Geometry

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UCF

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Vectors

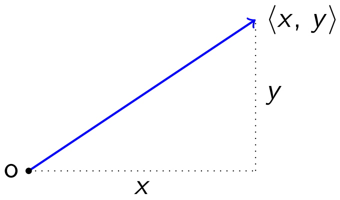
Vectors

- Geometric object
- Length and direction
- OR x component and y component in angle brackets

Notable vector functions

- Addition
- Scale
- Dot product
- Cross product

All of this will work in 3D

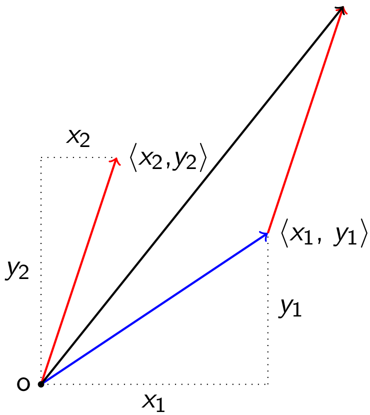


Vector Addition

When adding 2 vectors

- Vectors don't need an anchor
- Vectors can slide in the 2D plane
- Move vector end to end
- Add like components

$\langle x_1, y_1 \rangle + \langle x_2, y_2 \rangle =$
$\langle x_1 + x_2, y_1 + y_2 \rangle$

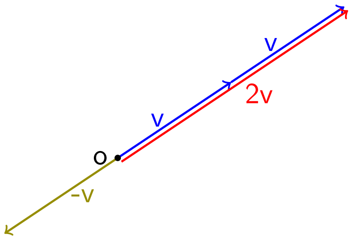


Vector Scaling

Multiplying a vector by number (a)

- Adding the vector a times
- When $a < 0$ the direction is flipped
- Scale each component by a

$a\langle x_1, y_1\rangle = \langle ax_1, ay_1\rangle$

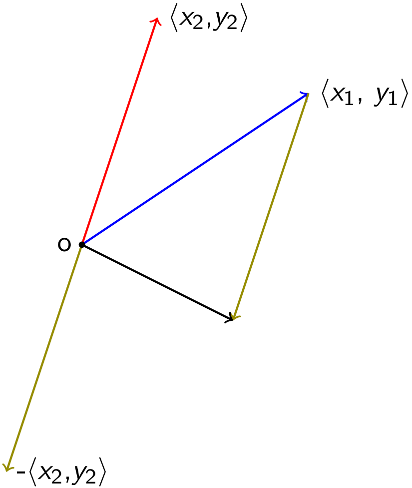


Vector Difference

Vectors, $\overrightarrow{v_1}$ and $\overrightarrow{v_2}$, can be subtracted

Add negative of the second to the first

$$\overrightarrow{v_1} - \overrightarrow{v_2} = \overrightarrow{v_1} + (-1)\overrightarrow{v_2}$$
$$\overrightarrow{v_1} - \overrightarrow{v_2} = \langle x_1 + (-1)x_2,\ y_1 + (-1)y_2 \rangle$$
$$\overrightarrow{v_1} - \overrightarrow{v_2} = \langle x_1 - x_2,\ y_1 - y_2 \rangle$$



Vector Information Extraction (Magnitude)

Vectors are a magnitude and a direction BUT representation is via x and y components

How is extraction of magnitude and direction information performed?

For magnitude, Pythagorean Theorem

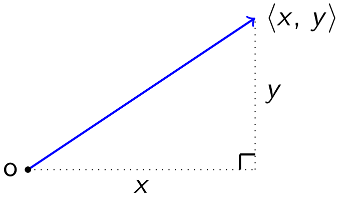
The x component and y component are orthogonal

$\text{Mag}(\langle x, y \rangle) = \sqrt{x^2 + y^2}$

Magnitude is denoted using vertical bars

For example, $|\overrightarrow{v}|$

Note, $|a\overrightarrow{v}| = a|\overrightarrow{v}|$



Vector Information Extraction (Direction)

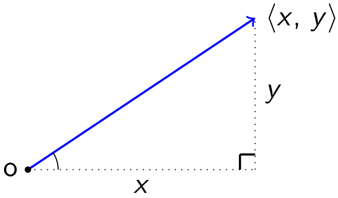
For direction trig is useful...

