EECS 203: Discrete Mathematics
Winter 2024
Homework 3

Due **Thursday, Feb. 8**, 10:00 pm

No late homework accepted past midnight.

Number of Problems: $7 + 1$ Total Points: $100 + 30$

- **Match your pages!** Your submission time is when you upload the file, so the time you take to match pages doesn't count against you.
- Submit this assignment (and any regrade requests later) on Gradescope.
- Justify your answers and show your work (unless a question says otherwise).
- By submitting this homework, you agree that you are in compliance with the Engineering Honor Code and the Course Policies for 203, and that you are submitting your own work.
- Check the syllabus for full details.

1. On the Contrary [12 points]

Let n be an integer. Prove that if $4 \mid (n^2 - 1)$, then n is odd using

(a) a proof by contraposition, and

(b) a proof by contradiction.

Then,

(c) compare your answers to parts (a) and (b). What is different? What is the same?

Solution:

(a) We will prove the contrapositive. Assume n is even. Then n can be expressed as $2m$, where m is an integer. Then $n^2 - 1 = (2m)^2 - 1 = 4m^2 - 1$. m^2 is an integer, so $n^2 - 1$ is not divisible by 4. Thus the contrapositive is true, and the original statement is also true.

(b) Assume $4 \mid (n^2 - 1)$ and n is even. Then n can be expressed as $2m$, where m is an integer. Then $n^2 - 1 = (2m)^2 - 1 = 4m^2 - 1$. m^2 is an integer, so $n^2 - 1$ is not divisible by 4. This contradicts our original assumption, so the assumption is false.

(c) The assumption for contradiction contains one more clause. The contrapositive proof is to prove a truth, while the contradiction proves falsity. However, most of the steps taken to prove these two are the same, such as using $n = 2m$.

2. An Even-Numbered Question about Even Numbers [16 points]

Prove or disprove the following statements:

(a) For all integers x, if x is even, then x^2 is even.

(b) For all integers x, if x^2 is even, then x is even.

(c) For all integers x, if x is even, then $2x$ is even.

(d) For all integers x, if $2x$ is even, then x is even.

Solution:

(a) Assume x is even. Then x can be expressed $2m$, where m is an integer. Then $x^2 = (2m)^2 = 4m^2 = 2(2m^2)$. $2m^2$ is an integer since m is an integer, so x^2 is even.

(b) We will prove the contrapositive. Assume x is odd. Then x can be written $2k+1$ where k is an integer. Then x^2 can be expressed $(2k+1)^2 = 4k^2+4k+1 = 2(2k^2+2k)+1$. So x^2 is not even when x is odd, and the given statement is true.

(c) $2x$ is always even by definition, so the implication is true.

(d) False. Consider $x = 1$. $2x = 2$ is even, but x is odd.

3. Even Stevens [16 points]

Prove or disprove the following statement: "There is a finite amount of even numbers."

Solution:

Assume the statement is true, and let $2n$ be the largest of these even numbers, where n is an integer. Then $2n+2 = 2(n+1)$ is even by definition, so $2n$ was not the largest even number and there is a contradiction. Thus the statement is false.

4. Pay it Forward (Or Don't, It's Up To You) [12 points]

Consider a centipede game, where there are two players: Ka-chun and Zyaire. The game starts by Ka-chun's decision of take or wait.

- If Ka-chun takes, Ka-chun earns \$1 while Zyaire earns nothing, and the game ends.
- If Ka-chun waits, then Zyaire can choose between take or wait. If Zyaire takes, Zyaire earns \$2 while Ka-chun earns nothing and the game ends. If Zyaire waits it becomes Ka-chun's turn to choose again.
- If they keep waiting the reward grows by \$1 each round, until Zyaire's choice of taking \$20 or waiting, when the game will end no matter what.

Both of Ka-chun and Zyaire want to maximize their rewards, and behave as perfect logicians.

(a) Suppose Ka-chun and Zyaire made it to round 20. What happens in round 20?

(b) Using your answer to (a), what would happen if they made it to round 19?

(c) Building off of parts (a) and (b), argue that Ka-chun should take \$1 in the very first round.

Solution:

(a) In round 20, Zyaire is required to take the \$20.

(b) In round 19, Ka-chun knows Zyaire will take the \$20 next round, so he will take \$19 in this round to maximize his reward, instead of getting nothing next round.

(c) In round 18, Zyaire will have the same reasoning as Ka-chun in round 19, so he would take the money in this round. Ka-chun, knowing this, would then take the money in round 17. This pattern extends similarly until the first round, where Ka-chun knows Zyaire will to take the money in the next round. So, Ka-chun should take the \$1 in the first round.

