

MATH 215 FALL 2023
Homework Set 7: §15.3 – 15.6
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1. Let E be the bounded region in the first octant bounded by the surface $z = 1 + 2(x - 3)^2 + y^2$ and the planes $y = 3$ and $y = x$. Sketch E and compute its volume.

Solution:

$$\begin{aligned}
 &0 \leq y \leq 3 \\
 &0 \leq x \leq y \\
 &1 \leq z \leq 1 + 2(x - 3)^2 + y^2 \\
 &V = \int \int \int_D dV \\
 &\int_0^3 \int_0^y \int_1^{1+2(x-3)^2+y^2} 1 \, dz \, dx \, dy \\
 &\int_0^3 \int_0^y (1 + 2(x - 3)^2 + y^2) - 1 \, dx \, dy \\
 &\int_0^3 \left(\frac{2}{3}(x - 3)^3 + xy^2 \right) \Big|_{x=0}^y \, dy \\
 &\int_0^3 \left(\frac{2}{3}(y - 3)^3 + y^3 + 18 \right) \, dy \\
 &\left(\frac{1}{6}(y - 3)^4 + \frac{y^4}{4} + 10y \right) \Big|_{y=0}^3 \\
 &\left(\frac{3^4}{4} + 10 \cdot 3 \right) - \left(\frac{3^4}{6} \right) = \boxed{\frac{243}{4}}
 \end{aligned}$$

□

2. Let D be the region given in polar coordinates by $0 \leq r \leq \sqrt{\sin(2\theta)}, 0 \leq \theta \leq \frac{\pi}{2}$. Sketch D and compute $\int \int_D x \sqrt{x^2 + y^2} dA$

Solution:

$$\begin{aligned}
 r^2 &= x^2 + y^2 \\
 \int \int_D x \sqrt{x^2 + y^2} dA &= \int_0^{\frac{\pi}{2}} \int_0^{\sqrt{\sin(2\theta)}} r^3 \cos(\theta) dr d\theta \\
 &= \int_0^{\frac{\pi}{2}} \left(\frac{r^4}{4} \cos(\theta) \right) \Big|_{r=0}^{\sqrt{\sin(2\theta)}} d\theta \\
 &= \frac{1}{4} \int_0^{\frac{\pi}{2}} \cos(\theta) \sin^2(2\theta) d\theta \\
 &= \int_0^{\frac{\pi}{2}} \sin^2(\theta) \cos^3(\theta) d\theta \\
 &= \int_0^{\frac{\pi}{2}} \sin^2(\theta) \cos^3(\theta) d\theta \\
 &= \int_0^{\frac{\pi}{2}} (1 - \sin^2(\theta)) \sin^2(\theta) \cos(\theta) d\theta \\
 u &= \sin(\theta); du = \cos(\theta) \\
 &= \int_0^1 (1 - u^2) u^2 du \\
 &= \frac{1}{3} - \frac{1}{5} \\
 &= \boxed{\frac{2}{15}}
 \end{aligned}$$

□

3. Use polar coordinates to evaluate the following integrals:

(a)

$$\int_0^4 \int_{-\sqrt{16-y^2}}^{\sqrt{16-y^2}} (y^3 + x^2 y) dx dy$$

Solution:

$$\begin{aligned} 0 \leq \theta \leq \pi, 0 \leq r \leq 4 \\ \int_0^\pi \int_0^4 r^4 \sin(\theta) dr d\theta \\ = \frac{1}{5} \int_0^\pi (r^5 \sin(\theta)) \Big|_{r=0}^4 d\theta \\ = \frac{4^5}{5} \int_0^\pi \sin(\theta) d\theta \\ = \frac{4^5}{5} (\cos(\theta)) \Big|_{\theta=0}^\pi \\ = \boxed{\frac{2048}{5}} \end{aligned}$$

□

(b)

$$\iint_D \frac{y}{x^2 + y^2} dA,$$

where D is the region outside the circle of radius 1 centered at the origin and inside the circle of radius 1 centered at $(0, 1)$.

