# MATH 217 - W24 - LINEAR ALGEBRA
# HOMEWORK 7, DUE Thursday, March 14 at 11:59pm

Submit Part A and Part B as two *separate* assignments in Gradescope as a **pdf file.** At the time of submission, Gradescope will prompt you to match each problem to the page(s) on which it appears. **You must match problems to pages in Gradescope so we know what page each problem appears on.** Failure to do so may result in not having the problem graded.

**A few words about solution writing:**

- Unless you are explicitly told otherwise for a particular problem, **you are always expected to show your work and to give justification for your answers.**
- Your solutions will be judged on precision and completeness and not merely on "basically getting it right".
- Cite every theorem or fact from the book that you are using (e.g. "By Theorem 1.10 ...").

## Part A (15 points)

Solve the following problems from the book:
**Section 4.3:** 14, 28, 60
**Section 5.1:** 6, 17, 26

## Part B (25 points)

**Problem 1.** Let $W$ be an $n$-dimensional vector space with ordered bases $\mathcal{A}, \mathcal{B}$, and $\mathcal{C}$.

(a) Prove that $S_{\mathcal{C}\to\mathcal{A}} = S_{\mathcal{B}\to\mathcal{A}}\ S_{\mathcal{C}\to\mathcal{B}}$.
(b) Show that $S_{\mathcal{C}\to\mathcal{A}}\ S_{\mathcal{B}\to\mathcal{C}}\ S_{\mathcal{A}\to\mathcal{B}} = I_n$.

**Problem 2.** Let $f_1, f_2, f_3$ be the smooth functions defined by

$$f_1(x) = \sin 2x, f_2(x) = \cos 2x, f_3(x) = e^{3x}$$

and consider the subspace $V \subseteq C^\infty(\mathbb{R})$ spanned by the basis $\mathcal{B} = (f_1, f_2, f_3)$. (You may assume without proof that these three functions are linearly independent.) Now consider the linear transformation $D : V \to V$ defined by differentiation, i.e. for any function $g \in V$, $D(g)(x) = \frac{dg}{dx}$.

(a) Find $[D]_{\mathcal{B}}$.
(b) Give a geometric interpretation of the matrix $[D]_{\mathcal{B}}$. That is, how does it act on $\mathbb{R}^3$?

**Problem 3.** Let $V$ be a vector space with ordered bases $\mathcal{B} = (b_1, \dots, b_n)$ and $\mathcal{C} = (c_1, \dots, c_n)$. Let $T : V \to V$ be a linear transformation, with $B = [T]_{\mathcal{B}}$ and $C = [T]_{\mathcal{C}}$. Give a proof or counterexample for each of the following statements:

(a) For all integers $k \geq 1$, $B^k$ and $C^k$ are similar.
(b) $\ker(B) = \ker(C)$.
(c) $\dim(\ker(B)) = \dim(\ker(C))$.

**Problem 4.** Let $T : U \to W$ be a linear transformation between vector spaces $U$ and $W$. Suppose that $\mathcal{B} = (u_1, u_2, \dots, u_k)$ is a basis for the source $U$ and $\mathcal{C} = (w_1, w_2, \dots, w_d)$ is a basis for the target $W$. As usual, let $L_{\mathcal{B}}$ denote the coordinate isomorphism $U \to \mathbb{R}^k$ and let $L_{\mathcal{C}}$ denote the coordinate isomorphism $W \to \mathbb{R}^d$.

(a) Show that there exists a linear transformation $T' : \mathbb{R}^k \to \mathbb{R}^d$ such that $T' \circ L_{\mathcal{B}} = L_{\mathcal{C}} \circ T$. [Hint: A diagram showing four vector spaces and four maps between them, similar to those immediately before and after Definition 4.3.1 in the textbook, might be useful.]

(b) Let $[T]_{(\mathcal{B},\mathcal{C})}$ denote the standard matrix of the transformation $T'$ you described in (a). Prove that for all $u \in U$,
$$[T(u)]_{\mathcal{C}} = [T]_{(\mathcal{B},\mathcal{C})}[u]_{\mathcal{B}}.$$

(c) Describe, with explanation, the columns of matrix $[T]_{(\mathcal{B},\mathcal{C})}$ in terms of the bases $\mathcal{B}$ and $\mathcal{C}$.

**Problem 5.** Let $f_1, f_2, f_3$ be the functions defined by
$$f_1(x) = \sin x,\ \ f_2(x) = \cos x,\ \ f_3(x) = e^x,$$
which you may assume without proof are linearly independent. Consider the subspace $V$ of $\mathcal{C}^\infty$ spanned by the set $\{f_1, f_2, f_3\}$. Recall from Calculus that every function in $V$ may be expressed as a Taylor series that converges for all real numbers. For example,
$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots,$$
$$\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \cdots,$$
$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots.$$
Let $T : V \to \mathcal{P}_3$ be the linear transformation that assigns to each function $f \in V$ the third-degree Taylor polynomial $f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3$ for $f$, a polynomial approximation to $f$.

(a) Find a basis $\mathcal{C}$ for $\mathcal{P}_3$ such that
$$[T(f_1)]_{\mathcal{C}} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix}, [T(f_2)]_{\mathcal{C}} = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}, [T(f_3)]_{\mathcal{C}} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}.$$

(b) Let $\mathcal{C}$ be as in (a), and let $\mathcal{B} = (f_1 + f_2, f_1 - f_2, f_3 + f_1)$. Find $[T]_{(\mathcal{B},\mathcal{C})}$ (see Problem 4).

**Problem 6.** Let $A = \begin{bmatrix} -6 & -30 \\ -30 & 19 \end{bmatrix}$ and let $V = \operatorname{span}\left(\begin{bmatrix} 3 \\ 2 \end{bmatrix}\right)$.

(a) Show that for all $\vec{v} \in V$, $A\vec{v} \in V$.

(b) Find a basis for $V^\perp$, and show that for all $\vec{w} \in V^\perp$, $A\vec{w} \in V^\perp$.

(c) Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation defined by $T(\vec{x}) = A\vec{x}$ for all $\vec{x} \in \mathbb{R}^2$. Find a basis $\mathcal{B}$ of $\mathbb{R}^2$ such that $[T]_{\mathcal{B}}$ is diagonal, and write the matrix $[T]_{\mathcal{B}}$ explicitly.

(d) Calculate $[T^{10}]_{\mathcal{B}}$. [Hint: Leave numbers like $7^{13}$ in that form; do not attempt to multiply them out.]

(e) Calculate $[T^{10}]_{\mathcal{E}}$. [Hint: Leave the entries as numerical expressions; do not attempt to simplify.]