AMSC 460 Homework 3 Part 1

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3.3.1 Use Eq. (3.10) or Algorithm 3.2 to construct interpolating polynomials of degree one, two, and three for the following data. Approximate the specified value using each of the polynomials.

a. $f(8.4)$ if $f(8.1) = 16.94410$, $f(8.3) = 17.56492$, $f(8.6) = 18.50515$, $f(8.7) = 18.82091$

First divided differences:

$$f(x_0, x_1) = f(8.1, 8.3) = \frac{f(8.3) - f(8.1)}{8.3 - 8.1} = \frac{17.56492 - 16.94410}{0.2} = \frac{0.62082}{0.2} = 3.1041$$

$$f(x_1, x_2) = f(8.3, 8.6) = \frac{f(8.6) - f(8.3)}{8.6 - 8.3} = \frac{18.50515 - 17.56492}{0.3} = \frac{0.94023}{0.3} = 3.1341$$

$$f(x_2, x_3) = f(8.6, 8.7) = \frac{f(8.7) - f(8.6)}{8.7 - 8.6} = \frac{18.82091 - 18.50515}{0.1} = \frac{0.31576}{0.1} = 3.1576$$

Second divided differences:

$$f(x_0, x_1, x_2) = \frac{3.1341 - 3.1041}{8.6 - 8.1} = \frac{0.03}{0.5} = 0.06$$

$$f(x_1, x_2, x_3) = \frac{3.1576 - 3.1341}{8.7 - 8.3} = \frac{0.0235}{0.4} = 0.05875$$

Third divided differences:

$$f(x_0, x_1, x_2, x_3) = \frac{0.05875 - 0.06}{8.7 - 8.1} = \frac{-0.00125}{0.6} = -0.00208333333 = \frac{-1}{480}$$

We know the equation 3.10 is:

$$P_n(x) = f\left[x_0\right] + \sum_{k=1}^{n} f\left[x_0, x_1, \ldots, x_k\right](x - x_0)\cdots(x - x_{k-1}) \tag{1}$$

Thus if we substitute x and n with values, we get:
x = 8.4, n = 3:

$$\begin{aligned} P_3(x) &= f\left[x_0\right] + \sum_{k=1}^{3} f\left[x_0, x_1, \ldots, x_k\right](x - x_0)\cdots(x - x_{k-1}) \\ &= f\left[x_0\right] + f\left[x_0, x_1\right](x - x_0) + f\left[x_0, x_1, x_2\right](x - x_0)(x - x_1) + \\ &f\left[x_0, x_1, x_2, x_3\right](x - x_0)(x - x_1)(x - x_2) \\ &= 16.94410 + 3.1041(x - 8.1) + 0.06(x - 8.1)(x - 8.3) \\ &- 0.00208333333(x - 8.1)(x - 8.3)(x - 8.6) \\ &= \boxed{17.2563225} \end{aligned} \tag{2}$$

x = 8.4, n = 2:

$$\begin{aligned}
P_2(x) &= f[x_0] + \sum_{k=1}^{2} f[x_0, x_1, \ldots, x_k](x - x_0)\cdots(x - x_{k-1}) \\
&= f[x_0] + f[x_0, x_1](x - x_0) + f[x_0, x_1, x_2](x - x_0)(x - x_1) \\
&= 16.94410 + 3.1041(x - 8.1) + 0.06(x - 8.1)(x - 8.3) \\
&= \boxed{17.25631}
\end{aligned} \tag{3}$$

x = 8.4, n = 1:

$$\begin{aligned}
P_1(x) &= f[x_0] + \sum_{k=1}^{1} f[x_0, x_1, \ldots, x_k](x - x_0)\cdots(x - x_{k-1}) \\
&= f[x_0] + f[x_0, x_1](x - x_0) \\
&= 16.94410 + 3.1041(x - 8.1) \\
&= \boxed{17.25451}
\end{aligned} \tag{4}$$

