A Multigraded $\mathfrak{S}_n$-equivariant Infinite Resolution

Jacob Zoromski

University of Notre Dame

Introduction

Minimal free resolutions contain lots of information about modules, and their study is a major research area in commutative algebra. For a graded ring R with homogeneous maximal ideal $\mathfrak{m}$, and a finitely generated graded R-module M, a **minimal free resolution** of M is a complex

$$0 \leftarrow F_0 \leftarrow F_1 \xleftarrow{\partial} F_2 \xleftarrow{\partial} \dots$$

where $H_0(F_\bullet) \cong M$, $H_{>0}(F_\bullet) = 0$, and $\partial(F_i) \subset \mathfrak{m}F_{i-1}$. When R is a regular ring, the minimal free resolution of M has finite length, but when R is not regular, the minimal free resolution has infinite length. The structure of infinite resolutions is intricate, and not much is known.

Equivariant Betti Numbers

The ranks of the modules in the minimal free resolution give **graded Betti numbers**

$$\beta_{i,j}(M) = \text{rank of degree } j \text{ piece of } F_i = \dim_k \operatorname{Tor}_i^R(M,k)_j$$

Graded Betti numbers are important invariants of the module, but when we have a group action, we have even finer invariants. If the free resolution is G**-equivariant**, i.e. $g\partial = \partial g$ for all $g \in G$, then each $\operatorname{Tor}_i(M,k)$ is a G-representation.

For example, if $F_\bullet$ equivariant under the torus $(k^*)^n$, then each $\operatorname{Tor}_i(M,k)$ is a $\mathbb{Z}^n$-graded vector space, in which case we have **multigraded Betti numbers** $\operatorname{Tor}_i(M,k)_\lambda$ for $\lambda \in \mathbb{Z}^n$. This occurs, for example, when R is a factor ring of a monomial ideal. If, in addition, we have an $\mathfrak{S}_n$ action (i.e. an $(k^*)^n \rtimes \mathfrak{S}_n$-equivariant resolution), then each vector space

$$\operatorname{Tor}_i(M,k)_{\langle\lambda\rangle} := \bigoplus_{\text{permutations } \mu \text{ of } \lambda} \operatorname{Tor}_i(M,k)_\mu$$

is an $\mathfrak{S}_n$-representation.

The Problem

Setup:

- $S = k[x_1, \dots, x_n]$, where k is algebraically closed, characteristic 0.
- $I = (x_i x_j : i \neq j) \subset S$, the ideal of squarefree monomials of degree 2.
- $R = S/I$

Since I is monomial, a minimal resolution will be multigraded. $\mathfrak{S}_n$ acts on S by permuting the variables. This makes I an $\mathfrak{S}_n$-invariant ideal as well. Thus, R-modules have a $(k^*)^n \rtimes \mathfrak{S}_n$-equivariant resolution.

Goals:

- Construct a free R-resolution of k that respects the multigraded and symmetric group structure. That is, construct a $(k^*)^n \rtimes \mathfrak{S}_n$-equivariant resolution.
- Describe $\operatorname{Tor}_i^R(k,k)$ in terms of its multigraded and symmetric group structure, i.e. describe $\operatorname{Tor}_i^R(k,k)_{\langle\lambda\rangle}$ as an $\mathfrak{S}_n$ representation.

Carlitz Multipermutations

Our construction relies on the notion of a Carlitz multipermutation. A **multipermutation** is a finite length sequence of the symbols $1, 2, \dots, n$. A multipermutation is **Carlitz** if it contains no neighbors.

For example, 132234 is a non-Carlitz multipermutation (there are two 2's in a row), while 421323 is Carlitz.

Resolution Construction

Let F_i be the free R-module with basis consisting of the Carlitz multipermutations of length i.

Multigraded structure: The multidegree $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n)$ component of F_i has basis consisting of those multipermutations with λ_1 1's, λ_2 2's, and so on. For example, for $n = 3$, the multidegree $(2,1,1)$ component of F_4 consists of the basis

$$\{1231, 1321, 1213, 1312, 2131, 3121\}$$

Symmetric group structure: $\mathfrak{S}_n$ acts on a multipermutation by permuting the symbols $1, 2, \dots, n$. For example, the $\mathfrak{S}_3$ orbit of 1231 is

$$\{1231, 2132, 3213, 1321, 2312, 3123\}$$

Differential: The differential removes the last number from a multipermutation and replaces it with the corresponding variable. For example, $\partial(1231) = x_1 123$.

