MATH 217 W24 - LINEAR ALGEBRA, Section 001 (Dr. Paul Kessenich)
Homework Set Part B due Thurs, Feb 8 at 11:59pm
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1. Let $\mathbb{R}^n \xrightarrow{T} \mathbb{R}^n$ be a linear transformation. As on HW 3, we define T^k to be the k-fold composition of T with itself. Let A be the standard matrix of T, by which we mean the unique $n \times n$ matrix such that $T(\vec{x}) = A\vec{x}$ for all $\vec{x} \in \mathbb{R}^n$.

   (a) Prove that for all k, the standard matrix for T^k is the matrix A^k. [Hint: induction works nicely.]

   Solution: We are given that the standard matrix $A^{(1)}$ represents the transformation $T^{(1)}$. Assume that the transformation T^n can be represented by the standard matrix A^n. We know by a theorem on the worksheets that the standard matrix of two linear transformations, both from $\mathbb{R}^n \to \mathbb{R}^n$, is equal to the product of their respective standard matrices. Then $(T^n \circ T)(x) = A^n A\vec{x}$. This is equal to $T^{n+1}(x) = A^{n+1}\vec{x}$. So by induction, $T^k(\vec{x}) = A^k\vec{x}$ for all $\vec{x} \in \mathbb{R}^n$ and $k \in \mathbb{N}$.

   (b) We define T to be nilpotent if there exists some $k \in \mathbb{N}$ such that T^k is the zero transformation. Prove that if T is nilpotent, then A is not invertible.

   Solution: Assume A is invertible. Let $k \in \mathbb{N}$ such that T^k is the zero transformation. We know by part (a) that the standard matrix of T^k is A^k. By problem 6c on Worksheet 6 (CHECK CITATION), the inverse of A^k is $(A^{-1})^k$. However, the zero transformation has no inverse, so there is a contradiction. Thus A cannot be invertible.

   (c) Prove that if T is nilpotent, then $A - I_n$ is invertible. [Hint: try multiplying out $(A - I_n)(-I_n - A - A^2 - \cdots - A^{k-1})$ and see what you get.]

   Solution: Let $k \in \mathbb{N}$ such that T^k is the zero transformation.
   Expanding $(A - I_n)(-I_n - A - A^2 - \cdots - A^{k-1})$:

$$\begin{aligned}
&= -A\left(I_n + A + \cdots + A^{k-1}\right) + I_n\left(I_n + A + \cdots + A^{k-1}\right) && \text{(distributivity)}\\
&= -\left(A + A^2 + \cdots + A^{k-1} + A^k\right) + \left(I_n + A + \cdots + A^{k-1}\right) && \text{(distributivity)}\\
&= \left(I_n + A + \cdots + A^{k-1}\right) - \left(A + A^2 + \cdots + A^{k-1} + 0_{n\times n}\right) && (A^k = 0_{n\times n})\\
&= I_n
\end{aligned}$$

   Swapping the order of multiplication,

$$\begin{aligned}
&(-I_n - A - A^2 - \cdots - A^{k-1})(A - I_n)\\
&\quad= -\left(I_n + A + \cdots + A^{k-1}\right)A + \left(I_n + A + \cdots + A^{k-1}\right)I_n && \text{(distributivity)}\\
&\quad= -\left(A + A^2 + \cdots + A^{k-1} + A^k\right) + \left(I_n + A + \cdots + A^{k-1}\right) && \text{(distributivity)}\\
&\quad= \left(I_n + A + \cdots + A^{k-1}\right) - \left(A + A^2 + \cdots + A^{k-1} + 0_{n\times n}\right) && (A^k = 0_{n\times n})\\
&\quad= I_n
\end{aligned}$$

   So the inverse of $A - I_n$ is $-(I_n + A + A^2 + \cdots + A^{k-1})$, and $A - I_n$ is invertible by definition.

2. Let V be any vector space, and let S be any set. Let $\mathcal{F}(S, V)$ denote the set of all functions from S to V . (Note: we are not assuming $S \subseteq V$ here, just that S is some set. S is not assumed to be a vector space, but it could be. Similarly, the functions in $F(S, V)$ are not assumed to be linear transformations, although it is possible that some of them might be.)

   For any functions f, $g \in \mathcal{F}(S, V)$ we can define their sum to be the function $f + g$ given by the formula $(f + g)(s) = f(s) + g(s)$, where s is any element in S. Similarly, for any scalar $c \in R$ and any function $f \in \mathcal{F}(S, V)$ we define the function cf to be given by the formula $(cf)(s) = c(f(s))$ for all $s \in S$.

   (a) Prove that $\mathcal{F}(S, V)$ is a vector space. Note: For this problem you must explicitly prove that each of the vector space properties VS1-8 from Worksheet 6 is true. (These proofs should be very short but are not skippable.)

   Solution: Let arbitrary $a, b \in \mathbb{R}$, arbitrary $f, g, h \in \mathcal{F}(S, V)$. Note that $+_{\mathcal{F}(S,V)}$ borrows the qualities of $+_V$ (the summation operation of V) through the definition; namely associativity and commutativity.
   VS-1: $((f + g) + h)(s) = (f(s) +_V g(s)) +_V h(s) = f(s) +_V (g(s) +_V h(s)) = (f + (g + h))(s)$ by associativity of vector space addition.
   VS-2: $(f + g)(s) = f(s) + g(s) = g(s) +_V f(s) = (g + f)(s)$ by commutativity of vector space addition.
   VS-3:

   (b) Is $0_{\mathcal{F}(S,V)}$ the same element as 0_V ? If not, explain how they are different.

   (c) We could similarly define $\mathcal{F}(V, S)$ to be the set of all functions from V to S. Would $\mathcal{F}(V, S)$ also a vector space? Why or why not?

   (d) The familiar vector spaces $\mathcal{P}$, $\mathcal{P}_n$ and $\mathcal{C}^\infty$ (all from Worksheet 6) are all subsets of $\mathcal{F}(S, V)$ for some S and V . What are S and V for each of these functions?