Mathematical Annotation for the food and energy prices' analysis

1. Time series and Data Generating Mechanism

Consider the food and energy prices (donated as $\{P_t\}, t = t_0, t_1, \dots, t_\infty$)as a time series generated by a Data Generating Mechanism, and $\{p_t\}, t = t_0, t_1, \dots, t_T)$ as a realization of$\{P_t\}$. And j(p_t) is drawn from a joint distribution j(P_t). Here the food prices are denoted as $\{P_t^f\}_{m*1}$ and the energy prices are denotes as$\{P_t^e\}_{l*1}$. Hence $P_t = \begin{pmatrix} P_t^f \\ P_t^e \end{pmatrix}_{(k+l)*1}$.

2. Stationarity

A strictly stationary process is a stochastic process whose joint probability distribution does not change when shifted in time or space. As for time series, if a time series $\{P_t\}, t = t_0, t_1, \dots, t_\infty$ is strictly stationary, then $\{P_1, \dots, P_n\}$ and $\{P_{1+h}, \dots, P_{n+h}\}$ have the same joint distribution for ∀ h and n>0. And if a process is strictly stationary, parameters such as mean and covariance, if they exist, should be independent of time, by which the weak stationarity is induced.

>weakly stationary: Xt is weakly stationary if

a.$\mu(p_t)$is independent of t
b.$\gamma_p(t+h, t)$is independent of t for ∀ h.

In the presence of non-stationary variable, there might be what Granger and Newbold (1974) call a spurious regression. A spurious regression appears to have significant relationship among variable but the results are in fact without any economic meaning. An example of spurious regression (Jesús Gonzalo) is the regression of US Defense Expenditure (Y) (1971-1990, annual data) on Population of South African (X): $\widehat{Y}$ = -368.99 + .0179 X, R^2= .940, Corr = .9694. The regression has a high R-square but no economic meaning.

3. Unit Root

A linear stochastic process has a unit root if 1 is a root of the process's characteristic equation. Consider a discrete time stochastic process AR (p). If m = 1 is a root of the characteristic equation:

$$m^p - m^{p-1}a_1 - m^{p-2}a_2 - m^{p-3}a_3 \dots . -a_p = 0$$

Then the stochastic process has a unit root or, alternatively, is integrated of order one, denoted I (1). If m = 1 is a root of multiplicity r, then the stochastic process is integrated of order r, denoted I(r).

4. Dickey-Fuller Unit Root Test (DF)

   Assume the food and energy prices (with no trend and drift) can be modeled as:

$$p_t = \beta p_{t-1} + \varepsilon_t, t = 1, \dots, T$$

Rewrite the equation we have,

$$\Delta p_t = (\beta - 1)p_{t-1} + \varepsilon_t$$
$$\Delta p_t = \gamma p_{t-1} + \varepsilon_t$$

where Δ is the first difference operator. Testing the hypothesis that there is unit root is to test $\gamma = 0$. Dickey and Fuller (1979) consider three different regression equations: with no drift and trend, with drift and with drift and trend. In each case the null hypothesis is Ho: $\gamma = 0\ (unit\ root)$. The test statistics and critical values are summarized in the following table.

Model	Hypothesis	Test Statistics	Critical Values for 95% and 99% C.I.
$\boldsymbol{\Delta y_t = \alpha_0 + \gamma y_{t-1} + \alpha_2 t + \varepsilon_t}$	$\gamma = 0$	τ_τ	-3.45 and -4.04
	$\gamma = \alpha_2 = 0$	ϕ_3	6.49 and 8.73
	$\alpha_0 = \gamma = \alpha_2 = 0$	ϕ_2	4.88 and 6.50
$\boldsymbol{\Delta y_t = \alpha_0 + \gamma y_{t-1} + \varepsilon_t}$	$\gamma = 0$	τ_μ	-2.89 and -3.51
	$\alpha_0 = \gamma = 0$	ϕ_1	4.71 and 6.70
$\boldsymbol{\Delta y_t = \gamma y_{t-1} + \varepsilon_t}$	$\gamma = 0$	τ	-1.95 and -2.60

$$where \quad \phi_i = \frac{[SSR(restricted) - SSR(unrestricted)]/r}{SSR(unrestricted)/(T-k)} \sim F$$

$$\tau_\tau = \tau = \tau_\mu = \frac{\gamma - 0}{SE(\gamma)} \sim T$$

5. Augmented Dickey-Fuller Test (ADF)
   If the error term has autocorrelation more than one period, the unit root test can be modified as

$$\Delta p_t = \alpha + \gamma p_{t-1} + \sum_{2}^{k} \lambda_i \Delta p_{t-i+1} + \varepsilon_t$$

   The number of lagged difference terms to be included can be chosen based on AIC (Greene, 1993). The null hypothesis and the test procedure for the ADF test are the same as DF test.

