A Simple Model of Price Jumps

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1 Introduction

As shown in the previous chapters, economic price series often present short-run sudden changes of noticeable magnitude which we label as price jumps. They differ from structural changes which represent permanent level shifts or variance changes and yet have a substantial effect in the economy. The importance of price jumps to the economy has been noted by many researchers and their effects on real economy examined. Hamilton (2003) investigated the effect of shocks[1] in oil prices on the U.S. GDP growth and concluded that GDP growth responds nonlinearly and asymmetrically to oil price shocks (see [Hamilton, 2003]). Kilian (2007) (see [Kilian, 2007]) had a detailed discussion about the linkage between energy price shocks and economic performance.

Price jumps have received continuing attention in the literature. From the source of price jumps to their (mathematical) representation have been investigated in the literature. For soft commodity markets, Trostle et al. (2008) discussed the importance of price jumps in explaining volatile soft commodity prices and discussed possible factors that lead to price jumps (see [Trostle et al., 2008]). Sumner (2009) in [Sumner, 2009] also noted such importance in explaining soft commodity price movements. For the electricity market, Huisman, Ronald, and Ronald Mahieu (2003) proposed a model of price jumps series resulting from regime switching processes (see [Huisman and Mahieu, 2003]). Barlow (2002) in his paper [Barlow, 2002] used a diffusion model to represent jumps in electricity prices. Deng in his book [Deng, 2000] proposed stochastic models of different energy commodity prices to integrate price jumps. Shafiee and Topal in [Shafiee and Topal, 2010] reviewed global gold market and suggested a jump-diffusion modelling of gold prices.

In a perfectly competitive market of a product of interest, the equilibrium price will fluctuate as exogenous factors vary. In a simple model (see, [Massell, 1969], [Helmberger and Weaver, 1977]) of perfect competitive market:

$$\begin{aligned} D_t(P_t) &= \alpha_0(Z_t) - \alpha_1 P_t + U_t \\ S_t(P_t^*) &= \beta_0(Z_t) + \beta_1 P_t^* + V_t \end{aligned} \tag{$*$}$$

where D_t is the demand quantity and S_t is the supply. P_t is the market price at time t. Z_t is a vector of exogenous factors that shift demand and supply curves. α_0 and β_0 are parameters that depend on Z_t, U_t and V_t are independent identical distributed (i.i.d. thereafter) random variable with 0 mean and variance σ_u^2 and σ_v^2. U_t and V_t represent independent demand (for example, unexpected electricity usage peak due to unanticipated events) and supply shocks (for example, grain harvest shortage due to extreme weather), respectively. P_t^* is the rational expectation of equilibrium price of time t formed at time of $t-1$ in the Muthian sense ([Muth, 1961]). That is, $P_t^* \equiv E_{t-1}P_t \equiv \mathbb{E}(P_t|\Phi_{t-1})$, where E_{t-1} denotes the conditional expectation of price at time $(t-1)$ as in [Muth, 1961] and Φ_{t-1} is the information available to the agent at time $(t-1)$ which includes the past history of observed Z,

[1] Note "shock" may seem a bit vague. Here it indicates some kind of short-run volatility which includes but is not limited to the case of price jumps. Recall to what extent a current price differs from its past so that such difference can be called a jump is explicitly defined in the previous chapter.

U and V up to time $t-1$. Then without storage[2], $D_t(P_t) = S_t(P_t^*)$ and the system $(*)$ can be easily solved algebraically by taking conditional expectation on both sides of the equation. Given U_t and V_t are i.i.d. over time, $\mathbb{E}(U_t|\Phi_{t-1}) = \mathbb{E}U_t = 0$. The rational expected price can be written as:

$$P_t^* = \frac{\alpha_0(Z_t) - \beta_0(Z_t)}{\alpha_1 + \beta_1}$$

This says the rational expectation is stable over time since the agents can not anticipate future shocks and P_t^* is affected only by Z_t. Then we have $S_t = \frac{\alpha_1\beta_0(Z)+\beta_1\alpha_0(Z)}{\alpha_1+\beta_1} + V_1$. Plug this back into the linear system $(*)$, we have

$$P_t = P_t^* + \frac{U_t - V_t}{\alpha_1} \qquad (**)$$

Thus the market price can be decomposed into a stable variable P_t^* and a random shock $\frac{U_1-V_1}{\alpha_1}$. That says, the actual market price (at time t) equals the expected price (formed at time *t-1*) plus a random shock due to contemporaneous demand or supply shock. In particular, either supply or demand shocks impact prices, e.g. a negative V_t in $(**)$ leads to an increase in the market price by $|\frac{V_t}{\alpha_1}|$. Importantly, given linearity, the price effect is determined by the magnitude of the shocks. Further, if demand or supply factors are consider as determinants of α_0 and β_0, $(**)$ clarifies their effects on price.

In any case, such market price variation induced by shocks can hardly be labeled as price jumps mainly for two reasons:

- Magnitude: We see from the model that the market equilibrium price P_t responds to supply shocks linearly (by some constant determined by demand price elasticity). This is inconsistent with the observation that price jumps usually exhibit nonlinearity and/or discontinuity of noticeable magnitude as shown in previous chapters.

- Frequency: In the competitive market model with no inventories, a large negative supply shock (or positive demand shock) generates a large contemporaneous increase in the market price. On the other hand, if in the next period $(t+1)$ the supply restores to its normal level, the market price P_{t+1} will immediately revert to its mean level. This implies contemporaneous demand or supply have no effect on subsequent prices. This is inconsistent with the observation that the response of market price to a single supply shock usually lasts over several periods before the effects of shock completely vanishes[3]. In a competitive market model with inventories, the equilibrium prices could show

[2]The case with storage is more complex. Weaver and Helmberger (1977) discussed in detail different situations when the market price falls into different price intervals. On the other hand, these interval are bounded by linear functions of demand and supply shocks. Thus the argument that the competitive market model can not generally characterize price jumps is valid for the case with storage.

[3]In some particular market, for example electricity market, price jumps exhibits mean reverting pattern. On the other hand, in general, prices do not immediately revert to their mean level after jumps.

> wider fluctuations ([Helmberger and Weaver, 1977]). But whether these fluctuations can be labelled as jumps is still subject to their magnitude, though as [Helmberger and Weaver, 1977] shows stock out could induce nonlinearity. With the linearity, it is likely that these fluctuations are not large enough to be characterized as jumps. In that case, the fluctuations are more likely to be interpreted as high conditional variance rather than outlier (jump) effects.

Thus, the perfect competitive market model does not fully capture key features of jumps in observed economic price series. This motivates a closer look into the mechanism that induces price jumps of the magnitude and frequency as observed in reality.

This paper is concerned with the micro-structure specification to identify origins of price jumps that can not be generally characterized by the competitive market models. In particular we propose a rather general model of procurement process where imperfectly informed buyers search for and place bids to suppliers to fulfill procurement demand. We show that in this process, search cost, market structure and market condition are crucial factors in generating price jumps. Later in the simulation part we show that the model proposed in this paper can generate jumps that resemble those in the observed economic price series.

Another strand of literature relating to our model focuses on the procurement process. The procurement problem in energy markets has received consistent attention in the literature. [Bonser and Wu, 2001] studied the procurement problem for electrical utilities under stochastic demand and market prices. The procurement plan balances contract and spot market purchases and is modeled as a stochastic programming problem and solved by a two-phase procedure, which performs better than a rolling-horizon, stochastic-programming heuristic. [Swider and Weber, 2007] proposed a methodology for profit maximized bidding under price uncertainty in a multi-unit and pay-as-bid procurement auction for power systems reserve. The model we propose in this paper resembles a multi-unit procurement auction (see [Swider and Weber, 2007], [Wolfram, 1997]) but is extended to accommodate different cases of buyers' learning and offer rule and suppliers' acceptance rule.

The paper is organized as follows. Section II presents an overview of basic settings of this model. Section III and section III introduces the notations used in the model. Section IV is the main body of the model. Section V presents two exemplary analytical solutions to the model under two specific cases. Section VI discusses extensions of the model. Section VII presents some simulation results. The paper ends with some concluding remarks.

2 Model Basics

In this model, we assume the product of interest is homogeneous, continuously divisible. We suppose there is an unordered finite set of risk neutral [4] buyers I and

[4] Risk premiums in the case of risk aversion are ignored for the simplicity of the argument. That is, we assume that an agent does not react to uncertainty in his payoff and his utility is a linear function of his expected "payoff", where payoff can be thought as *negative* cost in the case of buyers or profit in the case of supplier. On the other hand, since later in the context we

an unordered finite set of risk neutral suppliers K who compose what we label as a market of a particular (homogeneous) product. Each of these two sets does not possess natural ordering.

We assume buyers and suppliers are heterogeneous and imperfectly informed. We assume buyers do not know at what price suppliers are willing to sell. Further, we assume suppliers do not know what buyers might be willing to pay. Each buyer $i \in I$ has an initial fixed demand that buyer seeks to fulfill at lowest possible cost. To fulfill such demand, he will search for the product at each supplier $k \in K$ (or, location k) until some stopping time[5] determined by what we define as buyers' *optimal search stopping rule*. To each supplier k, the buyer offers a bid, which is a price-quantity pair. If the supplier k accepts his bid, the price in bid becomes (part of) the transaction price, and the supplier's accepted bid quantity can be subtracted from the buyer's demand to be fulfilled. We note this price is a bilateral transaction price between buyer willingness to pay and supplier willingness to sell. It is not a market transaction price.

To characterize the time scope of this model, we define a trading session as a time interval during which each buyer and each supplier meet at most once. Each trading session starts with an initial supply allocation to suppliers and demand allocation to buyers, and ends when either all buyers or all suppliers are out of market[6]. We assume there are three types of trading sessions:

- In a *type 1 trading session* (TS1), there is no *direct* interaction between buyers. They arrive at suppliers sequentially. Thus, suppliers receive one bid at a time and buyers can not observe or react to other buyer bids. Further, we assume there is no intertemporal dependence across trading sessions. That is, the demand or supply in current session will not be carried over to the next one. We assume in this type of trading session an agent (a buyer or supplier) has only *short memory* in a sense that he can only remember the history of that single trading session (although his unfulfilled demand or unsold supply can be carried over to the next trading session). And agents are *myopic* in that they fail to consider implications of history for future sessions.

- In a *type 2 trading session* (TS2), buyers are allowed to have *direct* interaction. Multiple buyers may arrive at a supplier at the same time (*coincidentally*) and may submit multiple bids to one supplier in a single trading session.

assume that the search cost is a nonlinear function of the market condition, even with the linear utility function, the model characterizes the buyers' risk aversion in the uncertainties in market condition. Later in equation (5) search cost satisfies

$$\frac{\partial \widetilde{SC_{ik}}^{i}}{\partial \widetilde{Z}^{i}} > 0, \frac{\partial^2 \widetilde{SC_{ik}}^{i}}{\partial \widetilde{Z}^{i2}} > 0$$

That means that the search cost is a convex function in the market condition. Hence the payoff is a concave function in the market condition and so is the buyer's utility function. Thus the buyer is risk averse in market condition.

[5]Here the term "stopping time" is a general term and not to be confused with the use in stochastic processes.

[6]In a trading session, a buyer is out-of-market if he has fulfilled all his procurement demand; and a supplier is out of market if he has sold all his supply capacity.

However, we continue to assume that there is no intertemporal dependence across sessions. That is, there is no carry-over inventory across sessions, and any agent is myopic and has short memory as in *type 1 trading session*.

- In a *type 3 trading session* (TS3), there is no *direct* interaction between buyers as in TS1. We add intertemporal dependence between sessions. Buyers' unfulfilled demand and suppliers' inventory can be carried over to the next one. In this type of trading session, we assume buyers are myopic and have short memory. But we assume that suppliers can look into future and make tradeoff between selling in current session or in future sessions. We call such suppliers as "*far-sighted*". We assume suppliers have "long memory" in a sense that they can recall what happened in all past sessions. In particular, suppliers know the equilibrium price and supply capacities in all past trading sessions.

- In a *type 4 trading session* (TS4), there is direct interaction between buyers as in TS2 and intertemporal dependence between trading sessions as in TS3.

