AMSC 460 Homework 6 Part 1

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5.1.4 For each choice of $f(t, y)$ given in parts (a)-(d):

i. Does f satisfy a Lipschitz condition on $D = \{(t, y) \mid 0 \leq t \leq 1, -\infty < y < \infty\}$?

ii. Can Theorem 5.6 be used to show that the initial-value problem

$$y' = f(t, y), \quad 0 \leq t \leq 1, \quad y(0) = 1$$

is well posed?

A function $f(t, y)$ is said to satisfy a Lipschitz condition in the variable y on a set $D \subset \mathbb{R}^2$ if a constant $L > 0$ exists with

$$|f(t, y_1) - f(t, y_2,)| \leq L |y_1 - y_2| \tag{1}$$

whenever (t, y_1) and (t, y_2) are in D. The constant L is called a Lipschitz constant for f.

Theorem 5.6: Suppose $D = \{(t, y) \mid a \leq t \leq b \text{ and } -\infty < y < \infty\}$. If f is continuous and satisfies a Lipschitz condition in the variable y on the set D, then the initial-value problem

$$\frac{dy}{dt} = f(t, y), \quad a \leq t \leq b, \quad y(a) = \alpha \tag{2}$$

is well posed.

(b) $f(t, y) = \frac{1+y}{1+t}$

Given initial value problem

$$y' = \frac{1+y}{1+t}, \quad 0 \leq t \leq 1, y(0) = 1$$

Let $D = \{(t, y) : 0 \leq t \leq 1, -\infty < y < \infty\}$ and $f(t, y) = \frac{1+y}{1+t}$. Let $y_1 \neq y_2$ in $\mathbb{R}$.

$$\begin{aligned} |f(t, y_1) - f(t, y_2)| &= \left| \frac{1+y_1}{1+t} - \frac{1+y_2}{1+t} \right| \\ &= \frac{1}{1+t} |y_1 - y_2| \\ &\leq |y_1 - y_2| \end{aligned}$$

Thus the given function $f(t, y)$ is Lipschitz continuous on D w.r.t. y. Since f is continuous on D with respect to both variables, by Theorem 5.6, the initial value problem is well-posed.

(d) $f(t,y) = \frac{y^2}{1+t}$

Given initial value problem

$$y' = \frac{y^2}{1+t}, \quad 0 \leq t \leq 1, y(0) = 1$$

Let $D = \{(t,y) : 0 \leq t \leq 1, -\infty < y < \infty\}$ and $f(t,y) = \frac{y^2}{1+t}$. Let $y_1 \neq y_2$ in $\mathbb{R}$.

$$\begin{aligned}
|f(t,y_1) - f(t,y_2)| &= \left| \frac{y_1^2}{1+t} - \frac{y_2^2}{1+t} \right| \\
&= \frac{1}{1+t} |(y_1+y_2)(y_1-y_2)| \\
&= \left(\frac{1}{1+t} |y_1+y_2| \right) |y_1 - y_2| \\
&\geq \left(\frac{1}{2} |y_1+y_2| \right) |y_1 - y_2|
\end{aligned}$$

For any $\alpha > 0, y_1, y_2 \in \mathbb{R}$ can be chosen such that $|y_1+y_2| > \alpha$ thus the function $f(t,y)$ is not Lipschitz. Theorem 5.6 cannot be applied to well-posed.

5.2.2d Use Euler's method to approximate the solutions for each of the following initial-value problems.

$y' = t^{-2}(\sin 2t - 2ty), \quad 1 \leq t \leq 2, \quad y(1) = 2$, with $h = 0.25$

Euler's Method

To approximate the solution of the initial-value problem

$$y' = f(t,y), \quad a \leq t \leq b, \quad y(a) = \alpha,$$

at $(N+1)$ equally spaced numbers in the interval $[a,b]$:

INPUT endpoints a, b; integer N; initial condition α.

OUTPUT approximation w to y at the $(N+1)$ values of t.

Step 1 Set $h = (b-a)/N$;
$t = a$;
$w = \alpha$;
OUTPUT (t, w).

Step 2 For $i = 1, 2, \ldots, N$ do Steps 3, 4.

Step 3 Set $w = w + hf(t,w)$; (*Compute* w_i.)
$t = a + ih$. (*Compute* t_i.)

Step 4 OUTPUT (t, w).

Step 5 STOP.

