**MATH 217 W24 - LINEAR ALGEBRA, Section 001 (Dr. Paul Kessenich)**
**Homework Set Part B due Thursday, January 25 at 11:59pm**
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1. In parts (a) - (d) below, determine whether the given function is injective, surjective, both, or neither. Justify your answers.

   (a) the function $f : [0, 4] \to [0, 18]$ defined by $f(x) = x^2 + 2$;

   **Solution:** Injective. If $f(x_1) = f(x_2)$, it follows that $x_1^2 = x_2^2$. Since the domain is positive, this also means $x_1 = x_2$, showing injectivity. There is no solution in the domain to $f(x) = 0$, so there exists a value in the codomain which is not in the image of $f$. Thus, the function is not surjective.

   (b) the function $g : \mathbb{R} \to \mathbb{R}$ defined by $g(x) = 2x - 5$;

   **Solution:** Bijective. If $g(x_1) = g(x_2)$, $2x_1 - 5 = 2x_2 - 5$. Therefore, $x_1 = x_2$, showing injectivity. Let $y \in \mathbb{R}$, and $x = \frac{y+5}{2}$. Then $x \in \mathbb{R}$, and $g(x) = y$. Thus $g$ is surjective.

   (c) the function $h : \mathbb{R}^2 \to \mathbb{R}$ defined by $h(x, y) = 2x^2 + 5y^2$;

   **Solution:** Neither. $10 = h\left(\sqrt{5}, 0\right) = h\left(0, \sqrt{2}\right)$, so $h$ is not injective. $h(x, y) = -2$ has no solutions in $\mathbb{R}^2$ since a square cannot be a negative number, therefore $h$ is not surjective.

   (d) the function $q : \mathbb{N} \to \mathbb{N}$ defined by $q(n) = \begin{cases} n, & \text{if n is odd} \\ n/2 & \text{if n is even.} \end{cases}$

   **Solution:** Surjective. $1 = q(1) = q(2)$, so $q$ is not injective. Let $m \in \mathbb{N}, n = 2m$. Then $n \in \mathbb{N}$, $n$ is even, and $q(n) = m$. So $q$ is surjective.

2. Determine whether each statement is true or false. If it is true, prove it. If it is false, prove this by giving a counterexample.

   (a) For every function $f : X \to Y$ and all $A, B \subseteq X$, if $A \cap B = \emptyset$, then $f[A] \cap f[B] = \emptyset$.

   **Solution:** False. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x^2$. Assign $A = \mathbb{R}^+$, $B = \mathbb{R}^-$. Then $A \cap B = \emptyset$, but $f(1 \in A) = f(-1 \in B) = 1$. Therefore $f[A] \cap f[B] \neq \emptyset$.

   (b) For every function $f : X \to Y$ and all $A, B \subseteq X$, if $f[A] \cap f[B] = \emptyset$, then $A \cap B = \emptyset$.

   **Solution:** True. We prove the contrapositive. Take any $f : X \to Y$ and $A, B \subseteq X$ such that $A \cap B$. Then $\exists a \in X$ such that $a \in A \cap B$. Since $a \in A, f(a) \in f[A]$. Similarly, $a \in B$, so $f(a) \in f[B]$. As $f(a) \in f[A]$ and $f(a) \in f[B]$, $f(a) \in f[A] \cap f[B]$. This means that for every function $f : X \to Y$ and all $A, B \subseteq X$, if $A \cap B \neq \emptyset$, then $f[A] \cap f[B] \neq \emptyset$. Thus the contrapositive is true, so the original statement is true.

(c) For every function $f : X \to Y$ and all $A \subseteq X$, we have $f^{-1}[f[A]] = A$.

**Solution:** False. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x^2$. Assign $A = \{1\}$. Then $f[A] = \{1\}$. However, $f(1) = f(-1) = 1$, so $f^{-1}[f[A]] = \{-1, 1\} \neq A$.

(d) For every function $f : X \to Y$ and all $A \subseteq X$, we have $f[X \backslash A] = Y \backslash f[A]$.

**Solution:** False. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x^2$. Assign $A = \{1\}$. Then $f[A] = \{1\}$, but $f(1) = f(-1) = 1$. Therefore $f[A] \subseteq f[X \backslash A]$, and $f[X \backslash A] \neq Y \backslash f[A]$.

(e) For every bijective function $f : X \to Y$ and all $A, B \subseteq X$, we have $f[A \cap B] = f[A] \cap f[B]$.

**Solution:** True.

Let $x \in f[A \cap B]$. Then let $a = f^{-1}(x)$. We know $a \in A \cap B$ is unique due to bijectivity. Since $a \in A \cap B, a \in A \wedge a \in B$. Thus $f[A \cap B] \subseteq f[A] \cap f[B]$.
Let $y \in f[A] \cap f[B]$. Then let $b = f^{-1}(y)$. We know $b$ is unique due to bijectivity. Additionally, $b \in A \wedge b \in B$ because $y \in f[A] \cap f[B]$. Since $b \in A \cap B, f(b) = y \in A \wedge a \in B$. Thus $f[A] \cap f[B] \subseteq f[A \cap B]$.
Both $f[A] \cap f[B] \subseteq f[A \cap B]$ and $f[A \cap B] \subseteq f[A] \cap f[B]$, so $f[A \cap B] = f[A] \cap f[B]$.

