Sidorenko's Conjecture and Graph Homomorphism Density

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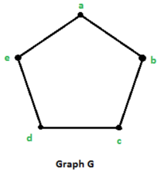
Review: Graph Homomorphism

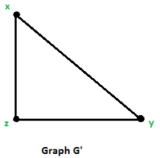
Definition

A graph homomorphism f from a graph $G = (V(G), E(G))$ to a graph $H = (V(H), E(H))$, written $f : G \to H$ is a function from $V(G)$ to $V(H)$ s.t. $(u, v) \in E(G)$ implies $(f(u), f(v)) \in E(H)$, for all pairs of vertices u, v in $V(G)$.

Example of Graph Homomorphism

$f : a \mapsto x, b \mapsto y, c \mapsto z, d \mapsto x, e \mapsto z$





Review: Graph Homomorphism Density

Definition

For graphs H, G, the homomorphism density of H in G is (the number of homomorphisms of H in G, normalized by the number of maps $V_H \to V_G$)

$$t(H, G) := \frac{|\operatorname{hom}(H, G)|}{|V(G)|^{|V(H)|}}.$$

$\operatorname{hom}(H, G)$ = set of graph homomorphisms $H \to G$

Statement of Sidorenko Conjecture

Theorem (Informal)

Let H bipartite. For all graphs G with $|V_G| = n$, if G has an average degree of pn, then $t(H, G) \geq p^{|E(H)|}$.$(0 \leq p \leq 1)$

Simple Exercise: Is statement still True?

For any ~~bipartite~~ graph H and for all graphs G , if G has n vertices and an average degree of pn, then $t(H, G) \geq p^{|E(H)|}$.$(0 \leq p \leq 1)$.

Statement of Sidorenko Conjecture

Theorem (Informal)

For any bipartite graph H and for all graphs G , if G has n vertices and an average degree of pn, then $t(H, G) \geq p^{|E(H)|}.(0 \leq p \leq 1)$

Simple Exercise: Is statement still True?

For any ~~bipartite~~ graph H and for all graphs G , if G has n vertices and an average degree of pn, then $t(H, G) \geq p^{|E(H)|}.(0 \leq p \leq 1)$.

- **No!**

Simple Example

Simple Exercise: Is statement still True?

False: For any ~~bipartite~~ graph H and for all graphs G , if G has n vertices and an average degree of pn, then $t(H, G) \geq p^{|E(H)|}.(0 \leq p \leq 1)$.

Proof

Consider when G is a bipartite graph, then $t\,(K_3, G) = 0$ since G is triangle-free. But $t\,(K_2, G)$=$2|E(G)|$, so Sidorenko's property does not hold for K_3.
Similarly, no graph with an odd cycle has Sidorenko's property. Since a graph is bipartite if and only if it has no odd cycles, this implies that the only possible graphs that can have Sidorenko's property are bipartite graphs.
Thus we know that we need H to be bipartite.

Graphon and Extended Homomorphism Density

Definition

A graphon (also known as a graph limit) is a symmetric measurable function $W : [0,1]^2 \to [0,1]$.

Definition

For every $W \in \mathcal{W}$ and multigraph $F = (V, E)$ (without loops), define the extended graph homomorphism density $t(F, W)$ as

$$t(F, W) = \int_{[0,1]^V} \prod_{ij \in E} W(x_i, x_j) \prod_{i \in V} dx_i$$

Example Computation of Extended Homomorphism Density

Example

Suppose W is a simple graphon defined by

$$W(x, y) = p \in (0, 1)$$

This represents a graphon where every pair of vertices is connected with the same probability p. Given the multigraph F with vertices $V = \{1, 2, 3\}$ and edges $E = \{(1, 2), (1, 3)\}$, Compute the extended graph homomorphism density $t(F, W)$ for a graphon W.

Example Computation of Extended Homomorphism Density

Answer

The integral formula for the homomorphism density is:

$$t(F, W) = \int_{[0,1]^{\mu}} W(x_1, x_2) \cdot W(x_1, x_3)\, dx_1 dx_2 dx_3$$

Since $W(x, y) = p$, the integral simplifies to:

$$t(F, W) = \int_0^1 \int_0^1 \int_0^1 p \cdot p dx_1 dx_2 dx_3$$

Example Computation of Extended Homomorphism Density

Answer(contd)

This can be simplified further because the integral of a constant over an interval is just the product of the constant and the length of the interval:

$$t(F, W) = p^2 \int_0^1 \int_0^1 \int_0^1 dx_1 dx_2 dx_3$$

Since each integral from 0 to 1 is just 1 , we have:

$$t(F, W) = p^2$$

Statement of Sidorenko Conjecture

Theorem (Informal)

For any bipartite graph H and for all graphs G , if G has n vertices and an average degree of pn, then $t(H, G) \geq p^{|E(H)|}.(0 \leq p \leq 1)$

Theorem

For all bipartite graph H and all graphons W, we have

$$t(H, W) \geq t\left(K_2, W\right)^{\mathbf{e}(H)}.$$

Remark

Okay if don't understand the graphon version at this point. One can just keep the informal statement of the conjecture in mind.

