# AMSC 460 Homework 4 Part 1

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8.1.1 Compute the linear least squares polynomial for the data of Example 2.

Table 8.3

| $i$ | $x_i$ | $y_i$ |
|---|---|---|
| 1 | 0 | 1.0000 |
| 2 | 0.25 | 1.2840 |
| 3 | 0.50 | 1.6487 |
| 4 | 0.75 | 2.1170 |
| 5 | 1.00 | 2.7183 |

For Linear Least Squares, we know that

$$\begin{aligned} a_0 &= \frac{\sum_{i=1}^m x_i^2 \sum_{i=1}^m y_i - \sum_{i=1}^m x_i y_i \sum_{i=1}^m x_i}{m\left(\sum_{i=1}^m x_i^2\right) - \left(\sum_{i=1}^m x_i\right)^2} \\ a_1 &= \frac{m\sum_{i=1}^m x_i y_i - \sum_{i=1}^m x_i \sum_{i=1}^m y_i}{m\left(\sum_{i=1}^m x_i^2\right) - \left(\sum_{i=1}^m x_i\right)^2} \end{aligned} \tag{1}$$

From above data, we have

| $i$ | $x_i$ | $y_i$ | $x_i y_i$ | $x_i^2$ |
|---|---|---|---|---|
| 1 | 0 | 1.0000 | 0.0 | 0.0 |
| 2 | 0.25 | 1.2840 | 0.321 | 0.0625 |
| 3 | 0.50 | 1.6487 | 0.82435 | 0.25 |
| 4 | 0.75 | 2.1170 | 1.58775 | 0.5625 |
| 5 | 1.00 | 2.7183 | 2.7183 | 1.0 |
| $\sum_{i=1}^5$ | 2.5 | 8.768 | 5.4514 | 1.875 |

Thus

$$\begin{aligned} a_0 &= \frac{1.875 \times 8.768 - 5.4514 \times 2.5}{5 \times 1.875 - (2.5)^2} \\ &= 0.89968 \\ a_1 &= \frac{5 \times 5.4514 - 2.5 \times 8.768}{5 \times 1.875 - (2.5)^2} \\ &= 1.70784 \end{aligned}$$

which means the linear least square fit is

$$\boxed{P(x) = 0.89968 + 1.70784x}$$

8.1.4 Find the least squares polynomials of degrees 1,2 , and 3 for the data in the following table. Compute the error $E$ in each case. Graph the data and the polynomials.

| $x_i$ | 0 | 0.15 | 0.31 | 0.5 | 0.6 | 0.75 |
|---|---|---|---|---|---|---|
| $y_i$ | 1.0 | 1.004 | 1.031 | 1.117 | 1.223 | 1.422 |

We know that for a polynomial of degree n, the set of equations for $a_0, \ldots, a_n$ is:

$$\begin{aligned}
a_0 \sum_{i=1}^{m} x_i^0 + a_1 \sum_{i=1}^{m} x_i^1 + a_2 \sum_{i=1}^{m} x_i^2 + \cdots + a_n \sum_{i=1}^{m} x_i^n &= \sum_{i=1}^{m} y_i x_i^0 \\
a_0 \sum_{i=1}^{m} x_i^1 + a_1 \sum_{i=1}^{m} x_i^2 + a_2 \sum_{i=1}^{m} x_i^3 + \cdots + a_n \sum_{i=1}^{m} x_i^{n+1} &= \sum_{i=1}^{m} y_i x_i^1 \\
&\vdots \\
a_0 \sum_{i=1}^{m} x_i^n + a_1 \sum_{i=1}^{m} x_i^{n+1} + a_2 \sum_{i=1}^{m} x_i^{n+2} + \cdots + a_n \sum_{i=1}^{m} x_i^{2n} &= \sum_{i=1}^{m} y_i x_i^n
\end{aligned} \tag{2}$$

