AMSC 460 Homework 1 Part 1

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1.2.19 Use the 64-bit-long real format to find the decimal equivalent of the following floating-point machine numbers.

To convert 64-bit-long real format to decimal equivalent, we use $(-1)^s 2^{(c-1023)}(1+f)$

(a) 0 10000001010 1001001100000000000000000000000000000000000000000000

$s = 0, c = 2^1 + 2^3 + 2^{10} = 2 + 8 + 1024 = 1034, f = (\frac{1}{2})^1 + (\frac{1}{2})^4 + (\frac{1}{2})^7 + (\frac{1}{2})^8 = \frac{147}{256} =$ 0.57421875. Hence the answer is $(-1)^s 2^{(c-1023)}(1+f) = \boxed{3224}$

(b) 1 10000001010 1001001100000000000000000000000000000000000000000000

Same as part a except the answer is negative $\implies \boxed{-3224}$

(c) 0 01111111111 0101001100000000000000000000000000000000000000000000

$s = 0, c = 2^0 + 2^1 + 2^2 + 2^3 + 2^4 + 2^5 + 2^6 + 2^7 + 2^8 + 2^9 = 1 + 2 + 4 + 8 + 16 + 32 + 64 + 128 + 256 + 512 = 1023, f = (\frac{1}{2})^2 + (\frac{1}{2})^4 + (\frac{1}{2})^7 + (\frac{1}{2})^8 = 83/256 = 0.32421875.$

Hence the answer is $(-1)^s 2^{(c-1023)}(1+f) = \boxed{1.32421875}$

(d) 0 01111111111 0101001100000000000000000000000000000000000000000001

$s = 0, c = 2^0 + 2^1 + 2^2 + 2^3 + 2^4 + 2^5 + 2^6 + 2^7 + 2^8 + 2^9 = 1 + 2 + 4 + 8 + 16 + 32 + 64 + 128 + 256 + 512 = 1023, f = (\frac{1}{2})^2 + (\frac{1}{2})^4 + (\frac{1}{2})^7 + (\frac{1}{2})^8 + (\frac{1}{2})^{52} =$ 0.3242187500000002220446049250313080847263336181640625.

Hence the answer is $(-1)^s 2^{(c-1023)}(1+f) =$

$\boxed{1.3242187500000002220446049250313080847263336181640625}$

1.2.21 Suppose two points (x_0, y_0) and (x_1, y_1) are on a straight line with $y_1 \neq y_0$. Two formulas are available to find the x-intercept of the line: $x = \frac{x_0 y_1 - x_1 y_0}{y_1 - y_0}$ and $x = x_0 - \frac{(x_1 - x_0)y_0}{y_1 - y_0}$

(a) Show that both formulas are algebraically correct.

The equation of the line that passes through the 2 points is $\frac{y - y_0}{x - x_0} = \frac{y_1 - y_0}{x_1 - x_0}$.

To isolate x we do $(x - x_0)(y - y_0) = (x_1 - x_0)(y - y_0) \implies (x - x_0) = \frac{(y - y_0)(x_1 - x_0)}{y - y_0} \implies x = \frac{(y - y_0)(x_1 - x_0)}{y - y_0} + x_0 \implies x = x_0 + \frac{(y - y_0)(x_1 - x_0)}{y - y_0}$.

Since the x-intercept is at $y = 0$ we can substitute y and get $x = x_0 + \frac{(0 - y_0)(x_1 - x_0)}{y - y_0} \implies$

$$\boxed{x = x_0 - \frac{(x_1 - x_0)y_0}{y - y_0}}$$

We can get the other result with some algebra: $x = x_0 - \frac{(x_1 - x_0)y_0}{y - y_0} \iff x = x_0 - \frac{x_1 y_0 - x_0 y_0}{y_1 - y_0} \iff x = \frac{x_0 y_1 - x_0 y_0 - x_1 y_0 + x_0 y_0}{y_1 - y_0} \iff \boxed{x = \frac{x_0 y_1 - x_1 y_0}{y_1 - y_0}}$

(b) Use the data $(x_0, y_0) = (1.31, 3.24)$ and $(x_1, y_1) = (1.93, 4.76)$ and three-digit rounding arithmetic to compute the x-intercept both ways. Which method is better, and why?