6. Phillips Perron test
   The Phillips Perron test is also a unit root test. It builds on the Dickey-Fuller test of the null hypothesis $\gamma = 0$ in the model:

$$\Delta p_t = \gamma p_{t-1} + \varepsilon_t\ , t = 1, \ldots, T$$

   The test makes a non-parametric correction to the t-test statistic generated by the DF test. Let the t-statistic of the DF test is the same as mentioned above:

$$\tau = \frac{\gamma - 0}{SE(\gamma)} \sim T$$

Phillips and Perron (1988) considered a different way to handle short-run memory and advocated nonparametric corrections of the test statistics:

$$Z(\hat{\gamma}) = T(\hat{\gamma}) - \frac{\widehat{\Omega} - \widehat{\Gamma}(0)}{2T^{-2}\sum x_{t-1}^2},$$

$$Z(\tau) = \frac{\sqrt{T^{-1}\sum \widehat{\varepsilon_t}^2}}{\sqrt{\widehat{\Omega}}}\tau - \frac{\widehat{\Omega} - \widehat{\Gamma}(0)}{2\sqrt{\widehat{\Omega}}\sqrt{T^{-2}\sum x_{t-1}^2}}$$

where $\widehat{\Gamma}(0)$ *and* $\widehat{\Omega}$ are consistent estimators of the variance and long-run variance of ε_t respectively.

7. Co-integration
If two or more series are individually integrated (in the time series sense) but some linear combination of them has a lower order of integration, then the series are said to be co-integrated. If two series are both I(1), there might be a linear combination of integrated variables that is stationary. Such variables are said to be co-integrated. That is, they share a common unit root and the sequence of stochastic shocks is common to both. The co-integration is a powerful concept that allows capturing the equilibrium relationship even between non- stationary series within a stationary model. Co-integration implies that prices move closely together in the long run, although in the short run they may drift apart

To test co-integration, Granger and Engle (1987) developed a simple procedure which comprises of estimating the static co-integration regression, and apply unit test, such as the ADF and PP to the estimated residuals, in order to test the null hypothesis of no co-integration.

$$p_t^f = a_0 + a_1 p_t^e + \nu_t$$

If $p_i^f, i = 1, \dots, k$ and $p_j^e, j = 1, \dots, l$ are both integrated of order one, and $\nu_t$is I(0), then p_t^f and p_t^e are co-integrated. The test procedure follows the Dickey-Fuller test.

8. Johansen co-integration test
The Johansen (1988) procedure relies on the relationship between the rank of a matrix and its characteristic roots. Johansen suggests starting with a traditional vector auto regression model (VAR), which is

$$\begin{bmatrix} p_t^f \\ p_t^e \end{bmatrix} = \begin{bmatrix} c_{1t} \\ \vdots \\ c_{2t} \end{bmatrix} + \sum_{i=1}^{p} \begin{bmatrix} a_{11,i} & \cdots & a_{1(k+l),i} \\ \vdots & \ddots & \vdots \\ a_{(k+l)1,i} & \cdots & a_{(k+l)(k+l),i} \end{bmatrix} \begin{bmatrix} p_t^f \\ p_t^e \end{bmatrix} + \begin{bmatrix} \varepsilon_{1,t} \\ \vdots \\ \varepsilon_{(k+l),t} \end{bmatrix}$$

$$where\ \Sigma = \begin{bmatrix} \sigma_{1,t}^2 & & 0 \\ & \ddots & \\ 0 & & \sigma_{(k+l),t}^2 \end{bmatrix}$$

Select appropriate number of lags (p) based on AIC in the VAR model, and then estimate the vector error correction model (VECM) and determine the rank of the matrix of parameters. The co-integration of the system is tested using the maximum likelihood $L_{max}(r)$ which is a function of the co-integration rank r. Johansen describes two test methods: Trace Test and Maximum Eigenvalue Test

9. Error correction model (ECM)
   An error correction model is a dynamic model in which the movement of the variables in any period is related to the previous period's gap from long-run equilibrium.

   A ECM(*p*) form is written as

$$\Delta p_t = \alpha + \gamma p_{t-1} + \sum_{2}^{k} \lambda_i \Delta p_{t-i+1} + \varepsilon_t$$

Where$\Delta$is the differencing operator, such that $\Delta y_t = y_t - y_{t-1}$

It has an equivalent VAR (*p*) representation as described in the preceding section.

$$p_t = \alpha + \left(I_{(k+l)} + \gamma + \lambda_i\right)p_{t-1} + \sum_{i=2}^{p-1} (\lambda_i - \lambda_{i-1})p_{t-i} - \lambda_{p-1}p_{t-p} + \varepsilon_t$$