**MATH 217 - W24 - LINEAR ALGEBRA**
**HOMEWORK 3, DUE Thursday, February 1 at 11:59pm**

Submit Part A and Part B as two *separate* assignments in Gradescope as a **pdf file.** At the time of submission, Gradescope will prompt you to match each problem to the page(s) on which it appears. **You must match problems to pages in Gradescope so we know what page each problem appears on.** Failure to do so may result in not having the problem graded.

**A few words about solution writing:**

- Unless you are explicitly told otherwise for a particular problem, **you are always expected to show your work and to give justification for your answers.**
- Your solutions will be judged on precision and completeness and not merely on "basically getting it right".
- Cite every theorem or fact from the book that you are using (e.g. "By Theorem 1.10 . . . ").

## Part A (10 points)

Solve the following problems from the book:
**Section 2.2:** 20, 38;
**Section 2.3:** 18, 34;
**Section 2.4:** 12, 34.

## Part B (25 points)

The definitions of *trace*, *determinant* and *transpose* will be needed in this part.

**Definition 1.** Given a square $n \times n$ matrix $C = \begin{bmatrix} c_{11} & \cdots & c_{1n} \\ \vdots & \ddots & \vdots \\ c_{n1} & \cdots & c_{nn} \end{bmatrix}$, we define the **trace** of $C$ to be the sum of the diagonal elements $c_{11} + \cdots + c_{nn} = \sum_{i=1}^{n} c_{ii}$, denoted $\operatorname{tr}(C)$.

**Definition 2.** The **determinant** of a square matrix $C$ will be denoted $\det(C)$. We define the determinant of a $1 \times 1$ matrix by $\det[a] = a$, and the determinant of a $2 \times 2$ matrix by $\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc$. (We will wait until Chapter 6 to define determinants of larger square matrices).

**Definition 3.** Consider an $m \times n$ matrix $A$. The **transpose** $A^\top$ of $A$ is the $n \times m$ matrix obtained from $A$ by rewriting all of the columns of $A$ as rows, and vice versa, so that the $(i, j)$-entry of $A^\top$ is the $(j, i)$-entry of $A$. Further, we say that the square matrix $A$ is **symmetric** if $A^\top = A$.

**Problem 1.** Determine whether the following statements are true or false, and justify your answer with a proof or a counterexample.

(a) For all $2 \times 2$ matrices $A$ and $B$, $(AB)^\top = A^\top B^\top$.
(b) For all $2 \times 2$ matrices $A$ and $B$, $(AB)^\top \neq A^\top B^\top$.

(c) For all matrices $A$ and $B$ such that the matrix product $AB$ exists, $(AB)^\top = B^\top A^\top$.
(d) If $A$ is a symmetric matrix, then for all $n \in \mathbb{N}$, $A^n$ is also symmetric.
(e) If $A$ is a square matrix and $A^2$ is symmetric, then so is $A$.

**Problem 2.** Determine whether the following statements are true or false, and justify your answer with a proof or a counterexample.
(a) Every 3-by-3 matrix that has a row of zeros is not invertible.
(b) Every square matrix with 1's down the main diagonal is invertible.
(c) For any matrix $A$, if $A$ is invertible, then so is $A^{-1}$.
(d) For any matrix $A$, if $A$ is invertible, then $A^n$ is invertible.

**Problem 3.** Let $A$ be an $m \times n$ matrix.
(a) Prove that if there exists an $n \times m$ matrix $B$ such that $BA = I_n$, then the system of linear equations $A\vec{x} = \vec{0}$ has a unique solution. (Note: a matrix $B$ with this property is called a *left-inverse* for $A$. Can you guess why?)
(b) **(Recreational)** State and prove the converse of the statement in (a).

**Problem 4.** Given two matrices $A$ and $B$ such that the product $AB$ is defined (say, $A$ is $n \times m$ and $B$ is $m \times k$), exactly one of the following two statements is true:
(a) Every column of $AB$ is a linear combination of columns of $A$,
(b) Every column of $AB$ is a linear combination of columns of $B$.

Prove the one that is true, and provide a counterexample for the one that is false.

**Problem 5.** Let $f\colon X \to X$ be a function. We let $f^n$ denote the function $f^n\colon X \to X$ given by composing $f$ iteratively, $n$ many times. In other words, $f^n(x) = \underbrace{(f \circ \cdots \circ f)}_{n \text{ times}}(x)$. Also, we define $f^0$ to be the identity function, i.e. $\forall x \in X, f^0(x) = x$.
(a) Assume that $X = \mathbb{R}^d$. Prove by induction that if $f$ is a linear transformation, then the $n$th iterate $f^n$ is also a linear transformation.
(b) Find an example of a function $f : \mathbb{R}^2 \to \mathbb{R}^2$ which is **not** a linear transformation, but for which there exists an $n$ such that the $n$th iterate $f^n$ is a linear transformation.
(c) Prove that for $X = \mathbb{R}^d$ and $f$ linear, if the equation $f(x) = 0$ has a unique solution, then the iterated equation $f^n(x) = 0$ also has a unique solution.