EECS 203: Discrete Mathematics
Winter 2024
Homework 4

Due **Thursday, Feb. 15th**, 10:00 pm
No late homework accepted past midnight.

Number of Problems: $8 + 2$ Total Points: $100 + 20$

- **Match your pages!** Your submission time is when you upload the file, so the time you take to match pages doesn't count against you.

- Submit this assignment (and any regrade requests later) on Gradescope.

- Justify your answers and show your work (unless a question says otherwise).

- By submitting this homework, you agree that you are in compliance with the Engineering Honor Code and the Course Policies for 203, and that you are submitting your own work.

- Check the syllabus for full details.

Individual Portion

1. Even Just One [12 points]

Prove that if $n^3 + 4$ is even or $3n + 3$ is odd, then n is even.

> **Solution:**
>
> We show the contrapositive, that is, if n is odd, then $n^3 + 4$ is odd and $3n + 3$ is even. Assume n is odd. Then n can be expressed $n = 2k + 1$, where k is an integer. Then for $n^3 + 4$,
>
> $$\begin{aligned} n^3 + 4 &= \\ &= (2k+1)^3 + 4 \\ &= 8k^3 + 12k^2 + 6k + 1 + 4 \\ &= 2(4k^3 + 6k^2 + 3k + 2) + 1 \end{aligned}$$
>
> So n^3 is odd, since $4k^3 + 6k^2 + 3k + 2$ is an integer.
>
> Similarly, for $3n + 3$:
>
> $$\begin{aligned} 3n + 3 &= \\ &= 3(2k+1) + 3 \\ &= 6k + 3 + 3 \\ &= 2(3k + 3) \end{aligned}$$
>
> Since $3k + 3$ is an integer, $3n + 3$ is even. So the contrapositive is true, and the original statement is true.

2. Odd^2 [20 points]

Prove the following for all integers x and y:

(a) If $x + y$ is even, then (x is even and y is even) or (x is odd and y is odd).

(b) Using your answer from part (a), show that if $(x - y)^2$ is odd, then $x + y$ is odd.

> **Solution:**

(a) We use casework, with all possible cases: x, y are even; x, y are odd; and x is even and y is odd. Note that the final case is equivalent to x being odd and y being even due to commutativity.

x, y are even: x, y can be expressed as $x = 2n$, $y = 2m$, where $n, m \in \mathbb{Z}$. So $x + y = 2n + 2m = 2(n + m)$. Since $n + m$ is an integer, $x + y$ is even.

x, y are odd: x, y can be expressed as $x = 2n + 1$, $y = 2m + 1$, where $n, m \in \mathbb{Z}$. So $x + y = 2n + 1 + 2m + 1 = 2(n + m + 1)$. Since $n + m + 1$ is an integer, $x + y$ is even.

x is even and y is odd: x and y can be expressed as $x = 2n$, $y = 2m + 1$, where $n, m \in \mathbb{Z}$. So $x + y = 2n + 2m + 1 = 2(n + m) + 1$. Since $n + m$ is an integer, $x + y$ is odd.

(b) We show the contrapositive, that is, if $x + y$ is even, then $(x - y)^2$ is even. From part (a), we know that if $x + y$ is even, then both x and y have the same parity. We use casework to find the parity of $x - y$.

If they are even, then $x = 2n$ and $y = 2m$, where $n, m \in \mathbb{Z}$. So $x - y = 2n - 2m = 2(n - m)$. Since $n - m$ is an integer, $x - y$ is even.

If x and y are odd, then $x = 2n + 1$ and $y = 2m + 1$, where $n, m \in \mathbb{Z}$. So $x - y = 2n + 1 - 2m - 1 = 2(n - m)$. Since $n - m$ is an integer, $x - y$ is even.

So, $x - y$ is always even when $x + y$ is even. Then by problem 2 (a) of HW 3, $(x - y)^2$ is even. Thus, the contrapositive is proven and the statement is true.

3. Do you $\exists$xist...? [8 points]

Prove or disprove the following: There exist integers x and y so that $20x + 4y = 1$.

Solution:

False. Assume such integers exist. Then $20x + 4y = 2(10x + 2y)$. So since x and y are integers, $20x + 4y$ is even. But 1 is odd, so there is a contradiction. Thus, such integers cannot exist.

