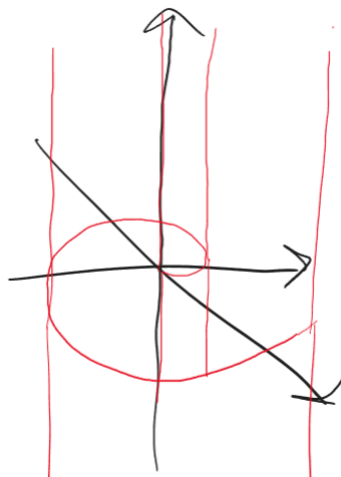


**MATH 215 FALL 2023**  
**Homework Set 8: §15.7 – 16.1**  
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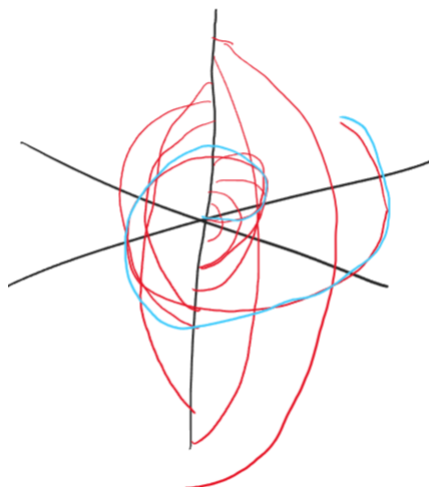
1. For the following problem, take  $r, \theta, \rho$ , and  $\phi$  to have the standard definitions in cylindrical and spherical coordinates. Describe (and try to sketch) the following surfaces:
- (a)  $r = \theta$

**Solution:** A cylinder through a spiral starting from the origin.



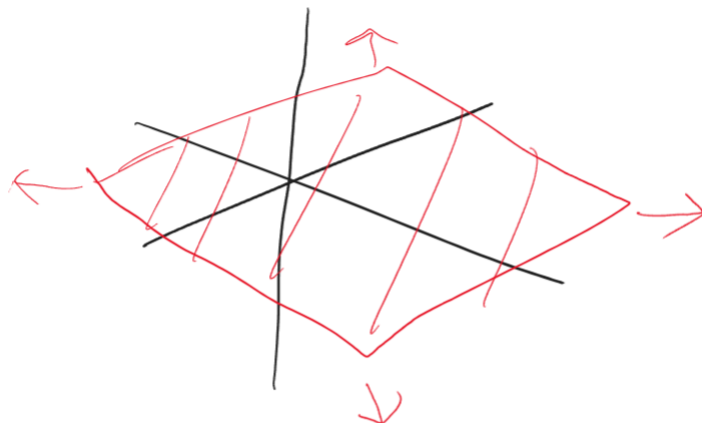
- (b)  $\rho = \theta$

**Solution:** A spiral in the  $xy$  plane, where each point has vertical arcs of circles passing through them to the line  $x = y = 0$ , each with the origin as their center.



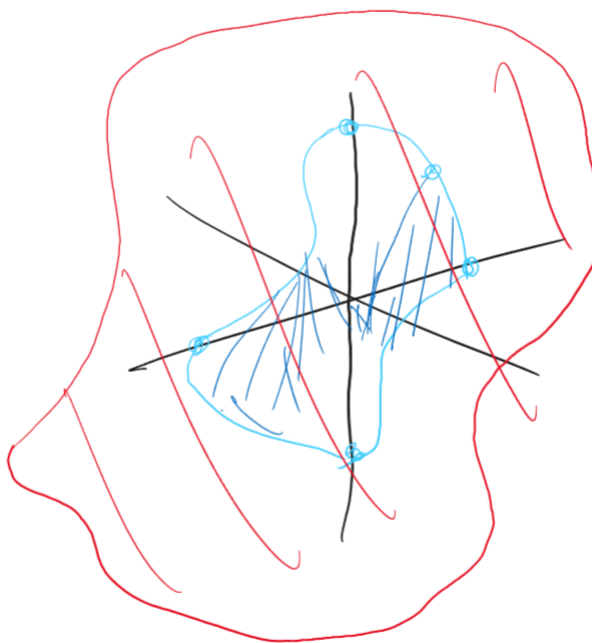
- (c)  $r = \rho$

**Solution:** The  $xy$  plane.



(d)  $\theta = \phi$

**Solution:** A curved surface. When a curve is drawn on this surface with  $\rho$  fixed, the curve looks similar to a sin curve when viewed from the y-axis.



2. Let  $E$  be the ball of radius 1 centered at the point  $(0, 0, 1)$ .

(a) Show that  $E$  is given in Cartesian coordinates by the equation  $x^2 + y^2 + z^2 - 2z \leq 0$ .

