

MATH 217 W24 - LINEAR ALGEBRA, Section 001 (Dr. Paul Kessenich)
Homework 11 Part B due SUNDAY, April 21 at 11:59pm
Zhengyu James Pan (jzpan@umich.edu)

1. (a) Let E_0 denote the 0-eigenspace of T . Explicitly describe E_0 (as a set).

Solution:

$$E_0 = \{(x_1, 0, x_2, 0, x_3, 0, \dots) \mid x_i \in \mathbb{R}\}$$

- (b) Prove that every real number λ is an eigenvalue of T . (Hint: explicitly construct an eigenvector $(x_1, x_2, x_3, \dots) \in V$. First consider x_i when i is a power of 2.)

Solution: Let $\lambda \in \mathbb{R}$. Then let

$$s = (1, \lambda, \lambda, \lambda^2, \lambda^2, \lambda^2, \lambda^2, \lambda^3, \lambda^3, \lambda^3, \lambda^3, \lambda^3, \lambda^3, \lambda^3, \lambda^3, \dots)$$

be an infinite sequence such that each consecutive power λ^n is repeated n times in the sequence, starting from $n = 0$. Then

$$\begin{aligned} T(s) &= (\lambda, \lambda^2, \lambda^2, \lambda^3, \lambda^3, \lambda^3, \lambda^3, \lambda^4, \lambda^4, \lambda^4, \lambda^4, \lambda^4, \lambda^4, \lambda^4, \lambda^4, \dots) \\ &= \lambda(s). \end{aligned}$$

So any real number is an eigenvalue of T .

2. (a) Let \mathcal{D} be a diagonal $n \times n$ matrix with distinct entries along the diagonal, and let \mathcal{C} be the subset of $\mathbb{R}^{n \times n}$ consisting of all diagonal matrices. Prove $\mathcal{C}(D) = \mathcal{D}$.

Solution: Let the diagonal entries of D be d_1, \dots, d_n . Let $A \in \mathcal{D}$, with diagonal entries a_1, \dots, a_n . Then the product

$$AD = \begin{bmatrix} a_1 d_1 & 0 & \dots & 0 \\ 0 & a_2 d_2 & 0 & \vdots \\ \vdots & 0 & \ddots & 0 \\ 0 & \dots & 0 & a_n d_n \end{bmatrix} = \begin{bmatrix} d_1 a_1 & 0 & \dots & 0 \\ 0 & d_2 a_2 & 0 & \vdots \\ \vdots & 0 & \ddots & 0 \\ 0 & \dots & 0 & d_n a_n \end{bmatrix} = DA.$$

So $\mathcal{D} \subset \mathcal{C}(D)$.

Let $B \in \mathcal{C}(D)$ with columns $\vec{b}_1, \dots, \vec{b}_n$, rows $\vec{c}_1, \dots, \vec{c}_n$, and element of i th row and j th column b_{ij} . Then

$$\begin{aligned} BD &= \begin{bmatrix} \left| \begin{array}{c} B(d_1 \vec{e}_1) \\ \vdots \end{array} \right| & \dots & \left| \begin{array}{c} B(d_n \vec{e}_n) \\ \vdots \end{array} \right| \end{bmatrix} \\ &= \begin{bmatrix} \left| \begin{array}{c} d_1 \vec{b}_1 \\ \vdots \end{array} \right| & \dots & \left| \begin{array}{c} d_n \vec{b}_n \\ \vdots \end{array} \right| \end{bmatrix} \end{aligned}$$

$$\begin{aligned} DB &= ((DB)^\top)^\top \\ &= (B^\top D)^\top \\ &= \begin{bmatrix} \left| \begin{array}{c} d_1 \vec{c}_1^\top \\ \vdots \end{array} \right| & \dots & \left| \begin{array}{c} d_n \vec{c}_n^\top \\ \vdots \end{array} \right| \end{bmatrix}^\top \\ &= \begin{bmatrix} \text{---} & d_1 \vec{c}_1 & \text{---} \\ & \vdots & \\ \text{---} & d_n \vec{c}_n & \text{---} \end{bmatrix} \end{aligned}$$

where \vec{e}_i is the i th standard basis vector. Since $B \in \mathcal{C}(D)$, $BD = DB$. Considering arbitrary b_{ij} , this means that $d_i b_{ij} = d_j b_{ij}$.

