MATH 217 W24 - LINEAR ALGEBRA, Section 001 (Dr. Paul Kessenich)
Homework Set 7 Part B due Thursday, March 14 at 11:59pm
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1. Let W be an n-dimensional vector space with ordered bases $\mathcal{A}, \mathcal{B}$, and $\mathcal{C}$.

   (a) Prove that $S_{C\to A} = S_{B\to A}S_{C\to B}$.

   Solution: Let w be an arbitrary vector in W, and let $[w]_\mathcal{C}$ be its representation in the $\mathcal{C}$-coordinate space $\mathbb{R}^n$. Then $S_{C\to B}[w]_\mathcal{C} = [w]_\mathcal{B}$ by definition of $S_{C\to B}$.

   So $S_{B\to A}S_{C\to B}[w]_\mathcal{C} = [w]_\mathcal{A}$ by definition.

   Additionally, $S_{C\to A}[w]_\mathcal{C} = [w]_\mathcal{A}$ by definition.

   These two matrices have the same dimensions ($n \times n$), and right-multiplying by the standard unit vectors would have the same results with either matrix, so their columns are identical by the Key Theorem. Thus, $S_{C\to A} = S_{B\to A}S_{C\to B}$.

   (b) Show that $S_{C\to A}S_{B\to C}S_{A\to B} = I_n$

   Solution: Let w be an arbitrary vector in W, and let $[w]_\mathcal{A}$ be its representation in the $\mathcal{A}$-coordinate space $\mathbb{R}^n$. Then $S_{C\to B}[w]_\mathcal{A} = [w]_\mathcal{B}$ by definition of $S_{A\to B}$.

   So $S_{B\to A}S_{C\to B}[w]_\mathcal{A} = [w]_\mathcal{A}$ by definition.

   Similarly, $S_{B\to C}S_{B\to A}S_{C\to B}[w]_\mathcal{A} = [w]_\mathcal{A}$ by definition.

   So $S_{B\to C}S_{B\to A}S_{C\to B}\vec{e}_1 = e_1; S_{B\to C}S_{B\to A}S_{C\to B}\vec{e}_2 = e_2; ...S_{B\to C}S_{B\to A}S_{C\to B}\vec{e}_n = e_n$. Thus by the Key Theorem, $S_{B\to C}S_{B\to A}S_{C\to B} = I_n$.

2. Let f_1, f_2, f_3 be the smooth functions defined by

$$f_1(x) = \sin 2x, f_2(x) = \cos 2x, f_3(x) = e^{3x}$$

and consider the subspace $V \subseteq C^\infty(\mathbb{R})$ spanned by the basis $\mathcal{B} = (f_1, f_2, f_3)$. (You may assume without proof that these three functions are linearly independent.) Now consider the linear transformation $D : V \to V$ defined by differentiation, i.e. for any function $g \in V, D(g)(x) = \frac{dg}{dx}$.

(a) Find $[D]_\mathcal{B}$.

Solution:

$$\begin{aligned}
[D]_B &= \begin{bmatrix} | & | & | \\ [D(f_1)]_\mathcal{B} & [D(f_2)]_\mathcal{B} & [D(f_3)]_\mathcal{B} \\ | & | & | \end{bmatrix} \\
&= \begin{bmatrix} | & | & | \\ [2\cos 2x]_\mathcal{B} & [-2\sin 2x]_\mathcal{B} & [3e^3 x]_\mathcal{B} \\ | & | & | \end{bmatrix} \\
&= \begin{bmatrix} 0 & -2 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & 3 \end{bmatrix}
\end{aligned}$$

(b) Give a geometric interpretation of the matrix $[D]_\mathcal{B}$. That is, how does it act on $\mathbb{R}^3$?

Solution: The matrix $[D]_\mathcal{B}$ will dilate vectors by 2 in the x and y-directions, and 3 in the z-direction. Then, it will flip the vectors over the yz-plane (making the x-coordinate negative). Finally, it will flip the result over the $x = y$ plane, switching the x and y-coordinates of the vector.