3.3.8a Use Algorithm 3.2 to construct the interpolating polynomial of degree four for the unequally spaced points given in the following table:

x	$f(x)$
0.0	-6.00000
0.1	-5.89483
0.3	-5.65014
0.6	-5.17788
1.0	-4.28172

First divided differences:

$$f(x_0, x_1) = f(0, 0.1) = \frac{f(0.1) - f(0)}{0.1 - 0} = \frac{-5.89483 - -6.00000}{0.1} = \frac{0.10517}{0.1} = 1.0517$$

$$f(x_1, x_2) = f(0.1, 0.3) = \frac{f(0.3) - f(0.1)}{0.3 - 0.1} = \frac{-5.65014 - -5.89483}{0.2} = \frac{0.24469}{0.2} = 1.22345$$

$$f(x_2, x_3) = f(0.3, 0.6) = \frac{f(0.6) - f(0.3)}{0.6 - 0.3} = \frac{-5.17788 - -5.65014}{0.3} = \frac{0.47226}{0.3} = 1.5742$$

$$f(x_3, x_4) = f(0.6, 1) = \frac{f(1) - f(0.6)}{1 - 0.6} = \frac{-4.28172 - -5.17788}{0.4} = \frac{0.89616}{0.4} = 2.2404$$

Second divided differences:

$$f(x_0, x_1, x_2) = \frac{1.22345 - 1.0517}{0.3 - 0} = \frac{0.17175}{0.3} = 0.5725$$

$$f(x_1, x_2, x_3) = \frac{1.5742 - 1.22345}{0.6 - 0.1} = \frac{0.35075}{0.4} = 0.876875$$

$$f(x_2, x_3, x_4) = \frac{2.2404 - 1.5742}{1 - 0.3} = \frac{0.6662}{0.7} = 0.95171428571$$

Third divided differences:

$$f(x_0, x_1, x_2, x_3) = \frac{0.876875 - 0.5725}{0.6 - 0} = \frac{0.304375}{0.6} = 0.50729166666$$

$$f(x_1, x_2, x_3, x_4) = \frac{0.95171428571 - 0.876875}{1 - 0.1} = \frac{0.07483928571}{0.9} = 0.0831547619$$

4th divided differences:

$$f(x_0, x_1, x_2, x_3, x_4) = \frac{0.0831547619 - 0.50729166666}{1 - 0} = \frac{-0.42413690476}{1} = -0.42413690476$$

Substituting the values found into equation 3.10 where n = 4, we get:

$$\begin{aligned} P_4(x) &= f[x_0] + \sum_{k=1}^{4} f[x_0, x_1, \ldots, x_k](x - x_0) \cdots (x - x_{k-1}) \\ &= f[x_0] + f[x_0, x_1](x - x_0) + f[x_0, x_1, x_2](x - x_0)(x - x_1) + \\ &\quad f[x_0, x_1, x_2, x_3](x - x_0)(x - x_1)(x - x_2) + \\ &\quad f[x_0, x_1, x_2, x_3, x_4](x - x_0)(x - x_1)(x - x_2)(x - x_3) \end{aligned} \tag{5}$$

Which is essentially:

$$\boxed{-6.00000 + 1.0517(x)(x - 0.1) + 0.5725(x)(x - 0.1)(x - 0.3) + 0.50729166666(x)(x - 0.1)(x - 0.3)(x - 0.6)} \tag{6}$$

3.1.1 For the given functions $f(a)$, let $x_0 = 0$, $x_1 = 0.6$, and $x_2 = 0.9$. Construct interpolation polynomials of degree at most one and at most two to approximate $f(0.45)$ and find the absolute error.

b. $f(x) = \sqrt{1 + x}$

The linear Langrange interpolating polynomial through $(x_0, f(x_0))$ and $(x_1, f(x_1))$ is given as

$$P_1(x) = L_0(x) f(x_0) + L_1(x) f(x_1) = \frac{x - x_1}{x_0 - x_1} f(x_0) + \frac{x - x_0}{x_1 - x_0} f(x_1)$$

At nodes $x_0 = 0$ and $x_1 = 0.6$, the corresponding function values are

$$f(x_0) = \sqrt{1 + 0} = 1 \quad \text{and} \quad f(x_1) = \sqrt{1 + 0.6} = 1.264911$$

Therefore, the polynomial is determined as

$$\begin{aligned} P_1(x) &= \frac{x - 0.6}{0 - 0.6} f(0) + \frac{x - 0}{0.6 - 0} f(0.6) \\ &= -\frac{1}{0.6}(x - 0.6) + \frac{1.264911}{0.6} \cdot x \\ &= \boxed{1 + 0.441518x} \end{aligned}$$

