MATH 215 FALL 2023
Homework Set 8: §15.7 – 16.1
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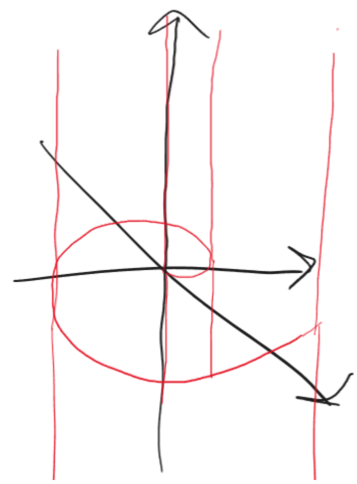
1. For the following problem, take r, θ, ρ, and ϕ to have the standard definitions in cylindrical and spherical coordinates. Describe (and try to sketch) the following surfaces:

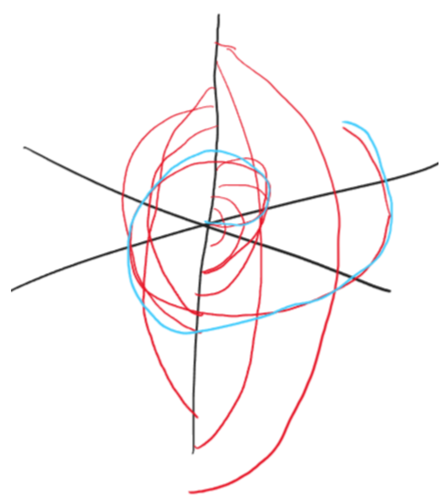
   (a) $r = \theta$

   Solution: A cylinder through a spiral starting from the origin.

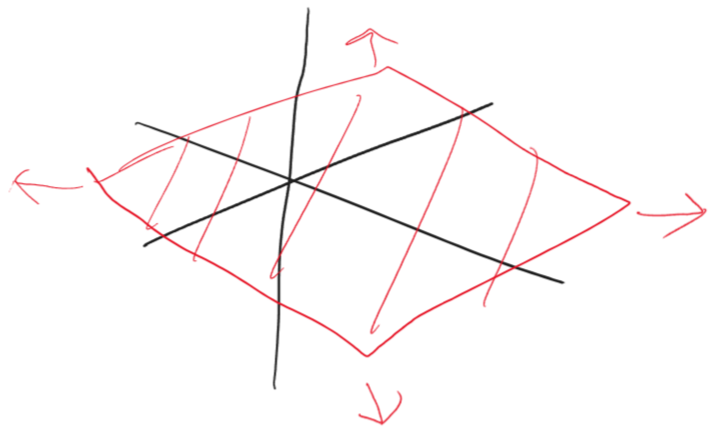
   (b) $\rho = \theta$

   Solution: A spiral in the xy plane, where each point has vertical arcs of circles passing through them to the line $x = y = 0$, each with the origin as their center.

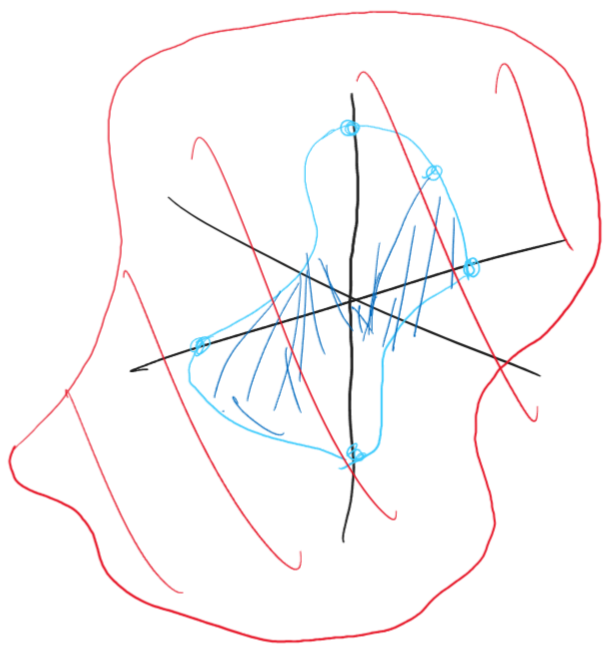
   (c) $r = \rho$

   Solution: The xy plane.



