Ridge regression

STAT 471

October 12, 2021

Where we are

- ✓ **Unit 1:** Intro to modern data mining
- ✓ **Unit 2:** Tuning predictive models
- **Unit 3:** Regression-based methods
- **Unit 4:** Tree-based methods
- **Unit 5:** Deep learning

Lecture 1: Logistic regression

Lecture 2: Regression in high dimensions

Lecture 3: Ridge regression

[**Fall break:** No class]

Lecture 4: Lasso regression

Lecture 5: Unit review and quiz in class

Homework 1 due the following Sunday.

Midterm exam following Monday (7-9pm).

Idea: Encourage coefficients not to be too large

First, recall linear regression:

$$\hat{\beta}^{\text{least squares}} = \arg\min_{\beta} \sum_{i=1}^{n} (Y_i - (\beta_0 + \beta_1 X_{i1} + \cdots + \beta_p X_{ip}))^2.$$

If there are too many features, the coefficients can get a little wild (high variance!).

To tame those wild coefficients, we add a penalty to disincentivize large values:

$$\hat{\beta}^{\text{ridge}} = \arg\min_{\beta} \sum_{i=1}^{n} (Y_i - (\beta_0 + \beta_1 X_{i1} + \cdots + \beta_p X_{ip}))^2 + \lambda \sum_{j=1}^{p} \beta_j^2$$

for some $\lambda \geq 0$.

Ridge regression is defined even if $p > n$, as long as $\lambda > 0$.

The effect of the penalty parameter λ

$$\hat{\beta}^{\text{ridge}} = \arg\min_{\beta} \sum_{i=1}^{n} (Y_i - (\beta_0 + \beta_1 X_{i1} + \cdots + \beta_p X_{ip}))^2 + \lambda \sum_{j=1}^{p} \beta_j^2$$

- The larger λ is, the more of a penalty there is.
- For $\lambda = 0$, we get back ordinary least squares.
- For $\lambda = \infty$, we get $\beta_1 = \cdots = \beta_p = 0$, leaving only the intercept (which is not penalized).

We should think of λ as controlling the flexibility of the ridge regression fit, like the degrees of freedom in a spline fit. However, larger λ means fewer degrees of freedom.

