

**MATH 217 W24 - LINEAR ALGEBRA, Section 001 (Dr. Paul Kessenich)**  
**Homework Set Part B due ??? at 11:59pm**  
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1. Question

- (a) Prove that  $F$  is alternating if and only if  $F(\vec{u}, \vec{v}) = -F(\vec{v}, \vec{u})$  for all  $\vec{u}, \vec{v} \in \mathbb{R}^2$ .

**Solution:** By bilinearity, we know

$$\begin{aligned} F(u + v, v + u) &= 0 \\ F(u, v + u) + F(v, v + u) &= 0 \\ F(u, v) + F(u, u) + F(v, v) + F(v, u) &= 0 \\ F(u, v) + 0 + 0 + F(v, u) &= 0 \\ F(u, v) + F(v, u) &= 0 \\ F(u, v) &= -F(v, u) \end{aligned}$$

- (b) Prove that if  $F$  is alternating and  $F(\vec{e}_1, \vec{e}_2) = 1$ , then  $F(\vec{u}, \vec{v}) = \det[\vec{u} \ \vec{v}]$  for all  $\vec{u}, \vec{v} \in \mathbb{R}^2$ .

**Solution:** Express  $\vec{u}$  and  $\vec{v}$  as linear combinations of  $e_1, e_2$  :

$$\vec{u} = u_1\vec{e}_1 + u_2\vec{e}_2 \text{ and } \vec{v} = v_1\vec{e}_1 + v_2\vec{e}_2$$

Then

$$\begin{aligned} F(\vec{u}, \vec{v}) &= F(u_1\vec{e}_1 + u_2\vec{e}_2, v_1\vec{e}_1 + v_2\vec{e}_2) \\ &= F(u_1\vec{e}_1, v_1\vec{e}_1 + v_2\vec{e}_2) + F(u_2\vec{e}_2, v_1\vec{e}_1 + v_2\vec{e}_2) && \text{(bilinearity)} \\ &= F(u_1\vec{e}_1, v_1\vec{e}_1) + F(u_1\vec{e}_1, v_2\vec{e}_2) + F(u_2\vec{e}_2, v_1\vec{e}_1) + F(u_2\vec{e}_2, v_2\vec{e}_2) \\ &= u_1v_1F(\vec{e}_1, \vec{e}_1) + u_1v_2F(\vec{e}_1, \vec{e}_2) + u_2v_1F(\vec{e}_2, \vec{e}_1) + u_2v_2F(\vec{e}_2, \vec{e}_2) \\ &= u_1v_1(0) + u_1v_2(1) + u_2v_1(-1) + u_2v_2(0) && \text{(alternating)} \\ &= u_1v_2 - u_2v_1 \\ &= \det \begin{bmatrix} u_1 & v_1 \\ u_2 & v_2 \end{bmatrix} \\ &= \det[\vec{u} \ \vec{v}] \end{aligned}$$

2. (a) Prove that  $T$  is a linear transformation.

**Solution:** Let  $A, B \in \mathbb{R}^{2 \times 2}$ , and  $c \in \mathbb{R}$ .

$T$  respects addition:

$$T(A + B) = (A + B)M = AM + BM = T(A) + T(B)$$

by distributivity of matrix multiplication.

$T$  respects scalar multiplication:

$$T(cA) = (cA)M = c(AM) = cT(A)$$

by properties of matrix multiplication.

Since  $T$  respects addition and scalar multiplication, it is linear.

- (b) Find the  $\mathcal{E}$ -matrix  $[T]_{\mathcal{E}}$  of  $T$ , where  $\mathcal{E}$  is the ordered basis

$$\mathcal{E} = (E_{11}, E_{12}, E_{21}, E_{22}) = \left( \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right)$$

of  $\mathbb{R}^{2 \times 2}$ . Your answer should be in terms of the entries of  $M$ .

