MATH 217 W24 - LINEAR ALGEBRA, Section 001 (Dr. Paul Kessenich)
Homework Set Part B due Sunday, February 18 at 11:59pm
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1. Let V and W be vector spaces, and let $T : V \to W$ be a linear transformation. Let $X = (\vec{x}_1, \dots, \vec{x}_k)$ be a list of vectors in V , and consider the list $Y = (T(\vec{x}_1), \dots, T(\vec{x}_k))$ of vectors in W. Determine whether the following statements are true or false. If true, provide a proof. If false, provide a counter-example.

   (a) If X is linearly independent, then Y is also linearly independent.

   Solution: False, consider $T_0 : \mathbb{R}^2 \to \mathbb{R}^2$, $\vec{v} \mapsto \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $X = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$. Then $Y = \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}$, which is not linearly independent since $\vec{y}_1 + \vec{y}_2 = 0$.

   (b) If Y is linearly independent, then X is also linearly independent.

   Solution: True. We show the contrapositive. Let X be a linearly dependent set of vectors in V. Then there exist nonzero scalars $c_1, c_2, \dots, c_k$ such that $c_1\vec{x}_1 + \cdots c_n\vec{x}_n = 0$. Then we know:

$$
\begin{aligned}
T(c_1\vec{x}_1 + \cdots + c_n\vec{x}_n) &= T(0) \\
T(c_1\vec{x}_1) + \cdots + T(c_n\vec{x}_n) &= 0_V && \text{(linearity)} \\
c_1T(\vec{x}_1) + \cdots + c_nT(\vec{x}_n) &= 0_V && \text{(linearity)}
\end{aligned}
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