When $i = j$, then $d_i = d_j$, so b_{ij} , a diagonal element of B , can be anything.

When $i \neq j$, then $d_i \neq d_j$ since D has distinct diagonal elements. But $d_i b_{ij} = d_j b_{ij}$. So $b_{ij} = 0$. Note that in this case, b_{ij} is any non-diagonal element. Since only diagonal elements of B can be nonzero, B is diagonal, and $B \in \mathcal{D}$. So $\mathcal{C}(D) \subset \mathcal{D}$. Thus $\mathcal{D} = \mathcal{C}(D)$.

- (b) Prove that if A and B are simultaneously diagonalizable $n \times n$ matrices, then

$B \in \mathcal{C}(A)$.

Solution: We know that $\mathcal{D} \subset \mathcal{C}(D)$ for any diagonal matrix D by part (a). (The $\mathcal{D} \subset \mathcal{C}(D)$ direction did not depend on the fact that D had distinct diagonal elements.) So then

$$\begin{aligned} S^{-1}BSS^{-1}AS &= S^{-1}ASS^{-1}BS \\ S^{-1}BAS &= S^{-1}ABS \\ BA &= AB \end{aligned}$$

Thus $B \in \mathcal{C}(A)$.

- (c) Prove that if A and B are $n \times n$ matrices such that A has n distinct eigenvalues and $B \in \mathcal{C}(A)$, then A and B are simultaneously diagonalizable.

Solution: Let S be an eigenbasis of A . Then $S^{-1}AS$, the diagonalized matrix of A , has distinct diagonal elements. We know that

$$\begin{aligned} BA &= AB \\ S^{-1}BAS &= S^{-1}ABS \\ S^{-1}BSS^{-1}AS &= S^{-1}ASS^{-1}BS \end{aligned}$$

So $S^{-1}BS \in \mathcal{C}(S^{-1}AS)$. Then by part (a), since $S^{-1}AS$ is diagonal with distinct entries, $S^{-1}BS$ is diagonal. So S diagonalizes both B and A . So A and B are simultaneously diagonalizable.

3. (a) Suppose that A has eigenvalue 0 but is not diagonalizable. Prove that $\text{im}(A) = E_0$, and conclude from this that $A^2 = 0$.

Solution: The givens imply that $\det(A - 0I_1) = 0$. So $\det(A) = 0$. But since it is not diagonalizable, there must be one eigenvalue λ of A which does not satisfy $\text{almu}(\lambda) = \text{gemu}(\lambda)$. Since both almu and gemu are always at least 1, $\text{almu} \geq \text{gemu}$, and the sum of $\text{almus} = n$; we know $\text{almu}(0) = 2$, and $\text{gemu}(0) = 1$. From this, we infer that the nullity is 1, and the rank is also 1. The almu also implies that the characteristic equation of A is $x^2 = 0$.

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Since

$$\begin{aligned} \det(A - 0I_1) &= (a - x)(d - x) - bc \\ &= ad - bc - (a + d)x + x^2 \end{aligned}$$

and the characteristic polynomial has no x^1 terms, we know that $a + d = 0$, or equivalently that $a = -d$. Additionally, $ad = bc$.