**Solution:**

$$\begin{aligned}
 x^2 + y^2 + (z - 1)^2 &\leq 1 \\
 x^2 + y^2 + z^2 - 2z + 1 &\leq 1 \\
 x^2 + y^2 + z^2 - 2z &\leq 0
 \end{aligned}$$

□

- (b) Write
- $E$
- in spherical coordinates. Make sure to specify the domain of
- $\rho$
- ,
- $\theta$
- , and
- $\phi$
- .

**Solution:**

$$x = \rho \sin(\phi) \cos(\theta), y = \rho \sin(\phi) \sin(\theta), z = \rho \cos(\phi)$$

$$\begin{aligned}
 (\rho \sin(\phi) \cos(\theta))^2 + (\rho \sin(\phi) \sin(\theta))^2 + (\rho \cos(\phi))^2 - 2(\rho \cos(\phi)) &\leq 0 \\
 \rho^2 \sin^2(\phi) \cos^2(\theta) + \rho^2 \sin^2(\phi) \sin^2(\theta) + \rho^2 \cos^2(\phi) - 2\rho \cos(\phi) &\leq 0 \\
 \rho^2 \sin^2(\phi) \cos^2(\theta) + \rho^2 \sin^2(\phi) \sin^2(\theta) + \rho^2 \cos^2(\phi) - 2\rho \cos(\phi) &\leq 0 \\
 \rho^2 \sin^2(\phi) (\cos^2(\theta) + \sin^2(\theta)) + \rho^2 \cos^2(\phi) - 2\rho \cos(\phi) &\leq 0 \\
 \rho^2 - 2\rho \cos(\phi) &\leq 0 \\
 \rho(\rho - 2 \cos(\phi)) &\leq 0
 \end{aligned}$$

$$0 \leq \rho \leq 2 \cos(\phi), 0 \leq \phi \leq \frac{\pi}{2}, 0 \leq \theta \leq 2\pi$$

□

- (c) Suppose the density on
- $E$
- is proportional to the distance to the origin, with the largest density being equal to 2. Use spherical coordinates to compute the mass and center of mass of
- $E$
- .

**Solution:** A density equal to  $\rho$  satisfies these conditions.

$$\begin{aligned}
 M &= 2\pi \int_0^{\frac{\pi}{2}} \int_0^{2 \cos(\phi)} \rho^3 \sin(\phi) d\rho d\phi \\
 &= 2\pi \int_0^{\frac{\pi}{2}} \sin(\phi) \int_0^{2 \cos(\phi)} \left[ \frac{\rho^4}{4} \right]_{\rho=0}^{2 \cos(\phi)} d\phi \\
 &= 2\pi \int_0^{\frac{\pi}{2}} \sin(\phi) 4 \cos^4(\phi) d\phi \\
 u &= \cos(\phi), du = -\sin(\phi) d\phi \\
 &= 2\pi \int_1^0 -4u^4 d\phi \\
 &= -8\pi \left( \frac{u^5}{5} \right)_{u=1}^0 \\
 M &= \boxed{\frac{8\pi}{5}}
 \end{aligned}$$

- (d) Suppose we tried to do this problem for the ball of radius 1 centered at the point  $(0, 1, 0)$ . Why is this problem harder with the new ball?

**Solution:**

3. Begin with a sphere of radius  $R$  and bore a hole into the sphere in the shape of a right circular cylinder, leaving only a band of height  $h$ . Find the volume of the resulting shape.

**Solution:** The radius of the cylinder will be  $r_c = \sqrt{R^2 - h^2}$ . We use cylindrical coordinates to perform the integration.

$$\begin{aligned}
 & 2\pi \int_{-h}^h \int_{\sqrt{R^2-h^2}}^{\sqrt{R^2-z^2}} r \, dr \, d\theta \\
 &= \pi \int_{-h}^h (r^2) \Big|_{\sqrt{R^2-h^2}}^{\sqrt{R^2-z^2}} dr \, d\theta \\
 &= \pi \int_{-h}^h R^2 - z^2 - R^2 + h^2 \, d\theta \\
 &= \pi \left( -\frac{z^3}{3} + h^2 z \right) \Big|_{z=-h}^h \\
 &= \boxed{\frac{4\pi h^3}{3}}
 \end{aligned}$$

□

4. Find the mass of a wedge cut from a sphere of radius  $R$  by two planes that intersect along a diameter and at an angle of  $\frac{\pi}{5}$ , assuming that the density is proportional to the distance from the origin in such a way that the maximum density is 2. (This shape should look like a segment of an orange.)

