X: observed data

Z: latent variable + parameter.

变分推断为了逼近后验分布 $P(\theta|X)$.

$(X, Z)$: complete data.

$$\log P(x) = \log P(X,Z) - \log P(Z|X)$$

$$= \log \frac{P(X,Z)}{q(Z)} - \log \frac{P(Z|X)}{q(Z)}$$

$$\int_Z q(Z) \log P(x)\, dZ = \log P(x) = \int_Z q(Z) \log \frac{P(X,Z)}{q(Z)} - \int_Z q(Z) \log \frac{P(Z|X)}{q(Z)}$$

$$= \underbrace{\int_Z q(Z) \log \frac{P(X,Z)}{q(Z)}}_{\text{ELBO}} \underbrace{- \int_Z q(Z) \log \frac{P(Z|X)}{q(Z)}}_{KL(q\|P)}$$

$$= L(q) + \underbrace{KL(q\|P)}_{\geq 0}$$

$L(q)$ ↓ 变分

要使 $q(Z) \approx P(Z|X)$

⇓

$$\underset{q(Z)}{\arg\max}\, L(q)$$

$$\hat{q}(Z) = \underset{q(Z)}{\arg\max}\, L(q) \Rightarrow \hat{q}(Z) \approx P(Z|X)$$

Vazyme

分成 M 个相互独立的组.

$$q(Z) = \prod_{i=1}^{M} q_i(Z_i)$$

$$L(q) = \int_Z q(Z) \log \frac{P(X,Z)}{q(Z)}$$

$$= \underbrace{\int_Z q(Z) \log P(X,Z)}_{①} - \underbrace{\int_Z q(Z) \log q(Z)}_{②}$$

$$① = \int_Z \prod_{i=1}^{M} q_i(Z_i) \log P(X,Z)\, dZ_1 dZ_2 \cdots dZ_M$$

$$= \int_{Z_j} q_j(Z_j) \underbrace{\left[ \int_{Z_1 \cdots Z_M\,(i \neq j)} \prod_{i \neq j}^{M} q_i(Z_i) \log P(X,Z)\, dZ_i \right]}_{\int_{Z_1 \cdots Z_M\,(i \neq j)} \log P(X,Z) \cdot \prod_{i \neq j}^{M} q_i(Z_i)\, dZ_i} dZ_j$$

$$= \int_{Z_j} q_j(Z_j) \cdot \underbrace{E_{\prod_{i \neq j}^{M} q_i(Z_i)} \left[ \log P(X,Z) \right]}_{\log \hat{P}(X, Z_j)} dZ_j$$

$$② \int_Z q(Z) \log q(Z)\, dZ = \sum_{i=1}^{M} \int_{Z_i} q_i(Z_i) \log q_i(Z_i)\, dZ_i$$

$$= \int_{Z_j} q_j(Z_j) \log q_j(Z_j)\, dZ_j + C$$

$$① - ② = \int_{Z_j} q_j(Z_j) \cdot \log \frac{\hat{P}(X,Z_j)}{\cdots}\, dZ_j = -KL(q_j \| \hat{P}(X, Z_j)) \cdots$$