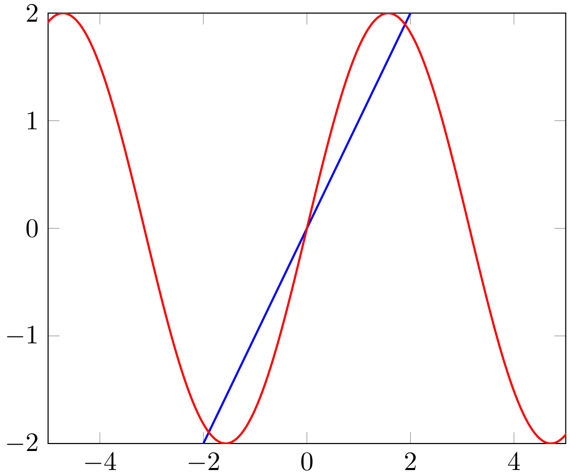
$x = \boxed{-0.01158}$

$$x = \frac{x_0 y_1 - x_1 y_0}{y_1 - y_0} = \frac{1.31(4.76) - 1.93(3.24)}{4.76 - 3.24} = \frac{6.24 - 6.25}{1.52} = \frac{-0.01}{1.52} = \boxed{-0.00658}$$

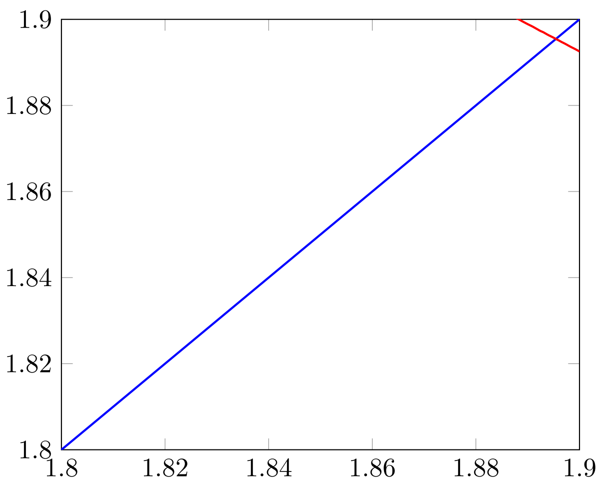
$$x = x_0 - \frac{(x_1 - x_0)y_0}{y - y_0} = 1.31 - \frac{(1.93 - 1.31)3.24}{4.76 - 3.24} = 1.31 - \frac{(0.62)3.24}{4.76 - 3.24} = 1.31 - \frac{2.01}{1.52} = 1.31 - 1.32 = \boxed{-0.0100}$$

Using $\boxed{x = x_0 - \frac{(x_1 - x_0)y_0}{y - y_0}}$ is better since it produces a more accurate result.

2.1.7 (a) Sketch the graphs $y = x$ and $y = 2sin(x)$



(b) Use the Bisection method to find an approximation to within 10^{-5} to the first positive value of x with $x = 2sin(x)$.



Based on this zoomed in graph, it appears that the positive intersection of the 2 lines lies on the interval [1.88,1.90]. Let n be the number of iterations needed, then $\frac{b-a}{2^n} < 10^{-5} \implies 2^n \geq (b-a) * 10^5 \implies n \geq log_2[(b-a) * 10^5] \implies n \geq log_2(0.02 * 10^5) \approx 10.96578428 \implies n = 11.$

n	a_n	b_n	p_n	$f(p_n)$
1	1.88	1.90	1.89	0.0089712297
2	1.89	1.90	1.895	0.0008904001202
3	1.895	1.90	1.8975	-0.0032892928
4	1.895	1.8975	1.89625	-0.0012384659
5	1.895	1.895625	1.8953125	0.0002977112765
6	1.8953125	1.895625	1.89546875	0.00004179743
7	1.89546875	1.895625	1.895546875	-0.00008617684
8	1.89546875	1.895546875	1.8955078125	-0.00002218825
9	1.89546875	1.8955078125	1.89548828125	0.00000980494
10	1.89548828125	1.8955078125	1.89549804688	-0.00000619157
11	1.89548828125	1.89549804688	1.89549316406	0.00000180671

Table 1: x = 2sin(x)

