MATH 217 W24 - LINEAR ALGEBRA, Section 001 (Dr. Paul Kessenich)
Homework 11 Part B due SUNDAY, April 21 at 11:59pm
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1. (a) Let E_0 denote the 0-eigenspace of T. Explicitly describe E_0 (as a set).

   Solution:
$$E_0 = \{(x_1, 0, x_2, 0, x_3, 0, ...) \mid x_i \in \mathbb{R}\}$$

   (b) Prove that every real number λ is an eigenvalue of T. (Hint: explicitly construct an eigenvector $(x_1, x_2, x_3, ...) \in V$. First consider x_i when i is a power of 2.)

   Solution: Let $\lambda \in \mathbb{R}$. Then let
$$s = (1,\ \lambda, \lambda,\ \lambda^2, \lambda^2, \lambda^2, \lambda^2,\ \lambda^3, \lambda^3, \lambda^3, \lambda^3, \lambda^3, \lambda^3, \lambda^3, \lambda^3,\ ...)$$
   be an infinite sequence such that each consecutive power λ^n is repeated n times in the sequence, starting from $n = 0$. Then
$$\begin{aligned} T(s) &= (\lambda,\ \lambda^2, \lambda^2,\ \lambda^3, \lambda^3, \lambda^3, \lambda^3,\ \lambda^4, \lambda^4, \lambda^4, \lambda^4, \lambda^4, \lambda^4, \lambda^4, \lambda^4, ...) \\ &= \lambda(s). \end{aligned}$$
   So any real number is an eigenvalue of T.

2. (a) Let $\mathscr{D}$ be a diagonal $n \times n$ matrix with distinct entries along the diagonal, and let $\mathscr{D}$ be the subset of $\mathbb{R}^{n\times n}$ consisting of all diagonal matrices. Prove $\mathscr{C}(D) = \mathscr{D}$.

   Solution: Let the diagonal entries of D be $d_1, ..., d_n$. Let $A \in \mathscr{D}$, with diagonal entries $a_1, ...a_n$. Then the product
$$AD = \begin{bmatrix} a_1d_1 & 0 & \dots & 0 \\ 0 & a_2d_2 & 0 & \vdots \\ \vdots & 0 & \ddots & 0 \\ 0 & \dots & 0 & a_nd_n \end{bmatrix} = \begin{bmatrix} d_1a_1 & 0 & \dots & 0 \\ 0 & d_2a_2 & 0 & \vdots \\ \vdots & 0 & \ddots & 0 \\ 0 & \dots & 0 & d_na_n \end{bmatrix} = DA.$$
   So $\mathscr{D} \subset \mathscr{C}(D)$.

   Let $B \in \mathscr{C}(D)$ with columns $\vec{b}_1, ..., \vec{b}_n$, rows $\vec{c}_1, ..., \vec{c}_n$, and element of ith row and

jth column b_{ij}. Then

$$\begin{aligned}
BD &= \begin{bmatrix} | & & | \\ B(d_1\vec{e}_1) & \cdots & B(d_n\vec{e}_n) \\ | & & | \end{bmatrix} \\
&= \begin{bmatrix} | & & | \\ d_1\vec{b}_1 & \cdots & d_n\vec{b}_n \\ | & & | \end{bmatrix}
\end{aligned}$$

$$\begin{aligned}
DB &= \left((DB)^\top\right)^\top \\
&= \left(B^\top D\right)^\top \\
&= \begin{bmatrix} | & & | \\ d_1\vec{c}_1^\top & \cdots & d_n\vec{c}_n^\top \\ | & & | \end{bmatrix}^\top \\
&= \begin{bmatrix} — & d_1\vec{c}_1 & — \\ & \vdots & \\ — & d_n\vec{c}_n & — \end{bmatrix}
\end{aligned}$$

where $\vec{e}_i$ is the ith standard basis vector. Since $B \in \mathscr{C}(D), BD = DB$. Considering arbitrary b_{ij}, this means that $d_i b_{ij} = d_j b_{ij}$.

When $i = j$, then $d_i = d_j$, so b_{ij}, a diagonal element of B, can be anything.

When $i \neq j$, then $d_i \neq d_j$ since D has distinct diagonal elements. But $d_i b_{ij} = d_j b_{ij}$. So $b_i j = 0$. Note that in this case, $b_i j$ is any non-diagonal element. Since only diagonal elements of B can be nonzero, B is diagonal, and $B \in \mathscr{D}$. So $\mathscr{C}(D) \subset \mathscr{D}$.

Thus $\mathscr{D} = \mathscr{C}(D)$.

(b) Prove that if A and B are simultaneously diagonalizable $n \times n$ matrices, then $B \in \mathscr{C}(A)$.

Solution: We know that $\mathscr{D} \subset \mathscr{C}(D)$ for any diagonal matrix D by part (a). (The $\mathscr{D} \subset \mathscr{C}(D)$ direction did not depend on the fact that D had distinct diagonal elements.) So then

$$\begin{aligned}
S^{-1}BSS^{-1}AS &= S^{-1}ASS^{-1}BS \\
S^{-1}BAS &= S^{-1}ABS \\
BA &= AB
\end{aligned}$$

Thus $B \in \mathscr{C}(A)$.