5. Proofs to the Max [12 points]

Prove that for all real numbers a, b, and c, if $\max\{a^2(b-c), -a\}$ is non-negative, then $a \leq 0$ or $b \geq c$.

Note: You can use the following facts in your proof:

- If x and y are positive, then $x \cdot y$ is positive.
- If x is positive and y is negative, then $x \cdot y$ is negative.
- If x and y are negative, then $x \cdot y$ is positive.

Solution:

Assume $\max\{a^2(b-c), -a\}$ is non-negative. We will use casework. There are 2 possible cases, $\max\{a^2(b-c), -a\} = a^2(b-c)$ or $\max\{a^2(b-c), -a\} = -a$.

- (Case 1) Assume $\max\{a^2(b-c), -a\} = a^2(b-c)$. Then $a^2(b-c)$ is nonnegative.
  - If $a^2(b-c) = 0$, one or both of the factors is 0 by the zero product property (0 can only be the product of 0 and some other number). If $a^2 = 0$, then $a = 0 \leq 0$ and the statement is true. If $b - c = 0$, then $b = c$ and the statement is true.
  - If $a^2(b-c) > 0$, both a^2 and $b-c$ are simultaneously positive or negative by the given facts. It is impossible for a^2 to be negative by the given facts; $a \cdot a$ is either positive·positive or negative·negative. This would result in a contradiction if both factors were negative, so they must both be positive.

    If they are positive, then $b - c > 0 \Rightarrow b \geq c$, which satisfies the original statement.

- (Case 2) Assume $\max\{a^2(b-c), -a\} = -a$. Then $-a \geq 0$. Adding a to both sides, we find $0 \geq a$, making the statement true.

We have shown that the statement holds for all cases, so it is true.

6. Let's All Be Rational [16 points]

Show that these statements about a real number x are equivalent to each other:

(i) x is rational
(ii) $\frac{x}{2}$ is rational
(iii) $3x - 1$ is rational.

Hint: One way to prove statements (i), (ii) and (iii) are equivalent is by proving (i) $\rightarrow$ (ii), (ii) $\rightarrow$ (iii), and (iii) $\rightarrow$ (i).

Solution:

Assume x is rational. Then it can be expressed $\frac{p}{q}$, where $p, q \in \mathbb{Z}$ are relatively prime and q is nonzero. Then $\frac{x}{2} = \frac{p}{2q}$. Since $2q \neq 0 \in \mathbb{Z}$, this is also a rational number. So (i) $\rightarrow$ (ii).

Assume $\frac{x}{2}$ is rational. Then it can be expressed $\frac{p}{q}$, where $p, q \in \mathbb{Z}$ are relatively prime and q is nonzero. Then $3x - 1 = \frac{6p}{q} - 1 = \frac{6p-q}{q}$. Since p and q are integers, $6p - q$ is an integer. Also, $q \neq 0$. Thus $3x - 1$ is rational by definition. So (ii) $\rightarrow$ (iii).

Assume $3x - 1$ is rational. Then it can be expressed $\frac{p}{q}$, where $p, q \in \mathbb{Z}$ are relatively prime and q is nonzero. Then $x = \frac{(3x-1)+1}{3} = \frac{\frac{p+q}{q}}{3} = \frac{p+q}{3q}$. Since p and q are integers, $p+q$ and $3q$ are integers, $3q \neq 0$, and x is rational by definition. So (iii) $\rightarrow$ (i).

We have proven (i) $\rightarrow$ (ii), (ii) $\rightarrow$ (iii), and (iii) $\rightarrow$ (i). Thus (i), (ii) and (iii) are equivalent.

7. Irrational Pr00f [16 points]

Prove or disprove that the product of a nonzero rational number and an irrational number is irrational.

Solution:

Let $r = \frac{a}{b}$ be a nonzero rational number with $a, b \in \mathbb{Z}$, with $a, b \neq 0$. Let t be an irrational number. Assume the product rt is rational and can be expressed $rt = \frac{p}{q}$. Then

$$\begin{aligned} t = \frac{rt}{r} &= \frac{\frac{p}{q}}{\frac{a}{b}} \\ &= \frac{bp}{aq} \end{aligned}$$

$bp, aq \in \mathbb{Z}$ because $b, p, a, q \in \mathbb{Z}$. This means t is rational, which is a contradiction. Thus, our assumption that rt is rational was impossible. So the statement is true.