Table 1: Types of Trading Sessions

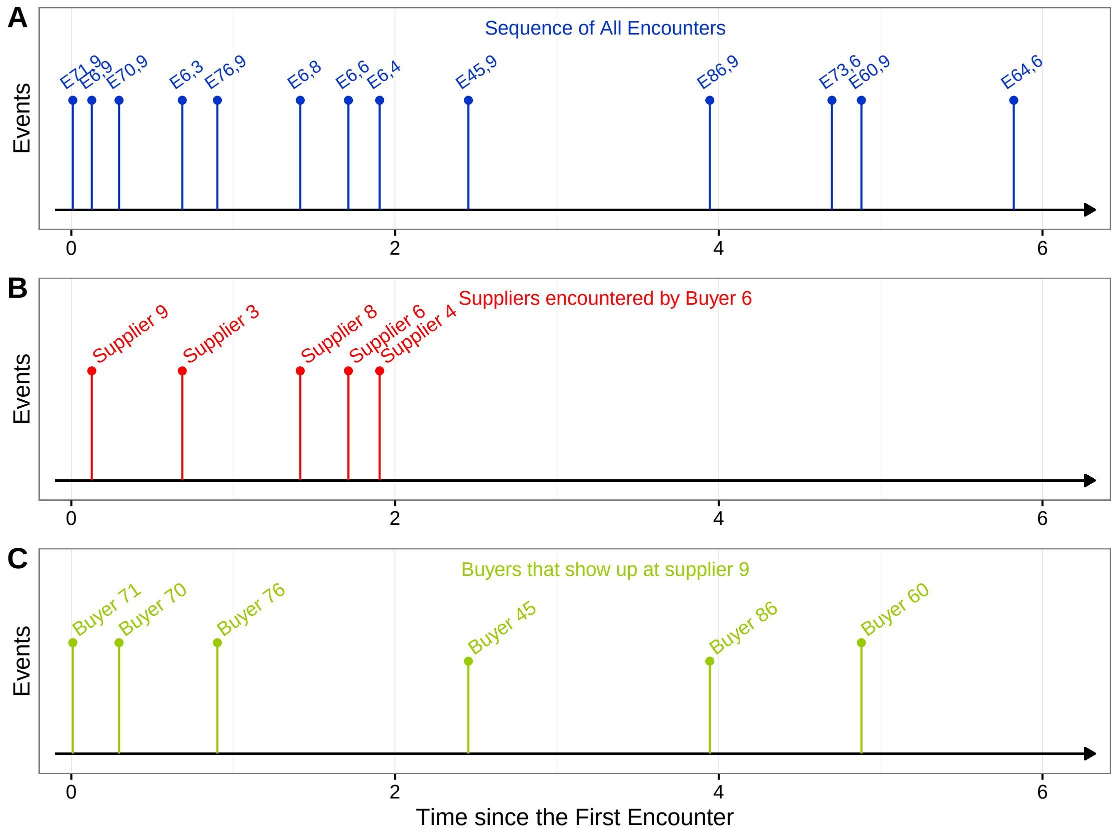
Assumptions	No Intertemporal Dependence across Trading Sessions	Intertemporal Dependence across Trading Sessions
No Buyers' Coincidental Arrival	TS1	TS3
With Buyers' Coincidental Arrival	TS2	TS4

These four types of trading sessions are summarized in table 1. It is worth noting that in each of these trading session cases, we assume the agents behave exactly the same in each single trading session. That is, they make decisions following the same set of rules (buyer's optimal bid rule, supplier's acceptance rule, etc.) in every session. Thus for each type of these trading sessions, we only concern with a single trading session in the model. We first present the model of agents' behaviour in a type 1 trading session as a simple case. Extensions to other types of trading sessions are discussed later in section VI.

The buyers visit the sellers in a random order, in a sense that a particular visiting order is a *realization* of some data generating mechanism (DGM). Each buyer i has a different and random order of visiting sellers. The DGM determines the arrival order of buyers at supplier k. Since the visiting order is random, the buyers' arrival order at supplier k is also random.

From another perspective, we can define each meeting between a buyer i and a supplier k as an *encounter*, which is a pair of a buyer and a supplier (i, k). Each encounter generates a bid. The arrival of the pairs (encounters) occurs in an order which we see as a *realization* of some random generating mechanism. A partition of these encounters with respect to a buyer i is the buyer i's visiting order. A partition of these encounters with respect to a supplier k can be seen as the sequence of buyers'

Figure 1: A Timeline of a *realized* sequence of encounters



Note: Figure 1 plots an exemplary realized sequence of events. The x-axis is the time since the first encounter. Since we are not concerned with the actual continuous time of arrivals but the order of the arrival, we index encounters by order position in the sequence. An event in subplot Figure-1-A represents an arbitrary encounter. For example, "E73,6" represents the encounter between buyer 73 and seller 6. Events in subplot Figure-1-B (as a partition of the encounters in Figure-1-A) represent suppliers encountered by buyer 6. Events in subplot Figure-1-C correspond to buyers that show up at supplier 9.

arrivals to the supplier k. Here of interest is not the continuous time of arrivals, but the order of the arrivals. In Figure 1 is an illustration of an realized encounter sequence.

In figure 1, the timeline is plotted with respect to the time of first encounter, i.e., the time of first encounter is labeled as time 0. Since we are not concerned with the actual continuous time of arrivals but the order of the arrival, we index encounters by order position in the sequence. A (partial) particular *realized* sequence of encounters is plotted in Figure-1-A. If we partition these encounters with respect to buyers, then a particular buyer (here buyer 6[7]) partition represents a buyer's visiting sequence

[7]Here the number "6" in "buyer 6" does not indicate any order of the buyer set (i.e., buyer 6 is not the sixth buyer). It only serves as a convenient label of that buyers. It is no different from "buyer Z" or "buyer Alex", etc. Such label can be given since the buyer set I is at most countable. On the other hand, later in the context, we see, with respect to a particular realized sequence of encounters, at supplier k, there is a natural ordering of buyer set I_k defined by the arrival order of buyers. For example, in Figure 1, at supplier 9, the first buyer is buyer 10, and the last buyer is buyer 74.

(to suppliers), which is plotted in Figure-1-B. If we partition these encounters with respect to suppliers, then a particular supplier (here supplier 9) partition is the buyers' arrival sequence of this particular supplier, which is plotted in Figure-1-C. Since the sequence of encounters is random generated, both buyers' visiting sequences and suppliers' buyer arrival sequences as partitions of the encounters sequence are random.

Given a particular *realized* sequence of all encounters, we can define an ordering of the buyer set with respect to a supplier k by the buyers' arriving at the supplier k. We denote the ordered set of buyers as I_k. Such ordering is k-specific since buyers' arrivals sequence at different suppliers are different and in its nature random as a partition of a realization of encounters generating mechanism. Similarly, we can also define an ordering of the supplier set with respect a buyer i by the buyer's visiting sequence to the suppliers. We define the ordered supplier set as K_i. It processes the same property the ordered buyer set.

At the beginning of each trading session, each supplier k has an initial endowment or allocation of supply S_k^* which is the maximal quantity he can supply to the market. His goal is to maximize profit[8]. As buyers come to the supplier one at a time, the supplier has no information concerning which buyer will arrive next, so the supplier views arrival of buyers as random. The supplier needs to choose whether he sells to each bid as it is received or wait for a potential higher bid with waiting cost. Thus, to maximize profit, the supplier k must use some criteria to make decision. In the model, such criteria is defined as the *supplier's acceptance rule*. Since there is uncertainty in arrival order and bids of buyers (and in other factors such as deterioration), the waiting cost is treated as a random variate. Such uncertainty implies randomness in the supplier's acceptance and implies acceptance must be represented with a probability driven acceptance rule.

Each buyer i has an initial demand D_i^* to fulfill at the beginning of each trading session. At each encounter, if the buyer does not fulfill the procurement requirement, there exists an encounter-specific or an instantaneous penalty cost. For the simplicity of analysis, we assume the marginal penalty cost is constant, but buyer-specific. The penalty cost motivates the buyer to search for the product at each supplier k until some stopping time determined by buyers' stopping rule. We assume the buyer i perceives that there is randomness in the supplier's acceptance. He knows that whether his bid will be accepted or not depends on the supplier's probabilistic acceptance rule. If accepted, the buyer pays the price of products; if rejected, the buyer faces *search costs* which we assume are a function of the buyer's unfulfilled demand and what we will define as the buyer's perception of market conditions. Thus if the buyer bids high, he will have a higher probability of acceptance and securing the product. This means he can search for less time or even stop searching (when all his procurement demand is fulfilled). In this case, his expected search cost is determined. In contrast, if the bid (b_{ik}, q_{ik}) is not accepted, he will search this quantity q_{ik} again and search cost will remain uncertain. However, if he bids high, and his bid is accepted, the cost of buying the product will be high. Since the total procurement cost includes both search cost and the cost of products, a buyer will be

[8]Later we see in a type 3 or type 4 trading sessions that the maximal quantity equals to s_k^* (new allocation) plus carry-over storage.

assumed to compose bids such as to minimize the total procurement cost, making a tradeoff between the search cost and the cost of products. We assume buyers have information about the function form of suppliers' acceptance probability but not other buyers' bids (which is a part of acceptance rule 1). The buyer makes this tradeoff by choosing an optimal bid price conditioned on his information about the supplier's acceptance rule.

Buyers who have imperfect information about market must search for the products in their procurement process and hence face search costs. Stigler (1961) modeled search cost as a constant in each additional search (i.e.constant marginal search cost) (see [Stigler, 1961]). De los Santos et al. (2013) more generally specified search cost as an stochastic variable (see [De los Santos et al., 2013]). In this paper, we model *marginal* search cost as an increasing function of market conditions defined by market excess demand. That is, we assume as market becomes tighter, it is more difficult for buyers to find an additional unit of product. Such an effect could be nonlinear for search cost can be rather high with low total supply as noted in [Ellison and Wolitzky, 2012]. Thus, we model search cost as a nonlinear increasing function of market conditions.

If the distribution of random variates (e.g. search cost) are unobservable, imperfectly informed buyers can only make their decision based on their personal estimate of the economic variates. Such personal estimates are made based on buyers' subjective belief (priors) in the market uncertainties. In later text, we show that buyers who make (personal) optimal decision based on the expectation of their personal estimate of variates are actually optimizing over their subjective expectation of the real variates. On the other hand, in the process of searching for suppliers, the buyer usually makes decisions based on information collected from past encounters. This means, an agent may learn as he searches. In our model, the buyer will learn (update his subjective belief) about the market's supply capacity (i.e. market condition) by Bayesian updating their belief on the supply capacity (and hence market condition).

The market condition plays a central role in this model. It affects the final transaction price through buyers' search costs (as stated above), suppliers' waiting costs, and their acceptance probabilities. The supplier's acceptance decision is also based on their interpretation of market conditions. If there is a large excess demand that is perceivable to both sides in the market, the buyers would bid high to secure product (since search cost is high). Suppliers who perceive such information will become reluctant to accepts bids. Hence, the probability of acceptance will decrease, and the bid(s) that supplier accepts will be high. As a result, the transaction price will be high. As shown later in this paper, if search cost increases nonlinearly in excess demand, a large excess demand yields a huge increase in search cost. Such high search cost motivates buyers to bid high to win (otherwise, they will have to face high search costs). The increases in bid prices should be about proportional to (usually higher than, since if there is shortage in supply, buyers tend to observe low stock in the first several suppliers and form a biased belief that supply is extremely short) those in search costs. This means the final transaction price as the price in the accepted bid will increase nonlinearly in excess demand. This will provide our explanation of jump in prices.

3 Notation

3.1 Notation on Uncertainty

As stated above, there is uncertainty in the supplier's waiting cost and in the supplier's supply capacity. To account for such uncertainty, we define:
ω: a scenario corresponding to a particular waiting cost and supply capacity pair;
Ω: the set of scenarios that generates all possible waiting cost and supply capacity pairs.

If both suppliers' waiting cost and supply capacity admit finite values, then it is easy to enumerate all possible scenarios (2^Ω). However, in our settings, both these random variates admit continuous values ($WC \in \mathbb{R}_+$, and $S^* \in \mathbb{R}_+$) and we can not enumerate all possible scenarios. In this case, to measure the possibility of a particular event (for example, $S^* \leq 6$ or $WC \in (2, 10]$), we need to define a $\sigma-$algebra. Each set in this $\sigma-$algebra is a measurable[9] event (for example, $\{\omega | WC(\omega) \in (2, 10]\}$ is an event).
$\mathscr{F}$: the $\sigma-$algebra generated by Ω; each set in $\mathscr{F}$ is an event, on which we can assign a probability measure[10];
$\mathbb{P}$: an objective probability measure defined on the measurable space $(\Omega, \mathscr{F})$.

Then the triple $(\Omega, \mathscr{F}, \mathbb{P})$ is the underlying probability space on which all random variables are defined. Let $\mathbb{E}$ be the expectation taken with respect to $\mathbb{P}$. Now we can define variates of both sides in the market. We further assume all random variables are defined on $C_b^1(\Omega)$ (the set of continuous, differentiable and bounded functions).[11]

3.2 Notation on Ordering and Indices

As discussed in section 2, the set of buyers I and the set of sellers K are both unordered. On the other hand, with respect to a particular seller k, we can define an order of the buyer set by their (*realized*) arrival sequence to supplier k. We denote this ordered set of buyers as I_k. In this ordered set, each buyer $i \in I$ has a unique ordered (integer) index $\tau_k \in I_k$ such that any other buyer i' with index $\tau_k' > \tau_k$ means that buyer i' come to the supplier k earlier than buyer i. It is worth noting that there is one-to-one mapping between i and τ_k. And the index set $\{\tau_k\}_{\tau=1}^{|I|}$ is k-specific. Every other index set $\{\tau_{k'}\}$ (for supplier k') is a permutation of $\{\tau_k\}$.

