MATH 215 FALL 2023
Homework Set 11: §16.4 – 16.9

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1. We define a flow line of a vector field $\overrightarrow{F}$ as a curve parametrized by $\overrightarrow{c}(t), t_0 \leq t \leq t_f$ for which

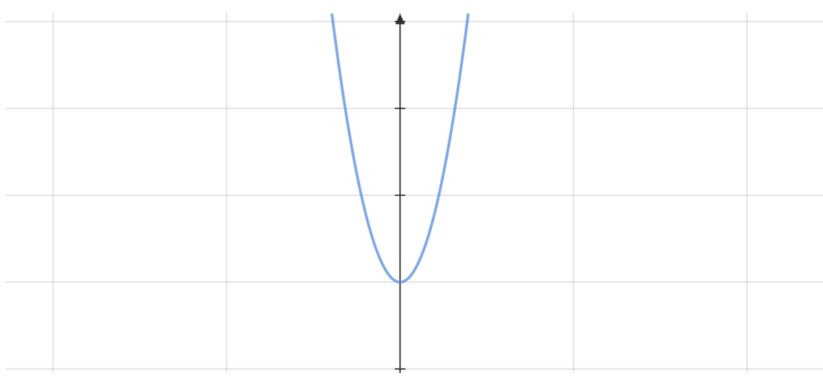
$$\frac{d\overrightarrow{c}}{dt} = \overrightarrow{F}(\overrightarrow{c}(t))$$

(a) For the vector field $\overrightarrow{F}(x, y, z) = \langle 0, x, 0 \rangle$, verify that lines parallel to the y-axis are flow lines, when $x \neq 0$.

Solution: Choosing any point with $x \neq 0$ as the beginning of a flow curve, the x- and z-values remain constant. Because of this, only the y-value of the curve changes, at a constant rate with respect to t since x remains constant. This will result in a curve parallel to the y-axis. $\square$

(b) Find the curl of $\overrightarrow{F}$ for any point for which $x \neq 0$. Even though there is no rotation in the vector-field, explain why the curl is non-zero. Where is the rotation in this vector field?

Solution: The curl is nonzero because of the flip of the sign of x across the y-axis. For instance, a particle traveling with constant x velocity across the y-axis will be pushed one direction by the vector field on the negative side of the x-axis, but then the opposite direction on the other side. $\square$



2. We are going to revisit the basic premise of the last problem, this time with two new vector fields.

   (a) For the vector field $\overrightarrow{F}(x, y) = \langle -y, x\rangle$, verify that a circle of arbitrary fixed radius, centered at the origin, is a flow line of $\overrightarrow{F}$.

   Solution: We need to verify that $\overrightarrow{F}$ is tangent to the circle. Let $x^2 + y^2 = r^2$ be the equation of the circle. This can be parameterized as $C(\theta) = (r\cos\theta, r\sin\theta)$. Then $\overrightarrow{F} = \langle -rsin\theta, r\cos\theta\rangle$. Since this is equal to $C'(\theta)$, $C(t)$ is a flow line of $\overrightarrow{F}$.

   (b) For the vector field $\overrightarrow{G}(x, y) = \langle \frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2}\rangle$, verify that a circle of arbitrary fixed radius, centered at the origin, is a flow line of $\overrightarrow{G}$. It may help to remember that you may need to choose the *frequency*, or *angular speed*, of motion around the circle to be a special value to ensure that your answer is a flow line.

   Solution: We need to verify that $\overrightarrow{G}$ is tangent to the circle. Let $x^2+y^2 = r^2$ be the equation of the circle. This can be parameterized as $C(\theta) = (r\cos(\frac{1}{r^2}\theta), r\sin(\frac{1}{r^2}\theta))$. Then $\overrightarrow{G} = \left\langle \frac{-\sin(\frac{1}{r^2}\theta)}{r}, \frac{\cos(\frac{1}{r^2}\theta)}{r}\right\rangle$. Since this is equal to $C'(\theta)$, $C(\theta)$ is a flow line of $\overrightarrow{F}$.

   (c) Treating the vector fields from the previous two parts as 3d-vector fields (with z-component identically equal to zero), compute the curls of $\overrightarrow{F}$ and $\overrightarrow{G}$.

   Solution:
$$\nabla\overrightarrow{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{j} \\ \frac{\delta}{\delta x} & \frac{\delta}{\delta y} & \frac{\delta}{\delta z} \\ -y & x & 0 \end{vmatrix} = \langle 0-0, 0-0, 1+1\rangle = \langle 0, 0, 2\rangle$$
$$\nabla\overrightarrow{G} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{j} \\ \frac{\delta}{\delta x} & \frac{\delta}{\delta y} & \frac{\delta}{\delta z} \\ \frac{-y}{x^2+y^2} & \frac{x}{x^2+y^2} & 0 \end{vmatrix} = \langle 0-0, 0-0, -\frac{x^2+y^2}{(x^2+y^2)^2} + \frac{x^2+y^2}{(x^2+y^2)^2}\rangle = \langle 0, 0, 0\rangle$$

   (d) One of your answers from the previous part should be the zero vector. Explain why, even though these two vector fields both have circles as flow lines, the curl is zero for one and non-zero for the other.

   Solution: This relies on the fact that we needed a different angular velocity for $\overrightarrow{G}$ than usual. This equates to an initial velocity traveling along the surface which $\overrightarrow{G}$ is the gradient of. This is just like planets orbiting another, as they have initial velocities relative to each other already which causes them to keep circling around. Meanwhile, this is not the case for $\overrightarrow{F}$.

3. Find a parametric representation for the part of the sphere (a surface, not a solid) of radius 4 centered at the origin that lies

   (a) inside the cone $z = \sqrt{3(x^2 + y^2)}$

   Solution:

$$\begin{cases} z = \sqrt{3(x^2 + y^2)} \\ x^2 + y^2 + z^2 = 16 \end{cases}$$
$$z = \sqrt{48 - 3z^2}$$
$$z^2 = 48 - 3z^2$$
$$z^2 = 12$$
$$z = \sqrt{12}$$
$$x^2 + y^2 = 4$$
$$0 \leq t \leq 2\pi$$
$$\boxed{C = \langle 2\cos(t), 2\sin(t), \sqrt{12} \rangle}$$

   (b) inside the cone $x = \sqrt{y^2 + z^2}$

   Solution:

$$\begin{cases} x = \sqrt{y^2 + z^2} \\ x^2 + y^2 + z^2 = 16 \end{cases}$$
$$x = \sqrt{16 - x^2}$$
$$z^2 = 16 - x^2$$
$$x = \sqrt{8}$$
$$y^2 + z^2 = 8$$
$$0 \leq t \leq 2\pi$$
$$\boxed{C = \langle 2\sqrt{2}, 2\sqrt{2}\sin(t), 2\sqrt{2}\cos(t) \rangle}$$

   (c) inside the cone $y = \sqrt{\frac{1}{3}(x^2 + z^2)}$

Solution:

$$\begin{cases} y = \sqrt{\frac{1}{3}(x^2+z^2)} \\ x^2+y^2+z^2 = 16 \end{cases}$$
$$y = \sqrt{\frac{16}{3} - \frac{y^2}{3}}$$
$$3y^2 = 16 - y^2$$
$$y = 2$$
$$x^2 + z^2 = 12$$
$$0 \le t \le 2\pi$$
$$\boxed{C = \langle 2\sqrt{3}\sin(t), 2, 2\sqrt{3}\cos(t)\rangle}$$

□

Hint: You may want to use *spherical-like* coordinates, but not actually spherical coordinates.

4. Prove that if S is the surface of any sphere of radius 2, then the surface integral over S of the function $f(x, y, z) = \cos(\pi z)$ is zero.

Solution:

$$f(x, y, z) = \cos(\pi z)$$
$$0 \leq \phi \leq \pi$$
$$0 \leq \theta \leq 2\pi$$
$$\overrightarrow{r} = \langle 2\sin\phi\cos\theta, 2\sin\phi\sin\theta, 2\cos\phi\rangle$$
$$\frac{\delta r}{\delta\theta} = \langle -2\sin\theta\sin\phi, 2\sin\phi\cos\theta, 0\rangle$$
$$\frac{\delta r}{\delta\phi} = \langle 2\cos\theta 2\cos\phi, 2\cos\phi\sin\theta, -2\sin\phi\rangle$$
$$\frac{\delta r}{\delta\theta} \times \frac{\delta r}{\delta\phi} = \begin{vmatrix} i & j & k \\ -2\sin\theta\sin\phi & 2\sin\phi\cos\theta & 0 \\ 2\cos\theta 2\cos\phi & 2\cos\phi\sin\theta & -2\sin\phi \end{vmatrix}$$
$$= \langle -4\sin^2(\phi)\cos(\theta), -4\sin^2(\phi)\sin(\theta), -4\sin\phi\cos\phi\rangle$$
$$\left|\frac{\delta r}{\delta\theta} \times \frac{\delta r}{\delta\phi}\right| = 4\sin\phi$$
$$\int\int_S \cos(2\pi\cos\phi) \cdot 4\sin\phi\, d\phi\, d\theta$$
$$= 2\pi\int_0^\pi \cos(2\pi\cos\phi) \cdot 4\sin\phi\, d\phi$$
$$u = 2\pi\cos\phi, du = -2\pi\sin\phi\, d\phi$$
$$= \int_{\phi=0}^{\pi} -\cos(u)4\, du$$
$$= -4\left[-\sin(2\pi\cos\phi)\right]_0^\pi$$
$$= -4(-\sin(-2\pi) + \sin(2\pi))$$
$$= -4 \cdot 0 = \boxed{0}$$

5. Compute directly the flux of the vector field $\overrightarrow{F}(x, y, z) = \langle 3z, 2x, y\rangle$ outward across the surface of the tetrahedron with vertices (0, 0, 0), (1, 0, 0), (0, 1, 0), and (0, 0, 2).

   Solution: The unit normal vectors of the sides are:

$$\begin{aligned} S_1(0,0,0),(1,0,0),(0,1,0) &: \hat{n}_{S_1} = -\langle 0,0,1\rangle \\ S_2(0,0,0),(1,0,0),(0,0,2) &: \hat{n}_{S_2} = -\langle 0,1,0\rangle \\ S_3(0,0,0),(0,1,0),(0,0,2) &: \hat{n}_{S_3} = -\langle 1,0,0\rangle \\ S_4(1,0,0),(0,1,0),(0,0,2) &: \hat{n}_{S_4} = \left\langle \frac{2}{3}, \frac{2}{3}, \frac{1}{3}\right\rangle \end{aligned}$$

   We can now compute the flux integral $\int\int_S F \cdot \hat{n} dA$ for each side.

$$\begin{aligned} \int\int_{S_1} F \cdot \hat{n} dA &= \int_0^1 \int_0^y y\,dx\,dy \\ &= \int_0^1 y^2\,dy = \frac{1}{3} \end{aligned}$$

$$\begin{aligned} \int\int_{S_2} F \cdot \hat{n} dA &= \int_0^2 \int_0^{\frac{z}{2}} 2x\,dx\,dz \\ &= \int_0^2 \frac{z^2}{4}\,dy = \frac{2}{3} \end{aligned}$$

$$\begin{aligned} \int\int_{S_3} F \cdot \hat{n} dA &= \int_0^2 \int_0^{\frac{z}{2}} 3z\,dx\,dz \\ &= \int_0^2 \frac{3z^2}{2}\,dx\,dz = 4 \end{aligned}$$

$$\int\int_{S_4} F \cdot \hat{n} dA = \int\int_{S_4} 2z + \frac{4x}{3} + \frac{y}{3}\,dA$$

$$z = 2 - 2x - 2y$$

$$\int_0^1 \int_0^1 \left(4 - 4x - 4y + \frac{4x}{3} + \frac{y}{3}\right) \sqrt{1 + 2^2 + 2^2}\,dx\,dy$$

$$\begin{aligned} &= \int_0^1 \int_0^1 (12 - 8x - 11y)\,dx\,dy \\ &= 12 - 4 - \frac{11}{2} = \frac{5}{2} \end{aligned}$$

$$\frac{2}{3} + \frac{2}{3} + 4 - 0 + \frac{5}{2} = \boxed{\frac{47}{6}}$$

6. Verify that Stokes' Theorem is true for the vector field $\overrightarrow{F}(x,y,z) = \langle x^2, z, y^2\rangle$ and the surface S that is the part of the paraboloid $y = x^2 + z^2$ that lies in the half-space $\{(x,y,z) \mid y \leq 1\}$. We assume that the surface is oriented in the positive direction of the y-axis.

Solution:

$$curl(\overrightarrow{F}) = \begin{vmatrix} i & j & k \\ \frac{\delta}{\delta x} & \frac{\delta}{\delta y} & \frac{\delta}{\delta z} \\ x^2 & z & y^2 \end{vmatrix} = \langle 2y-1, 0, 0\rangle$$

$$S(x,z) = \langle x, x^2+z^2, z\rangle$$

$$\overrightarrow{n} = \frac{\delta S}{\delta x} \times \frac{\delta S}{\delta z} = \langle 1, 2x, 0\rangle \times \langle 0, 2z, 1\rangle = \langle 2x, -1, 2z\rangle$$

$$\hat{n} = \frac{n}{\sqrt{1+4x^2+4z^2}}$$

$$\int\int_S (\nabla \times \overrightarrow{F}) \cdot \hat{n}\, dA = \int\int_S \frac{4yx - 2x}{\sqrt{1+4x^2+4z^2}}\, dx\, dz$$

$$x = r\cos\theta, z = r\sin\theta, y = r^2$$

$$\int_0^{2\pi}\int_0^1 \frac{4r^3\cos(\theta) - 2r\cos\theta}{\sqrt{1+4r^2}}\, dr\, d\theta$$

$$= 0 \text{ (symmetry of cos about } \theta)$$

$$\delta S = C(t) = \langle \cos t, 1, -\sin t\rangle$$

$$C'(t) = \langle -\sin t, 0, -\cos t\rangle$$

$$\int_C \overrightarrow{F} \cdot d\mathbf{r} = \int_0^{2\pi} \overrightarrow{F}(C(t)) \cdot C'(t)\, dt = \int_0^{2\pi} \sin^2 t\cos t - \cos t\, dt = \boxed{0}$$

7. Evaluate the integral $\int\int_S (\nabla \times \overrightarrow{F}) \cdot d\overrightarrow{S}$, where S is the portion of the surface of the sphere defined by $x^2+y^2+z^2=1$ and $x+y+z \geq 1$, and where $\overrightarrow{F} = \langle x,y,z\rangle \times \left(\hat{i}+\hat{j}+\hat{k}\right)$.

Solution: We can parameterize the curve of the edge of the surface as a circle within the plane of $x+y+z=1$. By symmetry, the center of this circle is $\left\langle \frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right\rangle$. Then, we choose two perpendicular vectors from the center to the edge of the circle. $\left\langle -\frac{1}{3}, \frac{2}{3}, -\frac{1}{3}\right\rangle$ and $\left\langle \frac{1}{\sqrt{3}}, 0, -\frac{1}{\sqrt{3}}\right\rangle$. Multiplying by $\cos(t)$ and $\sin(t)$ respectively and adding them to the center will result in the parameterization of a circle in the $x+y+z=1$ plane. This results in the equation

$$C(t) = \left\langle \frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right\rangle + \cos(t)\left\langle -\frac{1}{3}, \frac{2}{3}, -\frac{1}{3}\right\rangle + \sin(t)\left\langle \frac{1}{\sqrt{3}}, 0, -\frac{1}{\sqrt{3}}\right\rangle$$

or

$$C(t) = \left\langle \frac{1}{3} - \frac{1}{3}\cos(t) - \frac{1}{\sqrt{3}}\sin(t), \frac{1}{3} + \frac{2}{3}\cos(t), \frac{1}{3} - \frac{1}{3}\cos(t) - \frac{1}{\sqrt{3}}\sin(t)\right\rangle.$$

Now, we take the line integral with $\int_C F(C(t)) \cdot C'(t)\,dt$.

$$C'(t) = \langle -\frac{1}{3}\sin(t) - \frac{1}{\sqrt{3}}\cos(t), -\frac{2}{3}\sin(t), -\frac{1}{3}\sin(t) - \frac{1}{\sqrt{3}}\cos(t)\rangle$$

$$\begin{aligned}\nabla \times \overrightarrow{F} &= \langle \frac{\delta}{\delta x}, \frac{\delta}{\delta y}, \frac{\delta}{\delta z}\rangle \times \langle y-z, z-x, x-y\rangle \\ &= \langle -1-1, -1-1, -1-1\rangle \\ &= \langle -2, -2, -2\rangle\end{aligned}$$

$$\begin{aligned}\int_C F(C(t)) \cdot C'(t)\,dt &= \int_0^{2\pi} \frac{2}{3}\sin(t) + \frac{2}{\sqrt{3}}\cos(t) + \frac{4}{3}\sin(t) + \frac{2}{3}\sin(t) + \frac{2}{\sqrt{3}}\cos(t)\,dt \\ &= \boxed{0}\end{aligned}$$

8. Suppose that E is the unit cube in the first octant (i.e. the cube with the vectors $\hat{\imath}+\hat{\jmath}+\hat{k}$ as 3 of the edges) and $\overrightarrow{F}(x,y,z) = \langle 3x, -3y, 2z\rangle$. Let S be the surface obtained by taking the surface of E without its top (so S has five sides and is oriented "out" from E). Calculate $\int\int_S \overrightarrow{F} \cdot d\overrightarrow{S}$ directly and by using the divergence theorem.

Solution: Calculating directly:

$$\begin{aligned}
S_1 &: \overrightarrow{0}, \hat{\imath}, \hat{\jmath}, \hat{\imath}+\hat{\jmath} : \hat{n}_{S_1} = -\hat{k} \\
S_2 &: \overrightarrow{0}, \hat{\imath}, \hat{k}, \hat{\imath}+\hat{k} : \hat{n}_{S_2} = -\hat{\jmath} \\
S_3 &: \overrightarrow{0}, \hat{\jmath}, \hat{k}, \hat{\jmath}+\hat{k} : \hat{n}_{S_3} = -\hat{\imath} \\
S_4 &: \hat{\imath}+\hat{\jmath}+\hat{k}, \hat{\imath}, \hat{\jmath}, \hat{\imath}+\hat{\jmath} : \hat{n}_{S_4} = \hat{k} \\
\text{(removed) } S_5 &: \hat{\imath}+\hat{\jmath}+\hat{k}, \hat{\imath}, \hat{k}, \hat{\imath}+\hat{k} : \hat{n}_{S_5} = \hat{\jmath} \\
S_6 &: \hat{\imath}+\hat{\jmath}+\hat{k}, \hat{\jmath}, \hat{k}, \hat{\jmath}+\hat{k} : \hat{n}_{S_6} = \hat{\imath}
\end{aligned}$$

$$\begin{aligned}
S_4 + S_1 &= 1\cdot(3\cdot 1) - 1\cdot(3\cdot 0) = 3 \\
S_2 &= 1\cdot(0\cdot 1) = 0 \\
S_6 + S_3 &= 1\cdot(2\cdot 1) - 1\cdot(2\cdot 0) = 2 \\
&= \boxed{5}
\end{aligned}$$

By divergence theorem, we use the closed cube first and subtract the flux of the removed side.

$$\begin{aligned}
\int_0^1\int_0^1\int_0^1 3-3+2\,dz\,dy\,dx &= 2 \\
\int\int_{S_5} \overline{F}\times\hat{n}dA = 1\cdot(-3\cdot 1) &= -3 \\
2+3 &= \boxed{5}
\end{aligned}$$

9. Define $\overrightarrow{e_r}(x, y, z) = \langle \frac{x}{r}, \frac{y}{r}, \frac{z}{r} \rangle$, where $r = \sqrt{x^2 + y^2 + z^2}$ and $(x, y, z) \neq (0, 0, 0)$. This vector field is often called the unit radial vector field.

   (a) Show that the flux of $\overrightarrow{F} = \overrightarrow{e_r}/r^2$ outward through the surface of a sphere centered at the origin is independent of the radius of the sphere.