3. Let V be a vector space with ordered bases $\mathcal{B} = (b_1, ..., b_n)$ and $\mathcal{C} = (c_1, ..., c_n)$. Let $T : V \to V$ be a linear transformation, with $B = [T]_{\mathcal{B}}$ and $C = [T]_{\mathcal{C}}$. Give a proof or counterexample for each of the following statements:

   (a) For all integers $k \geq 1$, B^k and C^k are similar.

   Solution: This statement is true. By the Change of Basis Theorem, we know that $B = S^{-1}CS$, where S is the change-of-coordinates transformation from $\mathcal{B}$ coordinates to $\mathcal{C}$ coordinates, which is an isomorphism. Then

$$\begin{aligned} B^k &= (S^{-1}CS)^k = S^{-1}CSS^{-1}CSS^{-1}CS...S^{-1}CS \\ &= S^{-1}CI_nCI_nCS...I_nCS \\ &= S^{-1}C^kS \end{aligned}$$

   by trivial induction, when $k \geq 1$. So there exists invertible matrix S such that $B^k = S^{-1}C^kS$. Thus B^k and C^k are similar by definition.

   (b) $\ker(B) = \ker(C)$.

   Solution: This statement is false. Consider transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$, $T(x, y) = \begin{bmatrix} x - y \\ 0 \end{bmatrix}$, and bases $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$, $\mathcal{C} = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$. Then $\ker(C)$ includes $\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \left[\begin{bmatrix} 1 \\ 1 \end{bmatrix} \right]_{\mathcal{C}}$, but $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is not within the kernel of B; $B \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Thus $\ker(B) \neq \ker(C)$.

   (c) $\dim(\ker(B)) = \dim(\ker(C))$.

   Solution: This statement is true. Let S be the change-of-coordinates matrix from $\mathcal{B}$ to $\mathcal{C}$ coordinates.

   Let basis $K_B = \{k_{B1}, \dots k_{Bm}\}$ be a basis of the kernel of B with dimension m. Then let $K_C = \{Sk_{B1}, \dots Sk_{Bm}\}$. Note that since S is an isomorphism, the span of K_C has the same dimension as the kernel of B. We know by the Change of Basis Theorem that $C = SBS^{-1}$. So $CSk_{Bj} = SBS^{-1}Sk_{Bj} = S\vec{0} = \vec{0}$. Because $CSk_{Bj} = 0$ for all vectors $Sk_{Bj} \in K_C$, the span of K_C is within the kernel of C by linearity, and $\dim(\ker(B)) \leq \dim(\ker(C))$.

   Let basis $K_C = \{k_{C1}, \dots k_{Cm}\}$ be a basis of the kernel of C with dimension m. Then let $K_B = \{S^{-1}k_{C1}, \dots S^{-1}k_{Cm}\}$. Note that since S^{-1} is an isomorphism, the span of K_B has the same dimension as the kernel of C. We know by the Change of Basis Theorem that $B = S^{-1}CS$. So $BS^{-1}k_{Cj} = S^{-1}BSS^{-1}k_{Cj} = S^{-1}\vec{0} = \vec{0}$. Because $BS^{-1}k_{Cj} = 0$ for all vectors $Sk_{Cj} \in K_B$, the span of K_B is within the kernel of B by linearity, and $\dim(\ker(C)) \leq \dim(\ker(B))$.

   Because $\dim(\ker(B)) \leq \dim(\ker(C))$ and $\dim(\ker(C)) \leq \dim(\ker(B))$, then $\dim(\ker(C)) = \dim(\ker(B))$.