Solution:

$$\begin{aligned} \frac{\pi}{6} \leq \theta \leq \frac{5\pi}{6}, 1 \leq r \leq 2 \sin(\theta) \\ 0 = x^2 + y^2 - 2y \\ 0 = r^2 - 2r \sin(\theta) \\ r = 2 \sin(\theta) \\ \int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \int_1^{2 \sin(\theta)} \frac{r \sin(\theta)}{r^2} r dr d\theta \\ \int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \int_1^{2 \sin(\theta)} \sin(\theta) dr d\theta \\ \int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} 2 \sin^2(\theta) - \sin(\theta) d\theta \\ \int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} 1 - \cos(2\theta) - \sin(\theta) d\theta \end{aligned}$$

$$\begin{aligned} & \left(\theta - \frac{1}{2} \sin(2\theta) + \cos(\theta) \right) \Big|_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \\ &= \boxed{\frac{2\pi}{3} - \frac{\sqrt{3}}{2}} \end{aligned}$$

□

4. You are painting a room, and you need to transfer the paint from the can in which you bought it to a tray that you are going to use for the brush. To do this transfer, you pour the paint through a funnel comprised of a cylindrical section of radius 1 inch and height 2 inches attached to a cone segment of smaller radius 1 inch, larger radius 5 inches, and height 5 inches. Because of the way you pour the paint, assume that the inside of the funnel is evenly coated with a thin layer of paint. How many square inches of paint have you wasted?

Solution: Surface area of lateral surface of cylinder is $2\pi rh = 4\pi$. Surface area of lateral surface of truncated cone is $2\pi \frac{r_1+r_2}{2}l = 6\pi\sqrt{41}$. In total, the surface area is

$4\pi + 6\pi\sqrt{41} \approx 133.262419$ square inches of paint. □

5. Find the surface area of the part of the sphere $x^2 + y^2 + z^2 = 4z$ that lies above the paraboloid $3z = x^2 + y^2$.

Solution:

$$3z = r^2$$

$$r^2 + z^2 = 4z$$

$$3z + z^2 = 4z$$

$$z^2 - z = 0$$

$$z = 0, 1$$

$$r = \sqrt{3}$$

$$0 \leq \theta \leq 2\pi$$

$$(z - 2)^2 = 4 - x^2 - y^2$$

$$z = 2 + \sqrt{4 - x^2 - y^2}$$

$$S = \int \int_D \sqrt{1 + \left(\frac{\delta z}{\delta x}\right)^2 + \left(\frac{\delta z}{\delta y}\right)^2} dx$$

$$S_{upper} = \int \int_D \sqrt{1 + \left(-\frac{x}{\sqrt{4 - x^2 - y^2}}\right)^2 + \left(-\frac{y}{\sqrt{4 - x^2 - y^2}}\right)^2} dx$$

$$= \int \int_D \frac{2}{\sqrt{4 - r^2}} dx$$

$$\int_0^{2\pi} \int_0^2 \frac{2}{\sqrt{4 - r^2}} r dr d\theta$$

$$2 \int_0^{2\pi} 2 d\theta$$

$$8\pi$$

$$S_{lower} = \int_0^{2\pi} \int_0^{\sqrt{3}} r \frac{2}{\sqrt{4 - r^2}} dr d\theta$$

$$2 \int_0^{2\pi} 1 d\theta$$

$$= 4\pi$$

$$\boxed{S = 12\pi}$$

□

6. The small but edgy city of Dreieck is modeled by the region $D = \{(x, y) : 5|x| \leq y \leq 5\}$, where x and y are measured in km. The population density in people per km^2 is given by $\rho(x, y) = 10^{3-y}$. Find the total population of the city. (This should preferably be an integer.)

Solution:

$$\begin{aligned}
 0 &\leq y \leq 5, -\frac{y}{5} \leq x \leq \frac{y}{5} \\
 \iint_D \rho(x, y) dA &= \int_0^5 \int_{-\frac{y}{5}}^{\frac{y}{5}} 10^{3-y} dx dy \\
 &= \int_0^5 \left(10^{3-y} x \right) \Big|_{x=-\frac{y}{5}}^{\frac{y}{5}} dy \\
 &= \int_0^5 10^{3-y} \frac{2y}{5} dy \\
 &= \frac{2}{5} 10^3 \int_0^5 10^{-y} y dy \\
 &= \frac{2}{5} 10^3 \left(-\frac{10^{-y} y}{\ln(10)} - \int_0^5 -\frac{10^{-y}}{\ln(10)} \right) \\
 &= 400 \left(-\frac{y}{10^y \cdot \ln(10)} - \frac{1}{\ln^2(10) \cdot 10^y} \right) \Big|_{y=0}^5 \\
 &= 400 \cdot 0.1885880961705487 \approx \boxed{76 \text{ people}}
 \end{aligned}$$

□

7. Find the mass and center of mass of the tetrahedron bounded by the planes $x = 0, y = 0, z = 0, 2x + 3y + z = 6$, if the density function is given by $\rho(x, y, z) = z$.