(d) $\theta = \phi$

Solution: A curved surface. When a curve is drawn on this surface with ρ fixed, the curve looks similar to a sin curve when viewed from the y-axis.



2. Let E be the ball of radius 1 centered at the point (0, 0, 1).
   (a) Show that E is given in Cartesian coordinates by the equation $x^2+y^2+z^2-2z \le 0$.
   (b) Write E in spherical coordinates. Make sure to specify the domain of ρ, θ, and ϕ.
   (c) Suppose the density on E is proportional to the distance to the origin, with the largest density being equal to 2. Use spherical coordinates to compute the mass

and center of mass of E.

(d) Suppose we tried to do this problem for the ball of radius 1 centered at the point (0, 1, 0). Why is this problem harder with the new ball?

3. Begin with a sphere of radius R and bore a hole into the sphere in the shape of a right circular cylinder, leaving only a band of height h. Find the volume of the resulting shape.

Solution: The radius of the cylinder will be $r_c = \sqrt{R^2 - h^2}$. We use cylindrical coordinates to perform the integration.

$$\begin{aligned}
2\pi \int_{-h}^{h} \int_{\sqrt{R^2-h^2}}^{\sqrt{R^2-z^2}} r\, dr\, d\theta \\
= \pi \int_{-h}^{h} \left(r^2\right) |_{\sqrt{R^2-h^2}}^{\sqrt{R^2-z^2}} dr\, d\theta \\
= \pi \int_{-h}^{h} R^2 - z^2 - R^2 + h^2\, d\theta \\
= \pi \left(-\frac{z^3}{3} + h^2 z\right) |_{z=-h}^{h} \\
= \boxed{\frac{4\pi h^3}{3}}
\end{aligned}$$

□

4. Find the mass of a wedge cut from a sphere of radius R by two planes that intersect along a diameter and at an angle of $\frac{\pi}{5}$, assuming that the density is proportional to the distance from the origin in such a way that the maximum density is 2. (This shape should look like a segment of an orange.)

Solution: We use spherical coordinates for this problem, with (r, θ, ϕ). The density function will be $\rho(r) = \frac{2r}{R}$ to have a maximum density of 2 when the distance is equal to the radius.

$$
\begin{aligned}
&\frac{\pi}{5}\int_0^R\int_0^\pi \frac{2r}{R}r^2\sin(\phi)\,d\phi\,dr \\
&= \frac{\pi}{5R}\int_0^R 2r^3\int_0^\pi \sin(\phi)\,d\phi\,dr \\
&= \frac{\pi}{5R}\int_0^R 2r^3\left(-\cos(\phi)\right)\Big|_{\phi=0}^{\pi}\,dr \\
&= \frac{\pi}{5R}\int_0^R 4r^3\,dr \\
&= \frac{\pi}{5R}\left(r^4\right)\Big|_{r=0}^{R} \\
&= \boxed{\frac{\pi R^3}{5}}
\end{aligned}
$$

□

5. Find $\int\int_R f(x,y)\,dA$ where $f(x,y) = 3y^2 - 4xy - 4x^2$ and R is the quadrilateral with vertices (0, 2), (3, 0), (5, 4), and (2, 6). *Hint*: There may be a straightforward but tedious way to solve this problem, as well as a faster, more subtle, way to solve this problem.