We have

$$w_0 = y(t_0) = y(a) = y(1) = 2$$

From the step size

$$h = \frac{b-a}{N} = \frac{t_N - t_0}{N} = \frac{2-1}{N}$$
$$N = \frac{1}{h} = \frac{1}{0.25} = 4$$

with

$$t_i = t_0 + ih = a + ih = 1 + i(0.25)$$
$$t_i = 1 + 0.25i.$$

Then,

$$f(t,y) = t^{-2}(\sin(2t) - 2ty) \Rightarrow f(t_i, y_i) = t_i^{-2}\left(\sin(2t_i) - 2t_i y_i\right)$$
$$\Rightarrow f(t_i, w_i) = t_i^{-2}\left(\sin(2t_i) - 2t_i w_i\right)$$

Substituting for t_i:

$$\boxed{f(t_i, w_i) = (1+0.25i)^{-2}\left(\sin(2(1+0.25i)) - 2(1+0.25i)w_i\right)}$$

5.2.4d The actual solutions to the initial-value problems in Exercise 2 are given here. Compare the actual error at each step to the error bound if Theorem 5.9 can be applied.

Theorem 5.9: Suppose f is continuous and satisfies a Lipschitz condition with constant L on

$$D = \{(t,y) \mid a \le t \le b \text{ and } -\infty < y < \infty\}$$

and that a constant M exists with

$$|y''(t)| \le M, \quad \text{for all } t \in [a,b]$$

where $y(t)$ denotes the unique solution to the initial-value problem

$$y' = f(t,y), \quad a \le t \le b, \quad y(a) = \alpha.$$

Let $w_0, w_1, \ldots, w_N$ be the approximations generated by Euler's method for some positive integer N. Then, for each $i = 0, 1, 2, \ldots, N$,

$$|y(t_i) - w_i| \le \frac{hM}{2L}\left[e^{L(t_i - a)} - 1\right] \tag{3}$$

Given initial-value problem

$$y' = t^{-2}(\sin 2t - 2ty), \quad 1 \le t \le 2, \quad y(1) = 2, \text{ with } h = 0.25$$

has the unique solution

$$y(t) = \frac{4 + \cos 2 - \cos 2t}{2t^2}$$

Let $f = t^{-2}(\sin 2t - 2ty)$ then $\frac{\partial f}{\partial y} = -\frac{2}{t} \implies \left|\frac{\partial f}{\partial y}\right| \le 2$ so the function $f(t,y)$ satisfies Lipschitz condition with $L = 2$. Now,

$$y'' = \frac{3(-\cos(2t) + 4 + \cos(2))}{t^4} - \frac{4\sin(2t)}{t^3} + \frac{2\cos(2t)}{t^2}$$

which gives $|y''| \le 7.53052$ when $t = 1$. This gives us upper bound.

$$|y(t_i) - w_i| \le \frac{0.25 \cdot 7.53052}{2 \cdot 2}\left[e^{2(t_i - 1)} - 1\right]$$

From problem 2(d), we get

i	t_i	w_i	$y(t_i)$	$\lvert w_i - y(t_i)\rvert$	Error Bound
0	1	2	2	0	0
1	1.25	1.227324	1.403199	0.1758746	0.3053255
2	1.50	0.8321502	1.016410	0.1842600	0.8087222
3	1.75	0.5704468	0.7380098	0.1675630	1.638683
4	2	0.3788266	0.5296871	0.1508605	3.007057

5.4.27 The irreversible chemical reaction in which two molecules of solid potassium dichromate ($K_2Cr_2O_7$), two molecules of water (H_2O), and three atoms of solid sulfur (S) combine to yield three molecules of the gas sulfur dioxide (SO_2), four molecules of solid potassium hydroxide (KOH), and two molecules of solid chromic oxide (Cr_2O_3) can be represented symbolically by the stoichiometric equation:

$$2\ \mathrm{K_2Cr_2O_7} + 2\mathrm{H_2O} + 3\ \mathrm{S} \longrightarrow 4\mathrm{KOH} + 2\mathrm{Cr_2O_3} + 3\mathrm{SO_2}.$$

If n_1 molecules of $K_2Cr_2O_7, n_2$ molecules of H_2O, and n_3 molecules of S are originally available, the following differential equation describes the amount $x(t)$ of KOH after time t :

$$\frac{dx}{dt} = k\left(n_1 - \frac{x}{2}\right)^2\left(n_2 - \frac{x}{2}\right)^2\left(n_3 - \frac{3x}{4}\right)^3$$

where k is the velocity constant of the reaction. If $k = 6.22 \times 10^{-19}, n_1 = n_2 = 2 \times 10^3$, and $n_3 = 3 \times 10^3$, use the Runge-Kutta method of order four to determine how many units of potassium hydroxide will have been formed after 0.2 s.