The approximation of $f(0.45) = \sqrt{1 + 0.45} = 1.204159$ would be

$$P_1(0.45) = 1 + 0.441518 \cdot 0.45 = \boxed{1.198683}$$

So the absolute error is

$$\varepsilon = |1.204159 - 1.198683| = \boxed{0.005476}$$

In order to find the quadratic interpolating polynomial, we need all three nodes and the corresponding values of f,

$$f(x_0) = 1, \quad f(x_1) = 1.264911, \quad f(x_2) = \sqrt{1 + 0.9} = 1.378405$$

The polynomial is

$$P_2(x) = L_0(x) f(x_0) + L_1(x) f(x_1) + L_2(x) f(x_2)$$

where $L_0(x), L_1(x)$ and $L_2(x)$ are the same as in (a), as they only depend on the nodes and not the function we are approximating. Hence,

$$\begin{aligned} P_2(x) = & \left(1.851852x^2 - 2.777778x + 1\right) f(0) \\ & + \left(-5.555556x^2 + 5x\right) f(0.6) \\ & + \left(3.703704x^2 - 2.222222x\right) f(0.9) \\ = & \boxed{-0.0702278x^2 + 0.483655x + 1} \end{aligned}$$

and the approximate value of $f(0.45)$ is

$$P_2(0.45) = -0.0702278 \cdot 0.45^2 + 0.483655 \cdot 0.45 + 1 = \boxed{1.203424}$$

The absolute error is

$$\varepsilon = |1.204159 - 1.203424| = \boxed{0.000735}$$

3.1.13 Construct the Lagrange interpolating polynomials for the following functions and find a bound for the absolute error on the interval $[x_0, x_n]$.

a. $f(x) = e^{2x} \cos 3x, \quad x_0 = 0, x_1 = 0.3, x_2 = 0.6, n = 2$

The Lagrange interpolating polynomial is given as

$$P(x) = f(x_0) L_{n,0}(x) + \cdots + f(x_n) L_{n,n}(x) = \sum_{k=0}^{n} f(x_k) L_{n,k}(x) \tag{7}$$

where the polynomials $L_k(x)$ are:

$$\begin{aligned} L_{n,k}(x) &= \frac{(x - x_0)(x - x_1) \cdots (x - x_{k-1})(x - x_{k+1}) \cdots (x - x_n)}{(x_k - x_0)(x_k - x_1) \cdots (x_k - x_{k-1})(x_k - x_{k+1}) \cdots (x_k - x_n)} \\ &= \prod_{\substack{i=0 \\ i \neq k}}^{n} \frac{(x - x_i)}{(x_k - x_i)} \end{aligned} \tag{8}$$

For $k = 0, 1, 2$, the $L_k(x)$s are:

$$\begin{aligned} L_0(x) &= \frac{(x - x_1)(x - x_2)}{(x_0 - x_1)(x_0 - x_2)} \\ L_1(x) &= \frac{(x - x_0)(x - x_2)}{(x_1 - x_0)(x_1 - x_2)} \\ L_2(x) &= \frac{(x - x_0)(x - x_1)}{(x_2 - x_0)(x_2 - x_1)} \end{aligned}$$

The nodes are $x_0 = 0, x_1 = 0.3$ and $x_2 = 0.6$. Substitute into the expressions for L_k above to obtain

$$\begin{aligned} L_0(x) &= \frac{(x - 0.3)(x - 0.6)}{(0 - 0.3)(0 - 0.6)} = \frac{(x - 0.3)(x - 0.6)}{0.18} = \frac{50}{9}x^2 - 5x + 1 \\ L_1(x) &= \frac{(x - 0)(x - 0.6)}{(0.3 - 0)(0.3 - 0.6)} = -\frac{x(x - 0.6)}{0.09} = -\frac{100}{9}x^2 + \frac{20}{3}x \\ L_2(x) &= \frac{(x - 0)(x - 0.3)}{(0.6 - 0)(0.6 - 0.3)} = \frac{x(x - 0.3)}{0.18} = \frac{50}{9}x^2 - \frac{5}{3}x \end{aligned}$$

