AMSC 460 Homework 5 Part 1

Jeffrey Zhang

April 10, 2022

4.1.23 In Exercise 9 of Section 3.4, data were given describing a car traveling on a straight road. That problem asked to predict the position and speed of the car when $t = 10$ seconds. Use the following times and positions to predict the speed at each time listed.

Time	0	3	5	8	10	13
Distance	0	225	383	623	742	993

Forward difference:

$$f'(x_0) = \frac{f(x_0+h) - f(x_0)}{h} - \frac{h}{2} f''(\xi) \tag{1}$$

Backward difference:

$$f'(x_0) = \frac{f(x_0) - f(x_0-h)}{h} - \frac{h}{2} f''(\xi) \tag{2}$$

Central difference:

$$f'(x_0) = \frac{f(x_0+h) - f(x_0-h)}{2h} - \frac{h^2}{6} f^{(3)}(\xi) \tag{3}$$

Three-point endpoint formula

$$f'(x_0) = \frac{-3f(x_0) + 4f(x_0+h) - f(x_0+2h)}{2h} + \frac{h^2}{3} f^{(3)}(\xi) \tag{4}$$

We calculate the approximate speed at each time appropriately:

$$f'(0) \approx \frac{f(3) - f(0)}{3} = \frac{225-0}{3} = \frac{225}{3} = \boxed{75} \quad \text{Forward difference } h = 3$$

$$f'(0) \approx \frac{-3f(0) + 4f(5) - f(10)}{10} = \boxed{79} \quad \text{Three-point endpoint formula } h = 5$$

$$f'(3) \approx \frac{f(3) - f(0)}{3} = \frac{225-0}{3} = \frac{225}{3} = \boxed{75} \quad \text{Backward difference } h = 3$$

$$f'(3) \approx \frac{f(5) - f(3)}{2} = \frac{383-225}{2} = \frac{225}{3} = \boxed{79} \quad \text{Forward difference } h = 2$$

$$f'(3) \approx \frac{-3f(3) + 4f(8) - f(13)}{10} = \boxed{82.4} \quad \text{Three-point endpoint formula } h = 5$$

$$f'(5) \approx \frac{f(5) - f(3)}{2} = \frac{383-225}{2} = \frac{225}{3} = \boxed{79} \quad \text{Backward difference } h = 2$$

$$f'(5) \approx \frac{f(8) - f(5)}{3} = \frac{623-383}{3} = \frac{240}{3} = \boxed{80} \quad \text{Forward difference } h = 3$$

$$f'(5) \approx \frac{f(10) - f(0)}{10} = \frac{742-0}{10} = \boxed{74.2} \quad \text{Central difference } h = 5$$

$$f'(8) \approx \frac{f(8) - f(5)}{3} = \frac{623-383}{3} = \frac{240}{3} = \boxed{80} \quad \text{Backward difference } h = 3$$

$$f'(8) \approx \frac{f(10) - f(8)}{2} = \frac{742-623}{2} = \frac{119}{2} = \boxed{59.5} \quad \text{Forward difference } h = 2$$

$$f'(8) \approx \frac{f(13) - f(3)}{10} = \frac{993-225}{10} = \frac{768}{10} = \boxed{76.8} \quad \text{Central difference } h = 5$$

$$f'(10) \approx \frac{f(10)-f(8)}{2} = \frac{742-623}{2} = \frac{119}{2} = \boxed{59.5} \quad \text{Backward difference } h=2$$
$$f'(10) \approx \frac{f(13)-f(10)}{3} = \frac{993-742}{3} = \frac{251}{3} \approx \boxed{83.67} \quad \text{Forward difference } h=3$$
$$f'(10) \approx \frac{-3f(10)+4f(5)-f(0)}{-10} = \boxed{69.4} \quad \text{Three-point endpoint formula } h=-5$$
$$f'(13) \approx \frac{f(13)-f(10)}{3} = \frac{993-742}{3} = \frac{251}{3} \approx \boxed{83.67} \quad \text{Backward difference } h=3$$
$$f'(13) \approx \frac{-3f(13)+4f(8)-f(3)}{-10} = \boxed{71.2} \quad \text{Three-point endpoint formula } h=-5$$