[9] "Measurable" in a sense that one can assess the possibility of the event, objectively or subjectively.

[10] Here, we assume all the uncertainties in the model are those generated from waiting cost and supply capacity. A more technical formulation is: for some underlying metric space (Ω, ρ), where each ω represents some randomness, define two continuous and bounded mappings waiting cost WC and (local) supply capacity S_k^* from (Ω, ρ) to $(\mathbb{R}_+, \lambda)$, where λ is the Lebesgue measure. Denote $\sigma_{WC} = \{WC^{-1}(B) \text{ for } \forall B \in \mathscr{B}\}$ be the σ-algebra generated by waiting cost WC, and $\sigma_{S_k^*}$ the σ-algebra generated by supply capacity. Let $\mathscr{F} = \sigma(\sigma_{WC} \cup \sigma_{S_k^*})$. Then $(\Omega, \mathscr{F})$ is a measurable space and both WC and S_K^* are random variables on this space.

[11] It is worth noting that continuity and differentiability requires some metric on Ω. Indeed, if WC and S^* admit values in $\mathbb{R}_+$, then $\Omega = \mathbb{R}_+^2$. Let ρ be the euclidean distance on $\mathbb{R}_+^2$, then (Ω, ρ) is a metric space. And thus continuity and differentiability are properly defined.

Similarly, with respect to a particular buyer i, we can define an order of the supplier set by the buyer's visiting order. We denote this ordered set of suppliers as K_i. Then each supplier $k \in K$ has a unique ordered index $\tau_i \in K_i$ such that for any other supplier k' with index τ_i', the buyer i visits supplier k' prior to supplier k if $\tau_i' < \tau_i$. The set of index $\{\tau_i\}_{\tau=1}^{|K|}$ has similar properties as $\{\tau_k\}_{\tau=1}^{|I|}$.

In the context where supplier k is definite (for example, some variate $X_{\tau_k k}$ is clearly related to supplier k), for simplicity, we will write τ instead of τ_k (in this case, we write $X_{\tau k}$ instead of $X_{\tau_k k}$). Similar rule applies to τ_i.

In the case when both the set I and the buyer are numerically labeled from $i = 1$ to $i = |I|$ where $|\cdot|$ is the cardinality, the one-to-one mapping between I and I_k (for any k) is a permutation matrix and hence an isometry. Then it is essentially the same to write i or τ_k. Same thing applies to k and τ_i. On the other hand, for consistency, we will keep write τ_k and τ_i when ordering is involved.

3.3 Buyer Side Notation

There is an finite unordered set of buyers I. For each buyer $i \in I$, his characteristics are:
D_i^*: initial procurement demand that buyer i seeks to fulfill at the beginning of each trading session; if he fails to fulfill such demand, he will face a cost (or penalty);
(i, k): a buyer and supplier pair to denote there is an encounter between buyer i and supplier k (or, when the buyer i visit to supplier k); every encounter generates a price-quantity bid pair;
(b_{ik}, q_{ik}): the price-quantity bid pair he offers to supplier k, where q_{ik} is the bid price, and b_{ik} is the bid quantity;
c_i: per-unit instantaneous penalty cost of unsatisfied demand;
D_i^{*k}: the unfulfilled procurement demand (the amount he needs to search) before he bids at supplier k, that is the total demand minus the amount of product he already procured ($D_i^{*k} = D_i^* - \sum_{\tau'<\tau} q^a_{i\tau'}$), where τ is the index of supplier $k \in K$ in the ordered set K_i ;
$SC_{ik}(\omega, D_i^{*k}, \text{ market conditions, current supplier's supply})$: his search cost given k-th supplier which is unknown before hand and is an increasing function of quantity needed D_i^{*k} and the market conditions; it is a decreasing function of current supplier's supply;
$\mathbb{Q}_i$: his subjective probability measure on $(\Omega, \mathscr{F})$; it reflects the buyer's personal belief in the uncertainty in the system (waiting cost and supplier's capacity); let $\mathbb{E}_{\mathbb{Q}_i}$ be the expectation taken with respect to $\mathbb{Q}_i$. The buyer i generates his estimate of supplier's waiting cost and supply capacity based on his personal belief.

Thus at the beginning of each trading session, one buyer has an initial demand D_i^* to fulfill. At supplier k, his offers a bid of (b_{ik}, q_{ik}) a price-quantity pair. He holds his own subjective belief in the unknowns, which is characterized by his subjective probability $\mathbb{Q}_i$.

3.4 Supplier Side Notation

There is an finite unordered set of suppliers K. For each supplier $k \in K$, his characteristics are:
$S_k^*(\omega)$: the supplier k's supply capacity; we assume this supply capacity is random. Supply is identically distributed across different suppliers (locations) by some data generating mechanism $S^*(\omega)$. In this model each supplier can be seen to have a unique location, and thus later in this article, S_k^* is also referred as local supply.
s_k^*: the *realized* suppler k's capacity (or, realized local supply) and *observed* by buyers;
$WC_{ik}(\omega)$: his waiting cost before the buyer i offers a bid; it is a random variable that depends on market conditions;
$B_{ik}(\omega)$: B_{ik} is the supplier's choice or control. We define it as a binary indicator based on acceptance of the buyer i's bid (q_{ik}, b_{ik}); that is, B_{ik} is 1 if the bid (q_{ik}, b_{ik}) is accepted by supplier k and 0 if the bid is not accepted; it is a random variable since the acceptance rule is probability-driven and conditioned by market condition;
P_{ik}(market conditions): as B_{ik} is binary, we define P_{ik} as the probability that a bid is accepted at the encounter between i and k;
$q_{ik}^a(\omega)$: the quantity of fulfillment associated with the buyer i's bid *accepted* by supplier k; in a simple case, $q_{ik}^a = B_{ik} q_{ik}$;
$\mathbb{C}_k$: the supplier k's optimal choice set of bids, i.e. $\mathbb{C}_k = \{i \in I | B_{ik}^* = 1\}$;
$\mathbb{Q}_k$: his subjective probability measure on $(\Omega, \mathscr{F})$; it reflects the supplier k's personal belief in the uncertainty in the system (waiting cost and supplier's capacity); let $\mathbb{E}_{\mathbb{Q}_k}$ be the expectation taken with respect to $\mathbb{Q}_k$.

The sequence of buyers' arrivals to supplier k is generated from a random data generating mechanism. Each buyer's arrival to the supplier (or, encounter) generates a sequence of bids. Thus the supplier k faces a *realized* ordered sequence of bids $\{(q_{\tau k}, b_{\tau k})\}_{\tau \in I_k}$. The supplier k only observes a partially realized ordered sequence of bids. Upon the arrival of a particular buyer i with index $\tau \in I_k$, the supplier k only knows past arrivals and their bids prior to buyer i, i.e, $\{q_{\tau' k}, b_{\tau' k}\}_{\tau' < \tau}$. If we define the set of buyers that comes earlier than i (or, with index strictly smaller than τ) as $I_{<\tau}$[12], the supplier k's information about past bids can be written as $\{(q_{ik}, b_{ik})\}_{i \in I_{<\tau}}$. The supplier uses this information to assess whether there is a potential higher bid in the future and decide whether to accept the current bid.

3.5 Market Condition Notation

For the market conditions, we define:
$\sum_{i \in I} D_i^*$: total initial market demand at the beginning of each trading session;
$\sum_{k \in K} S_k^*(\omega)$: total market supply; this is a random variate since there is shock in supply, as discussed above;
$Z(\omega) = \sum_{i \in I} D_i^* - \sum_{k \in K} S_k^*(\omega)$: actual excess demand in market at the start of each trading session, unobservable to buyers.

[12] Similarly we denote the set of suppliers visited by buyer i earlier than supplier k (with index $\tau \in K_i$) as $K_{<\tau}$

3.6 Agent's Information and Estimate Notation

In this model, both sides in the market do not have perfect information. Their information evolves as they have encounters. That is, a buyer has a sequence of increasingly richer information set as he visits more suppliers; and a supplier has richer information as more buyers visit him.

Φ_i^k: buyer i's information set when he is at supplier k. Note each supplier k announce his supply capacity before the buyer offers a bid, thus his information set before offering bid to supplier k (with index τ_i). Φ_i^k includes:

1. all the supply from previous supplier including supplier k, $\{S_{1_i}, S_{2_i}, ..., S_{\tau_i}\}$;
2. all the accepted quantity by previous suppliers excluding supplier k, that is, $\{q_{i1}^a, q_{i2}^a, ..., q_{i(\tau-1)}^a\}$;
3. the function form of supplier's acceptance probability.

After offering the bid to supplier k, the supplier announces whether to accept the bid or not (or, accepted quantity). Then the buyer's information set (after the supplier's announcement of accepted quantity) becomes to $\Phi_i^k \cup \{q_{i\tau}^a\}$.

To define the relation between this information set and personal belief, we note a perfectly informed agent must have his belief $\mathbb{Q}_i$ agreeing with the true (objective) probability $\mathbb{P}$. That is, he knows all the waiting cost and supplier capacity pairs and is able to assess any event with correct probability. For an imperfectly informed individual, in a simple case, suppose the buyer i has partial but correct information, then for some two sets with different $\mathbb{P}$ probability, $\mathbb{Q}_i$ may admit the same probability[13]. In a more general case, buyer i may have some misinformation, which means $\mathbb{Q}_i$ and $\mathbb{P}$ do not agree for any set in $\mathscr{F}$ (the buyer hold a "wrong" subjective belief for an event, "wrong" in sense it is different than the *true* probability).

Since buyers do not have perfect information about market condition, waiting cost and other buyers' bids, he must estimate these variates, on which he forms his best *personal* optimal bid price.

$\widetilde{X}^i(\omega)$: for a random variable $X(\omega)$, we use $\widetilde{X}^i(\omega)$ to denote the buyer i's estimate of this random variable $X(\omega)$; here $\widetilde{X}^i(\omega)$ is a random variable which is (almost surely) defined by the distribution function $\mathbb{Q}_i \circ X^{-1}$, where $\circ$ is composition[14]. That is, the cumulative density function $F_{\widetilde{X}_i}(t) = \mathbb{Q}_i(X^{*-1}((-\infty, t]))$, for $\forall t \in \mathbb{R}$.[15]

$\widetilde{S_k}^i(\omega)$: for buyer i and his visiting sequence K_i, at the time he visits supplier k with index $\tau \in K_i$, he only knows about past suppliers' supply $\{S_{\tau'}^*\}_{\tau'<\tau}$; for the suppliers he has not visited, he must estimate the supply capacity; here we use $\widetilde{S_k}^i(\omega)$ to denote the buyer i's estimate of the k-th supplier's capacity;

[13]That is, $\sigma(\{\mathbb{Q}_i^{-1}(a) : \text{ for } \forall a \in [0,1]\}$, which is a strict subset of $\sigma(\{\mathbb{P}^{-1}(b) : \text{ for } \forall b \in [0,1]\}) = \mathscr{F}$.

[14]To simplify notation, I will write $\mathbb{P} \circ X^{-1}$ as $\mathbb{P}X^{-1}$ to denote the probability distribution of X on the probability space $(\Omega, \mathscr{F}, \mathbb{P})$ in the following text.

[15]More accurately, a buyer makes his estimate based on the *observed* value of random variates. Thus the estimate must be measurable with respect to the filtration $\mathscr{F}_k$, where $\mathscr{F}_k$ is the sigma algebra generated by supplier waiting cost and supply capacity up to supplier k.

$\widetilde{Z}^i(\omega) = \sum_{i \in I} D_i^* - \sum_{k \in K} \widetilde{S_k}^i(\omega)$: at the time of encounter ik, the buyer i estimates the available supply capacity in the market based on his information about past visit $\{S^*_{\tau'}\}_{\tau' < \tau}$; we use $\widetilde{Z}^i(\omega)$ to denote the buyer i's estimate of market condition (excess demand Z).

Similarly, a supplier k has richer information set as more buyers arrive. To characterize his information and personal estimate, we define:
Ψ_k^i: supplier k's information set when the buyer i (with index $\tau_k \in I_k$) arrives and makes a bid to him. In general Ψ_k^i includes:

1. all the past buyers' bids received by supplier k prior to (and including) buyer i, $\{(q_{ik}, b_{ik})\}_{i \in I_{\leq \tau_k}} = \{(q_{\tau' k}, b_{\tau' k})\}_{\tau' \leq \tau_k}$, where $I_{\leq \tau_k}$ is defined as the set of buyers that comes to the supplier k earlier than (including) buyer i;

2. the waiting costs for the most recently encountered buyers' waiting costs $\{WC_{ik}\}_{i \in I_{\leq \tau_k}}$, where $I_{\leq \tau_k}$ is defined as above.