$f(x_k)$ for $k = 0, 1, 2$ is:

$$\begin{aligned}
f(x_0) &= f(0) = e^{2\cdot 0}\cos(3\cdot 0) = 1 \\
f(x_1) &= f(0.3) = e^{2\cdot 0.3}\cos(3\cdot 0.3) = 1.13264721 \\
f(x_2) &= f(0.6) = e^{2\cdot 0.6}\cos(3\cdot 0.6) = -0.75433752
\end{aligned}$$

Thus:

$$\begin{aligned}
P_2(x) =& L_0(x)f(x_0) + L_1(x)f(x_1) + L_2(x)f(x_2) \\
=& \left(\frac{50}{9}x^2 - 5x + 1\right)\cdot f(0) + \left(-\frac{100}{9}x^2 + \frac{20}{3}x\right)\cdot f(0.3) + \\
& + \left(\frac{50}{9}x^2 - \frac{5}{3}x\right)\cdot f(0.6) \\
=& \boxed{-11.220177x^2 + 3.808211x + 1}
\end{aligned}$$

The absolute error is:

$$|f(x) - P_n(x)| = \left|\frac{f^{(n+1)}(\xi(x))}{(n+1)!}\prod_{k=0}^{n}(x - x_k)\right| \tag{9}$$

where $\xi(x) \in [0, 0.6]$. In our case, $n = 2$, so the absolute error is

$$\begin{aligned}
|f(x) - P_2(x)| &= \left|\frac{f^{(3)}(\xi(x))}{3!}(x - x_0)(x - x_1)(x - x_2)\right| \\
&= \left|\frac{f^{(3)}(\xi(x))}{6}(x - 0)(x - 0.3)(x - 0.6)\right| \\
&= \left|\frac{f^{(3)}(\xi(x))}{6}(x)(x - 0.3)(x - 0.6)\right|
\end{aligned}$$

Let's define:

$$p(x) = (x)(x - 0.3)(x - 0.6)$$

To find the max of this error, we need to find the greatest absolute value for both $f^{(3)}(\xi(x))$ and $p(x)$.

To find the greatest absolute value for $p(x)$, we find the 2nd derivative and find the value where it equals 0 in the range $x \in [0, 6]$:

$$\begin{aligned}
p(x) &= x(x - 0.3)(x - 0.6) = x^3 - 0.9x^2 + 0.18x \\
p'(x) &= 3x^2 - 1.8x + 0.18 \\
p'(x) &= 0 \Leftrightarrow x_{1,2} = \frac{1.8 \pm \sqrt{1.8^2 - 4\cdot 3\cdot 0.18}}{2\cdot 3}
\end{aligned}$$

Which is approximately $(0.1267949192, 0.4732050808)$. $p(0.1267949192) = 0.01039$ and $p(0.4732050808) = -0.01039$ which implies $\max|p(x)| = 0.01039$ for $x \in [0, 6]$

To find the greatest absolute value for $f^{(3)}(\xi(x))$, we find the 4th derivative of $f(x)$ and find the value where it equals 0 for $x \in [0,6]$:

$$\begin{aligned}
f(x) &= e^{2x}\cos 3x \\
f'(x) &= 2e^{2x}\cos 3x - e^{2x}\cdot \sin 3x \cdot 3 \\
&= e^{2x}(2\cos 3x - 3\sin 3x) \\
f''(x) &= 2e^{2x}(2\cos 3x - 3\sin 3x) + e^{2x}(-6\sin 3x - 9\cos 3x) \\
&= e^{2x}(-5\cos 3x - 12\sin 3x) \\
f'''(x) &= 2e^{2x}(-5\cos 3x - 12\sin 3x) + e^{2x}(15\sin 3x - 36\cos 3x) \\
&= e^{2x}(-46\cos 3x - 9\sin 3x) \\
f^4(x) &= \mathrm{e}^{2x}\left(138\sin(3x) - 27\cos(3x)\right) + 2\mathrm{e}^{2x}\left(-9\sin(3x) - 46\cos(3x)\right) \\
&= \mathrm{e}^{2x}\left(120\sin(3x) - 119\cos(3x)\right)
\end{aligned}$$

