MATH 217 W24 - LINEAR ALGEBRA, Section 001 (Dr. Paul Kessenich)
Homework 3 Part B due Thursday, February 1 at 11:59pm
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1. Determine whether the following statements are true or false, and justify your answer with a proof or a counterexample.

   (a) For all 2×2 matrices A and B, $(AB)^T = A^T B^T$.

   Solution: False. For example,

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}$$

$$\begin{aligned} (AB)^\top &= \begin{bmatrix} 19 & 22 \\ 43 & 50 \end{bmatrix}^\top \\ &= \begin{bmatrix} 1 & 22 \\ 43 & 50 \end{bmatrix} \\ \neq A^\top B^\top &= \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} 5 & 7 \\ 6 & 8 \end{bmatrix} \\ &= \begin{bmatrix} 23 & 31 \\ 34 & 46 \end{bmatrix} \end{aligned}$$

   (b) For all 2×2 matrices A and B, $(AB)^\top \neq A^\top B^\top$.

   Solution: False. For example:

$$A = \begin{bmatrix} 1 & 3 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

$$\begin{aligned} (AB)^\top &= \begin{bmatrix} 1 & 3 \\ 0 & 0 \end{bmatrix}^\top \\ &= \begin{bmatrix} 1 & 0 \\ 3 & 0 \end{bmatrix} \\ = A^\top B^\top &= \begin{bmatrix} 1 & 0 \\ 3 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ 3 & 0 \end{bmatrix} \end{aligned}$$

   (c) For all matrices A and B such that the matrix product AB exists, $(AB)^\top = B^\top A^\top$.

Solution: True. Computing with arbitrary matrices A and B with ijth element a_{ij}, bij respectively, we see the two products are identical.

$$
\begin{aligned}
(AB)^\top &= \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} \end{bmatrix}^\top \\
&= \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{21}b_{11} + a_{22}b_{21} \\ a_{11}b_{12} + a_{12}b_{22} & a_{21}b_{12} + a_{22}b_{22} \end{bmatrix}
\end{aligned}
$$

$$
\begin{aligned}
B^\top A^\top &= \begin{bmatrix} b_{11} & b_{21} \\ b_{12} & b_{22} \end{bmatrix} \begin{bmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{bmatrix} \\
&= \begin{bmatrix} b_{11}a_{11} + b_{21}a_{12} & b_{11}a_{21} + b_{21}a_{22} \\ b_{12}a_{11} + b_{22}a_{12} & b_{12}a_{21} + b_{22}a_{22} \end{bmatrix} \quad = \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{21}b_{11} + a_{22}b_{21} \\ a_{11}b_{12} + a_{12}b_{22} & a_{21}b_{12} + a_{22}b_{22} \end{bmatrix}
\end{aligned}
$$

(d) If A is a symmetric matrix, then for all $n \in \mathbb{N}$, An is also symmetric.

Solution: True. If A is symmetric, then $A = A^\top$. Multiplying by n on both sides results in $An = A^\top n$. Since scalar multiplication applies to all elements of a matrix, transpose clearly respects scalar multiplication; thus $An = (An)^\top$. This means An is symmetric.

(e) If A is a square matrix and A^2 is symmetric, then so is A.

Solution: False. The matrix $A = \begin{bmatrix} -1 & -2 \\ 2 & 1 \end{bmatrix}$ is not symmetric. However, $A^2 = \begin{bmatrix} -3 & 0 \\ 0 & -3 \end{bmatrix}$ is symmetric. This example demonstrates that A is not necessarily symmetric if A^2 is.

2. Determine whether the following statements are true or false, and justify your answer with a proof or a counterexample.

(a) Every 3-by-3 matrix that has a row of zeros is not invertible.

Solution: A matrix with a row of zeros can only have 2 pivots, for a rank of 2. Thus by theorem 2.4.3, it is not invertible.

(b) Every square matrix with 1's down the main diagonal is invertible.

Solution: False. For example, $A = \begin{bmatrix} 1 & 2 \\ 0.5 & 1 \end{bmatrix}$ is not invertible: both $A \begin{bmatrix} 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} = A \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

(c) For any matrix A, if A is invertible, then so is A^{-1}.

Solution: True. PROOF NEEDED

(d) For any matrix A, if A is invertible, then A^n is invertible.

Solution: True. $(A^n)^{-1} = (A^{-1})^n$ PROOF NEEDED

3. Let A be an $m \times n$ matrix. Prove that if there exists an $n \times m$ matrix B such that $BA = I_n$, then the system of linear equations $A\vec{x} = \vec{0}$ has a unique solution. (Note: a matrix B with this property is called a left-inverse for A. Can you guess why?)

4. Given two matrices A and B such that the product AB is defined (say, A is $n \times m$ and B is $m \times k$), exactly one of the following two statements is true:

   (a) Every column of AB is a linear combination of columns of A,

   (b) Every column of AB is a linear combination of columns of B.

   Prove the one that is true, and provide a counterexample for the one that is false.

5. Let $f : X \to X$ be a function. We let f^n denote the function $f^n : X \to X$ given by composing f iteratively, n many times. Also, we define f^0 to be the identity function, i.e. $\forall x \in X, f^0(x) = x$.

   (a) Assume that $X = \mathbb{R}^d$. Prove by induction that if f is a linear transformation, then the nth iterate f^n is also a linear transformation.

   (b) Find an example of a function $f : \mathbb{R}^2 \to \mathbb{R}^2$ which is not a linear transformation, but for which there exists an n such that the nth iterate f^n is a linear transformation.

   (c) Prove that for $X = Rd$ and f linear, if the equation $f(x) = 0$ has a unique solution, then the iterated equation $f^n(x) = 0$ also has a unique solution.