Solution: We begin by finding the total mass.

$$\begin{aligned}
 \int_0^6 \int_0^{\frac{6-z}{2}} \int_0^{\frac{6-z-2x}{3}} z \, dy \, dx \, dz &= \int_0^6 z \int_0^{\frac{6-z}{2}} \frac{6-z-2x}{3} \, dy \, dx \, dz \\
 &= \int_0^6 z \int_0^{\frac{6-z}{2}} \frac{6-z-2x}{3} \, dy \, dx \, dz \\
 &= \int_0^6 z \left(\frac{6-z}{2} \frac{6-z}{3} - \frac{(6-z)^2}{12} \right) dz \\
 &= \int_0^6 z \frac{(6-z)^2}{12} dz \\
 &= \frac{1}{12} \int_0^6 36z - 12z^2 + z^3 \, dz \\
 &= \frac{1}{12} \cdot \left(\frac{z^4}{4} + 18z^2 - 4z^3 \right) \Big|_{z=0}^{z=6} \\
 &= \frac{1}{12} (324 + 648 - 864) \\
 &= \boxed{9}
 \end{aligned}$$

Now, we perform a weighted sum of each the x, y , and z coordinates, then divide by the total mass to find the respective value for the center of mass. For x :

$$\begin{aligned}
 \int_0^6 \int_0^{\frac{6-z}{2}} \int_0^{\frac{6-z-2x}{3}} xz \, dy \, dx \, dz &= \int_0^6 z \int_0^{\frac{6-z}{2}} x \int_0^{\frac{6-z-2x}{3}} 1 \, dy \, dx \, dz \\
 &= \int_0^6 z \int_0^{\frac{6-z}{2}} x \int_0^{\frac{6-z-2x}{3}} 1 \, dy \, dx \, dz \\
 &= \int_0^6 z \int_0^{\frac{6-z}{2}} \frac{6x - zx - 2x^2}{3} \, dx \, dz \\
 &= \int_0^6 z \left(\frac{3x^2 - \frac{zx^2}{2} - \frac{2x^3}{3}}{3} \right) \Big|_{x=0}^{x=\frac{6-z}{2}} dz \\
 &= \int_0^6 z \left(\frac{3 \left(\frac{6-z}{2} \right)^2 - \frac{z \left(\frac{6-z}{2} \right)^2}{2} - \frac{2 \left(\frac{6-z}{2} \right)^3}{3}}{3} \right) dz \\
 &= \frac{1}{12} \int_0^6 z \left(3(6-z)^2 - \frac{z(6-z)^2}{2} - \frac{2(6-z)^3}{3} \right) dz \\
 &= \frac{1}{12} \int_0^6 z(6-z)^2 \left(3 - \frac{z}{2} - \frac{2(6-z)}{3} \right) dz
 \end{aligned}$$

$$= -\frac{1}{72} \int_0^6 z(6-z)^3 dz$$

$$\begin{aligned} u &= 6 - z \\ &= \frac{1}{72} \int_0^6 (6-u)(u)^3 du \\ &= \frac{1}{72} \int_0^6 6u^3 - u^4 du \\ &= \frac{1}{72} \left(\frac{3u^4}{2} - \frac{u^5}{5} \right) \Big|_{u=0}^6 \\ &= \frac{1}{72} \left(\frac{3u^4}{2} - \frac{u^5}{5} \right) \Big|_{u=0}^6 \\ &= 5.4 \\ \bar{x} &= \frac{3}{5} \end{aligned}$$

For y :

$$\begin{aligned} \int_0^6 z \int_0^{\frac{6-z}{3}} y \int_0^{\frac{6-z-3y}{2}} 1 dx dy dz &= \int_0^6 z \int_0^{\frac{6-z}{3}} y \frac{6-z-3y}{2} dy dz \\ &= \int_0^6 z \int_0^{\frac{6-z}{3}} y \frac{6-z-3y}{2} dy dz \\ &= \int_0^6 z \left(-\frac{2y^3 + (z-6)y^2}{4} \right) \Big|_{y=0}^{\frac{6-z}{3}} dz \\ &= \int_0^6 z \left(-\frac{2\left(\frac{6-z}{3}\right)^3 + (z-6)\left(\frac{6-z}{3}\right)^2}{4} \right) dz \\ &= \frac{18}{5} \\ \bar{y} &= \frac{2}{5} \end{aligned}$$

