

$$\log P(x|\theta) = \log P(x, z|\theta) - \log P(z|\theta)$$

$$P(x, z) = P(z|x) P(x)$$

$$\log P(x) = \log \frac{P(x, z)}{P(z|x)} = \log P(x, z) - \log P(z|x)$$

$$\log P(x|\theta) = \log P(x, z|\theta) - \log P(z|x, \theta)$$

$$= \log \frac{P(x, z|\theta)}{q(z)} - \log \frac{P(z|x, \theta)}{q(z)} \quad (q(z) \neq 0 \text{ 且 } q(z) \text{ 关于 } z \text{ 的分布函数})$$

$$\text{左边} = \int_z q(z) \cdot \log P(x|\theta) dz = \log P(x|\theta) \cdot \underbrace{\int_z q(z) dz}_{=1} = \log P(x|\theta)$$

$$\text{右边} = \underbrace{\int_z q(z) \log \frac{P(x, z|\theta)}{q(z)} dz}_{\int_z q(z) \log \frac{P(z|x, \theta)}{q(z)} dz}$$

$$\text{ELBO (Evidence Lower Bound)} \quad KL(q(z) || P(z|x, \theta))$$

$$\log P(x|\theta) = \text{ELBO} + KL(q || P)$$

$$(\text{KL}(q || P) \geq 0)$$

$$\log P(x|\theta) \geq \text{ELBO}$$

$$\theta = \arg \max_{\theta} \text{ELBO} = \arg \max_{\theta} \int q(z) \log \frac{P(x, z|\theta)}{q(z)} dz$$

$$= \arg \max_{\theta} \int P(z|x, \theta^t) \log \frac{P(x, z|\theta)}{P(z|x, \theta^t)} dz$$

$$= \arg \max_{\theta} \int P(z|x, \theta^t) \cdot \log P(x, z|\theta) dz$$

$$\hat{q} = P(z|x, \theta^t)$$

$$\text{E-step: } P(z|x, \theta^t) \rightarrow E_{z|x, \theta^t} [\log P(x, z|\theta)]$$

$$\text{M-step } \theta^{t+1} = \arg \max_{\theta} E_{z|x, \theta^t} [\log P(x, z|\theta)]$$

$$\log P(x|\theta) = \log \int_z P(x, z|\theta) dz = \log \int_z \frac{P(x, z|\theta)}{q(z)} \cdot q(z) dz$$

$$= \log E_{q(z)} \left[\frac{P(x, z|\theta)}{q(z)} \right]$$

$$\geq E_{q(z)} \left[\log \frac{P(x, z|\theta)}{q(z)} \right]$$

$$\parallel \text{ELBO} = \text{ELBO} \cdot P(z|x, \theta^t)$$

z 为隐变量

$$\text{EM Algorithm } \theta = \arg \max_{\theta} P(x|\theta) = \arg \max_{\theta} \log P(x|\theta)$$

$$\log P(x|\theta) = \text{ELBO} + KL(q || P)$$

$$\nearrow \text{ELBO} \\ L(q, \theta)$$

$q(z)$ 可能取不到 $P(z|x, \theta)$

因此, $\log P(x|\theta)$ 固定, $\text{ELBO} \uparrow$, $KL(q || P) \downarrow$

相当于求 $\arg \max_{q} KL(q || P) = \arg \max_{q} L(q, \theta)$

固定 $\hat{q}, \theta = \arg \max_{\theta} L(\hat{q}, \theta)$