Finite Element Formulation for the Pekeris Waveguide

Generated with Claude Code

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1 Introduction

This document describes the finite element formulation implemented in `pekeris_fem.py` for solving the acoustic wave propagation problem in a Pekeris waveguide. The Pekeris waveguide is a classical two-layer acoustic model consisting of:

- A water layer (depth $0 \leq z \leq H$) with sound speed c_1 and density ρ_1
- A sediment half-space ($z > H$) with sound speed $c_2 > c_1$ and density ρ_2

The problem is axisymmetric, so we solve in the (r, z) meridional plane where r is the radial distance from the axis of symmetry and z is depth (positive downward).

2 Governing Equation

2.1 Time-Harmonic Acoustic Wave Equation

For time-harmonic waves with angular frequency ω (assuming $e^{+i\omega t}$ time dependence), the acoustic pressure $p(r, z)$ satisfies the Helmholtz equation in a medium with spatially varying density ρ and sound speed c:

$$\nabla \cdot \left(\frac{1}{\rho}\nabla p\right) + \frac{\omega^2}{\rho c^2} p = 0 \tag{1}$$

2.2 Axisymmetric Form

In cylindrical coordinates (r, θ, z) with azimuthal symmetry ($\partial/\partial\theta = 0$), the divergence and gradient operators give:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(\frac{r}{\rho}\frac{\partial p}{\partial r}\right) + \frac{\partial}{\partial z}\left(\frac{1}{\rho}\frac{\partial p}{\partial z}\right) + \frac{\omega^2}{\rho c^2} p = 0 \tag{2}$$

This can be written more compactly as:

$$\frac{1}{r}\nabla \cdot_{rz} \left(\frac{r}{\rho}\nabla_{rz} p\right) + \frac{\omega^2}{\rho c^2} p = 0 \tag{3}$$

where $\nabla_{rz} = (\partial/\partial r, \partial/\partial z)^T$ is the gradient in the (r, z) plane.

3 Variational Formulation

3.1 Weak Form Derivation

Multiplying by a test function v and integrating over the domain Ω in the (r, z) plane, we account for the axisymmetric geometry by including the factor r in the measure (from the Jacobian of cylindrical coordinates):

$$\int_\Omega \left[\nabla \cdot_{rz} \left(\frac{r}{\rho} \nabla_{rz} p \right) + \frac{\omega^2 r}{\rho c^2} p \right] \overline{v} \, \mathrm{d}r \, \mathrm{d}z = 0 \tag{4}$$

Applying integration by parts (Green's first identity) to the divergence term:

$$-\int_\Omega \frac{r}{\rho} \nabla_{rz} p \cdot \nabla_{rz} \overline{v} \, \mathrm{d}r \, \mathrm{d}z + \int_{\partial\Omega} \frac{r}{\rho} \frac{\partial p}{\partial n} \overline{v} \, \mathrm{d}s + \int_\Omega \frac{\omega^2 r}{\rho c^2} p \, \overline{v} \, \mathrm{d}r \, \mathrm{d}z = 0 \tag{5}$$

where $\partial p / \partial n = \nabla p \cdot \boldsymbol{n}$ is the normal derivative on the boundary.

3.2 Sesquilinear Form

Rearranging, we obtain the variational problem: Find $p \in V$ such that

$$a(p, v) = L(v) \quad \forall v \in V \tag{6}$$

where the sesquilinear form is:

$$\boxed{a(p, v) = \int_\Omega \frac{r}{\rho} \nabla p \cdot \nabla \overline{v} \, \mathrm{d}r \, \mathrm{d}z - \int_\Omega \frac{\omega^2 r}{\rho c^2} p \, \overline{v} \, \mathrm{d}r \, \mathrm{d}z} \tag{7}$$

and the linear form (from boundary conditions) is:

$$L(v) = \int_{\partial\Omega_N} \frac{r}{\rho} g_N \overline{v} \, \mathrm{d}s \tag{8}$$

where $g_N = \partial p / \partial n$ is the prescribed Neumann data.

4 Boundary Conditions

4.1 Pressure-Release Surface ($z = 0$)

At the water surface, the acoustic pressure vanishes (Dirichlet condition):

$$p(r, 0) = 0 \tag{9}$$

4.2 Source Boundary Condition

The source is modeled as a small semicircular exclusion of radius r_s centered at the source location $(0, z_s)$. On this boundary, we prescribe the normal velocity v_n:

$$\frac{1}{\rho} \frac{\partial p}{\partial n} = -i\omega v_n \tag{10}$$

This gives the source contribution to the linear form:

$$L_{\text{source}}(v) = -i\omega \rho_1 v_n \int_{\Gamma_s} r \, \overline{v} \, \mathrm{d}s \tag{11}$$

4.3 Axis of Symmetry ($r = 0$)

On the axis of symmetry, the natural boundary condition $\partial p/\partial r = 0$ is automatically satisfied by the weak form (no explicit treatment needed).

4.4 PML Absorbing Boundaries

See Section 5 for the treatment of far-field boundaries using Perfectly Matched Layers.

