MATH 217 - W24 - LINEAR ALGEBRA
HOMEWORK 5, DUE Sunday, February 18 at 11:59pm

Submit Part A and Part B as two *separate* assignments in Gradescope as a **pdf file.** At the time of submission, Gradescope will prompt you to match each problem to the page(s) on which it appears. **You must match problems to pages in Gradescope so we know what page each problem appears on.** Failure to do so may result in not having the problem graded.

A few words about solution writing:

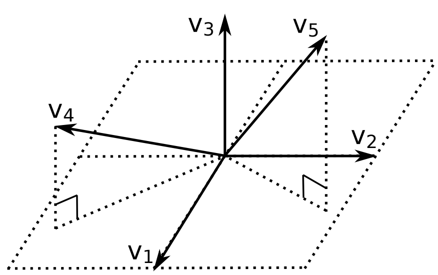
- Unless you are explicitly told otherwise for a particular problem, **you are always expected to show your work and to give justification for your answers.**
- Your solutions will be judged on precision and completeness and not merely on "basically getting it right".
- Cite every theorem or fact from the book that you are using (e.g. "By Theorem 1.10 . . . ").

Part A (10 points)

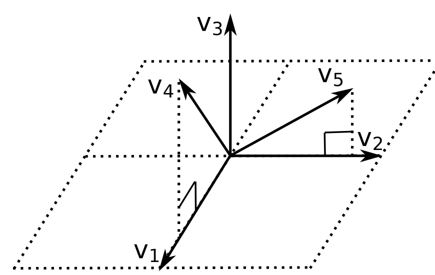
Solve the following problems from the book:

Section 3.2: 56
Section 3.3: 33, 63 (Hint: argue that every basis of V must also be a basis of W).
Section 4.1: 12, 28

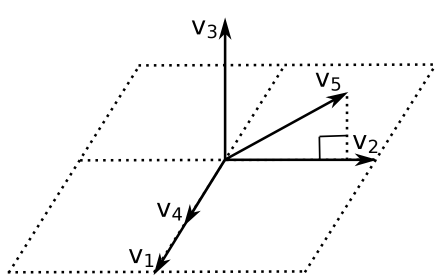
Part A Problem 6. Let $\mathbf{v}_1, \ldots, \mathbf{v}_5$ be vectors in $\mathbb{R}^3$, as shown in the four figures below. In each figure, find *all* linearly dependent sets consisting of three of these five vectors, or else state that there are none if this is the case. *No justification needed.* (Note that in each of these figures, $\mathbf{v}_1$ and $\mathbf{v}_2$ span the displayed plane, $\mathbf{v}_3$ points "up" and is perpendicular to this plane, and for any other vector *not* in the plane, we draw a dotted vertical line indicating its position above the plane.)



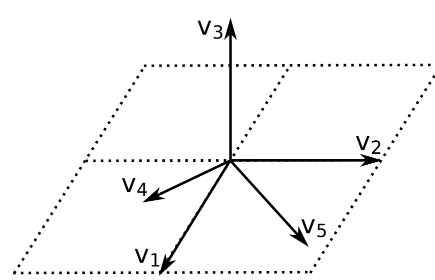
(a)



(b)



(c)



(d)

Part B (25 points)

Problem 1. Let V and W be vector spaces, and let $T : V \to W$ be a linear transformation. Let $X = (\mathbf{x}_1, \dots, \mathbf{x}_k)$ be a list of vectors in V, and consider the list $Y = (T(\mathbf{x}_1), \dots, T(\mathbf{x}_k))$ of vectors in W. Determine whether the following statements are true or false. If true, provide a proof. If false, provide a counter-example.

(a) If X is linearly independent, then Y is also linearly independent.
(b) If Y is linearly independent, then X is also linearly independent.

Problem 2.

(a) Find[1] a linear transformation $T : \mathbb{R}^5 \to \mathbb{R}^3$ such that

$$\ker(T) = \{\vec{x} \in \mathbb{R}^5 \ : \ x_1 = 5x_2 \text{ and } x_3 = 7x_4\} \qquad \text{and} \qquad \operatorname{im}(T) = \{\vec{x} \in \mathbb{R}^3 \ : \ x_1 = x_3\}.$$

(b) Is the linear transformation you found in part (a) unique? Justify your claim.

Problem 3. Let X and Y be vector spaces.

(a) Consider a basis $\mathfrak{B} = \{\mathbf{x}_1, \dots, \mathbf{x}_n\}$ of X. Let $\mathbf{y}_1, \dots, \mathbf{y}_n$ be any vectors (not necessarily a basis, or even distinct) in Y. Prove that there exists a unique linear transformation $T : X \to Y$ such that $T(\mathbf{x}_i) = \mathbf{y}_i$ for all $1 \le i \le n$.
(b) Let U and V be subspaces of X and Y respectively such that $\dim(U) + \dim(V) = \dim(X)$. Prove that there exists a linear transformation $T_{U,V} : X \to Y$ such that $\ker(T_{U,V}) = U$ and $\operatorname{im}(T_{U,V}) = V$. (Hint: use part (a). You might also want to try to generalize the method you used to solve Problem 2.)
(c) In the map $T_{U,V}$ that you found in part (b) unique? Justify your answer.

Problem 4. Let U, V, and W be finite-dimensional vector spaces, and let $T : U \to V$ and $S : V \to W$ be linear transformations. Determine whether the following statements are true or false, and provide a proof of your claim.

(a) $\operatorname{rank}(S \circ T) \le \operatorname{rank}(S)$.
(b) $\operatorname{rank}(S \circ T) \le \operatorname{rank}(T)$.
(c) $\operatorname{nullity}(S \circ T) \ge \operatorname{nullity}(T)$.
(d) $\operatorname{nullity}(S \circ T) \ge \operatorname{nullity}(S)$.

Problem 5. A matrix $A \in \mathbb{R}^{n \times n}$ is said to be *symmetric* if $A^\top = A$, and *skew-symmetric* if $A^\top = -A$. Let Sym_n and Skew_n denote the set of all symmetric matrices and the set of all skew-symmetric matrices in $\mathbb{R}^{n \times n}$, respectively.

(a) Let $T : \mathbb{R}^{n \times n} \to \mathbb{R}^{n \times n}$ be the map defined by $T(A) = A + A^\top$. Prove that T is linear.
(b) Prove that $\ker(T) = \mathrm{Skew}_n$ and $\operatorname{im}(T) = \mathrm{Sym}_n$.
(c) Prove that Sym_n and Skew_n are subspaces of $\mathbb{R}^{n \times n}$.
(d) Find $\dim(\mathrm{Sym}_n)$ and $\dim(\mathrm{Skew}_n)$.

[1]What does "find" mean? Should you go look in your closet? In this context, "find" means to explicitly describe or construct, as in "produce a concrete example and prove that it works." Here you're asked to find a function, and functions are typcially defined by specifying their source and target and a rule for converting inputs to outputs. In this case you're *given* the source and target, so you just need to specify a rule. So your answer should be something lke "Let T be the function defined by $T(\vec{x}) =$?? for all $\vec{x} \in \mathbb{R}^5$." Your job is to decide what ?? should be.