**MATH 215 FALL 2023**
**Homework Set 8: §15.7 – 16.1**
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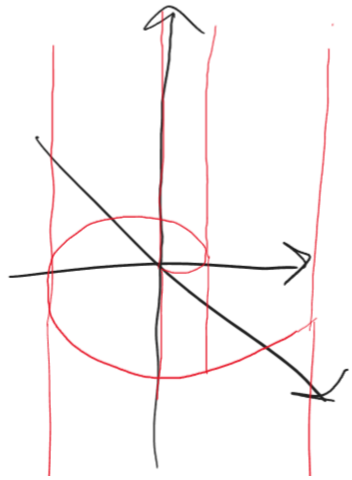
1. For the following problem, take $r, \theta, \rho$, and $\phi$ to have the standard definitions in cylindrical and spherical coordinates. Describe (and try to sketch) the following surfaces:

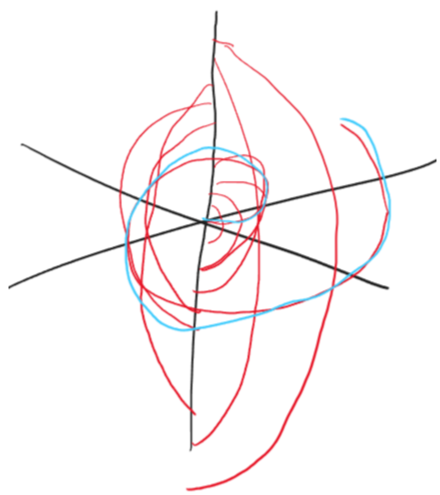
   (a) $r = \theta$

   **Solution:** A cylinder through a spiral starting from the origin.

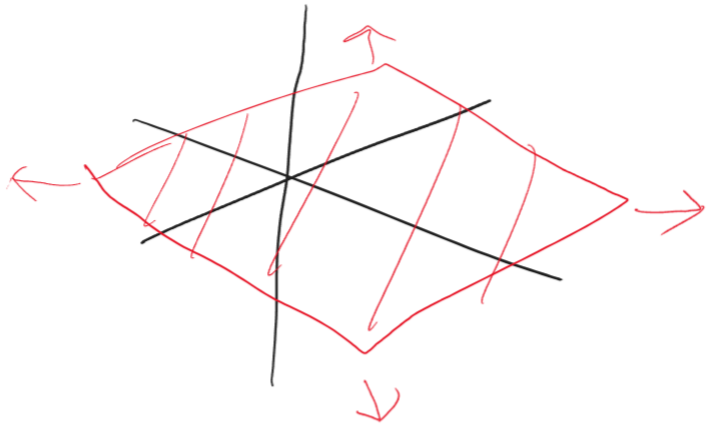
   (b) $\rho = \theta$

   **Solution:** A spiral in the $xy$ plane, where each point has vertical arcs of circles passing through them to the line $x = y = 0$, each with the origin as their center.

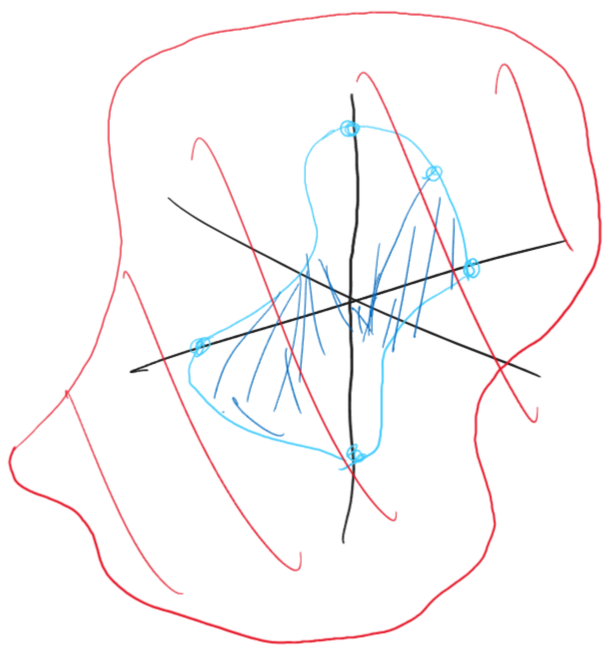
   (c) $r = \rho$

   **Solution:** The $xy$ plane.



(d) $\theta = \phi$

**Solution:** A curved surface. When a curve is drawn on this surface with $\rho$ fixed, the curve looks similar to a sin curve when viewed from the y-axis.



2. Let $E$ be the ball of radius 1 centered at the point (0, 0, 1).

   (a) Show that $E$ is given in Cartesian coordinates by the equation $x^2+y^2+z^2-2z \leq 0$.

**Solution:**

$$\begin{aligned} x^2 + y^2 + (z-1)^2 &\leq 1 \\ x^2 + y^2 + z^2 - 2z + 1 &\leq 1 \\ x^2 + y^2 + z^2 - 2z &\leq 0 \end{aligned}$$

□

(b) Write $E$ in spherical coordinates. Make sure to specify the domain of $\rho, \theta$, and $\phi$.

