# EECS 203: Discrete Mathematics
# Winter 2024
# Homework 2

## Due **Thursday, Feb. 1st**, 10:00 pm

No late homework accepted past midnight.

Number of Problems: $8 + 2$ Total Points: $100 + 40$

- **Match your pages!** Your submission time is when you upload the file, so the time you take to match pages doesn't count against you.
- Submit this assignment (and any regrade requests later) on Gradescope.
- Justify your answers and show your work (unless a question says otherwise).
- By submitting this homework, you agree that you are in compliance with the Engineering Honor Code and the Course Policies for 203, and that you are submitting your own work.
- Check the syllabus for full details.

# Individual Portion

## 1. Negation Transformation [12 points]

Find the negation of each statement below. If your thought process involves intermediate steps, show them. If not, simply writing the negation is sufficient.

Your answer should not contain the original proposition. That is, you shouldn't negate it as "It is not the case that ..." or something similar.

(a) Every student in this course is enrolled in exactly one Discussion section.

(b) There is a student in this class who is not on Gradescope or not on Piazza.

(c) For all integers $a$ and $b$, if $a + b > 0$, then $a - b < 0$.

(d) For every irrational number $x$, there is a rational number $y$ such that $x^y$ is rational.

**Solution:**

(a) There is at least 1 student in this course who is not enrolled in exactly 1 discussion section.

(b) Every student in this class is on both Gradescope and Piazza.

(c) There exist integers $a$ and $b$ such that $a + b > 0$ and $a - b \geq 0$.

(d) There is an irrational number $x$ for which there does not exist a rational number $y$ such that $x^y$ is rational.

## 2. Not It [12 points]

Negate the following statements so that all negation symbols immediately precede predicates. Make sure to show all intermediate steps.

**Note:** $\neg(P(x) \vee Q(x))$ would not be considered fully simplified since the negation ($\neg$) does not immediately come before $P(x)$ or $Q(x)$. However, $\neg P(x) \vee \neg Q(x)$ is fully simplified, for example.

(a) $\forall y[\exists x P(x, y) \vee \forall x Q(x, y)]$

(b) $\exists x \forall y[R(x, y) \rightarrow R(y, x)]$

**Solution:**

(a) $\exists y[\forall x \neg P(x,y) \wedge \exists x \neg Q(x,y)]$

(b) $\forall x \exists y[R(x,y) \wedge \neg R(y,x)]$

## 3. Order's Up! [12 points]

Let $P(x,y)$ be the statement "customer $x$ has ordered dish $y$," where the domain for $x$ consists of all customers and for $y$ consists of all dishes at a restaurant. Express each of these propositions in logic.

(a) Some customer has ordered some dish at this restaurant.

(b) Some customer has ordered all of the dishes at this restaurant.

(c) Each customer has ordered at least one dish at this restaurant.

(d) Some dish at this restaurant has been ordered by all customers.

(e) Each dish at this restaurant has been ordered by at least one customer.

(f) All customers have ordered every dish at this restaurant.

(g) Some dish at this restaurant has been ordered by a customer.

(h) Every dish at this restaurant has been ordered by every customer.

**Solution:**

(a) $\exists x \exists y[P(x,y)]$

(b) $\exists x \forall y[P(x,y)]$

(c) $\forall x \exists y[P(x,y)]$

(d) $\exists y \forall x[P(x,y)]$

(e) $\forall y \exists x[P(x,y)]$

(f) $\forall x \forall y[P(x,y)]$

(g) $\exists y \exists x[P(x,y)]$

(h) $\forall y \forall x[P(x,y)]$

## 4. Sports Statements [12 points]

Let $I(x)$ be the statement "$x$ has a favorite sport" and $C(x, y)$ be the statement "$x$ and $y$ have the same favorite sport," where the domain for the variables $x$ and $y$ consists of all students in your class. Use quantifiers and the logical connectives you learned in lecture to express each of the statements below.

**Hint:** You can use an $=$ sign to compare people.

(a) Someone in your class does not have a favorite sport.

(b) No one in the class has the same favorite sport as Chloe.

(c) Everyone except one student in your class has a favorite sport.

**Solution:**

(a) $\exists x[\neg I(x)]$

(b) $\forall x[\neg C(x, \text{Chloe})]$

(c) $\exists x \forall y[\neg I(x) \wedge (x = y \vee I(y))]$

## 5. Quantifier Quandary [12 points]

For each of the propositions below, write the negation, and determine whether the original proposition is true or if its negation is true. Your negation cannot contain the logical "not" symbol ($\neg$), but you may use the not-equals sign ($\neq$). The domain of discourse is all real numbers. **Briefly justify your answers.**

(a) $\exists x(x^3 = -1)$

(b) $\forall x(2x > x)$

(c) $\exists x \forall y(x + y = 0)$

(d) $\forall x \exists y(x + y = 0)$

**Solution:**

(a) $\forall x(x^3 \neq -1)$. The original is true: $x = -1$ satisfies the original.

(b) $\exists x(2x \leq x)$. The negation is true: $x = -1$ satisfies the negation.

