MATH 217 - W24 - LINEAR ALGEBRA
HOMEWORK 8, DUE SUNDAY, MARCH 24 at 11:59pm

Submit Part A and Part B as two *separate* assignments in Gradescope as a **pdf file.** At the time of submission, Gradescope will prompt you to match each problem to the page(s) on which it appears. **You must match problems to pages in Gradescope so we know what page each problem appears on.** Failure to do so may result in not having the problem graded.

A few words about solution writing:

- Unless you are explicitly told otherwise for a particular problem, **you are always expected to show your work and to give justification for your answers.**
- Your solutions will be judged on precision and completeness and not merely on "basically getting it right".
- Cite every theorem or fact from the book that you are using (e.g. "By Theorem 1.10 ...").

Part A (15 points)

Solve the following problems from the book:
Section 5.1: 45
Section 5.2: 14, 26
Section 5.3: 36
Section 5.4: 26, 32.

Part B (25 points)

Problem 1. Let W be a subspace of $\mathbb{R}^n$ and let $\mathcal{B} = (\vec{v}_1, \ldots, \vec{v}_d)$ be a basis for W. Consider the transformation $\mathbb{R}^n \xrightarrow{\pi} \mathbb{R}^n$ defined by

$$\pi(\vec{v}) = \sum_{i=1}^{d} \frac{\vec{v} \cdot \vec{v}_i}{\vec{v}_i \cdot \vec{v}_i}\, \vec{v}_i.$$

(a) Show that if $\vec{v}_i \cdot \vec{v}_j = 0$ for all $1 \leq i \neq j \leq d$, then the transformation π is the orthogonal projection onto W. (Note: this is almost, *but not quite*, the way we defined orthogonal projection. Make sure you understand how our definition is different from this before you start trying to prove it!)

(b) Give a counterexample to show that if the basis vectors in $\mathcal{B}$ are *not* perpendicular to each other, then the linear transformation π defined above π is *not* orthogonal projection onto W .

Problem 2. Let $\mathcal{O}_n \subseteq \mathbb{R}^{n\times n}$ denote the set of orthogonal $n \times n$ matrices. Determine whether each of the following statements is True or False, and provide a short proof (or a counter-example) of your claim.

(a) $\mathcal{O}_n$ is a subspace of $\mathbb{R}^{n\times n}$.
(b) If $A, B \in \mathcal{O}_n$, then $AB \in \mathcal{O}_n$.
(c) If $A \in \mathcal{O}_n$, then $A^2 \in \mathcal{O}_n$.
(d) If $A^2 \in \mathcal{O}_n$, then $A \in \mathcal{O}_n$.
(e) If $A \in \mathcal{O}_n$ and A^2 is the identity matrix, then A is symmetric.

Problem 3. (a) Suppose that $\mathcal{B} = (\vec{b}_1, \ldots, \vec{b}_r)$ is an orthonormal basis of a subspace V of $\mathbb{R}^n$. Prove that for all $\vec{v}, \vec{w} \in V$, $[\vec{v}]_{\mathcal{B}} \cdot [\vec{w}]_{\mathcal{B}} = \vec{v} \cdot \vec{w}$.

(b) Prove that if $\mathcal{B} = (\vec{b}_1, \ldots, \vec{b}_r)$ and $\mathcal{C} = (\vec{c}_1, \ldots, \vec{c}_r)$ are two orthonormal bases of V, then $S_{\mathcal{B}\to\mathcal{C}}$ is an orthogonal $r \times r$ matrix.

Problem 4. Let A be an $n \times m$ matrix. Prove or disprove each of the following statements:

(a) $(\ker A)^\perp = \operatorname{im} A^\top$.
(b) $\operatorname{Rank}(A) = \operatorname{Rank}(A^\top A)$.
(c) $\operatorname{Rank}(A) = \operatorname{Rank}(A^\top)$.
(d) $\operatorname{Rank}(A^\top A) = \operatorname{Rank}(AA^\top)$.
(e) $\ker A = \ker AA^\top$.

For Problem 5, you will need the following definitions:

Definition. If A and B are two subsets of $\mathbb{R}^n$, then we say $A \perp B$ if for all $\vec{x} \in A$ and for all $\vec{y} \in B$, $\vec{x} \cdot \vec{y} = 0$. (Note that in this definition that A and B do not need to be subspaces, just subsets.)

Definition. A subset $A \subseteq \mathbb{R}^n$ is called *pairwise orthogonal* if any two elements $\vec{x}, \vec{y} \in A$ are orthogonal. Such a pairwise orthogonal subset $A \subseteq \mathbb{R}^n$ is called *maximally pairwise orthogonal* if it is not possible to enlarge set A to obtain a pairwise orthogonal subset $A' \subseteq \mathbb{R}^n$ that strictly contains A.

Problem 5. Let $n \in \mathbb{N}$. We consider the vector space $\mathbb{R}^n$.

(a) Prove that for all $X, Y \subseteq \mathbb{R}^n$, if $X \perp Y$ then $\operatorname{Span}(X) \perp \operatorname{Span}(Y)$.
(b) Let X and Y each be a linearly independent subset of $\mathbb{R}^n$. Prove that if $X \perp Y$, then $X \cup Y$ is linearly independent.
(c) Prove that every maximally pairwise orthogonal set of vectors in $\mathbb{R}^n$ has $n+1$ elements.

Problem 6. Let A be an $n \times m$ matrix, with $m \leq n$.

(a) If $\operatorname{rank}(A) = m$, prove that it is always possible to write $A = QL$, where Q is an $n \times m$ matrix with orthonormal columns and L is a **lower** triangular $m \times m$ matrix with positive diagonal entries.
(b) Prove that if $\operatorname{rank}(A) < m$, it is still possible to obtain such a decomposition if we allow some diagonal entries to be zero.