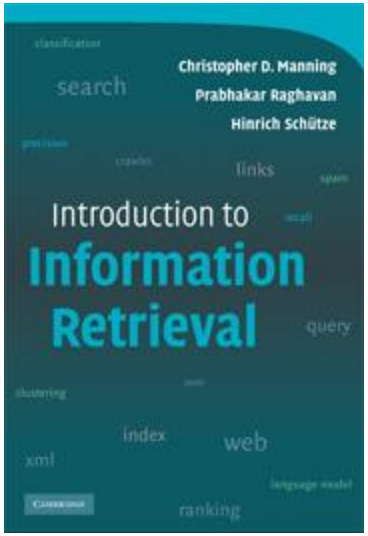
Information Retrieval and Organisation



A Brief Introduction to Probability and Statistics

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Probability theory is nothing but common sense reduced to calculation.

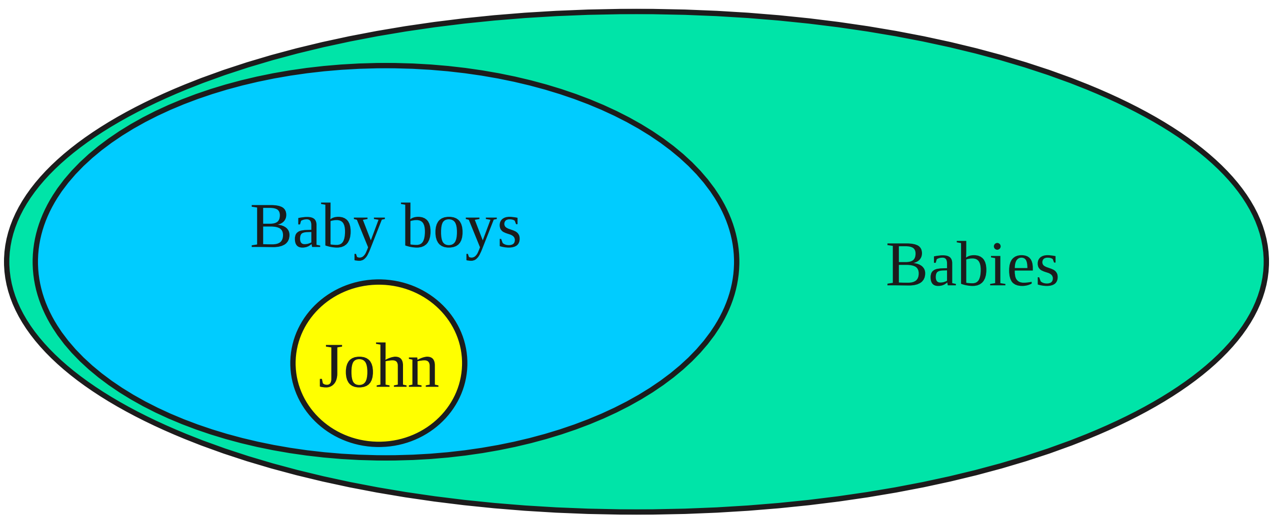
--- Pierre-Simon Laplace

The Bean Machine

- Wikipedia
  - http://en.wikipedia.org/wiki/Bean_machine
- Demonstration:
  - http://www.youtube.com/watch?v=9xUBhhM4vbM
- Simulation:
  - http://www.ms.uky.edu/~mai/java/stat/GaltonMachine.html

