

**MATH 217 W24 - LINEAR ALGEBRA, Section 001 (Dr. Paul Kessenich)**  
**Homework 3 Part B due Thursday, February 1 at 11:59pm**  
**Zhengyu James Pan (jzpan@umich.edu)**

1. Determine whether the following statements are true or false, and justify your answer with a proof or a counterexample.

(a) For all  $2 \times 2$  matrices  $A$  and  $B$ ,  $(AB)^T = A^T B^T$ .

**Solution:** False. For example,

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}$$

$$\begin{aligned} (AB)^T &= \begin{bmatrix} 19 & 22 \\ 43 & 50 \end{bmatrix}^T \\ &= \begin{bmatrix} 1 & 22 \\ 43 & 50 \end{bmatrix} \\ &\neq A^T B^T = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} 5 & 7 \\ 6 & 8 \end{bmatrix} \\ &= \begin{bmatrix} 23 & 31 \\ 34 & 46 \end{bmatrix} \end{aligned}$$

(b) For all  $2 \times 2$  matrices  $A$  and  $B$ ,  $(AB)^T \neq A^T B^T$ .

**Solution:** False. For example:

$$A = \begin{bmatrix} 1 & 3 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

$$\begin{aligned} (AB)^T &= \begin{bmatrix} 1 & 3 \\ 0 & 0 \end{bmatrix}^T \\ &= \begin{bmatrix} 1 & 0 \\ 3 & 0 \end{bmatrix} \\ &= A^T B^T = \begin{bmatrix} 1 & 0 \\ 3 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ 3 & 0 \end{bmatrix} \end{aligned}$$

(c) For all matrices  $A$  and  $B$  such that the matrix product  $AB$  exists,  $(AB)^T = B^T A^T$ .

**Solution:** True. Computing with arbitrary matrices  $A$  and  $B$  with  $ij$ th element  $a_{ij}, b_{ij}$  respectively, we see the two products are identical.

$$\begin{aligned}(AB)^\top &= \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} \end{bmatrix}^\top \\ &= \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{21}b_{11} + a_{22}b_{21} \\ a_{11}b_{12} + a_{12}b_{22} & a_{21}b_{12} + a_{22}b_{22} \end{bmatrix}\end{aligned}$$

$$\begin{aligned}B^\top A^\top &= \begin{bmatrix} b_{11} & b_{21} \\ b_{12} & b_{22} \end{bmatrix} \begin{bmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{bmatrix} \\ &= \begin{bmatrix} b_{11}a_{11} + b_{21}a_{12} & b_{11}a_{21} + b_{21}a_{22} \\ b_{12}a_{11} + b_{22}a_{12} & b_{12}a_{21} + b_{22}a_{22} \end{bmatrix} = \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{21}b_{11} + a_{22}b_{21} \\ a_{11}b_{12} + a_{12}b_{22} & a_{21}b_{12} + a_{22}b_{22} \end{bmatrix}\end{aligned}$$

- (d) If  $A$  is a symmetric matrix, then for all  $n \in \mathbb{N}$ ,  $A^n$  is also symmetric.

**Solution:** True. By definition, if  $A$  is symmetric, then  $A = A^\top$ . Assume  $A^n = (A^n)^\top$ . Then

$$A^{n+1} = A^n A = (A^n)^\top A^\top.$$

By part (c) of this problem,

$$(A^n)^\top A^\top = (AA^n)^\top = (A^{n+1})^\top.$$

So if  $A$  is symmetric,  $A^n$  is symmetric for all  $n$  by induction.

- (e) If  $A$  is a square matrix and  $A^2$  is symmetric, then so is  $A$ .

**Solution:** False. The matrix  $A = \begin{bmatrix} -1 & -2 \\ 2 & 1 \end{bmatrix}$  is not symmetric. However,  $A^2 = \begin{bmatrix} -3 & 0 \\ 0 & -3 \end{bmatrix}$  is symmetric. This example demonstrates that  $A$  is not necessarily symmetric if  $A^2$  is.

2. Determine whether the following statements are true or false, and justify your answer with a proof or a counterexample.

- (a) Every 3-by-3 matrix that has a row of zeros is not invertible.

**Solution:** A matrix with a row of zeros can have at most 2 pivots, for a rank of 2. Thus by Theorem 2.4.3, it is not invertible.

- (b) Every square matrix with 1's down the main diagonal is invertible.

**Solution:** False. For example,  $A = \begin{bmatrix} 1 & 2 \\ 0.5 & 1 \end{bmatrix}$  is not bijective:  $A \begin{bmatrix} 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} = A \begin{bmatrix} 0 \\ 1 \end{bmatrix}$  shows that there are multiple solutions to the same matrix equation. Thus it is not invertible.

- (c) For any matrix  $A$ , if  $A$  is invertible, then so is  $A^{-1}$ .

**Solution:** True. Since  $AA^{-1} = A^{-1}A = I_n$ ,  $A^{-1}$  is invertible by definition 2.16 from the Theory of Linear Algebra handout.