(c) Prove that if A and B are $n \times n$ matrices such that A has n distinct eigenvalues and $B \in \mathscr{C}(A)$, then A and B are simultaneously diagonalizable.

Solution: Let S be an eigenbasis of A. Then $S^{-1}AS$, the diagonalized matrix of A, has distinct diagonal elements. We know that

$$\begin{aligned} BA &= AB \\ S^{-1}BAS &= S^{-1}ABS \\ S^{-1}BSS^{-1}AS &= S^{-1}ASS^{-1}BS \end{aligned}$$

So $S^{-1}BS \in \mathscr{C}(S^{-1}AS)$. Then by part (a), since $S^{-1}AS$ is diagonal with distinct entries, $S^{-1}BS$ is diagonal. So S diagonalizes both B and A. So A and B are simultaneously diagonalizable.

3. (a) Suppose that A has eigenvalue 0 but is not diagonalizable. Prove that $\mathrm{im}(A) = E_0$, and conclude from this that $A^2 = 0$.

Solution: The givens imply that $\det(A - 0I_1) = 0$. So $\det(A) = 0$. But since it is not diagonalizable, there must be one eigenvalue λ of A which does not satisfy $\mathrm{almu}(\lambda) = \mathrm{gemu}(\lambda)$. Since both almu and gemu are always at least 1, almu $\geq$ gemu, and the sum of almus $= n$; we know $\mathrm{almu}(0) = 2$, and $\mathrm{gemu}(0) = 1$. From this, we infer that the nullity is 1, and the rank is also 1. The almu also implies that the characteristic equation of A is $x^2 = 0$.
Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Since

$$\begin{aligned} \det(A - 0I_1) &= (a - x)(d - x) - bc \\ &= ad - bc - (a + d)x + x^2 \end{aligned}$$

and the characteristic polynomial has no x^1 terms, we know that $a + d = 0$, or equivalently that $a = -d$. Additionally, $ad = bc$.
We know that A does not have linearly independent columns, so the columns must be scalar multiples of each other. In fact, since $a = -d$, the ratio of the second column to the first is $\dfrac{d}{c} = \dfrac{-a}{c}$. So then the rref form of A is

$$\begin{bmatrix} 1 & \dfrac{-a}{c} \\ 0 & 0 \end{bmatrix}$$

which has nullspace spanned by $\begin{bmatrix} a \\ b \end{bmatrix}$. But this is also the image of A, since both columns of A are scalar multiples of $\begin{bmatrix} a \\ b \end{bmatrix}$. So the image and nullity of A are the same. Thus

$$A^2\vec{x} = A(A\vec{x}) = 0$$

for any $\vec{x} \in \mathbb{R}^2$. Since $A\vec{x}$ is in $\ker(A)$. So $A^2 = 0$.

(b) Let $\lambda \in \mathbb{R}$ and suppose that A has eigenvalue λ but is not diagonalizable. Prove that we have $(A - \lambda I_2)^2 = 0$, and deduce from this that $A\vec{v} - \lambda\vec{v} \in E_\lambda$ for every $\vec{v} \in \mathbb{R}^2$. [Hint: apply part (a) to the matrix $A - \lambda I_2$].

Solution: Since both almu and gemu are always at least 1, almu $\geq$ gemu, and the sum of almus $= n$; we know almu$(\lambda) = 2$, and gemu$(\lambda) = 1$. So the matrix $A - \lambda I_2$ has an eigenvalue of 0, and nullity 1. So $A - \lambda I_2$ has an eigenvalue of 0 and is not diagonalizable.

Then by (a), $(A - \lambda I_2)^2 = 0$, and the image of $A - \lambda I_2$ is equal to its 0-eigenspace. The 0-eigenspace of $A - \lambda I_2$ is in turn equal to its own kernel. We know $\ker(A - \lambda I_2)$ is also the λ-eigenspace of A. So for all $\vec{v} \in \mathbb{R}^2$, $(A - \lambda I_n)\vec{v} = A\vec{v} - \lambda\vec{v} \in E_\lambda$.

(c) Prove that if A has eigenvalue λ but is not diagonalizable, then A is similar to $\begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$.

Solution: By (b), we know that $A\vec{v} - \lambda\vec{v} \in E_\lambda$ for any vector $\vec{v} \in \mathbb{R}^2$. Let $\mathcal{B} = \{\vec{b}_1, \vec{b}_2\}$, where $\vec{b}_2 \in E_\lambda^\perp$ and $\vec{b}_1 = A\vec{b}_2 - \lambda\vec{b}_2$, which is in E_λ by (b). We know that these are linearly independent since they are orthogonal.

Note that this definition of $\mathcal{B}$ ensures that $A\vec{b}_1 = \lambda\vec{b}_1$ and $A\vec{b}_2 = \vec{b}_1 + \lambda\vec{b}_2$.

