MATH 215 FALL 2023
Homework Set 8: §16.1 – 16.3
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1. Compute $\int_C x^2 y\, ds$, where C is the segment of the helix of radius 1 about the z-axis, oriented counter-clockwise in the xy-plane, starting at (1, 0, 0) and ending at $(0, 1, \frac{\pi}{2})$.

2. Compute $\int_C x^2\, dx + y^2\, dy$, where C is the circular arc starting at (2, 0) and ending at (0, 2) followed by the straight line segment from (0, 2) to (-1, 1).

3. Do Exercise 53 of §16.2 in *Stewart's Multivariable Calculus.*

4. A wire has the shape of a helix with parametrization $x = t, y = 2\cos t, z = 2\sin t$ for $0 \leq t \leq 6\pi$, where distances are measured in cm. Find the mass and the center of mass of the wire if the density (in grams/cm) of the wire at any point is equal to four times the square of the distance from the origin to the point.

5. Let $\overrightarrow{F} = \nabla f$ where $f(x, y) = \frac{y^{2002}}{1+x^{200002}+y^{2002}}$. Can you find a (smooth, simple, but not necessarily closed) curve C with the following property:

   (a) $\int_C \overrightarrow{F} \cdot d\overrightarrow{r} = \frac{1}{2}$

   Solution: By the Fundamental Theorem of Line Integrals, this integral is equal to the value of $f(x_1, y_1)$ at the beginning of the curve subtracted from $f(x_2, y_2)$ at the end of the curve. So, we find two points where $f(x_2, y_2) - f(x_1, y_1) = \frac{1}{2}$. (0, 0) and (0, 1) satisfy these conditions. So, the line segment from (0, 0) to (0, 1) fulfills this property.

   (b) $\int_C \overrightarrow{F} \cdot d\overrightarrow{r} = 1$

   Solution: The minimum value of f is 0 at (0, 0), and only the limit at $(0, \infty)$ is 1. Thus, there is no finite curve where this is true. If an infinite curve is allowed, then the infinite line beginning at the origin along the y-axis satisfies this property.

6. Do Exercises 11, 31, and 32 of §16.3 in *Stewart's Multivariable Calculus.*
   Solution

   11. (a) The vector field is conservative, the gradient of the function $f(x, y) = x^2 y$. Thus, the Fundamental Theorem of Line integrals applies. Since all these curves begin and end at the same point, the line integrals have the same value.

       (b) $f(3, 2) - f(1, 2) = \boxed{16}$

   31. This vector field is not conservative because any counterclockwise path will have positive work, whereas clockwise paths will have negative work. Thus, this field is not the gradient of a function. You could also verify this by drawing level curves perpendicular to the vectors, which would all pass through the origin. However, since they are all increasing counterclockwise, this would be impossible for a function to have.

32.

7. (a) Calculate $\oint_C \overrightarrow{F} \cdot d\overrightarrow{r}$ where

$$\overrightarrow{F}(x, y) = \langle \frac{2xy}{(x^2+y^2)^2}, \frac{y^2-x^2}{(x^2+y^2)^2} \rangle$$

for $(x, y) \neq (0, 0)$ and C is the circle of radius R centered at the origin, oriented clockwise.

Solution: We can parameterize this by $C(t) = \langle R\sin(t), R\cos(t)\rangle, 0 \leq t \leq \pi$. Then $C'(t) = \langle R\cos(t), -R\sin(t)\rangle$. The integral is then

$$\begin{aligned}
\oint_C \overrightarrow{F} \cdot d\overrightarrow{r} &= \int_0^{2\pi} \langle \frac{2\sin(t)\cos(t)}{R^2}, \frac{\cos^2(t) - \sin^2(t)}{R^2} \rangle \cdot \langle R\cos(t), -R\sin(t)\rangle \, dt \\
&= \frac{1}{R}\int_0^{2\pi} 2\sin(t)\cos^2(t) - \sin(t)\cos^2(t) + \sin^3(t)\, dt \\
&= \frac{1}{R}\int_0^{2\pi} \sin(t)\cos^2(t) + \sin^3(t)\, dt \\
&= \frac{1}{R}\int_0^{2\pi} \sin(t)(\sin^2(t) + \cos^2(t))\, dt \\
&= \frac{1}{R}\int_0^{2\pi} \sin(t)\, dt \\
&= \boxed{0}
\end{aligned}$$

We could also have noticed that $\overrightarrow{F}$ is the gradient of the function $f(x, y) = \frac{-y}{x^2+y^2}$. Following from the Fundamental Theorem of Line Integrals, a line integral of a conservative vector field on a closed curve is 0. $\square$

(b) Repeat the previous part, only this time take the curve C to be the ellipse defined by $4x^2 + 9y^2 = 36$, oriented counterclockwise. Hint: It may be possible to do this integration without parametrizing the ellipse.

Solution: $\overrightarrow{F}$ is the gradient of the function $f(x, y) = \frac{-y}{x^2+y^2}$. Following from the Fundamental Theorem of Line Integrals, a line integral of a conservative vector field on a closed curve is $\boxed{0}$. $\square$