4. What's Nunya? Nunya Products are Negative. [12 points]

Given any three real numbers, prove that the product of two of them will always be non-negative.

5. Element or Subset? [8 points]

Let $A = \{1, 2, \text{"a"}\}$. State whether each statement is true or false. Give a brief explanation if false (you do not need to justify why a statement is true).

(a) "a" $\in A$

(b) "a" $\subseteq A$

(c) $\{1, 2\} \in A$

(d) $\{1, 2\} \subseteq A$

Solution:

(a) True

(b) False, "a" is not a set.

(c) False, A has no element which is a set.

(d) True

6. Ready, $\{s, e, t\}$, go! [12 points]

Let $S = \{1, 2, 3, 4, 5\}$, $A = \{1, 2\}$, $B = \{2, 3\}$, and $C = \{4, 5\}$. Compute the following, where complements are taken within S. Show intermediate steps as part of your justification.

(a) $\mathcal{P}\left((A\cap B)\cap\overline{C}\right)$

(b) $\mathcal{P}\left((\overline{C}-B)\cap A\right)$

(c) $\{A\times B\}\cap\{S\times B\}$

(d) $(A\times B)\cap(S\times B)$

Solution:

(a)

$$\mathcal{P}\left((A\cap B)\cap\overline{C}\right)$$
$$= \mathcal{P}\left((\{1,2\}\cap\{2,3\})\cap\{1,2,3\}\right)$$
$$= \mathcal{P}\left(\{2\}\cap\{1,2,3\}\right)$$
$$= \mathcal{P}\left(\{2\}\right)$$
$$= \{\emptyset,\{2\}\}$$

(b)

$$\mathcal{P}\left((\overline{C}-B)\cap A\right)$$
$$= \mathcal{P}\left((\{1,2,3\}-\{2,3\})\cap\{1,2\}\right)$$
$$= \mathcal{P}\left(\{1\}\cap\{1,2\}\right)$$
$$= \mathcal{P}\left(\{1\}\right)$$
$$= \{\emptyset,\{1\}\}$$

(c)

$$\{A\times B\}\cap\{S\times B\}$$
$$= \{\{a,b\,|\,a\in A, b\in B\}\}\cap\{\{s,b\,|\,s\in S, b\in B\}\}$$
$$= \emptyset$$

(d)

$$(A\times B)\cap(S\times B)$$
$$= (A\cap S)\times B$$
$$= \{1,2\}\times\{2,3\}$$
$$= \{(1,2),(1,3),(2,2),(2,3)\}$$

7. Subset Proofs [16 points]

Prove that if A and B are sets, then $A \cup (A \cap B) = A$ by proving each side is a subset of the other. This set identity is known as an absorption law. Your answer should be a word proof, and not use any set equivalence laws.

Solution:

Let A and B be sets. Consider arbitrary element $x \in A \cup (A \cap B)$. By definition of the union operation, x is contained in A or $A \cap B$. If $x \in A \cap B$, then $x \in A$ by definition of the intersection operation. So x is always an element of A. So $A \cup (A \cap B) \subseteq A$.

Consider arbitrary element $y \in A$. By definition of the union operation, y is contained in any set $A \cup C$ for any set C. So $y \in A \cup (A \cap B)$. Thus $A \subseteq A \cup (A \cap B)$.

So A and $A \cup (A \cap B)$ are mutual subsets, and $A \cup (A \cap B) = A$.

8. IceCream-Exclusion [12 points]

Out of the 40 EECS 203 staff members, 21 like vanilla ice cream, 18 like chocolate ice cream, and 24 like strawberry ice cream. In addition, 13 like both strawberry and vanilla, and 7 like chocolate and vanilla.

(a) How many staff members like all three ice cream flavors if 9 staff members like both strawberry and chocolate ice cream, assuming everyone likes at least one type of ice cream?

(b) How many staff members don't like any of the ice cream flavors if 14 staff members like both strawberry and chocolate ice cream and 3 staff members like all three ice cream flavors?

Solution:

(a) 21 + 18 + 24 - 13 - 7 - 9 = 34. 40 - 34 = 6 people like all 3 ice cream flavors.

(b) 21 + 18 + 24 - 13 - 7 - 14 + 3 = 32. 40 - 32 = 8 people like no ice cream flavors.