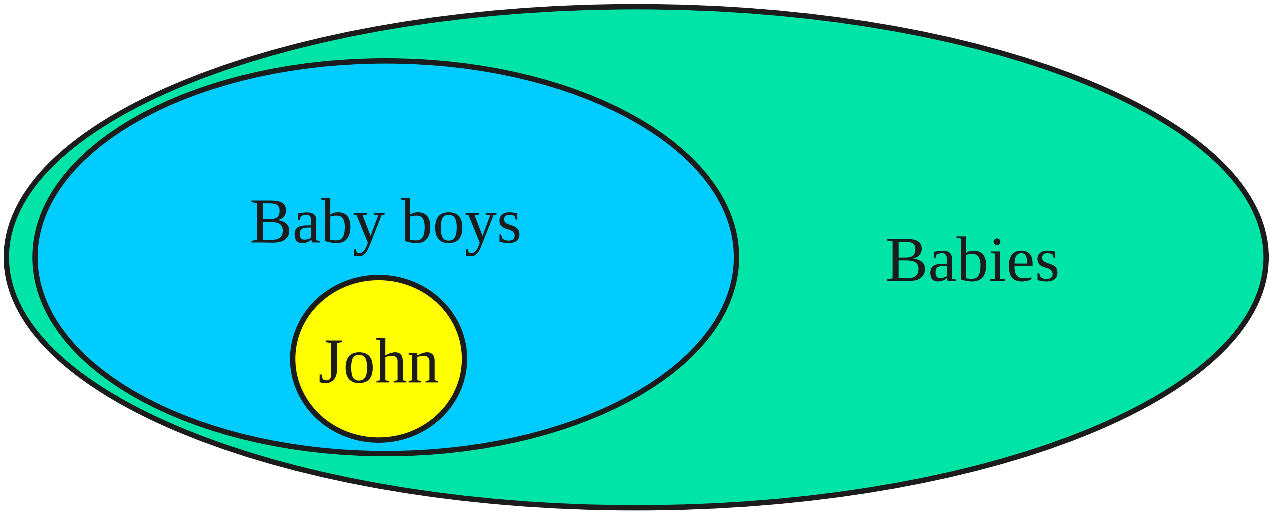
Probability

- $P(A)$ means **probability** that A is true
  - $P(\text{baby is a boy}) \approx 0.5$ (% of total that are boys)
  - $P(\text{baby is named John}) \approx 0.001$ (% of total named John)



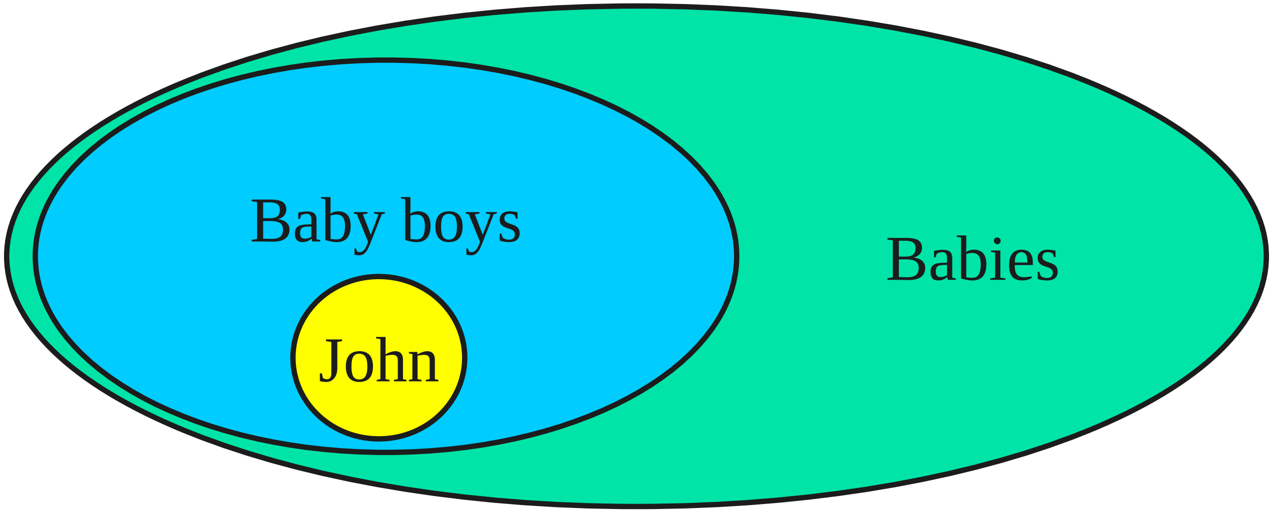
Odds

- The **odds** of an event *A*: $$O(A) = \frac{P(A)}{P(\overline{A})} = \frac{P(A)}{1 - P(A)}.$$
  - O(baby is a boy) = 0.5 / 0.5 = 1
  - O(baby is named John) = 0.001 / 0.999 = 1/999



