MATH 217 W24 - LINEAR ALGEBRA, Section 001 (Dr. Paul Kessenich)
Homework Set Part B due Thursday, January 25 at 11:59pm
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1. In parts (a) - (d) below, determine whether the given function is injective, surjective, both, or neither. Justify your answers.

   (a) the function $f : [0, 4] \to [0, 18]$ defined by $f(x) = x^2 + 2$;

   Solution: Injective. If $f(x_1) = f(x_2)$, it follows that $x_1^2 = x_2^2$. Since the domain is positive, this also means $x_1 = x_2$, showing injectivity. There is no solution in the domain to $f(x) = 0$, so there exists a value in the codomain which is not in the image of f. Thus, the function is not surjective.

   (b) the function $g : \mathbb{R} \to \mathbb{R}$ defined by $g(x) = 2x - 5$;

   Solution: Bijective. If $g(x_1) = g(x_2)$, $2x_1 - 5 = 2x_2 - 5$. Therefore, $x_1 = x_2$, showing injectivity. Let $y \in \mathbb{R}$, and $x = \frac{y+5}{2}$. Then $x \in \mathbb{R}$, and $g(x) = y$. Thus g is surjective.

   (c) the function $h : \mathbb{R}^2 \to \mathbb{R}$ defined by $h(x, y) = 2x^2 + 5y^2$;

   Solution: Neither. $10 = h\left(\sqrt{5}, 0\right) = h\left(0, \sqrt{2}\right)$, so h is not injective. $h(x, y) = -2$ has no solutions in $\mathbb{R}^2$ since a square cannot be a negative number, therefore h is not surjective.

   (d) the function $q : \mathbb{N} \to \mathbb{N}$ defined by $q(n) = \begin{cases} n, & \text{if n is odd} \\ n/2 & \text{if n is even.} \end{cases}$

   Solution: Surjective. $1 = q(1) = q(2)$, so q is not injective. Let $m \in \mathbb{N}, n = 2m$. Then $n \in \mathbb{N}$, n is even, and $q(n) = m$. So q is surjective.

2. Determine whether each statement is true or false. If it is true, prove it. If it is false, prove this by giving a counterexample.

   (a) For every function $f : X \to Y$ and all $A, B \subseteq X$, if $A \cap B = \emptyset$, then $f[A] \cap f[B] = \emptyset$.

   Solution: False. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x^2$. Assign $A = \mathbb{R}^+$, $B = \mathbb{R}^-$. Then $A \cap B = \emptyset$, but $f(1 \in A) = f(-1 \in B) = 1$. Therefore $f[A] \cap f[B] \neq \emptyset$.

   (b) For every function $f : X \to Y$ and all $A, B \subseteq X$, if $f[A] \cap f[B] = \emptyset$, then $A \cap B = \emptyset$.

   Solution: True. We prove the contrapositive. Take any $f : X \to Y$ and $A, B \subseteq X$ such that $A \cap B$. Then $\exists a \in X$ such that $a \in A \cap B$. Since $a \in A, f(a) \in f[A]$. Similarly, $a \in B$, so $f(a) \in f[B]$. As $f(a) \in f[A]$ and $f(a) \in f[B]$, $f(a) \in f[A] \cap f[B]$. This means that for every function $f : X \to Y$ and all $A, B \subseteq X$, if $A \cap B \neq \emptyset$, then $f[A] \cap f[B] \neq \emptyset$. Thus the contrapositive is true, so the original statement is true.

(c) For every function $f : X \to Y$ and all $A \subseteq X$, we have $f^{-1}[f[A]] = A$.

Solution: False. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x^2$. Assign $A = \{1\}$. Then $f[A] = \{1\}$. However, $f(1) = f(-1) = 1$, so $f^{-1}[f[A]] = \{-1, 1\} \neq A$.

(d) For every function $f : X \to Y$ and all $A \subseteq X$, we have $f[X \backslash A] = Y \backslash f[A]$.

Solution: False. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x^2$. Assign $A = \{1\}$. Then $f[A] = \{1\}$, but $f(1) = f(-1) = 1$. Therefore $f[A] \subseteq f[X \backslash A]$, and $f[X \backslash A] \neq Y \backslash f[A]$.

(e) For every bijective function $f : X \to Y$ and all $A, B \subseteq X$, we have $f[A \cap B] = f[A] \cap f[B]$.

Solution: True. PROOF NEEDED. Let $x \in f[A \cap B]$. Then let $a = f^{-1}(x)$. We know $a \in A \cap B$ is unique due to bijectivity. Since $a \in A \cap B, a \in A \wedge a \in B$.