

**MATH 217 W24 - LINEAR ALGEBRA, Section 001 (Dr. Paul Kessenich)**  
**Homework Set Part B due SUNDAY, April 21 at 11:59pm**  
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1. (a) Let  $E_0$  denote the 0-eigenspace of  $T$ . Explicitly describe  $E_0$  (as a set).

**Solution:**

$$E_0 = \{(x_1, 0, x_2, 0, x_3, 0, \dots) \mid x_i \in \mathbb{R}\}$$

- (b) Prove that every real number  $\lambda$  is an eigenvalue of  $T$ . (Hint: explicitly construct an eigenvector  $(x_1, x_2, x_3, \dots) \in V$ . First consider  $x_i$  when  $i$  is a power of 2.)

**Solution:** Let  $\lambda \in \mathbb{R}$ . Then let

$$s = (1, \lambda, \lambda, \lambda^2, \lambda^2, \lambda^2, \lambda^2, \lambda^3, \lambda^3, \lambda^3, \lambda^3, \lambda^3, \lambda^3, \lambda^3, \lambda^3, \dots)$$

be an infinite sequence such that each consecutive power  $\lambda^n$  is repeated  $n$  times in the sequence, starting from  $n = 0$ . Then

$$\begin{aligned} T(s) &= (\lambda, \lambda^2, \lambda^2, \lambda^3, \lambda^3, \lambda^3, \lambda^3, \lambda^4, \lambda^4, \lambda^4, \lambda^4, \lambda^4, \lambda^4, \lambda^4, \lambda^4, \dots) \\ &= \lambda(s). \end{aligned}$$

So any real number is an eigenvalue of  $T$ .

2. (a) Let  $\mathcal{D}$  be a diagonal  $n \times n$  matrix with distinct entries along the diagonal, and let  $\mathcal{D}$  be the subset of  $\mathbb{R}^{n \times n}$  consisting of all diagonal matrices. Prove  $\mathcal{C}(D) = \mathcal{D}$ .

**Solution:** Let the diagonal entries of  $D$  be  $d_1, \dots, d_n$ . Let  $A \in \mathcal{D}$ , with diagonal entries  $a_1, \dots, a_n$ . Then the product

$$AD = \begin{bmatrix} a_1 d_1 & 0 & \dots & 0 \\ 0 & a_2 d_2 & 0 & \vdots \\ \vdots & 0 & \ddots & 0 \\ 0 & \dots & 0 & a_n d_n \end{bmatrix} = \begin{bmatrix} d_1 a_1 & 0 & \dots & 0 \\ 0 & d_2 a_2 & 0 & \vdots \\ \vdots & 0 & \ddots & 0 \\ 0 & \dots & 0 & d_n a_n \end{bmatrix} = DA.$$

So  $\mathcal{D} \subset \mathcal{C}(D)$ .

Let  $B \in \mathcal{C}(D)$  with columns  $\vec{b}_1, \dots, \vec{b}_n$ , rows  $\vec{c}_1, \dots, \vec{c}_n$ , and element of  $i$ th row and

$j$ th column  $b_{ij}$ . Then

$$\begin{aligned} BD &= \left[ \begin{array}{c|ccc} & & & \\ B(d_1 \vec{e}_1) & \cdots & B(d_n \vec{e}_n) & \\ & & & \end{array} \right] \\ &= \left[ \begin{array}{c|ccc} & & & \\ d_1 \vec{b}_1 & \cdots & d_n \vec{b}_n & \\ & & & \end{array} \right] \end{aligned}$$

$$\begin{aligned} DB &= ((DB)^\top)^\top \\ &= (B^\top D)^\top \\ &= \left[ \begin{array}{c|ccc} & & & \\ d_1 \vec{c}_1^\top & \cdots & d_n \vec{c}_n^\top & \\ & & & \end{array} \right]^\top \\ &= \left[ \begin{array}{ccc} - & d_1 \vec{c}_1 & - \\ & \vdots & \\ - & d_n \vec{c}_n & - \end{array} \right] \end{aligned}$$

where  $\vec{e}_i$  is the  $i$ th standard basis vector. Since  $B \in \mathcal{C}(D)$ ,  $BD = DB$ . Considering arbitrary  $b_{ij}$ , this means that  $d_i b_{ij} = d_j b_{ij}$ .

