MATH 217 W24 - LINEAR ALGEBRA, Section 001 (Dr. Paul Kessenich)
Homework Set Part B due Thurs, Feb 8 at 11:59pm
Zhengyu James Pan (jzpan@umich.edu)

1. Let $\mathbb{R}^n \xrightarrow{T} \mathbb{R}^n$ be a linear transformation. As on HW 3, we define T^k to be the k-fold composition of T with itself. Let A be the standard matrix of T, by which we mean the unique $n \times n$ matrix such that $T(\vec{x}) = A\vec{x}$ for all $\vec{x} \in \mathbb{R}^n$.

   (a) Prove that for all k, the standard matrix for T^k is the matrix A^k. [Hint: induction works nicely.]

   Solution: We are given that the standard matrix $A^{(1)}$ represents the transformation $T^{(1)}$. Assume that the transformation T^n can be represented by the standard matrix A^n. We know by a theorem on the worksheets that the standard matrix of two linear transformations, both from $\mathbb{R}^n \to \mathbb{R}^n$, is equal to the product of their respective standard matrices. Then $(T^n \circ T)(x) = A^n A\vec{x}$. This is equal to $T^{n+1}(x) = A^{n+1}\vec{x}$. So by induction, $T^k(\vec{x}) = A^k\vec{x}$ for all $\vec{x} \in \mathbb{R}^n$ and $k \in \mathbb{N}$.

   (b) We define T to be nilpotent if there exists some $k \in \mathbb{N}$ such that T^k is the zero transformation. Prove that if T is nilpotent, then A is not invertible.

   Solution: We will prove the contrapositive. Assume A is invertible. We know zero transformation is surjective, so it is noninvertible, and We know by part (a) that the standard matrix of T^k is A^k for natural number k. By problem 5c on Worksheet 7, the inverse of A^k is $(A^{-1})^k$. So A^k can never be noninvertible, and is thus never the zero transformation. Thus if A is invertible, then T is not nilpotent, so the original statement is true.

   (c) Prove that if T is nilpotent, then $A - I_n$ is invertible. [Hint: try multiplying out $(A - I_n)(-I_n - A - A^2 - \cdots - A^{k-1})$ and see what you get.]

   Solution: Let $k \in \mathbb{N}$ such that T^k is the zero transformation.
   Expanding $(A - I_n)(-I_n - A - A^2 - \cdots - A^{k-1})$:

$$\begin{aligned}
&= -A\left(I_n + A + \cdots + A^{k-1}\right) + I_n\left(I_n + A + \cdots + A^{k-1}\right) && \text{(distributivity)}\\
&= -\left(A + A^2 + \cdots + A^{k-1} + A^k\right) + \left(I_n + A + \cdots + A^{k-1}\right) && \text{(distributivity)}\\
&= \left(I_n + A + \cdots + A^{k-1}\right) - \left(A + A^2 + \cdots + A^{k-1} + 0_{n\times n}\right) && (A^k = 0_{n\times n})\\
&= I_n
\end{aligned}$$

   Swapping the order of multiplication,

$$\begin{aligned}
&(-I_n - A - A^2 - \cdots - A^{k-1})(A - I_n)\\
&\quad= -\left(I_n + A + \cdots + A^{k-1}\right)A + \left(I_n + A + \cdots + A^{k-1}\right)I_n && \text{(distributivity)}\\
&\quad= -\left(A + A^2 + \cdots + A^{k-1} + A^k\right) + \left(I_n + A + \cdots + A^{k-1}\right) && \text{(distributivity)}\\
&\quad= \left(I_n + A + \cdots + A^{k-1}\right) - \left(A + A^2 + \cdots + A^{k-1} + 0_{n\times n}\right) && (A^k = 0_{n\times n})\\
&\quad= I_n
\end{aligned}$$

So the inverse of $A - I_n$ is $-(I_n + A + A^2 + \cdots + A^{k-1})$, and $A - I_n$ is invertible by definition.

2. Let V be any vector space, and let S be any set. Let $\mathcal{F}(S,V)$ denote the set of all functions from S to V . (Note: we are not assuming $S \subseteq V$ here, just that S is some set. S is not assumed to be a vector space, but it could be. Similarly, the functions in $F(S,V)$ are not assumed to be linear transformations, although it is possible that some of them might be.)

   For any functions f, $g \in \mathcal{F}(S,V)$ we can define their sum to be the function $f+g$ given by the formula $(f+g)(s) = f(s)+g(s)$, where s is any element in S. Similarly, for any scalar $c \in R$ and any function $f \in \mathcal{F}(S,V)$ we define the function cf to be given by the formula $(cf)(s) = c(f(s))$ for all $s \in S$.