4. Let $T : U \to W$ be a linear transformation between vector spaces U and W. Suppose that $\mathcal{B} = (u_1, u_2, ..., u_k)$ is a basis for the source U and $\mathcal{C} = (w_1, w_2, ..., w_d)$ is a basis for the target W. As usual, let $L_\mathcal{B}$ denote the coordinate isomorphism $U \to \mathbb{R}^k$ and let $L_\mathcal{C}$ denote the coordinate isomorphism $W \to \mathbb{R}^d$.

   (a) Show that there exists a linear transformation $T' : \mathbb{R}^k \to \mathbb{R}^d$ such that $T' \circ L_\mathcal{B} = L_\mathcal{C} \circ T$. [Hint: A diagram showing four vector spaces and four maps between them, similar to those immediately before and after Definition 4.3.1 in the textbook, might be useful.]

   Solution: Let T' be the composition of transformations $L_\mathcal{C} \circ T \circ L_\mathcal{B}^{-1}$. We know that this transformation exists since the domains and codomains of $L_\mathcal{C}, T$, and $L_\mathcal{B}^{-1}$ match by definition of the coordinate isomorphisms.

   $$T' : \mathbb{R}^k \xrightarrow{L_\mathcal{B}^{-1}} U \xrightarrow{T} W \xrightarrow{L_\mathcal{C}} \mathbb{R}^d$$

   Additionally, we know this composition is linear since all the component transformations are linear. Then, T' satisfies

   $$\begin{aligned} T' \circ L_\mathcal{B} &= L_\mathcal{C} \circ T \\ L_\mathcal{C} \circ T \circ L_\mathcal{B}^{-1} \circ L_\mathcal{B} &= L_\mathcal{C} \circ T \\ L_\mathcal{C} \circ T \circ I &= L_\mathcal{C} \circ T \\ L_\mathcal{C} \circ T &= L_\mathcal{C} \circ T \end{aligned}$$

   (b) Let $[T]_{(\mathcal{B},\mathcal{C})}$ denote the standard matrix of the transformation T' you described in (a). Prove that for all $u \in U$,

   $$[T(u)]_\mathcal{C} = [T]_{(\mathcal{B},\mathcal{C})}[u]_\mathcal{B}.$$

   Solution: $[T(u)]_\mathcal{C}$ is equivalent to $L_\mathcal{C}(T(u)) = (L_\mathcal{C} \circ T)(u)$. Additionally, $[T]_{(\mathcal{B},\mathcal{C})}[u]_\mathcal{B}$ is equivalent to $T'(L_\mathcal{B}(u)) = (T' \circ L_\mathcal{B})(u)$. Since we have shown in part (a) that $T' \circ L_\mathcal{B} = L_\mathcal{C} \circ T$, then $(T' \circ L_\mathcal{B})(u) = (L_\mathcal{C} \circ T)(u)$. So the given statement is true for all $u \in U$.

   (c) Describe, with explanation, the columns of matrix $[T]_{(\mathcal{B},\mathcal{C})}$ in terms of the bases $\mathcal{B}$ and $\mathcal{C}$.

   Solution: Column i of matrix $[T]_{(\mathcal{B},\mathcal{C})}$ will be $[T(u_i)]_\mathcal{C}$. This is the representation of the [result of basis element u_i of $\mathcal{B}$ after undergoing transformation T] as a $\mathcal{C}$-coordinate. This is very similar to the Key Theorem, except with differing dimensions between domain and codomain. Using the same strategy as the Key Theorem, plugging in a standard vector of $\mathbb{R}^k$ is like plugging in basis element u_i into T. Then, since the output is in W, we use $\mathcal{C}$-coordinates to represent it in $\mathbb{R}^d$.

5. Let f_1, f_2, f_3 be the functions defined by

$$f_1(x) = \sin x, f_2(x) = \cos x, f_3(x) = e^x,$$

which you may assume without proof are linearly independent. Consider the subspace V of C^∞ spanned by the set $\{f_1, f_2, f_3\}$. Recall from Calculus that every function in V may be expressed as a Taylor series that converges for all real numbers.

