we have that $f^{-1}|_X(U)$ is open in X for every open set U in C. By Definition 5.4 then we know that $f|_X\colon X \to C$ is continuous. $\square$

It is important that the definition of continuity of $f\colon X \to C$ states that the inverse image of an open set is *open in* X, but not necessarily open in C.

Exercise 5.7. Find a function $f\colon [0,1] \to \mathbb{R}$ that appears to be continuous (in the "not lifting your pencil" sense), but for which there exists an open set $U \subset \mathbb{R}$ such that $f^{-1}(U)$ is *not* open in $\mathbb{R}$. Show that, nevertheless, $f^{-1}(U)$ is open in $[0,1]$.

Proof. Let $f : [0,1] \to \mathbb{R}$ such that $f(x) = X$. Let U be the set $(-0.5, 0.5)$. U is a region, so it is open. It follows then that $f^{-1}(U) = [0, 0.5)$, which is not open in R because no region S can be constructed such that $0 \in S \subset f^{-1}(U)$ (Theorem 3.10). $f^{-1}(U)$ is open in $[0,1]$ however because $[0, 0.5) = [0,1] \cap U$, and U is open. $\square$

Remark 5.8. We review some properties of preimages from Homework 3. Let $X \subset C$ and $f\colon X \to C$. If $A, B \subset C$, then

$$\begin{aligned} f^{-1}(A \cup B) &= f^{-1}(A) \cup f^{-1}(B), \\ \text{and} \quad f^{-1}(A \cap B) &= f^{-1}(A) \cap f^{-1}(B). \end{aligned}$$

Exercise 5.9. Let $f\colon X \to C$. Let $A \subset X$ and $B \subset C$. Then

$$f(f^{-1}(B)) \subset B \quad \text{and} \quad A \subset f^{-1}(f(A)).$$

Proof. Let $f : X \to C$ with $A \subset X$ and $B \subset C$.
Let $y \in f(f^{-1}(B))$, then $\exists x \in f^{-1}(B)$ such that $f(x) = y$. Since for all $x \in f^{-1}(B)$, we know that $f(x) \in B$. Since $f(x) = y$, we know that $y \in B$. Thus we have that $f(f^{-1}(B)) \subset B$.
Let $x \in A$, then $f(x) \in f(A)$. We have that $f^{-1}(A) = \{x \in X \mid f(x) \in A\}$ so $f^{-1}(f(A)) = \{x \in X \mid f(x) \in f(A)\}$. $f(x) \in f(A)$, so $x \in f^{-1}(f(A))$. Thus it follows that $A \subset f^{-1}(f(A))$. $\square$

Exercise 5.10. Let $f\colon X \to C$. Show that f is continuous if, and only if, $f(\overline{A} \cap X) \subset \overline{f(A)}$, for all $A \subset X$.

Note: You should try to do this exercise without considering limit points explicitly. Instead use 5.5, 5.9 and the result from question 3(a) of Homework 4.

Proof. Suppose that f is continuous. then for a set $F \subset C$ such that F is closed, $f^{-1}(F)$ is closed in X by Exercise 5.5. We know that $\overline{f(A)}$ is closed, so choose $F = \overline{f(A)}$. Then we have that $f^{-1}(\overline{f(A)})$ is closed in X so $f^{-1}(\overline{f(A)}) = X \cap V$ for some closed set V. It follows then that $f^{-1}(f(A)) \cup f^{-1}(\{x \in C \mid x \text{ is a limit point of } f(A)\}) = X \cap V$. We have then that $f^{-1}(f(A)) \subset (X \cap V)$ and from Exercise 5.9 we have that $A \subset f^{-1}(f(A))$ so $A \subset (X \cap V)$. $A \subset V$ and V is closed, so by problem 3a on homework 4, $\overline{A} \subset V$. $\overline{A} \subset V$ so $\overline{A} \cap X \subset (V \cap X)$. $(V \cap X) = f^{-1}(\overline{f(A)})$ so we have $(\overline{A} \cap X) \subset f^{-1}(\overline{f(A)})$. It follows then that $f(\overline{A} \cap X) \subset f(f^{-1}(\overline{f(A)}))$. Using Exercise 5.9 again, we have that $f(f^{-1}(\overline{f(A)})) \subset \overline{f(A)}$ so $f(\overline{A} \cap X) \subset \overline{f(A)}$.
Suppose $f(\overline{A} \cap X) \subset \overline{f(A)}$ holds for $A \subset X$. Let $A = f(f^{-1}(F))$ for some closed set F. Then $f(f^{-1}(F)) \subset X$ because $f : X \to C$. Using Exercise 5.9, we know that $f(f^{1}(F)) \subset F$. F is closed, so applying homework 4, problem 3a, we get that $f(\overline{f^{-1}(F)}) \subset F$. Applying the initial assumption, we get that $f(\overline{f^{-1}(F)} \cap X) \subset F$. Taking the pre-image on both sides, we get $f^{-1}(f(\overline{F^{-1}(F)} \cap X)) \subset f^{-1}(F)$. We know that $\overline{f^{-1}(F)} \cap X \subset f^{-1}(f(\overline{F^{-1}(F)} \cap X))$ so $\overline{f^{-1}(F)} \cap X \subset f^{-1}(F)$. We also know $f^{-1}(F) \subset X$ and $f^{-1}(F) \subset \overline{f^{-1}(F)}$ by the definition of closure. So it follows that $f^{-1}(F) \subset \overline{f^{-1}(F)} \cap X$. We have shown then that $\overline{f^{-1}(F)} \cap X \subset f^{-1}(F)$ and $f^{-1}(F) \subset f^{-1}(F) \subset \overline{f^{-1}(F)} \cap X$, so we know $f^{-1}(F) = \overline{f^{-1}(F)} \cap X$. Thus we have shown that the pre-image of F is the intersection of a closed set and X, so by Definition 5.4 we have that f is continuous. $\square$