(c) $\forall x \exists y (x + y \neq 0)$. The negation is true: when $x$ is nonzero, $x + x = 2x \neq 0$. When $x = 0$, $x + 1 = 1 \neq 0$.

(d) $\exists x \forall y (x + y \neq 0)$. The original is true: $x + (-x) = 0$ for any $x$.

## 6. Even Stevens [8 points]

Prove that if $n$ is an even integer, then $\frac{n^2}{2}$ is also an even integer.

**Solution:**

By definition, $n = 2m$ for some $m \in \mathbb{Z}$. Then

$$\frac{n^2}{2} = \frac{(2m)^2}{2} = \frac{4m^2}{2} = 2m^2.$$

We know $m$ is an integer, so $m^2$ is an integer. Thus $2m^2 = \frac{n^2}{2}$ is even.

## 7. To Prove or Not To Prove [16 points]

**Prove or disprove** each of the following statements where the domain of discourse is all real numbers.

(a) For all $x$, $x^2 > 0$.

(b) There exists $x$ such that $x \leq 0$ and $2x > x$.

(c) There exists $x$ such that for all $y$, $x^2 + y^2 > 203$.

(d) There exists $x$ such that for all $y$, $(x + y)^2 > 203$.

**Solution:**

(a) False. The value $x = 0$ is a counterexample: $x^2 = 0 \not> 0$.

(b) False. Subtracting $x$ on both sides from the inequality, it becomes $x > 0$. However, this is a contradiction with the other inequality $x \leq 0$. Thus both $x \leq 0$ and $2x > x$ cannot be true.

(c) True. The value $x = 15$ satisfies this: the least value $y^2$ can take is 0. When $y^2 = 0$, $x^2 + y^2 = 225 > 203$.

(d) False. For any $x$, choose $y = -x$. Then $(x + y)^2 = 0^2 = 0 \not< 203$.

## 8. Mixed Quantifiers Proof [16 points]

For this problem, let the domain of discourse be positive integers.

(a) Consider the following predicate:

$$P(x, z) := (z > x) \land (x \mid z) \land (4 \nmid z)$$

Let $x = 10$. Find the three smallest values of $z$ which satisfy $P(10, z)$.

(b) Now prove the following proposition:

$$\forall x[4 \nmid x \rightarrow \exists z\, P(x, z)]$$

**Note:** The statement $a \mid b$ means "$a$ divides $b$," i.e. there exists some integer $q$ such that $b = aq$. Similarly, $a \nmid b$ means "$a$ does not divide $b$."

**Solution:**

(a) In English, the predicate means that $z$ is greater than $x$ and divisible by $x$ and not divisible by 4. When $x = 10$, the solutions $z = 30, 50, 70$ fulfill these requirements. These are the smallest values since $z$ must be greater than 10.

(b) Assuming that $4 \nmid x$, we construct a $z$ which fulfills the requirements. Let $z = 5x$. Since $x$ is positive, $z = 5x > x$. We also know $x|z$ since $z = 5x$. Additionally, since $5x = 4x + x$, $z$ is not divisible by 4 because $4|4x$ and $4 \nmid x$. Thus $z$ fulfills predicate $P(x, z)$.

# Grading of Groupwork 1

Using the solutions and Grading Guidelines, grade your Groupwork 1 Problems:

- Use the table below to grade your past groupwork submission and calculate scores.
- While grading, mark up your past submission. Include this with the table when you submit your grading.
- Write whether your submission achieved each rubric item. If it didn't achieve one, say why not.
- For extra credit, write positive comment(s) about your work.
- You don't have to redo problems correctly, but it is recommended!
- See "All About Groupwork" on Canvas for more detailed guidance, and what to do if you change groups.

| | (i) | (ii) | (iii) | (iv) | (v) | (vi) | (vii) | (viii) | (ix) | (x) | (xi) | Total: |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Problem 1 | | | | | | | | | | | | /20 |
| Total: | | | | | | | | | | | | /20 |

# Groupwork 2 Problems

## 1. Bézout's Identity [20 points]

In number theory, there's a simple yet powerful theorem called Bézout's identity, which states that for any two integers $a$ and $b$ (with $a$ and $b$ not both zero) there exist two integers $r$ and $s$ such that $ar + bs = \gcd(a, b)$. Use Bézout's identity to prove the following statements (you may assume all variables are integers):

(a) If $d \mid a$ and $d \mid b$, then $d \mid \gcd(a, b)$.

(b) If $a \mid bc$ and $\gcd(a, b) = 1$, then $a \mid c$.

**Note:** gcd is short for "greatest common divisor," so the value of $\gcd(a, b)$ is the largest integer that evenly divides $a$ and $b$. You won't need to apply this definition, just know that $\gcd(a, b)$ is an integer.

| **Solution:** |
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## 2. Proposition Michigan [20 points]

Translate each of the following English statements into logic. You may define predicates as necessary.

**Note:** Your predicates should not trivialize the problem.

(a) Each pair of students at UMich has at least two mutual friends at UMich. The domain of discourse is all students at UMich.

(b) Nobody knows everyone's Wolverine Access password except the Wolverine Access administrators, who know all passwords. The domain of discourse is all people who have a Wolverine Access account (the administrators have Wolverine Access accounts).

| **Solution:** |
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