   (a) Prove that $\mathcal{F}(S,V)$ is a vector space. Note: For this problem you must explicitly prove that each of the vector space properties VS1-8 from Worksheet 6 is true. (These proofs should be very short but are not skippable.)

   Solution: Let arbitrary $a,b \in \mathbb{R}$, arbitrary $f,g,h \in \mathcal{F}(S,V)$. Note that $+_{\mathcal{F}(S,V)}$ borrows the qualities of $+_V$ (the summation operation of V) through the definition; namely associativity and commutativity.

   VS-1: True, $(f+g)+h = (f(s)+g(s))+h(s) = f(s)+(g(s)+h(s)) = f+(g+h)$ by additive associativity of vector space V.
   VS-2: $f+g = f(s)+g(s) = g(s)+f(s) = g+f$ by addititve commutativity of vector space V.
   VS-3: True, $f(s) = 0_V$ satisfies this property. $g+f = g(s)+0_V = g$
   VS-4: True, for all values of $f(s)$, such a $-f(s)$ exists since V is a vector space. So the function $-f$ exists as well.
   VS-5: True, $a(f+g) = a(f(s)+g(s)) = af(s)+ag(s) = af+ag$ by distributivity of the vector space V.
   VS-6: True, $(a+b)f = (a+b)f(s) = af(s)+bf(s) = af+bf$. by scalar multiplicative distributivity of vector space V.
   VS-7: True, $a(bf) = a(bf(s)) = (ab)f(s) = (ab)f$ by scalar multiplicative associativity of vector space V.
   VS-8: True, $1f = 1f(s) = f(s) = f$ by the unitary law of vector space V.

   (b) Is $0_{\mathcal{F}(S,V)}$ the same element as 0_V ? If not, explain how they are different.

   Solution: No. $0_{\mathcal{F}(S,V)}$ maps any element of the set S to 0_V, while 0_V is only a vector in the space V. $0_{\mathcal{F}(S,V)}$ is a function and can take an input, while 0_V cannot take an input like a function.

   (c) We could similarly define $\mathcal{F}(V,S)$ to be the set of all functions from V to S. Would $\mathcal{F}(V,S)$ also a vector space? Why or why not?

   Solution: Not necessarily. We used the vector space properties of image V to prove the vector space axioms for $\mathcal{F}(S,V)$. However, when arbitrary set S is the image, those properties do not necessarily apply.

   (d) The familiar vector spaces $\mathcal{P}$, $\mathcal{P}_n$ and $\mathcal{C}^\infty$ (all from Worksheet 6) are all subsets of $\mathcal{F}(S,V)$ for some S and V . What are S and V for each of these functions?

Solution: All of these vector spaces are composed of functions which map from $\mathbb{R} \to \mathbb{R}$.

3. Let $\mathcal{P}$ be the vector space of all polynomial functions from $\mathbb{R}$ to $\mathbb{R}$ in the variable t, and for each $n \in \mathbb{N}$, let $\mathcal{P}_n$ be (as usual) the subset of $\mathcal{P}$ consisting of all polynomial functions of degree at most n. (We already know that $\mathcal{P}_n$ is also a vector space.) Also let $T : \mathcal{P} \to \mathcal{P}$ be the map defined by $T(p)(t) = p'(t) + p(0)$ for each $p \in \mathcal{P}$ and for all $t \in \mathbb{R}$.

   (a) Show that T is a linear transformation.

   Solution: Note that the derivative is linear by problem 5 of worksheet 6. Although the current domain and codomain are $\mathcal{P}$ and not $\mathcal{C}^\infty$, the derivative is closed in $\mathcal{P} \subset \mathcal{C}^\infty$, so it is still linear. First, we show T respects addition:

$$\begin{aligned}
T(p+q)(t) &= (p+q)'(t) + (p+q)(0) && \text{(definition of } T\text{)} \\
&= p'(t) + q'(t) + p(0) + q(0) && \text{(linearity of derivative)} \\
&= p'(t) + p(0) + q'(t) + q(0) && \text{(associativity and commutativity of } \mathcal{P}\text{)} \\
&= T(p) + T(q) && \text{(definition of } T(p)\text{)}
\end{aligned}$$

   Next, we show T respects scalar multiplication. Let $c \in \mathbb{R}$.

$$\begin{aligned}
T(cp) &= (cp)'(t) + (cp)(0) && \text{(definition of } T\text{)} \\
&= c(p'(t)) + cp(0) && \text{(linearity of derivative)} \\
&= c(p'(t) + p(0)) && \text{(distributivity of scalar multiplication in vector space } \mathcal{P}\text{)} \\
&= cT(p)
\end{aligned}$$

   Thus T is linear.