**Solution:**

$$[T]_{\mathcal{E}} = \begin{bmatrix} \begin{array}{c|c} & \\ \hline [T(E_{11})]_{\mathcal{E}} & [T(E_{12})]_{\mathcal{E}} \\ \hline & \end{array} & \begin{array}{c|c} & \\ \hline [T(E_{21})]_{\mathcal{E}} & [T(E_{22})]_{\mathcal{E}} \\ \hline & \end{array} \end{bmatrix}$$

$$[T(E_{11})]_{\mathcal{E}} = \left[ \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix} \right]_{\mathcal{E}} = \begin{bmatrix} a \\ b \\ 0 \\ 0 \end{bmatrix}$$

$$[T(E_{12})]_{\mathcal{E}} = \left[ \begin{bmatrix} c & d \\ 0 & 0 \end{bmatrix} \right]_{\mathcal{E}} = \begin{bmatrix} c \\ d \\ 0 \\ 0 \end{bmatrix}$$

$$[T(E_{21})]_{\mathcal{E}} = \left[ \begin{bmatrix} 0 & 0 \\ a & b \end{bmatrix} \right]_{\mathcal{E}} = \begin{bmatrix} 0 \\ 0 \\ a \\ b \end{bmatrix}$$

$$[T(E_{22})]_{\mathcal{E}} = \left[ \begin{bmatrix} 0 & 0 \\ c & d \end{bmatrix} \right]_{\mathcal{E}} = \begin{bmatrix} 0 \\ 0 \\ c \\ d \end{bmatrix}$$

$$[T]_{\mathcal{E}} = \begin{bmatrix} a & c & 0 & 0 \\ b & d & 0 & 0 \\ 0 & 0 & a & c \\ 0 & 0 & b & d \end{bmatrix}$$

(c) Compute  $\det[T]_{\mathcal{E}}$ .

**Solution:** Using the Laplace expansion on our result from (b),

$$\begin{aligned}
 \det[T]_{\mathcal{E}} &= \begin{vmatrix} a & c & 0 & 0 \\ b & d & 0 & 0 \\ 0 & 0 & a & c \\ 0 & 0 & b & d \end{vmatrix} \\
 &= a \begin{vmatrix} d & 0 & 0 \\ 0 & a & c \\ 0 & b & d \end{vmatrix} - c \begin{vmatrix} b & 0 & 0 \\ 0 & a & c \\ 0 & b & d \end{vmatrix} \\
 &= ad \begin{vmatrix} a & c \\ b & d \end{vmatrix} - bc \begin{vmatrix} a & c \\ b & d \end{vmatrix} \\
 &= (ad - bc)^2 \\
 &= a^2d^2 - 2abcd + b^2c^2
 \end{aligned}$$

(d) Compute  $\det[T]_{\mathcal{B}}$ .

**Solution:** The determinant of a transformation is the same in any basis. So it will be the same as part (c), or  $(ad - bc)^2 = a^2d^2 - 2abcd + b^2c^2$ .

(e) Either prove that  $T$  is always diagonalizable no matter what  $M$  is, or provide an explicit example of a matrix  $M$  for which  $T$  is not diagonalizable and briefly explain why your example works.

**Solution:**

3. (a) Prove that there exists a unique vector  $\vec{z} \in \mathbb{R}^4$  such that  $T(\vec{x}) = \vec{z} \cdot \vec{x}$  for all  $\vec{x} \in \mathbb{R}^4$ , and find the components of  $\vec{z}$  in terms of the vectors  $\vec{u}, \vec{v}$ , and  $\vec{w}$ . (Hint:  $\vec{x} = x_1\vec{e}_1 + x_2\vec{e}_2 + x_3\vec{e}_3 + x_4\vec{e}_4$ .)

**Solution:** We calculate the determinant by expanding the column of  $\vec{x}$ .

$$|\vec{x} \ \vec{u} \ \vec{v} \ \vec{w}| = x_1 \begin{vmatrix} u_2 & v_2 & w_2 \\ u_3 & v_3 & w_3 \\ u_4 & v_4 & w_4 \end{vmatrix} - x_2 \begin{vmatrix} u_1 & v_1 & w_1 \\ u_3 & v_3 & w_3 \\ u_4 & v_4 & w_4 \end{vmatrix} + x_3 \begin{vmatrix} u_1 & v_1 & w_1 \\ u_2 & v_2 & w_2 \\ u_4 & v_4 & w_4 \end{vmatrix} - x_4 \begin{vmatrix} u_1 & v_1 & w_1 \\ u_2 & v_2 & w_2 \\ u_3 & v_3 & w_3 \end{vmatrix}$$