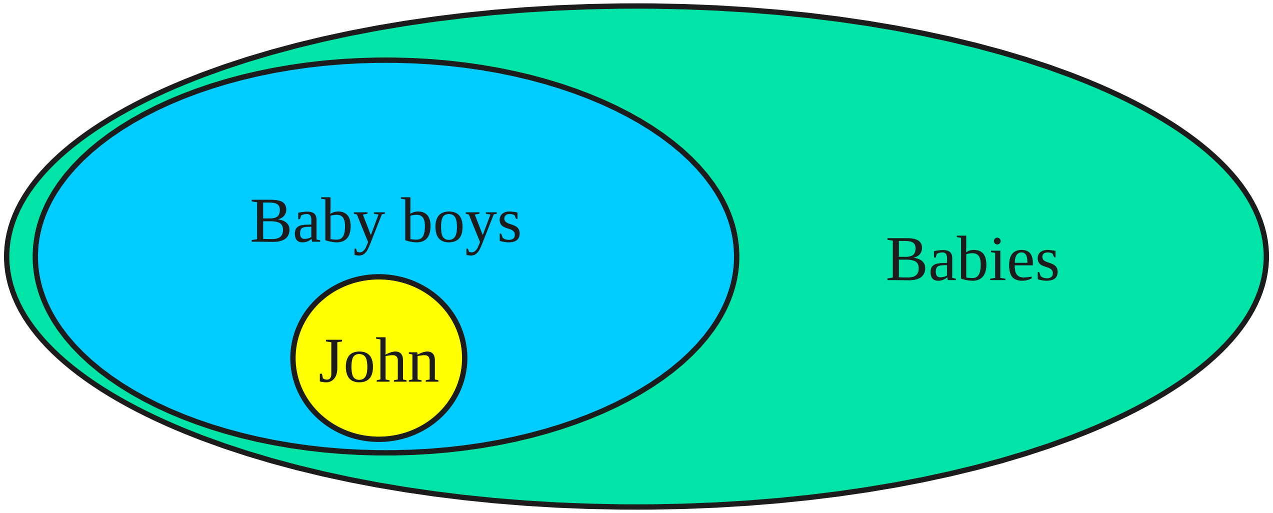
Joint Probability

- $P(A,B)$ means probability that A and B are both true
  - P(baby is named John, baby is a boy)



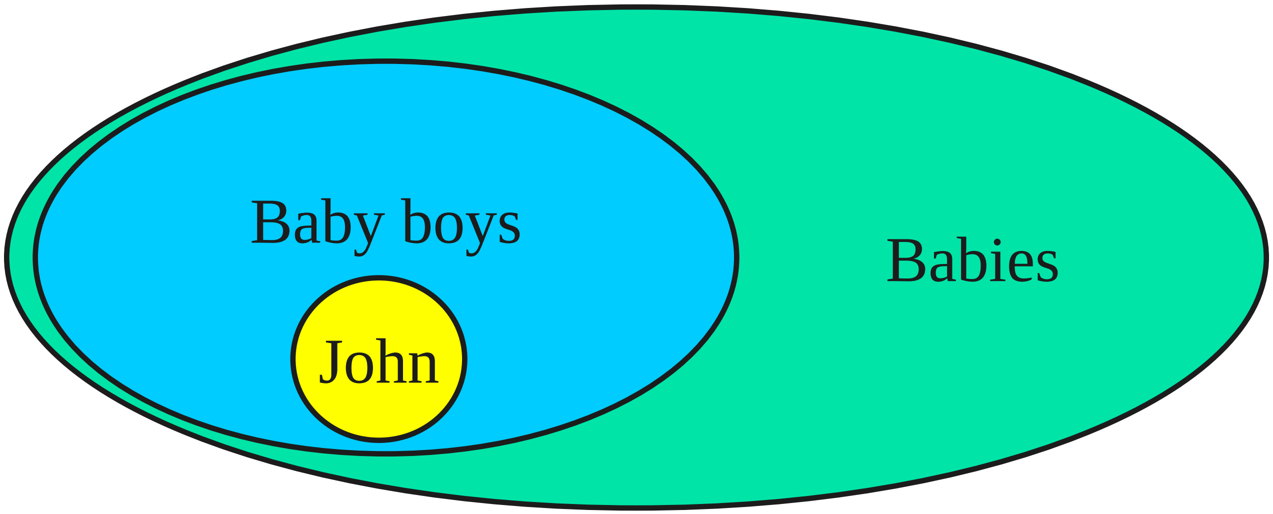
Conditional Probability

- $P(A|B)$ means probability that A is true when we already know B is true
  - P(baby is named John | baby is a boy) ≈ 0.002
  - P(baby is a boy | baby is named John) ≈ 1