The bias-variance tradeoff for ridge regression

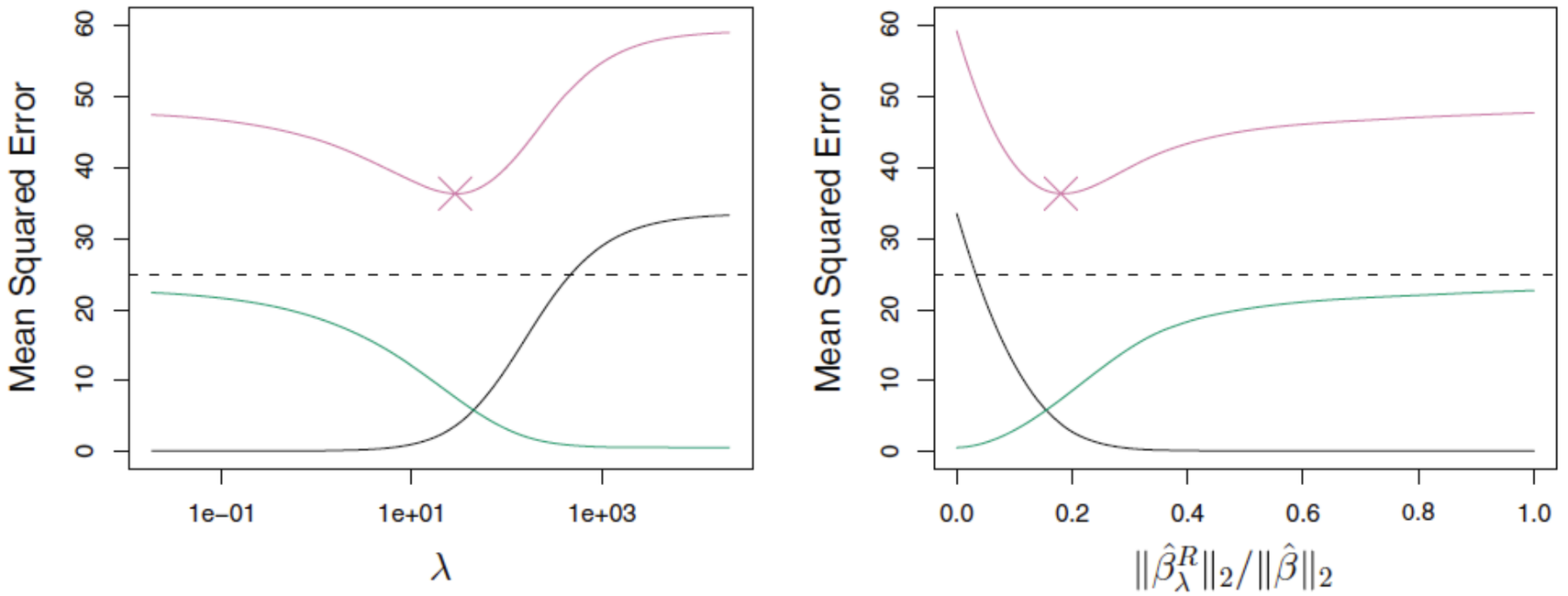
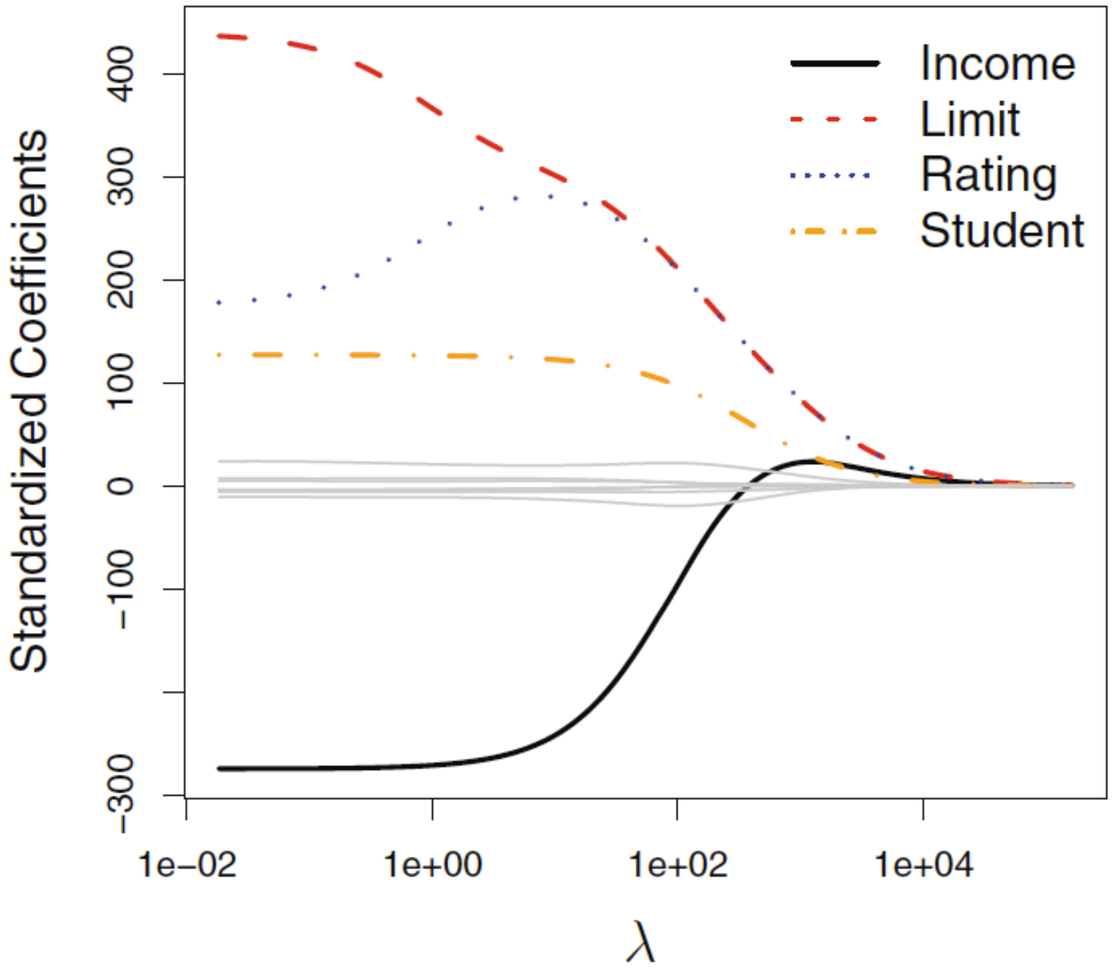


FIGURE 6.5. *Squared bias (black), variance (green), and test mean squared error (purple) for the ridge regression predictions on a simulated data set, as a function of* λ *and* $\|\hat{\beta}_\lambda^R\|_2/\|\hat{\beta}\|_2$. *The horizontal dashed lines indicate the minimum possible MSE. The purple crosses indicate the ridge regression models for which the MSE is smallest.*

In practice, λ is chosen by cross-validation.