$$\vec{z} = \begin{bmatrix} \begin{vmatrix} u_2 & v_2 & w_2 \\ u_3 & v_3 & w_3 \\ u_4 & v_4 & w_4 \end{vmatrix} \\ - \begin{vmatrix} u_1 & v_1 & w_1 \\ u_3 & v_3 & w_3 \\ u_4 & v_4 & w_4 \end{vmatrix} \\ \begin{vmatrix} u_1 & v_1 & w_1 \\ u_2 & v_2 & w_2 \\ u_4 & v_4 & w_4 \end{vmatrix} \\ - \begin{vmatrix} u_1 & v_1 & w_1 \\ u_2 & v_2 & w_2 \\ u_3 & v_3 & w_3 \end{vmatrix} \end{bmatrix}$$

- (b) Find the vector  $\vec{z}$  (as in part (a)) when  $\vec{u} = \vec{e}_1, \vec{v} = \vec{e}_2$ , and  $\vec{w} = \vec{e}_3$  are the first three standard basis vectors in  $\mathbb{R}^4$ .

**Solution:** When  $u, v, w$  are such, only the last element of  $\vec{z}$  is nonzero. So,  $\vec{z}$  is

$$\begin{bmatrix} 0 \\ 0 \\ 0 \\ -1 \end{bmatrix}.$$

- (c) When is  $\vec{z} = \vec{0}$ ? (Your answer should be in terms of  $\vec{u}, \vec{v}$ , and  $\vec{w}$ .)

**Solution:** The determinant of a set of vectors is zero if and only if it is linearly dependent. When  $\vec{z}$  is 0, this is an equivalent statement to the determinant  $|\vec{x} \ \vec{u} \ \vec{v} \ \vec{w}|$  being 0 for all  $\vec{x}$ . This can only happen when  $\vec{u}, \vec{v}, \vec{w}$  are linearly dependent.

Alternatively, we can realize that  $\vec{z} = 0$  means that  $\vec{u}, \vec{v}, \vec{w}$  are linearly dependent since their corresponding vectors with one element removed are linearly dependent

(e.g.  $\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}, \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}, \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}$  are linearly dependent since the last element of  $\vec{z}$ , the determinant of these three vectors, is 0).

- (d) Prove that  $\vec{z}$  is orthogonal to each of  $\vec{u}, \vec{v}$  and  $\vec{w}$ , and find  $\det([\vec{z} \ \vec{u} \ \vec{v} \ \vec{w}])$  in terms of  $\|\vec{z}\|$ .

**Solution:** We use  $\vec{u}$  to represent any of  $\vec{u}, \vec{v}, \vec{w}$  without loss of generality. The dot product  $\vec{u} \cdot \vec{z} = \det([\vec{u} \ \vec{u} \ \vec{v} \ \vec{w}])$  by our definition of  $\vec{z}$ . Since this matrix has two columns of  $\vec{u}$ , it is linearly dependent. So its determinant is 0, and the dot product

$\vec{u} \cdot \vec{v} = 0$ . So  $\vec{z}$  is orthogonal to  $\vec{u}, \vec{v}, \vec{w}$ .

By our definition of  $\vec{z}$ ,  $\det([\vec{z} \ \vec{u} \ \vec{v} \ \vec{w}]) = \vec{z} \cdot \vec{z} = \|\vec{z}\|^2$ .

4. (a) Prove that for every  $n \times n$  matrix  $A$  and for every eigenvalue  $\lambda$  of  $A$ , the real number  $p(\lambda)$  is an eigenvalue of the  $n \times n$  matrix  $p(A)$ .

**Solution:** Let  $\vec{v}$  be an eigenvector corresponding to  $\lambda$  of  $A$ . Let  $p(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \dots$ , where  $a_i \in \mathbb{R}$ . Then

$$\begin{aligned} p(A)\vec{v} &= a_0\vec{v} + a_1A\vec{v} + a_2A^2\vec{v} + a_3A^3\vec{v} + \dots \\ &= a_0\vec{v} + a_1\lambda\vec{v} + a_2\lambda^2\vec{v} + a_3\lambda^3\vec{v} + \dots && \text{(definition of eigenvector)} \\ &= (a_0 + a_1\lambda + a_2\lambda^2 + a_3\lambda^3 + \dots) \vec{v} \\ &= p(\lambda)\vec{v} \end{aligned}$$

So  $p(\lambda)$  is an eigenvalue of  $p(A)$ .

