MATH 217 W24 - LINEAR ALGEBRA, Section 001 (Dr. Paul Kessenich)
Homework Set Part B due Sunday, February 18 at 11:59pm

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1. Let V and W be vector spaces, and let $T : V \to W$ be a linear transformation. Let $X = (\vec{x}_1, \dots, \vec{x}_k)$ be a list of vectors in V , and consider the list $Y = (T(\vec{x}_1), \dots, T(\vec{x}_k))$ of vectors in W. Determine whether the following statements are true or false. If true, provide a proof. If false, provide a counter-example.

   (a) If X is linearly independent, then Y is also linearly independent.

   Solution: False, consider $T_0 : \mathbb{R}^2 \to \mathbb{R}^2$, $\vec{v} \mapsto \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $X = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$. Then $Y = \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}$, which is not linearly independent since $\vec{y}_1 + \vec{y}_2 = 0$.

   (b) If Y is linearly independent, then X is also linearly independent.

   Solution: True. We show the contrapositive. Let X be a linearly dependent set of vectors in V. Then by definition there exist nonzero scalars $c_1, c_2, \dots, c_k$ such that $c_1\vec{x}_1 + \cdots c_n\vec{x}_n = 0$. Then we know:

$$
\begin{aligned}
T(c_1\vec{x}_1 + \cdots + c_n\vec{x}_n) &= T(0) \\
T(c_1\vec{x}_1) + \cdots + T(c_n\vec{x}_n) &= 0_W && \text{(linearity)} \\
c_1T(\vec{x}_1) + \cdots + c_nT(\vec{x}_n) &= 0_W && \text{(linearity)}
\end{aligned}
$$

   So Y has a nontrivial relation, and hence is linearly dependent when X is linearly dependent. Thus the contrapositive is true, and the original statement is also true.

2. (a) Find a linear transformation $T : \mathbb{R}^5 \to \mathbb{R}^3$ such that
$\ker(T) = \{\vec{x} \in \mathbb{R}^5 : x_1 = 5x_2 \text{ and } x_3 = 7x_4\}$ and $\operatorname{im}(T) = \{\vec{x} \in \mathbb{R}^3 : x_1 = x_3\}$.

Solution: This can be satisfied by a linear transformation with standard matrix

$$A = \begin{bmatrix} 1 & -5 & 0 & 0 & 0 \\ 0 & 0 & 1 & -7 & 0 \\ 1 & -5 & 0 & 0 & 0 \end{bmatrix}$$

(b) Is the linear transformation you found in part (a) unique? Justify your claim.

Solution: No. As we know from class, elementary row operations do not change the solutions to the system, so performing any elementary row operations on A will maintain its properties with relation to kernel. As for image, scaling the entire matrix maintains the directions of the span, so it will have the same image. So, scalar multiples of A will have the same properties.

3. Let X and Y be vector spaces.

(a) Consider a basis $\mathcal{B} = \vec{x}_1, \ldots, \vec{x}_n$ of X. Let $\vec{y}_1, \ldots, \vec{y}_n$ be any vectors (not necessarily a basis, or even distinct) in Y. Prove that there exists a unique linear transformation $T : X \to Y$ such that $T(\vec{x}_i) = \vec{y}_i$ for all $1 \le i \le n$.

Solution: By problem 1(a), the set of y_i need not be linearly independent although $\mathcal{B}$ is. So there is nothing contradictory about defining a T by the given qualifications.

Let T_1, T_2 be linear transformations which satisfy $T_1(\vec{x}_i) = T_2(\vec{x}_i) = \vec{y}_i$ for all $1 \le i \le n$. Then, since $\mathcal{B}$ spans X, all $\vec{v} \in X$ can be expressed as a linear combination

$$\vec{v} = c_1\vec{x}_1 + \cdots + c_n\vec{x}_n,$$

where $c_1, \ldots, c_n \in \mathbb{R}$. So

$$T_1(\vec{v}) = c_1T_1(\vec{x}_1) + \cdots + c_nT_1(\vec{x}_n) = c_1T_2(\vec{x}_1) + \cdots + c_nT_2(\vec{x}_n) = T_2(\vec{v})$$

by linearity. Thus $T_1 = T_2$, and such a linear transformation is unique.

(b) Let U and V be subspaces of X and Y respectively such that $\dim(U) + \dim(V) = \dim(X)$. Prove that there exists a linear transformation $T_{U,V} : X \to Y$ such that $\ker(T_{U,V}) = U$ and $\operatorname{im}(T_{U,V}) = V$. (Hint: use part (a). You might also want to try to generalize the method you used to solve Problem 2.)

