AMSC 460 Homework 2 Part 1

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6.2.10b Use Gaussian elimination and three-digit chopping arithmetic to solve the following linear systems and compare the approximations to the actual solution:

$3.3330x_1 + 15920x_2 + 10.333x_3 = 7953,$

$2.2220x_1 + 16.710x_2 + 9.6120x_3 = 0.965,$

$-1.5611x_1 + 5.1792x_2 - 1.6855x_3 = 2.714.$

Actual solution [1, 0.5, — 1].

From three-digit chopping arithmetic, the given system has the augmented matrix

$$\left[\begin{array}{ccc|c} 3.33 & 15900 & 10.3 & 7950 \\ 2.22 & 16.7 & 9.61 & 0.965 \\ -1.56 & 5.17 & -1.68 & 2.71 \end{array}\right]$$

The multipliers are:

$$m_{21} = \frac{a_{21}}{a_{11}} = \frac{2.22}{3.33} \approx 0.666, m_{31} = \frac{a_{31}}{a_{11}} = \frac{-1.56}{3.33} \approx -0.468$$

With operations $E_2 - 0.666E_1 \to E_2, E_3 + 0.468E_1 \to E_3$, we get

$$a_{21} = 2.22 - 0.666 * 3.33 = 2.22 - 2.21778 \approx 2.22 - 2.21 = 0.01$$

$$a_{22} = 16.7 - 0.666 * 15900 = 16.7 - 10589.4 \approx 16.7 - 10500 \approx -10400$$

$$a_{23} = 9.61 - 0.666 * 10.3 = 9.61 - 6.8598 \approx 9.61 - 6.85 = 2.76$$

$$a_{24} = 0.965 - 0.666 * 7950 = 0.965 - 5294.7 \approx 0.965 - 5290 = -5289.035 \approx -5280$$

$$a_{31} = -1.56 + 0.468 * 3.33 = -1.56 + 1.5584 \approx -1.56 + 1.55 = -0.01$$

$$a_{32} = 5.17 + 0.468 * 15900 = 5.17 + 7441.2 \approx 5.17 + 7440 = 7445.17 \approx 7440$$

$$a_{33} = -1.68 + 0.468 * 10.3 = -1.68 + 4.8204 \approx -1.68 + 4.82 = 3.15$$

$$a_{34} = 2.71 + 0.468 * 7950 = 2.71 + 3720.6 \approx= 2.71 + 3720 = 3722.71 \approx 3720$$

$$\left[\begin{array}{ccc|c} 3.33 & 15900 & 10.3 & 7950 \\ 0.01 & -10400 & 2.76 & -5280 \\ -0.01 & 7440 & 3.14 & 3720 \end{array}\right]$$

The entries $E_{k,1}, k \geq 2$ are not zero due to three digit chopping. Therefore we round it to zero. This gives us matrix

$$\left[\begin{array}{ccc|c} 3.33 & 15900 & 10.3 & 7950 \\ 0.0 & -10400 & 2.76 & -5280 \\ 0.0 & 7440 & 3.14 & 3720 \end{array}\right]$$

The multiplier is:

$$m_{32} = \frac{a_{32}}{a_{22}} = \frac{440}{-10400} \approx -0.715$$

With operations $E_3 + 0.715E_2 \to E_3$, we get

$$a_{32} = 7740 + 0.715 * -10400 = 7740 - 7436 \approx 7740 - 7730 = 10.0$$

$$a_{33} = 3.14 + 0.715 * 2.76 = 3.14 + 1.9734 \approx= 3.14 + 1.97 = 5.11$$

$$a_{34} = 3720 + 0.715 * -5280 = 3720 - 3775.2 \approx 3720 - 3770 = -50.0$$

$$\left[\begin{array}{lll|l} 3.33 & 15900 & 10.3 & 7950 \\ 0.0 & -10400 & 2.76 & -5280 \\ 0.0 & 10.0 & 5.11 & -50.0 \end{array}\right]$$

The entry $E_{3,2}$ is not zero due to three digit chopping. Therefore we round it to zero. This gives us matrix

$$\left[\begin{array}{lll|l} 3.33 & 15900 & 10.3 & 7950 \\ 0.0 & -10400 & 2.76 & -5280 \\ 0.0 & 0.0 & 5.11 & -50.0 \end{array}\right]$$

