EECS 203: Discrete Mathematics
Winter 2024
Homework 7

Due **Thursday, Mar. 21**, 10:00 pm

No late homework accepted past midnight.

Number of Problems: $8 + 2$ Total Points: $100 + 18$

- **Match your pages!** Your submission time is when you upload the file, so the time you take to match pages doesn't count against you.
- Submit this assignment (and any regrade requests later) on Gradescope.
- Justify your answers and show your work (unless a question says otherwise).
- By submitting this homework, you agree that you are in compliance with the Engineering Honor Code and the Course Policies for 203, and that you are submitting your own work.
- Check the syllabus for full details.

Individual Portion

1. Growing your Growth Mindset [5 points]

(a) Watch the linked video about developing a growth mindset. This is a different video than the one you saw in lecture.

(b) Rewrite the last two fixed mindset statements as growth mindset statements.

(c) Write down one of your recurring fixed mindset thoughts, then write a thought you can replace it with that reflects a growth mindset.

Video: Developing a Growth Mindset (tinyurl.com/eecs203growthMindset)

What to submit: Your three pairs of fixed and growth mindset statements (the two from the table, and one that you came up with on your own).

Fixed Mindset Statement	**Growth Mindset Statement**
When I have to ask for help or get called on in lecture, I get anxious and feel like people will think I'm not smart.	The question I have is likely the same question someone else in lecture may have. It's important for me to ask so I can better understand what I am learning.
I'm jealous of other people's success.	I am inspired and encouraged by other people's success. They show me what is possible.
I didn't score as high on the exam as I expected. I'm not going to do well in this class and should drop it.	I learned from my mistakes on exam 1, and exam 2 will be a new opportunity for me to practice what I've learned.

Solution:

Fixed Mindset Statement	**Growth Mindset Statement**
This class is hard for me, so I am not fit for this major.	Perservering through this class will help me develop skills to succeed in this major.
Either I'm good at Discrete Math, or I'm not.	If I'm struggling with Discrete Math right now, I can learn from my mistakes to get a better understanding.
I'm bad at coding.	I'll get much better at coding as I keep doing it.

2. Sketchy Compositions [15 points]

Consider $f\colon X \to Y$ and $g\colon Y \to X$. **Prove or disprove** each of the following statements.

(a) If $f \circ g$ is one-to-one, then g must be one-to-one.

(b) If $g \circ f$ is one-to-one, then g must be one-to-one.

Solution:

(a) True. Let arbitrary $z \in X$. Since $f \circ g$ is injective, we know that if $f(g(x_1)) = f(g(x_2)) = z$, then $x_1 = x_2$, for $x_1, x_2 \in X$. Assume $g(x_1) = g(x_2)$. Then applying f to both sides, $f(g(x_1)) = f(g(x_2))$. Since this is true, we know that $x_1 = x_2$. So when $f \circ g$ is injective, $g(x_1) = g(x_2)$ implies $x_1 = x_2$. Thus if $f \circ g$ is injective, then g must be injective.

(b) False. Let $f : \{1\} \to \{1, 2\}; 1 \mapsto 1$, which is injective. Let $g : \{1, 2\} \to \{1\}; 1, 2 \mapsto 1$ which is not injective. Then $g \circ f : \{1\} \to \{1\}; 1 \mapsto 1$ is injective. But g is not injective. So $g \circ f$ is injective does not imply g is injective.

3. Flippy Function Fun! [15 points]

A function $f\colon A \to A$ is said to be *flippy* if for all $a \in A$, $f(f(a)) = a$. **Prove or disprove** each of the following statements

(a) If $f\colon A \to A$ is flippy, then f is bijective. (Either prove f is both onto and one-to-one using their respective definitions, or provide a counterexample.)

(b) If $f\colon A \to A$ and $g\colon A \to A$ are flippy, then $f \circ g$ must be flippy.

Solution:

(a) True. We know that f is injective, since $f(f(a)) = f(f(b))$ implies $a = b$ by definition of a flippy function. So by Problem 2, part (a), we know that f is injective. Additionally, for any $a \in A$, we know $f(a) \in A$ satisifies $f(f(a)) = a$. So every a in A is a possible output of f, meaning it is surjective. Since f is both surjective and injective, it is bijective.

(b) False. Let $f : \mathbb{R} \to \mathbb{R}; x \neq 0 \mapsto x^{-1}, 0 \mapsto 0$ and $g : \mathbb{R} \to \mathbb{R}; x \mapsto -x - 3$. Both of these functions are individually flippy. But

$$\begin{aligned}
(f \circ g) \circ (f \circ g)(1) &= f(g(f(g(1)))) \\
&= f(g(f(-4))) \\
&= f\left(g\left(-\frac{1}{4}\right)\right) \\
&= f\left(\frac{-11}{4}\right) \\
(f \circ g) \circ (f \circ g)(1) &= -\frac{4}{11} \neq 1
\end{aligned}$$

So the composition of individually flippy functions is not necessarily flippy.