2.1.17 Use Theorem 2.1 to find a bound for the number of iterations needed to achieve an approximation with accuracy 10^{-4} to the solution of $x^3 - x - 1 = 0$ lying in the interval [1,2]. Find an approximation to the root with this degree of accuracy.
Let n be the number of iterations needed, then $\frac{b-a}{2^n} < 10^{-4} \implies 2^n \geq (b-a)*10^4 \implies n \geq log_2[(b-a)*10^4] \implies n \geq log_2((2-1)*10^4) \implies n \geq log_2(10^4) \approx 13.28771238 \implies n = 14$

n	a_n	b_n	p_n	$f(p_n)$
1	1	2	1.5	0.875
2	1	1.5	1.25	-0.296875
3	1.25	1.5	1.375	0.224609375
4	1.25	1.375	1.3125	-0.051513672
5	1.3125	1.375	1.34375	0.082611084
6	1.3125	1.34375	1.328125	0.014575958
7	1.3125	1.328125	1.3203125	-0.018710613
8	1.3203125	1.328125	1.32421875	-0.002127945
9	1.32421875	1.328125	1.326171875	0.00620883
10	1.32421875	1.326171875	1.325195313	0.002036651
11	1.32421875	1.325195313	1.324707031	-0.0000465949
12	1.324707031	1.325195313	1.324951172	0.000994791
13	1.324707031	1.324951172	1.324829102	0.000474039
14	1.324707031	1.324829102	1.324768066	0.000213707

Table 2: $x^3 - x - 1 = 0$

2.2.3 Let $f(x) = x^3 - 2x + 1$. To solve f(x) = 0, the following four fixed-point problems are proposed.Derive each fixed point method and compute p_1, p_2, p_3, p_4. Which methods seem to be appropriate?

Let $p_{i+1} = g(p_i)$ where g(x) = x

(a) $x = \frac{1}{2}(x^3 + 1), p_0 = \frac{1}{2}$

i	p_i	$p_{i+1} = g(p_i)$
0	0.5	0.5625
1	0.5625	0.5889892578
2	0.5889892578	0.6021626446
3	0.6021626446	0.6091720425
4	0.6091720425	0.6130290025

(b) $x = \frac{2}{x} - \frac{1}{x^2}, p_0 = \frac{1}{2}$

i	p_i	$p_{i+1} = g(p_i)$
0	0.5	0
1	0	undefined
2	undefined	undefined
3	undefined	undefined
4	undefined	undefined

(c) $x = \sqrt{2 - \frac{1}{x}}, p_0 = \frac{1}{2}$

i	p_i	$p_{i+1} = g(p_i)$
0	0.5	0
1	0	undefined
2	undefined	undefined
3	undefined	undefined
4	undefined	undefined

(d) $x = -\sqrt[3]{1 - 2x}, p_0 = \frac{1}{2}$

i	p_i	$p_{i+1} = g(p_i)$
0	0.5	0
1	0	-1
2	-1	$-\sqrt[3]{3}$
3	$-\sqrt[3]{3}$	-1.571972738
4	-1.571972738	-1.606218699

Methods a and d seem appropriate as they do not create undefined values and do appear to converge to a fixed point.

2.2.10 Use Theorem 2.3 to show that $g(x) = 2^{-x}$ has a unique fixed point on $[\frac{1}{3}, 1]$. Use fixed-point iteration to find an approximation to the fixed point accurate to within 10^{-4}. Use Corollary 2.5 to estimate the number of iterations required to achieve 10^{-4} accuracy and compare this theoretical estimate to the number actually needed.

(i) To prove existence we know that $g(x) = 2^{-x}$ is a continuous decreasing function. $g(\frac{1}{3}) = 2^{-\frac{1}{3}} \approx 0.79370052598$ and $g(1) = 2^{-1} = 0.5$. Thus $g(\frac{1}{3}) > \frac{1}{3}$ and $g(1) < 1$. The function $h(x) = g(x) - x$ is continuous on $[\frac{1}{3}, 1]$, with $h(\frac{1}{3}) = g(\frac{1}{3}) - \frac{1}{3} \approx 0.46036719265 > 0$ and $h(1) = g(1) - 1 = -0.5 < 0$. The Intermediate Value Theorem implies that $\exists p \in (\frac{1}{3}, 1)$ for which $h(p) = 0$. This number p is a fixed point for g because $0 = h(p) = g(p) - p \implies g(p) = p$. Hence g(x) has at least one fixed point in $[\frac{1}{3}, 1]$.