For z :

$$\begin{aligned} \int_0^6 z^2 \int_0^{\frac{6-z}{3}} \int_0^{\frac{6-z-3y}{2}} 1 dx dy dz &= \int_0^6 z^2 \int_0^{\frac{6-z}{3}} \frac{6-z-3y}{2} dy dz \\ &= \int_0^6 z^2 \int_0^{\frac{6-z}{3}} \frac{6-z-3y}{2} dy dz \\ &= \int_0^6 z \frac{6-z-\frac{3\left(\frac{6-z}{3}\right)^2}{2}}{2} dz \end{aligned}$$

$$\begin{aligned} &= \frac{162}{5} \\ \bar{z} &= \frac{18}{5} \end{aligned}$$

Thus, the mass is $\boxed{9}$ and the center of mass is $(x, y, z) = \boxed{\left(\frac{3}{5}, \frac{2}{5}, \frac{18}{5}\right)}$.

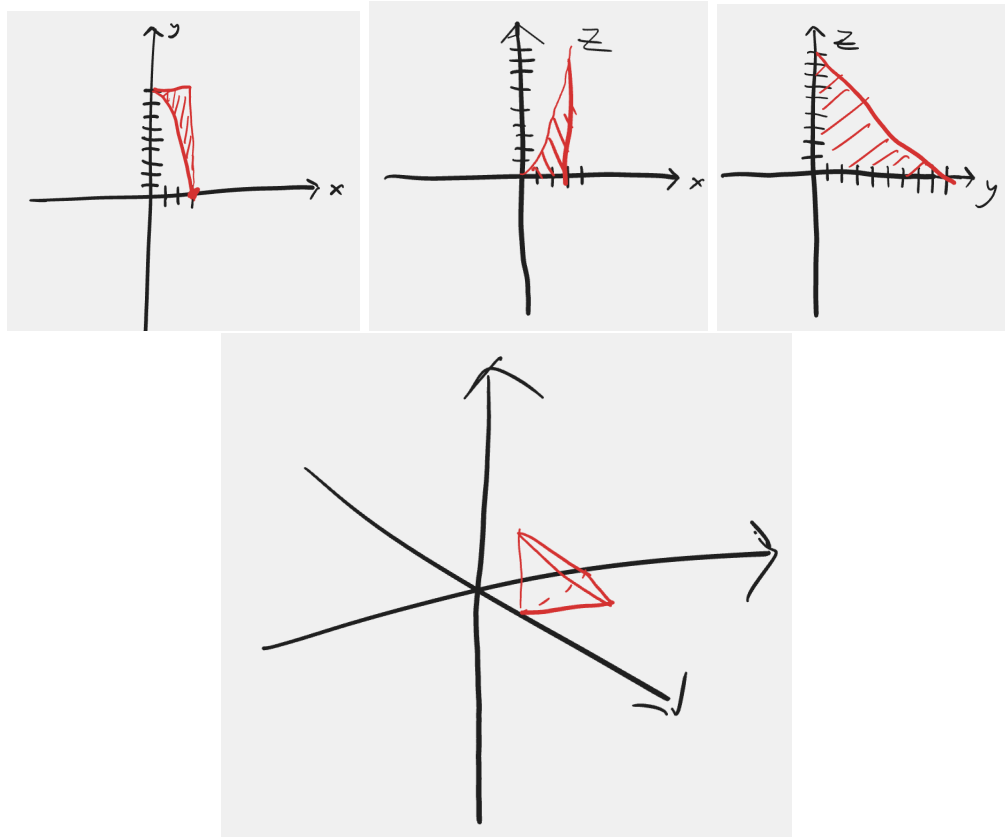
□

8. Sketch the region of integration for the integral

$$\int_0^3 \int_{9-x^2}^9 \int_0^{9-y} f(x, y, z) dz dy dx$$

Rewrite this integral as an equivalent iterated integral in three of the five possible other orders.

Solution:



$$\int_0^3 \int_0^{x^2} \int_{9-x^2}^{9-z} f(x, y, z) dy dz dx \quad (1)$$

$$\int_0^9 \int_{\sqrt{z}}^3 \int_{9-z}^{9-x^2} f(x, y, z) dy dx dz \quad (2)$$

$$\int_0^9 \int_{\sqrt{9-y}}^3 \int_0^{9-y} f(x, y, z) dz dx dy \quad \square$$

9. Find the region E for which the triple integral

$$\int \int \int_E (9 - 4x^2 - 4y^2 - 4z^2) dV$$

is a maximum, and compute this maximum value.

Solution: We will use spherical coordinates. First, we find the critical point, then the maximum value of the integral.