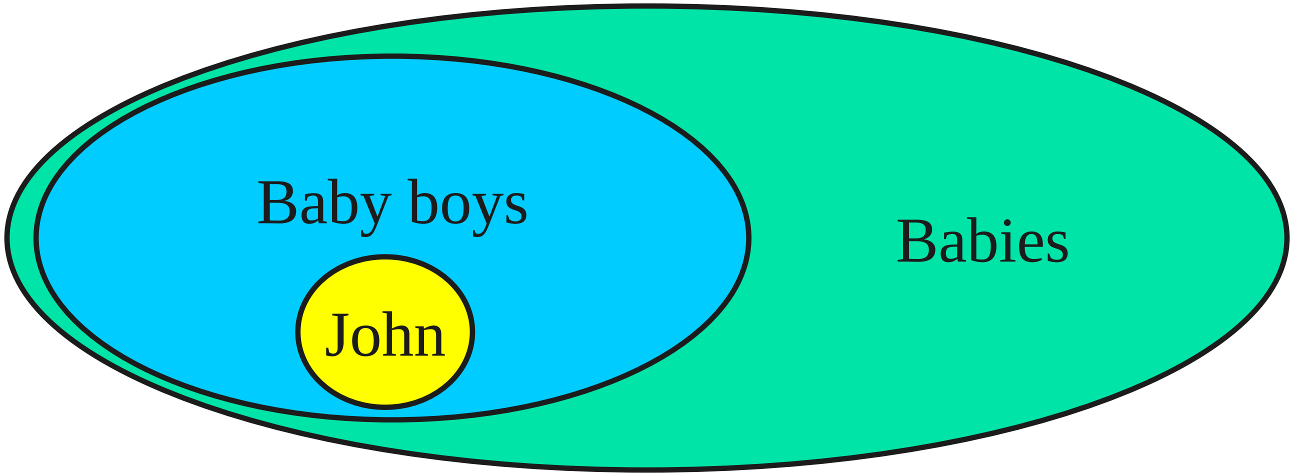
Basic Rules of Probability

- Chain Rule: $P(A, B) = P(A|B)P(B)$
  - P(named John, boy)
    = P(named John | boy) $\times$ P(boy)
    = 0.002 * 0.5 = 0.001



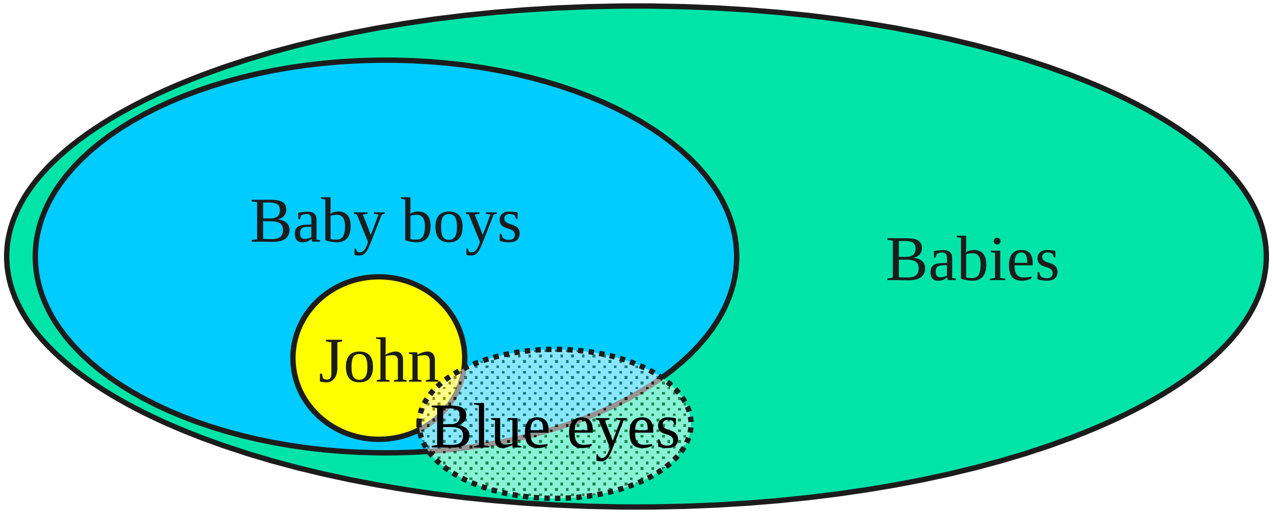
Basic Rules of Probability

- Partition Rule: $P(B) = P(A, B) + P(\overline{A}, B).$
  - P(boy)
    $=P$(named John, boy) + P(*not* named John, boy)
    $= 0.001 + 0.499 = 0.5$



