MATH 217 - W24 - LINEAR ALGEBRA
HOMEWORK 4, DUE Thursday, February 8 at 11:59pm

CORRECTED VERSION

Submit Part A and Part B as two *separate* assignments in Gradescope as a **pdf file.** At the time of submission, Gradescope will prompt you to match each problem to the page(s) on which it appears. **You must match problems to pages in Gradescope so we know what page each problem appears on.** Failure to do so may result in not having the problem graded.

A few words about solution writing:

- Unless you are explicitly told otherwise for a particular problem, **you are always expected to show your work and to give justification for your answers.**
- Your solutions will be judged on precision and completeness and not merely on "basically getting it right".
- Cite every theorem or fact from the book that you are using (e.g. "By Theorem 1.10 ...").

Part A (15 points)

Solve the following problems from the book:
Section 2.4: 28, 30, 42
Section 3.1: 6, 14

Part B (25 points)

Problem 1. Let $\mathbb{R}^n \xrightarrow{T} \mathbb{R}^n$ be a linear transformation. As on HW 3, we define T^k to be the k-fold composition of T with itself,

$$= \underbrace{T \circ T \circ T \circ \cdots\cdots \circ T}_{k \text{ times}}.$$

Let A be the standard matrix of T, by which we mean the unique $n \times n$ matrix such that $T(\vec{x}) = A\vec{x}$ for all $\vec{x} \in \mathbb{R}^n$.

(a) Prove that for all k, the standard matrix for T^k is the matrix A^k. [Hint: induction works nicely.]

(b) We define T to be **nilpotent** if there exists some $k \in \mathbb{N}$ such that T^k is the zero transformation. Prove that if T is nilpotent, then A is not invertible.

(c) Prove that if T is nilpotent, then $A - I_n$ **is** invertible.
[Hint: try multiplying out $(A - I_n)(I_n + A + A^2 + \cdots + A^{k-1})$ and see what you get.]

Problem 2. Let V be any vector space, and let S be any set. Let $\mathcal{F}(S, V)$ denote the set of all functions from S to V. (Note: we are not assuming $S \subseteq V$ here, just that S is some set. S is not assumed to be a vector space, but it could be. Similarly, the functions in $\mathcal{F}(S, V)$ are not assumed to be linear transformations, although it is possible that some of them might be.)

For any functions $f, g \in \mathcal{F}(S, V)$ we can define their *sum* to be the function $f + g$ given by the formula $(f + g)(s) = f(s) + g(s)$, where s is any element in S. Similarly, for any scalar $c \in \mathbb{R}$ and any function $f \in \mathcal{F}(S, V)$ we define the function cf to be given by the formula $(cf)(s) = c(f(s))$ for all $s \in S$.

(a) Prove that $\mathcal{F}(S,V)$ is a vector space. **Note:** For this problem you must *explicitly prove* that each of the vector space properties VS1-8 from Worksheet 6 is true. (These proofs should be very short but are not skippable.)
(b) Is $0_{\mathcal{F}(S,V)}$ the same element as 0_V? If not, explain how they are different.
(c) We could similarly define $\mathcal{F}(V,S)$ to be the set of all functions from V to S. Would $\mathcal{F}(V,S)$ also a vector space? Why or why not?
(d) The familiar vector spaces $\mathcal{P}, \mathcal{P}_n$ and $\mathcal{C}^\infty$ (all from Worksheet 6) are all subsets of $\mathcal{F}(S,V)$ for some S and V. What are S and V for each of these functions?

Problem 3. Let $\mathcal{P}$ be the vector space of all polynomial functions from $\mathbb{R}$ to $\mathbb{R}$ in the variable t, and for each $n \in \mathbb{N}$, let $\mathcal{P}_n$ be (as usual) the subset of $\mathcal{P}$ consisting of all polynomial functions of degree at most n. (We already know that $\mathcal{P}_n$ is also a vector space.) Also let $T : \mathcal{P} \to \mathcal{P}$ be the map defined by $T(p)(t) = p'(t) + p(0)$ for each $p \in \mathcal{P}$ and for all $t \in \mathbb{R}$.
(a) Show that T is a linear transformation.
(b) Let $n \in \mathbb{N}$, and let $T_n : \mathcal{P}_n \to \mathcal{P}_n$ be defined by $T_n(p)(t) = p'(t) + p(0)$, so that T_n is just T with both domain and codomain restricted to $\mathcal{P}_n$. Is T_n injective? Is T_n surjective?
(c) Is T injective? Is T surjective?

Problem 4. We denote by $\mathbb{R}^{n\times n}$ the vector space of all $n \times n$ matrices. Let A be an $n \times n$ matrix, and define the function $L_A : \mathbb{R}^{n\times n} \to \mathbb{R}^{n\times n}$ by $L_A(B) = AB$ for all $B \in \mathbb{R}^{n\times n}$. (Note carefully: this is *not* the same function as T_A. While both can be described informally as "multiplication by A", the two functions L_A and T_A have different domains and codomains. Make sure you understand this distinction before beginning to work on this problem!)
(a) Show that L_A is a linear transformation.
(b) Show that the matrix A is invertible if and only if the linear transformation L_A is invertible.

Now let $\mathcal{F}$ be the set of all functions from $\mathbb{R}^{n\times n}$ to $\mathbb{R}^{n\times n}$, and define the function $L : \mathbb{R}^{n\times n} \to \mathcal{F}$ by $L(A) = L_A$.
(c) Show that L is injective.
(d) Is L surjective? Be sure to justify your claim.

Problem 5. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation defined as follows:

$$T = \mathrm{Rot}_{-80^\circ} \circ \mathrm{Proj}_y \circ \mathrm{Rot}_{35^\circ},$$

where Rot_θ is counter-clockwise rotation by θ, and Proj_y is projection onto the y-axis.
(a) Sketch $\mathrm{im}(T)$ in $\mathbb{R}^2$. Indicate the angle between $\mathrm{im}(T)$ and the x-axis.
(b) Sketch $\ker(T)$ in $\mathbb{R}^2$. Indicate the angle between $\ker(T)$ and the x-axis.
(c) Let $T_{\phi,\theta} := \mathrm{Rot}_\phi \circ \mathrm{Proj}_y \circ \mathrm{Rot}_\theta$. For which ϕ and θ is $\mathrm{im}(T_{\phi,\theta}) = \ker(T_{\phi,\theta})$?