MATH 215 FALL 2023
Homework Set 4: §14.1 – 14.5
Zhengyu James Pan (jzpan@umich.edu)

1. Do Exercise 32 of §14.1 of Stewart's Multivariable Calculus.

2. Do Exercises 61-66 of §14.1 of Stewart's Multivariable Calculus.

3. Do Exercise 6 of §14.3 of Stewart's Multivariable Calculus.

4. (a) Suppose $g(x, y) = \sqrt{9 - 9x^2 - y^2}$. Draw a contour map for g and then sketch the graph of g.

   (b) Draw a contour map of the function $m(x, y) = \frac{x}{(x2+3y2)}$, showing and labelling several level curves. curves.

5. (a) Use a linear approximation to estimate $(0.99)3 + (2.01)3 - 6(0.99)(2.01)$.

   (b) Let $f(x, y) = xe^{y^2} - ye^{x^2}$ and find the equation for the tangent plane to the graph of f at (1, 2).

   (c) What point on the surface $z = x^2 - y^2$ has a tangent plane parallel to the plane found in the previous part?

6. The wave heights h in the open sea depend on the speed v of the wind and the length of time t that the wind has been blowing at that speed. Values of the function $h = f(v, t)$ are recorded in feet in the following table:

Duration (hours)

Wind speed (knots)	v \ t	5	10	15	20	30	40	50
	10	2	2	2	2	2	2	2
	15	4	4	5	5	5	5	5
	20	5	7	8	8	9	9	9
	30	9	13	16	17	18	19	19
	40	14	21	25	28	31	33	33
	50	19	29	36	40	45	48	50
	60	24	37	47	54	62	67	69

   (a) What are the meanings of the partial derivatives $\frac{\delta h}{\delta v}$ and $\frac{\delta h}{\delta t}$?

(b) Estimate the values of $f_v(40, 15)$ and $f_t(40, 15)$. What are the practical interpretations of these values?

(c) Estimate the values of $f_{vv}(30, 20)$, $f_{tt}(30, 20)$, $f_{vt}(30, 20)$, and $f_{tv}(30, 20)$. Are your answers for f_{tv} the same as for f_{vt}? Should they be? Explain. Hint: This problem might be trickier than it looks.

7. Determine which of the following functions is a solution to Laplace's equation uxx + uyy = 0:

(a)

8.

9. Consider the function

$$f(x, y) = \begin{cases} \frac{x^3y-xy^3}{x^2+y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) \neq (0, 0) \end{cases}$$

(a) Based on the plot of the level curves above, does it appear that f is continuous at (0, 0)? Explain.

Solution: No, it does not appear to be continuous because the contour lines are not continued at (0, 0). Thus, it looks like an open point or sudden jump in the function.

(b) Find $f_x(x, y)$ and $f_y(x, y)$ when $(x, y) \neq (0, 0)$.

Solution:

$$\begin{aligned} f_x(x, y) &= \frac{(x^2+y^2)(3x^2y-y^3)-(x^3y-xy^3)(2x)}{(x^2+y^2)^2} \\ &= \frac{3x^4y-x^2y^3+3x^2y^3-y^5-2x^4y+2x^2y^3}{(x^2+y^2)} \\ &= \boxed{\frac{x^4y+4x^2y^3-y^5}{(x^2+y^2)^2}} \end{aligned}$$

$$\begin{aligned} f_y(x, y) &= \frac{(x^2+y^2)(x^3-3xy^2)-(x^3y-xy^3)(2y)}{(x^2+y^2)^2} \\ &= \frac{x^5-3x^3y^2+x^3y^2-3xy^4-2x^3y^2+2xy^4}{(x^2+y^2)^2} \\ &= \boxed{\frac{x^5-4x^3y^2-xy^4}{(x^2+y^2)^2}} \end{aligned}$$

□

(c) Can you use your answers from the previous part to find $f_x(0, 0)$ and $f_y(0, 0)$? Explain.

Solution: Yes, since the piecewise $f(0,0) = 0$ patches the hole in f, so we can take the derivative at (0, 0).

(d) Using the definition of the partial derivative, find $f_x(0,0)$ and $f_y(0,0)$.

Solution:

$$\begin{aligned}
f_x(0,0) &= \lim_{h\to 0} \frac{f(0+h,0) - f(0,0)}{h} \\
&= \lim_{h\to 0} \frac{\frac{(x+h)^3y-(x+h)y^3}{(x+h)^2+y^2} - f(0,0)}{h} \\
&= \lim_{h\to 0} \frac{\frac{0-0}{x^2+2hx+h^2} - 0}{h} \\
&= \lim_{h\to 0} \frac{\frac{0}{h^2}}{h} \\
&= \boxed{0}
\end{aligned}$$

$$\begin{aligned}
f_y(0,0) &= \lim_{h\to 0} \frac{f(0,0+h) - f(0,0)}{h} \\
&= \lim_{h\to 0} \frac{\frac{x^3(y+h)-x(y+h)^3}{x^2+(y+h)^2} - 0}{h} \\
&= \lim_{h\to 0} \frac{\frac{0}{h^2} - 0}{h} \\
&= \boxed{0}
\end{aligned}$$

(e) Using the definition of the partial derivative, show $f_{xy}(0,0) = -1$ and $f_{yx}(0,0) = 1$.

(f) Does the result from the previous part contradiction Clairaut's Theorem? Justify you reasoning. Hint: Contour plots, possible generated using something similar to how we plotted a contour from MatLab above, might be a way to help bolster your explanation.