南方科技大学
SOUTHERN UNIVERSITY OF SCIENCE AND TECHNOLOGY

DDL: 14:00 Thursday of the sixteenth academic week (June 5th) .

The homework contains 4 questions and the score is 100 in total.

1. (25 marks) The following data were obtained for the growth of a sheep population introduced into a new environment on the island of Tasmania (adapted from J. Davidson, "On the Growth of the Sheep Population in Tasmania," *Trans. R. Soc. S. Australia* **62**(1938): 342–346).

t (year)	1814	1824	1834	1844	1854	1864
$P(t)$	125	275	830	1200	1750	1650

   (a) (15 marks) Make an estimate of M by graphing $P(t)$.

   (b) (10 marks) Plot $\ln\left[\frac{P}{M-P}\right]$ against t. If a logistic curve seems reasonable, estimate rM and t^*.

2. (25 marks) Consider the spreading of a highly communicable disease on an isolated island with population size N. A portion of the population travels abroad and returns to the island infected with the disease. You would like to predict the number of people X who will have been infected by some time t. Consider the following model, where $k > 0$ is constant:

$$\frac{dX}{dt} = kX(N - X)$$

   (a) List two major assumptions implicit in the preceding model. How reasonable are your assumptions?

   (b) Graph $\frac{dX}{dt}$ versus X.

   (c) Graph X versus t if the initial number of infections is $X_1 < \frac{N}{2}$. Graph X versus t if the initial number of infections is $X_2 > \frac{N}{2}$.

   (d) Solve the model given earlier for X as a function of t.

(e) From part (d), find the limit of X as t approaches infinity.

(f) Consider an island with a population of 5000. At various times during the epidemic the number of people infected was recorded as follows:

t (days)	2	6	10
X (people infected)	1887	4087	4853
$\ln\left(\frac{X}{N-X}\right)$	-0.5	1.5	3.5

Do the collected data support the given model?

(g) Use the results in part (f) to estimate the constants in the model, and predict the number of people who will be infected by $t = 12$ days.

3. (20 marks) The fish and game department in a certain state is planning to issue hunting permits to control the deer population(one deer per permit). It is known that if the deer population falls below a certain level m, the deer will become extinct. It is also known that if the deer population rises above the carrying capacity M, the population will decrease back to M through disease and malnutrition.

   (a) (5 marks) Discuss the reasonableness of the following model for the growth rate of the deer population as a function of time:

   $$\frac{\mathrm{d}P}{\mathrm{d}t} = rP(M-P)(P-m)$$

   where P is the population of the deer and r is a positive constant of proportionality. Include a phase line.

   (b) (5 marks) Explain how this model differs from the logistic model $\mathrm{d}P/\mathrm{d}t = rP(M-P)$. Is it better or worse than the logistic model?

   (c) (5 marks) Show that if $P > M$ for all t, then $\lim_{t\to\infty} P(t) = M$.

   (d) (5 marks) Discuss the solutions to the differential equation. What are the equilibrium points of the model? Explain the dependence of the steady-state value of P on the initial values of P. About how many permits should be issued?

4. (25 marks) A patient is given a dosage Q of a drug at regular intervals of time T. The concentration of the drug in the blood has been shown experimentally to obey the law:

$$\frac{dC}{dt} = -ke^{C}$$

(a) If the first dose is administered at $t = 0$ hr, show that after T hr have elapsed, the residual

$$R_1 = -\ln(kT + e^{-Q})$$

remains in the blood.

(b) Assuming an instantaneous rise in concentration whenever the drug is administered, show that after the second dose and T hr have elapsed again, the residual

$$R_2 = -\ln\left[kT(1 + e^{-Q}) + e^{-2Q}\right]$$

(c) Show that the limiting value R of the residual concentrations for doses of Q mg/ml repeated at intervals of T hr is given by the formula

$$R = -\ln\left(\frac{kT}{1 - e^{-Q}}\right)$$

(d) Assuming the drug is ineffective below a concentration L and harmful above some higher concentration H, show that the dose schedule T for a safe and effective concentration of the drug in the blood satisfies the formula

$$T = \frac{1}{k}(e^{-L} - e^{-H})$$

where k is a positive constant.

Hint: You can reasonably use any AI tools to assist you in completing your homework. Attention: Please submit ONLY the PDF of your homework to

jzlisustc@gmail.com to keep record.