MATH 217 - W24 - LINEAR ALGEBRA
HOMEWORK 11, DUE SUNDAY, April 21 at 11:59pm

Submit Part A and Part B as two *separate* assignments in Gradescope as a **pdf file.** At the time of submission, Gradescope will prompt you to match each problem to the page(s) on which it appears. **You must match problems to pages in Gradescope so we know what page each problem appears on.** Failure to do so may result in not having the problem graded.

A few words about solution writing:

- Unless you are explicitly told otherwise for a particular problem, **you are always expected to show your work and to give justification for your answers.**
- Your solutions will be judged on precision and completeness and not merely on "basically getting it right".
- Cite every theorem or fact from the book that you are using (e.g. "By Theorem 1.10 . . . ").

Part A

Solve the following problems from the book:
Section 7.2: 16;
Section 7.3: 16, 22, 24;
Section 7.4: 48, 64;
Section 7.5: 30;
Section 8.1: 14.

Part B

Problem 1. Let V be the infinite-dimensional vector space of all infinite sequences $(x_1, x_2, x_3, \dots)$ of real numbers indexed by $\mathbb{N}$. Consider the linear transformation $T : V \to V$ which deletes all the components with an odd index, i.e.,

$$T(x_1, x_2, x_3, x_4, x_5, x_6, \dots) = (x_2, x_4, x_6, \dots) \quad \text{for all } (x_1, x_2, x_3, \dots) \in V.$$

(a) Let E_0 denote the 0-eigenspace of T. Explicitly describe E_0 (as a set).
(b) Prove that every real number λ is an eigenvalue of T. (Hint: explicitly construct an eigenvector $(x_1, x_2, x_3, \dots) \in V$. First consider x_i when i is a power of 2.)

Problem 2. If A is an $n \times n$ matrix, define

$$\mathscr{C}(A) = \{B \in \mathbb{R}^{n\times n} \mid AB = BA\}.$$

(a) Let D be a diagonal $n \times n$ matrix with distinct entries along the diagonal, and let $\mathscr{D}$ be the subset of $\mathbb{R}^{n\times n}$ consisting of all diagonal matrices. Prove $\mathscr{C}(D) = \mathscr{D}$.

Two $n \times n$ matrices A and B are said to be *simultaneously diagonalizable* if there exists an invertible $n \times n$ matrix S such that $S^{-1}AS$ and $S^{-1}BS$ are both diagonal.

(b) Prove that if A and B are simultaneously diagonalizable $n \times n$ matrices, then $B \in \mathscr{C}(A)$.
(c) Prove that if A and B are $n \times n$ matrices such that A has n distinct eigenvalues and $B \in \mathscr{C}(A)$, then A and B are simultaneously diagonalizable.

Problem 3. *(Classifying non-diagonalizable[1] 2×2 matrices.)* Let $A \in \mathbb{R}^{2\times 2}$ be a 2×2 matrix.

(a) Suppose that A has eigenvalue 0 but is not diagonalizable. Prove that[2] $\operatorname{im}(A) = E_0$, and conclude from this that $A^2 = 0$.

(b) Let $\lambda \in \mathbb{R}$ and suppose that A has eigenvalue λ but is not diagonalizable. Prove that we have $(A - \lambda I_2)^2 = 0$, and deduce from this that $A\vec{v} - \lambda\vec{v} \in E_\lambda$ for every $\vec{v} \in \mathbb{R}^2$.

[Hint: apply part (a) to the matrix $A - \lambda I_2$.]

(c) Prove that if A has eigenvalue λ but is not diagonalizable, then A is similar to $\begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$.

[Hint: consider the basis $\mathcal{B} = (A\vec{v} - \lambda\vec{v}, \vec{v})$ where $\vec{v} \notin E_\lambda$.]

(d) Prove that if A does not have any real eigenvalues, then A is similar to a matrix of the form λQ where Q is an orthogonal matrix and $\lambda > 0$.

Problem 4. Consider the sequence of real numbers defined by the recursive formula

$$x_0 = 0, \quad x_1 = 2, \quad x_{n+2} = 4x_{n+1} - 13x_n \text{ for all } n \geq 0.$$

Thus, the sequence starts like this: $0, 2, 8, 6, -80, \ldots$.
In this problem we will use linear algebra to find an explicit formula for x_n.

(a) Find a matrix $A \in \mathbb{R}^{2\times 2}$ such that $A \begin{bmatrix} x_n \\ x_{n+1} \end{bmatrix} = \begin{bmatrix} x_{n+1} \\ x_{n+2} \end{bmatrix}$ for every integer $n \geq 0$.

(b) Use part (a) to prove by induction that your matrix A satisfies $A^n \begin{bmatrix} 0 \\ 2 \end{bmatrix} = \begin{bmatrix} x_n \\ x_{n+1} \end{bmatrix}$ for every $n \geq 0$.