**Solution:** We use spherical coordinates for this problem, with  $(r, \theta, \phi)$ . The density function will be  $\rho(r) = \frac{2r}{R}$  to have a maximum density of 2 when the distance is equal

to the radius.

$$\begin{aligned}
 & \frac{\pi}{5} \int_0^R \int_0^\pi \frac{2r}{R} r^2 \sin(\phi) d\phi dr \\
 &= \frac{\pi}{5R} \int_0^R 2r^3 \int_0^\pi \sin(\phi) d\phi dr \\
 &= \frac{\pi}{5R} \int_0^R 2r^3 (-\cos(\phi)) \Big|_{\phi=0}^\pi dr \\
 &= \frac{\pi}{5R} \int_0^R 4r^3 dr \\
 &= \frac{\pi}{5R} (r^4) \Big|_{r=0}^R \\
 &= \boxed{\frac{\pi R^3}{5}}
 \end{aligned}$$

□

5. Find  $\int_R f(x, y) dA$  where  $f(x, y) = 3y^2 - 4xy - 4x^2$  and  $R$  is the quadrilateral with vertices  $(0, 2)$ ,  $(3, 0)$ ,  $(5, 4)$ , and  $(2, 6)$ . *Hint:* There may be a straightforward but tedious way to solve this problem, as well as a faster, more subtle, way to solve this problem.

**Solution:** We can factor  $f(x, y) = (3y + 2x)(y - 2x)$ . Then, we can use change of variables to change both the function and the bounds. Let  $u = 3y + 2x$ ,  $v = y - 2x$ . Then  $f(u, v) = uv$ ,  $d(x, y) = (2 - 3(-2))^{-1} d(u, v) = \frac{1}{8} d(u, v)$ . Also,  $R$  has vertices at  $(u, v) = (6, 2)$ ,  $(6, -6)$ ,  $(22, -6)$ ,  $(22, 2)$ .

$$\begin{aligned}
 & \frac{1}{8} \int_{-6}^2 \int_6^{22} uv du dv \\
 &= \frac{1}{24} \int_{-6}^2 v [(22)^2 - (6)^2] dv \\
 &= \frac{1}{16} \int_{-6}^2 448v dv \\
 &= \frac{1}{16} (224v^2) \Big|_{v=-6}^2 \\
 &= \frac{1}{16} \cdot (-7168) \\
 &= \boxed{-448}
 \end{aligned}$$

6. Let  $E$  be the region in the first quadrant that is above the line  $y = \frac{x}{3}$ , below the line  $y = 3x$ , and between the curves defined by  $xy = 3$  and  $xy = 27$ .

- (a) Sketch the region.
  - (b) Evaluate  $\int \int (\frac{x^2}{y^2} + x^2 y^2) dA$ . (Hint: Try  $u = xy$  and  $v = \frac{y}{x}$ .)
  - (c) Why was the hint a reasonable guess for a change of coordinates?
7. Do Exercises 13-18 of §16.1 in *Stewart's Multivariable Calculus*.

**Solution:**

- 13.  $\boxed{IV}$  – vectors with direction and magnitude equal to displacement, except flipped vertically.
  - 14.  $\boxed{V}$  – downward direction when  $x < y$ , upward when  $y < x$ , horizontal when  $x = y$ .
  - 15.  $\boxed{I}$  – when  $y = -2$ , vectors are horizontal.
  - 16.  $\boxed{VI}$  – magnitude increases more with  $x$  than  $y$ .
  - 17.  $\boxed{III}$  – the magnitude/direction oscillates when either coordinate is fixed.
  - 18.  $\boxed{II}$  – direction becomes more vertical when  $x$  increases, while horizontal component oscillates.
8. Do Exercises 19-22 of §16.1 in *Stewart's Multivariable Calculus*.

**Solution:**

- 19.  $\boxed{IV}$  – only constant vector field.
  - 20.  $\boxed{I}$  – the vector field is constant when  $z$  is fixed.
  - 21.  $\boxed{III}$  – always positive vertical direction, same direction as displacement from origin for  $x$  and  $y$ .
  - 22.  $\boxed{II}$  – same direction/magnitude as displacement from origin.
9. Do Exercises 31-34 of §16.1 in *Stewart's Multivariable Calculus*.

**Solution:**

- 31.  $\boxed{III}$  – gradient is  $(2x, 2y)$ , so linearly increasing magnitude and same direction as displacement from origin.
- 32.  $\boxed{IV}$  – gradient is  $(2x + y, x)$ , thus the direction is close to horizontal near the  $y$ -axis and becomes more vertical as  $x$  increases.

33.  $\boxed{II}$  – gradient is  $(2x + 2y, 2y + 2x)$ . Since the  $x$  and  $y$  coordinates are the same, the direction is always the same  $\langle 1, 1 \rangle$ , except with positive or negative magnitude.
34.  $\boxed{I}$  – Gradient will include something with  $\cos$  for both  $f_x$  and  $f_y$  coordinates, thus the magnitude will oscillate.