(ii) To prove uniqueness, we must show that $|g'(x)| \leq k < 1$ on the given interval. The reasoning for why follows from the proof for part ii of Theorem 2.3. $g'(x) = -\ln(2) * 2^{-x}$ which is a continuous decreasing function which implies it's maximum and minimum values will be at $\frac{1}{3}$ and 1 respectfully on the given interval. Thus $g'(\frac{1}{3}) = -\ln(2) * 2^{-\frac{1}{3}} \approx -0.55015128179 > -1$ and $g'(1) = -\ln(2) * 2^{-1} \approx -0.34657359028 < 1$. We can set $k = |g'(\frac{1}{3})| < 1$ and thus $|g'(x)| \leq k < 1$ on $[\frac{1}{3}, 1]$. Thus $g(x)$ has a unique fixed point on the interval $[\frac{1}{3}, 1]$.

(iii) To find the bounds we use the equation $|p_n - p| \leq \frac{k^n}{1-k}|p_1 - p_0|$ so that $|p_n - p| < 10^{-4}$. We set $k = |g'(\frac{1}{3})| \approx 0.55015128179$. From the table below, we see that $|p_1 - p_0| \approx 0.03670614172$. Thus $10^{-4} \leq \frac{0.55015128179^n}{1 - 0.55015128179} * 0.03670614172 \implies 10^{-4} \leq 0.55015128179^n * 0.08159663512 \implies \frac{10^{-4}}{0.08159663512} \leq 0.55015128179^n \implies n \geq 11.2195443$. Thus we need a lower bound of 12 iterations for $|p_n - p| < 10^{-4}$, however, from the table, we see that we only need 8 iterations, thus the estimate is more than satisfactory.

i	p_i	$p_{i+1} = g(p_i)$	$\|p_{i+1} - p_i\|$
0	$\frac{2}{3}$	0.62996052494	0.03670614172
1	0.62996052494	0.64619409625	0.01623357131
2	0.64619409625	0.63896371138	0.00723038487
3	0.63896371138	0.64217405712	0.00321034574
4	0.64217405712	0.64074665312	0.001427404
5	0.64074665312	0.64138092226	0.00063426914
6	0.64138092226	0.64109900633	0.00028191593
7	0.64109900633	0.64122429523	0.0001252889
8	0.64122429523	0.6411686114	0.00005568383

2.3.1 Let $f(x) = x^2 - 6$ and $p_0 = 1$. Use Newton's method to find p_2.

The equation for Newton's method is $p_n = p_{n-1} - \frac{f(p_{n-1})}{f'(p_{n-1})}$. The base case in this instance is $p_0 = 1$. $f'(x) = 2x$. Thus $p_1 = p_0 - \frac{f(p_0)}{f'(p_0)} = 1 - \frac{f(1)}{f'(1)} = 1 - \frac{-5}{2} = \frac{7}{2} = 3.5 \implies p_2 = p_1 - \frac{f(p_1)}{f'(p_1)} = 1 - \frac{f(3.5)}{f'(3.5)} = 3.5 - \frac{6.25}{7} = \boxed{\frac{73}{28}}$

2.3.20 Use Newton's method to approximate, to within 10^{-4}, the value of x that produces the point on the graph of $y = \frac{1}{x}$ that is closest to (2, 1).

Since any point on the curve can be written as (x,$\frac{1}{x}$), we can find a formula for the squared distance of any point on the curve to the point (2,1) as $f(x) = (x-2)^2 + (\frac{1}{x} - 1)^2 = x^2 + \frac{1}{x^2} - 4x - \frac{2}{x} + 5$. The derivative of this function is $f'(x) = -\frac{2}{x^3} + \frac{2}{x^2} + 2x - 4$. Since we want to find the x-value that minimizes the function, we set the derivative $f'(x) = 0$. To use Newton's method to find when $f'(x) = 0$, we need to find the 2nd derivative which is $f''(x) = -\frac{4}{x^3} + \frac{6}{x^4} + 2$. Based on the table below the value to approximate within 10^{-4} for x is $\boxed{1.866760399174}$ after 2 iterations.

i	p_i	$p_{i+1} = p_i - \frac{f'(p_i)}{f''(p_i)}$	$f'(p_i)$
0	2	$\frac{28}{15}$	0.25
1	$\frac{28}{15}$	1.866760399009	-0.00017614188533
2	1.866760399009	1.866760399174	$-3.1002516 * 10^{-10}$