Let $T : V \to \mathcal{P}^3$ be the linear transformation that assigns to each function $f \in V$ the third-degree Taylor polynomial $f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3$ for f, a polynomial approximation to f.

(a) Find a basis $\mathcal{C}$ for $\mathcal{P}^3$ such that

$$[T(f_1)]_{\mathcal{C}} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix}, [T(f_2)]_{\mathcal{C}} = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}, [T(f_3)]_{\mathcal{C}} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}.$$

Solution: $\mathcal{C} = \left\{1, x, \frac{x^2}{2}, \frac{x^3}{3!}\right\}$.

(b) Let C be as in (a), and let $B = (f_1 + f_2, f_1 - f_2, f_3 + f_1)$. Find $[T]_{(\mathcal{B},\mathcal{C})}$ (see Problem 4).

Solution: As we saw in problem 4, column i of matrix $[T]_{(\mathcal{B},\mathcal{C})}$ will be $[T(f_i)]_{\mathcal{C}}$. We were already given these in part (a), so finding the standard matrix is simple:

$$[T]_{(\mathcal{B},\mathcal{C})} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & -1 & 1 \\ -1 & 0 & 1 \end{bmatrix}$$

6. Let $A = \begin{bmatrix} -6 & -30 \\ -30 & 19 \end{bmatrix}$ and let $V = \operatorname{span}\left(\begin{bmatrix} 3 \\ 2 \end{bmatrix}\right)$.

(a) Show that for all $\vec{v} \in V, A\vec{v} \in V$.

Solution: Let arbitrary $\vec{v} \in V$. Then since $\vec{v}$ is in the span of $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$, it can be expressed $v = a\begin{bmatrix} 3 \\ 2 \end{bmatrix}$ for some $a \in \mathbb{R}$. Then

$$\begin{aligned} A\vec{v} &= \begin{bmatrix} -6 & -30 \\ -30 & 19 \end{bmatrix}\left(a\begin{bmatrix} 3 \\ 2 \end{bmatrix}\right) \\ &= 3a\begin{bmatrix} -6 \\ -30 \end{bmatrix} + 2a\begin{bmatrix} -30 \\ 19 \end{bmatrix} \\ &= a\begin{bmatrix} 3 \cdot (-6) + 2 \cdot (-30) \\ 3 \cdot (-30) + 2 \cdot 19 \end{bmatrix} \\ &= a\begin{bmatrix} -78 \\ -52 \end{bmatrix} \\ &= -\frac{a}{26}\begin{bmatrix} 3 \\ 2 \end{bmatrix} \end{aligned}$$

We know $-\frac{a}{26} \in \mathbb{R}$ by closure of nonzero real division, so $A\vec{v} = -\frac{a}{26}\begin{bmatrix} 3 \\ 2 \end{bmatrix} \in V$.

(b) Find a basis for $V^\perp$, and show that for all $\vec{w} \in V^\perp, A\vec{w} \in V^\perp$.

Solution: Let the basis for $V^\perp$ be $\{\begin{bmatrix} 2 \\ -3 \end{bmatrix}\}$. Let arbitrary $\vec{w} \in V^\perp$. Then since $\vec{w}$ is in the span of $\begin{bmatrix} 2 \\ -3 \end{bmatrix}$, it can be expressed $\vec{w} = a\begin{bmatrix} 2 \\ -3 \end{bmatrix}$ for some $a \in \mathbb{R}$. Then

$$\begin{aligned} A\vec{w} &= \begin{bmatrix} -6 & -30 \\ -30 & 19 \end{bmatrix}\left(a\begin{bmatrix} 2 \\ -3 \end{bmatrix}\right) \\ &= 2a\begin{bmatrix} -6 \\ -30 \end{bmatrix} - 3a\begin{bmatrix} -30 \\ 19 \end{bmatrix} \\ &= a\begin{bmatrix} 2 \cdot (-6) - 3 \cdot (-30) \\ 2 \cdot (-30) - 3 \cdot 19 \end{bmatrix} \\ &= a\begin{bmatrix} 78 \\ -117 \end{bmatrix} \\ &= \frac{a}{39}\begin{bmatrix} 2 \\ -3 \end{bmatrix} \end{aligned}$$

We know $\frac{a}{39} \in \mathbb{R}$ by closure of nonzero real division, so $A\vec{w} = \frac{a}{39}\begin{bmatrix} 2 \\ -3 \end{bmatrix} \in V$.

