EECS 203: Discrete Mathematics
Winter 2024
Homework 10

Due **Thursday, April 18**, 10:00 pm

No late homework accepted past midnight.

Number of Problems: $8 + 2$ Total Points: $100 + 35$

- **Match your pages!** Your submission time is when you upload the file, so the time you take to match pages doesn't count against you.
- Submit this assignment (and any regrade requests later) on Gradescope.
- Justify your answers and show your work (unless a question says otherwise).
- By submitting this homework, you agree that you are in compliance with the Engineering Honor Code and the Course Policies for 203, and that you are submitting your own work.
- Check the syllabus for full details.

Individual Portion

1. Color Conundrum [14 points]

Each day Donovan Edwards either wears a T-shirt or a tank top. On a given day, there is a 70% chance he wears a T-shirt and a 30% chance he wears a tank top. If he wears a T-shirt, he randomly picks one of 4 pink T-shirts, 3 blue T-shirts, and 2 black T-shirts (he is equally likely to pick any particular shirt). If he wears a tank top, he randomly picks one of 2 pink tank tops, 3 white tank tops, or 2 blue tank tops.

(a) What is the probability that he is wearing pink or white on a given day?

(b) Given that Donovan is wearing pink or white on a given day, what is the probability that he is wearing a T-shirt?

You **do not** need to simplify your answers.

Solution:

(a) T-shirt ($70\% = \frac{7}{10}$):

$$\frac{4}{4+3+2} = \frac{4}{9}$$

Tank top ($30\% = \frac{3}{10}$):

$$\frac{2+3}{2+3+2} = \frac{5}{7}$$

In total:

$$\boxed{\frac{7}{10} \cdot \frac{4}{9} + \frac{3}{10} \cdot \frac{5}{7}}$$

(b) By Bayes' theorem, is joint probability divided by probability that he is wearing pink or white. So it equals

$$\boxed{\frac{\frac{7}{10} \cdot \frac{4}{9}}{\frac{7}{10} \cdot \frac{4}{9} + \frac{3}{10} \cdot \frac{5}{7}}}$$

2. Bayes' $\times 3$ [8 points]

Suppose that E, F_1, F_2, and F_3 are events from a sample space S. Furthermore, suppose that F_1, F_2, and F_3 are each mutually exclusive, and that their union is S. Find $P(F_2 \mid E)$

if

$$P(E \mid F_2) = \frac{3}{8} \qquad P(F_1) = \frac{1}{6}$$
$$P(E \mid F_3) = \frac{1}{2} \qquad P(F_2) = \frac{1}{2}$$
$$P(E \mid F_1) = \frac{2}{7} \qquad P(F_3) = \frac{1}{3}$$

Express your final answer as a **single, fully-simplified** number.

Solution:

By Bayes',

$$P(F_2|E) = \frac{P(F_2 \cap E)}{P(E)}$$

From the givens, by Bayes' we know that

$$P(E \mid F_2) = \frac{P(F_2 \cap E)}{P(F_2)} = \frac{3}{8}, \quad P(F_2) = \frac{1}{2} \quad \Rightarrow \quad P(F_2 \cap E) = \frac{3}{8} \cdot \frac{1}{2} = \frac{3}{16}$$
$$P(E \mid F_3) = \frac{P(F_3 \cap E)}{P(F_3)} = \frac{1}{2}, \quad P(F_3) = \frac{1}{3} \quad \Rightarrow \quad P(F_2 \cap E) = \frac{1}{2} \cdot \frac{1}{3} = \frac{1}{6}$$
$$P(E \mid F_1) = \frac{P(F_1 \cap E)}{P(F_1)} = \frac{2}{7}, \quad P(F_1) = \frac{1}{6} \quad \Rightarrow \quad P(F_3 \cap E) = \frac{2}{7} \cdot \frac{1}{6} = \frac{1}{21}$$

Since $F_1 \cup F_2 \cup F_3 = S$, then

$$(F_1 \cap E) \cup (F_2 \cap E) \cup (F_3 \cap E) = S \cap E = E.$$

The probability of a union between disjoint events is just the sum of the individual probabilities. So the probabilities behave in an analagous way:

$$\begin{aligned} P(E) &= P(F_1 \cap E) + P(F_2 \cap E) + P(F_3 \cap E) \\ &= \frac{45}{112} \end{aligned}$$

$$P(F_2 \mid E) = \frac{P(F_2 \cap E)}{P(E)} = \frac{\frac{3}{16}}{\frac{45}{112}} = \boxed{\frac{7}{15} = 0.46667}$$

3. There Snow Way I'm Running In This [12 points]

Ishaan likes to run, but he hates running in the snow. If it snows, the probability of Ishaan going for a run is $\frac{1}{10}$. If it doesn't snow, the probability of Ishaan running is $\frac{8}{10}$. If Ishaan goes for run, then the probability that it snowed is $\frac{1}{9}$. What is the probability that it snows? Express your final answer as a **single, fully-simplified** number.

Solution:

Let R be the event that Ishaan runs, and S be the event that it snows. Then

$$P(R \mid S) = \frac{P(S \cap R)}{P(S)} = \frac{1}{10}$$
$$P(R \mid \neg S) = \frac{P(\neg S \cap R)}{P(\neg S)} = \frac{P(R) - P(S \cap R)}{1 - P(S)} = \frac{8}{10}$$
$$P(S \mid R) = \frac{P(S \cap R)}{P(R)} = \frac{1}{9}$$

From this we can infer:

$$P(S \cap R) = \frac{P(S)}{10}$$
$$10P(R) = 9P(S)$$
$$P(R) = \frac{9}{10}P(S)$$

Substituting into the second equation:

$$\frac{\frac{9}{10}P(S) - \frac{P(S)}{10}}{1 - P(S)} = \frac{8}{10}$$
$$\frac{9}{10}P(S) - \frac{P(S)}{10} = \frac{8}{10} - \frac{8}{10}P(S)$$
$$\frac{16}{10}P(S) = \frac{8}{10}$$
$$\boxed{P(S) = \frac{1}{2} = 0.5}$$

4. What did you expect? [12 points]

The EECS 203 staff is going on a road trip! The 36 staff members have decided to split up into 6 different cars with 9, 8, 6, 6, 4, 3 people in each of the respective cars.

(a) Suppose we pick a car uniformly at random, and consider X to be the random variable defined by the number of staff members in that car. What is the expected value of X?

(b) Now suppose we pick one of the staff members uniformly at random. Let Y be the random variable defined by the number of people in the car that staff member is in. What is the expected value of Y?

Express your final answers as **single, fully-simplified** numbers.

Solution:

(a) Since the probability of each car is the same, this expected value is just the average number of people in each car.

$$X = \frac{9+8+6+6+4+3}{6} = \boxed{6}$$

(b) The probability of a car being picked is equal to the number of people within the car, divided by the total number of people.

$$\begin{aligned} Y &= \frac{9}{36} \cdot 9 + \frac{8}{36} \cdot 8 + \frac{6}{36} \cdot 6 + \frac{6}{36} \cdot 6 + \frac{4}{36} \cdot 4 + \frac{3}{36} \cdot 3 \\ &= \boxed{6.7222} \end{aligned}$$

5. Zero-sum game...or is it? [12 points]

Your friend proposes to play the following game. You roll a fair, 6-sided dice twice and record the result. Let X be the random variable defined as twice the value of the first roll, minus three times the value of the second roll. For example, if you rolled 3 then 4, then X would equal $2 \cdot 3 - 3 \cdot 4 = -6$. You win X dollars if X is positive, but have to give your friend $|X|$ dollars if X is negative. If X is zero then you neither win nor lose money. How much money do you expect to win or lose?

Express your final answer as a **single, fully-simplified** number.

Solution:

Let the random variable dice be Y_1, Y_2 respectively. So we can apply linearity of expected values. So the expected value of X is equal to

$$\overline{X} = 2 \cdot \overline{Y}_1 - 3 \cdot \overline{Y}_2$$

The expected value of any fair dice roll is 3.5, the average of all possible values of the roll. So

$$\overline{X} = 2 \cdot 3.5 - 3 \cdot 3.5 = \boxed{-3.5}$$

So you should expect to lose 3.5 dollars on average when playing this game.

6. Rollie Pollie [15 points]

Rohit recently became super passionate about rolling dice. He decides to roll a single fair 6-sided die 100 times. What is the expected number of times he rolls a 5 followed by a 6?

Express your final answer as a **single, fully-simplified** number.

Solution:

Let the expected number of 5-6 sequences within n rolls be $\overline{X}_n$. Then for $n+1$ dice rolls, the expected value X_{n+1} is $\overline{X}_n + \overline{D}$ by linearity, where $\overline{D}$ is the expected value of the additional dice roll. The additional dice roll has a $\dfrac{1}{36} = \dfrac{1}{6} \cdot \dfrac{1}{6}$ probability to add a 5-6 sequence: by uniformity, the 2nd to last roll has a $\dfrac{1}{6}$ chance to be 5, and the last roll has a $\dfrac{1}{6}$ chance to be 6. So, this is a $\frac{1}{36}$ probability of a single 5-6 sequence due to the additional roll, for an expected value of $\overline{D} = \dfrac{1}{36}$. Thus, we have a recurrence relation

$$\overline{X}_{n+1} = \overline{X}_n + \frac{1}{36}$$

The base case for this is $X_1 = 0$, so the explicit formula is

$$X_n = \frac{n-1}{36}$$

by the explicit formula for an arithmetic sequence.

Thus, for 100 dice rolls, the expected number of times Rohit rolls a 5 followed by a 6 is

$$\frac{99}{36} = \boxed{\frac{11}{4} = 2.75}$$

7. Bernoulli trials, binomial distribution [15 points]

You roll a fair six-sided die 12 times. Find:

(a) The probability that exactly two rolls come up as a 6.

(b) The probability that exactly two rolls come up as a 6, given that the first four rolls each came up as 3.

(c) The probability that at least two rolls come up as a 6.

(d) The expected number of rolls that come up as 6.

You **do not** need to simplify your answers.

Solution:

(a) This is a Binomial experiment with success probability of $\frac{1}{6}$, and 12 trials. So this probability is equal to the probability of rolling a 6, squared; multiplied by the probability of not rolling a 6, to the power of 10; multiplied by the number of ways to choose 2 items from 12.

$$\boxed{\left(\frac{1}{6}\right)^2 \left(\frac{5}{6}\right)^{10} \binom{12}{2}}$$

(b) The last 8 rolls are independent of the first 4, so we can model this as a Binomial experiment with $p = \frac{1}{6}$ and 8 trials. So the probability is

$$\boxed{\left(\frac{1}{6}\right)^2 \left(\frac{5}{6}\right)^{6} \binom{8}{2}}$$

(c) We can find this with the same model as part (a), using complementary counting. That is, subtract the probability that less than two rolls come up as 6 from 1. This is

$$\boxed{1 - \left(\frac{1}{6}\right) \left(\frac{5}{6}\right)^{11} \binom{12}{1} - \left(\frac{5}{6}\right)^{12} \binom{12}{0}}$$

(d) As a Binomial experiment, the expected value of rolls that come up as 6 is $12 \cdot \frac{1}{6}$ by linearity. This is equal to $\boxed{2}$.

8. Fastest Draw in the Midwest [12 points]

Suppose Grace has a standard deck of 52 cards. Grace expects she can draw all 52 cards in order (defined below) in 1300 draws. Yunsoo expects 1600 draws. Explain why Yunsoo is further away from the real expected value.

Note: The order of cards goes Ace, 2, 3, . . . , King and ♣, ♢, ♡, ♠. If the next card in order is not drawn, then it is placed back into the deck at random. If the next card in order is drawn, then Grace sets it aside, removing it from the deck.

Note: The cards do not have to be selected consecutively. For example, <u>A♣</u>, 3♢, J♠, <u>2♣</u> is a valid start, and there would only be 50 cards left in the deck at this point.

Solution:

By linearity, this is equal to the expected number of draws for each card in the sequence. For each card in the sequence, the probability of drawing the card can be modeled as a geometric distribution. For instance, for A$\clubsuit$, there is always a $\frac{1}{52}$ chance that the card is the next one drawn, since there will be 52 cards in the deck at that point. As a geometric distribution, there will be an expected value of 52 draws before the A$\clubsuit$. Then, the 2$\clubsuit$ has a probability of $\frac{1}{51}$, and thus an expected value of 51 draws. This pattern continues to the last card in the deck, which has an expected value of 1 draw.

The sum of the expected values is equal to an arithmetic series 1 + 2 + 3 + ... + 52. This is equal to

$$\frac{52 \cdot 51}{2} = 1326.$$

So the expected value of draws is 1326, which makes Grace's guess closer than Yunsoo's.