MATH 217 W24 - LINEAR ALGEBRA, Section 001 (Dr. Paul Kessenich)
Homework 11 Part B due SUNDAY, April 21 at 11:59pm
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1. (a) Let E_0 denote the 0-eigenspace of T. Explicitly describe E_0 (as a set).

   Solution:
   $$E_0 = \{(x_1, 0, x_2, 0, x_3, 0, ...) \mid x_i \in \mathbb{R}\}$$

   (b) Prove that every real number λ is an eigenvalue of T. (Hint: explicitly construct an eigenvector $(x_1, x_2, x_3, ...) \in V$. First consider x_i when i is a power of 2.)

   Solution: Let $\lambda \in \mathbb{R}$. Then let
   $$s = (1,\ \lambda, \lambda,\ \lambda^2, \lambda^2, \lambda^2, \lambda^2,\ \lambda^3, \lambda^3, \lambda^3, \lambda^3, \lambda^3, \lambda^3, \lambda^3, \lambda^3,\ ...)$$
   be an infinite sequence such that each consecutive power λ^n is repeated n times in the sequence, starting from $n = 0$. Then
   $$\begin{aligned} T(s) &= (\lambda,\ \lambda^2, \lambda^2,\ \lambda^3, \lambda^3, \lambda^3, \lambda^3,\ \lambda^4, \lambda^4, \lambda^4, \lambda^4, \lambda^4, \lambda^4, \lambda^4, \lambda^4, ...) \\ &= \lambda(s). \end{aligned}$$
   So any real number is an eigenvalue of T.

2. (a) Let $\mathscr{D}$ be a diagonal $n \times n$ matrix with distinct entries along the diagonal, and let $\mathscr{D}$ be the subset of $\mathbb{R}^{n\times n}$ consisting of all diagonal matrices. Prove $\mathscr{C}(D) = \mathscr{D}$.

Solution: Let the diagonal entries of D be $d_1, ..., d_n$. Let $A \in \mathscr{D}$, with diagonal entries $a_1, ...a_n$. Then the product

$$AD = \begin{bmatrix} a_1d_1 & 0 & \dots & 0 \\ 0 & a_2d_2 & 0 & \vdots \\ \vdots & 0 & \ddots & 0 \\ 0 & \dots & 0 & a_nd_n \end{bmatrix} = \begin{bmatrix} d_1a_1 & 0 & \dots & 0 \\ 0 & d_2a_2 & 0 & \vdots \\ \vdots & 0 & \ddots & 0 \\ 0 & \dots & 0 & d_na_n \end{bmatrix} = DA.$$

So $\mathscr{D} \subset \mathscr{C}(D)$.

Let $B \in \mathscr{C}(D)$ with columns $\vec{b}_1, ..., \vec{b}_n$, rows $\vec{c}_1, ..., \vec{c}_n$, and element of ith row and jth column b_{ij}. Then

$$\begin{aligned} BD &= \begin{bmatrix} | & & | \\ B(d_1\vec{e}_1) & \cdots & B(d_n\vec{e}_n) \\ | & & | \end{bmatrix} \\ &= \begin{bmatrix} | & & | \\ d_1\vec{b}_1 & \cdots & d_n\vec{b}_n \\ | & & | \end{bmatrix} \end{aligned}$$

$$\begin{aligned} DB &= \left((DB)^\top\right)^\top \\ &= \left(B^\top D\right)^\top \\ &= \begin{bmatrix} | & & | \\ d_1\vec{c}_1^\top & \cdots & d_n\vec{c}_n^\top \\ | & & | \end{bmatrix}^\top \\ &= \begin{bmatrix} — & d_1\vec{c}_1 & — \\ & \vdots & \\ — & d_n\vec{c}_n & — \end{bmatrix} \end{aligned}$$

where $\vec{e}_i$ is the ith standard basis vector. Since $B \in \mathscr{C}(D), BD = DB$. Considering arbitrary b_{ij}, this means that $d_ib_{ij} = d_jb_{ij}$.

When $i = j$, then $d_i = d_j$, so b_{ij}, a diagonal element of B, can be anything.

When $i \neq j$, then $d_i \neq d_j$ since D has distinct diagonal elements. But $d_ib_{ij} = d_jb_{ij}$. So $b_ij = 0$. Note that in this case, b_ij is any non-diagonal element. Since only diagonal elements of B can be nonzero, B is diagonal, and $B \in \mathscr{D}$. So $\mathscr{C}(D) \subset \mathscr{D}$.