- A right triangle is formed
- The y component is opposite
- The x component is adjacent
- Arc-tangent can be used on $\frac{y}{x}$

Division by 0 is a problem

Built-in atan2 can

- Corner case this out
- Find the correct quadrant



Vector Multiplication (Dot Product)

Numbers can be multiplied together...

Vectors can be multiplied together as well!

The first way

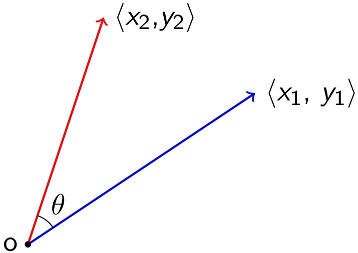
- Multiply each component together
- Add the products

$\langle x_1,\, y_1\rangle \cdot \langle x_2,\, y_2\rangle = x_1(x_2) + y_1(y_2)$

The result is not a vector, but a scalar

For example
$\langle 3,\, 2\rangle \cdot \langle -4,\, 5\rangle = 3(-4) + 2(5) = -2$

It should be noted that
$\overrightarrow{v_1} \cdot \overrightarrow{v_2} = |\overrightarrow{v_1}||\overrightarrow{v_2}|\cos(\theta)$
where θ is the difference of directions



Dot Product Notes

The dot product is

- Commutative
- $|\overrightarrow{v}|^2$ when used on $\overrightarrow{v}$ and $\overrightarrow{v}$
- Maximized when θ is 0
- Minimized when θ is π radians
- 0 when θ is $\frac{\pi}{2}$ radians (orthogonal)

When using the dot product on a negative vector, the negative sign can be moved outside

I.E. $\overrightarrow{a} \cdot (-\overrightarrow{b}) = -(\overrightarrow{a} \cdot \overrightarrow{b})$

Vector Multiplication (Cross Product)

A second vector product exists

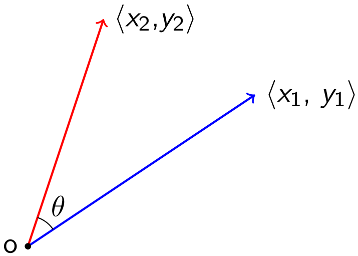
Referred to as cross ($\times$) product

- Write components in a 2 by 2 grid
- First vector over second vector
- Add product of forward diagonal
- Subtract product of back diagonal

$\overrightarrow{v_1} \times \overrightarrow{v_2} = |\overrightarrow{v_1}||\overrightarrow{v_2}|\sin(\theta)$
where θ is the difference of directions

For example
$\langle 3,\, 2\rangle \times \langle -4,\, 5\rangle = 3(5) - 2(-4) = 23$

This produces a scalar, BUT
in 3D produces a vector



Cross Product Notes

The cross product is

- NOT Commutative
- 0 when used vectors that are parallel
- Positive when θ is in (0, π radians); left turn
- Negative when θ is in (π radians, 2π radians); right turn

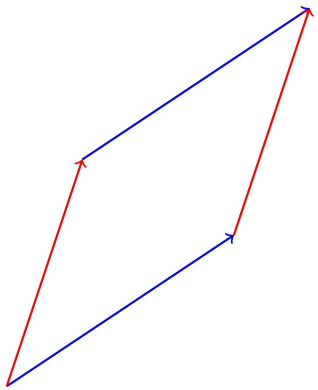
When using the dot product on a negative vector, the negative sign can be moved outside

I.E. $\overrightarrow{a} \times (-\overrightarrow{b}) = -(\overrightarrow{a} \times \overrightarrow{b})$

Absolute value of the cross product also results in the parallelogram area defined by two vectors

In 3D the cross product produce a perpendicular vector

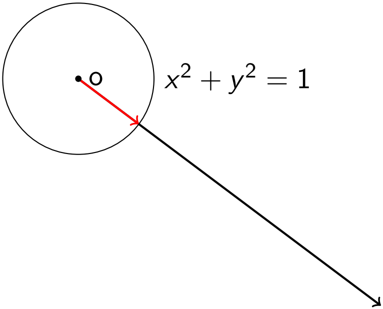
Cross Product Area



Unit Vectors

A unit vector is

- A vector with magnitude 1
- Has direction of some other vector
- Good when thinking about projections



Vector Projection

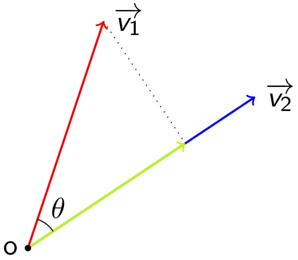
How is a vector projected onto another?