   (b) Let $n \in \mathbb{N}$, and let $T_n : \mathcal{P}_n \to \mathcal{P}_n$ be defined by $T_n(p)(t) = p'(t) + p(0)$, so that T_n is just T with both domain and codomain restricted to $\mathcal{P}_n$. Is T_n injective? Is T_n surjective?

   Solution: T is not surjective in $\mathcal{P}_n$: no $p \in \mathcal{P}_n$ satisfies $T(p) = x^n$. T is not injective in $\mathcal{P}_n$: $T(2x) = 2 = T(2)$.

   (c) Is T injective? Is T surjective?

   Solution: T is surjective: for any polynomial $p(x) \in \mathcal{P}$, let $q = \int p(x)\,dx$. Then $T(q) = p(x)$, so T is surjective. However, T is not injective: $T(2x) = 2 = T(2)$.

4. We denote by $\mathbb{R}^{n\times n}$ the vector space of all $n \times n$ matrices. Let A be an $n \times n$ matrix, and define the function $L_A : \mathbb{R}^{n\times n} \to \mathbb{R}^{n\times n}$ by $L_A(B) = AB$ for all $B \in \mathbb{R}^{n\times n}$. (Note carefully: this is *not* the same function as T_A. While both can be described informally as "multiplication by A", the two functions L_A and T_A have different domains and codomains. Make sure you understand this distinction before beginning to work on this problem!)

   (a) Show that L_A is a linear transformation.

   Solution: Let $B, C \in \mathbb{R}^{n\times n}$, and $k \in \mathbb{R}$. We first show L_A respects addition:

$$\begin{aligned} L_A(B+C) &= A(B+C) && \text{(definition of } L_A\text{)} \\ &= AB + AC && \text{(distributivity of matrix multiplication)} \\ &= L_A(B) + L_A(C) \end{aligned}$$

   Next, we show L_A respects scalar multiplication.

$$\begin{aligned} L_A(kB) &= A(kB) && \text{(definition of } L_A\text{)} \\ &= (Ak)B && \text{(scalar associativity of matrix multiplication)} \\ &= (kA)B && \text{(scalar commutativity of matrices)} \\ &= k(AB) \\ &= kL_A(B) && \text{(definition of } L_A\text{)} \end{aligned}$$

   So L_A is linear.

   (b) Show that the matrix A is invertible if and only if the linear transformation L_A is invertible.

   Solution: Assume the linear transformation L_A is invertible. Then there exists L_A^{-1} such that $L_A \circ L_A^{-1} = L_A^{-1} \circ L_A$ is the identity function. Let $C \in \mathbb{R}^{n\times n}$ such that $C = L_A^{-1}(I_n)$. Then $A(C) = L_A\left(L_A^{-1}(I_n)\right) = I_n$. So A is invertible by corollary 2.19 from the Theory of Linear Algebra handout.

   Assume A is invertible. Let its inverse be C. Then $L_A(L_C(B)) = A(CB) = (AC)B = I_nB$ is the identity function. Similarly, $L_C(L_A(B)) = C(AB) = (CA)B = I_nB$ is the identity function. So L_A is invertible with inverse L_C.

Now let $\mathcal{F}$ be the set of all functions from $\mathbb{R}^{n\times n}$ to $\mathbb{R}^{n\times n}$, and define the function $L : \mathbb{R}^{n\times n} \to \mathcal{F}$ by $L(A) = L_A$.

   (c) Show that L is injective.

   Solution: Let $A_1, A_2 \in \mathbb{R}^{n\times n}$ be such that $L(A_1) = L(A_2)$. Then $L_{A_1}(I_n) = A_1I_n = A_1$ is equal to $L_{A_2} = A_2I_n = A_2$. So $A_1 = A_2$, and L is injective.

   (d) Is L surjective? Be sure to justify your claim.

   Solution: No, L cannot map to functions which are nonlinear by part (a), even though such functions assuredly exist for $\mathbb{R}^{n\times n} \to \mathbb{R}^{n\times n}$.

5. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation defined as follows:

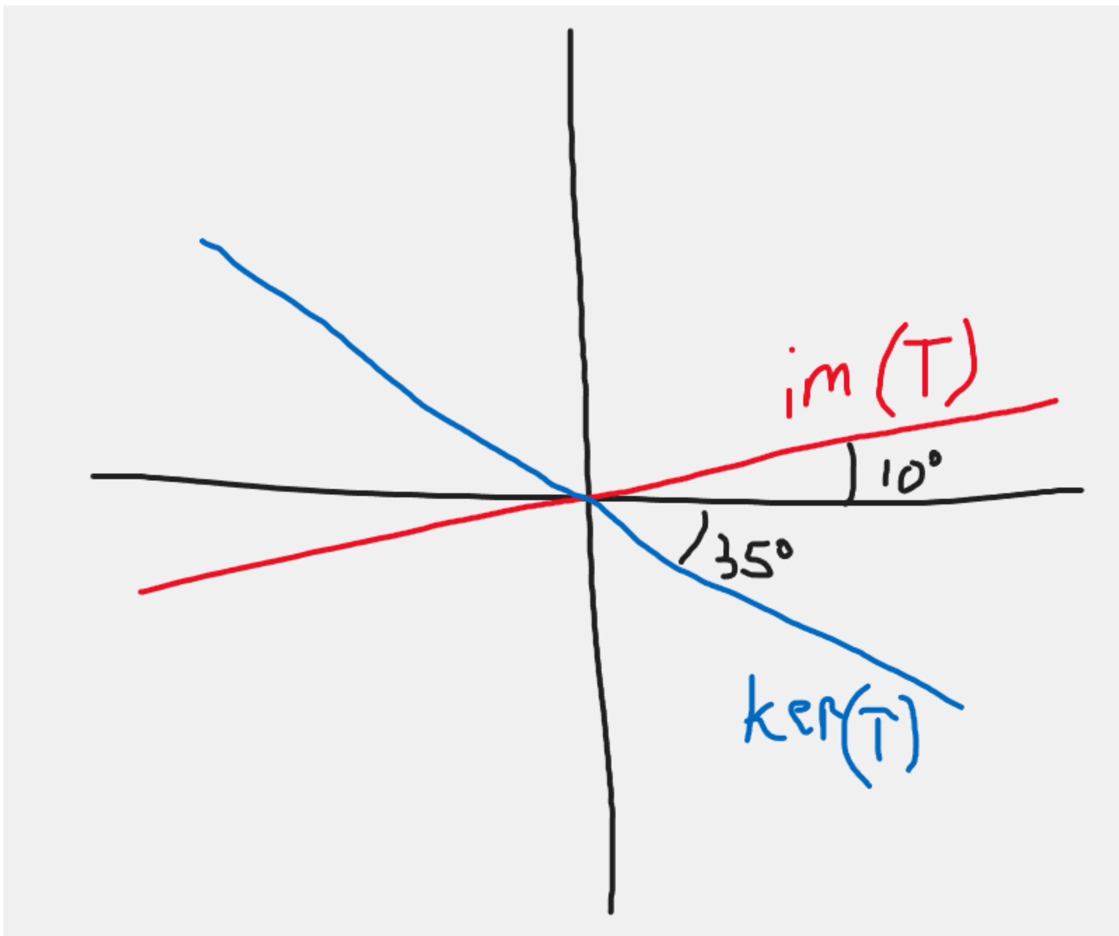
$$T = \mathrm{Rot}_{-80^\circ} \circ \mathrm{Proj}_y \circ \mathrm{Rot}_{35^\circ},$$

where Rot_θ is counter-clockwise rotation by θ, and Proj_y is projection onto the y-axis.

(a) Sketch $\mathrm{im}(T\ )$ in $\mathbb{R}^2$. Indicate the angle between $\mathrm{im}(T\ )$ and the x-axis.

(b) Sketch $\ker(T\ )$ in $\mathbb{R}^2$. Indicate the angle between $\ker(T\ )$ and the x-axis.

Solution:



The image is the image of the y-axis rotated by 80 degrees clockwise, which forms an angle of 10 degrees with the x-axis.

The kernel of the rotation is the origin. The kernel of the projection is the x-axis. So the line 35 degrees clockwise of the x-axis is the kernel of the compositions.

(c) Let $T_{\phi,\theta} := \mathrm{Rot}_\theta \circ \mathrm{Proj}_y \circ \mathrm{Rot}_\phi$. For which ϕ and θ is $\mathrm{im}(T_{\phi,\theta}) = \ker(T_{\phi,\theta})$?

Solution: Using the logic from the previous part, the y-axis rotated by θ counter-clockwise should match the x-axis rotated by ϕ clockwise. So $\theta + 90^\circ = -\phi$.