We know that A does not have linearly independent columns, so the columns must be scalar multiples of each other. In fact, since $a = -d$, the ratio of the second column to the first is $\frac{d}{c} = \frac{-a}{c}$. So then the rref form of A is

$$\begin{bmatrix} 1 & \frac{-a}{c} \\ 0 & 0 \end{bmatrix}$$

which has nullspace spanned by $\begin{bmatrix} a \\ b \end{bmatrix}$. But this is also the image of A , since both columns of A are scalar multiples of $\begin{bmatrix} a \\ b \end{bmatrix}$. So the image and nullity of A are the same. Thus

$$A^2\vec{x} = A(A\vec{x}) = 0$$

for any $\vec{x} \in \mathbb{R}^2$. Since $A\vec{x}$ is in $\ker(A)$. So $A^2 = 0$.

- (b) Let $\lambda \in \mathbb{R}$ and suppose that A has eigenvalue λ but is not diagonalizable. Prove that we have $(A - \lambda I_2)^2 = 0$, and deduce from this that $A\vec{v} - \lambda\vec{v} \in E_\lambda$ for every $\vec{v} \in \mathbb{R}^2$. [Hint: apply part (a) to the matrix $A - \lambda I_2$].

Solution: Since both almu and gemu are always at least 1, $\text{almu} \geq \text{gemu}$, and the sum of $\text{almus} = n$; we know $\text{almu}(\lambda) = 2$, and $\text{gemu}(\lambda) = 1$. So the matrix $A - \lambda I_2$ has an eigenvalue of 0, and nullity 1. So $A - \lambda I_2$ has an eigenvalue of 0 and is not diagonalizable.

Then by (a), $(A - \lambda I_2)^2 = 0$, and the image of $A - \lambda I_2$ is equal to its 0-eigenspace. The 0-eigenspace of $A - \lambda I_2$ is in turn equal to its own kernel. We know $\ker(A - \lambda I_2)$ is also the λ -eigenspace of A . So for all $\vec{v} \in \mathbb{R}^2$, $(A - \lambda I_n)\vec{v} = A\vec{v} - \lambda\vec{v} \in E_\lambda$.

- (c) Prove that if A has eigenvalue λ but is not diagonalizable, then A is similar to $\begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$.

Solution: By (b), we know that $A\vec{v} - \lambda\vec{v} \in E_\lambda$ for any vector $\vec{v} \in \mathbb{R}^2$. Let $\mathcal{B} = \{\vec{b}_1, \vec{b}_2\}$, where $\vec{b}_2 \in E_\lambda^\perp$ and $\vec{b}_1 = A\vec{b}_2 - \lambda\vec{b}_2$, which is in E_λ by (b). We know that these are linearly independent since they are orthogonal.

Note that this definition of \mathcal{B} ensures that $A\vec{b}_1 = \lambda\vec{b}_1$ and $A\vec{b}_2 = \vec{b}_1 + \lambda\vec{b}_2$.

Then

$$\begin{aligned} [A]_{\mathcal{B}} &= \begin{bmatrix} [A\vec{b}_1]_{\mathcal{B}} & [A\vec{b}_2]_{\mathcal{B}} \end{bmatrix} \\ &= \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix} \end{aligned}$$

So A is similar to $\begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$ by the Change-of-basis of Theorem.

- (d) Prove that if A does not have any real eigenvalues, then A is similar to a matrix of the form λQ where Q is an orthogonal matrix and $\lambda > 0$.

Solution: We know A has a pair of complex eigenvalues of the form $a \pm bi$. By Worksheet 26 problem 9, A is then similar to the matrix $B = \begin{bmatrix} a & b \\ -b & a \end{bmatrix}$. Since the columns are already orthogonal, we simply need to normalize them to make the matrix orthogonal, which can be done by dividing by $\lambda = \sqrt{a^2 + b^2}$, which is a positive number. So $Q = \frac{1}{\lambda}B$, and the statement is true.

4. (a) Find a matrix $A \in \mathbb{R}^{2 \times 2}$ such that $A \begin{bmatrix} x_n \\ x_{n+1} \end{bmatrix} = \begin{bmatrix} x_{n+1} \\ x_{n+2} \end{bmatrix}$ for every integer $n \geq 0$.

