MATH 215 FALL 2023
Homework Set 8: §16.1 – 16.3
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1. Compute $\int_C x^2 y\, ds$, where C is the segment of the helix of radius 1 about the z-axis, oriented counter-clockwise in the xy-plane, starting at (1, 0, 0) and ending at $(0, 1, \frac{\pi}{2})$.

2. Compute $\int_C x^2\, dx + y^2\, dy$, where C is the circular arc starting at (2, 0) and ending at (0, 2) followed by the straight line segment from (0, 2) to (-1, 1).

3. Do Exercise 53 of §16.2 in *Stewart's Multivariable Calculus*.

4. A wire has the shape of a helix with parametrization $x = t, y = 2\cos t, z = 2\sin t$ for $0 \le t \le 6\pi$, where distances are measured in cm. Find the mass and the center of mass of the wire if the density (in grams/cm) of the wire at any point is equal to four times the square of the distance from the origin to the point.

5. Let $\overrightarrow{F} = \nabla f$ where $f(x, y) = \frac{y^{2002}}{1+x^{200002}+y^{2002}}$. Can you find a (smooth, simple, but not necessarily closed) curve C with the following property:

   (a) $\int_C \overrightarrow{F} \cdot d\overrightarrow{r} = \frac{1}{2}$

   Solution: By the Fundamental Theorem of Line Integrals, this integral is equal to the value of $f(x_1, y_1)$ at the beginning of the curve subtracted from $f(x_2, y_2)$ at the end of the curve. So, we find two points where $f(x_2, y_2) - f(x_1, y_1) = \frac{1}{2}$. (0, 0) and (0, 1) satisfy these conditions. So, the line segment from (0, 0) to (0, 1) fulfills this property.

   (b) $\int_C \overrightarrow{F} \cdot d\overrightarrow{r} = 1$

   Solution: The minimum value of f is 0 at (0, 0), and only the limit at $(0, \infty)$ is 1. Thus, there is no finite curve where this is true. If an infinite curve is allowed, then the infinite line beginning at the origin along the y-axis satisfies this property.

6. Do Exercises 11, 31, and 32 of §16.3 in *Stewart's Multivariable Calculus*.
   Solution

   11. (a) The vector field is conservative, the gradient of the function $f(x, y) = x^2 y$. Thus, the Fundamental Theorem of Line integrals applies. Since all these curves begin and end at the same point, the line integrals have the same value.

       (b) $f(3, 2) - f(1, 2) = \boxed{16}$

   31. This vector field is not conservative because any counterclockwise path will have positive work, whereas clockwise paths will have negative work. Thus, this field is not the gradient of a function.

   32.

7. (a) Calculate $\oint_C \overrightarrow{F} \cdot d\overrightarrow{r}$ where

$$\overrightarrow{F}(x,y) = \langle \frac{2xy}{(x^2+y^2)^2}, \frac{y^2-x^2}{(x^2+y^2)^2} \rangle$$

for $(x,y) \neq (0,0)$ and C is the circle of radius R centered at the origin, oriented clockwise.

Solution: We can parameterize this by $C(t) = \langle R\sin(t), R\cos(t)\rangle, 0 \leq t \leq \pi$. Then $C'(t) = \langle R\cos(t), -R\sin(t)\rangle$. The integral is then

$$\begin{aligned}
\oint_C \overrightarrow{F} \cdot d\overrightarrow{r} &= \int_0^{2\pi} \langle \frac{2\sin(t)\cos(t)}{R^2}, \frac{\cos^2(t)-\sin^2(t)}{R^2} \rangle \cdot \langle R\cos(t), -R\sin(t)\rangle \, dt \\
&= \frac{1}{R}\int_0^{2\pi} 2\sin(t)\cos^2(t) - \sin(t)\cos^2(t) + \sin^3(t)\, dt \\
&= \frac{1}{R}\int_0^{2\pi} \sin(t)\cos^2(t) + \sin^3(t)\, dt \\
&= \frac{1}{R}\int_0^{2\pi} \sin(t)(\sin^2(t) + \cos^2(t))\, dt \\
&= \frac{1}{R}\int_0^{2\pi} \sin(t)\, dt \\
&= \boxed{0}
\end{aligned}$$

We could also have noticed that $\overrightarrow{F}$ is the gradient of the function $f(x,y) = \frac{-y}{x^2+y^2}$. Following from the Fundamental Theorem of Line Integrals, a line integral of a conservative vector field on a closed curve is 0. □

(b) Repeat the previous part, only this time take the curve C to be the ellipse defined by $4x^2 + 9y^2 = 36$, oriented counterclockwise. Hint: It may be possible to do this integration without parametrizing the ellipse.

Solution: $\overrightarrow{F}$ is the gradient of the function $f(x,y) = \frac{-y}{x^2+y^2}$. Following from the Fundamental Theorem of Line Integrals, a line integral of a conservative vector field on a closed curve is $\boxed{0}$. □