- (b) Let  $p$  be a polynomial and let  $n \in \mathbb{N}$ . Prove that if  $S$  is an invertible  $n \times n$  matrix, then for every  $A \in \mathbb{R}^{n \times n}$  we have  $p(S^{-1}AS) = S^{-1}p(A)S$ .

**Solution:** Let  $p(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \dots$ , where  $a_i \in \mathbb{R}$ . Then

$$\begin{aligned} p(S^{-1}AS) &= a_0I_n + a_1(S^{-1}AS) + a_2(S^{-1}AS)^2 + a_3(S^{-1}AS)^3 + \dots \\ &= a_0I_n + a_1(S^{-1}AS) + a_2S^{-1}A^2S + a_3S^{-1}A^3S + \dots \\ &\quad \text{(since } (S^{-1}AS)^n = S^{-1}A^nS \ \forall n \in \mathbb{N}) \\ &= a_0S^{-1}I_nS + a_1(S^{-1}AS) + a_2S^{-1}A^2S + a_3S^{-1}A^3S + \dots \\ &= S^{-1}(a_0I_n + a_1A + a_2A^2 + a_3A^3 + \dots)S \\ &= S^{-1}p(A)S \end{aligned}$$

- (c) Let  $p$  be a polynomial and let  $A$  be an  $n \times n$  matrix. Prove that if  $A$  is diagonalizable, then every eigenvalue of  $p(A)$  is of the form  $p(\lambda)$  for some eigenvalue  $\lambda$  of  $A$ .

**Solution:** Assume  $A$  is diagonalizable, and let  $A = S^{-1}DS$  for some  $n \times n$  diagonal matrix  $D$  and invertible  $n \times n$  matrix  $S$ .

5. (a) Let  $A \in \mathbb{R}^{2 \times 2}$  be a  $2 \times 2$  matrix such that  $A^2 = I_2$ . Prove that  $A$  is diagonalizable. (Hint: try factoring  $A^2 - I_2$ , and consider the possible ranks of the factors.)

**Solution:** We know the rank of  $A$  must be 2, since  $I_2$  has a rank of 2 and the composition of two linear transformations has less than or equal rank than the minimum rank of the two. Also,  $A^2 = I_2$  is equivalent to  $A^2 - I_2 = 0$ . So

$$\begin{aligned} A^2 - I_2 &= 0_2 \\ A^2 - I_2^2 &= 0_2 \\ (A + I_2)(A - I_2) &= 0_2 \end{aligned}$$

Since the nullity of a matrix composition is at most the sum of the nullities of the components,  $A$  has eigenvalues of 1 and/or  $-1$  with a total geometric multiplicity of 2. Since its geometric multiplicities add up to 2 and the dimensions of  $A$  are  $2 \times 2$ , then  $A$  is diagonalizable.

- (b) Does the same result hold for larger matrices? That is, if  $A \in \mathbb{R}^{n \times n}$  is an  $n \times n$  matrix for which  $A^2 = I_n$ , must  $A$  be diagonalizable? Either prove this or give a counterexample.

**Solution:** Yes, the same result holds. We know the rank of  $A$  must be  $n$ , since  $I_n$  has a rank of  $n$  and the composition of two linear transformations is at least the minimum rank of the two. Also,  $A^2 = I_n$  is equivalent to  $A^2 - I_n = 0$ . So

$$\begin{aligned} A^2 - I_n &= 0_n \\ A^2 - I_n^2 &= 0_n & (I_n^2 = I_n) \\ (A + I_n)(A - I_n) &= 0_n \end{aligned}$$

Since the nullity of a matrix composition is at most the sum of the nullities of the components,  $A$  has eigenvalues of 1 and/or  $-1$  with a total geometric multiplicity of  $n$ . Since its geometric multiplicities add up to  $n$  and the dimensions of  $A$  are  $n \times n$ , then  $A$  is diagonalizable.