MATH 217 - W24 - LINEAR ALGEBRA
HOMEWORK 10, DUE Thursday, April 11 at 11:59pm

Submit Part A and Part B as two *separate* assignments in Gradescope as a **pdf file.** At the time of submission, Gradescope will prompt you to match each problem to the page(s) on which it appears. **You must match problems to pages in Gradescope so we know what page each problem appears on.** Failure to do so may result in not having the problem graded.

A few words about solution writing:

- Unless you are explicitly told otherwise for a particular problem, **you are always expected to show your work and to give justification for your answers.**
- Your solutions will be judged on precision and completeness and not merely on "basically getting it right".
- Cite every theorem or fact from the book that you are using (e.g. "By Theorem 1.10 ...").

Part A (10 points)

Solve the following problems from the book:
6.1: 20, 54
6.2: 42, 50
6.3: 14
7.1: 12, 18, 42

Part B (25 points)

Problem 1. If V and W are vector spaces, a function $F : V \times W \to \mathbb{R}$ is said to be *bilinear* if all of the following hold:

- for all $\vec{x}, \vec{y} \in V$ and $\vec{z} \in W$, $F(\vec{x}+\vec{y}, \vec{z}) = F(\vec{x}, \vec{z}) + F(\vec{y}, \vec{z})$;
- for all $\vec{x} \in V$ and $\vec{y}, \vec{z} \in W$, $F(\vec{x}, \vec{y}+\vec{z}) = F(\vec{x}, \vec{y}) + F(\vec{x}, \vec{z})$;
- for all $\vec{x} \in V$ and $\vec{y} \in W$ and for all $a \in \mathbb{R}$, $F(a\vec{x}, \vec{y}) = aF(\vec{x}, \vec{y})$ and $F(\vec{x}, a\vec{y}) = aF(\vec{x}, \vec{y})$.

Furthermore, if $F : V \times V \to \mathbb{R}$ is a bilinear function, we say that F is *alternating* if $F(\vec{v}, \vec{v}) = 0$ for all $\vec{v} \in V$. Throughout this problem, let $F : \mathbb{R}^2 \times \mathbb{R}^2 \longrightarrow \mathbb{R}$ be a bilinear function.

(a) Prove that F is alternating if and only if $F(\vec{u}, \vec{v}) = -F(\vec{v}, \vec{u})$ for all $\vec{u}, \vec{v} \in \mathbb{R}^2$.

(b) Prove that if F is alternating and $F(\vec{e}_1, \vec{e}_2) = 1$, then

$$F(\vec{u}, \vec{v}) = \det[\vec{u}\ \vec{v}] \qquad \text{for all } \vec{u}, \vec{v} \in \mathbb{R}^2.$$

Problem 2. Let $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathbb{R}^{2\times 2}$, and consider the map $T : \mathbb{R}^{2\times 2} \to \mathbb{R}^{2\times 2}$ defined by $T(A) = AM$.

(a) Prove that T is a linear transformation.

(b) Find the $\mathcal{E}$-matrix $[T]_{\mathcal{E}}$ of T, where $\mathcal{E}$ is the ordered basis

$$\mathcal{E} = (E_{11}, E_{12}, E_{21}, E_{22}) = \left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right)$$

of $\mathbb{R}^{2\times 2}$. Your answer should be in terms of the entries of M.

(c) Compute $\det[T]_{\mathcal{E}}$.

(d) Compute $\det[T]_{\mathcal{B}}$, where $\mathcal{B}$ is the ordered basis of $\mathbb{R}^{2\times 2}$ given by

$$\mathcal{B} = \left(\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ \pi & 1 \end{bmatrix}, \begin{bmatrix} 0 & 7 \\ 8 & 9 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \right).$$

(e) Either prove that T is always diagonalizable no matter what M is, or provide an explicit example of a matrix M for which T is *not* diagonalizable and briefly explain why your example works.

Problem 3. Let $\vec{u}, \vec{v}, \vec{w}$ be vectors in $\mathbb{R}^4$. Define the linear transformation $T : \mathbb{R}^4 \to \mathbb{R}$ by the rule $T(\vec{x}) = \det([\vec{x}\ \vec{u}\ \vec{v}\ \vec{w}])$ for all $\vec{x} \in \mathbb{R}^4$. (You do not have to prove that T is linear.)

(a) Prove that there exists a unique vector $\vec{z} \in \mathbb{R}^4$ such that $T(\vec{x}) = \vec{z}\cdot\vec{x}$ for all $\vec{x} \in \mathbb{R}^4$, and find the components of $\vec{z}$ in terms of the vectors $\vec{u}, \vec{v}$, and $\vec{w}$. (Hint: $\vec{x} = x_1\vec{e}_1 + x_2\vec{e}_2 + x_3\vec{e}_3 + x_4\vec{e}_4$.)

(b) Find the vector $\vec{z}$ (as in part (a)) when $\vec{u} = \vec{e}_1$, $\vec{v} = \vec{e}_2$, and $\vec{w} = \vec{e}_3$ are the first three standard basis vectors in $\mathbb{R}^4$.

(c) When is $\vec{z} = \vec{0}$? (Your answer should be in terms of $\vec{u}$, $\vec{v}$, and $\vec{w}$.)

(d) Prove that $\vec{z}$ is orthogonal to each of $\vec{u}, \vec{v}$ and $\vec{w}$, and find $\det([\vec{z}\ \vec{u}\ \vec{v}\ \vec{w}])$ in terms of $\|\vec{z}\|$.

Problem 4. For a polynomial $p(x)$ and an $n \times n$ matrix A, let $p(A)$ denote the matrix obtained by plugging in A for x. For example, if $p(x) = x^3 + 2x^2 + 3$, then $p(A) = A^3 + 2A^2 + 3I_n$. (Note that I_n behaves like the constant "1" in $\mathbb{R}^{n\times n}$.)

(a) Prove that for every $n \times n$ matrix A and for every eigenvalue λ of A, the real number $p(\lambda)$ is an eigenvalue of the $n \times n$ matrix $p(A)$.

(b) Let p be a polynomial and let $n \in \mathbb{N}$. Prove that if S is an invertible $n \times n$ matrix, then for every $A \in \mathbb{R}^{n\times n}$ we have $p(S^{-1}AS) = S^{-1}p(A)S$.

(c) Let p be a polynomial and let A be an $n \times n$ matrix. Prove that if A is diagonalizable, then every eigenvalue of $p(A)$ is of the form $p(\lambda)$ for some eigenvalue λ of A.

Problem 5.

(a) Let $A \in \mathbb{R}^{2\times 2}$ be a 2×2 matrix such that $A^2 = I_2$. Prove that A is diagonalizable. (Hint: try factoring $A^2 - I_2$, and consider the possible ranks of the factors.)

(b) Does the same result hold for larger matrices? That is, if $A \in \mathbb{R}^{n\times n}$ is an $n \times n$ matrix for which $A^2 = I_n$, must A be diagonalizable? Either prove this or give a counterexample.