MATH 215 FALL 2023
Homework Set 2: §12.4 – 13.1
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1. (a) Find the equations of all planes parallel to the plane $y = 2$ and 4 units away from it.

   Solution: The plane $y = 2$ has a normal of $\langle 0, 1, 0\rangle$, and so will parallel planes. Adding 4 times this normal gives the plane $\boxed{y = 6}$, and subtracting 4 times the normal gives $\boxed{y = -2}$. □

   (b) Find the equations of all planes parallel to the plane $2x - y + z = 0$ and 3 units away from it. Hint: Think about what it means for two planes to be parallel, and how to find the distance between two planes.

   Solution: Using the same approach as part b, we can find the normal direction and add or subtract it (scaled by the distance) from the coordinate vector in th vector equation. The normal of the given plane is $\mathbf{n} = \langle 2, -1, 1\rangle$, and the unit vector is $\mathbf{e_n} = \langle \frac{2}{\sqrt{6}}, -\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}\rangle$. Adding 3 times this direction to $\langle x, y, z\rangle$ grants the equation

$$\boxed{2x - y + z = -\frac{18}{\sqrt{6}}}.$$

   Subtracting grants

$$\boxed{2x - y + z = \frac{18}{\sqrt{6}}}$$ □

2. Last week you found the surface equidistant between two points and arrived at an equation for a plane. This week, find an equation for the surface consisting of all points equidistant from the point (5, 0, 0) and the plane y = 2. Identify what type of surface this is and sketch it (do not use a plotting tool to generate the surface – I want to see that you can sketch a surface from an equation. This might be a good skill to practice for a test, *hint hint.*).

   Solution: Similar to how the points equidistant from a line and a point in $\mathbb{R}^2$ is a parabola, points equidistant from a point and a plane will form a paraboloid in $\mathbb{R}^3$.

$$\{\langle x, y, z\rangle : y - 2 = \pm\sqrt{(x-5)^2 + y^2 + z^2}\}$$
$$\{\langle x, y, z\rangle : y^2 - 4y + 4 = (x-5)^2 + y^2 + z^2\}$$
$$\boxed{\frac{(x-5)^2}{4} + y + \frac{z^2}{4} = 1}$$

□

3. Find parametric equations which describe the curve defined by intersecting the cylinder $x^2 + y^2 = 64$ with the paraboloid of revolution $x^2 + y^2 + 18z = 0$.

Solution: The cylinder can be parameterized as $(8\sin(t), 8\cos(t), z)$. Since the intersection must satisfy $x^2 + y^2 = 64$, then $64 + 18z = 0 \Rightarrow z = -\frac{9}{32}$.
Thus the equation of the intersection is

$$\boxed{(8\sin(t), 8\cos(t), -\frac{9}{32})}$$

□

4. Find the equation of the line consisting of the points equidistant from the three points $(2, 1, 1)$, $(-1, -1, 10)$, and $(1, 3, -4)$.

   Solution: Since the equidistant points between pairs of points is a plane, we can find the intersection of two of the planes formed.
   Between the points $(2, 1, 1)$ and $(-1, -1, 10)$ is the plane $\langle 3, 2, 9\rangle \cdot (x-0.5, y, z-4.5) = 0$.
   Between the points $(-1, -1, 10)$ and $(1, 3, -4)$ is the plane $\langle 2, 4, -14\rangle \cdot (x, y-1, z-3) = 0$.
   Crossing the two normals, we find the vector of the line is $\langle -64, 60, 8\rangle$. A single point on it would be the median of these three points, $(\frac{1}{3}, 1, \frac{7}{3})$. Thus, the vector equation of the line is:
   $$\boxed{(x, y, z) = \left(\frac{1}{3}, 1, \frac{7}{3}\right) + \langle -16, 30, 4\rangle}$$
   □

5. (a) Describe and sketch the part of the first octant where $x + 2y + 3z \leq 4$. (The first octant is the region where $x, y, z \geq 0$).

   (b) Describe and sketch the region give**n** by $|x| + 2, |y| + 3, |z| \leq 4$. It may help to look at the equation in each of the eight octants separately.

6. Do Exercises 23-30 of section 12.6 of Stewart's *Multivariable Calculus*.

   Solutions:

   23.

7. (a) Sketch the curve defined by $r(t) = \langle 5t\sin(3t), 6t\cos(3t), t^2\rangle$. Find a quadratic surface on which this curve lives.

   (b) Sketch the curve defined by $r(t) = \langle 3 + 5\sin(t^4), 2 - 6\sin(t^4), 3 + 7\sin(t^4)\rangle$. Find a two-dimensional surface on which this curve lives (there are an infinite number).

8. A cooling tower for a power plant is to be constructed in the shape of a hyperboloid of one sheet, with equation given by $x^2/a^2 + y^2/b^2 - z^2/c^2 = 1$. The diameter of the circular base of the tower is 80m, and the minimum diameter is is 40m, located 30m above the ground.

   (a) Sketch the tower. Make sure to label your sketch with relevant diameters, heights, and the location of the origin.

   (b) Find an equation of the tower, i.e., find a, b, and c.

9. Two jets travel simultaneously according to the vector equations

$$\mathbf{r_1}(t) = \langle 2, -1, 3\rangle + t\langle 4, 2, 8\rangle \text{ and } \mathbf{r_2}(t) = \langle -1, -3, 1\rangle + t\langle 1, 4, -2\rangle$$

Let time t be measured in seconds and positions measured in kilometers.

(a) Find the shortest distance between the jets, and the time at which this occurs.

(b) Find the shortest distance between the trajectories of the jets.

10. Two people are located in space at the points $(0, -2, 1)$ and $(1, 0, 2)$, respectively.

    (a) Find a parametric equation for the line of sight between the two people.

    (b) Between the two people is a hot air balloon in the shape of a sphere, centered at $(0, -1, 1)$ and with radius 2. Can the two people see each other, or is the line of sight blocked by the sphere? If the line of sight is blocked, find the points on the sphere that each person sees.