4.1.27 All calculus students know that the derivative of a function f at x can be defined as

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}.$$

Choose your favorite function f, nonzero number x, and computer or calculator. Generate approximations $f_n'(x)$ to $f'(x)$ by

$$f_n'(x) = \frac{f\left(x+10^{-n}\right)-f(x)}{10^{-n}}$$

for $n = 1, 2, \ldots, 20$, and describe what happens.

Let's take the function $f(x) = \sin x$ and try to find approximations to $f'(\pi)$. The following table gives the values of $f_n'(\pi)$ for $n = 1, 2, \ldots, 20$.

n	$f\left(\pi+10^{-n}\right)$	$f_n'(\pi) = \frac{f\left(\pi+10^{-n}\right)-f(\pi)}{10)^{-n}}$
1	−0.09983341664682810000	−0.99833416646828200000
2	−0.00999983333416633000	−0.99998333341664500000
3	−0.00099999983333310900	−0.99999983333323200000
4	−0.00009999999983342190	−0.99999999833544400000
5	−0.00000999999999977633	−0.99999999998988400000
6	−0.00000100000000001710	−1.00000000013961000000
7	−0.00000009999999971383	−0.99999999836341900000
8	−0.00000000999999981671	−0.99999999392252900000
9	−0.00000000099999996023	−1.00000008274037000000
10	−0.00000000009999988576	−1.00000008274037000000
11	−0.00000000000999987831	−1.00000008274037000000
12	−0.00000000000099996639	−1.00008890058234000000
13	−0.00000000000009979756	−0.99920072216264100000
14	−0.00000000000001009154	−1.02140518265514000000
15	−0.00000000000000076566	−0.88817841970012500000
16	0.00000000000000012251	0.00000000000000000000
17	0.00000000000000012251	0.00000000000000000000
18	0.00000000000000012251	0.00000000000000000000
19	0.00000000000000012251	0.00000000000000000000
20	0.00000000000000012251	0.00000000000000000000

For $n = 1, \ldots, 15$ the value of $f_n'(\pi)$ is approximately -1 which is exactly $\cos \pi$. The error appears to be "curved", as in it becomes more accurate from $n = 1, \ldots, 6$ and less accurate from $n = 7, \ldots, 15$. For $n \geq 16$, it appears that the difference was too small to detect, so the calculator used assumed the answer is 0.

4.3.6d Repeat Exercise 2 using Simpson's rule.

$$\int_e^{e+1} \frac{1}{x \ln x} dx$$

Simpson's Rule:

$$\int_{x_0}^{x_2} f(x)dx = \frac{h}{3}\left[f\left(x_0\right) + 4f\left(x_1\right) + f\left(x_2\right)\right] - \frac{h^5}{90} f^{(4)}(\xi). \tag{5}$$

$$\int_a^b f(x)dx \approx \frac{b-a}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right]$$

Thus:

$$\begin{aligned}\int_e^{e+1} \frac{1}{x \ln x} dx &\approx \frac{1}{6}\left[\frac{1}{e\ln(e)} + \frac{4}{\frac{2e+1}{2}\ln(\frac{2e+1}{2})} + \frac{1}{(e+1)\ln(e+1)}\right] \\ &= \frac{1}{6}\left[\frac{1}{e} + \frac{4}{\frac{2e+1}{2}\ln(\frac{2e+1}{2})} + \frac{1}{(e+1)\ln(e+1)}\right] \\ &\approx \boxed{0.2726704523}\end{aligned}$$

4.3.8d Repeat Exercise 4 using Simpson's rule and the results of Exercise 6:

Find a bound for the error in Exercise 2 using the error formula and compare this to the actual error.