Using backward substitution, we get

$$5.11x_3 = -50.0 \implies \boxed{x_3 = \frac{-50.0}{5.11} \approx -9.78}$$

$$-10400x_2 + 2.76(-9.78) = -5280 \implies -10400x_2 + -26.9 \approx -5280 \implies$$

$$-10400x_2 \approx -5250 \implies \boxed{x_2 \approx \frac{-5250}{-10400} \approx -0.504}$$

$$3.33x_1 + 15900(0.504) + 10.3(-9.78) = 7950 \implies 3.33x_1 + 8010 - 100 \approx 7950 \implies$$

$$3.33x_1 + 7910 \approx 7950 \implies 3.33x_1 \approx 40 \implies \boxed{x_1 \approx \frac{40}{3.33} \approx 12.0}$$

$$x_1 = 12.0, x_2 = 0.504, x_3 = -9.78 \implies \boxed{[12.0, 0.504, -9.78]}$$

The solution is very different from the actual solution.

6.2.14b Repeat Exercise 10 using Gaussian elimination with partial pivoting.

Answer is literally the exact same as above since the pivots don't imply any row swaps

From three-digit chopping arithmetic, the given system has the augmented matrix

$$\left[\begin{array}{lll|l} 3.33 & 15900 & 10.3 & 7950 \\ 2.22 & 16.7 & 9.61 & 0.965 \\ -1.56 & 5.17 & -1.68 & 2.71 \end{array}\right]$$

Out of the numbers in the row $a_{k,1}$ where $1 \leq k \leq 3$, $|a_{11}|$ is the greatest so we do not swap any rows. Thus the multipliers are:

$$m_{21} = \frac{a_{21}}{a_{11}} = \frac{2.22}{3.33} \approx 0.666, m_{31} = \frac{a_{31}}{a_{11}} = \frac{-1.56}{3.33} \approx -0.468$$

With operations $E_2 - 0.666E_1 \to E_2, E_3 + 0.468E_1 \to E_3$, we get

$$a_{21} = 2.22 - 0.666 * 3.33 = 2.22 - 2.21778 \approx 2.22 - 2.21 = 0.01$$

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$$\left[\begin{array}{lll|l} 3.33 & 15900 & 10.3 & 7950 \\ 0.01 & -10400 & 2.76 & -5280 \\ -0.01 & 7440 & 3.14 & 3720 \end{array}\right]$$

The entries $E_{k,1}, k \geq 2$ are not zero due to three digit chopping. Therefore we round it to zero. This gives us matrix

$$\left[\begin{array}{lll|l} 3.33 & 15900 & 10.3 & 7950 \\ 0.0 & -10400 & 2.76 & -5280 \\ 0.0 & 7440 & 3.14 & 3720 \end{array}\right]$$

Out if the numbers in the row $a_{2,k}$, where $2 \leq k \leq 3$, $|a_{22}|$ is the greatest so we do not swap any rows. Thus the multiplier is:

$$m_{32} = \frac{a_{32}}{a_{22}} = \frac{440}{-10400} \approx -0.715$$

With operations $E_3 + 0.715E_2 \to E_3$, we get

$$a_{32} = 7740 + 0.715 * -10400 = 7740 - 7436 \approx 7740 - 7730 = 10.0$$

$$a_{33} = 3.14 + 0.715 * 2.76 = 3.14 + 1.9734 \approx= 3.14 + 1.97 = 5.11$$

$$a_{34} = 3720 + 0.715 * -5280 = 3720 - 3775.2 \approx 3720 - 3770 = -50.0$$

$$\left[\begin{array}{lll|l} 3.33 & 15900 & 10.3 & 7950 \\ 0.0 & -10400 & 2.76 & -5280 \\ 0.0 & 10.0 & 5.11 & -50.0 \end{array}\right]$$

The entry $E_{3,2}$ is not zero due to three digit chopping. Therefore we round it to zero. This gives us matrix

$$\left[\begin{array}{lll|l} 3.33 & 15900 & 10.3 & 7950 \\ 0.0 & -10400 & 2.76 & -5280 \\ 0.0 & 0.0 & 5.11 & -50.0 \end{array}\right]$$

Using backward substitution, we get

$$5.11x_3 = -50.0 \implies \boxed{x_3 = \frac{-50.0}{5.11} \approx -9.78}$$

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$$-10400x_2 \approx -5250 \implies \boxed{x_2 \approx \frac{-5250}{-10400} \approx -0.504}$$

$$3.33x_1 + 15900(0.504) + 10.3(-9.78) = 7950 \implies 3.33x_1 + 8010 - 100 \approx 7950 \implies$$

$$3.33x_1 + 7910 \approx 7950 \implies 3.33x_1 \approx 40 \implies \boxed{x_1 \approx \frac{40}{3.33} \approx 12.0}$$

$$x_1 = 12.0, x_2 = 0.504, x_3 = -9.78 \implies \boxed{[12.0, 0.504, -9.78]}$$

The solution is very different from the actual solution.