Upon plotting in a calculator, we see that $x \approx 0.2451298$. Where if we substitute in for x in $f'''(x)$, we get -65.55464914. Thus applying the Cauchy-Schwartz Inequality we get:

$$\begin{aligned}
|f(x) - P_2(x)| &= \left|\frac{f^{(3)}(\xi(x))}{6}x(x-0.3)(x-0.6)\right| \\
&= \left|\frac{f^{(3)}(\xi(x))}{6}p(x)\right| \\
&\leq \left|\frac{-65.55464914}{6}\right|\cdot|0.01039| \\
&= \boxed{0.1135188008}
\end{aligned}$$

3.4.5a Use the following values and five-digit rounding arithmetic to construct the Hermite interpolating polynomial to approximate $\sin 0.34$.

x	$\sin x$	$D_x \sin x = \cos x$
0.30	0.29552	0.95534
0.32	0.31457	0.94924
0.35	0.34290	0.93937

Note: Not all computations are shown to save time from typing, but for every computation, the round off error is applied and if it appears not, it is purely a computation error.

We first compute the Lagrange polynomials and their derivatives. This gives:

$$\begin{aligned}
L_{2,0}(x) &= \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} = 1000x^2 - 670x + 112, \quad L'_{2,0}(x) = 2000x - 670 \\
L_{2,1}(x) &= \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} = -1666.6x^2 + 1083.3x - 175, \quad L'_{2,1}(x) = -3333.2x + 1083.3 \\
L_{2,2}(x) &= \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} = 666.66x^2 - 413.33x + 64, \quad L'_{2,2}(x) = 1333.3x - 413.33
\end{aligned}$$

We know that:

$$H_{n,j}(x) = \left[1 - 2(x-x_j)L'_{n,j}(x_j)\right]L^2_{n,j}(x) \quad \text{and} \quad \hat{H}_{n,j}(x) = (x-x_j)L^2_{n,j}(x) \tag{10}$$

Thus the polynomials $H_{2,j}(x)$ and $\hat{H}_{2,j}(x)$ are then:

$$\begin{aligned}
H_{2,0}(x) &= [1-2(x-0.3)L'_{2,0}(0.3)]L^2_{2,0}(x) \\
&= [1-2(x-0.3)L'_{2,0}(0.3)]\left(1000x^2-670x+112\right)^2 \\
&= [1-2(x-0.3)(-70)]\left(1000x^2-670x+112\right)^2 \\
&= (140x-41)\left(1000x^2-670x+112\right)^2
\end{aligned}$$

$$\begin{aligned}
H_{2,1}(x) &= [1-2(x-0.32)L'_{2,1}(0.32)]L^2_{2,1}(x) \\
&= [1-2(x-0.32)L'_{2,1}(0.32)]\left(-1666.6x^2+1083.3x-175\right)^2 \\
&= [1-2(x-0.32)(16.7)]\left(-1666.6x^2+1083.3x-175\right)^2 \\
&= (-33.4x+11.688)\left(-1666.6x^2+1083.3x-175\right)^2
\end{aligned}$$

$$\begin{aligned}
H_{2,2}(x) &= [1-2(x-0.35)L'_{2,2}(0.35)]L^2_{2,2}(x) \\
&= [1-2(x-0.35)L'_{2,2}(0.35)]\left(666.66x^2-413.33x+64\right)^2 \\
&= [1-2(x-0.35)(53.27)]\left(666.66x^2-413.33x+64\right)^2 \\
&= (-106.54x+38.289)\left(666.66x^2-413.33x+64\right)^2
\end{aligned}$$

$$\begin{aligned}
\hat{H}_{2,0}(x) &= (x-0.3)\left(1000x^2-670x+112\right)^2 \\
\hat{H}_{2,1}(x) &= (x-0.32)\left(-1666.6x^2+1083.3x-175\right)^2 \\
\hat{H}_{2,2}(x) &= (x-0.35)\left(666.66x^2-413.33x+64\right)^2
\end{aligned}$$

The Hermite interpolating polynomial is:

$$H_{2n+1}(x) = \sum_{j=0}^{n} f\left(x_j\right) H_{n,j}(x) + \sum_{j=0}^{n} f'\left(x_j\right) \hat{H}_{n,j}(x) \tag{11}$$

Thus: