MATH 217 W24 - LINEAR ALGEBRA, Section 001 (Dr. Paul Kessenich)
Homework Set Part B due ??? at 11:59pm

Zhengyu James Pan (jzpan@umich.edu)

1. Let W be a subspace of $\mathbb{R}^n$ and let $\mathcal{B} = (\vec{v}_1, ..., \vec{v}_d)$ be a basis for W. Consider the transformation $\mathbb{R}^n \xrightarrow{\pi} \mathbb{R}^n$ defined by

$$\pi(\vec{v}) = \sum_{i=1}^{d} \frac{\vec{v} \cdot \vec{v}_i}{\vec{v}_i \cdot \vec{v}_i} \vec{v}_i.$$

   (a) Show that if $\vec{v}_i \cdot \vec{v}_j = 0$ for all $1 \leq i \neq j \leq d$, then the transformation π is the orthogonal projection onto W. (Note: this is almost, but not quite, the way we defined orthogonal projection. Make sure you understand how our definition is different from this before you start trying to prove it!)

   Solution:

   (b) Give a counterexample to show that if the basis vectors in $\mathcal{B}$ are not perpendicular to each other, then the linear transformation π defined above π is not orthogonal projection onto W.

   Solution: HELPER DEFINITION: Orthogonal Projection. If W is a subspace of an inner product space V and if $\vec{v} \in V$, the orthogonal projection of $\vec{v}$ onto W is the unique vector $\vec{w} \in W$ such that $\vec{v} - \vec{w} \in W^{\perp}$. The orthogonal projection of $\vec{v}$ onto W is sometimes denoted $\text{proj}_W(\vec{v})$.

2. Let $\mathcal{O}_n \subseteq \mathbb{R}^{n\times n}$ denote the set of orthogonal $n \times n$ matrices. Determine whether each of the following statements is True or False, and provide a short proof (or a counterexample) of your claim.
   (a) $\mathcal{O}_n$ is a subspace of $\mathbb{R}^{n\times n}$.
   (b) If $A, B \in \mathcal{O}_n$, then $AB \in \mathcal{O}_n$.
   (c) If $A \in \mathcal{O}_n$, then $A^2 \in \mathcal{O}_n$.
   (d) If $A^2 \in \mathcal{O}_n$, then $A \in \mathcal{O}_n$.
   (e) If $A \in \mathcal{O}_n$ and A^2 is the identity matrix, then A is symmetric.

3. (a) Suppose that $\mathcal{B} = (\vec{b}_1, ..., \vec{b}_r)$ is an orthonormal basis of a subspace V of $\mathbb{R}^n$. Prove that for all $\vec{v}, \vec{w} \in V$, $[\vec{v}]_\mathcal{B} \cdot [\vec{w}]_\mathcal{B} = \vec{v} \cdot \vec{w}$

   (b) Prove that if $\mathcal{B} = (\vec{b}_1, ..., \vec{b}_r)$ and $\mathcal{C} = (\vec{c}_1, ..., \vec{c}_r)$ are two orthonormal bases of V, then $S_{\mathcal{B}\to\mathcal{C}}$ is an orthogonal $r \times r$ matrix.