Solution:

$$A = \begin{bmatrix} 0 & 1 \\ -13 & 4 \end{bmatrix}$$

$$A \begin{bmatrix} x_n \\ x_{n+1} \end{bmatrix} = \begin{bmatrix} 0x_n + x_{n+1} \\ -13x_n + 4x_{n+1} \end{bmatrix} = \begin{bmatrix} x_{n+1} \\ x_{n+2} \end{bmatrix}$$

- (b) Use part (a) to prove by induction that your matrix A satisfies $A^n \begin{bmatrix} 0 \\ 2 \end{bmatrix} = \begin{bmatrix} x_n \\ x_{n+1} \end{bmatrix}$ for every $n \geq 0$.

Solution: Base case: $A^0 \begin{bmatrix} 0 \\ 2 \end{bmatrix} = I_2 \begin{bmatrix} 0 \\ 2 \end{bmatrix}$

Inductive step: Assume that for some integer $n \geq 0$,

$$A^n \begin{bmatrix} 0 \\ 2 \end{bmatrix} = \begin{bmatrix} x_n \\ x_{n+1} \end{bmatrix}.$$

Then by part (a),

$$\begin{aligned} AA^n \begin{bmatrix} 0 \\ 2 \end{bmatrix} &= A^{n+1} \begin{bmatrix} 0 \\ 2 \end{bmatrix} \\ &= A \begin{bmatrix} x_n \\ x_{n+1} \end{bmatrix} \\ &= \begin{bmatrix} x_{n+1} \\ x_{n+2} \end{bmatrix} \end{aligned}$$

So if n satisfies the statement, then $n + 1$ satisfies the statement. Since $n = 0$ satisfies the statement, the statement is true for all $n \geq 0$.

- (c) Find all (real or complex) eigenvalues and corresponding eigenvectors for A .

Solution:

$$\begin{aligned} \begin{vmatrix} 0 - \lambda & 1 \\ -13 & 4 - \lambda \end{vmatrix} &= (-\lambda)(4 - \lambda) + 13 \\ &= \lambda^2 - 4\lambda + 13 \end{aligned}$$

$$\lambda = \frac{4 \pm \sqrt{16 - 52}}{2} = 2 \pm 3i$$

$$\begin{aligned} &2 + 3i : \\ \begin{bmatrix} -2 - 3i & 1 \\ -13 & 2 - 3i \end{bmatrix} &\rightarrow \begin{bmatrix} -2 - 3i & 1 \\ 0 & 0 \end{bmatrix} \\ E_{2+3i} &= \text{span} \left(\begin{bmatrix} 1 \\ 2 + 3i \end{bmatrix} \right) \end{aligned}$$

$$\begin{aligned} &2 - 3i : \\ \begin{bmatrix} -2 + 3i & 1 \\ -13 & 2 + 3i \end{bmatrix} &\rightarrow \begin{bmatrix} -2 + 3i & 1 \\ 0 & 0 \end{bmatrix} \\ E_{2-3i} &= \text{span} \left(\begin{bmatrix} 1 \\ 2 - 3i \end{bmatrix} \right) \end{aligned}$$

- (d) Find an invertible (real or complex) matrix P such that $A = PDP^{-1}$ where D is a diagonal matrix.

Solution:

$$P = \begin{bmatrix} 1 & 1 \\ 2 + 3i & 2 - 3i \end{bmatrix}$$

- (e) First give an explicit formula for D_n , and then use this to give an explicit formula for A_n .