Solution: Using basis $\mathcal{B}$ and $\vec{y}_1, \ldots, \vec{y}_n$ from part (a), let $y_1, y_2, \ldots, y_k$ be a basis for U, where $k = dim(U)$. Additionally, let $y_{k+1} = y_{k+2} = \cdots = y_n = 0_Y$. Also, let $x_{k+1}, \ldots, x_n$ be the basis of U.

Then by part (a), there exists a unique linear transformation from $\mathcal{B}$ to $y_1, \ldots, y_n$. Additionally, since $x_{k+1}, \ldots, x_n$ all map to 0_V, the kernel of T has dimension equal to $\dim(\operatorname{span}(\{x_{k+1}, \ldots, x_n\})) = \dim(U) = n - k$.

Meanwhile, the image is $\operatorname{span}(\{y_1, y_2, \ldots, y_k\}) = V$, which has dimension $\dim(\operatorname{span}(\{y_1, y_2, \ldots, y_k\})) = \dim(V) = k$. So T is a linear transformation with image V, kernel U, and satisfies $\dim(U) + \dim(V) = \dim(X)$.

(c) Is the map $T_{U,V}$ that you found in part (b) unique? Justify your answer.

Solution: No. Similarly to problem 2(b), a scalar multiple of T would have the same image and kernel.

4. Let U, V, and W be finite-dimensional vector spaces, and let $T : U \to V$ and $S : V \to W$ be linear transformations. Determine whether the following statements are true or false, and provide a proof of your claim.

   (a) $\text{rank}(S \circ T) \leq \text{rank}(S)$.

   Solution: True. The image of any $\vec{v} \in V$ under S cannot lie outside the image, of S itself, so neither can $S(T(\vec{u}))$ for $T(\vec{u}) \in V$ with any $\vec{u} \in U$. Thus the image of S contains the image of $S \circ T$, and $S \circ T$ has rank at most the rank of S.

   (b) $\text{rank}(S \circ T) \leq \text{rank}(T)$.

   Solution: True. Let the basis of the image of T be $\mathcal{T} = \{\vec{t}_1, \ldots, \vec{t}_n\}$, where n is $\text{rank}(T)$. Then any vector $\vec{v} \in \text{im}(T) \subseteq V$ can be represented as a linear combination $\vec{v} = c_1\vec{t}_1 + \cdots + c_n\vec{t}_n$. Applying S to this:

$$\begin{aligned} S(\vec{v}) &= S(c_1\vec{t}_1 + \cdots + c_n\vec{t}_n) \\ &= c_1 S(\vec{t}_1) + \cdots + c_n S(\vec{t}_n) && \text{(linearity)} \end{aligned}$$

   So any vector in the image of $S \circ T$ can be represented as a linear combination of at most n vectors. This means the rank of $S \circ T$ is at most $\text{rank}(T)$, and the statement is true.

   (c) $\text{nullity}(S \circ T) \geq \text{nullity}(T)$.

   Solution: True. Anything in the kernel of T is already mapped to 0_V, which is, by the property of linear transformations, then mapped to 0_W by S. So the kernel of $S \circ T$ contains at least the kernel of T, meaning its nullity is greater than or equal to $\text{nullity}(T)$.

   (d) $\text{nullity}(S \circ T) \geq \text{nullity}(S)$.

   Solution: False. Take $T : \mathbb{R} \to \mathbb{R}^2$ with standard matrix $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$, and $S : \mathbb{R}^2 \to \mathbb{R}^3$ with standard matrix $\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$ and $\text{nullity}(S) = 2$. Then $S \circ T$ can be represented $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$, which has a nullity of $1 < \text{nullity}(S)$.

5. A matrix $A \in \mathbb{R}^{n \times n}$ is said to be symmetric if $A^\top = A$, and skew-symmetric if $A^\top = -A$. Let Sym_n and Skew_n denote the set of all symmetric matrices and the set of all skew-symmetric matrices in $\mathbb{R}^{n \times n}$, respectively.

   (a) Let $T : \mathbb{R}^{n \times n} \to \mathbb{R}^{n \times n}$ be the map defined by $T(A) = A + A^\top$. Prove that T is linear.