Contents

Methods used to Study Homomorphism Density

- Motivation: C_4 is Sidorenko
- Graph Entropy
- Flag Algebras

Motivation: C_4 is Sidorenko

First, we examine the 4-cycle C_4, which is known to be Sidorenko.

Theorem

For all graphs G, if G has edge density $p \in [0, 1]$, then
$t(C_4, G) \geq p^{|E(C_4)|} = p^4$.

Motivation: C_4 is Sidorenko

Proof.

For vertices u, v, define $codegree(u, v) = \#$ common neighbors.

$$t(C_4, G) = \sum_{u,v \in V_G} cod(u, v)^2 \geq \frac{1}{n^2}\Big(\sum_{u,v} cod(u, v)\Big)^2 \text{ by Cauchy-Schwarz} \tag{1}$$

By double-counting:
$\sum_{\text{pairs of vertices}}$ common neighbors $= \sum_{\text{vertices}}$ pairs of neighbors.

$$t(C_4, G) \geq \frac{1}{n^2}\Big(\sum_{x \in V_G} deg(x)^2\Big)^2 \geq \dots \text{ (by C-S) } \dots \geq p^4 n^4. \tag{2}$$

$\square$

Motivation: C_4 is Sidorenko

Keeping track: We used Cauchy-Schwarz 2 times and double-counting 1 time. It would be nice to have some mechanisms to make these arguments easier...

Graph Entropy

- Uses probabilistic "entropy" of graphs to achieve desirable bounds
- Uses techniques from probability and information theory

Flag Algebras

- Studies an algebraic structure of partially labeled graphs ("flags")
- Uses techniques from algebra
- From Razborov (2007)

Contents

Entropy

Entropy

The (Shannon) entropy of random variable X is defined by

$$\mathrm{H}(X) = \sum_{x \in \mathcal{X}} p(x) \log \frac{1}{p(x)} = \mathop{\mathbb{E}}_{X} \left[\log \frac{1}{p(X)} \right]$$

with the convention that $0 \log 0 = 0$.

Known Sidorenko Graphs Proven by Entropy

- Tree
- Complete Bipartite Graph

Contents

Overview of Flag Algebra

- Motivation: Graphons and sequences of graphs
- Classifying graph limits
- The Algebra A^σ
- Example: Mantel's Theorem

Motivation: Graphons and sequences of graphs

- In extremal graph theory, we often want to study the limit behavior of a sequence of graphs with large numbers of vertices.
- Graphons allow "partial" edges.
- Intuitively, a graphon represents a "limit" of a "sequence" of graphs.

Sequences of graphs

Definition

A sequence of graphs (G_n) is convergent if the size of G_n diverges and for all graphs F, $lim_{n\to\infty} t(F, G_n)$ exists.

In this case, we write $\phi(F) = lim_{n\to\infty} t(F, G_n)$.

- Let $(G_n) = (K_n)$. Convergent because $\phi(F) = 1$.
- Let (G_n) converge to a random graph, as in Sidorenko's Conjecture

Classifying Graph Limits

Guiding question: We want to study how we can manipulate these "limit functionals" ϕ.

Classifying Graph Limits

How can we define operations $+, \cdot$ such that for graphs F, F', sequence (G_n),

$$lim_{n\to\infty} t(F, G_n) + t(F', G_n) = lim_{n\to\infty} t(F + F', G_n) \tag{3}$$

$$lim_{n\to\infty} t(F, G_n) t(F', G_n) = lim_{n\to\infty} t(F \cdot F', G_n) \tag{4}$$

The Algebra A^σ

Addition is easy: Allow $\mathbb{R}$-linear combinations of graphs.
Multiplication is harder: We define $F \cdot F' \in \mathbb{R}M_l$ for some l, where M_l $= \{$ graphs of size l $\}$

Definition

This gives us a formal algebraic structure A^0 of graphs with **addition, multiplication, scalar multiplication**.

The Algebra A^σ

Theorem

Denote the set of positive homomorphisms from A^0 to $\mathbb{R}$ as $Hom^+(A^0, \mathbb{R})$. Then, $Hom^+(A^0, \mathbb{R}) = \{$ limit functionals $\phi\}$.

This theory generalizes very easily to algebraic structures of **partially labelled graphs**.

Flags and Flag Densities

Definition

A type σ is a fully labeled graph. A σ-flag F is a partially labeled graph with labeled part isomorphic to σ.

The Algebra A^σ

Definition

We can use the same definitions to construct the algebra A^σ of σ-flags. This is called the **flag algebra of σ**.

All of the properties of A^0 also apply to A^σ.

Example: Mantel's Theorem

Flag algebra gives a much cleaner way to formulate many extremal graph theoretic problems.

Theorem

Mantel's Theorem Every n-vertex triangle-free graph has $\leq \lfloor \frac{n^2}{4} \rfloor$ *edges.*

Theorem

Mantel. Let H *be the triangle,* $\Phi =$ *the set of algebraic homomorphisms on* G*, the set of triangle-free graphs. Then* $sup\{\phi(edge) : \phi \in \Phi\} = \frac{1}{2}$.

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