From the given data, we obtain

| $i$ | 1 | 2 | 3 | 4 | 5 | 6 | $\sum_{i=1}^{6}$ |
|---|---|---|---|---|---|---|---|
| $x_i$ | 0.0 | 0.15 | 0.31 | 0.5 | 0.6 | 0.75 | 2.31 |
| $x_i^2$ | 0.0 | 0.0225 | 0.0961 | 0.25 | 0.36 | 0.5625 | 1.2911 |
| $x_i^3$ | 0.0 | 0.003375 | 0.029791 | 0.125 | 0.216 | 0.421875 | 0.796041 |
| $x_i^4$ | 0.0 | $5.0625 \times 10^{-4}$ | 0.0092351 | 0.0625 | 0.1296 | 0.31640625 | 0.5182476 |
| $x_i^5$ | 0.0 | $7.59375 \times 10^{-5}$ | 0.0028629151 | 0.03125 | 0.07776 | 0.2373046875 | 0.3492535401 |
| $x_i^6$ | 0.0 | $1.1390625 \times 10^{-5}$ | 0.00088750368 | 0.015625 | 0.046656 | 0.17797851562 | 0.24115840992 |
| $y_i$ | 1.0 | 1.004 | 1.031 | 1.117 | 1.223 | 1.422 | 6.797 |
| $y_i x_i$ | 0.0 | 0.1506 | 0.31961 | 0.5585 | 0.7338 | 1.0665 | 2.82901 |
| $y_i x_i^2$ | 0.0 | 0.02259 | 0.0990791 | 0.27925 | 0.44028 | 0.799875 | 1.6410741 |
| $y_i x_i^3$ | 0.0 | 0.0033885 | 0.030714521 | 0.139625 | 0.264168 | 0.59990625 | 1.037802271 |

For the least squares polynomials of degree 1, we know it's a linear least square fit. From equation (1) above, we can substitute the variables with the values calculated above to obtain:

$$\begin{aligned}
a_0 &= \frac{1.2911 \times 6.797 - 2.82901 \times 2.31}{6 \times 1.2911 - (2.31)^2} \\
&= 0.9294277951 \\
a_1 &= \frac{6 \times 2.82901 - 2.31 \times 6.797}{6 \times 1.2911 - (2.31)^2} \\
&= 0.5281020535
\end{aligned}$$

Thus:

$$\boxed{P_1(x) = 0.9294277951 + 0.5281020535x}$$

For the least squares polynomials of degree 2, $P_2(x) = a_0 + a_1x + a_2x^2$, we obtain:

$$a_0 \sum_{i=1}^{6} 1 + a_1 \sum_{i=1}^{6} x_i + a_2 \sum_{i=1}^{6} x_i^2 = \sum_{i=1}^{6} y_i$$
$$a_0 \sum_{i=1}^{6} x_i + a_1 \sum_{i=1}^{6} x_i^2 + a_2 \sum_{i=1}^{6} x_i^3 = \sum_{i=1}^{6} x_i y_i$$
$$a_0 \sum_{i=1}^{6} x_i^2 + a_1 \sum_{i=1}^{6} x_i^3 + a_2 \sum_{i=1}^{6} x_i^4 = \sum_{i=1}^{6} x_i^2 y_i$$

This gives

$$6a_0 + 2.31a_1 + 1.2911a_2 = 6.797$$
$$2.31a_0 + 1.2911a_1 + 0.796041a_2 = 2.82901$$
$$1.2911a_0 + 0.796041a_1 + 0.5182476a_2 = 1.6410741$$