Thus $\mathscr{D} = \mathscr{C}(D)$.

(b) Prove that if A and B are simultaneously diagonalizable $n \times n$ matrices, then

$B \in \mathscr{C}(A)$.

Solution: We know that $\mathscr{D} \subset \mathscr{C}(D)$ for any diagonal matrix D by part (a). (The $\mathscr{D} \subset \mathscr{C}(D)$ direction did not depend on the fact that D had distinct diagonal elements.) So then

$$\begin{aligned} S^{-1}BSS^{-1}AS &= S^{-1}ASS^{-1}BS \\ S^{-1}BAS &= S^{-1}ABS \\ BA &= AB \end{aligned}$$

Thus $B \in \mathscr{C}(A)$.

(c) Prove that if A and B are $n \times n$ matrices such that A has n distinct eigenvalues and $B \in \mathscr{C}(A)$, then A and B are simultaneously diagonalizable.

Solution: Let S be an eigenbasis of A. Then $S^{-1}AS$, the diagonalized matrix of A, has distinct diagonal elements. We know that

$$\begin{aligned} BA &= AB \\ S^{-1}BAS &= S^{-1}ABS \\ S^{-1}BSS^{-1}AS &= S^{-1}ASS^{-1}BS \end{aligned}$$

So $S^{-1}BS \in \mathscr{C}(S^{-1}AS)$. Then by part (a), since $S^{-1}AS$ is diagonal with distinct entries, $S^{-1}BS$ is diagonal. So S diagonalizes both B and A. So A and B are simultaneously diagonalizable.

3. (a) Suppose that A has eigenvalue 0 but is not diagonalizable. Prove that $\operatorname{im}(A) = E_0$, and conclude from this that $A^2 = 0$.

Solution: The givens imply that $\det(A - 0I_1) = 0$. So $\det(A) = 0$. But since it is not diagonalizable, there must be one eigenvalue λ of A which does not satisfy $\text{almu}(\lambda) = \text{gemu}(\lambda)$. Since both almu and gemu are always at least 1, almu $\geq$ gemu, and the sum of almus $= n$; we know $\text{almu}(0) = 2$, and $\text{gemu}(0) = 1$. From this, we infer that the nullity is 1, and the rank is also 1. The almu also implies that the characteristic equation of A is $x^2 = 0$.

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Since

$$\begin{aligned} \det(A - 0I_1) &= (a - x)(d - x) - bc \\ &= ad - bc - (a + d)x + x^2 \end{aligned}$$

and the characteristic polynomial has no x^1 terms, we know that $a + d = 0$, or equivalently that $a = -d$. Additionally, $ad = bc$.

We know that A does not have linearly independent columns, so the columns must be scalar multiples of each other. In fact, since $a = -d$, the ratio of the second column to the first is $\frac{d}{c} = \frac{-a}{c}$. So then the rref form of A is

$$\begin{bmatrix} 1 & \frac{-a}{c} \\ 0 & 0 \end{bmatrix}$$

which has nullspace spanned by $\begin{bmatrix} a \\ b \end{bmatrix}$. But this is also the image of A, since both columns of A are scalar multiples of $\begin{bmatrix} a \\ b \end{bmatrix}$. So the image and nullity of A are the same. Thus

$$A^2\vec{x} = A(A\vec{x}) = 0$$

for any $\vec{x} \in \mathbb{R}^2$. Since $A\vec{x}$ is in $\ker(A)$. So $A^2 = 0$.

(b) Let $\lambda \in \mathbb{R}$ and suppose that A has eigenvalue λ but is not diagonalizable. Prove that we have $(A - \lambda I_2)^2 = 0$, and deduce from this that $A\vec{v} - \lambda\vec{v} \in E_\lambda$ for every $\vec{v} \in \mathbb{R}^2$. [Hint: apply part (a) to the matrix $A - \lambda I_2$].

Solution: Since both almu and gemu are always at least 1, almu $\geq$ gemu, and the sum of almus $= n$; we know $\text{almu}(\lambda) = 2$, and $\text{gemu}(\lambda) = 1$. So the matrix $A - \lambda I_2$ has an eigenvalue of 0, and nullity 1. So $A - \lambda I_2$ has an eigenvalue of 0 and is not diagonalizable.