3. (a) Prove that for every function $f : \mathbb{R} \to \mathbb{R}$, if $f(cx) = cf(x)$ for all $c \in \mathbb{R}$ and $x \in \mathbb{R}$, then $f(x + y) = f(x) + f(y)$ for all $x, y \in \mathbb{R}$. (In other words, prove that every function $f : \mathbb{R} \to \mathbb{R}$ that preserves scalar multiplication is a linear transformation from $\mathbb{R}$ to $\mathbb{R}$.)

**Solution:** We use 3 cases: $\begin{cases} (1) & x \neq 0 \in \mathbb{R}, y \in \mathbb{R} \\ (2) & x = 0, y \neq 0 \in \mathbb{R} \\ (3) & x = 0, y = 0 \end{cases}$

(1) Let $x \neq 0 \in \mathbb{R}$, $y \in \mathbb{R}$. Then $y = cx$, where $c \in \mathbb{R}$, by closure of nonzero division in the reals. Additionally, $f(y) = f(cx) = cf(x)$. Thus

$$f(x + y) = f(x + cx) = f((1 + c)x) = (1 + c)f(x) = f(x) + cf(x) = f(x) + f(y)$$

(2) The variables are analagously switched from case (1) if $x = 0$ and $y$ is nonzero.
(3) Note that

$$f(0) = 0f(k) = 0k \in \mathbb{R}.$$

So, if $x = y = 0$,

$$f(0 + 0) = f(0) + f(0) = 0.$$

Thus, linearity holds in all cases for all $f : \mathbb{R} \to \mathbb{R}$ if $f(cx) = cf(x)$ for all $c \in \mathbb{R}$ and $x \in \mathbb{R}$.

(b) Give an example to show that the argument you gave in part (a) cannot work in 2 dimensions. That is, explicitly describe a function $f : \mathbb{R}^2 \to \mathbb{R}^2$ that is not a linear transformation but has the property that $f(c\vec{x}) = cf(\vec{x})$ for all $\vec{x} \in \mathbb{R}^2$ and $c \in \mathbb{R}$. Remember to prove that your example works!

**Solution:** Define $f(x, y) = (\sqrt{x^2 + y^2}, 0)$.

Scalar multiplication: $cf(x, y) = \left(c\sqrt{x^2 + y^2}, 0\right) = \left(\sqrt{(cx)^2 + (cy)^2}, 0\right) = f(cx, cy)$

$$f(1, 0) = (1, 1)$$
$$f(0, 1) = (1, 1)$$
$$f(1, 1) = (1, 2) \neq (1, 1) + (1, 1) = f(1, 0) + f(0, 1)$$

Thus linearity does not hold here, even though scalar multiplication is preserved through the function.

4. Let $f : \mathbb{R} \to \mathbb{R}$ be a function, and suppose that $f(x + y) = f(x) + f(y)$ for all $x, y \in \mathbb{R}$. (In other words, suppose that $f$ preserves addition).

   (a) Prove that $f(0) = 0$.

   **Solution:** Let $x \in R$. Then

   $$f(x) = f(x + 0) = f(x) + f(0) = f(x) + 0.$$

   Thus $f(0) = 0$.

   (b) Prove that for all $x \in \mathbb{R}$, $f(-x) = -f(x)$.

   **Solution:** Let $x \in \mathbb{R}$. Then

   $$f(x + (-x)) = f(0) = f(x) + f(-x) = 0.$$

   Hence $f(-x) = -f(x)$.

   (c) Use induction to prove that for all $n \in \mathbb{N}$ and $x \in \mathbb{R}$, $f(nx) = nf(x)$.

   **Solution:** Base case: $n = 1$, $f(1 \cdot x) = 1 \cdot f(x)$.
   Inductive step: Assume $f(nx) = nf(x)$. Then

   $$\begin{aligned} f((n+1)x) &= f(nx + x) \\ &= f(nx) + f(x) && (f \text{ preserves addition}) \\ &= nf(x) + f(x) && (\text{assumption}) \\ &= (n+1)f(x) \end{aligned}$$

   Thus the statement holds for all $n$.

   (d) Prove that for all $m \in \mathbb{Z}$ and $x \in \mathbb{R}$, $f(mx) = mf(x)$.

   **Solution:** We split possibilities for $m \in \mathbb{Z}$ into 3 cases: $\begin{cases} (1) & x \in \mathbb{Z}^+ = \mathbb{N} \\ (2) & x \in 0 \\ (3) & x \in \mathbb{Z}^- = -\mathbb{N} \end{cases}$

(1) We have shown this in part (c).
(2) We know from part (a) that

$$f(0x) = f(0) = 0 = 0f(x).$$

(3) We know from part (b) that $f(-x) = -f(x)$. Then

$$f(mx) = -(-f(mx) = -f(-mx).$$

By the case assumption, $-m$ is positive, so it falls under case 1. Then we know

$$-f(-mx) = -(-mf(x)) = mf(x).$$

Thus the statement is true for all cases of $m$ and $x$.