MATH 217 W24 - LINEAR ALGEBRA, Section 001 (Dr. Paul Kessenich)
Homework 3 Part B due Thursday, February 1 at 11:59pm

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1. Determine whether the following statements are true or false, and justify your answer with a proof or a counterexample.

   (a) For all 2×2 matrices A and B, $(AB)^T = A^T B^T$.

   Solution: False. For example,

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}$$

$$\begin{aligned} (AB)^\top &= \begin{bmatrix} 19 & 22 \\ 43 & 50 \end{bmatrix}^\top \\ &= \begin{bmatrix} 1 & 22 \\ 43 & 50 \end{bmatrix} \\ \neq A^\top B^\top &= \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} 5 & 7 \\ 6 & 8 \end{bmatrix} \\ &= \begin{bmatrix} 23 & 31 \\ 34 & 46 \end{bmatrix} \end{aligned}$$

   (b) For all 2×2 matrices A and B, $(AB)^\top \neq A^\top B^\top$.

   Solution: False. For example:

$$A = \begin{bmatrix} 1 & 3 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

$$\begin{aligned} (AB)^\top &= \begin{bmatrix} 1 & 3 \\ 0 & 0 \end{bmatrix}^\top \\ &= \begin{bmatrix} 1 & 0 \\ 3 & 0 \end{bmatrix} \\ = A^\top B^\top &= \begin{bmatrix} 1 & 0 \\ 3 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ 3 & 0 \end{bmatrix} \end{aligned}$$

   (c) For all matrices A and B such that the matrix product AB exists, $(AB)^\top = B^\top A^\top$.

Solution: True. Computing with arbitrary matrices A and B with ijth element a_{ij}, bij respectively, we see the two products are identical.

$$\begin{aligned}(AB)^\top &= \begin{bmatrix} a_{11}b_{11}+a_{12}b_{21} & a_{11}b_{12}+a_{12}b_{22} \\ a_{21}b_{11}+a_{22}b_{21} & a_{21}b_{12}+a_{22}b_{22}\end{bmatrix}^\top \\ &= \begin{bmatrix} a_{11}b_{11}+a_{12}b_{21} & a_{21}b_{11}+a_{22}b_{21} \\ a_{11}b_{12}+a_{12}b_{22} & a_{21}b_{12}+a_{22}b_{22}\end{bmatrix}\end{aligned}$$

$$\begin{aligned}B^\top A^\top &= \begin{bmatrix} b_{11} & b_{21} \\ b_{12} & b_{22}\end{bmatrix}\begin{bmatrix} a_{11} & a_{21} \\ a_{12} & a_{22}\end{bmatrix} \\ &= \begin{bmatrix} b_{11}a_{11}+b_{21}a_{12} & b_{11}a_{21}+b_{21}a_{22} \\ b_{12}a_{11}+b_{22}a_{12} & b_{12}a_{21}+b_{22}a_{22}\end{bmatrix} \quad = \begin{bmatrix} a_{11}b_{11}+a_{12}b_{21} & a_{21}b_{11}+a_{22}b_{21} \\ a_{11}b_{12}+a_{12}b_{22} & a_{21}b_{12}+a_{22}b_{22}\end{bmatrix}\end{aligned}$$

(d) If A is a symmetric matrix, then for all $n \in \mathbb{N}$, A^n is also symmetric.

Solution: True. By definition, if A is symmetric, then $A = A^\top$. Assume $A^n = (A^n)^\top$. Then

$$A^{n+1} = A^n A = (A^n)^\top A^\top.$$

By part (c) of this problem,

$$(A^n)^\top A^\top = (AA^n)^\top = (A^{n+1})^\top.$$

So if A is symmetric, A^n is symmetric for all n by induction.

(e) If A is a square matrix and A^2 is symmetric, then so is A.

Solution: False. The matrix $A = \begin{bmatrix} -1 & -2 \\ 2 & 1\end{bmatrix}$ is not symmetric. However, $A^2 = \begin{bmatrix} -3 & 0 \\ 0 & -3\end{bmatrix}$ is symmetric. This example demonstrates that A is not necessarily symmetric if A^2 is.

2. Determine whether the following statements are true or false, and justify your answer with a proof or a counterexample.

(a) Every 3-by-3 matrix that has a row of zeros is not invertible.

Solution: A matrix with a row of zeros can have at most 2 pivots, for a rank of 2. Thus by Theorem 2.4.3, it is not invertible.

(b) Every square matrix with 1's down the main diagonal is invertible.

Solution: False. For example, $A = \begin{bmatrix} 1 & 2 \\ 0.5 & 1\end{bmatrix}$ is not bijective: $A\begin{bmatrix}2\\0\end{bmatrix} = \begin{bmatrix}2\\1\end{bmatrix} = A\begin{bmatrix}0\\1\end{bmatrix}$ shows that there are multiple solutions to the same matrix equation. Thus it is not invertible.