Definition 5.11. The function $f\colon X \to C$ is *continuous at* $x \in X$ if, for every region R containing $f(x)$, there exists a region S containing x such that $f(S \cap X) \subset R$.

Theorem 5.12. *The function $f\colon X \to C$ is continuous if and only if it is continuous at every $x \in X$.*

Proof. Let $f : X \to C$ be continuous. TIt folllows then that for all open sets $U \subset C$, $f^{-1}(U) = X \cap A$ where A is an open set. Let $x \in X$, and let R be a region such that $f(x) \in R$. $R \subset C$ is an open set, so $f^{-1}(R)$ must be open in X by Definition 5.4. So we get $f^{-1}(R) = X \cap S$ where S is some open set. S is open, so we know there exists a region P such that $x \in P \subset S$. Then $X \cap P \subset f^{-1}(R)$, so $f(X \cap P) \subset f(f^{-1}(R))$. Applying Exercise 5.9, we get $f(f^{-1}(R)) \subset R$, so $f(X \cap P) \subset R$. We also have that f is continuous, so $f(X \cap P) \subset R$ for a region R containing $f(x)$ and a region P containing x. Then it follows by Definition 5.11 that f is continuous at all $x \in X$.
Let f be continuous at every $x \in X$, so for every region R such that $f(x) \in R$ there must exist a region S such that $x \in S$ and $f(S \cap X) \subset R$. By taking the pre-image and applying Exercise 5.9 we get that $S \cap X \subset f^{-1}(R)$. A union of the preimages of the regions R will form any open set $U \subset C$¡ so $(\bigcup_\lambda S_\lambda) \cap X \subset f^{-1}(U)$, for any open set U. $f : X \to C$, so $f^{-1}(U) \subset\subset X$. Assume that there exists some $f(x) \in U$, then since f is continuous at every point, x is contained within some S_λ. So $x \in \bigcup_\lambda S_\lambda$. Then we have $f^{-1}(U) \subset \bigcup_\lambda S_\lambda$. It follows then that $f^{-1}(U) \subset \bigcup_\lambda S_\lambda \cap X$. So we know that for every open set $U \subset C, f^{-1}(U) = \bigcup_\lambda S_\lambda \cap X$. Since $\bigcup_\lambda S_\lambda$ is a union of open sets (regions), it is an open set. So we know that the pre-image of any open set U is open in X. Thus we have that if f is continuous at every $x \in X$, then f is continuous. $\square$

First, we discuss the relationship between continuity and connectedness. Now that we have defined the subspace topology, Definition 4.1 tells us that a set $X \subset C$ is *connected* if it cannot be written as the union $X = A \cup B$ of disjoint, non-empty sets A and B that are open in X.

Theorem 5.13. *Let $X \subset C$. Then X is connected if, and only if, for all $a, b \in X$ with $a < b$, $\underline{ab} \subset X$.*

