MATH 217 W24 - LINEAR ALGEBRA, Section 001 (Dr. Paul Kessenich)
Homework Set Part B due ??? at 11:59pm
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1. Let V be a vector space, and let $(\vec{v}_1, ..., \vec{v}_n)$ be a list of vectors in V. Define the function $T : \mathbb{R}^n \to V$ by

$$T\left(\begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}\right) = c_1\vec{v}_1 + \cdots + c_n\vec{v}_n \text{ for all } \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} \in \mathbb{R}^n$$

(a) Prove that T is a linear transformation.

Solution: Let $b = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}, c = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} \in \mathbb{R}^n, k \in \mathbb{R}$

$$\begin{aligned} T(b + c) &= (b_1 + c_1)v_1 + \cdots + (b_n + c_n)v_n \\ &= b_1v_1 + \cdots + b_nv_n + c_1v_1 + \cdots + c_nv_n \\ &= T(b) + T(c) \\ T(kc) &= (kc_1)v_1 + \cdots + (kc_n)v_n \\ &= k(c_1v_1) + \cdots + k(c_nv_n) \\ &= k(c_1v_1 + \cdots + c_nv_n) \\ &= kT(c) \end{aligned}$$

(b) Prove that T is injective if and only if $(\vec{v}_1, \ldots, \vec{v}_n)$ is linearly independent.

Solution: We know T is injective if and only if its kernel is 0_V. By definition, if $(\vec{v}_1, \ldots, \vec{v}_n)$ is linearly independent, then only $c_1 = \cdots = c_n = 0 \in \mathbb{R}$ satisfies $c_1v_1 + \cdots + c_nv_n = 0_V$, so the kernel of T is $\{0_n\}$. Meanwhile, if $(\vec{v}_1, \ldots, \vec{v}_n)$ is linearly dependent, there exists $c_1, \cdots, c_n \in \mathbb{R}$ not all zero which satisfy $c_1v_1 + \cdots + c_nv_n = 0_V$, meaning the kernel of T would not only be $\{0_n\}$. So linear independence of $(\vec{v}_1, \ldots, \vec{v}_n)$ is equivalent to T satisfying $\ker[T] = 0_n$, which in turn is equivalent to T being injective.

(c) Prove that T is surjective if and only if $(\vec{v}_1, \ldots, \vec{v}_n)$ spans V.

Solution: If $(\vec{v}_1, \ldots, \vec{v}_n)$ spans V, any vector $\vec{v} \in V$ can be represented as $c_1v_1 + \cdots + c_nv_n = \vec{v}$, with $c_1, \ldots, c_n \in \mathbb{R}$. Then $T\left(\begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}\right) = v$. So $(\vec{v}_1, \ldots, \vec{v}_n)$ spanning V implies that T is surjective.

If T is surjective, then for all $v \in V$, there exists $\begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} \in \mathbb{R}^n$ such that $T\left(\begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}\right) =$

$\vec{v}$. Then $v = c_1\vec{v}_1 + \cdots + c_n\vec{v}_n$ is a linear combination of $(\vec{v}_1, \ldots, \vec{v}_n)$. So every $v \in V$ is in the span of $(\vec{v}_1, \ldots, \vec{v}_n)$.

So we have shown that the surjectivity of T is equivalent to if $(\vec{v}_1, \ldots, \vec{v}_n)$ spans V.

(d) Prove that T is an isomorphism if and only if $(\vec{v}_1, \ldots, \vec{v}_n)$ is an ordered basis of V.

Solution: By parts (b) and (c), $(\vec{v}_1, \ldots, \vec{v}_n)$ must be both linearly independent and span V in order for linear transformation T to be bijective. Then by Theorem B of Worksheet 11, the minimal spanning subset and maximal linearly independent ordered set $(\vec{v}_1, \ldots, \vec{v}_n)$ must be an ordered basis.

2. For a 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, define the **transpose** of A to be the matrix

$$A^\top = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$$

Consider the linear transformation

$$T : \mathbb{R}^{2\times 2} \to \mathbb{R}^{2\times 2} \qquad\qquad T(A) = \frac{1}{2}(A + A^\top).$$

(a) Find the $\mathcal{E}$-matrix $[T]_\mathcal{E}$ of T, where

$$\mathcal{E} = \left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \right)$$