(c) For any matrix A, if A is invertible, then so is A^{-1}.

Solution: True. Since $AA^{-1} = A^{-1}A = I_n$, A^{-1} is invertible by definition 2.16 from the Theory of Linear Algebra handout.

(d) For any matrix A, if A is invertible, then A^n is invertible.

Solution: True. This is a special case of Problem 6c on Worksheet 7, where $A_1 = \ldots = A_n = A$. The inverse is thus $(A^n)^{-1} = (A^{-1})^n$.

3. Let A be an $m \times n$ matrix. Prove that if there exists an $n \times m$ matrix B such that $BA = I_n$, then the system of linear equations $A\vec{x} = \vec{0}$ has a unique solution. (Note: a matrix B with this property is called a left-inverse for A. Can you guess why?)

Solution: Left multiplying the matrix equation by B,

$$\begin{aligned} BA\vec{x} &= B\vec{0} \\ I_n\vec{x} &= \vec{0} \\ \vec{x} &= \vec{0} \end{aligned}$$

Then the only solution to the system is $\vec{x} = \vec{0}$.

4. Given two matrices A and B such that the product AB is defined (say, A is $n \times m$ and B is $m \times k$), exactly one of the following two statements is true:

(a) Every column of AB is a linear combination of columns of A,

(b) Every column of AB is a linear combination of columns of B.

Prove the one that is true, and provide a counterexample for the one that is false.

Solution: By theorem 2.3.2, the matrix product $AB = \begin{bmatrix} | & | & | \\ \vec{a_1} & \ldots & \vec{a_m} \\ | & | & | \end{bmatrix} \begin{bmatrix} | & | & | \\ \vec{b_1} & \ldots & \vec{b_k} \\ | & | & | \end{bmatrix} = \begin{bmatrix} | & | & | \\ A\vec{b_1} & \ldots & A\vec{b_k} \\ | & | & | \end{bmatrix}$. =By Example 13, we know that each product $A\vec{b_j}$ is the linear combination $\vec{a_1}b_{1j} + \ldots + \vec{a_m}b_{mj}$, where b_{ij} is the ijth element of B. Thus statement (a) is true. For a counterexample to (b), take $\begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 3 & 6 \\ 2 & 4 \end{bmatrix}$.

5. Let $f : X \to X$ be a function. We let f^n denote the function $f^n : X \to X$ given by composing f iteratively, n many times. Also, we define f^0 to be the identity function, i.e. $\forall x \in X, f^0(x) = x$.

(a) Assume that $X = \mathbb{R}^d$. Prove by induction that if f is a linear transformation, then the nth iterate f^n is also a linear transformation.

Solution: Base case: $f^1 = f$ is a linear transformation.
Induction hypothesis: Assume f^n is a linear transformation.
Inductive step: $f^{n+1} = f^n \circ f$ by definition. By Theorem 2.12 from the Theory of Linear Algebra handout, the composition of two linear transformations is also linear. Thus f^{n+1} is linear. Thus the statement is true for all n.

(b) Find an example of a function $f : \mathbb{R}^2 \to \mathbb{R}^2$ which is not a linear transformation, but for which there exists an n such that the nth iterate f^n is a linear transformation.

Solution: $f(x, y) = (1 - x, y)$ is not linear, but $f^2(x, y) = f(f(x, y)) = (1 - (1 - x), y) = (x, y)$ is linear.

(c) Prove that for $X = \mathbb{R}^d$ and f linear, if the equation $f(x) = 0$ has a unique solution, then the iterated equation $f^n(x) = 0$ also has a unique solution.

Solution: Let A be the unique standard matrix of f using the Key Theorem. Then the composition f^n can be represented by the standard matrix A^n. Since $f(x) = 0$ has a unique solution, the matrix equation $A\vec{x} = \vec{0}$ has a unique solution. By the linear combination definition of matrix multiplication, $B\vec{0} = \vec{0}$ for all compatible matrices B, so the unique solution for A must be $\vec{0}$. The matrix equation $A^n\vec{x} = \vec{0}$ represents the iterated equation $f^n(x) = 0$.
When $\vec{x} = \vec{0}$,

$$A^n\vec{0} = \vec{0}.$$

Meanwhile, when $\vec{x} \neq \vec{0}$, we know $A\vec{x} \neq \vec{0}$. Assuming $A^n\vec{x} \neq \vec{0}$, then

$$A^{n+1}\vec{x} = A(A^n\vec{x}) = A(\text{nonzero vector}) \neq 0.$$

So by induction, $A^n\vec{x} = \vec{0}$ has no solutions when $\vec{x}$ is nonzero. Thus $\vec{0}$ is the unique solution to f.