Solving the system of equations with a calculator gives $a_0 = 1.01134$, $a_1 = -0.325704$, and $a_2 = 1.14734$. Thus:

$$\boxed{P_2(x) = 1.01134 - 0.325704x + 1.14734x^2}$$

**Note: The calculator used has a 6 digit round off error when calculating systems of equations**

For the least squares polynomials of degree 3, $P_3(x) = a_0 + a_1x + a_2x^2 + a_3x^3$, we obtain:

$$a_0 \sum_{i=1}^{6} 1 + a_1 \sum_{i=1}^{6} x_i + a_2 \sum_{i=1}^{6} x_i^2 + a_3 \sum_{i=1}^{6} x_i^3 = \sum_{i=1}^{6} y_i$$
$$a_0 \sum_{i=1}^{6} x_i + a_1 \sum_{i=1}^{6} x_i^2 + a_2 \sum_{i=1}^{6} x_i^3 + a_3 \sum_{i=1}^{6} x_i^4 = \sum_{i=1}^{6} x_i y_i$$
$$a_0 \sum_{i=1}^{6} x_i^2 + a_1 \sum_{i=1}^{6} x_i^3 + a_2 \sum_{i=1}^{6} x_i^4 + a_3 \sum_{i=1}^{6} x_i^5 = \sum_{i=1}^{6} x_i^2 y_i$$
$$a_0 \sum_{i=1}^{6} x_i^3 + a_1 \sum_{i=1}^{6} x_i^4 + a_2 \sum_{i=1}^{6} x_i^5 + a_3 \sum_{i=1}^{6} x_i^6 = \sum_{i=1}^{6} x_i^3 y_i$$

This gives

$$6a_0 + 2.31a_1 + 1.2911a_2 + 0.796041a_3 = 6.797$$
$$2.31a_0 + 1.2911a_1 + 0.796041a_2 + 0.5182476a_3 = 2.82901$$
$$1.2911a_0 + 0.796041a_1 + 0.5182476a_2 + 0.3492535401a_3 = 1.6410741$$
$$0.796041a_0 + 0.5182476a_1 + 0.3492535401a_2 + 0.24115840992a_3 = 1.037802271$$

Solving the system of equations with a calculator gives $a_0 = 1.00044$, $a_1 = -0.00152253$, $a_2 = -0.0115628$, and $a_3 = 1.02107$. Thus:

$$\boxed{P_3(x) = 1.00044 - 0.00152253x - 0.0115628x^2 + 1.02107x^3}$$

**Note: The calculator used has a 6 digit round off error when calculating systems of equations**

8.2.1e Find the linear least squares polynomial approximation to f(x) on the indicated interval if: $f(x) = \frac{1}{2}cos(x) + \frac{1}{3}sin(2x)$, $[0, 1]$ (using the monomial basis)

If $P_n(x) = \sum_{i=0}^{n} a_i x^i$ is the least square polynomial of degree $n$ of function $f(x)$ then the coefficients $a_i, i = 0, 1, \ldots, n$ are given by the normal equations

$$\sum_{k=0}^{n} a_k \int_a^b x^{j+k} dx = \int_a^b x^j f(x) dx \tag{3}$$

for each $j = 0, 1, 2, \ldots, n$. Given $f(x) = \frac{1}{2}cos(x) + \frac{1}{3}sin(2x)$, $[0, 1]$ , the coefficients of the polynomial $P_1(x) = a_0 + a_1 x$ can be found as:

$$a_0 \int_0^1 x^0 + a_1 \int_0^1 x^1 dx = \int_0^1 x^0 f(x) dx$$
$$a_0 \int_0^1 x^1 + a_1 \int_0^1 x^2 dx = \int_0^1 x^1 f(x) dx$$

These gives us

$$a_0 + \frac{1}{2}a_1 = \int_0^1 \left(\frac{1}{2}cos(x) + \frac{1}{3}sin(2x)\right) dx = -\frac{\cos(2) - 3\sin(1) - 1}{6} \approx 0.6567599651618053$$
$$\frac{1}{2}a_0 + \frac{1}{3}a_1 = \int_0^1 x \left(\frac{1}{2}cos(x) + \frac{1}{3}sin(2x)\right) dx = \frac{\sin(2) - 2\cos(2) + 6\sin(1) + 6\cos(1) - 6}{12}$$
$$\approx 0.3360192369980153$$