When suppliers can be assumed to be able to observe other supplier decisions and their available supply (e.g. with a small set of suppliers), supplier k will have rich information about market conditions. It follows the supplier information set Ψ_k^i can be assumed to include (in addition to all information defined as above) : market condition Z, other suppliers' supply $\{S_j^*\}_{j \neq k}$ and market equilibrium price b^e.We use $\widetilde{X}^k(\omega)$ to denote the supplier k's estimate of a random variate $X(\omega)$ such that its distribution is defined as $\mathbb{Q}_k \circ X^{-1}$, where $\mathbb{Q}_k$ is his personal belief.

The reason to use such formulation to model buyer's personal estimate of supply is three fold. First, there are no subjective random variables. To represent a buyer's personal estimate of some economic (random) variate, we need a proper definition. Second, a person does not estimate or predict economic variates independently; instead, he holds a fundamental belief in the market uncertainties, on which he forms his estimates of random variable. Last, but most importantly, this is a consistent notation, in a sense that the objective (true) expectation of the estimate equals to the personal expectation of the true random variable. To see this, for any random variable X on $(\Omega, \mathscr{F})$, define $\tilde{X}$ by the distribution $\mathbb{Q}X^{-1}$, then $\mathbb{P}\tilde{X}^{-1} = \mathbb{Q}X^{-1}$, since they both are the distribution function (equivalent measure on $(\mathbb{R}, \mathscr{B})$). In further, for any continuous function f such that $f(X)$ has finite first moment, we have:

$$\begin{aligned}
\mathbb{E}_{\mathbb{P}} f(\tilde{X}) = \int_{\Omega} f(\tilde{X}(\omega))\mathbb{P}(d\omega) &= \int_{\mathbb{R}} f(x)\mathbb{P}\tilde{X}^{-1}(dx) \\
&= \int_{\mathbb{R}} f(x)\mathbb{Q}X^{-1}(dx) = \int_{\Omega} f(X(\omega))\mathbb{Q}(d\omega) = \mathbb{E}_{\mathbb{Q}} f(X)
\end{aligned} \tag{1}$$

That is, the expectation of the personal estimate of the variate equals to the personal expectation of the real variate. Thus buyers who make optimal decisions based on the expectation of their estimate of variates are actually optimizing over their personal expectation of real variates[16].

[16] $\max_c \mathbb{E}_{\mathbb{P}} f(c; \widetilde{X}) = \max_c \mathbb{E}_{\mathbb{Q}_i} f(c; X)$ where X is a random variate, $\widetilde{X}$ is the estimate of X and c is some deterministic variate.

4 Model

4.1 Buyer Side

In this section we present a model of the buyer's choice problem. As stated in section 3, each buyer i visits suppliers in a random order which can be seen as a realization of some data generating mechanism. A particular *realized* visiting sequence gives the unordered supplier set K an ordering K_i. Then for a supplier k, suppose he is the τ-th supplier that buyer i encounters, then we say his index is τ in the ordered set K_i. At the time that buyer i visits supplier k, his information set Φ_i^k includes his price-quantity bid offered to previous suppliers, the suppliers' accepted quantity and their supply capacity., i.e. $\{(b_{i\tau'}, q_{i\tau'}), S^*_{\tau'}, q^a_{i\tau'}\}_{\tau'<\tau}$. Recall, we label the set of suppliers visited by buyer i earlier than supplier k (with index $\tau \in K_i$) as $K_{<\tau}$. Thus, the buyer's information set Φ_i^k includes $\{(b_{ik'}, q_{ik'}), S^*_{k'}, q^a_{ik'}\}_{k'\in K_{<\tau}}$ [17]. At each supplier k, the buyer i must choose the optimal price-quantity bid to minimize his total cost conditioned [18]on his information set Φ_i^k, i.e. :

$$\begin{aligned} \min_{(q_{ik}, b_{ik})} \quad & \mathbb{E}_{\mathbb{Q}_i}\Big\{q^a_{ik}b_{ik} + SC_{ik} + c_i(D_i^{*k} - q^a_{ik})^+\Big\} \text{ for } \forall k \\ \text{subject to} \quad & q^a_{ik} \leq S^*_k \quad \forall k \in K \end{aligned} \tag{2}$$

where D_i^{*k} is the unsatisfied procurement demand (the amount he needs to search) at the ik-th encounter, i.e. before he bids at supplier k, that is the total demand minus the amount of product he already procured (recall $D_i^{*k} = D_i^* - \sum_{\tau'<\tau} q^a_{i\tau'}$). $q^a_{i\tau}$ is the accepted quantity by the τ-th supplier (in the ordered set K_i), i.e. $q^a_{ik} = B_{ik}q_{ik}$. If the buyer i has full information about supplier k's acceptance rule and the supply S^* (and hence $\mathbb{Q}_i$ and $\mathbb{P}$ agree)[19], the expected accepted quantity $\mathbb{E}_{\mathbb{Q}_i}(q^a_{ik}) = \mathbb{E}_{\mathbb{Q}_i}\Big\{B_{ik}q_{ik}\Big\} = \mathbb{E}\Big\{B_{ik}q_{ik}\Big\} = P_{ik}q_{ik}$ and hence the buyer best *subjective* response is indeed the optimal one . In a more general case, the buyer knows only partially about supplier k's acceptance rule and market condition. Then $\mathbb{E}_i f(\tilde{X}) \neq \mathbb{E}f(X)$. Thus the buyer's best response shifts away from the objectively best one, to what extent his subjective belief is biased.

Problem (2) says at every supplier k, buyer i tries to minimize the total expected cost. Such cost includes the money he pays for the bid $q^a_{ik}b_{ik}$, search cost for supplier k and instantaneous penalty cost $c_i(D_i^* - q^a_{ik})^+$. Note $c_i(D_i^* - q^a_{ik})^+ \equiv c_i \max(0, D_i^* - q^a_{ik})$ in the penalty cost means we only count the positive part of

[17] Similarly, we write set of supplier encountered by buyer i later than (including) supplier k as $K_{\geq\tau}$.

[18] Intuitively, the objective of problem (2) is to minimize the expectation conditioned on his current information $\mathbb{E}_{\mathbb{Q}_i}\Big\{q^a_{ik}b_{ik} + SC_{ik} + c_i(D_i^* - q^a_{ik})^+|\Phi_i^k\Big\}$. Since buyer i's personal belief $\mathbb{Q}_i$ evolves as his information set Φ_i^k becomes richer and hence has integrated the effect of information, we do not use the conditional expectation notation.

[19] Actually, in this case, the buyer i forms a "correct" expectation, in a sense that for $\forall\tilde{X}$ defined by $\mathbb{Q}_i X^{-1}$, and for $\forall$ continuous function f, $\mathbb{E}_{\mathbb{P}} f(\tilde{X}) = \int_\Omega f(\tilde{X}(\omega))\mathbb{P}(d\omega) = \int_{\mathbb{R}} f(x)\mathbb{P}\tilde{X}^{-1}(dx) = \int_{\mathbb{R}} f(x)\mathbb{Q}X^{-1}(dx) = \int_\Omega f(X(\omega))\mathbb{Q}(d\omega) = \int_\Omega f(X(\omega))\mathbb{P}(d\omega) = \mathbb{E}_{\mathbb{P}} f(X)$. In particular, the subjective expectation in the objective function in problem (2) becomes to objective expectation, and hence the buyer's best subjective response is indeed the best objective one.

penalty cost. On the other hand, since once $D_i^* - q_{ik}^a$ hits zero the buyer will be out of market, we will drop + sign from now on.

To solve the problem (2), at supplier k^*, the risk neutral buyer i needs to decide whether to bid or not at supplier k^*. If he does not bid at supplier k^*, his procurement cost minimization problem becomes to:

$$
\begin{aligned}
\min_{(q_{ik}, b_{ik})} \quad & \mathbb{E}_{\mathbb{Q}_i}\Big\{ q_{ik}^a b_{ik} + SC_{ik} + c_i(D_i^{*k} - q_{ik}^a)^+ \Big\} \text{ for } \forall k \\
\text{subject to} \quad & q_{ik^*} = 0 \\
& q_{ik}^a \le S_k^* \quad \forall k \in K \setminus \{k^*\}
\end{aligned}
\tag{3}
$$

Note problem (3) is equivalent to problem (2) with an additional constraints $q_{ik^*} = 0$. Then the solution to problem (3) must be no lower than that to problem (2). Thus, to minimize his procurement cost, the buyer places a bid at each supplier k.

4.1.1 Buyer's Search Cost and Searching Rule

Each buyer must search for suppliers and products, in what process he faces search cost (SC_{ik} in equation (2)). Intuitively, a buyer's search cost depends on market condition and the amount of products he is searching. On the other hand, a buyer does not know his search cost beforehand and hence he has to estimate it. Such estimate of search cost should also depend on the supply capacity at current encounter. If the buyer observes a low supply at current supplier (say, suppose the buyer i is now at supplier k), he will expect a higher search cost in the future. Thus the buyer's *estimated* search cost is a function of his estimate of market condition, the amount of products for search, and *observed* supplier k's supply s_k^*:

$$
\widetilde{SC_{ik}}^{\,i} = f(D_i^{*k} - q_{ik}^a, \widetilde{Z}^i, s_k^*) \tag{4}
$$

where $\widetilde{SC_{ik}}^{\,i}$ is the buyer i's estimated search cost, D_i^{*k} and q_{ik}^a are as defined above, $\widetilde{Z}^i$ is the buyer's estimate of market condition (we will specify this later in context) and f is a function. To derive some properties of the functional form of f, we assume f is smooth function that is twice differentiable with respect to Z and s_k^* to characterize the curvature (nonlinearity). We note that search cost should be an increasing function of the quantity to be searched after current encounter ($D_i^{*k} - q_{ik}^a$) and the market excess demand. In addition, as discussed in section II, search cost should increase nonlinearly in market excess demand. Thus, given the differentiability assumption of search cost, the first- and second-order derivatives of search cost with respect to market condition and the first-order derivative of search cost with respect to the quantity to be search should be positive.

Further, as discussed above, search costs should be able to characterize the effect of current supplier's supply on the buyer's personal estimate of search cost. That is, if the buyer i observes that the current supplier k's stock is "surprisingly" low, then the buyer's estimate of search cost will increase nonlinearly. These assumptions imply that buyer search cost is a decreasing function of supply capacity at current encounter with a decreasing second derivative[20]. In summary, these assumptions

[20]That is, as $s_k^* \to 0$, $\frac{\partial \widetilde{SC_{ik}}^{\,i}}{\partial s_k^*}$ increases.

regarding to search cost can be summarized as follows:

$$\frac{\partial \widetilde{SC_{ik}}^{i}}{\partial \widetilde{Z}^{i}} > 0, \frac{\partial \widetilde{SC_{ik}}^{i}}{\partial (D_i^{*k} - q_{ik}^{a})} > 0, \frac{\partial \widetilde{SC_{ik}}^{i}}{\partial s_k^{*}} < 0, \frac{\partial^2 \widetilde{SC_{ik}}^{i}}{\partial \widetilde{Z}^{i2}} > 0, \frac{\partial^2 \widetilde{SC_{ik}}^{i}}{\partial s_k^{*2}} > 0 \tag{5}$$

An exemplary specification of the search cost which we use later for analytical solution is

$$\widetilde{SC_{ik}}^{i} = (D_i^{*k} - q_{ik}^{a})(\widetilde{Z}^{i})^2/\beta s_k^{*}. \tag{6}$$

where β is the parameter specified later in the simulation section. It is easy to check that this formulation agrees with the desired properties of search cost specified in equation (5).

A risk neutral buyer will keep searching as long as his estimated search cost does not exceed his expected marginal benefits from search. Recall in this model, a buyer is assumed to remember only what happened in a single trading session ("short memory") and concerns only what will happen in that session, which we label as "myopic". Since a myopic buyer has an instantaneous penalty cost at every encounter with supplier in current trading session, every unit of unfulfilled demand admits an penalty cost for every encounter left in that trading session. As a result, he will not stop searching as long as his estimated search cost does not exceed the corresponding penalty cost[21] [22]. These results can be summarized as the buyer's optimal stopping rule.

Optimal Stopping Rule: A risk neutral buyer will stop searching if and only if his expected marginal benefits from search is smaller than his estimated search cost. In our settings (equation 2), a myopic (and risk neutral) buyer will stop searching if and if only his estimated search cost exceeds the corresponding penalty cost.

4.1.2 Buyers' Learning Rule

A buyer has imperfect information about market conditions. As the buyer i encounters suppliers, the buyer usually makes decisions based on not only prior information including learning information and past search information, but also information he got in the process of search. This means, an agent learns as he searches. In this setting, the buyer will learn (updates his subjective belief) about the supplier's individual supply capacity S^* (and hence market condition[23]) by Bayesian updating their belief on the supply capacity (and hence market condition).

Most "search with learning" models assumes the agent have complete knowledge about the distribution of prices. With such assumption, it can be shown that there

[21]For a buyer that is not myopic, he will not stop searching until all his demand is fulfilled since every unit of unfulfilled demand admits an infinite penalty cost in the infinite remote future. If we in further allows for buyers' hoarding, then he will never stop search another case since every *potential* unit of unfulfilled demand admits an infinite penalty cost in the infinite remote future.