Proof. Let $X \subset C$ and let X be connected. Assume that for $a, b \in X$ with $a < b$, there exists a point c such that $a < c < b$, $c \notin X$, $c \in \underline{ab}$. Then $X = \{x \in X \mid x < c\} \cup \{x \in X \mid c < x\}$, so X is the disjoint union of two non-empty open sets and thus X is disconnected. These two sets are non-empty because $a \in \{x \in X|midx < c\}$ and $b \in \{x \in X \mid c < x\}$. These sets are open by Corollary 3.12. This is a contradiction, so for all $c \in \underline{ab}$, $\underline{ab} \subset X$.
Let it be true that for all $a, b \in X$ with $a < b$, $\underline{ab} \subset X$. Assume X is disconnected, then we can write $X = A \cup B$ where A, B are non-empty disjoint and open in X. We choose $a \in A$ and $b \in B$ such that $a < b$. Let $c = \sup(A \cap [a, b])$. We know c exists by Theorem 4.18 because $a \in (A \cap [a, b])$ so $(A \cap [a, b])$ is non-empty and $(A \cap [a, b])$ is bounded by $[a, b]$. We have three cases here, either $c \in A$, $c \in B$, or $c \notin A, B$. We consider the first case by assuming $c \in A$. A is open in X, so $A = X \cap U$ where U is some open set. Then by Theorem 3.10 we know that there exists a region R such that $c \in R \subset U$. We then have $(R \cap X) \subset (U \cap X)$. $U \cap X = A$, so $(R \cap X) \subset A$. We write R as $\underline{de}$ for $d, e \in C$. Then $c \in R$ so $d < c < e$. By Theorem 4.3, we have that there exists a point f such that $d < c < f < e$. $f \in R$, and $(R \cap X) \subset A$ so $f \in A$. Note that $c \in (A \cap [a, b])$ because the intersection of two closed sets is closed and A is closed in X ($B = X \setminus A$ is open in X) and $[a, b]$ is closed in X. So $c \in [a, b]$. This means that $c < f < e < b$, so $f \in (A \cap [a, b])$. This is a contradiction, as $c = \sup(A \cap [a, b])$ and $f \in (A \cap [a, b])$, $c < f$. Note that if $f \notin X$, $f \in \underline{ab}$ so $\underline{ab} \not\subset X$ which is also a contradiction. We now consider the second case, where $c \in B$. We know that B is open in X, so $c \in S \subset B$ for some region S. Similarly we have $S \cap X \subset B \cap X$, and $B \subset X$ so $S \cap X \subset B$. We write S as $S = \underline{gh}$ where $g, h \in C$. Using Lemma 4.28, we know that there exists some $x \in A$ such that $x \in S$. This is a contradiction because $x \in S \subset B$, so $x \in B$, but we have A, B as disjoint so $x \in A$ and $x \in B$ contradicts X being disconnected. Note that if there is no $x \in A$ such that $g < x < c$, then g is an upper bound of A and $c \neq \sup(A \cap [a, b])$. Finally we consider the third case, where $c \notin A$ and $c \notin B$. It follows then that $c \notin X$, but $c \in \underline{ab}$ so $\underline{ab} \not\subset X$, which contradicts the initial assumption. □

Corollary 5.14. *Every region $R \subset C$ is connected.*

Proof. Let R be a region $\underline{\text{ ab }}$. Let $a', b' \in R$, then $a < a' < b' < b$. It follows then that $\underline{a'b'} \subset \underline{ab}$, so R is connected. □

Theorem 5.15 (Intermediate Value Theorem)**.** *Suppose that $f\colon X \to C$ is continuous. If X is connected, then $f(X)$ is connected.*

Proof. Let f be continuous and X be connected. Assume that $f(X)$ is disconnnected, so $f(X) = A \cup B$ where A, B are non-empty, disjoint, and open in X. From Exercise 5.9 we have that $X \subset f^{-1}(f(X))$. Substituting $f(X) = A \cup B$ we get $X \subset f^{-1}(A \cup B)$. It foollows then that $X \subset f^{-1}(A) \cup f^{-1}(B)$. We then have that $X = (f^{-1}(A) \cap X) \cup (f^{-1}(B) \cap X)$. But we know that $f : X \to C$, so $f^{-1}(A) \subset X$, $f^{-1}(B) \subset X$. Thus we end up with $X = f^{-1}(A) \cup f^{-1}(B)$. To show that $f^{-1}(A)$ and $f^{-1}(B)$ are disjoint, we assume that they are not, that is there exists some y such that $y \in f^{-1}(A)$ and $y \in f^{-1}(B)$. Then we have $f(y) \in A$, $f(y) \in B$. This is a contradiction, as A, B are disjoint, so $f^{-1}(A)$ and $f^{-1}(B)$ are disjoint. To show that $f^{-1}(A)$ and $f^{-1}(B)$ are non-empty, we assume that $f^{-1}(A)$ is empty. Then either A or B covers all of $f(X)$, so $f(X)$ is not disconnected which is a contradiction. Without loss of generality it can be similarly shown that $f^{-1}(B)$ must be non-empty. We know that f is continuous, so $f^{-1}(A)$ and $f^{-1}(B)$ must be open in X by Exercise 5.5. Thus we have shown that $f^{-1}(A)$ and $f^{-1}(B)$ are non-empty disjoint sets that are both open in X. So we have that X is disconnected, which is a contradiction to the given that X is connected. As a result, $f(X)$ must be connected if $f : X \to C$ is continuous and X is connected. □

Exercise 5.16. Use Theorem 5.15 to prove that if $f\colon [a,b] \to C$ is continuous, then for every point p between $f(a)$ and $f(b)$ there exists c such that $a < c < b$ and $f(c) = p$.