Then by (a), $(A - \lambda I_2)^2 = 0$, and the image of $A - \lambda I_2$ is equal to its 0-eigenspace. The 0-eigenspace of $A - \lambda I_2$ is in turn equal to its own kernel. We know $\ker(A - \lambda I_2)$ is also the λ-eigenspace of A. So for all $\vec{v} \in \mathbb{R}^2$, $(A - \lambda I_n)\vec{v} = A\vec{v} - \lambda\vec{v} \in E_\lambda$.

(c) Prove that if A has eigenvalue λ but is not diagonalizable, then A is similar to $\begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$.

Solution: By (b), we know that $A\vec{v} - \lambda\vec{v} \in E_\lambda$ for any vector $\vec{v} \in \mathbb{R}^2$. Let $\mathcal{B} = \{\vec{b}_1, \vec{b}_2\}$, where $\vec{b}_2 \in E_\lambda^\perp$ and $\vec{b}_1 = A\vec{b}_2 - \lambda\vec{b}_2$, which is in E_λ by (b). We know that these are linearly independent since they are orthogonal.

Note that this definition of $\mathcal{B}$ ensures that $A\vec{b}_1 = \lambda\vec{b}_1$ and $A\vec{b}_2 = \vec{b}_1 + \lambda\vec{b}_2$.

Then

$$\begin{aligned} [A]_\mathcal{B} &= \begin{bmatrix} [A\vec{b}_1]_\mathcal{B} & [A\vec{b}_2]_\mathcal{B} \end{bmatrix} \\ &= \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix} \end{aligned}$$

So A is similar to $\begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$ by the Change-of-basis of Theorem.

(d) Prove that if A does not have any real eigenvalues, then A is similar to a matrix of the form λQ where Q is an orthogonal matrix and $\lambda > 0$.

Solution: We know A has a pair of complex eigenvalues of the form $a \pm bi$. By Worksheet 26 problem 9, A is then similar to the matrix $B = \begin{bmatrix} a & b \\ -b & a \end{bmatrix}$. Since the columns are already orthogonal, we simply need to normalize them to make the matrix orthogonal, which can be done by dividing by $\lambda = \sqrt{a^2 + b^2}$, which is a positive number. So $Q = \frac{1}{\lambda}B$, and the statement is true.

4. (a) Find a matrix $A \in \mathbb{R}^{2\times 2}$ such that $A \begin{bmatrix} x_n \\ x_{n+1} \end{bmatrix} = \begin{bmatrix} x_{n+1} \\ x_{n+2} \end{bmatrix}$ for every integer $n \geq 0$.

Solution:

$$\boxed{A = \begin{bmatrix} 0 & 1 \\ -13 & 4 \end{bmatrix}}$$

$$A \begin{bmatrix} x_n \\ x_{n+1} \end{bmatrix} = \begin{bmatrix} 0x_n + x_{n+1} \\ -13x_n + 4x_{n+1} \end{bmatrix} = \begin{bmatrix} x_{n+1} \\ x_{n+2} \end{bmatrix}$$

(b) Use part (a) to prove by induction that your matrix A satisfies $A^n \begin{bmatrix} 0 \\ 2 \end{bmatrix} = \begin{bmatrix} x_n \\ x_{n+1} \end{bmatrix}$ for every $n \geq 0$.

Solution: Base case: $A^0 \begin{bmatrix} 0 \\ 2 \end{bmatrix} = I_2 \begin{bmatrix} 0 \\ 2 \end{bmatrix}$

Inductive step: Assume that for some integer $n \geq 0$,

$$A^n \begin{bmatrix} 0 \\ 2 \end{bmatrix} = \begin{bmatrix} x_n \\ x_{n+1} \end{bmatrix}.$$

Then by part (a),

$$\begin{aligned} AA^n \begin{bmatrix} 0 \\ 2 \end{bmatrix} &= A^{n+1} \begin{bmatrix} 0 \\ 2 \end{bmatrix} \\ &= A \begin{bmatrix} x_n \\ x_{n+1} \end{bmatrix} \\ &= \begin{bmatrix} x_{n+1} \\ x_{n+2} \end{bmatrix} \end{aligned}$$

So if n satisfies the statement, then $n+1$ satisfies the statement. Since $n = 0$ satisfies the statement, the statement is true for all $n \geq 0$.

(c) Find all (real or complex) eigenvalues and corresponding eigenvectors for A.