When  $i = j$ , then  $d_i = d_j$ , so  $b_{ij}$ , a diagonal element of  $B$ , can be anything.

When  $i \neq j$ , then  $d_i \neq d_j$  since  $D$  has distinct diagonal elements. But  $d_i b_{ij} = d_j b_{ij}$ . So  $b_{ij} = 0$ . Note that in this case,  $b_{ij}$  is any non-diagonal element. Since only diagonal elements of  $B$  can be nonzero,  $B$  is diagonal, and  $B \in \mathcal{D}$ . So  $\mathcal{C}(D) \subset \mathcal{D}$ . Thus  $\mathcal{D} = \mathcal{C}(D)$ .

- (b) Prove that if  $A$  and  $B$  are simultaneously diagonalizable  $n \times n$  matrices, then  $B \in \mathcal{C}(A)$ .

**Solution:** We know that  $\mathcal{D} \subset \mathcal{C}(D)$  for any diagonal matrix  $D$  by part (a). (The  $\mathcal{D} \subset \mathcal{C}(D)$  direction did not depend on the fact that  $D$  had distinct diagonal elements.) So then

$$\begin{aligned} S^{-1}BSS^{-1}AS &= S^{-1}ASS^{-1}BS \\ S^{-1}BAS &= S^{-1}ABS \\ BA &= AB \end{aligned}$$

Thus  $B \in \mathcal{C}(A)$ .

- (c) Prove that if  $A$  and  $B$  are  $n \times n$  matrices such that  $A$  has  $n$  distinct eigenvalues and  $B \in \mathcal{C}(A)$ , then  $A$  and  $B$  are simultaneously diagonalizable.

**Solution:** Let  $S$  be an eigenbasis of  $A$ . Then  $S^{-1}AS$ , the diagonalized matrix of  $A$ , has distinct diagonal elements. We know that

$$\begin{aligned} BA &= AB \\ S^{-1}BAS &= S^{-1}ABS \\ S^{-1}BSS^{-1}AS &= S^{-1}ASS^{-1}BS \end{aligned}$$

So  $S^{-1}BS \in \mathcal{C}(S^{-1}AS)$ . Then by part (a), since  $S^{-1}AS$  is diagonal with distinct entries,  $S^{-1}BS$  is diagonal. So  $S$  diagonalizes both  $B$  and  $A$ . So  $A$  and  $B$  are simultaneously diagonalizable.

3. (a) Suppose that  $A$  has eigenvalue 0 but is not diagonalizable. Prove that  $\text{im}(A) = E_0$ , and conclude from this that  $A^2 = 0$ .

**Solution:** The givens imply that  $\det(A - 0I_1) = 0$ . So  $\det(A) = 0$ . But since it is not diagonalizable, there must be one eigenvalue  $\lambda$  of  $A$  which does not satisfy  $\text{almu}(\lambda) = \text{gemu}(\lambda)$ . Since both  $\text{almu}$  and  $\text{gemu}$  are always at least 1,  $\text{almu} \geq \text{gemu}$ , and the sum of  $\text{almu} = n$ ; we know  $\text{almu}(0) = 2$ , and  $\text{gemu}(0) = 1$ .

So  $\text{nullity}(A - 0I_2) = \text{nullity}(A) = 1$ .

- (b) Let  $\lambda \in \mathbb{R}$  and suppose that  $A$  has eigenvalue  $\lambda$  but is not diagonalizable. Prove that we have  $(A - \lambda I_2)^2 = 0$ , and deduce from this that  $A\vec{v} - \lambda\vec{v} \in E_\lambda$  for every  $\vec{v} \in \mathbb{R}^2$ . [Hint: apply part (a) to the matrix  $A - \lambda I_2$ ].