MATH 217 - W24 - LINEAR ALGEBRA
HOMEWORK 9, DUE SUNDAY, MARCH 31 at 11:59pm

Submit Part A and Part B as two *separate* assignments in Gradescope as a **pdf file.** At the time of submission, Gradescope will prompt you to match each problem to the page(s) on which it appears. **You must match problems to pages in Gradescope so we know what page each problem appears on.** Failure to do so may result in not having the problem graded.

A few words about solution writing:

- Unless you are explicitly told otherwise for a particular problem, **you are always expected to show your work and to give justification for your answers.**
- Your solutions will be judged on precision and completeness and not merely on "basically getting it right".
- Cite every theorem or fact from the book that you are using (e.g. "By Theorem 1.10 ...").

Part A

Solve the following problems from Bretscher:
5.4: 27, 31.
5.5: 15, 23, 32(a, b, c, d).

Part B

Problem 1. Consider the four points $(2,4,6), (1,3,2), (1,1,0)$ and $(1,2,3)$ in $\mathbb{R}^3$.

(a) Write a matrix equation that, *if it were consistent*, could be used to find the coefficients A, B, C in the equation of a plane of the form $z = Ax + By + C$ that contains all four points.
(b) Show that the matrix equation from (a) is, in fact, inconsistent.
(c) Now write a matrix equation that can be used to find the least-squares solution to the equation you wrote in (a). Fully simplify any matrix products that occur in your equation, but *do not (yet) attempt to solve the equation.*
(d) Now, solve your equation using methods taught in this course. (You can use a matrix calculator to check your answer, but you must be able to solve this problem by hand.)
(e) **(Recreational:)**[1] Use 3-D graphing software such as GeoGebra or Desmos 3D to plot the four points and graph the "plane of best fit" through them.

Problem 2.

(a) Which of the following is an inner product in $\mathcal{P}_2$? Explain.
  (i) $\langle f, g\rangle = f(1)g(2) + f(2)g(1) + f(3)g(3)$
  (ii) $\langle f, g\rangle = f(1)g(1) + f(2)g(2) + f(3)g(3)$
(b) Let $V = C^\infty[-1,1]$, the vector space of smooth functions on the interval $[-1,1]$. Which of the following is an inner product in V? Explain.
  (i) $\langle f, g\rangle = \displaystyle\int_{-1}^{1} x f(x)g(x)\,dx$

[1]Recreational problems are for your own interest only; you are not required to submit your solutions, and if you do, they will not be graded. However, you may find that doing these problems helps you gain a better understanding of what we are doing in this somewhat abstract unit.

(ii) $\langle f, g\rangle = \displaystyle\int_{-1}^{1} x^2 f(x)g(x)\,dx$

Problem 3. Let $V = C^\infty\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, the vector space of smooth functions on the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, and consider the inner product defined by $\langle f, g\rangle = \displaystyle\int_{-\pi/2}^{\pi/2} f(x)g(x)\sin^2(x)\,dx$. (You do not need to show that this is an inner product, but make sure that you would be able to do so if it were an exam question!) Let $W = \operatorname{span}(1, x, x^2)$.

In what follows, you may feel free to use an online integral calculator (e.g. Wolfram Alpha) to evaluate any difficult integrals[2], but make sure that your work shows clearly *what integrals* you are computing, and how you are making use of the results. Results may be expressed using either exact expressions (e.g., $\pi/\sqrt{2}$) or decimal approximations (e.g., 2.2214), but if you use decimal approximations, please retain at least four digits' worth of precision.

(a) Compute each of the following.
  (i) $\langle 1, x\rangle$
  (ii) $\|1\|$
  (iii) $\|x\|$
(b) Find a basis $\mathcal{U}$ for the subspace W that is orthonormal relative to the given inner product.
(c) Let $h \in C^\infty\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ be the function defined by $h(x) = e^x$ for all $x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. Compute $\operatorname{proj}_W h$.
(d) **(Recreational:)** Repeat parts (a)–(c), this time using the simpler inner product
$$\langle f, g\rangle = \int_{-1}^{1} f(x)g(x)\,dx.$$
(e) **(Recreational:)** Use graphing software (e.g., Desmos) to plot the function h and the two different projections you found in (c) and (d) over the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, all on the same axes. How do these three functions compare? Which of the two projections does a "better job" of approximating h (and in what sense is it "better"?) What are some situations in which you might choose to use one inner product rather than the other?

[2]Despite what it may seem sometimes, the goal of the Inner Products unit is not to test your integration skills.