6.5.2a Solve the following linear system:

$$\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & -1 \\ 0 & 4 & 2 \\ 0 & 0 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -5 \end{bmatrix}$$

Let us assume $A = LU$ and $LUx = b$. Let $y = Ux$:

$$\begin{bmatrix} 2 & 1 & -1 \\ 0 & 4 & 2 \\ 0 & 0 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

Since $b = L(Ux) = Ly$ which means $Ly = b$:

$$\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -5 \end{bmatrix}$$

Multiplying, we get

$$y_1 = 1$$

$$-2y_1 + y_2 = 0$$

$$3y_1 + y_3 = -5$$

This gives us $y_1 = -1, y_2 = 2, y_3 = -8$. Since $Ux = y$, we get:

$$\begin{bmatrix} 2 & 1 & -1 \\ 0 & 4 & 2 \\ 0 & 0 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -8 \end{bmatrix}$$

This can be solved using backward substitution. Last equation gives

$$5x_3 = -8 \implies x_3 = -\frac{8}{5}$$

$$4x_2 + 2x_3 = 2 \implies 4x_2 = \frac{26}{5} \implies x_2 = \frac{13}{10}$$

$$2x_1 + x_2 - x_3 = 1 \implies 2x_1 = 1 - \frac{8}{5} - \frac{13}{10} \implies x_1 = -\frac{19}{20}$$

Thus, $\boxed{x_1 = -\frac{19}{20}, x_2 = \frac{13}{10}, x_3 = \frac{8}{5}}$

6.5.5a Factor the following matrices into the LU decomposition using the LU Factorization Algorithm with $I_{ii} = 1$ for all i.

$$\begin{bmatrix} 2 & -1 & 1 \\ 3 & 3 & 9 \\ 3 & 3 & 5 \end{bmatrix}$$

We first find m_{21} and m_{31}:

$$m_{21} = \frac{a_{21}}{a_{11}} = \frac{3}{2} = 1.5$$
$$m_{31} = \frac{a_{31}}{a_{11}} = \frac{3}{2} = 1.5$$

We then do $E_2 - 1.5E1 \to E_2$, $E_3 - 1.5E1 \to E_3$ on A to find $A^{(1)}$ and put m_{21} and m_{31} for a_{21} and a_{31} respectfully in I_3 to find $L^{(1)}$

$$L^{(1)} = \begin{bmatrix} 1 & 0 & 0 \\ 1.5 & 1 & 0 \\ 1.5 & 0 & 1 \end{bmatrix} A^{(1)} = \begin{bmatrix} 2 & -1 & 1 \\ 0 & 4.5 & 7.5 \\ 0 & 4.5 & 3.5 \end{bmatrix}$$

We then find m_{32}:

$$m_{32} = \frac{a_{32}}{a_{22}} = \frac{4.5}{4.5} = 1$$

We then do $E_3 - 1E2 \to E_2$ on $A^{(1)}$ to find U and put m_{32} for a_{32} respectfully in $L^{(1)}$ to find L

$$\boxed{L = \begin{bmatrix} 1 & 0 & 0 \\ 1.5 & 1 & 0 \\ 1.5 & 1 & 1 \end{bmatrix} U = \begin{bmatrix} 2 & -1 & 1 \\ 0 & 4.5 & 7.5 \\ 0 & 0 & -4 \end{bmatrix}}$$

6.6.5a Use the Cholesky Algorithm to find a factorization of the form A = LL' for the matrices in Exercise 3.

$$A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

We know that

$$A = \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{21} & a_{22} & a_{32} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} l_{11} & l_{21} & l_{31} \\ 0 & l_{22} & l_{32} \\ 0 & 0 & l_{33} \end{bmatrix}$$
$$= \begin{bmatrix} l_{11}^2 & l_{11}l_{21} & l_{11}l_{31} \\ l_{11}l_{21} & l_{21}^2 + l_{22}^2 & l_{21}l_{31} + l_{22}l_{32} \\ l_{11}l_{31} & l_{21}l_{31} + l_{22}l_{32} & l_{31}^2 + l_{32}^2 + l_{33}^2 \end{bmatrix}$$