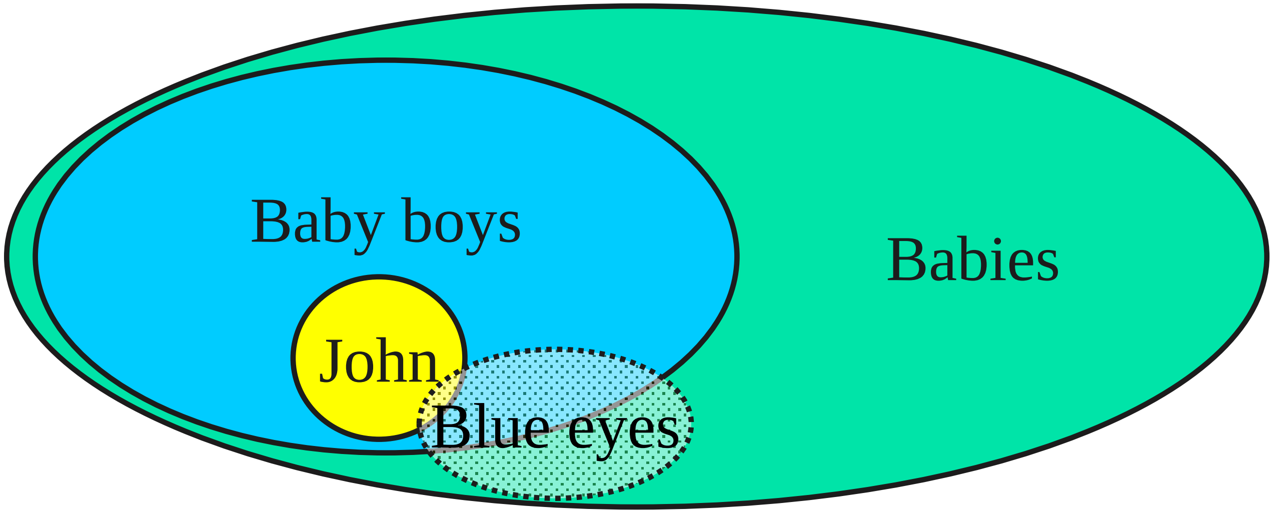
Independence

- $P(A,B) = P(A)P(B)$: A and B are **independent**
  - $P(\text{blue eyes, boy}) = P(\text{blue eyes}) \times P(\text{boy})$



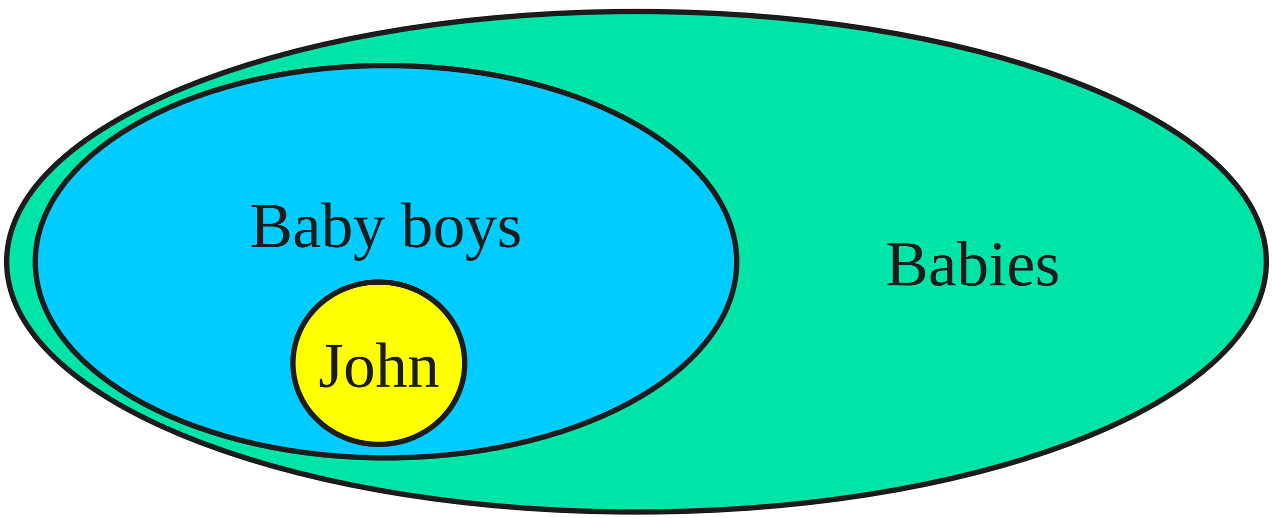
Conditional Independence

- $P(A,B|C) = P(A|C)P(B|C)$: A and B are **conditionally independent** given C
  - P(named John, blue eyes | boy)
    = P(named John | boy) × P(blue eyes | boy)



Bayes' Rule

- $P(A|B) = P(B|A)\ P(A)\ /\ P(B)$
  - P(named John | boy)
    $= P$(boy | named John) $\times$ P(named John) / P(boy)