   Solution: Let the spherical shell S have radius R. We calculate the flux directly.

$$\hat{n} = \left\langle \frac{x}{R}, \frac{y}{R}, \frac{z}{R} \right\rangle$$
$$\int\int_S \overrightarrow{F} \cdot \hat{n}\, dA = \int\int_S \frac{x^2 + y^2 + z^2}{} dS$$
$$= \int\int_S \frac{r^2}{r^3 R} = \int\int_S \frac{1}{rR}$$

   On the surface of the sphere, the radius is always equal to R, so the value of this integral will be equal to $\frac{1}{R^2}$ multiplied by the surface area. The surface area is $4\pi R^2$, so the flux evaluates to 4π, which is indeed independent of the radius of the sphere.

   (b) Calculate the flux of $\overrightarrow{F}$ through the disk D of radius 2 parallel to the yz-plane and centered at (3, 0, 0). The orientation of D is chosen so that the normal vector points in the direction of the positive x-axis.

   Solution:

$$\hat{n} = \langle 1, 0, 0 \rangle$$
$$\int\int_S \overrightarrow{F} \cdot \hat{n}\, dA = \int\int_S \frac{x}{r^3} dA$$
$$y = \cos\theta, z = \sin\theta = \int_0^{2\pi}\int_0^2 \frac{3}{\sqrt{9 + r^2}} r\, dr\, d\theta = 2\pi \int_0^2 \frac{3}{9 + r^2} r\, dr$$
$$u = 9 + r^2, du = 2r\, dr$$
$$= \pi \int_0^2 \frac{3}{u^{\frac{3}{2}}}\, du$$
$$= \pi \left[\frac{-6}{\sqrt{u}}\right]_{r=0}^2 = \boxed{-6\pi\left(\frac{1}{3} - \frac{1}{\sqrt{13}}\right)}$$

   (c) Compute the outward flux of the vector field $\overrightarrow{F} = \overrightarrow{e_r}/r^2$ through the ellipsoid $4x^2 + 9y^2 + z^2 = 36$.

   Solution: Let S be an arbitrary sphere centered at the origin. Then the region between S and the ellipsoid has 0 divergence, since it does not contain the sole singularity of the function, which is at the origin.
   Then the flux on the ellipsoid is the same as the flux on S. As we established in part (a), the flux on S is 4π, independent of radius. Thus, the flux through the ellipsoid is also 4π.

10. Suppose $\overrightarrow{F}$ is a smooth vector field. Let S_1 be the sphere of radius 1 centered at the origin and oriented outward. Let S_2 be the sphere of radius 2 centered at the origin, oriented outward. Let E be the volume between S_1 and S_2. Suppose that the outward flux of $\overrightarrow{F}$ across S_1 is 3π, and the divergence of $\overrightarrow{F}$ on E is 4. If possible, evaluate the following integrals:

$$(a) \int\int\int_E (\nabla \cdot F)\, dV \qquad (b) \int\int_{S_2} \overrightarrow{F} \cdot d\overrightarrow{S} \qquad \int\int_{S_1} \left(\nabla \times \overrightarrow{F}\right) \cdot d\overrightarrow{S}$$

If it is not possible to evaluate any of the integrals, explain why.

Solution:

(a)

$$\int\int\int_E \text{div} F\, dV = \int\int\int_E 4\, dV = V \cdot 4 = \frac{16}{3}\pi(2^3 - 1^3) = \frac{112\pi}{3}$$

(b)

$$\int\int\int_E \text{div} F\, dV = \int\int_{S_2} \overrightarrow{F} \cdot d\overrightarrow{S} - \int\int_{S_1} \overrightarrow{F} \cdot d\overrightarrow{S}$$
$$\int\int_{S_2} \overrightarrow{F} \cdot d\overrightarrow{S} = \frac{112\pi}{3} + 3\pi = \frac{121\pi}{3}$$

(c) By divergence theorem,

$$\int\int_{S_2} (\nabla \times \overrightarrow{F}) \cdot d\overrightarrow{S} = \int\int\int_{S_2} \text{div}(\nabla \times \overrightarrow{F}) \cdot dV$$
$$= \int\int\int_{S_2} 0 \cdot dV = 0$$