4. A Hairy Situation [12 points]

Assume that nobody on Earth has more than 1,000,000 hairs on their head. Assume that the population of New York City in 2024 is 8,468,000 people. As of 2024, what is the maximum number of people in New York City that we can guarantee all have the same number of hairs on their heads?

Your explanation should use the Pigeonhole Principle. Make sure to state what the pigeons and holes are, as well as how many of each you have.

Solution:

In this problem, the pigeons are people (8,468,000 people), and the holes are the number of hairs one has on their head (1,000,000 possibilities). Since the number of pigeons is more than 8 times the number of holes, there must be at 1 hole with at least 9 pigeons. So we can guarantee that at least 9 people in New York City have the same number of hairs on their heads.

5. A Pairy Situation [14 points]

Suppose that 52 integers are chosen among the set of natural numbers less than 100. In other words, suppose that 52 integers are chosen from $\{0, 1, 2, 3, \dots, 99\}$. **Prove or disprove** that there must exist at least one pair of integers among those chosen whose difference is equal to 7.

Your proof or disproof should use the Pigeonhole Principle. Make sure to state what the pigeons and holes are, as well as how many of each you have.

Solution:

Sort the set of natural numbers less than 100 by their remainder when divided by 7, creating 7 groups: $0, 1, 2, 3, 4, 5, 6 \pmod 7$. These are holes for the 52 integers, the pigeons. Since the number of pigeons is greater than 7 times the number of pigeons, there will be 1 hole with at least 8 pigeons.

Since $99 = 14 * 7 + 1$, we know there are either 13 or 14 integers to choose in each "remainder hole." Pair these 14 integers into 7 (disjoint) "difference holes," each with two consecutive integers with a difference of 7 between them (e.g. 4 and 11). Then, using the 8 pigeons we know will enter the "remainder hole," we know one of these 7 "difference holes" must contain 2 pigeons by the pigeonhole principle. Thus there must be 2 chosen integers with a difference of 7 between them. Note that when there are 13 integers in the "remainder hole," this logic still applies, as the "difference hole" with only one integer cannot have more than 1 chosen integer occupying it. So a "difference hole" with 2 spaces must have been filled. Thus there must be 2 chosen integers with a difference of 7 between them.

6. Super Sets [15 points]

Let A be the set of prime numbers less than 203. The universe of discourse is $\mathbb{R}$. State whether each of the following sets are empty, finite but nonempty, countably infinite, or uncountable. Briefly justify your answers.

(a) $\mathbb{Z} \times \mathbb{Z}$

(b) $(\mathbb{Z} \times \mathbb{Z}) - (\mathbb{Q} \times \mathbb{Q})$

(c) $\mathbb{R} - \mathbb{Q}$

(d) $\mathbb{Q} - \mathbb{R}$

(e) $A \cap \mathbb{Q}$

(f) $\overline{A} \cap \overline{\mathbb{Q}}$

Solution:

(a) Countably infinite: they can be shown to have a one-to-one correspondence with $\mathbb{N}$ in an analogous way to the rational numbers, except with zero as a possible denominator.

(b) Empty. Every integer is rational, so the difference will be the empty set.

(c) Uncountable. Subtracting rational numbers still leaves infinitesimal gaps where real numbers can exist uncountably.

(d) Empty. Every rational number is a real number.

(e) Finite but nonempty. Every number in A is rational, so the result of the intersection is just A.

(f) Uncountable. By DeMorgan's, this is equivalent to $\overline{A \cup \mathbb{Q}} = \overline{\mathbb{Q}} = \mathbb{R} - \mathbb{Q}$, so equivalent to part (d).

7. Cardinal Construction [12 points]

For each part, give *uncountable* sets A and B such that $A - B$ is

(a) uncountable.

(b) countably infinite.

(c) finite but nonempty.

(d) empty.

Solution:

(a) $A = \mathbb{R}$ and B is the set of reals with first nonzero digit 6.

(b) $A = \mathbb{R}$ and $B = \mathbb{R} - \mathbb{Z}$.

(c) $A = \mathbb{R}$ and $B = \mathbb{R} - \{1\}$.

(d) $A = \mathbb{R}$ and $B = \mathbb{R}$.

8. Interesting Intervals [12 points]

Prove that $|[0, 3]| = |(2, 5) \cup (6, 7)|$. If you construct functions in your solution with certain properties, you may assert that they have those properties without proof.

Solution:

Let $A = [0, 3]$ and $B = (2, 5) \cup (6, 7)$. Let $f : A \to B; a \mapsto \frac{a}{3} + 3$ and $g : B \to A; b \mapsto \frac{b}{3}$. Then both f and g are injective. So the cardinality of A and B are equal.