5 Perfectly Matched Layers (PML)

5.1 Complex Coordinate Stretching

To absorb outgoing waves and simulate an unbounded domain, we use Perfectly Matched Layers (PML) based on complex coordinate stretching. The idea is to transform the real coordinates (r, z) to complex coordinates $(\tilde{r}, \tilde{z})$ in the PML region.

For the radial PML (active for $r > r_{\max}$):

$$\tilde{r} = r\left(1 + i\sigma_r(r)\right), \quad \sigma_r(r) = \frac{\alpha}{k_0}\left(\frac{r - r_{\max}}{\Delta r}\right)^2 \tag{12}$$

For the vertical PML (active for $z > z_{\max}$):

$$\tilde{z} = z\left(1 + i\sigma_z(z)\right), \quad \sigma_z(z) = \frac{\alpha}{k_0}\left(\frac{z - z_{\max}}{\Delta z}\right)^2 \tag{13}$$

where:

- α is the PML absorption strength parameter
- $k_0 = \omega/c_1$ is the reference wavenumber
- $\Delta r = r_{\text{pml}} - r_{\max}$ and $\Delta z = z_{\text{pml}} - z_{\max}$ are the PML thicknesses

5.2 PML Stretching Functions

Define the stretching functions:

$$s_r = 1 + i\sigma_r, \quad s_z = 1 + i\sigma_z \tag{14}$$

The coordinate transformation has Jacobian:

$$\boldsymbol{J} = \begin{pmatrix} \partial\tilde{r}/\partial r & 0 \\ 0 & \partial\tilde{z}/\partial z \end{pmatrix} \approx \begin{pmatrix} s_r & 0 \\ 0 & s_z \end{pmatrix} \tag{15}$$

5.3 Modified Variational Form in PML

The PML can be interpreted as an anisotropic medium with modified material properties. The sesquilinear form in the PML region becomes:

$$a_{\text{PML}}(p, v) = \int_{\Omega_{\text{PML}}} \frac{r}{\rho}\left(\boldsymbol{A}\nabla p\right) \cdot \nabla\overline{v}\, \mathrm{d}r\, \mathrm{d}z - \int_{\Omega_{\text{PML}}} \frac{\omega^2 r}{\rho c^2} B\, p\, \overline{v}\, \mathrm{d}r\, \mathrm{d}z \tag{16}$$

where the PML tensors are:

$$\boxed{\boldsymbol{A} = \begin{pmatrix} s_z/s_r & 0 \\ 0 & s_r/s_z \end{pmatrix}, \quad B = s_r \cdot s_z} \tag{17}$$

5.4 PML Regions

The computational domain includes four PML regions with different stretching:

Region	Radial stretch	Vertical stretch
Water right PML	s_r	1
Sediment right PML	s_r	1
Bottom PML	1	s_z
Corner PML	s_r	s_z

6 Material Properties

The domain consists of two layers with different acoustic properties:

Layer	Sound speed	Density	Domain
Water	c_1	ρ_1	$0 \leq z < H$
Sediment	c_2	ρ_2	$z \geq H$

The wavenumber in each layer is $k = \omega/c$, and the material properties are discontinuous across the water-sediment interface at $z = H$.

7 Finite Element Discretization

7.1 Function Space

The pressure field is approximated using Lagrange finite elements of degree p:

$$p_h(r, z) = \sum_{j=1}^{N} p_j \phi_j(r, z) \tag{18}$$

where ϕ_j are the nodal basis functions and $p_j \in \mathbb{C}$ are the (complex) degrees of freedom.

7.2 Discrete System

The discrete variational problem leads to a linear system:

$$\boldsymbol{K}\boldsymbol{p} = \boldsymbol{f} \tag{19}$$

where:

- $\boldsymbol{K}$ is the complex-valued stiffness matrix combining the gradient and mass terms
- $\boldsymbol{p}$ is the vector of nodal pressure values
- $\boldsymbol{f}$ is the load vector from the source boundary condition

The matrix entries are:

$$K_{ij} = \int_\Omega \frac{r}{\rho} \nabla\phi_j \cdot \nabla\overline{\phi_i} \, \mathrm{d}r \, \mathrm{d}z - \int_\Omega \frac{\omega^2 r}{\rho c^2} \phi_j \overline{\phi_i} \, \mathrm{d}r \, \mathrm{d}z \tag{20}$$

with appropriate modifications in PML regions.

7.3 Implementation Notes

- The factor r in the integrals is handled by including it explicitly in the UFL forms
- Material properties (ρ, c) are represented using DG0 (piecewise constant) functions
- PML tensors are computed symbolically using UFL expressions
- The linear system is solved using a direct LU solver (MUMPS)

8 Summary

The complete variational formulation combines contributions from the physical domain and PML regions:

$$a(p, v) = a_{\text{water}}(p, v) + a_{\text{sediment}}(p, v) + \sum_{\text{PML regions}} a_{\text{PML}}(p, v) \tag{21}$$

with boundary conditions:

- Dirichlet: $p = 0$ at $z = 0$ (pressure-release surface)
- Neumann: $\frac{1}{\rho}\frac{\partial p}{\partial n} = -i\omega v_n$ at source boundary
- Natural: $\frac{\partial p}{\partial r} = 0$ at $r = 0$ (axis of symmetry)
- PML absorption at far-field boundaries