Solution: $D = \begin{bmatrix} 2 + 3i & 0 \\ 0 & 2 - 3i \end{bmatrix}$, so

$$D^n = \begin{bmatrix} (2 + 3i)^n & 0 \\ 0 & (2 - 3i)^n \end{bmatrix}$$

So then

$$A^n = P^{-1}D^nP = \begin{bmatrix} 1 & 1 \\ 2 + 3i & 2 - 3i \end{bmatrix}^{-1} \begin{bmatrix} (2 + 3i)^n & 0 \\ 0 & (2 - 3i)^n \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 + 3i & 2 - 3i \end{bmatrix}.$$

- (f) Using parts (b) and (e), give an explicit formula for x_n , the n th term in the sequence.

(Your formula may involve complex numbers, and need not be fully simplified.)

Solution:

$$\begin{aligned}
 \begin{bmatrix} x_n \\ x_{n+1} \end{bmatrix} &= A^n \begin{bmatrix} 0 \\ 2 \end{bmatrix} \\
 &= \begin{bmatrix} 1 & 1 \\ 2+3i & 2-3i \end{bmatrix}^{-1} \begin{bmatrix} (2+3i)^n & 0 \\ 0 & (2-3i)^n \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2+3i & 2-3i \end{bmatrix} \begin{bmatrix} 0 \\ 2 \end{bmatrix} \\
 &= \begin{bmatrix} 1 & 1 \\ 2+3i & 2-3i \end{bmatrix}^{-1} \begin{bmatrix} (2+3i)^n & 0 \\ 0 & (2-3i)^n \end{bmatrix} \begin{bmatrix} 2 \\ 4-3i \end{bmatrix} \\
 &= \begin{bmatrix} 1 & 1 \\ 2+3i & 2-3i \end{bmatrix}^{-1} \begin{bmatrix} 2(2+3i)^n \\ (4-3i)(2-3i)^n \end{bmatrix} \\
 &= \begin{bmatrix} \frac{3+2i}{6} & \frac{-i}{6} \\ \frac{3-2i}{6} & \frac{i}{6} \end{bmatrix} \begin{bmatrix} 2(2+3i)^n \\ (4-3i)(2-3i)^n \end{bmatrix}
 \end{aligned}$$

$$x_n = \frac{3+2i}{6} \cdot 2(2+3i)^n - \frac{i}{6}(4-3i)(2-3i)^n$$

5. (a) Find $[T]_{\mathcal{A}}$.

Solution:

$$\begin{aligned}
 [T]_{\mathcal{A}} &= \begin{bmatrix} [T(e^{3x})]_{\mathcal{A}} & [T(\cos 2x)]_{\mathcal{A}} & [T(\sin 2x)]_{\mathcal{A}} \end{bmatrix} \\
 [T(e^{3x})]_{\mathcal{A}} &= [3e^{3x}]_{\mathcal{A}} = \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix} \\
 [T(\cos 2x)]_{\mathcal{A}} &= [-2 \sin 2x]_{\mathcal{A}} = \begin{bmatrix} 0 \\ 0 \\ -2 \end{bmatrix} \\
 [T(\sin 2x)]_{\mathcal{A}} &= [2 \cos 2x]_{\mathcal{A}} = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}
 \end{aligned}$$

$$[T]_{\mathcal{A}} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & -2 & 0 \end{bmatrix}$$

- (b) Find all (real or complex) eigenvalues of the matrix $[T]_{\mathcal{A}}$.

Solution: We find the characteristic polynomial.

$$\begin{vmatrix} 3 - \lambda & 0 & 0 \\ 0 & 0 - \lambda & 2 \\ 0 & -2 & 0 - \lambda \end{vmatrix} = (3 - \lambda)(-\lambda)^2 + 4(3 - \lambda)$$

$$(3 - \lambda)(\lambda^2 + 4) = 0$$

$$\boxed{\lambda = 3, \pm 2i}$$

- (c) Viewing the matrix $[T]_{\mathcal{A}}$ as a linear transformation of the complex vector space \mathbb{C}^3 , find a complex eigenvector for $[T]_{\mathcal{A}}$ for each of the eigenvalues you found in (b).

