MATH 217 W24 - LINEAR ALGEBRA, Section 001 (Dr. Paul Kessenich)
Homework Set Part B due Thurs, Feb 8 at 11:59pm
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1. Let $\mathbb{R}^n \xrightarrow{T} \mathbb{R}^n$ be a linear transformation. As on HW 3, we define T^k to be the k-fold composition of T with itself. Let A be the standard matrix of T, by which we mean the unique $n \times n$ matrix such that $T(\vec{x}) = A\vec{x}$ for all $\vec{x} \in \mathbb{R}^n$.

   (a) Prove that for all k, the standard matrix for T^k is the matrix A^k. [Hint: induction works nicely.]

   Solution: We are given that the standard matrix $A^{(1)}$ represents the transformation $T^{(1)}$. Assume that the transformation T^n can be represented by the standard matrix A^n. We know by a theorem on the worksheets that the standard matrix of two linear transformations, both from $\mathbb{R}^n \to \mathbb{R}^n$, is equal to the product of their respective standard matrices. Then $(T^n \circ T)(x) = A^n A\vec{x}$. This is equal to $T^{n+1}(x) = A^{n+1}\vec{x}$. So by induction, $T^k(\vec{x}) = A^k\vec{x}$ for all $\vec{x} \in \mathbb{R}^n$ and $k \in \mathbb{N}$.

   (b) We define T to be nilpotent if there exists some $k \in \mathbb{N}$ such that T^k is the zero transformation. Prove that if T is nilpotent, then A is not invertible.

   Solution:

   (c) Prove that if T is nilpotent, then $A - I_n$ is invertible. [Hint: try multiplying out $(A - I_n)(I_n + A + A + 2 + \cdots + A_{k-1})$ and see what you get.]