**Solution:**

$$x = \rho\sin(\phi)\cos(\theta), y = \rho\sin(\phi)\sin(\theta), z = \rho\cos(\phi)$$

$$\begin{aligned} (\rho\sin(\phi)\cos(\theta))^2 + (\rho\sin(\phi)\sin(\theta))^2 + (\rho\cos(\phi))^2 - 2(\rho\cos(\phi)) &\leq 0 \\ \rho^2\sin^2(\phi)\cos^2(\theta) + \rho^2\sin^2(\phi)\sin^2(\theta) + \rho^2\cos^2(\phi) - 2\rho\cos(\phi) &\leq 0 \\ \rho^2\sin^2(\phi)\cos^2(\theta) + \rho^2\sin^2(\phi)\sin^2(\theta) + \rho^2\cos^2(\phi) - 2\rho\cos(\phi) &\leq 0 \\ \rho^2\sin^2(\phi)(\cos^2(\theta) + \sin^2(\theta)) + \rho^2\cos^2(\phi) - 2\rho\cos(\phi) &\leq 0 \\ \rho^2 - 2\rho\cos(\phi) &\leq 0 \\ \rho(\rho - 2\cos(\phi)) &\leq 0 \end{aligned}$$

$$\boxed{0 \leq \rho \leq 2\cos(\phi), 0 \leq \phi \leq -\frac{\pi}{2}, 0 \leq \theta \leq 2\pi}$$

□

(c) Suppose the density on $E$ is proportional to the distance to the origin, with the largest density being equal to 2. Use spherical coordinates to compute the mass and center of mass of $E$.

**Solution:** A density equal to $\rho$ satisfies these conditions.

$$\begin{gathered} M = 2\pi \int_0^{\frac{\pi}{2}} \int_0^{2\cos(\phi)} \rho^3 \sin(\phi)\, d\rho\, d\phi \\ 2\pi \int_0^{\frac{\pi}{2}} \sin(\phi) \left[\frac{\rho^4}{4}\right]_{\rho=0}^{2\cos(\phi)} d\phi \\ 2\pi \int_0^{\frac{\pi}{2}} \sin(\phi) 4\cos^4(\phi)\, d\phi \\ u = \cos(\phi), du = -\sin(\phi)\, d\phi \\ 2\pi \int_1^0 -4u^4\, d\phi \\ -8\pi \left(\frac{u^5}{5}\right)_{u=1}^{0} \\ M = \boxed{\frac{8\pi}{5}} \end{gathered}$$

By symmetry, $\overline{x} = \overline{y} = \overline{\phi} = \overline{\theta} = 0$. So, to find $\overline{z}$, we can actually find $\overline{\rho}$:

$$\overline{\rho} = \frac{5}{8\pi} 2\pi \int_0^{\frac{\pi}{2}} \int_0^{2\cos(\phi)} \rho^4 \sin(\phi)\, d\rho\, d\phi$$

$$\frac{5}{4} \int_0^{\frac{\pi}{2}} \sin(\phi) \left[\frac{\rho^5}{5}\right]_{\rho=0}^{2\cos(\phi)} d\phi$$

$$\frac{1}{4} \int_0^{\frac{\pi}{2}} \sin(\phi) 32 \cos^5(\phi)\, d\phi$$

$$u = \cos(\phi), du = -\sin(\phi)\, d\phi$$

$$\int_1^0 -8u^5\, d\phi$$

$$-\frac{8}{6}\left[u^6\right]_{u=1}^0$$

$$\overline{z} = \overline{\rho} = \frac{4}{3}$$

$$\boxed{(\overline{x}, \overline{y}, \overline{z} = (0, 0, \frac{4}{3})}$$

□

(d) Suppose we tried to do this problem for the ball of radius 1 centered at the point (0, 1, 0). Why is this problem harder with the new ball?

**Solution:** This is harder because the region of integration is not as simply described by any coordinate system. For instance, in spherical the region would be $0 \leq \rho \leq 2\sqrt{\sin^2(\theta) + \sin^2(\phi)}, 0 \leq \theta \leq \pi, 0 \leq \phi \leq \pi$. These bounds are much more annoying to integrate due to the square root for the upper bound of $\rho$.

3. Begin with a sphere of radius $R$ and bore a hole into the sphere in the shape of a right circular cylinder, leaving only a band of height $h$. Find the volume of the resulting shape.

**Solution:** The radius of the cylinder will be $r_c = \sqrt{R^2 - h^2}$. We use cylindrical coordinates to perform the integration.

$$\begin{aligned}
&2\pi \int_{-h}^{h} \int_{\sqrt{R^2-h^2}}^{\sqrt{R^2-z^2}} r\,dr\,d\theta \\
&= \pi \int_{-h}^{h} \left(r^2\right) |_{\sqrt{R^2-h^2}}^{\sqrt{R^2-z^2}}\,dr\,d\theta \\
&= \pi \int_{-h}^{h} R^2 - z^2 - R^2 + h^2\,d\theta \\
&= \pi \left(-\frac{z^3}{3} + h^2 z\right) |_{z=-h}^{h} \\
&= \boxed{\frac{4\pi h^3}{3}}
\end{aligned}$$

□

4. Find the mass of a wedge cut from a sphere of radius $R$ by two planes that intersect along a diameter and at an angle of $\frac{\pi}{5}$, assuming that the density is proportional to the distance from the origin in such a way that the maximum density is 2. (This shape should look like a segment of an orange.)