Solution: Plugging in each basis vector into T gives

$$T\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$
$$T\left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{bmatrix}$$
$$T\left(\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right) = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$
$$T\left(\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Using these results, we can find the $\mathcal{E}$-matrix of T

$$[T]_{\mathcal{E}}\left(\begin{bmatrix}1\\0\\0\\0\end{bmatrix}\right)=\left[\begin{bmatrix}1&0\\0&0\end{bmatrix}\right]_{\mathcal{E}}=\begin{bmatrix}1\\0\\0\\0\end{bmatrix}$$

$$[T]_{\mathcal{E}}\left(\begin{bmatrix}0\\1\\0\\0\end{bmatrix}\right)=\left[\begin{bmatrix}0&\frac{1}{2}\\\frac{1}{2}&0\end{bmatrix}\right]_{\mathcal{E}}=\begin{bmatrix}0\\1\\0\\-\frac{1}{2}\end{bmatrix}$$

$$[T]_{\mathcal{E}}\left(\begin{bmatrix}0\\0\\1\\0\end{bmatrix}\right)=\left[\begin{bmatrix}0&0\\0&1\end{bmatrix}\right]_{\mathcal{E}}=\begin{bmatrix}0\\0\\1\\0\end{bmatrix}$$

$$[T]_{\mathcal{E}}\left(\begin{bmatrix}1\\0\\0\\0\end{bmatrix}\right)=\left[\begin{bmatrix}0&0\\0&0\end{bmatrix}\right]_{\mathcal{E}}=\begin{bmatrix}0\\0\\0\\0\end{bmatrix}$$

(b) Find the $\mathcal{C}$-matrix of T, where $\mathcal{C}$ is the ordered basis

$$\mathcal{C}=\left(\begin{bmatrix}0&1\\1&0\end{bmatrix},\begin{bmatrix}1&0\\0&0\end{bmatrix},\begin{bmatrix}0&0\\0&1\end{bmatrix},\begin{bmatrix}0&1\\-1&0\end{bmatrix}\right)$$

Solution: Plugging in each basis vector into T gives

$$T\left(\begin{bmatrix}0&1\\1&0\end{bmatrix}\right)=\begin{bmatrix}0&1\\1&0\end{bmatrix}$$

$$T\left(\begin{bmatrix}1&0\\0&0\end{bmatrix}\right)=\begin{bmatrix}1&0\\0&0\end{bmatrix}$$

$$T\left(\begin{bmatrix}0&0\\0&1\end{bmatrix}\right)=\begin{bmatrix}0&0\\0&1\end{bmatrix}$$

$$T\left(\begin{bmatrix}0&1\\-1&0\end{bmatrix}\right)=\begin{bmatrix}0&-1\\1&0\end{bmatrix}$$

Using these results, we can find the $\mathcal{C}$-matrix of T

$$[T]_{\mathcal{C}}\left(\begin{bmatrix}1\\0\\0\\0\end{bmatrix}\right) = \left[\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}\right]_{\mathcal{C}} = \begin{bmatrix}1\\0\\0\\0\end{bmatrix}$$
$$[T]_{\mathcal{C}}\left(\begin{bmatrix}0\\1\\0\\0\end{bmatrix}\right) = \left[\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}\right]_{\mathcal{C}} = \begin{bmatrix}0\\1\\0\\0\end{bmatrix}$$
$$[T]_{\mathcal{C}}\left(\begin{bmatrix}0\\0\\1\\0\end{bmatrix}\right) = \left[\begin{bmatrix}0 & 0\\0 & 1\end{bmatrix}\right]_{\mathcal{C}} = \begin{bmatrix}0\\0\\1\\0\end{bmatrix}$$
$$[T]_{\mathcal{C}}\left(\begin{bmatrix}1\\0\\0\\0\end{bmatrix}\right) = \left[\begin{bmatrix}0 & -1\\1 & 0\end{bmatrix}\right]_{\mathcal{C}} = \begin{bmatrix}0\\0\\0\\-1\end{bmatrix}$$

(c) Compute the kernel of $[T]_{\mathcal{E}}$. This will be a subspace of the $\mathcal{E}$-coordinate space $\mathbb{R}^4$ for $\mathbb{R}^{2\times 2}$.

Solution:

(d) Find a basis for the corresponding subspace of $\mathbb{R}^{2\times 2}$-that is, for the image of $\ker[T]_{\mathcal{E}}$ under the coordinate isomorphism $L_{\mathcal{E}}^{-1} : \mathbb{R}^4 \to \mathbb{R}2 \times 2$.

(e) Compute the kernel of the $\mathcal{C}$-matrix. This will be a subspace of the $\mathcal{C}$-coordinate space $\mathbb{R}^4$ for $\mathbb{R}^{2\times 2}$.

(f) Compute the image of the subspace $\ker[T]_{\mathcal{C}}$ under the coordinate isomorphism $L_{\mathcal{C}}^{-1}$: $\mathbb{R}^4 \to \mathbb{R}2 \times 2$.

(g) Compare your answers in (d) and (f). How are they related to $\ker T$?

(h) Find a basis for the image of T using **either** $\mathcal{E}$-coordinates or $\mathcal{C}$-coordinates (which seems easier?) Don't forget to reinterpret vectors in the coordinate space as elements in $\mathbb{R}2 \times 2$!