Bayes' Rule

likelihood prior probability

$$\underbrace{P(A|B)}_{\text{posterior probability}} = \frac{\overbrace{P(B|A)}^{\text{likelihood}}\overbrace{P(A)}^{\text{prior probability}}}{P(B)} = \left[\frac{P(B|A)}{\sum_{X\in\{A,\overline{A}\}} P(B|X)P(X)}\right]P(A).$$

posterior probability

Example: British Weather

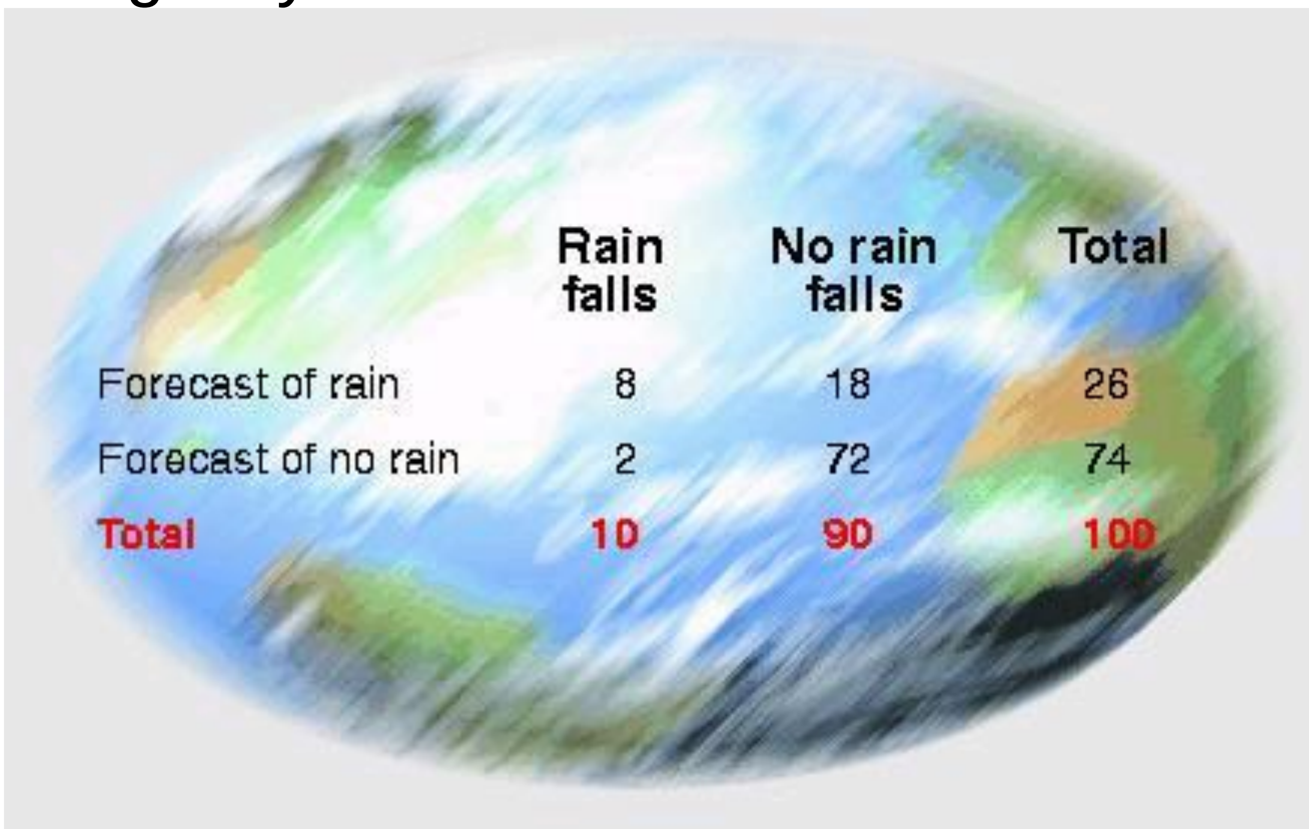
- You are about to set off into town to do some shopping.
- You will only be out for an hour or so, but rain has been forecasted, so what are you going to do?
- You know the forecasts are pretty good-around 80% accurate, in fact.
- So the chances that you will need an umbrella are 80%, right?

Example: British Weather

- Wrong, they're actually more like 30%.
- On the hourly timescales relevant to shopping trips, Britain's base-rate of rain is about 10%. That is, there is only a 1 in 10 chance of rain falling in any particular hour, and thus a 9 in 10 chance of rain not falling. And this has a significant impact on how much trust we can put in even an 80% reliable forecast.