Note that this is the statement of the Intermediate Value Theorem given in calculus texts (with C replaced by $\mathbb{R}$).

Proof. Let $f : [a,b] \to C$ be continuous. Then we know by Theorem 5.15 that $f([a,b])$ is connected. Let $f(a) < p < f(b)$. By Theorem 5.13, we have that $p \in f([a,b])$. We have then that there exists c such that $f(c) = p$. $c \in [a,b]$ and $p \neq f(a)$, $p \neq f(b)$, so $c \neq a$, $c \neq b$. Thus there exists c such that $a < c < b$ and $f(c) = p$. □

Second, we discuss the relationship between continuity and compactness.

Theorem 5.17. *Suppose that $f\colon X \to C$ is continuous. If X is compact, then $f(X)$ is compact.*

Proof. Let X be compact and $\mathcal{U}$ be an open cover of $f(X)$. Then we can write $\mathcal{U} = \bigcup_\lambda U_\lambda$. We have that $f^{-1}(f(X)) \subset f^{-1}(\bigcup_\lambda U_\lambda)$. Using Exercise 5.9, we can write that $X \subset f^{-1}(\bigcup_\lambda U_\lambda)$ because $X \subset f^{-1}(f(X))$. So it follows that $X \subset \bigcup_\lambda f^{-1}(U_\lambda)$. We know by Exercise 5.5 that $f^{-1}(U_\lambda)$ is open in X, so $f^{-1}(U_\lambda) = V_\lambda \cap X$ where V_λ is an open set. So we have $X \subset \bigcup_\lambda (V_\lambda \cap X)$. $X \subset X$, so it follows that $X \subset \bigcup_\lambda V_\lambda$. We have that X is compact, so we know that there must exist a finite subcover such that $X \subset \bigcup_{i=1}^n (V_{\lambda_i})$. We can rewrite this as $X \subset \bigcup_{i=1}^n (V_{\lambda_i} \cap X)$ because again $X \subset X$. So we then substitute and get $X \subset \bigcup_{i=1}^n (f^{-1}(U_{\lambda_i}))$. It follows directly that we can write this as $X \subset f^{-1} \bigcup_{i=1}^n U_{\lambda_i}$. It then follows that $f(X) \subset f(f^{-1} \bigcup_{i=1}^n U_{\lambda_i})$. Using Exercise 5.9 again, we get $f(X) \subset \bigcup_{i=1}^n U_{\lambda i}$, which is a finite subcover of $\mathcal{U}$. So we have $f(X)$ is compact. □

Corollary 5.18 (Extreme Value Theorem)**.** *If $X \subset C$ is non-empty, closed, and bounded and $f\colon X \to C$ is continuous, then $f(X)$ has a first and last point.*

Proof. Let X be non-empty, closed, and bounded. Then by Heine-Borel we have that X is compact. X is compact, so by Theorem 5.17 we have that $f(X)$ is compact. Again by

Heine-Borel, this gives us that $f(X)$ is closed and bounded. $f(X)$ is non-empty because X is non-empty, so $sup(f(X))$ exists by Theorem 4.18. Also, $sup(f(X)) \in f(X)$ because $f(X)$ is closed. So $sup(f(X))$ is the last point of $f(X)$. Without loss of generality it can be similarly shown that $inf(f(X))$ is the first point of $f(X)$. $\square$

Exercise 5.19. Use Corollary 5.19 to prove that if $f\colon [a,b] \to C$ is continuous, then there exists a point $c \in [a,b]$ such that $f(c) \geq f(x)$ for all $x \in [a,b]$. Similarly, there exists a point $d \in [a,b]$ such that $f(d) \leq f(x)$ for all $x \in [a,b]$.

Note that this is the statement of the Extreme Value Theorem given in calculus texts (with C replaced by $\mathbb{R}$).

Proof. Let $f : [a,b] \to C$ be continuous. $[a,b]$ is non-empty, closed, and bounded, so by Corollary 5.18 $f([a,b])$ has a first and last point. Let m be the first point of $f([a,b])$ and n be the last point of $f([a,b])$. $[a,b]$ is compact, so $f([a,b])$ is also compact by Theorem 5.17. Then by Heine-Borel we have that $f([a,b])$ is closed which means it contains all of its limit points, so by Theorem 4.14 $m \in f([a,b])$ and $n \in f([a,b])$. Then it follows by Exercise 5.16 that there exist $c, d \in [a,b]$ such that $f(c) = m$ and $f(d) = n$. $\square$