Solving the system of equations with a calculator gives $a_0 = 0.6109244387$, $a_1 = 0.091671053$
Thus:

$$\boxed{P_1(x) = 0.6109244387 + 0.091671053x}$$

8.2.4b Find the least squares polynomial approximation of degree 2 on the interval [-1,1] for the functions in Exercise 3. $f(x) = x^3$ (using the Legendre basis)

For any orthogonal basis, the least squares polynomial $Pn(x)$ can be found using these systems of equations:

$$\begin{bmatrix} (\phi_0, \phi_0)_w & \cdots & (\phi_0, \phi_n)_w \\ \vdots & & \\ (\phi_n, \phi_0)_w & \cdots & (\phi_n, \phi_n)_w \end{bmatrix} \begin{bmatrix} a_0 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} (f, \phi_0)_w \\ \vdots \\ (f, \phi_n)_w \end{bmatrix} \tag{4}$$

where:

$$(\phi_i, \phi_j)_w = 0, i \neq j, \quad (\phi_i, \phi_i)_w = \alpha_i$$
$$a_k = \frac{(f, \phi_k)_w}{\alpha_k}, \quad P(x) = \sum_{k=0}^{n} a_k \phi_k(x)$$

The first 6 polynomials of the Legendre basis are:

| $k$ | $\phi_k(x)$ |
|---|---|
| 0 | $1$ |
| 1 | $x$ |
| 2 | $\left(3x^2-1\right)/2$ |
| 3 | $\left(5x^3-3x\right)/2$ |
| 4 | $\left(35x^4-30x^2+3\right)/8$ |
| 5 | $\left(63x^5-70x^3+15x\right)/8$ |

The inner product $(f,\phi_k(x))_w = \int_a^b w(x)f(x)\phi_k(x)dx$ which in this instance is:

$$(f,\phi_k(x))_w = \int_{-1}^{1} x^3\phi_k(x)dx$$

since $w(x) = 1$. Thus given all this we can calculate $P_2(x)$ (Some calculation steps are omitted for typing simplicity):

$$\alpha_0 = \int_{-1}^{1} \phi_0^2 dx = \int_{-1}^{1} 1^2 dx = \int_{-1}^{1} 1dx = [x]_{-1}^{1} = 1 - -1 = 2$$

$$\alpha_1 = \int_{-1}^{1} \phi_1^2 dx = \int_{-1}^{1} x^2 dx = \left[\frac{x^3}{3}\right]_{-1}^{1} = \frac{1}{3} - -\frac{1}{3} = \frac{2}{3}$$

$$\alpha_2 = \int_{-1}^{1} \phi_2^2 dx = \int_{-1}^{1} \left(\frac{\left(3x^2-1\right)}{2}\right)^2 dx = \frac{2}{5}$$

$$(f,\phi_0(x))_w = \int_{-1}^{1} x^3\phi_0(x)dx = \int_{-1}^{1} x^3 dx = \left[\frac{x^4}{4}\right]_{-1}^{1} = 0$$

$$(f,\phi_1(x))_w = \int_{-1}^{1} x^3\phi_1(x)dx = \int_{-1}^{1} x^4 dx = \left[\frac{x^5}{5}\right]_{-1}^{1} = \frac{2}{5}$$

$$(f,\phi_2(x))_w = \int_{-1}^{1} x^3\phi_2(x)dx = \int_{-1}^{1} \frac{x^3\left(3x^2-1\right)}{2}dx = \left[\frac{2x^6-x^4}{8}\right]_{-1}^{1} = 0$$

$$a_0 = \frac{(f,\phi_0)_w}{\alpha_0} = \frac{0}{2} = 0$$

$$a_1 = \frac{(f,\phi_1)_w}{\alpha_1} = \frac{2/5}{2/3} = \frac{3}{5}$$

$$a_2 = \frac{(f,\phi_2)_w}{\alpha_2} = \frac{0}{2/5} = 0$$

$$\boxed{P_2(x) = \frac{3}{5}x}$$

8.5.5 Find the general continuous least squares trigonometric polynomial $S_n(x)$ for

$$f(x) = \begin{cases} 0, & \text{if } -\pi < x \le 0 \\ 1, & \text{if } 0 < x < \pi \end{cases}$$