Projection is found using dot product

Project $\overrightarrow{v_1}$ onto $\overrightarrow{v_2}$

The direction is the same as the $\overrightarrow{v_2}$

$|\text{Proj}|$ is $|\overrightarrow{v_1}|\sin(\theta)$



Vector Projection (cont.)

Consider $(\overrightarrow{v_1} \cdot \overrightarrow{v_2})\overrightarrow{v_2}$

Direction is the same as $\overrightarrow{v_2}$

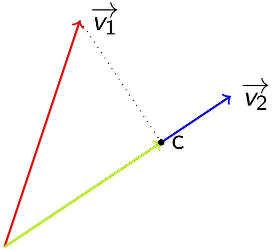
$$|(\overrightarrow{v_1} \cdot \overrightarrow{v_2})\overrightarrow{v_2}| = ||\overrightarrow{v_1}||\overrightarrow{v_2}|\sin(\theta)\overrightarrow{v_2}|$$
$$|(\overrightarrow{v_1} \cdot \overrightarrow{v_2})\overrightarrow{v_2}| = |\overrightarrow{v_1}||\overrightarrow{v_2}|\sin(\theta)|\overrightarrow{v_2}|$$

Divide by $|\overrightarrow{v_2}|^2$ for correct mag

Proj $\overrightarrow{v_1}$ onto $\overrightarrow{v_2} = \frac{(\overrightarrow{v_1} \cdot \overrightarrow{v_2})}{(\overrightarrow{v_2} \cdot \overrightarrow{v_2})}\overrightarrow{v_2}$

Project is useful for finding distance from points to lines

The projection is the closest point on a line to the end of the other vector



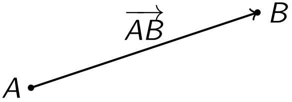
Points $\rightarrow$ Vectors

Point ($P = (x, y)$) in 2D

- Represented as a 2D vector
- Direction from origin to P
- Distance from origin to P
- $\langle x_1, y_1 \rangle$

Points A and B have a vector between them denoted as $\overrightarrow{AB}$

Found using vector differences



Lines

A unique line exists between any pair of non-identical points

It's likely know about the slope intercept form

$$y = mx + b$$

You might know the point slope form

$$(y - y_1) = m(x - x_1)$$

Rarely is the point-point form (cringe) known

$$(y - y_1) = \frac{(y_1 - y_2)}{(x_1 - x_2)}(x - x_1)$$

General form is most useful for competitive programming

$$ax + by + c = 0$$

It can represent vertical lines

2D $\rightarrow$ 3D

Projective Geometry is

- Black magic
- Representation that turns 2D into 3D points
- Allows easy conversion between points and lines

Points in a 3D virtual world (video game) are put onto a screen

Treat z as the depth from the camera

When objects are farther from the camera (higher z) they shrink

When objects are closer to the camera (lower z) they grow

Treat camera at origin and screen at z of 1

All points on the 2D plane are at $(x, y, 1)$

To find the 2D point from the 3D vector divide by z
$(x, y, z) \rightarrow (\frac{x}{z}, \frac{y}{z}, 1)$

Projective Geometry Dot Product

Take the point $(x, y, 1)$, and dot product with a vector $\langle a, b, c \rangle$

We get $ax + by + c1$

Setting the product to 0 gives the general line formula

When a dot product is 0, the vectors are perpendicular

Lines are 3D vectors in 2D Projective Geometry

A point is on a line if their dot product is 0

Cross Product 3D

$$\langle x_1, y_1, z_1 \rangle \times \langle x_2, y_2, z_2 \rangle = \det \begin{pmatrix} \hat{i} & \hat{j} & \hat{k} \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{pmatrix}$$