A pictorial representation of ridge regression



Ridge regression in a simple case

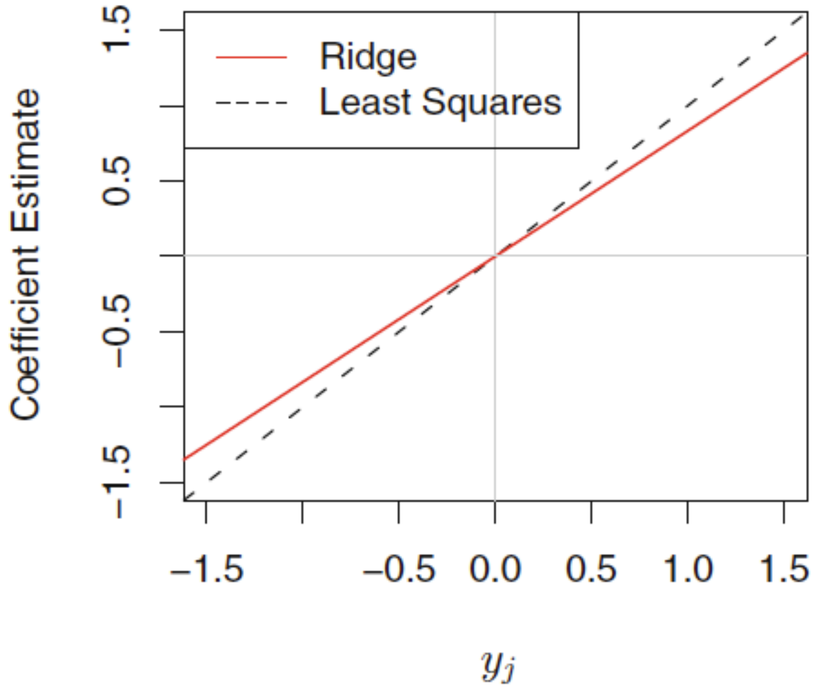
Suppose that $n = p$ and $X_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \,. \end{cases}$

Consider fitting ridge regression without intercept:

$$\hat{\beta}^{\text{ridge}} = \arg\min_{\beta} \sum_{j=1}^{p} (Y_j - \beta_j)^2 + \lambda \sum_{j=1}^{p} \beta_j^2.$$

In this simple case, $\hat{\beta}_j^{\text{OLS}} = Y_j$ and $\hat{\beta}_j^{\text{ridge}} = Y_j/(1+\lambda)$ (OLS stands for ordinary least squares).

So $\hat{\beta}^{\text{ridge}} = \dfrac{1}{1+\lambda}\hat{\beta}^{\text{OLS}}$, i.e. the ridge estimate is obtained by *shrinking* the OLS estimate by a factor of $1+\lambda$.



The importance of feature scaling

Suppose X_1 is height. Does it matter if it's measured in inches or feet?

For least squares, does not matter. If $X_1 \to 12X_1$, then $\widehat{\beta}_1 \to \frac{1}{12}\widehat{\beta}_1$.

$$\widehat{\beta}^{\text{least squares}} = \arg\min_{\beta} \sum_{i=1}^{n} (Y_i - (\beta_0 + \beta_1 X_{i1} + \cdots + \beta_p X_{ip}))^2$$

For ridge regression, it does matter! Implicitly, all features assumed on the same scale.

$$\widehat{\beta}^{\text{ridge}} = \arg\min_{\beta} \sum_{i=1}^{n} (Y_i - (\beta_0 + \beta_1 X_{i1} + \cdots + \beta_p X_{ip}))^2 + \lambda \sum_{j=1}^{p} \beta_j^2$$

Feature standardization

To put features on the same scale, center each feature and divide by its std. dev.:

$$X_{ij}^{\text{std}} = \frac{X_{ij} - \overline{X}_j}{\widehat{\sigma}_j}; \qquad \widehat{\sigma}_j^2 = \frac{1}{n}\sum_{i=1}^{n}(X_{ij} - \overline{X}_j)^2.$$

Feature standardization is recommended before applying ridge regression.

Treatment of correlated features

Linear regression coefficients for correlated features tend to be unstable.

Ridge regression is more stable, “splitting the credit” among correlated features.

For example, consider the linear regression

$$y = \beta_1 X_1 + \beta_2 X_1 + \epsilon,$$

where we’ve accidentally added the same feature twice.

- Linear regression is undefined because (β_1, β_2) and $(\beta_1 - c, \beta_2 + c)$ give the same RSS for each c.
- Ridge regression will obtain $\widehat{\beta}$ from $y = \beta X_1 + \epsilon$, and set $\widehat{\beta}_1 = \widehat{\beta}_2 = \frac{1}{2}\widehat{\beta}$.

Logistic regression with ridge penalty

Logistic regression can be penalized, just like linear regression!

Recall $\mathscr{L}(\beta)$, the logistic regression likelihood. We can view $-\log \mathscr{L}(\beta)$ as analogous to the linear regression RSS. Continuing the analogy, we can define

$$\hat{\beta}^{\text{ridge}} = \arg\min_{\beta} \left\{ -\log \mathscr{L}(\beta) + \lambda \sum_{j=1}^{p} \beta_j^2 \right\}.$$

Subtle point: While $\hat{\beta}^{\text{ridge}}$ is trained based on a (penalized) log-likelihood, during cross-validation we should choose λ based on whatever measure of test error we care about (e.g. misclassification error).

Can we get p-values from a ridge regression?

Unfortunately, we can’t. Ridge regression is exclusively a prediction method.