

MATH 217 W24 - LINEAR ALGEBRA, Section 001 (Dr. Paul Kessenich)
Homework Set Part B due ??? at 11:59pm
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1. Question

- (a) Prove that F is alternating if and only if $F(\vec{u}, \vec{v}) = -F(\vec{v}, \vec{u})$ for all $\vec{u}, \vec{v} \in \mathbb{R}^2$.

Solution: By bilinearity, we know

$$\begin{aligned} F(u+v, v+u) &= 0 \\ F(u, v+u) + F(v, v+u) &= 0 \\ F(u, v) + F(u, u) + F(v, v) + F(v, u) &= 0 \\ F(u, v) + 0 + 0 + F(v, u) &= 0 \\ F(u, v) + F(v, u) &= 0 \\ F(u, v) &= -F(v, u) \end{aligned}$$

- (b) Prove that if F is alternating and $F(\vec{e}_1, \vec{e}_2) = 1$, then $F(\vec{u}, \vec{v}) = \det[\vec{u} \ \vec{v}]$ for all $\vec{u}, \vec{v} \in \mathbb{R}^2$.

Solution: Express \vec{u} and \vec{v} as linear combinations of e_1, e_2 :

$$\vec{u} = u_1\vec{e}_1 + u_2\vec{e}_2 \text{ and } \vec{v} = v_1\vec{e}_1 + v_2\vec{e}_2$$

Then

$$\begin{aligned} F(\vec{u}, \vec{v}) &= F(u_1\vec{e}_1 + u_2\vec{e}_2, v_1\vec{e}_1 + v_2\vec{e}_2) \\ &= F(u_1\vec{e}_1, v_1\vec{e}_1 + v_2\vec{e}_2) + F(u_2\vec{e}_2, v_1\vec{e}_1 + v_2\vec{e}_2) && \text{(bilinearity)} \\ &= F(u_1\vec{e}_1, v_1\vec{e}_1) + F(u_1\vec{e}_1, v_2\vec{e}_2) + F(u_2\vec{e}_2, v_1\vec{e}_1) + F(u_2\vec{e}_2, v_2\vec{e}_2) \\ &= u_1v_1F(\vec{e}_1, \vec{e}_1) + u_1v_2F(\vec{e}_1, \vec{e}_2) + u_2v_1F(\vec{e}_2, \vec{e}_1) + u_2v_2F(\vec{e}_2, \vec{e}_2) \\ &= u_1v_1(0) + u_1v_2(1) + u_2v_1(-1) + u_2v_2(0) && \text{(alternating)} \\ &= u_1v_2 - u_2v_1 \\ &= \det \begin{bmatrix} u_1 & v_1 \\ u_2 & v_2 \end{bmatrix} \\ &= \det[\vec{u} \ \vec{v}] \end{aligned}$$

2. (a) Prove that T is a linear transformation.

Solution: Let $A, B \in \mathbb{R}^{2 \times 2}$, and $c \in \mathbb{R}$.

T respects addition:

$$T(A + B) = (A + B)M = AM + BM = T(A) + T(B)$$

by distributivity of matrix multiplication.

T respects scalar multiplication:

$$T(cA) = (cA)M = c(AM) = cT(A)$$

by properties of matrix multiplication.

Since T respects addition and scalar multiplication, it is linear.

- (b) Find the \mathcal{E} -matrix $[T]_{\mathcal{E}}$ of T , where \mathcal{E} is the ordered basis

$$\mathcal{E} = (E_{11}, E_{12}, E_{21}, E_{22}) = \left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right)$$

of $\mathbb{R}^{2 \times 2}$. Your answer should be in terms of the entries of M .

Solution:

$$[T]_{\mathcal{E}} = \begin{bmatrix} \begin{array}{c|c} & \\ \hline [T(E_{11})]_{\mathcal{E}} & [T(E_{12})]_{\mathcal{E}} \\ \hline & \end{array} & \begin{array}{c|c} & \\ \hline [T(E_{21})]_{\mathcal{E}} & [T(E_{22})]_{\mathcal{E}} \\ \hline & \end{array} \end{bmatrix}$$

$$[T(E_{11})]_{\mathcal{E}} = \left[\begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix} \right]_{\mathcal{E}} = \begin{bmatrix} a \\ b \\ 0 \\ 0 \end{bmatrix}$$

$$[T(E_{12})]_{\mathcal{E}} = \left[\begin{bmatrix} c & d \\ 0 & 0 \end{bmatrix} \right]_{\mathcal{E}} = \begin{bmatrix} c \\ d \\ 0 \\ 0 \end{bmatrix}$$

$$[T(E_{21})]_{\mathcal{E}} = \left[\begin{bmatrix} 0 & 0 \\ a & b \end{bmatrix} \right]_{\mathcal{E}} = \begin{bmatrix} 0 \\ 0 \\ a \\ b \end{bmatrix}$$

$$[T(E_{22})]_{\mathcal{E}} = \left[\begin{bmatrix} 0 & 0 \\ c & d \end{bmatrix} \right]_{\mathcal{E}} = \begin{bmatrix} 0 \\ 0 \\ c \\ d \end{bmatrix}$$

$$[T]_{\mathcal{E}} = \begin{bmatrix} a & c & 0 & 0 \\ b & d & 0 & 0 \\ 0 & 0 & a & c \\ 0 & 0 & b & d \end{bmatrix}$$

(c) Compute $\det[T]_{\mathcal{E}}$.

Solution: Using the Laplace expansion on our result from (b),

$$\begin{aligned}
 \det[T]_{\mathcal{E}} &= \begin{vmatrix} a & c & 0 & 0 \\ b & d & 0 & 0 \\ 0 & 0 & a & c \\ 0 & 0 & b & d \end{vmatrix} \\
 &= a \begin{vmatrix} d & 0 & 0 \\ 0 & a & c \\ 0 & b & d \end{vmatrix} - c \begin{vmatrix} b & 0 & 0 \\ 0 & a & c \\ 0 & b & d \end{vmatrix} \\
 &= ad \begin{vmatrix} a & c \\ b & d \end{vmatrix} - bc \begin{vmatrix} a & c \\ b & d \end{vmatrix} \\
 &= (ad - bc)^2 \\
 &= a^2d^2 - 2abcd + b^2c^2
 \end{aligned}$$

(d) Compute $\det[T]_{\mathcal{B}}$.

Solution: The determinant of a transformation is the same in any basis. So it will be the same as part (c), or $(ad - bc)^2 = a^2d^2 - 2abcd + b^2c^2$.

(e) Either prove that T is always diagonalizable no matter what M is, or provide an explicit example of a matrix M for which T is not diagonalizable and briefly explain why your example works.

Solution:

3. (a) Prove that there exists a unique vector $\vec{z} \in \mathbb{R}^4$ such that $T(\vec{x}) = \vec{z} \cdot \vec{x}$ for all $\vec{x} \in \mathbb{R}^4$, and find the components of \vec{z} in terms of the vectors \vec{u}, \vec{v} , and \vec{w} . (Hint: $\vec{x} = x_1\vec{e}_1 + x_2\vec{e}_2 + x_3\vec{e}_3 + x_4\vec{e}_4$.)

Solution: We calculate the determinant by expanding the column of \vec{x} .

$$|\vec{x} \ \vec{u} \ \vec{v} \ \vec{w}| = x_1 \begin{vmatrix} u_2 & v_2 & w_2 \\ u_3 & v_3 & w_3 \\ u_4 & v_4 & w_4 \end{vmatrix} - x_2 \begin{vmatrix} u_1 & v_1 & w_1 \\ u_3 & v_3 & w_3 \\ u_4 & v_4 & w_4 \end{vmatrix} + x_3 \begin{vmatrix} u_1 & v_1 & w_1 \\ u_2 & v_2 & w_2 \\ u_4 & v_4 & w_4 \end{vmatrix} - x_4 \begin{vmatrix} u_1 & v_1 & w_1 \\ u_2 & v_2 & w_2 \\ u_3 & v_3 & w_3 \end{vmatrix}$$

$$\vec{z} = \begin{bmatrix} \begin{vmatrix} u_2 & v_2 & w_2 \\ u_3 & v_3 & w_3 \\ u_4 & v_4 & w_4 \end{vmatrix} \\ - \begin{vmatrix} u_1 & v_1 & w_1 \\ u_3 & v_3 & w_3 \\ u_4 & v_4 & w_4 \end{vmatrix} \\ \begin{vmatrix} u_1 & v_1 & w_1 \\ u_2 & v_2 & w_2 \\ u_4 & v_4 & w_4 \end{vmatrix} \\ - \begin{vmatrix} u_1 & v_1 & w_1 \\ u_2 & v_2 & w_2 \\ u_3 & v_3 & w_3 \end{vmatrix} \end{bmatrix}$$

- (b) Find the vector \vec{z} (as in part (a)) when $\vec{u} = \vec{e}_1, \vec{v} = \vec{e}_2$, and $\vec{w} = \vec{e}_3$ are the first three standard basis vectors in \mathbb{R}^4 .

Solution: When u, v, w are such, only the last element of \vec{z} is nonzero. So, \vec{z} is

$$\begin{bmatrix} 0 \\ 0 \\ 0 \\ -1 \end{bmatrix}.$$

- (c) When is $\vec{z} = \vec{0}$? (Your answer should be in terms of \vec{u}, \vec{v} , and \vec{w} .)

Solution: The determinant of a set of vectors is zero if and only if it is linearly dependent. When \vec{z} is 0, this is an equivalent statement to the determinant $|\vec{x} \ \vec{u} \ \vec{v} \ \vec{w}|$ being 0 for all \vec{x} . This can only happen when $\vec{u}, \vec{v}, \vec{w}$ are linearly dependent.

Alternatively, we can realize that $\vec{z} = 0$ means that $\vec{u}, \vec{v}, \vec{w}$ are linearly dependent since their corresponding vectors with one element removed are linearly dependent

(e.g. $\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}, \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}, \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}$ are linearly dependent since the last element of \vec{z} , the determinant of these three vectors, is 0).

- (d) Prove that \vec{z} is orthogonal to each of \vec{u}, \vec{v} and \vec{w} , and find $\det([\vec{z} \ \vec{u} \ \vec{v} \ \vec{w}])$ in terms of $\|\vec{z}\|$.

Solution: We use \vec{u} to represent any of $\vec{u}, \vec{v}, \vec{w}$ without loss of generality. The dot product $\vec{u} \cdot \vec{z} = \det([\vec{u} \ \vec{u} \ \vec{v} \ \vec{w}])$ by our definition of \vec{z} . Since this matrix has two columns of \vec{u} , it is linearly dependent. So its determinant is 0, and the dot product

$\vec{u} \cdot \vec{v} = 0$. So \vec{z} is orthogonal to $\vec{u}, \vec{v}, \vec{w}$.

By our definition of \vec{z} , $\det([\vec{z} \ \vec{u} \ \vec{v} \ \vec{w}]) = \vec{z} \cdot \vec{z} = \|\vec{z}\|^2$.

4. (a) Prove that for every $n \times n$ matrix A and for every eigenvalue λ of A , the real number $p(\lambda)$ is an eigenvalue of the $n \times n$ matrix $p(A)$.

Solution: Let \vec{v} be an eigenvector corresponding to λ of A . Let $p(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \dots$, where $a_i \in \mathbb{R}$. Then

$$\begin{aligned} p(A)\vec{v} &= a_0\vec{v} + a_1A\vec{v} + a_2A^2\vec{v} + a_3A^3\vec{v} + \dots \\ &= a_0\vec{v} + a_1\lambda\vec{v} + a_2\lambda^2\vec{v} + a_3\lambda^3\vec{v} + \dots && \text{(definition of eigenvector)} \\ &= (a_0 + a_1\lambda + a_2\lambda^2 + a_3\lambda^3 + \dots) \vec{v} \\ &= p(\lambda)\vec{v} \end{aligned}$$

So $p(\lambda)$ is an eigenvalue of $p(A)$.

- (b) Let p be a polynomial and let $n \in \mathbb{N}$. Prove that if S is an invertible $n \times n$ matrix, then for every $A \in \mathbb{R}^{n \times n}$ we have $p(S^{-1}AS) = S^{-1}p(A)S$.

Solution: Let $p(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \dots$, where $a_i \in \mathbb{R}$. Then

$$\begin{aligned} p(S^{-1}AS) &= a_0I_n + a_1(S^{-1}AS) + a_2(S^{-1}AS)^2 + a_3(S^{-1}AS)^3 + \dots \\ &= a_0I_n + a_1(S^{-1}AS) + a_2S^{-1}A^2S + a_3S^{-1}A^3S + \dots \\ &\quad \text{(since } (S^{-1}AS)^n = S^{-1}A^nS \ \forall n \in \mathbb{N}) \\ &= a_0S^{-1}I_nS + a_1(S^{-1}AS) + a_2S^{-1}A^2S + a_3S^{-1}A^3S + \dots \\ &= S^{-1}(a_0I_n + a_1A + a_2A^2 + a_3A^3 + \dots)S \\ &= S^{-1}p(A)S \end{aligned}$$

- (c) Let p be a polynomial and let A be an $n \times n$ matrix. Prove that if A is diagonalizable, then every eigenvalue of $p(A)$ is of the form $p(\lambda)$ for some eigenvalue λ of A .

Solution:

5. (a) Let $A \in \mathbb{R}^{2 \times 2}$ be a 2×2 matrix such that $A^2 = I_2$. Prove that A is diagonalizable. (Hint: try factoring $A^2 - I_2$, and consider the possible ranks of the factors.)
- (b) Does the same result hold for larger matrices? That is, if $A \in \mathbb{R}^{n \times n}$ is an $n \times n$ matrix for which $A^2 = I_n$, must A be diagonalizable? Either prove this or give a counterexample.