Solution:

$$\begin{aligned} [T]_{\mathcal{A}} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} &= 3 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \\ [T]_{\mathcal{A}} \begin{bmatrix} 0 \\ -i \\ 1 \end{bmatrix} &= (2i) \begin{bmatrix} 0 \\ -i \\ 1 \end{bmatrix} \\ [T]_{\mathcal{A}} \begin{bmatrix} 0 \\ i \\ 1 \end{bmatrix} &= (-2i) \begin{bmatrix} 0 \\ i \\ 1 \end{bmatrix} \end{aligned}$$

(d) Interpret the eigenvectors you found in (c) as a set of three complex-valued functions

$$\mathcal{B} = (f_1(x), f_2(x), f_3(x))$$

with the property that any complex linear combination of the vectors in \mathcal{A} (that is, a linear combination with coefficients in \mathbb{C}) can be written as a complex linear combination of the vectors in \mathcal{B} , and vice versa.

Solution:

$$\boxed{\mathcal{B} = (e^{3x}, \sin(2x) - i \cos(2x), i \cos(2x) + \sin(2x))}$$

6. (a) Show that if I has a real eigenvalue λ then there exists an axis around which the solid object can rotate without wobbling.

Solution: Assume I has a real eigenvalue λ . Then let $\vec{\omega}$ be in the λ -eigenspace of I . Then $L = I\vec{\omega} = \lambda\vec{\omega}$. So L and $\vec{\omega}$ point in the same direction, since they are scalar multiples. Thus along the axis of rotation of the λ -eigenspace of I , the solid object will not wobble.

- (b) Show that I always has at least one real eigenvalue λ (and hence by (a) there always exists an axis around which a solid object can rotate without wobbling).

Solution: By the intermediate value theorem, an odd polynomial must always pass through the origin. Thus, it always has one real root. We know that the characteristic polynomial of I has degree 3 by Theorem 7.2.5. So I always has at least one real root, and thus one non-wobbling axis of rotation.

- (c) Show that if $\text{gemu}(\lambda) = 3$ then the solid object can rotate around any axis without wobbling.

Solution: If $\text{gemu}(\lambda) = 3$, then $I - \lambda I_3$ has nullity 3, so $I - \lambda I_3 = 0$. Thus $I = \lambda I_3$. So for any $\vec{\omega} \in \mathbb{R}^3$, $L = \lambda I_3 \vec{\omega} = \lambda \vec{\omega}$. So for any axis of angular velocity, the angular momentum will be in the same direction, and no wobbling will occur.

- (d) Show that if I has three distinct real eigenvalues then there exist three axes around which the solid object can rotate without wobbling.

Solution: Let I have distinct eigenvalues $\lambda_1, \lambda_2, \lambda_3$. Then the eigenspaces corresponding to these eigenvalues are all axes about which the solid object can rotate without wobbling. WLOG (between $\lambda_1, \lambda_2, \lambda_3$) let $\vec{\omega}_1 \in E_{\lambda_1}$. Then $L = I\vec{\omega}_1 = \lambda_1\vec{\omega}_1$. So the object will not wobble about the axis of the λ_1 -eigenspace.

Similarly for the λ_2 - and λ_3 -eigenspaces. So there exist 3 axes about which the solid object does not wobble.

- (e) Prove that for any solid object, there exist three perpendicular axes of rotation around which the object will not wobble.

Solution: From the formula for the (i, j) -component, the (i, j) component is the same as the (j, i) component (by commutativity of multiplication). So I is symmetric. Then by the Spectral Theorem, there exists an orthogonal S such that $S^{-1}IS$ is diagonal. Then S is an orthogonal eigenbasis of I .

By part (d) we know that the eigenspaces of I correspond to the axes around which the object will not wobble. From the characteristics of S , we know that the eigenspaces are orthogonal. So the axes of rotation around which the object will not wobble are orthogonal.