MATH 217 W24 - LINEAR ALGEBRA, Section 001 (Dr. Paul Kessenich)
Homework Set Part B due ??? at 11:59pm
Zhengyu James Pan (jzpan@umich.edu)

1. Question

   (a) Prove that F is alternating if and only if $F(\vec{u}, \vec{v}) = -F(\vec{v}, \vec{u})$ for all $\vec{u}, \vec{v} \in \mathbb{R}^2$.

   Solution: By bilinearity, we know

   $$
   \begin{aligned}
   F(u + v, v + u) &= 0 \\
   F(u, v + u) + F(v, v + u) &= 0 \\
   F(u, v) + F(u, u) + F(v, v) + F(v, u) &= 0 \\
   F(u, v) + 0 + 0 + F(v, u) &= 0 \\
   F(u, v) + F(v, u) &= 0 \\
   F(u, v) &= -F(v, u)
   \end{aligned}
   $$

   (b) Prove that if F is alternating and $F(\vec{e}_1, \vec{e}_2) = 1$, then $F(\vec{u}, \vec{v}) = \det[\vec{u}\ \vec{v}]$ for all $\vec{u}, \vec{v} \in \mathbb{R}^2$.

   Solution: Express $\vec{u}$ and $\vec{v}$ as linear combinations of e_1, e_2 :

   $$\vec{u} = u_1\vec{e}_1 + u_2\vec{e}_2 \text{ and } \vec{v} = v_1\vec{e}_1 + v_2\vec{e}_2$$

   Then

   $$
   \begin{aligned}
   F(\vec{u}, \vec{v}) &= F(u_1\vec{e}_1 + u_2\vec{e}_2, v_1\vec{e}_1 + v_2\vec{e}_2) \\
   &= F(u_1\vec{e}_1, v_1\vec{e}_1 + v_2\vec{e}_2) + F(u_2\vec{e}_2, v_1\vec{e}_1 + v_2\vec{e}_2) && \text{(bilinearity)} \\
   &= F(u_1\vec{e}_1, v_1\vec{e}_1) + F(u_1\vec{e}_1, v_2\vec{e}_2) + F(u_2\vec{e}_2, v_1\vec{e}_1) + F(u_2\vec{e}_2, v_2\vec{e}_2) \\
   &= u_1v_1 F(\vec{e}_1, \vec{e}_1) + u_1v_2 F(\vec{e}_1, \vec{e}_2) + u_2v_1 F(\vec{e}_2, \vec{e}_1) + u_2v_2 F(\vec{e}_2, \vec{e}_2) \\
   &= u_1v_1(0) + u_1v_2(1) + u_2v_1(-1) + u_2v_2(0) && \text{(alternating)} \\
   &= u_1v_2 - u_2v_1 \\
   &= \det \begin{bmatrix} u_1 & v_1 \\ u_2 & v_2 \end{bmatrix} \\
   &= \det[\vec{u}\ \vec{v}]
   \end{aligned}
   $$

2. (a) Prove that T is a linear transformation.

   Solution: Let $A, B \in \mathbb{R}^{2\times 2}$, and $c \in \mathbb{R}$.
   T respects addition:

   $$T(A + B) = (A + B)M = AM + BM = T(A) + T(B)$$

   by distributivity of matrix multiplication.
   T respects scalar multiplication:

   $$T(cA) = (cA)M = c(AM) = cT(A)$$

   by properties of matrix multiplication.
   Since T respects addition and scalar multiplication, it is linear.

   (b) Find the $\mathcal{E}$-matrix $[T]_{\mathcal{E}}$ of T, where $\mathcal{E}$ is the ordered basis

   $$\mathcal{E} = (E_{11}, E_{12}, E_{21}, E_{22}) = \left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right)$$

   of $\mathbb{R}^{2\times 2}$. Your answer should be in terms of the entries of M.

Solution:

$$[T]_{\mathcal{E}} = \begin{bmatrix} | & | & | & | \\ [T(E_{11})]_{\mathcal{E}} & [T(E_{12})]_{\mathcal{E}} & [T(E_{21})]_{\mathcal{E}} & [T(E_{22})]_{\mathcal{E}} \\ | & | & | & | \end{bmatrix}$$

$$[T(E_{11})]_{\mathcal{E}} = \left[\begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix}\right]_{\mathcal{E}} = \begin{bmatrix} a \\ b \\ 0 \\ 0 \end{bmatrix}$$

$$[T(E_{12})]_{\mathcal{E}} = \left[\begin{bmatrix} c & d \\ 0 & 0 \end{bmatrix}\right]_{\mathcal{E}} = \begin{bmatrix} c \\ d \\ 0 \\ 0 \end{bmatrix}$$

$$[T(E_{21})]_{\mathcal{E}} = \left[\begin{bmatrix} 0 & 0 \\ a & b \end{bmatrix}\right]_{\mathcal{E}} = \begin{bmatrix} 0 \\ 0 \\ a \\ b \end{bmatrix}$$

$$[T(E_{22})]_{\mathcal{E}} = \left[\begin{bmatrix} 0 & 0 \\ c & d \end{bmatrix}\right]_{\mathcal{E}} = \begin{bmatrix} 0 \\ 0 \\ c \\ d \end{bmatrix}$$

$$[T]_{\mathcal{E}} = \begin{bmatrix} a & c & 0 & 0 \\ b & d & 0 & 0 \\ 0 & 0 & a & c \\ 0 & 0 & b & d \end{bmatrix}$$

(c)