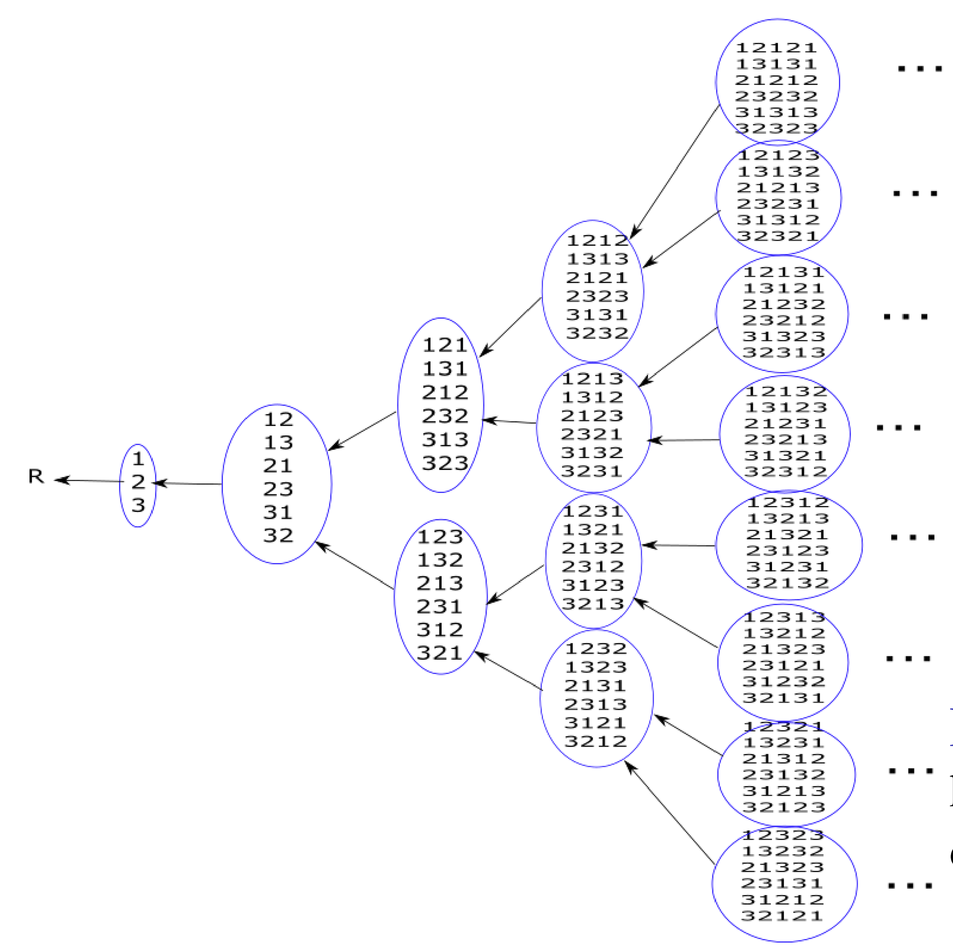

Figure: $n = 3$. Grouped by $\mathfrak{S}_3$-orbits

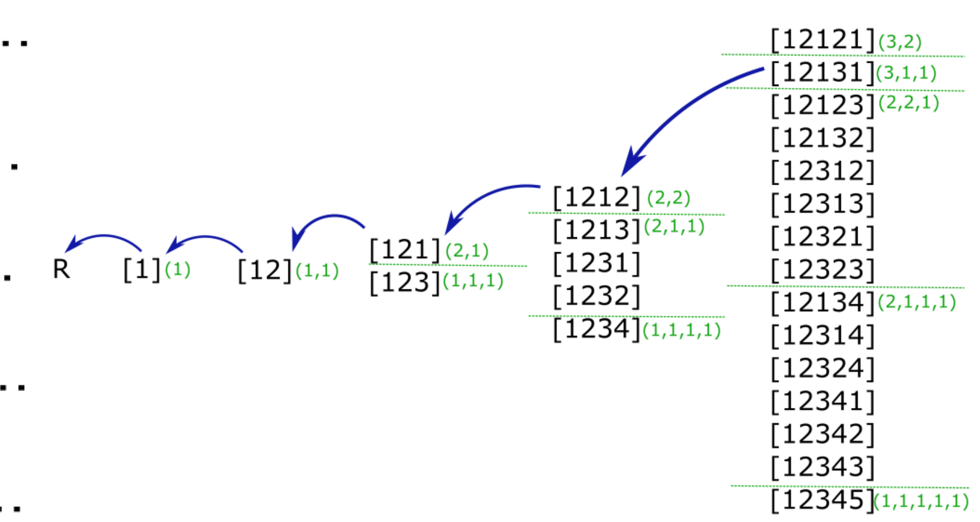

Figure: First 5 steps for $n \geq 5$. Each $\mathfrak{S}_n$-orbit has one representative. Grouped by multidegree denoted in green.