Example: British Weather

- Contingency Table



	Rain falls	No rain falls	Total
Forecast of rain	8	18	26
Forecast of no rain	2	72	74
Total	10	90	100

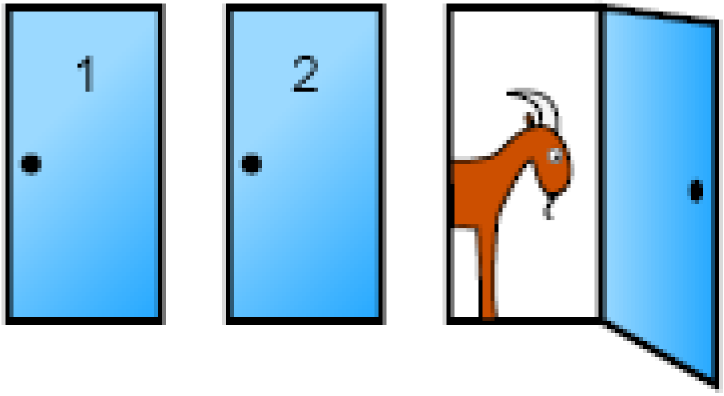
Robert Matthews, How right can you be?, New Scientist, 08 March 1997.

Example: British Weather

- In the language of probabilities
  - we can estimate P(rain-falls | forecast-of-rain)
  - by using Bayes' rule to combine the knowledge of P(rain-falls) and P(forecast-of-rain | rain-falls)!

Example: Monty Hall Problem

- YouTube videos on this problem
  - Cartoon
  - "21"
  - Numb3rs
- Wikipedia article on this problem



monty hall

September 10, 1990

Mr. Lawrence A. Denenberg
Harvard University Center for
Research in Computing Technology
Aiken Computation Laboratory, Room 102
Harvard University
Cambridge, MA 02138

Dear Larry:

In sending you my okay for the use of "The Monty Hall Paradox," I should like to ask you a question. You mention that in part (a), the player should switch doors even without additional compensation -- indeed the player should be willing to pay Monty up to $21,845 for the privilege of switching.

Now, I am not well versed in algorithms; but as I see it, it wouldn't make any difference after the player has selected Door A, and having been shown Door C - why should he then attempt to switch to Door B? The major prize could only be in one of the three doors. He has made his selection of one of the doors.

Example: Discriminatory Drugs

- What is the probability that Drug I will treat a man successfully?
- Is the success of Drug I independent from gender?

	Women		Men	
	Drug I	Drug II	Drug I	Drug II
Success	200	10	19	1000
Failure	1800	190	1	1000

Example: Discriminatory Drugs

- Which drug is more effective for women?
- Which drug is more effective for men?
- Which drug is more effective overall?
  - Simpson's Paradox

	Women		Men	
	Drug I	Drug II	Drug I	Drug II
Success	200	10	19	1000
Failure	1800	190	1	1000

"There are three kinds of lies: lies, damned lies, and statistics."

Example: Biased Coins

- Consider you have three coins
  - C_1, C_2, C_3
- Alex picked up one of the coins and flipped it 6 times.
- You didn't see which coin he picked out, but you observed the results of flipping coins
  - `THTHTT`
- How to guess which coin Alex chose?

Example: Biased Coins

- You experimented with the three coins, say 6 times
  - C_1: `HHHTHH`
  - C_2: `TTHTHH`
  - C_3: `THTTTH`
- Given the observation
  - O: `THTHTT`
- Which coin do you think Alex chose?