[22]In another case, if a myopic buyer does not have instantaneous penalty cost but penalty cost at the end of each trading session, then the buyer will not stop search as long as his marginal penalty cost at the end of each session does not exceed his estimated search cost

[23]This can be done since supply is assumed to be identically distributed.

exists an optimal stopping rule that controls and truncated further sampling from the distribution. And such stopping rule is myopic and has the reservation price property. The existence of optimal stopping rule which is myopic and admits an reservation price was extended to the case in which an agent does not have complete information of the distribution of prices. Rothschild (1974) by assuming price admits only finite values (multinomial Dirichlet distribution), showed that the optimal stopping rule also has the reservation price under unknown distribution (see [Rothschild, 1974]). Bikhchandani and Sharma (1996) in [Bikhchandani and Sharma, 1996] further relaxed the assumption and assumed prices can be any real number. Using Dirichlet process, they provided sufficient conditions for the existence of optimal stopping rules with the reservation property. Santos et al. (2013) applied such method to the case where the unknown opportunity' is an utility function instead of price (see [De los Santos et al., 2013]).

In our model, the buyer is searching for products (or, supply). It is too restrictive to assume the quantity of products for search can take only finite values following a multinomial Dirichlet distribution. In particular, we allow for supply to take any positive integer values. Since Dirichlet process is conjugate to any prior distribution (of supply), we follow [Bikhchandani and Sharma, 1996] and [De los Santos et al., 2013] and use Dirichlet process to model the buyer's Bayesian learning rule.

Formally, let $(\Omega^\ddagger, \mathscr{F}^\ddagger, \mathbb{P}^\ddagger)$ be a underlying probability space representing the randomness in the random measure. Suppose the random variable $S^*(\omega)$ admits value in $\mathbb{R}_+$ (suppliers do not have negative stock and cannot sell short). Let $\mathscr{B}$ be the Borel σ-algebra of $\mathbb{R}_+$. Then $(\mathbb{R}_+, \mathscr{B})$ is a measurable space. A mapping $P(\omega^\ddagger, B)$ from $\Omega^\ddagger \times \mathscr{B}$ to [0,1] is called a *random probability measure* on $(R_+, \mathscr{B})$ if:
(1) for fixed $\omega^\ddagger \in \Omega^\ddagger$, P is a probability measure on $(R_+, \mathscr{B})$;
(2) for fixed $B \in \mathscr{B}$, P is a random variable on $(\Omega^\ddagger, \mathscr{F})$.
A random measure D is distributed according to a Dirichlet process with base distribution $\mathbb{G}$ and concentration parameter α if for any finite measurable partition $(A_1, A_2, ..., A_n)$ of $\mathbb{R}_+$, the random vector

$$(D(A_1), D(A_2), ..., D(A_n)) \sim \mathrm{Dir}(\alpha\mathbb{G}(A_1), \alpha\mathbb{G}(A_2), ..., \alpha\mathbb{G}(A_n)), \tag{7}$$

where $\mathrm{Dir}(\alpha\mathbb{G}(A_1), \alpha\mathbb{G}(A_2), ..., \alpha\mathbb{G}(A_n))$ is a Dirichlet distribution with parameters $\alpha\mathbb{G}(A_1), \alpha\mathbb{G}(A_2), ..., \alpha\mathbb{G}(A_n)$, denoted as $D \sim DP(\alpha, \mathbb{G})$ (see [Ferguson, 1973]).

A Dirichlet process prior is conjugate for any distribution. Formally, for $D \sim DP(\alpha, \mathbb{G})$, D is a random distribution, from which we can draw a sample. In our setting, we assume each supplier's capacity $S_k^* \sim D$. A buyer who has observed $s_1^*, s_2^*, ..., s_\tau^*$ [24]has a posterior distribution

$$D | s_1^*, s_2^*, ..., s_\tau^* \sim DP(\alpha + \tau, \frac{\alpha}{\alpha + \tau}\mathbb{G} + \frac{\tau}{\alpha + \tau}\frac{\sum_{j=1}^{\tau} \delta_{s_j^*}}{\tau}) \tag{8}$$

where $\delta_{s_j^*}$ is the point mass at s_j^* and τ is the number of suppliers that buyer i has already encountered. That is, the buyer i is currently bidding at supplier k with index $\tau \in K_i$. Note that the posterior base distribution is a convex combination of

[24]Note here the indices $\{1, 2, ..., n\}$ in $s_1^*, s_2^*, ..., s_\tau^*$ are indices of s_τ in the ordered set K_i

the prior base distribution $\mathbb{G}$ and the empirical distribution $\frac{1}{\tau}\sum_{k=1}^{\tau}\delta_{S_j^*}$. The posterior base distribution is also the predictive distribution of S_{n+1}^* given $s_1^*, s_2^*, ..., s_\tau^*$ and can be written as[25]:

$$S_{n+1}^*|s_1^*, s_2^*, ..., s_\tau^* \sim \frac{\alpha}{\alpha+\tau}\mathbb{G} + \frac{\tau}{\alpha+\tau}\frac{\sum_{j=1}^{\tau}\delta_{s_j^*}}{\tau} \tag{9}$$

Thus, under perfect information, the best (objective) estimator of $S_{\tau+1}^*$ conditioning on $S_1^*, ..., S_\tau^*$ is its posterior mean:

$$\widehat{S_{\tau+1}^*}|S_1^*, S_2^*, ..., S_\tau^* = \mathbb{E}(S_{\tau+1}^*|S_1^*, S_2^*, ..., S_\tau^*) = \frac{\alpha}{\alpha+\tau}\mathbb{E}\mathbb{G} + \frac{\tau}{\alpha+\tau}\frac{\sum_j^\tau S_j^*}{\tau} \tag{11}$$

where $\mathbb{E}\mathbb{G}$ denotes the expectation taken with respect to $\mathbb{G}$.[26]

On the other, an imperfect informed buyer does not know the true base distribution $\mathbb{G}$ and the concentration parameter α. Suppose his personal estimate of $\mathbb{G}$ is $\widetilde{\mathbb{G}}^i$ and his estimate of α is $\widetilde{\alpha}^i$. Then his best personal estimator of $S_{\tau+1}^*$ given his observation $S_1^*, ..., s_\tau^*$ is:

$$\widetilde{S_{\tau+1}^*}|S_1^*, S_2^*, ..., S_\tau^* = \mathbb{E}_{\mathbb{Q}_i}(S_{\tau+1}^*|S_1^*, S_2^*, ..., S_\tau^*) = \frac{\widetilde{\alpha}^i}{\widetilde{\alpha}^i+\tau}\mathbb{E}\widetilde{\mathbb{G}}^i + \frac{\tau}{\widetilde{\alpha}^i+\tau}\frac{\sum_i^\tau S_j^*}{\tau} \tag{12}$$

To verify this learning process is consistent with our model, we need to examine the asymptotic behaviour of $\widetilde{S^*}$. In order to do so, suppose we have countable infinite suppliers ($|K| = \infty$), then we note as the observations $\tau \to \infty$ the total number of suppliers, in equation (7) the first part of (rhs) $\mathbb{E}\mathbb{G}$ converges pointwise to 0, the second part of (rhs) converges to S^* in distribution (see [Lo, 1983]). As a result, $\widehat{S_{\tau+1}^*} \xrightarrow{d} S^*$ as $\tau \to \infty$. By similar argument, we have

$$\widetilde{S_{\tau+1}^*} \xrightarrow{d} S^* \text{ as } \tau \to \infty \tag{13}$$

This means for whatever initial belief of α and $\mathbb{G}$ the buyer holds, after he observes enough suppliers, he will have a good estimate of each supplier's supply (or, local supply). We also note (10) is equivalent to $P\widetilde{S_{\tau+1}^*}^{-1} \Rightarrow PS^{*-1}$, where $\Rightarrow$ is *weak convergence* of measures on $(\mathbb{R}_+, \mathscr{B})$. On the other hand, since we constructed the buyer i's estimate $\widetilde{S^*}$ by the distribution $\mathbb{Q}_i S^{-1}$, then $\mathbb{Q}_i S^{*-1}$ and $\mathbb{P}(\widetilde{S}^i)^{-1}$ must coincide. Thus we have:

$$\mathbb{Q}_i^\tau S^{*-1} \Rightarrow \mathbb{P}S^{*-1} \text{ on } (\mathbb{R}_+, \mathscr{B}). \tag{14}$$

[25]More accurately, the $S_{\tau+1}^*|s_1^*, s_2^*, ..., s_\tau^*$ follows the base distribution of the posterior distribution of Dirichlet Process.

$$\begin{aligned}\mathbb{P}(S_{\tau+1}^*(\omega) \in B|s_1^*, s_2^*, ..., s_\tau^*) &= \mathbb{E}^\ddagger\{D(B)|s_1^*, s_2^*, ..., s_\tau^*\} \\ &= \frac{\alpha}{\alpha+\tau}\mathbb{G}(B) + \frac{\tau}{\alpha+\tau}\frac{\sum_{j=1}^{\tau}\delta_{s_j^*}(B)}{\tau}, \ \forall B \in \mathscr{B}\end{aligned} \tag{10}$$

where $\mathbb{E}^\ddagger$ is the expectation taken with respect to $P^\ddagger$ on $\Omega^\ddagger$.

[26]To simplify notation, define $\mathbb{E}\mathbb{G} \equiv \mathbb{E}X$, for X a random variable distributed as $\mathbb{G}$ (or $\mathbb{P}X^{-1}((-\infty, x]) = \mathbb{G}((-\infty, x])$, for $\forall x \in \mathbb{R}_+$).

where $\mathbb{Q}_i^\tau \equiv \mathbb{Q}_i$ when buyer i is at supplier τ. This says given enough observations, the buyer's estimate of (the distribution of) S_k^* will become rather accurate (say). In further, if there is no uncertainty in the waiting cost, then the buyer's belief $\mathbb{Q}_i$ weakly converges to the objective measure $\mathbb{P}$[27]. This means $\mathbb{E}_{\mathbb{Q}_i} X = \mathbb{E}_{\mathbb{P}} X$, for any continuous bounded X. This means the optimization problem (2) admits the true optimal bid strategy.

These results can be summarized as the buyer's learning rule.

Learning Rule of Local Supply: Recall the local supply is each supplier's individual supply S_k^* and it follows the random distribution D. The random distribution is distributed as Dirichlet process $DP(\alpha, \mathbb{G})$, where $\mathbb{G}$ is prior base distribution which represents prior knowledge or belief about the supply capacity and the random distribution. A imperfectly informed buyer i holds personal estimates $\widetilde{\alpha}^i$ and $\widetilde{\mathbb{G}}^i$. Thus, the buyer's estimate of D is $\widetilde{D}^i \sim DP(\widetilde{\alpha}^i, \widetilde{\mathbb{G}}^i)$. After the buyer observes a sequence of supply capacities $s_1^*, s_2^*, ..., s_\tau^*$, his posterior estimate about the distribution D will become to $\widetilde{D}^i | s_1^*, s_2^*, ..., s_\tau^* \sim DP(\widetilde{\alpha}^i + \tau, \frac{\widetilde{\alpha}^i}{\widetilde{\alpha}^i+\tau}\widetilde{\mathbb{G}}^i + \frac{\tau}{\widetilde{\alpha}^i+\tau}\frac{\sum_{j=1}^{\tau}\delta_{S_j^*}}{\tau})$. His (subjective) prediction about $S_{\tau+1}$ is distributed as the posterior base distribution of the posterior Dirichlet process. That is:

$$\widetilde{S_{\tau+1}^*} | s_1^*, s_2^*, ..., s_\tau^* = \frac{\widetilde{\alpha}^i}{\widetilde{\alpha}^i + \tau}\mathbb{E}\widetilde{\mathbb{G}}^i + \frac{\tau}{\widetilde{\alpha}^i + \tau}\frac{\sum_j^\tau s_j^*}{\tau} \tag{15}$$

Further, as the number of encounters go to infinity, the buyer's personal estimate converges to the true supply S^* in distribution, no matter what initial belief the buyer holds.