The actual answer for the above equation is given by:

$$\int_e^{e+1} \frac{1}{x \ln x} dx = \ln(\ln(e+1)) - \ln(\ln(e)) = \ln(\ln(e+1)) = 0.2725138805$$

Thus, the actual error between the exact integral and the approximate from Simpson's Rule is:

$$\text{error} \approx |0.2725138805 - 0.2726704523| = 0.0001565718$$

The upper bound for the error for Simpson's rule is defined as:

$$\left|\int_a^b f(x)dx - (b-a)\frac{f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)}{6}\right| \leq \frac{\max_{x\in[a,b]} f^{(4)}(x)}{24} \frac{(b-a)^5}{32} \frac{4}{15} \tag{6}$$

Thus we need to find the max of $f^{(4)}(x)$ where $e < \xi < e+1$. Thus:

$$\begin{aligned} f'(x) &= \frac{\ln(x) + 1}{x^2(\ln(x))^2} \\ f''(x) &= \frac{2\ln^2(x) + 3\ln(x) + 2}{x^3 \ln^3(x)} \\ f^{(3)}(x) &= -\frac{6 + 12\ln(x) + 11\ln^2(x) + 6\ln^3(x)}{x^4 \ln^4(x)} \\ f^{(4)}(x) &= \frac{24 + 60\ln(x) + 70\ln^2(x) + 50\ln^3(x) + 24\ln^4(x)}{x^5 \ln^5(x)}. \end{aligned}$$

We see that $f^{(4)}(x)$ is a monotonically decreasing function so the max occurs at e. Thus, the upper error bound is:

$$\begin{aligned}
\frac{\max_{x\in[a,b]} f^{(4)}(x)}{24}\frac{(b-a)^5}{32}\frac{4}{15} &= \frac{(b-a)^5}{2880}f^{(4)}(\xi) \\
&= \frac{1}{2880}f^{(4)}(e) \\
&\approx \frac{1}{2880} * 1.53625 \\
&= \boxed{5.334201389 \times 10^{-4}}
\end{aligned}$$

4.4.8c Approximate $\int_0^2 x^2 e^{-x^2} dx$ using $h = 0.25$. Use Composite Midpoint rule.

Let $f \in C^2[a,b], n$ be even, $h = (b-a)/(n+2)$, and $x_j = a + (j+1)h$ for each $j = -1, 0, \ldots, n+1$. There exists a $\mu \in (a,b)$ for which the Composite Midpoint rule for $n+2$ subintervals can be written with its error term as

$$\int_a^b f(x)dx = 2h\sum_{j=0}^{n/2} f(x_{2j}) + \frac{b-a}{6}h^2 f''(\mu) \tag{7}$$

Thus:

$$0.25 = \frac{2}{n+2} \implies 0.25n + 0.5 = 2 \implies 0.25n = 1.5 \implies n = 6$$

$$\begin{aligned}
\int_0^2 x^2 e^{-x^2} dx &\approx 2 \cdot 0.25 \sum_{j=0}^{6/2} x_{2j}^2 e^{-x_{2j}^2} \\
&= \frac{1}{2}\sum_{j=0}^{3} x_{2j}^2 e^{-x_{2j}^2} \\
&= \frac{1}{2}\sum_{j=0}^{3}((2j+1)\cdot 0.25)^2 e^{-((2j+1)\cdot 0.25)^2} \\
&= \frac{1}{2}\left(f\left(\frac{1}{4}\right) + f\left(\frac{3}{4}\right) + f\left(\frac{5}{4}\right) + f\left(\frac{7}{4}\right)\right) \\
&\approx \frac{1}{2}(0.058713316 + 0.320502838 + 0.327517792 + 0.143235031) \\
&= \boxed{0.424984488}
\end{aligned}$$

4.4.11c Determine the values of n and h required to approximate

$$\int_0^2 e^{2x}\sin 3x dx$$

to within 10^{-4}. Use Composite Midpoint rule.