(c) Find all (real or complex) eigenvalues and corresponding eigenvectors for A.

(d) Find an invertible (real or complex) matrix P such that $A = PDP^{-1}$ where D is a diagonal matrix.

(e) First give an explicit formula for D^n, and then use this to give an explicit formula for A^n.

(f) Using parts (b) and (e), give an explicit formula for x_n, the nth term in the sequence. (Your formula may involve complex numbers, and need not be fully simplified.)

Problem 5. Let $V = C^\infty(\mathbb{R})$, let $\mathcal{A} = (e^{3x}, \cos 2x, \sin 2x)$, and let $W = \operatorname{span} \mathcal{A}$. Let $T : W \to W$ be the linear transformation defined by $T(f) = f'$.

(a) Find $[T]_\mathcal{A}$.

(b) Find all (real or complex) eigenvalues of the matrix $[T]_\mathcal{A}$.

(c) Viewing the matrix $[T]_\mathcal{A}$ as a linear transformation of the complex vector space $\mathbb{C}^3$, find a complex eigenvector for $[T]_\mathcal{A}$ for each of the eigenvalues you found in (b).

(d) Interpret the eigenvectors you found in (c) as a set of three complex-valued functions

$$\mathcal{B} = (f_1(x), f_2(x), f_3(x))$$

with the property that any complex linear combination of the vectors in $\mathcal{A}$ (that is, a linear combination with coefficients in $\mathbb{C}$) can be written as a complex linear combination of the vectors in $\mathcal{B}$, and vice versa.

(e) **(Recreational).** *Euler's formula* allows us to work with complex exponential functions via the definition $e^{i\theta} = \cos\theta + i\sin\theta$. Find three constants $a, b, c \in \mathbb{C}$ such that $\mathcal{C} = (e^{ax}, e^{bx}, e^{cx})$ has the same span over $\mathbb{C}$ as does $\mathcal{B}$, and such that $[T]_\mathcal{C}$ is a diagonal matrix.

[1]We work over $\mathbb{R}$ throughout this problem. So "eigenvalue" means *real eigenvalue*, "diagonalizable" means *diagonalizable over* $\mathbb{R}$, and "similar" means *similar over* $\mathbb{R}$.

[2]Recall that for each $\lambda \in \mathbb{R}$, $E_\lambda = \{\vec{v} \in \mathbb{R}^2 \;:\; A\vec{v} = \lambda\vec{v}\}$.

Problem 6. In this problem we apply some of the theory we have learned to Physics. Consider a solid three-dimensional object with mass density given by a function $\rho(\vec{r})$, where $\vec{r} = \langle r_2, r_2, r_3 \rangle$ is the standard position vector in $\mathbb{R}^3$. When such an object rotates in space, it has a nonzero *angular velocity*, which is represented as a vector $\vec{\omega} \in \mathbb{R}^3$ pointing along the axis of rotation. The rotating object also has an *angular momentum*, which is represented by a vector $\vec{L} \in \mathbb{R}^3$, and which is related to $\vec{\omega}$ by the equation $\vec{L} = I\vec{\omega}$, where I is a fixed 3×3 real matrix called the *moment of inertia tensor* for the solid object. The rotating object will wobble (that is, its axis of rotation will precess) if and only if $\vec{L}$ and $\vec{\omega}$ point along different lines.

(a) Show that if I has a real eigenvalue λ than there exists an axis around which the solid object can rotate without wobbling.
(b) Show that I always has at least one real eigenvalue λ (and hence by (a) there always exists an axis around which a solid object can rotate without wobbling).
(c) Show that if $\operatorname{gemu}(\lambda) = 3$ then the solid object can rotate around any axis without wobbling.
(d) Show that if I has three distinct real eigenvalues then there exist three axes around which the solid object can rotate without wobbling.
(e) It can be shown (although you do not have to worry about the proof of this!) that the (i, j)-component of the moment of inertia tensor is given by a volume integral:

$$I_{ij} = \begin{cases} -\iiint r_i r_j \, \rho(\vec{r}) \, dV, & i \neq j \\ \iiint \|\vec{r} - \operatorname{proj}_{\vec{e}_i} \vec{r}\|^2 \, \rho(\vec{r}) dV, & i = j \end{cases}$$

where $\vec{r} = \langle r_2, r_2, r_3 \rangle$ is the standard position vector in $\mathbb{R}^3$, and $\rho(\vec{r})$ is the mass density of the object at $\vec{r}$. Prove that for any solid object, there exist three **perpendicular** axes of rotation around which the object will not wobble. (These are called the *principal axes* of the object.)
[*Hint:* compare I_{ij} and I_{ji}, and consider the Spectral Theorem.]