Given $x(t)$ determines the amount of KOH available in the chemical reaction

$$2\ \mathrm{K_2Cr_2O_7} + 2\mathrm{H_2O} + 3\ \mathrm{S} \longrightarrow 4\mathrm{KOH} + 2\mathrm{Cr_2O_3} + 3\mathrm{SO_2}$$

which is given by

$$\frac{dx}{dt} = k\left(n_1 - \frac{x}{2}\right)^2\left(n_2 - \frac{x}{2}\right)^2\left(n_3 - \frac{3x}{4}\right)^3$$

With $k = 6.22 \times 10^{-19}, n_1 = n_2 = 2 \times 10^3$ and $n_2 = 3 \times 10^3$. We get

$$\frac{dx}{dt} = 6.22 \times 10^{-19}\left(2 \times 10^3 - \frac{x}{2}\right)^2\left(2 \times 10^3 - \frac{x}{2}\right)^2\left(3 \times 10^3 - \frac{3x}{4}\right)^3$$

Since we are dealing with very large and very small numbers, it is appropriate to scale the problem which would be easier to compute. Let $\bar{x} = \frac{x}{2^4 3^3 \times 10^3}$ then the problem is scaled as

$$\frac{d\bar{x}}{dt} = 0.622(1 - 108\bar{x})^4(1 - 108\bar{x})^3$$

We will use the initial-value problem with above differential equation with $\bar{x}(0) = 0, h = 0.01$ and $f(\bar{x}, t) = 0.622(1 - 108\bar{x})^4(1 - 108\bar{x})^3$. The Runge-Kutta procedure can be recursively (or iteratively) calculated as follows

$$\begin{aligned}
w_0 &= 0 \\
k_1 &= hf\left(t_i, w_i\right) \\
k_2 &= hf\left(t_i + \frac{h}{2}, w_i + \frac{1}{2}k_1\right) \\
k_3 &= hf\left(t_i + \frac{h}{2}, w_i + \frac{1}{2}k_2\right) \\
k_4 &= hf\left(t_{i+1}, w_i + k_3\right) \\
w_{i+1} &= w_i + \frac{1}{6}\left(k_1 + 2k_2 + 2k_3 + k_4\right)
\end{aligned}$$

where $t_i = ih, i = 0, 1 \ldots, 20$. With $w_0 = 0, t_0 = 0$ and $h = 0.01$, we get the following result.

i	t_i	k_1	k_2	k_3	k_4	w_i
1	0.01	0.00622	0.0003544	0.0054331	1.28×10^{-05}	0.002968
2	0.02	0.0004159	0.0003287	0.0003455	0.00028	0.0033087
3	0.03	0.0002816	0.0002382	0.0002445	0.00021	0.0035515
4	0.04	0.0002104	0.0001847	0.0001877	0.0001665	0.0037384
5	0.05	0.0001666	0.0001498	0.0001514	0.0001372	0.0038895
6	0.06	0.0001372	0.0001254	0.0001264	0.0001162	0.0040156
7	0.07	0.0001162	0.0001075	0.0001081	0.0001004	0.0041236
8	0.08	0.0001004	9.38×10^{-05}	9.42×10^{-05}	8.82×10^{-05}	0.0042177
9	0.09	8.82×10^{-05}	8.3×10^{-05}	8.33×10^{-05}	7.85×10^{-05}	0.0043009
10	0.1	7.85×10^{-05}	7.43×10^{-05}	7.45×10^{-05}	7.06×10^{-05}	0.0043754
11	0.11	7.06×10^{-05}	6.71×10^{-05}	6.73×10^{-05}	6.41×10^{-05}	0.0044427
12	0.12	6.41×10^{-05}	6.12×10^{-05}	6.13×10^{-05}	5.86×10^{-05}	0.004504
13	0.13	5.86×10^{-05}	5.61×10^{-05}	5.62×10^{-05}	5.39×10^{-05}	0.0045602
14	0.14	5.39×10^{-05}	5.18×10^{-05}	5.19×10^{-05}	4.99×10^{-05}	0.004612
15	0.15	4.99×10^{-05}	4.81×10^{-05}	4.81×10^{-05}	4.64×10^{-05}	0.0046602
16	0.16	4.64×10^{-05}	4.48×10^{-05}	4.48×10^{-05}	4.33×10^{-05}	0.004705
17	0.17	4.33×10^{-05}	4.19×10^{-05}	4.19×10^{-05}	4.06×10^{-05}	0.0047469
18	0.18	4.06×10^{-05}	3.93×10^{-05}	3.94×10^{-05}	3.82×10^{-05}	0.0047863
19	0.19	3.82×10^{-05}	3.71×10^{-05}	3.71×10^{-05}	3.6×10^{-05}	0.0048234
20	0.2	3.6×10^{-05}	3.5×10^{-05}	3.5×10^{-05}	3.41×10^{-05}	0.0048584

