MATH 217 - W24 - LINEAR ALGEBRA
HOMEWORK 2, DUE Thursday, January 25 at 11:59pm

Submit Part A and Part B as two *separate* assignments. Include the following information in the top left corner of every assignment:

- your full name,
- instructor's last name and section number,
- homework number,
- whether they are Part A problems or Part B problems.

A few words about solution writing:

- **Unless you are explicitly told otherwise for a particular problem, you are always expected to show your work and to give justification for your answers.** In particular, when asked if a statement is true or false, you will need to explain *why* it is true or false to receive full credit.
- Write down your solutions in full, as if you were writing them for another student in the class to read and understand.
- Don't be sloppy, since your solutions will be judged on precision and completeness and not merely on "basically getting it right".
- Cite every theorem or fact from the book that you are using (e.g. "By Theorem 1.10 ...").
- If you compute something by observation, say so and make sure that your imaginary fellow student who is reading your proof can also clearly see what you are claiming.
- Justify each step in writing and leave nothing to the imagination.

Part A (10 points)

Solve the following problems from the book:

Section 1.3: 26, 34, 48;
Section 2.1: 6, 38, 44, 46.

Part B (25 points)

Let X and Y be sets. Recall that a **function** f **from** X **to** Y is a rule which assigns a unique element $f(x) \in Y$ to each element $x \in X$. We call X the **domain** or **source** of f, we call Y the **codomain** or **target space** of f, and we write $f : X \to Y$ to indicate that f is a function from X to Y. The **image** of f is defined to be the set $\operatorname{im}(f) = \{f(x) \mid x \in X\}$. (Note that the codomain and the image of f are not necessarily equal.) We say that the function $f : X \to Y$ is:

- **surjective** or **onto** if for all $y \in Y$, there exists at least one $x \in X$ such that $f(x) = y$;
- **injective** or **one-to-one** if for all $y \in \operatorname{im}(f)$ there exists at most one $x \in X$ such that $f(x) = y$;
- **bijective** if f is both injective and surjective.

Problem 1. In parts (a) – (d) below, determine whether the given function is injective, surjective, both, or neither. Justify your answers.

(a) the function $f : [0, 4] \to [0, 18]$ defined by $f(x) = x^2 + 2$;
(b) the function $g : \mathbb{R} \to \mathbb{R}$ defined by $g(x) = 2x - 5$;
(c) the function $h : \mathbb{R}^2 \to \mathbb{R}$ defined by $h(x, y) = 2x^2 + 5y^2$;
(d) the function $q : \mathbb{N} \to \mathbb{N}$ defined by $q(n) = \begin{cases} n, & \text{if } n \text{ is odd} \\ n/2 & \text{if } n \text{ is even.} \end{cases}$

Given a function $f : X \to Y$ and a subset $A \subseteq X$, we define the **direct image** or **forward image of A under** f to be the set

$$f[A] = \{f(x) \mid x \in A\}.$$

Similarly, if $B \subseteq Y$ then we define the **preimage** or **inverse image of B under** f to be the set

$$f^{-1}[B] = \{x \in X \mid f(x) \in B\}.$$

Note that $f[X] = \mathrm{im}(f)$.

Problem 2. Determine whether each statement is true or false. If it is true, prove it. If it is false, prove this by giving a counterexample.

(a) For every function $f : X \to Y$ and all $A, B \subseteq X$, if $A \cap B = \emptyset$, then $f[A] \cap f[B] = \emptyset$.
(b) For every function $f : X \to Y$ and all $A, B \subseteq X$, if $f[A] \cap f[B] = \emptyset$, then $A \cap B = \emptyset$.
(c) For every function $f : X \to Y$ and all $A \subseteq X$, we have $f^{-1}[f[A]] = A$.
(d) For every function $f : X \to Y$ and all $A \subseteq X$, we have $f[X \setminus A] = Y \setminus f[A]$.
(e) For every bijective function $f : X \to Y$ and all $A, B \subseteq X$, we have $f[A \cap B] = f[A] \cap f[B]$.

We call a function $T : \mathbb{R}^m \to \mathbb{R}^n$ a *linear transformation* if it satisfies:

(1) $T(\vec{x} + \vec{y}) = T(\vec{x}) + T(\vec{y})$ for all vectors $\vec{x}, \vec{y} \in \mathbb{R}^m$; and
(2) $T(k\vec{x}) = kT(\vec{x})$ for all vectors $\vec{x} \in \mathbb{R}^m$ and all scalars $k \in \mathbb{R}$.

(Note that this definition differs from the one given in Section 2.1 of the textbook.)

Problem 3.

(a) Prove that for every function $f : \mathbb{R} \to \mathbb{R}$, if $f(cx) = cf(x)$ for all $c \in \mathbb{R}$ and $x \in \mathbb{R}$, then $f(x + y) = f(x) + f(y)$ for all $x, y \in \mathbb{R}$. (In other words, prove that every function $f : \mathbb{R} \to \mathbb{R}$ that preserves scalar multiplication is a linear transformation from $\mathbb{R}$ to $\mathbb{R}$.)
(b) Give an example to show that the argument you gave in part (a) cannot work in 2 dimensions. That is, explicitly describe a function $f : \mathbb{R}^2 \to \mathbb{R}^2$ that is *not* a linear transformation but has the property that $f(c\vec{x}) = cf(\vec{x})$ for all $\vec{x} \in \mathbb{R}^2$ and $c \in \mathbb{R}$. Remember to prove that your example works!

Problem 4. Let $f : \mathbb{R} \to \mathbb{R}$ be a function, and suppose that $f(x + y) = f(x) + f(y)$ for all $x, y \in \mathbb{R}$. (In other words, suppose that f preserves addition).

(a) Prove that $f(0) = 0$.
(b) Prove that for all $x \in \mathbb{R}$, $f(-x) = -f(x)$.
(c) Use induction to prove that for all $n \in \mathbb{N}$ and $x \in \mathbb{R}$, $f(nx) = nf(x)$.
(d) Prove that for all $m \in \mathbb{Z}$ and $x \in \mathbb{R}$, $f(mx) = mf(x)$.
(e) (**RECREATIONAL**) Prove that for all $q \in \mathbb{Q}$ and $x \in \mathbb{R}$, $f(qx) = qf(x)$.

Remark: It will perhaps come as a surprise that the property in (c)-(e) cannot be extended to include arbitrary real scalars. That is, there exist functions $f : \mathbb{R} \to \mathbb{R}$ such that $f(x+y) = f(x) + f(y)$ for all $x, y \in \mathbb{R}$ but also $f(cx) \neq cf(x)$ for some $c, x \in \mathbb{R}$. Put yet another way, there exist functions $f : \mathbb{R} \to \mathbb{R}$ that preserve addition but not scalar multiplication. *This* fact is actually rather difficult to prove!)