Structure of $\operatorname{Tor}_i^R(k,k)$

Let λ be a partition of i, i.e. $\lambda_1 \geq \lambda_2 \geq \dots \geq 0$ and $\sum \lambda_j = i$. Denote this by $\lambda \vdash i$. Let $\ell(\lambda)$ denote the number of non-zero entries of λ.

Theorem

$$\operatorname{Tor}_i^R(k,k) \cong \bigoplus_{\lambda} \operatorname{Tor}_i^R(k,k)_{\langle\lambda\rangle} \cong \bigoplus_{\substack{\lambda \vdash i \\ \lambda_1 \leq \lceil \frac{i}{2} \rceil}} \operatorname{Ind}_{\mathfrak{S}_{\ell(\lambda)}}^{\mathfrak{S}_n}(\text{Regular representation})^{\oplus c'_\lambda}$$

Counting Carlitz Multipermutations

We are interested in counting Carlitz multipermutations. For $\lambda \in \mathbb{Z}^n_{\geq 0}$, a multipermutation corresponding with λ consists of λ_1 1's, λ_2 2's, etc. Let c_λ denote the number of Carlitz multipermutations corresponding to λ. As an example, $c_{(2,1,1)} = 6$. Let c'_λ denote the ordered Carlitz multipermutations corresponding to λ (two multipermutations have the same order if they are equivalent under swapping elements appearing the same number of times).

Theorem:

$$c_\lambda = \sum_{i_1<i_2} c_{\lambda - \mathbf{e}_{i_1 i_2}} + \sum_{i_1<i_2<i_3} 2c_{\lambda - \mathbf{e}_{i_1 i_2 i_3}} + \dots + (n-1)c_{\lambda - \mathbf{e}_{1\dots n}}$$

where $\mathbf{e}_i \in \mathbb{Z}^n$ is the standard basis vector. This generalizes previous work by [1] which gave formulas for $\lambda = (a, a, \dots, a)$. It can be proved by induction, but what's interesting is that it can be seen by techniques of infinite resolutions.

The information from a free resolution can be encoded in a power series, called the (multigraded) **Poincare series**

$$P_M^R(\mathbf{x}, z) = \sum_{\substack{i \geq 0 \\ \lambda \in \mathbb{Z}^n_{\geq 0}}} \dim_k \operatorname{Tor}_i(M,k)_\lambda \mathbf{x}^\lambda z^i$$

A ring R is **Golod**, as it is in our case, if and only if Poincare series is as large as possible, and is determined by resolving k and I over S:

$$P_k^R(\mathbf{x}, z) = \frac{P_k^S(\mathbf{x}, z)}{1 - z^2 P_I^S(\mathbf{x}, z)}$$

We know the Betti numbers of I over S by a theorem in [2] so, by our resolution,

$$1 + \sum_{\lambda \in \mathbb{Z}^n_{\geq 0}} c_\lambda \mathbf{x}^\lambda z^{|\lambda|} = \frac{1 + \sum x_i z + \sum x_{i_1} x_{i_2} z^2 + \dots + x_1 \dots x_n z^n}{1 - \sum x_i x_j z^2 - \sum 2 x_{i_1} x_{i_2} x_{i_3} z^3 - \dots - (n-1) x_1 \dots x_n z^n}$$

which implies the counting formula (look at the denominator!).

λ	c_λ	λ	c_λ	λ	c_λ	λ	c_λ
(1)	1	(1,1,1)	6	(1,1,1,1)	24	(5,5,2)	222
(1,1)	2	(2,1,1)	6	(2,1,1,1)	36	(6,5,2)	206
(k,k)	2	(3,1,1)	2	(3,1,1,1)	24	(6,4,4)	1700
$(k,k-1)$	1	(2,2,1)	12	(2,2,1,1)	84	(6,6,3)	1882
$(k+1,k-1,1)$	k	(3,2,1)	10	(4,1,1,1)	6	(6,5,5)	11492
$(k+1,k-a,a)$	$\binom{k}{a}$	(2,2,2)	30	(3,2,1,1)	96	(3,2,2,2)	1686
		(3,3,1)	18	(2,2,2,1)	246	(4,2,2,1)	336
		(3,2,2)	38	(4,2,1,1)	54	(4,3,2,1)	1074
		(4,3,1)	14	(3,3,1,1)	184	(3,3,2,2)	4204

Table: c_λ, the number of Carlitz multipermutations with λ_1 1's, λ_2 2's, etc. for various λ

References

[1] Henrik Eriksson and Alexis Martin.
Enumeration of carlitz multipermutations.
arXiv.org, 2017.

[2] Federico Galetto.
On the ideal generated by all squarefree monomials of a given degree.
Journal of commutative algebra, 12(2), 2020.