(c) Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation defined by $T(\vec{x}) = A\vec{x}$ for all $\vec{x} \in \mathbb{R}^2$. Find a basis $\mathcal{B}$ of $\mathbb{R}^2$ such that $[T]_\mathcal{B}$ is diagonal, and write the matrix $[T]_\mathcal{B}$ explicitly.

Solution: Let $\mathcal{B} = \left\{ \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \end{bmatrix} \right\}$. Then by the Key Theorem, $[T]_\mathcal{B} = \begin{bmatrix} [T(b_1)]_\mathcal{B} & [T(b_2)]_\mathcal{B} \end{bmatrix} = \begin{bmatrix} -\frac{1}{26} & 0 \\ 0 & \frac{1}{39} \end{bmatrix}$

(d) Calculate $[T^{10}]_\mathcal{B}$. [Hint: Leave numbers like 7^{13} in that form; do not attempt to multiply them out.]

Solution: $[T^{10}]_\mathcal{B} = [T]_\mathcal{B}^1 0$ since the matrix identifies the transformation. So

$$[T^{10}]_\mathcal{B} = \begin{bmatrix} -\frac{1}{26} & 0 \\ 0 & \frac{1}{39} \end{bmatrix}^{10} = \begin{bmatrix} \frac{1}{26^{10}} & 0 \\ 0 & \frac{1}{39^{10}} \end{bmatrix}$$

(e) Calculate $[T^{10}]_\mathcal{E}$. [Hint: Leave the entries as numerical expressions; do not attempt to simplify.]

Solution: We use the change of basis theorem for transformations, which tells us that $[T^{10}]_\mathcal{E} = S^{-1}[T^{10}]_\mathcal{B} S$, where S is the change of coordinates transformation from $\mathcal{E}$ to $\mathcal{B}$ coordinates. We find that $S = \begin{bmatrix} 3 & 2 \\ 2 & -3 \end{bmatrix}$ and $S^{-1} = \begin{bmatrix} \frac{3}{13} & \frac{2}{13} \\ \frac{2}{13} & \frac{-3}{13} \end{bmatrix}$. So

$$\begin{aligned}
[T^{10}]_\mathcal{E} &= \begin{bmatrix} \frac{3}{13} & \frac{2}{13} \\ \frac{2}{13} & \frac{-3}{13} \end{bmatrix} \begin{bmatrix} \frac{1}{26^{10}} & 0 \\ 0 & \frac{1}{39^{10}} \end{bmatrix} \begin{bmatrix} 3 & 2 \\ 2 & -3 \end{bmatrix} \\
&= \begin{bmatrix} \frac{3}{13} & \frac{2}{13} \\ \frac{2}{13} & \frac{-3}{13} \end{bmatrix} \begin{bmatrix} \frac{3}{26^{10}} & \frac{2}{26^{10}} \\ \frac{2}{39^{10}} & -\frac{3}{39^{10}} \end{bmatrix} \\
&= \frac{1}{13} \begin{bmatrix} \frac{9}{26^{10}} + \frac{4}{39^{10}} & \frac{6}{26^{10}} - \frac{6}{39^{10}} \\ \frac{9}{26^{10}} - \frac{6}{39^{10}} & \frac{4}{26^{10}} + \frac{9}{39^{10}} \end{bmatrix}
\end{aligned}$$