From the definition of the Composite Midpoint rule, we know that the error is:

$$\left|\frac{b-a}{6}h^2 f''(\mu)\right| \quad \mu \in (a,b)$$

Thus we need to find the max of the 2nd derivative:

$$\begin{aligned}
f'(x) &= \mathrm{e}^{2x}(2\sin(3x) + 3\cos(3x)) \\
f''(x) &= -\mathrm{e}^{2x}(5\sin(3x) - 12\cos(3x)) \\
f'''(x) &= -\mathrm{e}^{2x}(46\sin(3x) - 9\cos(3x))
\end{aligned}$$

From a graphing calculator, we see that the max for $f''(x)$ in $[0, 2]$ occurs at $x = 2$, thus:

$$|f''(\mu)| \le e^4(12\cos(6) - 5\sin(6)) \approx 705.3601029, \quad \forall \mu \in [0, 2]$$

The error can be bounded by

$$\left|\frac{b-a}{6}h^2 f''(\mu)\right| = \left|\frac{2-0}{6}h^2 f''(\mu)\right| = \left|\frac{1}{3}h^2 f''(\mu)\right| \le \frac{705.3601029}{3}h^2 < 10^{-4}$$

Thus, to find h:

$$\frac{705.3601029}{3}h^2 < 10^{-4}$$
$$h^2 < \frac{3 \cdot 10^{-4}}{705.3601029}$$
$$h < \sqrt{\frac{3 \cdot 10^{-4}}{705.3601029}} \approx \boxed{6.251615374 \times 10^{-4}}$$

To find n:

$$h = \frac{b-a}{n+2} = \frac{2-0}{n+2}$$
$$n + 2 = \frac{2}{h} \implies n = \frac{2}{h} - 2 \approx 3064.724861$$
$$\implies n \ge \boxed{3065}$$

4.7.6b Repeat Exercise 2 with n = 4.

$$\int_1^{1.6} \frac{2x}{x^2 - 4}dx$$

Theorem 4.17: Suppose that $x_1, x_2, \ldots, x_n$ are the roots of the nth Legendre polynomial $P_n(x)$ and that for each $i = 1, 2, \ldots, n$, the numbers c_i are defined by

$$c_i = \int_{-1}^{1} \prod_{\substack{j=1 \\ j \ne i}}^{n} \frac{x - x_j}{x_i - x_j} dx \tag{8}$$

If $P(x)$ is any polynomial of degree less than $2n$, then

$$\int_{-1}^{1} P(x)dx = \sum_{i=1}^{n} c_i P(x_i)\,. \tag{9}$$

n	Roots $r_{n,i}$	Coefficients $c_{n,i}$
2	0.5773502692	1.0000000000
	−0.5773502692	1.0000000000
3	0.7745966692	0.5555555556
	0.0000000000	0.8888888889
	−0.7745966692	0.5555555556
4	0.8611363116	0.3478548451
	0.3399810436	0.6521451549
	−0.3399810436	0.6521451549
	−0.8611363116	0.3478548451
5	0.9061798459	0.2369268850
	0.5384693101	0.4786286705
	0.0000000000	0.5688888889
	−0.5384693101	0.4786286705
	−0.9061798459	0.2369268850

An integral $\int_a^b f(x)dx$ over an arbitrary $[a,b]$ can be transformed into an integral over $[-1,1]$ by using the change of variables (see Figure 4.17):

$$t = \frac{2x-a-b}{b-a} \iff x = \frac{1}{2}[(b-a)t+a+b]. \tag{10}$$

This permits Gaussian quadrature to be applied to any interval $[a,b]$ because

$$\int_a^b f(x)dx = \int_{-1}^1 f\left(\frac{(b-a)t+(b+a)}{2}\right)\frac{(b-a)}{2}dt \tag{11}$$