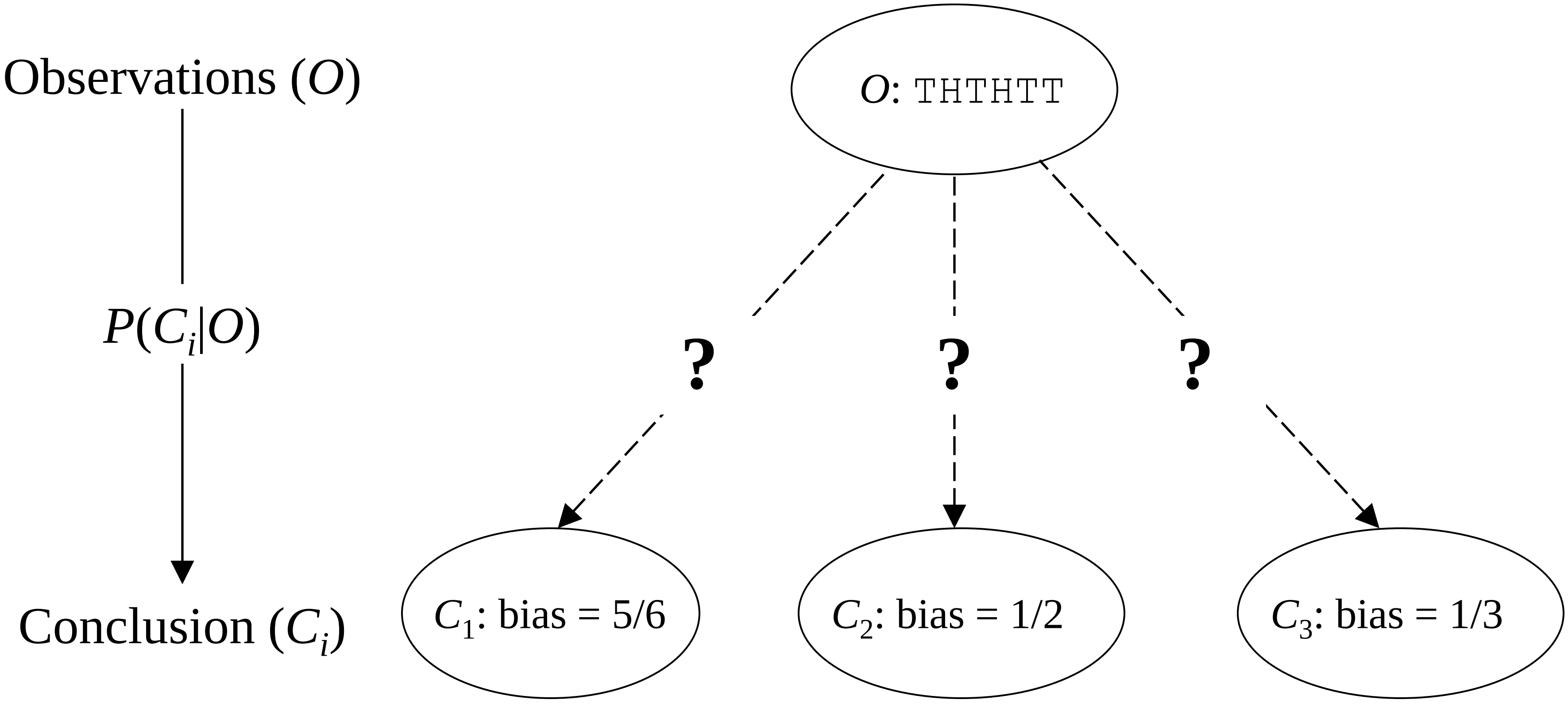
Example: Biased Coins

- A principled approach
  - Compare the posterior probability $P(C_i|O)$
  - It is not obvious how the posterior probability $P(C_i|O)$ can be computed directly
  - It is easy to compute the prior probability $P(C_i)$ and the likelihood $P(O|C_i)$.
- Build a model for each coin
  - C_1: `HHHTHH` → bias $P(\mathtt{H}|C_1) = 5/6$
  - C_2: `TTHTHH` → bias $P(\mathtt{H}|C_2) = 1/2$
  - C_3: `THTTTH` → bias $P(\mathtt{H}|C_3) = 1/3$

Example: Biased Coins

- Prior probability
  - $P(C_1) = P(C_2) = P(C_3) = 1/3$
- Likelihood
  - $P(O|C_1) = P(\mathtt{THTHTT}|C_1)$
    $= P(\mathtt{T}|C_1)P(\mathtt{H}|C_1)P(\mathtt{T}|C_1)P(\mathtt{H}|C_1)P(\mathtt{T}|C_1)P(\mathtt{T}|C_1)$
    $= P(\mathtt{H}|C_1)^2 P(\mathtt{T}|C_1)^4 = (5/6)^2 * (1/6)^4 \approx 0.0005$
  - $P(O|C_2) \approx 0.0156$
  - $P(O|C_3) \approx 0.0219$
- Posterior probability
  - Which coin has the largest posterior probability $P(C_i|O)$?

Example: Biased Coins



Example: Biased Coins

Observations (O)

$P(C_i|O)$

➔ $P(O|C_i)P(C_i)$

Conclusion (C_i)

O: THTHTT

Bayes' rule helps us convert the computation of $P(C_i|O)$ to the computation of $P(O|C_i)$ and $P(C_i)$.

C_1: bias = 5/6

C_2: bias = 1/2

C_3: bias = 1/3