   Solution: Let $A, B \in \mathbb{R}^{n \times n}$, and $c \in \mathbb{R}$. We know that tranpose respects scalar multiplication, since the

   scalar multiplication applies to every element of the matrix, regardless of the transpose or not. Likewise with addition, $(A+B)^\top = A^\top + B^\top$ since the addition occurs between elements with the same index. So

$$\begin{aligned} T(A+B) &= A + B + (A+B)^\top \\ &= A + A^\top + B + B^\top \\ &= T(A) + T(B) \end{aligned}$$

$$\begin{aligned} T(cA) &= cA + (cA)^\top \\ &= cA + c(A^\top) \\ &= c(A + A^\top) \\ &= cT(A) \end{aligned}$$

   Thus T is linear.

   (b) Prove that $\ker(T) = \mathrm{Skew}_n$ and $\mathrm{im}(T) = \mathrm{Sym}_n$.

   Solution: First we show $\ker(T) = \mathrm{Skew}_n$. Let $A \in \ker(T) \subseteq \mathbb{R}^{n \times n}$. Then by definition, $A + A^\top = 0_{\mathbb{R}^{n \times n}}$. Subtracting by A on both sides, $A^\top = -A$. So $A \in \mathrm{Skew}_n$, and $\ker(T) \subseteq \mathrm{Skew}_n$.

   Next, let $A \in \mathrm{Skew}_n \subset \mathbb{R}^{n \times n}$. Then by definition, $A^\top = -A$. Adding A to both sides grants $A + A^\top = 0_{\mathbb{R}^{n \times n}}$. So A is within the kernel of T, and $\mathrm{Skew}_n \subseteq \ker(T)$. Since $\ker(T)$ and Skew_n are mutual subsets, they are equal.

   To show $\mathrm{im}(T) = \mathrm{Sym}_n$, let $A \in \mathrm{im}(T) \subseteq \mathbb{R}^{n \times n}$. Then A can be expressed $A = B + B^\top$ for some $B \in \mathbb{R}^{n \times n}$. Taking the transpose, $A^\top = (B + B^\top)^\top = B^\top + B = A$. So $A \in \mathrm{Sym}_n$, and $\mathrm{im}(T) \subseteq \mathrm{Sym}_n$.

   Next let $A \in \mathrm{Sym}_n \subseteq \mathbb{R}^{n \times n}$, and let $B \in \mathbb{R}^{n \times n}$ be $\frac{1}{2}A$. Then A can be expressed $A = B + B^\top$, since B is also symmetrical. So $A \in \mathrm{im}(T)$, $\mathrm{im}(T) \subseteq \mathrm{Sym}_n$. Since $\mathrm{im}(T)$ and Sym_n are mutual subsets, they are equal.

   (c) Prove that Sym_n and Skew_n are subspaces of $\mathbb{R}^{n \times n}$.

   Solution: Zero: Since $0_{\mathbb{R}^{n \times n}}$ is a matrix with all elements equal to 0, it is both symmetric and skew-symmetric.

   Symmetric: Let $A, B \in \mathbb{R}^{n \times n}$ be symmetric. That is, $A = A^\top$ and $B = B^\top$. Let

c be an arbitrary scalar.

Additive closure: $(A+B)^\top = A^\top + B^\top = A + B$. So $A+B$ fulfills the definition of symmetric.

Closure under scalar multiplication: $(cA)^\top = cA^\top = cA$. So cA fulfills the definition of symmetric.

Skew-symmetric: Let $A, B \in \mathbb{R}^{n\times n}$ be skew-symmetric. That is, $-A = A^\top$ and $-B = B^\top$. Let c be an arbitrary scalar.

Additive closure: $(A+B)^\top = A^\top + B^\top = -A - B = -(A+B)$. So $A+B$ fulfills the definition of skew-symmetric.

Closure under scalar multiplication: $(cA)^\top = cA^\top = c(-A) = -(cA)$. So cA fulfills the definition of skew-symmetric.

(d) Find $\dim(\mathrm{Sym}_n)$ and $\dim(\mathrm{Skew}_n)$.

Solution: Symmetric matrices can be created by "reflecting" an upper triangular matrix across its diagonal. Similarly, a skew-symmetric matrix would be a strictly upper triangular matrix "reflected" across its diagonal, except with the negatives of the upper elements. So, the dimension is simply the number of upper triangular or strictly upper triangular elements of a $\mathbb{R}^{n\times n}$ matrix.

Upper triangular and symmetric matrices would have a dimension of $\frac{n(n+1)}{2}$ (sum of $1 + ... + n$). Meanwhile, strictly upper triangular and skew-symmetric matrices would have $\frac{n(n-1)}{2}$ (sum of $1 + ... + n - 1$).