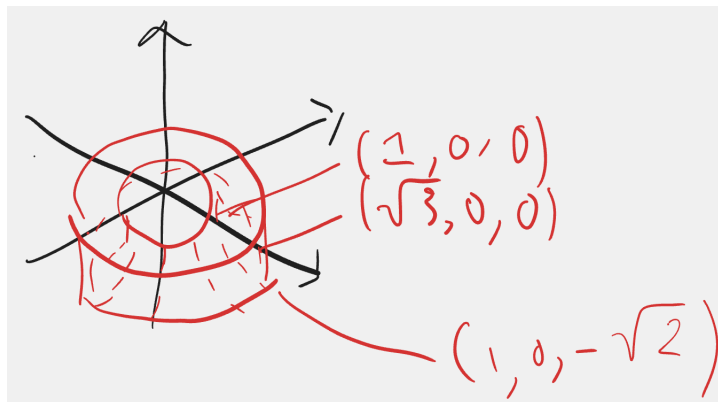
$$\begin{aligned} r &= \sqrt{x^2 + y^2 + z^2} \\ -4x^2 - 4y^2 - 4z^2 + 9 &= 0 \\ \frac{9}{4} &= x^2 + y^2 + z^2 \\ r &= \frac{3}{2} \\ \int \int \int_E (9 - 4x^2 - 4y^2 - 4z^2) dV &= \int \int \int_E (9 - 4r^2) r^2 \sin(\phi) dr d\phi d\theta \\ &= \int_0^{2\pi} \int_0^\pi \int_0^{\frac{3}{2}} (9 - 4r^2) r^2 \sin(\phi) dr d\phi d\theta \\ &= \int_0^{2\pi} \int_0^\pi \left(3r^3 \sin(\phi) - \frac{4}{5} r^5 \sin(\phi) \right) \Big|_{r=0}^{\frac{3}{2}} d\phi d\theta \\ &= \int_0^{2\pi} \int_0^\pi \left(\frac{81}{8} \sin(\phi) - \frac{243}{40} \sin(\phi) \right) d\phi d\theta \\ &= \int_0^{2\pi} \int_0^\pi \left(-\frac{81}{20} \cos(\phi) \right) \Big|_{\phi=0}^\pi d\theta \\ &= \int_0^{2\pi} \frac{81}{10} d\theta \\ &= \boxed{\frac{81\pi}{5}} \end{aligned}$$

□

10. A team of archeologists excavating the ancient settlement of Osmia has unearthed a previously unknown type of artifact. This can be modeled as the region E that lies inside the cylinder $x^2 + y^2 = 3$, is bounded above by the plane $z = 0$, and is bounded below by the hyperboloid $x^2 + y^2 - z^2 = 1$.

(a) Sketch E .

Solution:



- (b) Find the volume of E .

Solution:

$$\begin{aligned}
 & \int_{-\sqrt{2}}^0 \int_0^{2\pi} \int_{\sqrt{z^2+1}}^{\sqrt{3}} r \, dr \, d\theta \, dz \\
 &= \frac{1}{2} \int_{-\sqrt{2}}^0 \int_0^{2\pi} (3 - z^2 - 1) \, d\theta \, dz \\
 &= \pi \int_{-\sqrt{2}}^0 (2 - z^2) \, dz \\
 &= 2\sqrt{2}\pi - \frac{2\sqrt{2}\pi}{3} \\
 &= \boxed{\frac{4\pi\sqrt{2}}{3}}
 \end{aligned}$$

□

- (c) Assuming that its density is constant (equal to 22.59g/cm^3), find the center of mass of E .

Solution: Mass = $\frac{4\pi\sqrt{2}}{3} \cdot 22.59 = 133.8196341$

$$\begin{aligned}
 & \int_{-\sqrt{2}}^0 \int_0^{2\pi} \int_{\sqrt{z^2+1}}^{\sqrt{3}} rz \, dr \, d\theta \, dz \\
 &= \frac{1}{2} \int_{-\sqrt{2}}^0 z \int_0^{2\pi} (3 - z^2 - 1) \, d\theta \, dz
 \end{aligned}$$

$$\begin{aligned} &= \pi \int_{-\sqrt{2}}^0 2z - z^3 dz \\ &= 2\pi - \frac{4\pi}{4} \\ &= \pi \end{aligned}$$

$$\bar{z} = \frac{\pi}{133.8196341} \approx 0.02347632076$$

$$(\bar{x}, \bar{y}, \bar{z}) = (0, 0, 0.02347632076)$$

□