It's like 3 2D cross products

$$\langle x_1, y_1, z_1 \rangle \times \langle x_2, y_2, z_2 \rangle =$$
$$\langle y_1(z_2) - y_2(z_1), z_1(x_2) - z_2(x_1), x_1(y_2) - x_2(y_1) \rangle$$

The resulting vector is going to be perpendicular to both vectors

The magnitude of the resulting vector can represent the area of the parallelogram defined by the 2 vectors

Projective Geometry Cross Product

Points and lines have perpendicular vectors

For pairs of 3D vectors (2D points) a perpendicular vector can be found using cross product

A line between $\overrightarrow{p_1}$ and $\overrightarrow{p_2}$ is $\overrightarrow{p_1} \times \overrightarrow{p_2}$
A point on $\overrightarrow{l_1}$ and $\overrightarrow{l_2}$ is $\overrightarrow{l_1} \times \overrightarrow{l_2}$

Above is called point line duality

2 Line Segments Intersection

Intersection of 2 line segments can be found easily using Projective Geometry

Given l_1 defined by p_1 and p_2 and l_2 define by p_3 and p_4

The intersection is $(p_1 \times p_2) \times (p_3 \times p_4)$

Special cases exists!!!

If the lines are the same, then the resulting vector will be all 0s
If the lines are parallel, then the z will be 0

To check if a point is on both line segments
The sign of the dot product (in 2D) can be checked on both lines

Back to 2D

I recommend using projective geom only when dealing with lines and points

Things that are easier to do in 2D includes

- Circles (IMO)
- Finding the area of a polygon
- Checking for point in polygon
- Convex Hull

Circles

Defined by a point and a radius

Defined by the equation

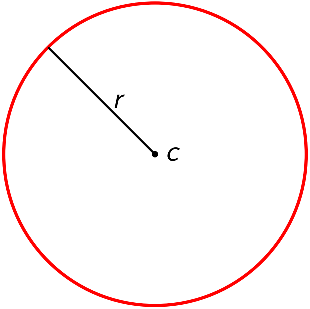
$$(x - c_x)^2 + (y - c_y)^2 = r^2$$

Every point on the circle is exactly r distance away

Intersect lines at 0, 1, or 2 points

Intersect circles at 0, 1, or 2 points

NOTE: Circles are symmetric about every line through their center



Circle-Line Intersection

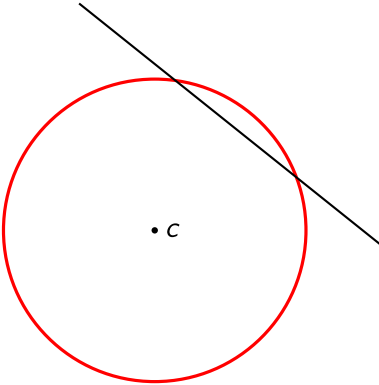
Consider a line (L) passing through (or by) a circle ($C = (c, r)$)

If the L is contained in the C, then the L must enter and exit

The closest point (p) on L to c would be in C (if any)

The p can be found using vector projection

Any point is in C, if the distance to c is less than r

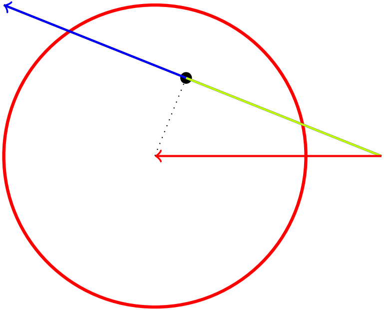


Center Projection

Grab 2 points (A, B) on L

Project $\overrightarrow{AC}$ onto $\overrightarrow{AB}$

If p is exactly r units from c, then L is tangent to C



Intersection Points

Intersection points (p_1, p_2) exist, assuming p is close enough

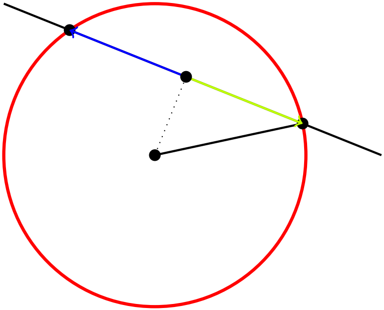
Found using a right triangle

- c to p is one leg
- c to p_1 is hypotenuse
- p to p_1 is other leg

The distance from p to p_1 can be found

Points are along the vector formed by L

L's vector should be normalized then scaled by the computed distance to find the offset from p