Solution:

$$\begin{vmatrix} 0-\lambda & 1 \\ -13 & 4-\lambda \end{vmatrix} = (-\lambda)(4-\lambda)+13$$
$$= \lambda^2 - 4\lambda + 13$$

$$\lambda = \frac{4 \pm \sqrt{16-52}}{2} = 2 \pm 3i$$

$$2+3i:$$
$$\begin{bmatrix} -2-3i & 1 \\ -13 & 2-3i \end{bmatrix} \to \begin{bmatrix} -2-3i & 1 \\ 0 & 0 \end{bmatrix}$$
$$E_{2+3i} = \text{span}\left(\begin{bmatrix} 1 \\ 2+3i \end{bmatrix}\right)$$

$$2-3i:$$
$$\begin{bmatrix} -2+3i & 1 \\ -13 & 2+3i \end{bmatrix} \to \begin{bmatrix} -2+3i & 1 \\ 0 & 0 \end{bmatrix}$$
$$E_{2+3i} = \text{span}\left(\begin{bmatrix} 1 \\ 2-3i \end{bmatrix}\right)$$

(d) Find an invertible (real or complex) matrix P such that $A = PDP^{-1}$ where D is a diagonal matrix.

Solution:

$$P = \begin{bmatrix} 1 & 1 \\ 2+3i & 2-3i \end{bmatrix}$$

(e) First give an explicit formula for D_n, and then use this to give an explicit formula for A_n.

Solution: $D = \begin{bmatrix} 2+3i & 0 \\ 0 & 2-3i \end{bmatrix}$, so

$$D^n = \begin{bmatrix} (2+3i)^n & 0 \\ 0 & (2-3i)^n \end{bmatrix}$$

So then

$$A^n = P^{-1}D^nP = \begin{bmatrix} 1 & 1 \\ 2+3i & 2-3i \end{bmatrix}^{-1} \begin{bmatrix} (2+3i)^n & 0 \\ 0 & (2-3i)^n \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2+3i & 2-3i \end{bmatrix}.$$

(f) Using parts (b) and (e), give an explicit formula for x_n, the nth term in the sequence.

(Your formula may involve complex numbers, and need not be fully simplified.)

Solution:

$$\begin{aligned}
\begin{bmatrix} x_n \\ x_{n+1} \end{bmatrix} &= A^n \begin{bmatrix} 0 \\ 2 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 1 \\ 2+3i & 2-3i \end{bmatrix}^{-1} \begin{bmatrix} (2+3i)^n & 0 \\ 0 & (2-3i)^n \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2+3i & 2-3i \end{bmatrix} \begin{bmatrix} 0 \\ 2 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 1 \\ 2+3i & 2-3i \end{bmatrix}^{-1} \begin{bmatrix} (2+3i)^n & 0 \\ 0 & (2-3i)^n \end{bmatrix} \begin{bmatrix} 2 \\ 4-3i \end{bmatrix} \\
&= \begin{bmatrix} 1 & 1 \\ 2+3i & 2-3i \end{bmatrix}^{-1} \begin{bmatrix} 2(2+3i)^n \\ (4-3i)(2-3i)^n \end{bmatrix} \\
&= \begin{bmatrix} \frac{3+2i}{6} & \frac{-i}{6} \\ \frac{3-2i}{6} & \frac{i}{6} \end{bmatrix} \begin{bmatrix} 2(2+3i)^n \\ (4-3i)(2-3i)^n \end{bmatrix}
\end{aligned}$$

$$x_n = \frac{3+2i}{6} \cdot 2(2+3i)^n - \frac{i}{6}(4-3i)(2-3i)^n$$

5. (a) Find $[T]_{\mathcal{A}}$.

Solution:

$$[T]_{\mathcal{A}} = \begin{bmatrix} [T(e^{3x})]_{\mathcal{A}} & [T(\cos 2x)]_{\mathcal{A}} & [T(\sin 2x)]_{\mathcal{A}} \end{bmatrix}$$

$$[T(e^{3x})]_{\mathcal{A}} = [3e^{3x}]_{\mathcal{A}} = \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix}$$

$$[T(\cos 2x)]_{\mathcal{A}} = [-2\sin 2x]_{\mathcal{A}} = \begin{bmatrix} 0 \\ 0 \\ -2 \end{bmatrix}$$

$$[T(\sin 2x)]_{\mathcal{A}} = [2\cos 2x]_{\mathcal{A}} = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}$$

$$[T]_{\mathcal{A}} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & -2 & 0 \end{bmatrix}$$

(b) Find all (real or complex) eigenvalues of the matrix $[T]_{\mathcal{A}}$.

Solution: We find the characteristic polynomial.

$$\begin{vmatrix} 3-\lambda & 0 & 0 \\ 0 & 0-\lambda & 2 \\ 0 & -2 & 0-\lambda \end{vmatrix} = (3-\lambda)(-\lambda)^2 + 4(3-\lambda)$$

$$(3-\lambda)(\lambda^2+4) = 0$$

$$\boxed{\lambda = 3, \pm 2i}$$

(c) Viewing the matrix $[T]_{\mathcal{A}}$ as a linear transformation of the complex vector space $\mathbb{C}^3$, find a complex eigenvector for $[T]_{\mathcal{A}}$ for each of the eigenvalues you found in (b).

Solution:

$$[T]_{\mathcal{A}} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$
$$[T]_{\mathcal{A}} \begin{bmatrix} 0 \\ -i \\ 1 \end{bmatrix} = (2i) \begin{bmatrix} 0 \\ -i \\ 1 \end{bmatrix}$$
$$[T]_{\mathcal{A}} \begin{bmatrix} 0 \\ i \\ 1 \end{bmatrix} = (-2i) \begin{bmatrix} 0 \\ i \\ 1 \end{bmatrix}$$

(d) Interpret the eigenvectors you found in (c) as a set of three complex-valued functions

$$\mathcal{B} = (f_1(x), f_2(x), f_3(x))$$

with the property that any complex linear combination of the vectors in A (that is, a linear combination with coefficients in $\mathbb{C}$) can be written as a complex linear combination of the vectors in $\mathcal{B}$, and vice versa.

Solution:

$$\boxed{\mathcal{B} = \left(e^{3x}, \sin(2x) - i\cos(2x), i\cos(2x) + \sin(2x)\right)}$$

6. (a) Show that if I has a real eigenvalue λ than there exists an axis around which the solid object can rotate without wobbling.

   Solution: Assume I has a real eigenvalue λ. Then let $\vec{\omega}$ be in the λ-eigenspace of I. Then $L = I\vec{\omega} = \lambda\vec{\omega}$. So L and $\vec{\omega}$ point in the same direction, since they are scalar multiples. Thus along the axis of rotation of the λ-eigenspace of I, the solid object will not wobble.

   (b) Show that I always has at least one real eigenvalue λ (and hence by (a) there always exists an axis around which a solid object can rotate without wobbling).

   Solution: By the intermediate value theorem, an odd polynomial must always pass through the origin. Thus, it always has one real root. We know that the characteristic polynomial of I has degree 3 by Theorem 7.2.5. So I always has at least one real root, and thus one non-wobbling axis of rotation.

   (c) Show that if $\text{gemu}(\lambda) = 3$ then the solid object can rotate around any axis without wobbling.

   Solution: If $\text{gemu}(\lambda) = 3$, then $I - \lambda I_3$ has nullity 3, so $I - \lambda I_3 = 0$. Thus $I = \lambda I_3$. So for any $\vec{\omega} \in \mathbb{R}^3$, $L = \lambda I_3\vec{\omega} = \lambda\omega$. So for any axis of angular velocity, the angular momentum will be in the same direction, and no wobbling will occur.

   (d) Show that if I has three distinct real eigenvalues then there exist three axes around which the solid object can rotate without wobbling.

   Solution: Let I have distinct eigenvalues $\lambda_1, \lambda_2, \lambda_3$. Then the eigenspaces corresponding to these eigenvalues are all axes about which the solid object can rotate without wobbling. WLOG (between $\lambda_1, \lambda_2, \lambda_3$) let $\vec{\omega}_1 \in E_{\lambda_1}$. Then $L = I\vec{\omega}_1 = \lambda\vec{\omega}_1$. So the object will not wobble about the axis of the λ_1-eigenspace.

   Similarly for the λ_2- and λ_3-eigenspaces. So there exist 3 axes about which the solid object does not wobble.

   (e) Prove that for any solid object, there exist three perpendicular axes of rotation around which the object will not wobble.

   Solution: From the formula for the (i, j)-component, the (i, j) component is the same as the (j, i) component (by commutativity of multiplication). So I is symmetric. Then by the Spectral Theorem, there exists an orthogonal S such that $S^{-1}IS$ is diagonal. Then S is an orthogonal eigenbasis of I.

   By part (d) we know that the eigenspaces of I correspond to the axes around which the object will not wobble. From the characteristics of S, we know that the eigenspaces are orthogonal. So the axes of rotation around which the object will not wobble are orthogonal.