MATH 215 FALL 2023
Homework Set 8: §16.3 – 16.6
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1. Calculate $\oint_C \overrightarrow{F} \cdot d\overrightarrow{r}$ where $F(x, y) = \langle x^3 + 3y, y^2 + 2x\rangle$ and C is the negatively oriented smooth boundary curve of some region D that has area 1324.

2. Let C be the part of the circle with radius 2 and center (0, 4) that lies in the left half-plane $x \leq 0$. Suppose C is oriented so that the starting point is (0, 2) and the endpoint is (0, 6). Compute the line integral $\int_C (\sin x + y)\, dx + (3x + y)\, dy$. *Hint*: This problem would be much easier if we could find a way to apply Green's theorem.

3. Let's revisit the shoelace theorem from Homework 0.
   (a) Let C be the line segment starting at (x_1, y_1) and ending at (x2, y2). Let $\overrightarrow{F} = \langle -\frac{y}{2}, \frac{x}{2}\rangle$. Compute directly the integral $\oint_C \overrightarrow{F} \cdot d\overrightarrow{r}$
   (b) Now let D be the triangle with vertices $(x_1, y_1), (x_2, y_2)$ and (x_3, y_3), oriented counterclockwise in the plane. Use Green's theorem to find the area of this triangle.
   (c) Explain how to use Green's theorem to extend the shoelace theorem to any polygon in the plane.

4. The gradient, curl, and divergence operators can be combine in some, but not all, orders. Explore this by doing Exercise 14 of §16.5 in Stewart's Multivariable Calculus.

   Solution:

   (a) No, cannot take dot product of a vector and a scalar.

   (b) Yes, this is a vector field.

   (c) Yes, this is a scalar field.

   (d) Yes, this is a vector field.

   (e) No, you cannot take the gradient of a vector field.

   (f) Yes, this is a vector field.

   (g) Yes, this is a scalar field.

   (h) No, you cannot take the gradient of a scalar field (dot product of vector with scalar).

   (i) Yes, this is a vector field.

   (j) No, you cannot take the divergence of a scalar field (dot product of vector with scalar).

   (k) No, this is the cross product of a vector with a scalar.

   (l) Yes, this is a scalar field.

5. Let $\overrightarrow{r} = \langle x, y, z\rangle$ and $r = |\overrightarrow{r}|$. Verify the following identities:
   (a) $\nabla \cdot \overrightarrow{r} = 3$

   Solution:

$$\frac{dx}{dx} + \frac{dy}{dy} + \frac{dz}{dz} = 1 + 1 + 1 = 3$$

   (b) $\nabla r = \frac{\overrightarrow{r}}{r}$

   Solution:

$$\begin{aligned}
\nabla r &= \left\langle \frac{1}{2\sqrt{x^2+y^2+z^2}} \cdot 2x, \frac{1}{2\sqrt{x^2+y^2+z^2}} \cdot 2y, \frac{1}{2\sqrt{x^2+y^2+z^2}} \cdot 2z \right\rangle \\
&= \left\langle \frac{x}{\sqrt{x^2+y^2+z^2}}, \frac{y}{\sqrt{x^2+y^2+z^2}}, \frac{z}{\sqrt{x^2+y^2+z^2}} \right\rangle \\
&= \frac{\overrightarrow{r}}{r} = \frac{\langle x, y, z\rangle}{\sqrt{x^2+y^2+z^2}}
\end{aligned}$$

   (c) $\nabla\left(\frac{1}{r}\right) = -\frac{\overrightarrow{r}}{r^3}$

   Solution:

$$\begin{aligned}
\nabla \frac{1}{r} &= \left\langle -\frac{x}{(x^2+y^2+z^2)^{3/2}}, -\frac{y}{(x^2+y^2+z^2)^{3/2}}, -\frac{z}{(x^2+y^2+z^2)^{3/2}} \right\rangle \\
&= -\frac{\overrightarrow{r}}{r^3} = -\frac{\langle x, y, z\rangle}{(x^2+y^2+z^2)^{3/2}}
\end{aligned}$$

   (d) $\nabla \times \overrightarrow{r} = \overrightarrow{0}$

   Solution:

$$\begin{aligned}
\nabla \times \overrightarrow{r} &= \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \frac{\delta}{\delta x} & \frac{\delta}{\delta y} & \frac{\delta}{\delta z} \\ x & y & z \end{vmatrix} \\
&= \left\langle \frac{\delta z}{\delta y} - \frac{\delta y}{\delta z}, \frac{\delta x}{\delta z} - \frac{\delta z}{\delta x}, \frac{\delta y}{\delta x} - \frac{\delta x}{\delta y} \right\rangle \\
&= \langle 0, 0, 0\rangle
\end{aligned}$$

6. Is there a vector field $\overrightarrow{F}(x, y, z)$ such that

$$\nabla \times \overrightarrow{F} = \langle xe^y + \cos\left(\frac{e^{y^2+z^2}}{1+y^2+z^2}\right), -e^y + \frac{\sin^- 1(x)}{1+e^z}, \cos z + (1+y^2)^{1+x^2}$$

Justify your answer and explain your reasoning.

7. Please do Exercises 25-31 of §16.5 in Stewart's *Multivariable Calculus*.

8. Please do Exercises 13-18 of §16.6 in Stewart's *Multivariable Calculus*.

Solution:

13. $\boxed{II}$ – creates circles of radius u when projected to xy plane, and is helical in 3D because $z = v$. The circular curves have u constant, while the radial ones have v constant.

14. $\boxed{VI}$ – The line $x = y$ has $z = 0$ since $u = v$. The grid curves "parallel" to the y-axis have v constant, and the curves "parallel" to the x axis have u constant.

15. $\boxed{VI}$ – A cylinder because x and z are not related to y at all. Curves parallel to y axis have u constant, while those parallel to x have v constant.

16. $\boxed{V}$ – If $u = 0$, $x = 3 + \cos v, y = 0, z = \sin(v)$. This forms a circle in the xz plane, which only graph V contains. The circles around the surface have u fixed, while the curves along the surface have v constant.

17. $\boxed{III}$ – This figure should not have circles, and also the xy cross section shrinks as z increases or decreases from 0. The grid curves within a plane parallel to the xy plane have z constant, and those which are not have u constant.

18. $\boxed{II}$ – Cross sections parallel to xy plane are circles scaled by z in the y direction. Curves in a parallel to xy plane have v constant, and the vertical curves have u constant.

19. Sketch the surface defined by the parametrization $\overrightarrow{r}(u, v) = <u, v \cos u, v \sin u\rangle$, for $0 \leq u \leq 6\pi$ and $2 \leq v \leq 4$. Find the area of this surface. You may use without proof the fact that
$$f(x) = x\sqrt{1+x} + \ln(x + \sqrt{1+x})$$
is the antiderivative of the function $g(x) = 2\sqrt{1+x^2}$. *Hint*: This problem covers material from §16.6 of the textbook, but if you want to start it before Friday, it is really an extension of how we found surface area earlier in the course. Construct two tangent vectors, $\delta\overrightarrow{r}/\delta u$ and $\delta\overrightarrow{r}/\delta v$, use them to construct a normal vector to the given surface, and then integrate the magnitude of this normal vector over the relevant rectangle in the uv-plane.