- (d) For any matrix  $A$ , if  $A$  is invertible, then  $A^n$  is invertible.

**Solution:** True. This is a special case of Problem 6c on Worksheet 7, where  $A_1 = \dots = A_n = A$ . The inverse is thus  $(A^n)^{-1} = (A^{-1})^n$ .

3. Let  $A$  be an  $m \times n$  matrix. Prove that if there exists an  $n \times m$  matrix  $B$  such that  $BA = I_n$ , then the system of linear equations  $A\vec{x} = \vec{0}$  has a unique solution. (Note: a matrix  $B$  with this property is called a left-inverse for  $A$ . Can you guess why?)

**Solution:** Left multiplying the matrix equation by  $B$ ,

$$BA\vec{x} = B\vec{0}$$

$$I_n\vec{x} = \vec{0}$$

$$\vec{x} = \vec{0}$$

Then the only solution to the system is  $\vec{x} = \vec{0}$ .

4. Given two matrices  $A$  and  $B$  such that the product  $AB$  is defined (say,  $A$  is  $n \times m$  and  $B$  is  $m \times k$ ), exactly one of the following two statements is true:

- (a) Every column of  $AB$  is a linear combination of columns of  $A$ ,
- (b) Every column of  $AB$  is a linear combination of columns of  $B$ .

Prove the one that is true, and provide a counterexample for the one that is false.

**Solution:** By theorem 2.3.2, the matrix product  $AB = \begin{bmatrix} | & | & | \\ \vec{a}_1 & \dots & \vec{a}_m \\ | & | & | \end{bmatrix} \begin{bmatrix} | & | & | \\ \vec{b}_1 & \dots & \vec{b}_k \\ | & | & | \end{bmatrix} =$

$\begin{bmatrix} | & | & | \\ A\vec{b}_1 & \dots & A\vec{b}_k \\ | & | & | \end{bmatrix}$ . By Example 13, we know that each product  $A\vec{b}_j$  is the linear combination  $a_1b_{1j} + \dots + a_mb_{mj}$ , where  $b_{ij}$  is the  $ij$ th element of  $B$ . Thus statement (a) is true.

For a counterexample to (b), take  $\begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 3 & 6 \\ 2 & 4 \end{bmatrix}$ .

5. Let  $f : X \rightarrow X$  be a function. We let  $f^n$  denote the function  $f^n : X \rightarrow X$  given by composing  $f$  iteratively,  $n$  many times. Also, we define  $f^0$  to be the identity function, i.e.  $\forall x \in X, f^0(x) = x$ .

- (a) Assume that  $X = \mathbb{R}^d$ . Prove by induction that if  $f$  is a linear transformation, then the  $n$ th iterate  $f^n$  is also a linear transformation.

**Solution:** Base case:  $f^1 = f$  is a linear transformation.

Induction hypothesis: Assume  $f^n$  is a linear transformation.

Inductive step:  $f^{n+1} = f^n \circ f$  by definition. By Theorem 2.12 from the Theory of Linear Algebra handout, the composition of two linear transformations is also linear. Thus  $f^{n+1}$  is linear. Thus the statement is true for all  $n$ .

- (b) Find an example of a function  $f : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  which is not a linear transformation, but for which there exists an  $n$  such that the  $n$ th iterate  $f^n$  is a linear transformation.

**Solution:**  $f(x) = 1 - x$  is not linear, but  $f^2(x) = f(f(x)) = 1 - (1 - x) = x$  is linear.

- (c) Prove that for  $X = \mathbb{R}^d$  and  $f$  linear, if the equation  $f(x) = 0$  has a unique solution, then the iterated equation  $f^n(x) = 0$  also has a unique solution.

**Solution:** Let  $A$  be the unique standard matrix of  $f$  using the Key Theorem. Then the composition  $f^n$  can be represented by the standard matrix  $A^n$ . Since  $f(x) = 0$  has a unique solution, the matrix equation  $A\vec{x} = \vec{0}$  has a unique solution. By the linear combination definition of matrix multiplication,  $B\vec{0} = \vec{0}$  for all compatible matrices  $B$ , so the unique solution for  $A$  must be  $\vec{0}$ . The matrix equation  $A^n\vec{x} = \vec{0}$  represents the iterated equation  $f^n(x) = 0$ .

When  $\vec{x} = \vec{0}$ ,

$$A^n\vec{0} = \vec{0}.$$

Meanwhile, when  $\vec{x} \neq \vec{0}$ , we know  $A\vec{x} \neq \vec{0}$ . Assuming  $A^n\vec{x} \neq \vec{0}$ , then

$$A^{n+1}\vec{x} = A(A^n\vec{x}) = A(\text{nonzero vector}) \neq 0.$$

So by induction,  $A^n\vec{x} = \vec{0}$  has no solutions when  $\vec{x}$  is nonzero. Thus  $\vec{0}$  is the unique solution to  $f$ .