The general continuous least squares trigonometric polynomial $S_n(x)$ is defined as:

$$S_n(x) = \frac{a_0}{2} + a_n\cos(nx) + \sum_{k=1}^{n-1}\left(a_k\cos(kx) + b_k\sin(kx)\right) \tag{5}$$

Continuous least squares approximation on $[-\pi, \pi]$:

$$\begin{array}{cc} a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(kx)dx & b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(kx)dx \\ k = 0\ldots n & k = 1\ldots n-1 \end{array}$$

Thus the general equation for $S_n(x)$ is:

$$\begin{aligned}
a_0 &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx \\
&= \frac{1}{2\pi}\left(\int_{-\pi}^{0} f(x)dx + \int_{0}^{\pi} f(x)dx\right) \\
&= \frac{1}{2\pi}(0+\pi) \\
&= \frac{1}{2} \\
a_k &= \frac{1}{\pi}\int_{-\pi}^{\pi} \cos(kx)f(x)dx \\
&= \frac{1}{\pi}\left(\int_{-\pi}^{0} \cos(kx)f(x)dx + \int_{0}^{\pi} \cos(kx)f(x)dx\right) \\
&= \frac{1}{\pi}\int_{0}^{\pi} \cos(kx)dx \\
&= 0 \\
b_k &= \frac{1}{\pi}\int_{-\pi}^{\pi} \sin(kx)f(x)dx \\
&= \frac{1}{\pi}\left(\int_{-\pi}^{0} \sin(kx)f(x)dx + \int_{0}^{\pi} \sin(kx)f(x)dx\right) \\
&= \frac{1}{\pi}\int_{0}^{\pi} \sin(kx)dx \\
&= \frac{1}{\pi}\left(\frac{1-(-1)^k}{k}\right)
\end{aligned}$$

The above holds since $a_k$ for $k = 1\ldots n$ always equals 0. Thus:

$$\boxed{S_n(x) = \frac{1}{4} + \frac{1}{\pi}\sum_{k=1}^{n-1}\left(\frac{1-(-1)^k}{k}\right)\sin(kx)}$$

8.5.7b Determine the discrete least squares trigonometric polynomial $S_n(x)$ on the interval $[-\pi, \pi]$ for the following functions, using the given values of m and n: $f(x) = cos(3x)$, $m = 4$, $n = 2$.

The general continuous least squares trigonometric polynomial $S_n(x)$ is defined by equation (5). The discrete least squares approximation for $2m$ equally-spaced $x_i$ on $[-\pi, \pi]$ for $a_k$ and $b_k$ is defined as:

$$\begin{array}{cc} a_k = \frac{1}{m}\sum_{i=0}^{2m-1} f(x_i)\cos(kx_i) & b_k = \frac{1}{m}\sum_{i=0}^{2m-1} f(x_i)\sin(kx_i) \\ k = 0\ldots n & k = 1\ldots n-1 \end{array} \tag{6}$$

For this problem, since $m = 4$, $x_{i+1} - x_i = \frac{\pi}{4}$ for $i = 0\ldots 2m-1$. Thus the general equation for $S_n(x)$ is:

$$\begin{aligned}
a_0 =& \frac{1}{4}\sum_{j=0}^{7} f\left(x_j\right) \\
=& \frac{1}{4}\left[\cos(-3\pi) + \cos\left(-\frac{9}{4}\pi\right) + \cos\left(-\frac{3}{2}\pi\right) + \cos\left(-\frac{3}{4}\pi\right)\right. \\
& \left. + \cos(0) + \cos\left(\frac{3}{4}\pi\right) + \cos\left(\frac{3}{2}\pi\right) + \cos\left(\frac{9}{4}\pi\right)\right] \\
=& \frac{1}{4}\left[-1 + \frac{\sqrt{2}}{2} + 0 - \frac{\sqrt{2}}{2} + 1 - \frac{\sqrt{2}}{2} + 0 + \frac{\sqrt{2}}{2}\right] = 0 \\
a_1 =& \frac{1}{4}\sum_{j=0}^{7} f\left(x_j\right)\cos\left(x_j\right) \\
=& \frac{1}{4}\sum_{j=0}^{7} \cos\left(3x_j\right)\cos\left(x_j\right) \\
=& \frac{1}{4}\left[\cos(-3\pi)\cos(-\pi) + \cos\left(-\frac{9}{4}\pi\right)\cos\left(-\frac{3}{4}\pi\right) + \cos\left(-\frac{3}{2}\pi\right)\cos\left(-\frac{1}{2}\pi\right) + \cos\left(-\frac{3}{4}\pi\right)\cos\left(-\frac{1}{4}\pi\right)\right. \\
& \left. + \cos(0)\cos(0) + \cos\left(\frac{3}{4}\pi\right)\cos\left(\frac{1}{4}\pi\right) + \cos\left(\frac{3}{2}\pi\right)\cos\left(\frac{1}{2}\pi\right) + \cos\left(\frac{9}{4}\pi\right)\cos\left(\frac{3}{4}\pi\right)\right] \\
=& \frac{1}{4}\left[1 - \frac{1}{2} + 0 - \frac{1}{2} + 1 - \frac{1}{2} + 0 - \frac{1}{2}\right] = 0 \\
a_2 =& \frac{1}{4}\sum_{j=0}^{7} f\left(x_j\right)\cos\left(2x_j\right) \\
=& \frac{1}{4}\sum_{j=0}^{7} \cos\left(3x_j\right)\cos\left(2x_j\right) \\
=& \frac{1}{4}\left[\cos(-3\pi)\cos(-2\pi) + \cos\left(-\frac{9}{4}\pi\right)\cos\left(-\frac{3}{2}\pi\right) + \cos\left(-\frac{3}{2}\pi\right)\cos\left(-\pi\right) + \cos\left(-\frac{3}{4}\pi\right)\cos\left(-\frac{1}{2}\pi\right)\right. \\
& \left. + \cos(0)\cos(0) + \cos\left(\frac{3}{4}\pi\right)\cos\left(\frac{1}{2}\pi\right) + \cos\left(\frac{3}{2}\pi\right)\cos\left(\pi\right) + \cos\left(\frac{9}{4}\pi\right)\cos\left(\frac{3}{2}\pi\right)\right] \\
=& \frac{1}{4}\left[-1 + 0 + 0 + 0 + 1 + 0 + 0 + 0\right] = 0 \\
b_1 =& \frac{1}{4}\sum_{j=0}^{7} f\left(x_j\right)\sin\left(x_j\right) \\
=& \frac{1}{4}\sum_{j=0}^{7} \cos\left(3x_j\right)\sin\left(x_j\right) \\
=& \frac{1}{4}\left[\cos(-3\pi)\sin(-\pi) + \cos\left(-\frac{9}{4}\pi\right)\sin\left(-\frac{3}{4}\pi\right) + \cos\left(-\frac{3}{2}\pi\right)\sin\left(-\frac{1}{2}\pi\right) + \cos\left(-\frac{3}{4}\pi\right)\sin\left(-\frac{1}{4}\pi\right)\right. \\
& \left. + \cos(0)\sin(0) + \cos\left(\frac{3}{4}\pi\right)\sin\left(\frac{1}{4}\pi\right) + \cos\left(\frac{3}{2}\pi\right)\sin\left(\frac{1}{2}\pi\right) + \cos\left(\frac{9}{4}\pi\right)\sin\left(\frac{3}{4}\pi\right)\right] \\
=& \frac{1}{4}\left[0 - \frac{1}{2} + 0 + \frac{1}{2} + 0 - \frac{1}{2} + 0 + \frac{1}{2}\right] = 0
\end{aligned}$$

Since all the values above are 0, we can conclude that:

$$\boxed{S_2(x) = 0}$$