From the information given above, we can rewrite the integral for this problem as:

$$x = \frac{1}{2}\left[(1.6-1)t+(1.6+1)\right] = \frac{1}{2}\left[0.6t+2.6\right] = 0.3t+1.3$$

$$\int_1^{1.6} \frac{2x}{x^2-4}dx = \int_{-1}^1 \frac{2(0.3t+1.3)}{(0.3t+1.3)^2-4}dt = 0.3\int_{-1}^1 \frac{0.6t+2.6}{(0.3t+1.3)^2-4}dt$$

where: $\displaystyle\int_{-1}^1 \frac{0.6t+2.6}{(0.3t+1.3)^2-4}dt$

$$\approx 0.3478548451\cdot f(x) + 0.6521451549\cdot f(x) + 0.6521451549\cdot f(x) + 0.3478548451\cdot f(x)$$

$$= 0.3478548451\cdot f(0.3t+1.3) + 0.6521451549\cdot f(0.3t+1.3) +$$

$$0.6521451549\cdot f(0.3t+1.3) + 0.3478548451\cdot f(0.3t+1.3)$$

$$= 0.3478548451\cdot f(0.3(0.8611363116)+1.3) + 0.6521451549\cdot f(0.3(0.3399810436)+1.3) +$$

$$0.6521451549\cdot f(0.3(-0.3399810436)+1.3) + 0.3478548451\cdot f(0.3(-0.8611363116)+1.3)$$

$$= 0.3478548451\cdot f(1.55834089348) + 0.6521451549\cdot f(1.40199431308) +$$

$$0.6521451549\cdot f(1.19800568692) + 0.3478548451\cdot f(1.04165910652)$$

$$= 0.3478548451\cdot -1.98315985236 + 0.6521451549\cdot -1.37827967022 +$$

$$0.6521451549\cdot -0.93419675604 + 0.3478548451\cdot -0.71470208864$$

$$= -2.44653464479$$

If we multiply by 0.3 we get $\boxed{-0.73396039343}$

4.7.11 Determine constants a, b, c, and d that will produce a quadrature formula

$$\int_{-1}^{1} f(x)dx = af(-1) + bf(1) + cf'(-1) + df'(1)$$

that has degree of precision three.

We want the formula above to hold for polynomials $1, x, x^2, \ldots$ Plugging these into the formula, we obtain:

$$f(x) = x^0 \quad \int_{-1}^{1} 1dx = x|_{-1}^{1} = 2 = a\cdot 1 + b\cdot 1 + c\cdot 0 + d\cdot 0$$

$$f(x) = x^1 \quad \int_{-1}^{1} xdx = \left.\frac{x^2}{2}\right|_{-1}^{1} = 0 = a\cdot(-1) + b\cdot 1 + c\cdot 1 + d\cdot 1$$

$$f(x) = x^2 \quad \int_{-1}^{1} x^2dx = \left.\frac{x^3}{3}\right|_{-1}^{1} = \frac{2}{3} = a\cdot 1 + b\cdot 1 + c\cdot(-2) + d\cdot 2$$

$$f(x) = x^3 \quad \int_{-1}^{1} x^3dx = \left.\frac{x^4}{4}\right|_{-1}^{1} = 0 = a\cdot(-1) + b\cdot 1 + c\cdot 3 + d\cdot 3$$

Thus:

$$\begin{aligned} a + b &= 2 \\ -a + b + c + d &= 0 \\ a + b - 2c + 2d &= \frac{2}{3} \\ -a + b + 3c + 3d &= 0 \end{aligned}$$

Solving this system, we obtain:

$$\boxed{a = 1, b = 1, c = \frac{1}{3}, d = -\frac{1}{3}}$$

Thus, the quadrature formula with accuracy $n = 3$ is:

$$\int_{-1}^{1} f(x)dx = f(-1) + f(1) + \frac{1}{3}f'(-1) - \frac{1}{3}f'(1)$$