Thus:

$$l_{11}^2 = 2 \implies l_{11} = \sqrt{2}$$
$$l_{11}l_{21} = -1 \implies l_{21} = \frac{-1}{\sqrt{2}}$$
$$l_{21}^2 + l_{22}^2 = 2 \implies l_{22} = \sqrt{2 - \frac{1}{2}} = \sqrt{\frac{3}{2}}$$

$$l_{11}l_{31} = 0 \implies l_{31} = 0$$

$$l_{21}l_{31} + l_{22}l_{32} = -1 \implies l_{32} = -\sqrt{\frac{2}{3}}$$

$$l_{31}^2 + l_{32}^2 + l_{33}^2 = 2 \implies l_{33} = \frac{2}{\sqrt{3}}$$

$$\boxed{L = \begin{bmatrix} \sqrt{2} & 0 & 0 \\ \frac{-1}{\sqrt{2}} & \sqrt{\frac{3}{2}} & 0 \\ 0 & -\sqrt{\frac{2}{3}} & \frac{2}{\sqrt{3}} \end{bmatrix} L^t = \begin{bmatrix} \sqrt{2} & \frac{-1}{\sqrt{2}} & 0 \\ 0 & \sqrt{\frac{3}{2}} & -\sqrt{\frac{2}{3}} \\ 0 & 0 & \frac{2}{\sqrt{3}} \end{bmatrix}}$$

7.5.1a Compute the condition numbers of the following matrices relative to $|| \cdot ||_\infty$.

The conditionnumber of the non-singular matrix A relative to norm $\| \cdot \|$ is given by

$$K(A) = \|A\| \cdot \left\|A^{-1}\right\|$$

The norm of the matrix A with respect to norm $\| \cdot \|_\infty$ is given by

$$\|A\|_\infty = \max_{1 \le i \le n} \left(\sum_{j=1}^{n} |a_{ij}| \right)$$

Given matrix

$$A = \begin{bmatrix} \frac{1}{2} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{4} \end{bmatrix}$$

and it's inverse is given by

$$A^{-1} = \begin{bmatrix} 18 & -24 \\ -24 & 36 \end{bmatrix}$$

The norms are given by

$$\begin{aligned} \|A\|_\infty &= \max\left\{\left|\tfrac{1}{2}\right| + \left|\tfrac{1}{3}\right|, \left|\tfrac{1}{3}\right| + \left|\tfrac{1}{4}\right|\right\} = \tfrac{5}{6} \\ \left\|A^{-1}\right\|_\infty &= \max\{|18| + |-24|, |-24| + |36|\} = 60 \end{aligned}$$

This gives

$$\boxed{K(A) = 50}$$

7.5.3a The following linear systems $Ax = b$ have $\tilde{x}$ as the actual solution and x as an approximate solution. Using the results of Exercise 1, compute $\|\mathbf{x} - \tilde{\mathbf{x}}\|_\infty$ and $K_\infty(A)\frac{\|\mathbf{b} - A\tilde{\mathbf{x}}\|_\infty}{\|A\|_\infty}$

$$\frac{1}{2}x_1 + \frac{1}{3}x_2 = \frac{1}{63}$$

$$\frac{1}{3}x_1 + \frac{1}{4}x_2 = \frac{1}{168}$$

$$\mathbf{x} = \left(\frac{1}{7}, -\frac{1}{6}\right)^t$$

$$\tilde{\mathbf{x}} = (0.142, -0.166)^t$$

From the values above we can deduce that:

$$\|\mathbf{x} - \tilde{\mathbf{x}}\|_\infty = \max\left\{\left|0.142 - \frac{1}{7}\right|, \left|-0.166 + \frac{1}{6}\right|\right\} = \left|0.142 - \frac{1}{7}\right| = \boxed{\frac{3}{3500}}$$

From problem 1a, we know that

$$K_\infty(A) = 50, \quad \|A\|_\infty = \frac{5}{6}$$

Calculating $A\tilde{x}$ gives us:

$$A\tilde{\mathbf{x}} = \begin{bmatrix} \frac{1}{2} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{4} \end{bmatrix} \begin{bmatrix} 0.142 \\ -0.166 \end{bmatrix} = \begin{bmatrix} \frac{47}{3000} \\ \frac{7}{1200} \end{bmatrix}$$

which implies

$$\|\mathbf{b} - A\tilde{\mathbf{x}}\|_\infty = \max\left\{\left|\frac{47}{3000} - \frac{1}{63}\right|, \left|\frac{7}{1200} - \frac{1}{168}\right|\right\} = \left|\frac{47}{3000} - \frac{1}{63}\right| = \frac{13}{63000}$$

Thus:

$$K_\infty(A)\frac{\|\mathbf{b} - A\tilde{\mathbf{x}}\|_\infty}{\|A\|_\infty} = 50 \times \frac{\frac{13}{63000}}{\frac{5}{6}} = \boxed{\frac{13}{1050}}$$