Then

$$\begin{aligned} [A]_\mathcal{B} &= \begin{bmatrix} [A\vec{b}_1]_\mathcal{B} & [A\vec{b}_2]_\mathcal{B} \end{bmatrix} \\ &= \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix} \end{aligned}$$

So A is similar to $\begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$ by the Change-of-basis of Theorem.

(d) Prove that if A does not have any real eigenvalues, then A is similar to a matrix of the form λQ where Q is an orthogonal matrix and $\lambda > 0$.

Solution: We know A has a pair of complex eigenvalues of the form $a \pm bi$. By Worksheet 26 problem 9, A is then similar to the matrix $B = \begin{bmatrix} a & b \\ -b & a \end{bmatrix}$. Since the columns are already orthogonal, we simply need to normalize them to make the matrix orthogonal, which can be done by dividing by $\lambda = \sqrt{a^2 + b^2}$, which is a positive number. So $Q = \frac{1}{\lambda}B$, and the statement is true.

4. (a) Find a matrix $A \in \mathbb{R}^{2\times 2}$ such that $A\begin{bmatrix} x_n \\ x_{n+1} \end{bmatrix} = \begin{bmatrix} x_{n+1} \\ x_{n+2} \end{bmatrix}$ for every integer $n \geq 0$.

Solution:

$$\boxed{A = \begin{bmatrix} 0 & 1 \\ -13 & 4 \end{bmatrix}}$$

$$A\begin{bmatrix} x_n \\ x_{n+1} \end{bmatrix} = \begin{bmatrix} 0x_n + x_{n+1} \\ -13x_n + 4x_{n+1} \end{bmatrix} = \begin{bmatrix} x_{n+1} \\ x_{n+2} \end{bmatrix}$$

(b) Use part (a) to prove by induction that your matrix A satisfies $A^n \begin{bmatrix} 0 \\ 2 \end{bmatrix} = \begin{bmatrix} x_n \\ x_{n+1} \end{bmatrix}$ for every $n \geq 0$.

Solution: Base case: $A^0 \begin{bmatrix} 0 \\ 2 \end{bmatrix} = I_2 \begin{bmatrix} 0 \\ 2 \end{bmatrix}$

Inductive step: Assume that for some integer $n \geq 0$,

$$A^n \begin{bmatrix} 0 \\ 2 \end{bmatrix} = \begin{bmatrix} x_n \\ x_{n+1} \end{bmatrix}.$$

Then by part (a),

$$\begin{aligned} AA^n \begin{bmatrix} 0 \\ 2 \end{bmatrix} &= A^{n+1} \begin{bmatrix} 0 \\ 2 \end{bmatrix} \\ &= A \begin{bmatrix} x_n \\ x_{n+1} \end{bmatrix} \\ &= \begin{bmatrix} x_{n+1} \\ x_{n+2} \end{bmatrix} \end{aligned}$$

So if n satisfies the statement, then $n+1$ satisfies the statement. Since $n = 0$ satisfies the statement, the statement is true for all $n \geq 0$.

(c) Find all (real or complex) eigenvalues and corresponding eigenvectors for A.

Solution:

$$\begin{vmatrix} 0-\lambda & 1 \\ -13 & 4-\lambda \end{vmatrix} = (-\lambda)(4-\lambda)+13$$
$$= \lambda^2 - 4\lambda + 13$$

$$\lambda = \frac{4 \pm \sqrt{16-52}}{2} = 2 \pm 3i$$

$$2+3i:$$
$$\begin{bmatrix} -2-3i & 1 \\ -13 & 2-3i \end{bmatrix} \to \begin{bmatrix} -2-3i & 1 \\ 0 & 0 \end{bmatrix}$$
$$E_{2+3i} = \text{span}\left(\begin{bmatrix} 1 \\ 2+3i \end{bmatrix}\right)$$

$$2-3i:$$
$$\begin{bmatrix} -2+3i & 1 \\ -13 & 2+3i \end{bmatrix} \to \begin{bmatrix} -2+3i & 1 \\ 0 & 0 \end{bmatrix}$$
$$E_{2+3i} = \text{span}\left(\begin{bmatrix} 1 \\ 2-3i \end{bmatrix}\right)$$

(d) Find an invertible (real or complex) matrix P such that $A = PDP^{-1}$ where D is a diagonal matrix.

Solution:

$$P = \begin{bmatrix} 1 & 1 \\ 2+3i & 2-3i \end{bmatrix}$$

(e) First give an explicit formula for D_n, and then use this to give an explicit formula for A_n.

Solution: $D = \begin{bmatrix} 2+3i & 0 \\ 0 & 2-3i \end{bmatrix}$, so

$$D^n = \begin{bmatrix} (2+3i)^n & 0 \\ 0 & (2-3i)^n \end{bmatrix}$$

So then

$$A^n = P^{-1}D^nP = \begin{bmatrix} 1 & 1 \\ 2+3i & 2-3i \end{bmatrix}^{-1} \begin{bmatrix} (2+3i)^n & 0 \\ 0 & (2-3i)^n \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2+3i & 2-3i \end{bmatrix}.$$

(f) Using parts (b) and (e), give an explicit formula for x_n, the nth term in the sequence.

(Your formula may involve complex numbers, and need not be fully simplified.)

Solution: