MATH 217 W24 - LINEAR ALGEBRA, Section 001 (Dr. Paul Kessenich)
Homework Set Part B due ??? at 11:59pm
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1. Question
   (a) Prove that F is alternating if and only if $F(\vec{u},\vec{v}) = -F(\vec{v},\vec{u})$ for all $\vec{u},\vec{v} \in \mathbb{R}^2$.

   Solution: By bilinearity, we know

$$\begin{aligned} F(u+v, v+u) &= 0 \\ F(u, v+u) + F(v, v+u) &= 0 \\ F(u,v) + F(u,u) + F(v,v) + F(v,u) &= 0 \\ F(u,v) + 0 + 0 + F(v,u) &= 0 \\ F(u,v) + F(v,u) &= 0 \\ F(u,v) &= -F(v,u) \end{aligned}$$

   (b) Prove that if F is alternating and $F(\vec{e}_1,\vec{e}_2) = 1$, then $F(\vec{u},\vec{v}) = \det[\vec{u}\ \vec{v}]$ for all $\vec{u},\vec{v} \in \mathbb{R}^2$.

   Solution: Express $\vec{u}$ and $\vec{v}$ as linear combinations of e_1, e_2 :

$$\vec{u} = u_1\vec{e}_1 + u_2\vec{e}_2 \text{ and } \vec{v} = v_1\vec{e}_1 + v_2\vec{e}_2$$

   Then

$$\begin{aligned} F(\vec{u},\vec{v}) &= F(u_1\vec{e}_1 + u_2\vec{e}_2, v_1\vec{e}_1 + v_2\vec{e}_2) \\ &= F(u_1\vec{e}_1, v_1\vec{e}_1 + v_2\vec{e}_2) + F(u_2\vec{e}_2, v_1\vec{e}_1 + v_2\vec{e}_2) && \text{(bilinearity)} \\ &= F(u_1\vec{e}_1, v_1\vec{e}_1) + F(u_1\vec{e}_1, v_2\vec{e}_2) + F(u_2\vec{e}_2, v_1\vec{e}_1) + F(u_2\vec{e}_2, v_2\vec{e}_2) \\ &= u_1v_1F(\vec{e}_1,\vec{e}_1) + u_1v_2F(\vec{e}_1,\vec{e}_2) + u_2v_1F(\vec{e}_2,\vec{e}_1) + u_2v_2F(\vec{e}_2,\vec{e}_2) \\ &= u_1v_1(0) + u_1v_2(1) + u_2v_1(-1) + u_2v_2(0) && \text{(alternating)} \\ &= u_1v_2 - u_2v_1 \\ &= \det\begin{bmatrix} u_1 & v_1 \\ u_2 & v_2 \end{bmatrix} \\ &= \det[\vec{u}\ \vec{v}] \end{aligned}$$

2. (a) Prove that T is a linear transformation.

Solution: Let $A, B \in \mathbb{R}^{2\times 2}$, and $c \in \mathbb{R}$.
T respects addition:

$$T(A+B) = (A+B)M = AM + BM = T(A) + T(B)$$

by distributivity of matrix multiplication.
T respects scalar multiplication:

$$T(cA) = (cA)M = c(AM) = cT(A)$$

by properties of matrix multiplication.

Since T respects addition and scalar multiplication, it is linear.

(b) Find the $\mathcal{E}$-matrix $[T]_{\mathcal{E}}$ of T, where $\mathcal{E}$ is the ordered basis

$$\mathcal{E} = (E_{11}, E_{12}, E_{21}, E_{22}) = \left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right)$$

of $\mathbb{R}^{2\times 2}$. Your answer should be in terms of the entries of M.

Solution:

$$[T]_{\mathcal{E}} = \begin{bmatrix} | & | & | & | \\ [T(E_{11})]_{\mathcal{E}} & [T(E_{12})]_{\mathcal{E}} & [T(E_{21})]_{\mathcal{E}} & [T(E_{22})]_{\mathcal{E}} \\ | & | & | & | \end{bmatrix}$$

$$[T(E_{11})]_{\mathcal{E}} = \left[\begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix}\right]_{\mathcal{E}} = \begin{bmatrix} a \\ b \\ 0 \\ 0 \end{bmatrix}$$

$$[T(E_{12})]_{\mathcal{E}} = \left[\begin{bmatrix} c & d \\ 0 & 0 \end{bmatrix}\right]_{\mathcal{E}} = \begin{bmatrix} c \\ d \\ 0 \\ 0 \end{bmatrix}$$

$$[T(E_{21})]_{\mathcal{E}} = \left[\begin{bmatrix} 0 & 0 \\ a & b \end{bmatrix}\right]_{\mathcal{E}} = \begin{bmatrix} 0 \\ 0 \\ a \\ b \end{bmatrix}$$

$$[T(E_{22})]_{\mathcal{E}} = \left[\begin{bmatrix} 0 & 0 \\ c & d \end{bmatrix}\right]_{\mathcal{E}} = \begin{bmatrix} 0 \\ 0 \\ c \\ d \end{bmatrix}$$

$$[T]_{\mathcal{E}} = \begin{bmatrix} a & c & 0 & 0 \\ b & d & 0 & 0 \\ 0 & 0 & a & c \\ 0 & 0 & b & d \end{bmatrix}$$

(c) Compute $\det[T]_{\mathcal{E}}$.

Solution: Using the Laplace expansion on our result from (b),

$$\begin{aligned}
\det[T]_{\mathcal{E}} &= \begin{vmatrix} a & c & 0 & 0 \\ b & d & 0 & 0 \\ 0 & 0 & a & c \\ 0 & 0 & b & d \end{vmatrix} \\
&= a \begin{vmatrix} d & 0 & 0 \\ 0 & a & c \\ 0 & b & d \end{vmatrix} - c \begin{vmatrix} b & 0 & 0 \\ 0 & a & c \\ 0 & b & d \end{vmatrix} \\
&= ad \begin{vmatrix} a & c \\ b & d \end{vmatrix} - bc \begin{vmatrix} a & c \\ b & d \end{vmatrix} \\
&= (ad - bc)^2 \\
&= a^2d^2 - 2abcd + b^2c^2
\end{aligned}$$

(d) Compute $\det[T]_{\mathcal{B}}$.

Solution: The determinant of a transformation is the same in any basis. So it will be the same as part (c), or $(ad - bc)^2 = a^2d^2 - 2abcd + b^2c^2$.

(e) Either prove that T is always diagonalizable no matter what M is, or provide an explicit example of a matrix M for which T is not diagonalizable and briefly explain why your example works.

Solution:

3. (a) Prove that there exists a unique vector $\vec{z} \in \mathbb{R}^4$ such that $T(\vec{x}) = \vec{z} \cdot \vec{x}$ for all $\vec{x} \in \mathbb{R}^4$, and find the components of $\vec{z}$ in terms of the vectors $\vec{u}, \vec{v}$, and $\vec{w}$. (Hint: $\vec{x} = x_1\vec{e}_1 + x_2\vec{e}_2 + x_3\vec{e}_3 + x_4\vec{e}_4$.)

Solution: We calculate the determinant by by expanding the column of $\vec{x}$.

$$\begin{vmatrix}\vec{x} & \vec{u} & \vec{v} & \vec{w}\end{vmatrix} = x_1\begin{vmatrix}u_2 & v_2 & w_2\\ u_3 & v_3 & w_3\\ u_4 & v_4 & w_4\end{vmatrix} - x_2\begin{vmatrix}u_1 & v_1 & w_1\\ u_3 & v_3 & w_3\\ u_4 & v_4 & w_4\end{vmatrix} + x_3\begin{vmatrix}u_1 & v_1 & w_1\\ u_2 & v_2 & w_2\\ u_4 & v_4 & w_4\end{vmatrix} - x_4\begin{vmatrix}u_1 & v_1 & w_1\\ u_2 & v_2 & w_2\\ u_3 & v_3 & w_3\end{vmatrix}$$

$$\vec{z} = \begin{bmatrix}\begin{vmatrix}u_2 & v_2 & w_2\\ u_3 & v_3 & w_3\\ u_4 & v_4 & w_4\end{vmatrix}\\ -\begin{vmatrix}u_1 & v_1 & w_1\\ u_3 & v_3 & w_3\\ u_4 & v_4 & w_4\end{vmatrix}\\ \begin{vmatrix}u_1 & v_1 & w_1\\ u_2 & v_2 & w_2\\ u_4 & v_4 & w_4\end{vmatrix}\\ -\begin{vmatrix}u_1 & v_1 & w_1\\ u_2 & v_2 & w_2\\ u_3 & v_3 & w_3\end{vmatrix}\end{bmatrix}$$

(b) Find the vector $\vec{z}$ (as in part (a)) when $\vec{u} = \vec{e}_1, \vec{v} = \vec{e}_2$, and $\vec{w} = \vec{e}_3$ are the first three standard basis vectors in $\mathbb{R}^4$.

Solution: When u, v, w are such, only the last element of $\vec{z}$ is nonzero. So, $\vec{z}$ is $\begin{bmatrix}0\\0\\0\\-1\end{bmatrix}$.

(c) When is $\vec{z} = \vec{0}$? (Your answer should be in terms of $\vec{u}, \vec{v}$, and $\vec{w}$.)

Solution: The determinant of a set of vectors is zero if and only if it is linearly dependent. When $\vec{z}$ is 0, this is an equivalent statement to the determinant $|\vec{x}\ \vec{u}\ \vec{v}\ \vec{w}|$ being 0 for all $\vec{x}$. This can only happen when $\vec{u}, \vec{v}, \vec{w}$ are linearly dependent.

Alternatively, we can realize that $\vec{z} = 0$ means that $\vec{u}, \vec{v}, \vec{w}$ are linearly dependent since their corresponding vectors with one element removed are linearly dependent (e.g. $\begin{bmatrix}u_1\\u_2\\u_3\end{bmatrix}, \begin{bmatrix}v_1\\v_2\\v_3\end{bmatrix}, \begin{bmatrix}w_1\\w_2\\w_3\end{bmatrix}$ are linearly dependent since the last element of $\vec{z}$, the determinant of these three vectors, is 0).

(d) Prove that $\vec{z}$ is orthogonal to each of $\vec{u}, \vec{v}$ and $\vec{w}$, and find $\det([\vec{z}\ \vec{u}\ \vec{v}\ \vec{w}])$ in terms of $||\vec{z}||$.

Solution: We use $\vec{u}$ to represent any of $\vec{u}, \vec{v}, \vec{w}$ without loss of generality. The dot product $\vec{u} \cdot \vec{z} = \det([\vec{u}\ \vec{u}\ \vec{v}\ \vec{w}])$ by our definition of $\vec{z}$. Since this matrix has two columns of $\vec{u}$, it is linearly dependent. So its determinant is 0, and the dot product

$\vec{u} \cdot \vec{v} = 0$. So $\vec{z}$ is orthogonal to $\vec{u}, \vec{v}, \vec{w}$.
By our definition of $\vec{z}$, $\det([\vec{z}\ \vec{u}\ \vec{v}\ \vec{w}]) = \vec{z} \cdot \vec{z} = ||\vec{z}||^2$.

4. (a) Prove that for every $n \times n$ matrix A and for every eigenvalue λ of A, the real number $p(\lambda)$ is an eigenvalue of the $n \times n$ matrix $p(A)$.

Solution: Let $\vec{v}$ be a eigenvector corresponding to λ of A. Let $p(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$, where $a_i \in \mathbb{R}$. Then

$$\begin{aligned} p(A)\vec{v} &= a_0\vec{v} + a_1 A\vec{v} + a_2 A^2\vec{v} + a_3 A^3\vec{v} + \cdots \\ &= a_0\vec{v} + a_1\lambda\vec{v} + a_2\lambda^2\vec{v} + a_3\lambda^3\vec{v} + \cdots && \text{(definition of eigenvector)} \\ &= \left(a_0 + a_1\lambda + a_2\lambda^2 + a_3\lambda^3 + \cdots\right)\vec{v} \\ &= p(\lambda)\vec{v} \end{aligned}$$

So $p(\lambda)$ is an eigenvalue of $p(A)$.

(b) Let p be a polynomial and let $n \in N$. Prove that if S is an invertible $n \times n$ matrix, then for every $A \in \mathbb{R}^{n\times n}$ we have $p(S^{-1}AS) = S^{-1}p(A)S$.

Solution: Let $p(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$, where $a_i \in \mathbb{R}$. Then

$$\begin{aligned} p(S^{-1}AS) &= a_0 I_n + a_1(S^{-1}AS) + a_2(S^{-1}AS)^2 + a_3(S^{-1}AS)^3 + \cdots \\ &= a_0 I_n + a_1(S^{-1}AS) + a_2 S^{-1}A^2 S + a_3 S^{-1}A^3 S + \cdots \\ &\qquad \text{(since } (S^{-1}AS)^n = S^{-1}A^n S\ \forall n \in \mathbb{N}) \\ &= a_0 S^{-1} I_n S + a_1(S^{-1}AS) + a_2 S^{-1}A^2 S + a_3 S^{-1}A^3 S + \cdots \\ &= S^{-1}\left(a_0 I_n + a_1 A + a_2 A^2 + a_3 A^3 + \cdots\right) S \\ &= S^{-1}p(A)S \end{aligned}$$

(c) Let p be a polynomial and let A be an $n \times n$ matrix. Prove that if A is diagonalizable, then every eigenvalue of $p(A)$ is of the form $p(\lambda)$ for some eigenvalue λ of A.

Solution: Assume A is diagonalizable, and let $A = S^{-1}DS$ for some $n \times n$ diagonal matrix D and invertible $n \times n$ matrix S.

5. (a) Let $A \in \mathbb{R}^{2\times 2}$ be a 2×2 matrix such that $A^2 = I_2$. Prove that A is diagonalizable. (Hint: try factoring $A^2 - I_2$, and consider the possible ranks of the factors.)

Solution: We know the rank of A must be 2, since I_2 has a rank of 2 and the composition of two linear transformations has less than or equal rank than the minimum rank of the two. Also, $A^2 = I_2$ is equivalent to $A^2 - I_2 = 0$. So

$$\begin{aligned} A^2 - I_2 &= 0_2 \\ A^2 - I_2^2 &= 0_2 \\ (A + I_2)(A - I_2) &= 0_2 \end{aligned}$$

Since the nullity of a matrix composition is at most the sum of the nullities of the components, A has eigenvalues of 1 and/or -1 with a total geometric multiplicity of 2. Since its geometric multiplicities add up to 2 and the dimensions of A are 2×2, then A is diagonalizable.

(b) Does the same result hold for larger matrices? That is, if $A \in \mathbb{R}^{n\times n}$ is an $n \times n$ matrix for which $A^2 = I_n$, must A be diagonalizable? Either prove this or give a counterexample.

Solution: Yes, the same result holds. We know the rank of A must be n, since I_n has a rank of n and the composition of two linear transformations is at least the minimum rank of the two. Also, $A^2 = I_n$ is equivalent to $A^2 - I_n = 0$. So

$$\begin{aligned} A^2 - I_n &= 0_n \\ A^2 - I_n^2 &= 0_n && (I_n^2 = I_n) \\ (A + I_n)(A - I_n) &= 0_n \end{aligned}$$

Since the nullity of a matrix composition is at most the sum of the nullities of the components, A has eigenvalues of 1 and/or -1 with a total geometric multiplicity of n. Since its geometric multiplicities add up to n and the dimensions of A are $n \times n$, then A is diagonalizable.