   Solution: We can factor $f(x,y) = (3y+2x)(y-2x)$. Then, we can use change of variables to change both the function and the bounds. Let $u = 3y + 2x$, $v = y - 2x$. Then $f(u,v) = uv$, $d(x,y) = (2 - 3(-2)))^{-1}\,d(u,v) = \frac{1}{8}d(u,v)$. Also, R has vertices at $(u,v) = (6,2), (6,-6), (22,-6), (22,2)$.

$$\frac{1}{8}\int_{-6}^{2}\int_{6}^{22} uv\sqrt{1+u^2+v^2}\,du\,dv$$
$$t = 1+u^2+v^2,\, dt = 2u\,du$$
$$= \frac{1}{16}\int_{-6}^{2} v \int_{u=6}^{u=22} \sqrt{t}\,dt\,dv$$
$$= \frac{1}{16}\int_{-6}^{2} v \left(\frac{2t^{3/2}}{3}\right)_{u=6}^{u=22} dv$$
$$= \frac{1}{16}\int_{-6}^{2} v\left(\frac{2(485+v^2)^{3/2}}{3}\right) - v\left(\frac{2(37+v^2)^{3/2}}{3}\right) dv$$
$$= \frac{1}{16}\int_{-6}^{2} v\left(\frac{2(485+v^2)^{3/2}}{3}\right) - v\left(\frac{2(37+v^2)^{3/2}}{3}\right) dv$$
$$w = 485+v^2,\, dw = 2v\,dv;\, z = 37+v^2,\, dz = 2v\,dv$$
$$= \frac{1}{16}\int_{v=-6}^{v=2}\left(\frac{(w)^{3/2}}{3}\right) dw - \frac{1}{16}\int_{v=-6}^{v=2}\left(\frac{(z)^{3/2}}{3}\right) dz$$
$$= \frac{1}{8}\left(\frac{(w)^{5/2}}{15}\right)_{-6}^{v=2} - \frac{1}{8}\left(\frac{(z)^{5/2}}{15}\right)_{v=-6}^{2}$$
$$= \boxed{-7276.863857}$$

□

6. Let E be the region in the first quadrant that is above the line $y = \frac{x}{3}$, below the line $y = 3x$, and between the curves defined by $xy = 3$ and $xy = 27$.
   (a) Sketch the region.
   (b) Evaluate $\int\int(\frac{x^2}{y^2} + x^2y^2)\,dA$. (Hint: Try $u = xy$ and $v = \frac{y}{x}$.)
   (c) Why was the hint a reasonable guess for a change of coordinates?

7. Do Exercises 13-18 of §16.1 in *Stewart's Multivariable Calculus*.

Solution:

13. $\boxed{IV}$ – vectors with direction and magnitude equal to displacement, except flipped vertically.

14. $\boxed{V}$ – downward direction when $x < y$, upward when $y < x$, horizontal when $x = y$.

15. $\boxed{I}$ – when $y = -2$, vectors are horizontal.

16. $\boxed{VI}$ – magnitude increases more with x than y.

17. $\boxed{III}$ – the magnitude/direction oscillates when either coordinate is fixed.

18. $\boxed{II}$ – direction becomes more vertical when x increases, while horizontal component oscillates.

8. Do Exercises 19-22 of §16.1 in *Stewart's Multivariable Calculus*.

Solution:

19. $\boxed{IV}$ – only constant vector field.

20. $\boxed{I}$ – the vector field is constant when z is fixed.

21. $\boxed{III}$ – always positive vertical direction, same direction as displacement from origin for x and y.

22. $\boxed{II}$ – same direction/magnitude as displacement from origin.

9. Do Exercises 31-34 of §16.1 in *Stewart's Multivariable Calculus*.

Solution:

31. $\boxed{III}$ – gradient is $(2x, 2y)$, so linearly increasing magnitude and same direction as displacement from origin.

32. $\boxed{IV}$ – gradient is $(2x+y, x)$, thus the direction is close to horizontal near the y-axis and becomes more vertical as x increases.

33. $\boxed{II}$ – gradient is $(2x + 2y, 2y + 2x)$. Since the x and y coordinates are the same, the direction is always the same $\langle 1, 1\rangle$, except with positive or negative magnitude.

34. $\boxed{I}$ – Gradient will include something with cos for both f_x and f_y coordinates, thus the magnitude will oscillate.