We are also interested in the buyer's estimate of the market condition Z (excess demand). To derive the buyer i's best estimate of Z after he encounters supplier k[28], we note

$$\begin{aligned}
\mathbb{E}_{\mathbb{Q}_i}(Z|S_1^*, S_2^*, ..., S_\tau^*) &= \mathbb{E}_{\mathbb{Q}_i}\Big\{(\sum_{i\in I} D_i^* - \sum_{\tau=1}^{|K|} S_\tau^*)|S_1^*, S_2^*, ..., S_\tau^*\Big\} \\
&= \sum_{i\in I} D_i^* - \mathbb{E}_{\mathbb{Q}_i}(\sum_{\tau=1}^{|K|} S_k^* | S_1^*, S_2^*, ..., S_\tau^*) \\
&= \sum_{i\in I} D_i^* - (S_1^* + S_2^* + ... + S_\tau^*) - \sum_{j=k+1}^{|K|} \mathbb{E}_{\mathbb{Q}_i}(S_j^*|S_1^*, S_2^*, ..., S_\tau^*)
\end{aligned} \tag{16}$$

It can be shown[29] that $\mathbb{E}_{\mathbb{Q}_i}(S_j|S_1^*, S_2^*, ..., S_\tau^*) = \mathbb{E}_{\mathbb{Q}_i}(S_{j+h}|S_1^*, S_2^*, ..., S_\tau^*)$ for $\forall j \geq \tau+1$

[27] *Proof*: For $\forall\mathscr{F}$-measurable set C_Ω closed in (Ω, ρ), since $\mathscr{F} \subseteq \sigma_S^*$, $C_\Omega \in \sigma_S^*$. Then there exist a set $C_{\mathbb{R}_+} \in \mathscr{B}$ such that ${S^*}^{-1}(C_{\mathbb{R}_+}) = C_\Omega$. By ${S^*}^{-1}$ is continuous, such set $C_\mathbb{R}$ must be closed in $(\mathbb{R}_+, \rho)$. By $\mathbb{Q}_i^\tau Z^{-1} \Rightarrow \mathbb{P}Z^{-1}$, we know $\limsup_\tau \mathbb{Q}_i^\tau Z^{-1}(C_\mathbb{R}) \leq \mathbb{P}Z^{-1}(C_\mathbb{R})$. Thus $\limsup_\tau \mathbb{Q}_i^\tau(C_\Omega) \leq \mathbb{P}(C_\Omega)$, i.e. $\mathbb{Q}_i \Rightarrow \mathbb{P}$ weakly.

[28] Recall we assume the supplier k announces his supply before buyer i place the bid.

and $\forall h \geq 0$, thus the best personal estimator of Z given observation $s_1^*, s_2^*, ..., s_\tau^*$ is

$$
\begin{aligned}
\widetilde{Z}^i|S_1^*, S_2^*, ..., S_\tau^* &= \mathbb{E}_{\mathbb{Q}_i}(Z|S_1^*, S_2^*, ..., S_\tau^*)\\
&= \sum_{i\in I} D_i^* - (S_1^* + S_2^* + ... + S_\tau^*) - (|K| - \tau)\mathbb{E}_{\mathbb{Q}_i}(S_{k+1}^*|S_1^*, S_2^*, ..., S_\tau^*)\\
&= \sum_{i\in I} D_i^* - \sum_{j=1}^{\tau} S_j^* - (|K| - \tau)\frac{\widetilde{\alpha}^i \mathbb{E}\widetilde{\mathbb{G}}^i + \sum_{j=1}^{\tau} S_j^*}{\widetilde{\alpha}^i + \tau}\\
&= \sum_{i\in I} D_i^* - \frac{\widetilde{\alpha}^i(|K| - \tau)}{\widetilde{\alpha}^i + \tau}\mathbb{E}\widetilde{\mathbb{G}}^i - \frac{|K| + \widetilde{\alpha}^i}{\widetilde{\alpha}^i + \tau}\sum_{j=1}^{\tau} S_j^*
\end{aligned}
\tag{17}
$$

We note as $\tau \to |K|$, $(\widetilde{Z}^i|S_1^*, S_2^*, ..., S_\tau^*) \xrightarrow{d} Z$ in distribution. This says after enough many observations on suppliers, the buyer i will form a good estimate of the total market condition. These results can be summarized as the buyer's learning rule.

Learning Rule of Market Condition: Under assumptions of previous learning rule, at supplier k with index $\tau \in K_i$, the buyer i has observed $s_1^*, s_2^*, ..., s_\tau^*$ (since s_τ^* is announced right before he bid, but no earlier), his best personal estimate of excess demand is

$$
\widetilde{Z}^i|s_1^*, s_2^*, ..., s_\tau^* = \sum_{i\in I} D_i^* - \frac{\widetilde{\alpha}^i(|K| - \tau)}{\widetilde{\alpha}^i + \tau}\mathbb{E}\widetilde{\mathbb{G}}^i - \frac{|K| + \widetilde{\alpha}^i}{\widetilde{\alpha}^i + \tau}\sum_{j=1}^{\tau} s_j^* \tag{18}
$$

In further, as observations goes to $|K|$, the buyer's personal estimate of excess demand $\widetilde{Z}^i$ converges to the real demand Z in distribution, no matter what initial belief the buyer holds.

[29] *Proof*: (By induction) It suffices to show $\mathbb{E}_{\mathbb{Q}_i}(S_{j+h}|S_1^*, S_2^*, ..., S_\tau^*) = \mathbb{E}_{\mathbb{Q}_i}(S_{\tau+1}^*|S_1^*, S_2^*, ..., S_\tau^*)$ for $\forall j \geq \tau+1$ and $\forall h \geq 0$. For $j = \tau+1$, the conclusion trivially holds. Suppose for $j = \tau+h+1$, the conclusion holds. Then it is must be true that

$$
\begin{aligned}
&\int ... \int (S_{\tau+1}^* + S_{\tau+2}^* + ... + S_{\tau+h}^*) f(S_{\tau+h}^*|S_1^*, \quad S_2^*, ..., S_{\tau+h-1}^*) f(S_{\tau+h-1}^*|S_1^*, S_2^*, ..., S_{\tau+h-2}^*)\\
&\qquad\qquad ... f(S_{\tau+1}^*|S_1^*, ..., S_\tau^*) dS_{\tau+h}^* dS_{\tau+h-1}^* ... dS_{\tau+1}^*\\
=&h\mathbb{E}_{\mathbb{Q}_i}(S_{\tau+1}^*|S_1^*, S_2^*, ..., S_\tau^*)
\end{aligned}
$$

Then for $j = \tau + h + 2$, we have,

$$
\begin{aligned}
&\mathbb{E}_{\mathbb{Q}_i}(S_{\tau+h+2}^*|S_1^*, S_2^*, ..., S_\tau^*)\\
=&\int ... \int \mathbb{E}_{\mathbb{Q}_i}(S_{\tau+h+2}^*|S_1^*, S_2^*, ..., S_\tau^*, S_{\tau+1}^*, ..., S_{\tau+h+1}^*) f(S_{\tau+h+1}^*|S_1^*, S_2^*, ..., S_{\tau+h}^*)\\
&... f(S_{\tau+1}^*|S_1^*, ..., S_\tau^*) dS_{\tau+h+1}^* dS_{\tau+h}^* ... dS_{\tau+1}^*\\
=&\Big\{\frac{\widetilde{\alpha}^i + \tau}{\widetilde{\alpha}^i + \tau + h + 1} + \frac{1}{\widetilde{\alpha}^i + \tau + h + 1}\Big(\frac{\widetilde{\alpha}^i + \tau}{\widetilde{\alpha}^i + \tau + h} + \frac{(\widetilde{\alpha}^i + \tau + h + 1)h}{\widetilde{\alpha}^i + \tau + h}\Big)\Big\}\mathbb{E}_{\mathbb{Q}_i}(S_{\tau+1}^*|S_1^*, S_2^*, ..., S_\tau^*)\\
=&\mathbb{E}_{\mathbb{Q}_i}(S_{\tau+1}^*|S_1^*, S_2^*, ..., S_\tau^*)
\end{aligned}
$$

Thus the conclusion holds for $j = \tau + h + 2$.

4.1.3 Buyers' Offer Rule

To determine buyer's optimal bid price, we first note that in this bidding framework, the constraints $q_{ik} \leq S_k^* \ \forall k \in K$ are automatically satisfied. Thus we drop the constraints in problem (2). Then at each encounter k, conditioning on his information Φ_i^k, the buyer's optimization problem becomes to:

$$\min_{(q_{ik}, b_{ik})} \mathbb{E}_{\mathbb{Q}_i} \Big\{ q_{ik}^a b_{ik} + SC_{ik} + c_i (D_i^{*k} - q_{ik}^a) \Big\} \tag{19}$$

To minimize the procurement cost based on his information, the buyer's personal optimal bid price must satisfy[30]:

$$\begin{aligned} \frac{\partial \mathscr{L}}{\partial b_{ik}} =& \mathbb{E}_{\mathbb{Q}_i} \Big\{ \frac{\partial q_{ik}^a}{\partial b_{ik}} b_{ik} + q_{ik}^a + \frac{\partial SC_{ik}}{\partial q_{ik}^a} \frac{\partial q_{ik}^a}{\partial b_{ik}} - c_i \frac{\partial q_{ik}^a}{\partial b_{ik}} \Big\} \\ =& \mathbb{E}_{\mathbb{Q}_i} \Big\{ \frac{\partial q_{ik}^a}{\partial b_{ik}} (b_{ik} - c_i + \frac{\partial SC_{ik}}{\partial q_{ik}^a}) + q_{ik}^a \Big\} = 0 \end{aligned} \tag{20}$$

To determine the bid quantity q_{ik}, we first note that bid quantity does not affect supplier's acceptance which only depends on bid price. On the other hand, if there is shortage in supply, the buyer i might bid up to S_k^*. Since the supplier can determine q_{ik}^a on his own, the buyer's bid quantity generally does not affect the acceptance probability[31]. These results can be summarized into the following rule:

***Offer Rule*:** To minimize procurement cost, an imperfect informed buyer i will offer a bid (q_{ik}, b_{ik}) to some supplier k. The buyer's personal optimal bid price satisfies the following equation:

$$\mathbb{E}_{\mathbb{Q}_i} \Big\{ \frac{\partial q_{ik}^a}{\partial b_{ik}} (b_{ik} - c_i + \frac{\partial SC_{ik}}{\partial q_{ik}^a}) + q_{ik}^a \Big\} = 0 \tag{21}$$

4.1.4 Relation with Auction Model

In this section, we will show if a buyer i bids at supplier k, then under some conditions his optimization problem (16) resembles an first-price sealed-price auction model. Suppose there are finite suppliers ($|K| < \infty$), and each supplier's accepted bid quantity follows $q_{ik}^a = B_{ik} q_{ik}$. In addition, suppose there is shortage in supply and each buyer bid quantity is s_k^*[32] (now $q_{ik}^a = B_{ik} q_{ik}$). Then at each supplier k, the buyer i's benefit from winning the auction is

$$c_i q_{ik}^a + \mathbb{E}_{\mathbb{Q}_i} SC(q_{ik}^a) \tag{22}$$

where c_i is the marginal penalty cost and $SC(q_{ik}^a) \equiv f(q_{ik}^a, \widetilde{Z}^i, s_k^*)$ with f defined as in equation (4). Equation (22) says, if he wins the auction, he will benefit from a

[30] Assume search cost and accepted bid quantity are independent and all variates are continuously differentiable as stated in section 3.1, then apply Leibniz integral rule.

[31] On the other hand, in some special cases, q_{ik} does have an effect on the acceptance probability. For example, if $q_{ik}^a = B_{ik} q_{ik}$ and $q_{ik} = S_k^*$, then in order for the bid to be accepted, buyer i's bid price must be higher than all other buyers' bid prices.

[32] Note in this here the auction is essentially a single-unit auction.

reduced penalty cost of q^a_{ik}. In addition, he will not need to search for q_{ik} at a cost of $\mathbb{E}_{\mathbb{Q}_i} SC(q^a_{ik})$ in this case. The buyer i's per unit cost of wining the auction is b_{ik}. Then his optimization problem can be written as:

$$\max_{b_{ik}} \mathbb{E}_{\mathbb{Q}_i} \Big\{ (c_i q_{ik} + SC(q^a_{ik}) - b_{ik} q_{ik}) \mathbb{1}(B_{ik} = 1) \Big\} \tag{23}$$

If we further assume search cost and acceptance rule are independent as in (16), then problem (21) becomes to

$$\max_{b_{ik}} \Big(c_i q_{ik} + \mathbb{E}_{\mathbb{Q}_i} SC(q^a_{ik}) - b_{ik} q_{ik} \Big) \mathbb{Q}_i (B_{ik} = 1) \tag{24}$$

In the first or second sealed price auction model, $\mathbb{Q}_i(B_{ik} = 1) = \mathbb{P}(B_{ik} = 1) = F^{n-1}(b^{-1}(b_{ik}))$ depending the distribution of the largest order statistics, where b is the optimal bidding function. In our model, who wins auction is not determined by the bids' relative ranking, but by the supplier's acceptance rule (as described in next section). In the supplier's acceptance case 1 (SAC1), the supplier's acceptance rule reflects bids' ranking as well as their arriving time which translates to supplier's waiting cost.