Scaling back, we get $x(0.2) \approx w_{20} = 0.0048584 \times 432 \times 1000 \approx \boxed{2099}$.

5.6.14 The Gompertz differential equation

$$N'(t) = \alpha \ln \frac{K}{N(t)} N(t)$$

serves as a model for the growth of tumors where $N(t)$ is the number of cells in a tumor at time t. The maximum number of cells that can be supported is K, and α is constant related to the proliferative ability of the cells.

In a particular type of cancer, $\alpha = 0.0439, k = 12000$, and t is measured in months. At the time ($t = 0$) the tumor is detected, $N(0) = 4000$. Using the Adams predictor-corrector method with $h = 0.5$, find the number of months it takes for $N(t) = 11000$ cells, which is the lethal number of cells for this cancer.

Adams Predictor-Corrector method of order 4
To approximate the solution of the initial-value problem

$$y' = f(t, y), \quad a \le t \le b, \quad y(a) = \alpha,$$

at $(N+1)$ equally spaced numbers in the interval $[a, b]$:
1. Set $h = (b-a)/N, t_0 = a, w_0 = \alpha.$
2. Calculate w_1, w_2, w_3 using Runge-Kutta method of order 4.
3. Set $t_{i+1} = t_i + h$
4. Predict by

$$w_{i+1}^* = w_i + \frac{h}{24}\left[55f(t_i, w_i) - 59f(t_{i1}, w_{i-1}) + 37f(t_{i-2}, w_{i-2}) - 9f(t_{i-3}, w_{i-3})\right]$$

Correct by

$$w_{i+1} = w_i + \frac{h}{24}\left[9f\left(t_i, w_{i+1}^*\right) + 19f(t_{i-1}, w_{i-1}) - 5f(t_{i-2}, w_{i-2}) + f(t_{i-3}, w_{i-3})\right]$$

To find b, we use an alternative version of this algorithm where we don't initially calculate h, and set a "target" for w_i and return the t_i.

Listing 1: Adams Fourth-Order Predictor-Corrector (find b given target)

```
def AFOPC_Time_Given_Target(h, a, alpha, target):
    t = [a]
    w = [alpha]
    K = [0,0,0,0]
    for i in range(1,4):
        K[0] = h*f(t[i - 1], w[i - 1])
        K[1] = h*f(t[i - 1] +h/2, w[i - 1] + K[0]/2)
        K[2] = h*f(t[i - 1] +h/2, w[i - 1] + K[1]/2)
        K[3] = h*f(t[i - 1] +h, w[i - 1] + K[2])
        w.append(w[i-1] + (K[0] + 2*K[1] + 2*K[2] + K[3])/6)
        t.append(a + i*h)

    i = 4
    #change from original since we are solving a different problem
    while (w[3] < target):
        time = a + i*h

        #included for "ease of access" (avoid recalculation)
        t3w3 = f(t[3],w[3])
        t2w2 = f(t[2],w[2])
        t1w1 = f(t[1],w[1])
```

```
        temp = w[3] + (h/24)*(55*t3w3 - 59*t2w2 + 37*t1w1 - 9*f(t[0],w[0]))
        temp = w[3] + (h/24)*(9*f(time,temp) + 19*t3w3 - 5*t2w2 + t1w1)
        for j in range(3):
            t[j] = t[j + 1]
            w[j] = w[j + 1]
        t[3] = time
        w[3] = temp
        i += 1
    return t[3]
```

Using this algorithm, the end result obtained is $\boxed{58}$. The code is given in the txt file.