**Solution:** We use spherical coordinates for this problem, with $(r, \theta, \phi)$. The density function will be $\rho(r) = \frac{2r}{R}$ to have a maximum density of 2 when the distance is equal to the radius.

$$\begin{aligned}
&\frac{\pi}{5} \int_0^R \int_0^\pi \frac{2r}{R} r^2 \sin(\phi)\,d\phi\,dr \\
&= \frac{\pi}{5R} \int_0^R 2r^3 \int_0^\pi \sin(\phi)\,d\phi\,dr \\
&= \frac{\pi}{5R} \int_0^R 2r^3 \left(-\cos(\phi)\right) |_{\phi=0}^{\pi}\,dr \\
&= \frac{\pi}{5R} \int_0^R 4r^3\,dr \\
&= \frac{\pi}{5R} \left(r^4\right) |_{r=0}^{R} \\
&= \boxed{\frac{\pi R^3}{5}}
\end{aligned}$$

□

5. Find $\int\int_R f(x, y)\,dA$ where $f(x, y) = 3y^2 - 4xy - 4x^2$ and $R$ is the quadrilateral with vertices (0, 2), (3, 0), (5, 4), and (2, 6). *Hint*: There may be a straightforward but tedious way to solve this problem, as well as a faster, more subtle, way to solve this problem.

**Solution:** We can factor $f(x,y) = (3y+2x)(y-2x)$. Then, we can use change of variables to change both the function and the bounds. Let $u = 3y+2x$, $v = y-2x$. Then $f(u,v) = uv$, $d(x,y) = (2-3(-2)))^{-1}\, d(u,v) = \frac{1}{8}d(u,v)$. Also, $R$ has vertices at $(u,v) = (6,2), (6,-6), (22,-6), (22,2)$.

$$\begin{aligned}
&\frac{1}{8}\int_{-6}^{2}\int_{6}^{22} uv\,du\,dv \\
&= \frac{1}{24}\int_{-6}^{2} v\left[(22)^2 - (6)^2\right]\,dv \\
&= \frac{1}{16}\int_{-6}^{2} 448v dv \\
&= \frac{1}{16}\left(224v^2\right)\big|_{v=-6}^{2} \\
&= \frac{1}{16}\cdot(-7168) \\
&= \boxed{-448}
\end{aligned}$$

6. Let $E$ be the region in the first quadrant that is above the line $y = \frac{x}{3}$, below the line $y = 3x$, and between the curves defined by $xy = 3$ and $xy = 27$.
   (a) Sketch the region.
   (b) Evaluate $\int\int(\frac{x^2}{y^2} + x^2y^2)\,dA$. (Hint: Try $u = xy$ and $v = \frac{y}{x}$.)
   (c) Why was the hint a reasonable guess for a change of coordinates?

7. Do Exercises 13-18 of §16.1 in *Stewart's Multivariable Calculus.*

**Solution:**

13. $\boxed{IV}$ – vectors with direction and magnitude equal to displacement, except flipped vertically.

14. $\boxed{V}$ – downward direction when $x < y$, upward when $y < x$, horizontal when $x = y$.

15. $\boxed{I}$ – when $y = -2$, vectors are horizontal.

16. $\boxed{VI}$ – magnitude increases more with $x$ than $y$.

17. $\boxed{III}$ – the magnitude/direction oscillates when either coordinate is fixed.

18. $\boxed{II}$ – direction becomes more vertical when x increases, while horizontal component oscillates.

8. Do Exercises 19-22 of §16.1 in *Stewart's Multivariable Calculus*.

**Solution:**

19. $\boxed{IV}$ – only constant vector field.

20. $\boxed{I}$ – the vector field is constant when z is fixed.

21. $\boxed{III}$ – always positive vertical direction, same direction as displacement from origin for $x$ and $y$.

22. $\boxed{II}$ – same direction/magnitude as displacement from origin.

9. Do Exercises 31-34 of §16.1 in *Stewart's Multivariable Calculus*.

**Solution:**

31. $\boxed{III}$ – gradient is $(2x, 2y)$, so linearly increasing magnitude and same direction as displacement from origin.

32. $\boxed{IV}$ – gradient is $(2x+y, x)$, thus the direction is close to horizontal near the y-axis and becomes more vertical as x increases.

33. $\boxed{II}$ – gradient is $(2x + 2y, 2y + 2x)$. Since the $x$ and $y$ coordinates are the same, the direction is always the same $\langle 1, 1\rangle$, except with positive or negative magnitude.

34. $\boxed{I}$ – Gradient will include something with cos for both $f_x$ and $f_y$ coordinates, thus the magnitude will oscillate.