4.1.5 Deviation from the Optimal Bid Price

Buyer's Optimal Bid Case 1

If a buyer is perfectly informed of the actual distributions of waiting cost WC and suppliers' supply capacity S^*, then his personal optimal bid price is the objective optimal bid price, which satisfies

$$\mathbb{E}_{\mathbb{Q}_i} \Big\{ \frac{\partial q^a_{ik}}{\partial b_{ik}} (b_{ik} - c_i + \frac{\partial SC_{ik}}{\partial q^a_{ik}}) + q^a_{ik} \Big\} = 0 \tag{25}$$

Later in this article it is shown if $SC = \frac{Z^2(D^{*k}_i - q^a_{ik})}{\beta s^*_k}$ increases nonlinearly in excess demand, and suppliers take logit acceptance rule ($\mathbb{P}(B_{ik} = 1) = \frac{e^{b_i *}}{\sum_{j \in I} e^{b_j}}$), then the (objective) optimal bid price is

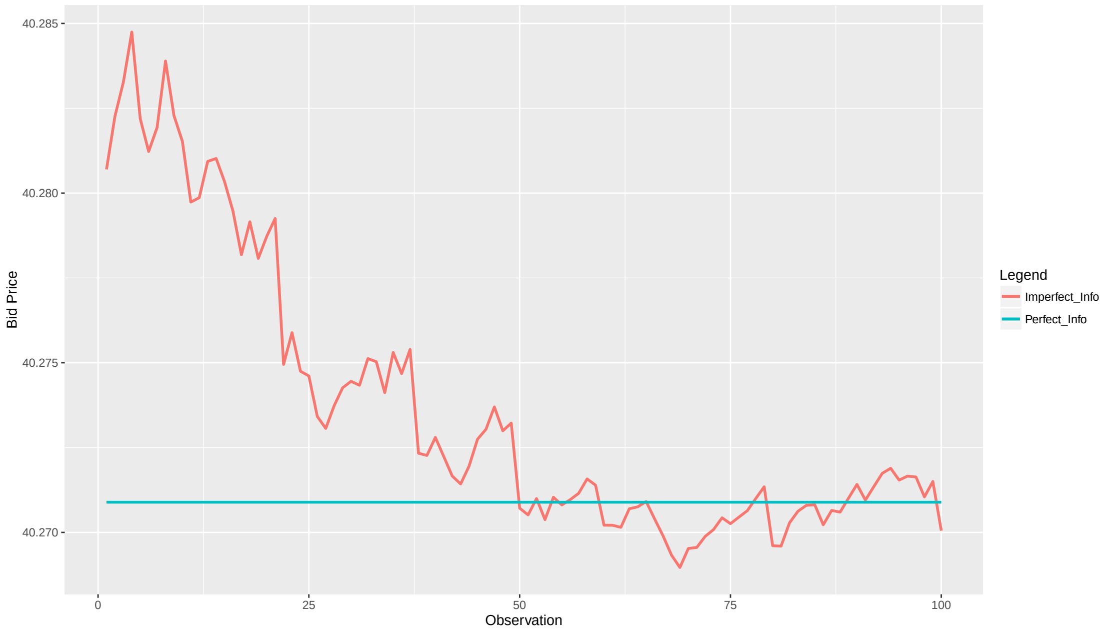
$$b^*_{ik} = c_i + \frac{\mathbb{E}_{\mathbb{P}} Z^2}{\beta s^*_k} - 1 - \mathscr{W}(\frac{e^{c_i + \mathbb{E}_{\mathbb{P}} \frac{Z^2}{\beta S^*_k} - 1}}{K}) \tag{26}$$

where $K = \sum_{j \neq i} e^{\widetilde{b_j}}$ and $\mathscr{W}$ is Lambert W function: for $f(z) = ze^z$, we have $z = f^{-1}(ze^z) = \mathscr{W}(ze^z)$.
Similarly, a buyer i with imperfect information has his personal bid price:

$$\widetilde{b^*_{ik}}^i = c_i + \frac{\mathbb{E}_{\mathbb{Q}_i} Z^2}{\beta s^*_k} - 1 - \mathscr{W}(\frac{e^{c_i + \frac{\mathbb{E}_{\mathbb{Q}_i} Z^2}{\beta s^*_k} - 1}}{\widetilde{K}^i}) \tag{27}$$

where $\widetilde{K}^i = \sum_{j \neq i} e^{b_j}$ and $\mathscr{W}$ is defined as above. We know from above that when the buyer i has enough many observations, his estimate of market condition converges to

Figure 2: Simulated Personal Optimal Bid Price under Imperfect and Perfect Information about Market Condition



Note: Figure 2 shows how an imperfectly informed buyer's personal (subjective) optimal bid price converges to the true (objective) optimal one. The x-axis is the number of suppliers the buyer has encountered and observed. The buyer's subjective optimal bid price is plotted in red. The true optimal bid price is plotted in blue.

the real one. Thus, as $k \to |K|$, $\mathbb{E}_{\mathbb{Q}_i} Z^2 = \mathbb{E}_{\mathbb{P}} \widetilde{Z^2}^i \to \mathbb{E}_{\mathbb{P}} Z^2$ (pointwise). If a buyer has perfect information about other bidders' price, then the deviation between personal optimal bid price and the true optimal one is

$$\triangle_{b_{ik}} = \widetilde{b_{ik}^*}^i - b_{ik}^* = \frac{\mathbb{E}_{\mathbb{Q}_i} Z^2}{\beta s_k^*} - \frac{\mathbb{E}_{\mathbb{P}} Z^2}{\beta s_k^*} - \mathscr{W}(\frac{e^{c_i + \frac{\mathbb{E}_{\mathbb{P}} Z^2}{\beta s_k^*} - 1}}{K}) - \mathscr{W}(\frac{e^{c_i + \frac{\mathbb{E}_{\mathbb{Q}_i} Z^2}{\beta s_k^*} - 1}}{K}) \tag{28}$$

A simulation result is plotted in Figure 2. It shows, if buyer has perfect information about others' bid prices, his personal optimal bid price will converge to the real optimal one as the number of observations goes up (since his estimate of market condition becomes more accurate). This is consistent with analytical result that $\triangle_{b_{ik}} \to 0$ pointwise as number of observations goes to $|K|$.

Buyer's Optimal Bid Case 2

In another case, if the buyer does not know others' bids, and he estimates others' bids using his own bid price $\widetilde{b_j}^i = b_i$, then his best personal bid price is

$$\begin{aligned}\widetilde{b_{ik}^*}^i =& \frac{(N-1)(c_i + \frac{\mathbb{E}_{\mathbb{Q}_i} Z^2}{\beta s_k^*} - 1) - 1}{N-1} \\ =& c_i + \frac{\mathbb{E}_{\mathbb{Q}_i} Z^2}{\beta s_k^*} + \frac{N-2}{N-1}\end{aligned} \tag{29}$$

And its deviation from the optimal bid pirce is:

$$\triangle_{b_{ik}} = \frac{\mathbb{E}_{\mathbb{Q}_i} Z^2}{\beta s_k^*} - \frac{\mathbb{E}_{\mathbb{P}} Z^2}{\beta s_k^*} - \mathscr{W}\left(\frac{e^{c_i + \frac{\mathbb{E}_{\mathbb{P}} Z^2}{\beta s_k^*} - 1}}{K}\right) + \frac{1}{N} \tag{30}$$

Note, in this case, as observation goes up, the deviations can be not eliminated ($\triangle_{b_{ik}} \to -\mathscr{W}(\frac{e^{c_i + \frac{\mathbb{E}_{\mathbb{P}} Z^2}{\beta s_k^*} - 1}}{K}) + \frac{1}{N} \neq 0$). The deviation in the limit case reflects buyer i's biased estimate of other buyers' bids.

4.2 Supplier Side

Similar to buyer's optimization problem, to maximize his profit, at encounter (i, k) each supplier k chooses to accept whether to accept or reject a bid. Thus his (optimal) decision is a sequence of 0 and 1's (indicators B_{ik}) indexed by the buyer's arrival. Thus his optimal *choice* problem can be written in a 0-1 integer programming form[33]. And the optimal choice problem can be written as [34]:

$$\begin{aligned} \underset{\{B_{ik}\}_{i\in I}}{\text{maximize}} \quad & \mathbb{E}_{\mathbb{Q}_k}\Big\{\sum_{i\in I} B_{ik} q_{ik}(b_{ik} - WC_{ik})\Big\} \\ \text{subject to} \quad & \sum_{i\in I} B_{ik} q_{ik} \leq S_k^* \end{aligned} \tag{31}$$

Problem (31) reads each supplier k maximizes his profit by choosing the best bids offered until his capacity is sold out. In a bid without recall setting, not accepting the bid (q_{i^*k}, b_{i^*k}) means the supplier may miss a good offer. Thus the supplier must compare the profit $q_{i^*k} b_{i^*k}$ made from accepting the bid (q_{i^*k}, b_{i^*k}) and the expected maximal profit difference. In a bid with recall framework, not accepting the bid immediately only means he will compare the profit with the waiting cost WC_{ik}. Here we use a bid without recall setting.

If each supplier k has perfect information about what bid prices will be and their time of arrival, he can have an explicit value ranking of all bids based on the ranking $b_{ik} - WC_{ik}$. Then to maximize profit as in equation (31), he will accept the bids with highest values $(b_{ik} - WC_{ik})$ until his supply is sold out. This is a deterministic acceptance rule.

On the other hand, the supplier k does not have perfect information on either the values or their time of arrival or both. Then his acceptance based on the uncertain waiting time. And hence his acceptance rule is stochastic. For example, if the supplier k knows about the values of the bid but does not know their arrival, each supplier k only accepts highest bids until his supply capacity is reached. Let $\mathbb{C}_k$ denote the supplier k's optimal choice set of bids. That is, $\mathbb{C}_k = \{i \in I | B_{ik}^* = 1\}$,

[33] On the other, we will use discrete choice methods to approach this problem. Thus equation (31) is more of a general conceptual formulation.

[34] It is worth noting that in equation (31), the decision variables $\{B_{ik}\}_{i\in I}$ are deterministic; however, the optimal solutions $\{B_{ik}\}_{i\in I}^*$ depend on the uncertainty ω, which means the solutions to (29), $\{B_{ik}\}_{i\ inI}^*$, are random variables

where B_{ik}^* is the optimal solution to his profit maximization problem (31). Then the supplier k will choose to accept the bid (q_{ik}, b_{ik}) if and only if

$$b_{ik} - WC_{ik} > b_{jk} - WC_{jk}, \text{ for } \forall j \notin \mathbb{C}_k \tag{32}$$

Note since waiting cost is a random variate, thus the supplier's acceptance rule as in (32) is a stochastic rule.

As a special case of equation (32), if there is some extreme shortage in supply (that is, $D_i^* > S_k^*$, for all i, k.), the supplier will only accept the highest bid with consideration of waiting cost (the accepted set $\mathbb{C}_k$ is a singleton). Then the supplier will choose to accept the bid (q_{ik}, b_{ik}) if

$$b_{ik} - WC_{ik} > b_{jk} - WC_{jk}, \text{ for } \forall j \neq i \tag{33}$$

Note here the acceptance rule is also stochastic since waiting cost is uncertain.

4.2.1 Waiting Cost and Acceptance Rule

The waiting cost WC_{ik} is the k-th supplier's future and uncertain waiting cost before the buyer i offers bid. As stated above, waiting cost is in its nature stochastic since there is uncertainty in the time when buyers bid and other factors. Meanwhile, it also depends on the market conditions Z.

The supplier is viewed as static, like a shop keeper who waits for the random arrival of buyers. As a buyer arrives, a bid is received. The supplier must decide whether to accept or reject the bid. If accepted, the volume of the sale must be determined by the supplier. In this specification we assume the buyer offer pair (b_{ik}, q_{ik}) must be accepted or rejected. Thus an acceptance rule must be specified. Here we focus on specifications of waiting cost. That is, if the supplier rejects the bid, the supplier must wait for the next buyer's bid to be received. Waiting cost involves both an expectation of feasible bids that might arrive as well as time cost of waiting (WC). Here, we focus on the former as a metric of the outside option of the supplier when faced with a bid from buyer i. We view the information available to the supplier in forming the WC as including perceived or true market conditions (Z), and the sequence of past bids received, $\{b_{\tau' k}\}_{\tau' \in I_{\leq \tau}}$. If only Z is considered, we specify Z as either determining WC as a simple linear projection , or conditioning a distribution from which a draw is taken for WC. A third alternative is to consider past bids. Suppliers use Bayesian learning (or, in a simple case, *adaptive expectation* as in [Muth, 1961]) to estimate future bid prices, based on which they form their acceptance rule. A final specification of interest would be to consider an observable, small set of suppliers. In this case, suppliers base their acceptance rule on their rational expectation of future bids that would clear the market.

Supplier's Waiting Cost and Acceptance Rule Case 1

A possible way to model WC_{ik} is to decompose WC_{ik} as sum of a deterministic component and a stochastic component:

$$WC_{ik} = \delta_k + e_{ik} \tag{34}$$

where δ_k is the deterministic component representing the systematic part of waiting cost, and e_{ik} is the stochastic component representing the uncertainty. Since the waiting cost is stochastic and depends on market conditions Z, we assume WC_{ik} follows a distribution f_Z depending on Z the market conditions. A simple example is when WC_{ik} is distributed as a scale location family with only its expectation depending on Z, and e_{ik} is zero mean disturbance term, then only the deterministic (systematic) part δ_k depends on Z ($\delta_k = \delta_k(Z)$. A more general case is though to assume e_{ik} follows a distribution f_Z.

