**MATH 217 W24 - LINEAR ALGEBRA, Section 001 (Dr. Paul Kessenich)**
**Homework Set Part B due ??? at 11:59pm**
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1. Question

(a) Prove that $F$ is alternating if and only if $F(\vec{u},\vec{v}) = -F(\vec{v},\vec{u})$ for all $\vec{u},\vec{v} \in \mathbb{R}^2$.

**Solution:** By bilinearity, we know

$$\begin{aligned}
F(u+v, v+u) &= 0 \\
F(u, v+u) + F(v, v+u) &= 0 \\
F(u,v) + F(u,u) + F(v,v) + F(v,u) &= 0 \\
F(u,v) + 0 + 0 + F(v,u) &= 0 \\
F(u,v) + F(v,u) &= 0 \\
F(u,v) &= -F(v,u)
\end{aligned}$$

(b) Prove that if $F$ is alternating and $F(\vec{e}_1,\vec{e}_2) = 1$, then $F(\vec{u},\vec{v}) = \det[\vec{u}\ \vec{v}]$ for all $\vec{u},\vec{v} \in \mathbb{R}^2$.

**Solution:** Express $\vec{u}$ and $\vec{v}$ as linear combinations of $e_1, e_2$ :

$$\vec{u} = u_1\vec{e}_1 + u_2\vec{e}_2 \text{ and } \vec{v} = v_1\vec{e}_1 + v_2\vec{e}_2$$

Then

$$\begin{aligned}
F(\vec{u},\vec{v}) &= F(u_1\vec{e}_1 + u_2\vec{e}_2, v_1\vec{e}_1 + v_2\vec{e}_2) \\
&= F(u_1\vec{e}_1, v_1\vec{e}_1 + v_2\vec{e}_2) + F(u_2\vec{e}_2, v_1\vec{e}_1 + v_2\vec{e}_2) && \text{(bilinearity)} \\
&= F(u_1\vec{e}_1, v_1\vec{e}_1) + F(u_1\vec{e}_1, v_2\vec{e}_2) + F(u_2\vec{e}_2, v_1\vec{e}_1) + F(u_2\vec{e}_2, v_2\vec{e}_2) \\
&= u_1v_1F(\vec{e}_1,\vec{e}_1) + u_1v_2F(\vec{e}_1,\vec{e}_2) + u_2v_1F(\vec{e}_2,\vec{e}_1) + u_2v_2F(\vec{e}_2,\vec{e}_2) \\
&= u_1v_1(0) + u_1v_2(1) + u_2v_1(-1) + u_2v_2(0) && \text{(alternating)} \\
&= u_1v_2 - u_2v_1 \\
&= \det\begin{bmatrix} u_1 & v_1 \\ u_2 & v_2 \end{bmatrix} \\
&= \det[\vec{u}\ \vec{v}]
\end{aligned}$$

2. (a) Prove that $T$ is a linear transformation.

**Solution:** Let $A, B \in \mathbb{R}^{2\times 2}$, and $c \in \mathbb{R}$.
$T$ respects addition:

$$T(A+B) = (A+B)M = AM + BM = T(A) + T(B)$$

by distributivity of matrix multiplication.
$T$ respects scalar multiplication:

$$T(cA) = (cA)M = c(AM) = cT(A)$$

by properties of matrix multiplication.

Since $T$ respects addition and scalar multiplication, it is linear.

(b) Find the $\mathcal{E}$-matrix $[T]_{\mathcal{E}}$ of $T$, where $\mathcal{E}$ is the ordered basis

$$\mathcal{E} = (E_{11}, E_{12}, E_{21}, E_{22}) = \left( \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right)$$

of $\mathbb{R}^{2\times 2}$. Your answer should be in terms of the entries of $M$.

**Solution:**

$$[T]_{\mathcal{E}} = \begin{bmatrix} | & | & | & | \\ [T(E_{11})]_{\mathcal{E}} & [T(E_{12})]_{\mathcal{E}} & [T(E_{21})]_{\mathcal{E}} & [T(E_{22})]_{\mathcal{E}} \\ | & | & | & | \end{bmatrix}$$

$$[T(E_{11})]_{\mathcal{E}} = \left[\begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix}\right]_{\mathcal{E}} = \begin{bmatrix} a \\ b \\ 0 \\ 0 \end{bmatrix}$$

$$[T(E_{12})]_{\mathcal{E}} = \left[\begin{bmatrix} c & d \\ 0 & 0 \end{bmatrix}\right]_{\mathcal{E}} = \begin{bmatrix} c \\ d \\ 0 \\ 0 \end{bmatrix}$$

$$[T(E_{21})]_{\mathcal{E}} = \left[\begin{bmatrix} 0 & 0 \\ a & b \end{bmatrix}\right]_{\mathcal{E}} = \begin{bmatrix} 0 \\ 0 \\ a \\ b \end{bmatrix}$$

$$[T(E_{22})]_{\mathcal{E}} = \left[\begin{bmatrix} 0 & 0 \\ c & d \end{bmatrix}\right]_{\mathcal{E}} = \begin{bmatrix} 0 \\ 0 \\ c \\ d \end{bmatrix}$$

$$[T]_{\mathcal{E}} = \begin{bmatrix} a & c & 0 & 0 \\ b & d & 0 & 0 \\ 0 & 0 & a & c \\ 0 & 0 & b & d \end{bmatrix}$$

(c) Compute $\det[T]_{\mathcal{E}}$.

**Solution:** Using the Laplace expansion on our result from (b),

$$\begin{aligned}
\det[T]_{\mathcal{E}} &= \begin{vmatrix} a & c & 0 & 0 \\ b & d & 0 & 0 \\ 0 & 0 & a & c \\ 0 & 0 & b & d \end{vmatrix} \\
&= a \begin{vmatrix} d & 0 & 0 \\ 0 & a & c \\ 0 & b & d \end{vmatrix} - c \begin{vmatrix} b & 0 & 0 \\ 0 & a & c \\ 0 & b & d \end{vmatrix} \\
&= ad \begin{vmatrix} a & c \\ b & d \end{vmatrix} - bc \begin{vmatrix} a & c \\ b & d \end{vmatrix} \\
&= (ad - bc)^2 \\
&= a^2d^2 - 2abcd + b^2c^2
\end{aligned}$$

(d) Compute $\det[T]_{\mathcal{B}}$.

**Solution:** The determinant of a transformation is the same in any basis. So it will be the same as part (c), or $(ad - bc)^2 = a^2d^2 - 2abcd + b^2c^2$.

(e)

3. Prove that there exists a unique vector z R4 such that T ( x) = z · x for all x R4, and find the components of z in terms of the vectors u, v, and w. (Hint: x = x1 e1 +x2 e2 +x3 e3 +x4 e4.)