Circle-Circle Intersection

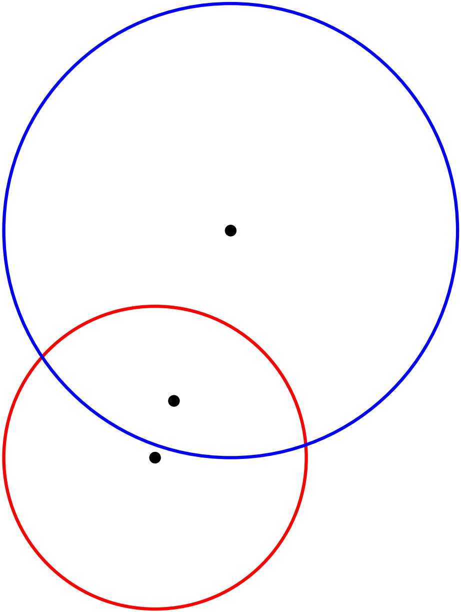
The intersection between Circle $C_1 = (c_1, r_1)$ and Circle $C_2 = (c_2, r_2)$

Heron's triangle formula and the distance d between centers can be used to derive

If the 2 intersections exist they lie on a line L

The distance d_1 from L to c_1 can be found with the following formula

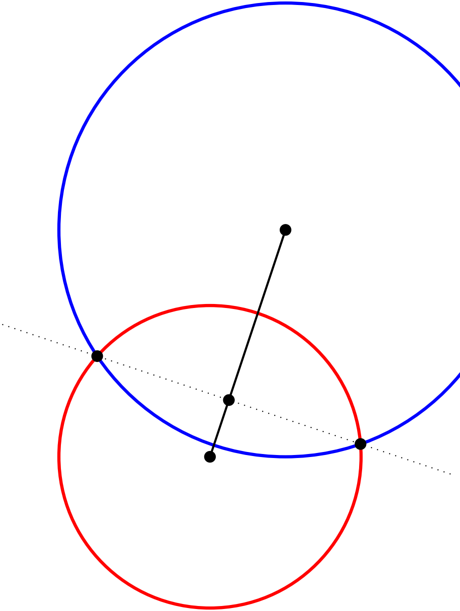
$$d_1 = \frac{d^2 - r_2^2 + r_1^2}{2d}$$



Circle-Circle Intersection (cont.)

Find the line that the intersections lie on using an orthogonal vector

Do line intersection with the circle



Pulleys

Does not cross

angle $= \arccos(\frac{r_1 - r_2}{d})$

Belts

Pulleys that cross

angle $= \arccos(\frac{r_1+r_2}{d})$

Does not exist is $r_1 + r_2 > d$

Polygons

A sequence of points

Simple - no line segments cross

Convex - Simple no internal angle is more than 180 degrees

Concave - Simple some angle is more than 180 degrees

Polygon Area

Cross products find a signed area

Sum of 2D cross products of each pair of adjacent points is twice the total signed area

Point in Polygon

Concave Polygon Method Sum the angles that go around the point between all adjacent points in the polygon O(N)

Convex Polygon Method Binary search around the triangulated polygon O(log(N))

Convex Hull

The convex hull of a set of points is the smallest convex polygon that contains all the points

Monotone Chaining

- Sort by x
- Store hull on a stack
- Ensure to only make right turns

The above method finds the lower hull

Upper hull is computed by reversing the order of the points

It can be found in O(Nlog(N)) time.

TODO

- Line Sweeps
- Checking for point in polygon O(log(N))
- Reflection trick
- Half Plane Intersection
- Circle Union
- Pick's Theorm
- Calipers