

**MATH 217 W24 - LINEAR ALGEBRA, Section 001 (Dr. Paul Kessenich)**  
**Homework Set Part B due Thursday, January 25 at 11:59pm**  
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1. In parts (a) - (d) below, determine whether the given function is injective, surjective, both, or neither. Justify your answers.

(a) the function  $f : [0, 4] \rightarrow [0, 18]$  defined by  $f(x) = x^2 + 2$ ;

**Solution:** Injective. If  $f(x_1) = f(x_2)$ , it follows that  $x_1^2 = x_2^2$ . Since the domain is positive, this also means  $x_1 = x_2$ , showing injectivity. There is no solution in the domain to  $f(x) = 0$ , so there exists a value in the codomain which is not in the image of  $f$ . Thus, the function is not surjective.

(b) the function  $g : \mathbb{R} \rightarrow \mathbb{R}$  defined by  $g(x) = 2x - 5$ ;

**Solution:** Bijective. If  $g(x_1) = g(x_2)$ ,  $2x_1 - 5 = 2x_2 - 5$ . Therefore,  $x_1 = x_2$ , showing injectivity. Let  $y \in \mathbb{R}$ , and  $x = \frac{y+5}{2}$ . Then  $x \in \mathbb{R}$ , and  $g(x) = y$ . Thus  $g$  is surjective.

(c) the function  $h : \mathbb{R}^2 \rightarrow \mathbb{R}$  defined by  $h(x, y) = 2x^2 + 5y^2$ ;

**Solution:** Neither.  $10 = h(\sqrt{5}, 0) = h(0, \sqrt{2})$ , so  $h$  is not injective.  $h(x, y) = -2$  has no solutions in  $\mathbb{R}^2$  since a square cannot be a negative number, therefore  $h$  is not surjective.

(d) the function  $q : \mathbb{N} \rightarrow \mathbb{N}$  defined by  $q(n) = \begin{cases} n, & \text{if } n \text{ is odd} \\ n/2 & \text{if } n \text{ is even.} \end{cases}$

**Solution:** Surjective.  $1 = q(1) = q(2)$ , so  $q$  is not injective. Let  $m \in \mathbb{N}$ ,  $n = 2m$ . Then  $n \in \mathbb{N}$ ,  $n$  is even, and  $q(n) = m$ . So  $q$  is surjective.

2. Determine whether each statement is true or false. If it is true, prove it. If it is false, prove this by giving a counterexample.

(a) For every function  $f : X \rightarrow Y$  and all  $A, B \subseteq X$ , if  $A \cap B = \emptyset$ , then  $f[A] \cap f[B] = \emptyset$ .

**Solution:** False. Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be defined by  $f(x) = x^2$ . Assign  $A = \mathbb{R}^+$ ,  $B = \mathbb{R}^-$ . Then  $A \cap B = \emptyset$ , but  $f(1 \in A) = f(-1 \in B) = 1$ . Therefore  $f[A] \cap f[B] \neq \emptyset$ .

(b) For every function  $f : X \rightarrow Y$  and all  $A, B \subseteq X$ , if  $f[A] \cap f[B] = \emptyset$ , then  $A \cap B = \emptyset$ .

**Solution:** True. We prove the contrapositive. Take any  $f : X \rightarrow Y$  and  $A, B \subseteq X$  such that  $A \cap B \neq \emptyset$ . Then  $\exists a \in X$  such that  $a \in A \cap B$ . Since  $a \in A$ ,  $f(a) \in f[A]$ . Similarly,  $a \in B$ , so  $f(a) \in f[B]$ . As  $f(a) \in f[A]$  and  $f(a) \in f[B]$ ,  $f(a) \in f[A] \cap f[B]$ . This means that for every function  $f : X \rightarrow Y$  and all  $A, B \subseteq X$ , if  $A \cap B \neq \emptyset$ , then  $f[A] \cap f[B] \neq \emptyset$ . Thus the contrapositive is true, so the original statement is true.

- (c) For every function  $f : X \rightarrow Y$  and all  $A \subseteq X$ , we have  $f^{-1}[f[A]] = A$ .

**Solution:** False. Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be defined by  $f(x) = x^2$ . Assign  $A = \{1\}$ . Then  $f[A] = \{1\}$ . However,  $f(1) = f(-1) = 1$ , so  $f^{-1}[f[A]] = \{-1, 1\} \neq A$ .

- (d) For every function  $f : X \rightarrow Y$  and all  $A \subseteq X$ , we have  $f[X \setminus A] = Y \setminus f[A]$ .

**Solution:** False. Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be defined by  $f(x) = x^2$ . Assign  $A = \{1\}$ . Then  $f[A] = \{1\}$ , but  $f(1) = f(-1) = 1$ . Therefore  $f[A] \subseteq f[X \setminus A]$ , and  $f[X \setminus A] \neq Y \setminus f[A]$ .

- (e) For every bijective function  $f : X \rightarrow Y$  and all  $A, B \subseteq X$ , we have  $f[A \cap B] = f[A] \cap f[B]$ .

**Solution:** True.

Let  $x \in f[A \cap B]$ . Then let  $a = f^{-1}(x)$ . We know  $a \in A \cap B$  is unique due to bijectivity. Since  $a \in A \cap B$ ,  $a \in A$  and  $a \in B$ . Thus  $f[A \cap B] \subseteq f[A] \cap f[B]$ .

Let  $y \in f[A] \cap f[B]$ . Then let  $b = f^{-1}(y)$ . We know  $b$  is unique due to bijectivity. Additionally,  $b \in A$  and  $b \in B$  because  $y \in f[A] \cap f[B]$ . Since  $b \in A \cap B$ ,  $f(b) = y \in f[A \cap B]$ . Thus  $f[A] \cap f[B] \subseteq f[A \cap B]$ .

Both  $f[A] \cap f[B] \subseteq f[A \cap B]$  and  $f[A \cap B] \subseteq f[A] \cap f[B]$ , so  $f[A \cap B] = f[A] \cap f[B]$ .

3. (a) Prove that for every function  $f : \mathbb{R} \rightarrow \mathbb{R}$ , if  $f(cx) = cf(x)$  for all  $c \in \mathbb{R}$  and  $x \in \mathbb{R}$ , then  $f(x + y) = f(x) + f(y)$  for all  $x, y \in \mathbb{R}$ . (In other words, prove that every function  $f : \mathbb{R} \rightarrow \mathbb{R}$  that preserves scalar multiplication is a linear transformation from  $\mathbb{R}$  to  $\mathbb{R}$ .)

**Solution:** Let  $x \neq 0 \in \mathbb{R}$ ,  $y \in \mathbb{R}$ . Then  $y = cx$ , where  $c \in \mathbb{R}$ , by closure of nonzero division in the reals. Additionally,  $f(y) = f(cx) = cf(x)$ . Thus  $f(x + y) = f(x + cx) = f((1 + c)x) = (1 + c)f(x) = f(x) + cf(x) = f(x) + f(y)$ . NEEDS MORE

- (b) Give an example to show that the argument you gave in part (a) cannot work in 2 dimensions. That is, explicitly describe a function  $f : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  that is not a linear transformation but has the property that  $f(c\vec{x}) = cf(\vec{x})$  for all  $\vec{x} \in \mathbb{R}^2$  and  $c \in \mathbb{R}$ . Remember to prove that your example works!