In the case when supplier knows bid price values but not their arrival time, based on the decomposition on WC_{ik} in equation (34), the supplier problem (32) is equivalent to:

$$e_{ik} - e_{jk} < b_{ik} - b_{jk} + \delta_k - \delta_k = b_{ik} - b_{jk}, \text{ for } \forall j \notin \mathbb{C}_k \tag{35}$$

where $\mathbb{C}_k$ is the supplier's optimal choice set. Note in this case, the acceptance rule is dependent of the market condition Z.

Recall B_{ik} is the indicator of the supplier accepting the bid (q_{ik}, b_{ik}), then the probability of supplier accepting the bid (q_{ik}, b_{ik}) is:

$$\mathbb{P}(\text{supplier k accepts bid } i|Z) = \mathbb{P}(B_{ik} = 1|Z) \tag{36}$$

Equation (36) says the probability of acceptance depends on market conditions Z. An example is if the excess demand increases, then there exist a higher probability that someone bids high price. And if such increasing excess demand is perceived by supplier k, then he will expect such possibility of getting a high price offer and hence becomes reluctant to sell at the old price.

In the case of severe supply shortage ($D_i^* > S_k^*$, we have some nice analytical form of acceptance probability. Under the first specification of waiting cost $WC_{ik} = \delta_k + e_{ik}$, the acceptance probability $\mathbb{P}(B_{i^*k} = 1|Z)$ can be written as:

$$\mathbb{P}(B_{ik} = 1|Z) = \int \mathbb{1}(e_{ik} - e_{jk} < b_{ik} - b_{jk}, \forall j \neq i^*) f_Z(\mathbf{e}_k) d\mathbf{e}_k \tag{37}$$

where $\mathbf{e}_k = (e_{1k}, e_{2k}, ..., e_{Ik})$ is an I by 1 vector. In the case of $\mathbf{e}_k \sim MN(\vec{0}, \Sigma)$, the analytical form resembles a multinomial probit problem. Note in this case, $\delta_k = \delta_k(Z) = \mathbb{E}(WC_{ik}|Z)$ is a function of Z.

***Acceptance Rule 1*:** Each supplier k chooses whether to accept a bid or not based on his interpretation of market conditions Z. The supplier will only accept the highest bid(s) i with consideration of cost. If the supplier k is informed about what the bids will be but does not know their arrival time, then under specification of equation (34), he will accept bid i if and only if

$$e_{ik} - e_{jk} < b_{ik} - b_{jk}, \text{ for } \forall j \notin \mathbb{C}_k \tag{38}$$

where $\mathbb{C}_k$ is the supplier k's optimal choice set defined as before. We label this acceptance rule as SAC1.

Supplier's Waiting Cost and Acceptance Rule Case 2

In case 1, the waiting time is calculated since time 0 [35]. Thus at time when buyer i came and bid at supplier k, the waiting time of next (potential higher) bid is the difference between two waiting time (note the buyer i's waiting time is already known, or realized random variable). And hence waiting cost is defined as the difference of the two waiting cost. In this case, we assume past waiting time has no influence on supplier's acceptance of buyer i's bid. Thus to determine whether to accept bid i^* to not, the supplier k only considers the waiting cost from the time when the bid i^* is received. In this case, the waiting time is calculated since the time when the bid i^* is received [36]. In particular, we assume that the waiting time (from the current bid) follows an exponential distribution [37]with parameter as a function of market condition Z, i.e.

$$WC_{ik} \sim exp(f(Z)) \tag{39}$$

[35] For each supplier k, we define time 0 can be seen as the time when the first buyer came to the supplier k. Thus time 0 is k specific.

[36] Or, we can think in this case the time 0 is the time when the buyer i submits his bid to the supplier k. Thus in this case, time 0 is i^* and k specific.

[37] This is different from assuming buyers' arrivals follow Poisson process. It is worth noting that since in this model buyers come to the supplier randomly, a Poisson process modelling seems natural. However, such formulation has two drawbacks. First, in Poisson process, the order of each arrival is exogenously determined (since Poisson process assumes a natural order). In contrast, in this model, the set of buyers is an unordered set and the order of arrival is *endogenously* determined by their "waiting time" (more accurately, the Poisson process modelling of customer arrivals assume every customer is homogeneous, thus any ordering does not matter). Secondly, the Poisson process modelling does not admit genernal analytical solution. To see this, we assume the set of buyers can be indexed by their arrival to supplier k and the interval time between two arrivals (the interval time between buyer 1 and buyer 2 is denoted as T_1) is distributed as exponential random variable with parameter Z (note as Z increases, the interval time has a higher probability to be a small number). That is, the interval time $T_1, T_2, ...T_N \sim i.i.d.\ exp(Z)$. Since Poisson process is memory-less, we denote T_i as the waiting time of next buyer from buyer i. Suppose the waiting cost is proportional to the waiting time ($WC_i = c\sum_{i'<i} T_{i'}$), then the acceptance rule becomes to: the supplier only accepts the i^* bid if

$$b_{i^*} > b_j - c\sum_{k=i^*+1}^{j} T_k, \text{for } \forall j \geq i^*$$

Then the probability of supplier accepting the bid i^* $\mathbb{P}(\text{supplier k accepts bid } i^*|Z)$ is

$$\mathbb{P}(b_{i^*} > b_{i^*+1} - cT_{i^*}, b_i^* > b_{i^*+2} - c(T_{i^*} + T_{i^*+1}), \ldots, b_{i^*} > b_N - c\sum_{k=i^*+1}^{N} T_k|Z)$$

For $i^* = N-1$, the acceptance probability in the equation above can be written as $\mathbb{P}(T_{N-1} > \frac{1}{c}(b_N - b_{N-1})) = e^{-\frac{Z(b_N - b_{N-1})}{c}}$ for $b_N > b_{N-1}$.
For $i^* = N-2$, such probability becomes to $\mathbb{P}(T_{N-1} > \frac{1}{c}(b_N - b_{N-1}), T_{N-1} + T_{N-2} > \frac{1}{c}(b_N - b_{N-2})) = \int_{T_{N-1}} \mathbb{1}(T_{N-1} > \frac{1}{c}(b_N - b_{N-1}) \int_{T_{N-2}} \mathbb{1}(T_{N-2} > \frac{1}{c}(b_N - b_{N-2}) - T_{N-1}) dT_{T_{N-2}} dT_{T_{N-1}} = e^{-\frac{Z(b_N - b_{N-2})}{c}} \left(\frac{Z(b_N - b_{N-1})}{c} + 1\right)$ for $b_N > b_{N-1} > b_{N-2}$ (otherwise it is a degenerate case similar to $i^* = N-1$).
For $i^* = N-3$, the acceptance probability becomes to $\mathbb{P}(T_{N-1} > \frac{1}{c}(b_N - b_{N-1}), T_{N-1} + T_{N-2} >$

Under the second specification of waiting cost where $WC_{ik} \sim exp(f(Z))$, the supplier k will accept the bid τ if $b_{\tau k} > \widetilde{b_{\tau' k}}^k - \widetilde{WC_{\tau' k}}^k$ for future bid $\tau' > \tau$, where τ is the index for current bid in the ordered set I_k, and τ' is in the index for some future bid j in the ordered set I_k. $\widetilde{b_{\tau' k}}^k$ is his estimate of some future bid $b_{\tau' k}$ and $\widetilde{WC_{\tau' k}}^k$ is corresponding estimated waiting cost. In this case, the acceptance probability can be written as:

$$\begin{aligned} \mathbb{P}(B_{\tau k} = 1|Z) =& \mathbb{P}(\widetilde{WC_{\tau' k}}^k > \widetilde{b_{\tau' k}}^k - b_{\tau k} \text{ for } \forall \tau' > \tau | Z) \\ =& \prod_{\tau' > \tau}^{N} e^{-z(\widetilde{b_{\tau' k}}^k - b_{\tau k})} = e^{-f(Z) \sum_{\tau' > \tau}^{N} (\widetilde{b_{\tau' k}}^k - b_{\tau k})} \end{aligned} \tag{40}$$

These results can be summarised into acceptance rule 2.

***Acceptance Rule 2*:** Let waiting cost be specified as in case 2 (equation (39)). Under the same condition as in acceptance rule 1, the supplier will accept bid i with index $\tau \in I_k$ if the following conditions is satisfied:

$$b_{\tau k} > \widetilde{b_{\tau' k}}^k - \widetilde{WC_{\tau' k}}^k \text{ for } \forall \tau' > \tau \tag{41}$$

where τ, τ', $\widetilde{b_{\tau' k}}^k$ and $\widetilde{WC_{\tau' k}}^k$ are defined as above. We label this acceptance rule as SAC2.

Supplier's Waiting Cost and Acceptance Rule Case 3

In the case when the suppliers are not informed about the bid values, the supplier needs to estimate future bid price (and their arrival time). When past bids are considered, we use Bayesian learning or adaptive rule based learning on bid prices.

In the setting of the supplier k Bayesian learning on bid price, we assume the supplier has some prior information about bid price. Suppose the his prior mean of bid price is $\widetilde{b_k}$. Then after observing a sequence of bids $(b_{1k}, b_{2k}, \ldots, b_{\tau k})$ in the ordered set I_k, his best estimate of bid price is $\frac{\alpha}{\alpha+\tau}\widetilde{b_k} + \frac{\tau}{\alpha+\tau}\bar{b}$, where α is the concentration factor as defined in section 3, τ is the number of bid before and including the bid $b_{\tau k}$ and $\bar{b}$ is the average bid receive before (and including) the bid $b_{\tau k}$. Thus at the time of buyer τ offers his bid to supplier k, supplier k's best estimate of future price is

$$\widetilde{b_{\tau' k}}^k = \frac{\alpha}{\alpha + \tau}\widetilde{b_k} + \frac{\tau}{\alpha + \tau}\bar{b} \text{ for } \forall \tau' > \tau \tag{42}$$

where $\widetilde{b_k}$ is the prior mean of bid price and $\bar{b}$ is the average bid receive before (and including) the bid $b_{\tau k}$.

$\frac{1}{c}(b_N - b_{N-2}), T_{N-1} + T_{N-2} + T_{N-3} > \frac{1}{c}(b_N - b_{N-3})) = e^{-\frac{Z(b_N - b_{N-3})}{c}} \left(\frac{Z^2}{2c^2} \left((b_N - b_{N-2})^2 + (b_{N-1} - b_{N-2})^2 \right) + Z(b_B - b_{N-2}) + 1 \right)$ for $b_N > b_{N-1} > b_{N-2} > b_{N-3}$ (otherwise it is a degenerate case similar to previous cases).

Hence we see there is no clear formula for a general i^*. This reason together with the endogenous ordering problem make the Poisson process formulation an inappropriate one in the context of this paper.

As a special case of Bayesian learning on bid price, the supplier k has no prior information on bid priceand only uses the last bid to estimate future bid price. That is,

$$\widetilde{b_{\tau' k}}^{k} = b_{\tau k} \text{ for } \forall \tau' > \tau \tag{43}$$

This is the adaptive rule based learning on bid price, where the meaning of "adaptive" is consistent with that in adaptive expectation ([Muth, 1961])[38].

Supplier's Waiting Cost and Acceptance Rule Case 4

A fourth alternative is to consider a rational expectation of future bids that would clear the market. In this case we assume an observable and small set of suppliers. Each supplier k has rich information about market condition and other suppliers. To make decisions, the supplier k considers his competitors reaction to market conditions. In particular, we assume supplier k's information set at the time of receiving bid i^* $\Psi_k^{i^*}$ includes:

- market condition Z;
- other suppliers' supply $\{S_j^*\}_{j \neq k}$;
- other suppliers' acceptance rules are the same as his own one;
- buyers know about suppliers' capacity and acceptance rule.

With such information, the supplier k can calculate the equilibrium price b^e that clears the market ($\sum_{k \in K} S_K^* = \sum_{i \in I} D_i(b^e)$ [39]). Then the suppliers can use this equilibrium price to estimate the future bid price, based on which he constructs his acceptance rule.

There is a good property of the market equilibrium price b^e in this settings. b^e is the supplier k's rational expectation of the future bid price. Given other suppliers use this b^e to estimate future bid price for the acceptance rule (supplier j accepts the bid i if $b_{ij} > b^e - WC_j$, for $\forall j \neq k$), the supplier k "should" use the same acceptance rule:

$$\text{Accept bid } i \text{ if } b_{ik} > b^e - WC_k \tag{44}$$

where "should" means that it is the most profitable strategy of supplier k that guarantees he can sell all his products. To see this, we note if supplier k choose a higher threshold $b^t > b^e$, then there is a positive probability that he can not sell all

[38]In the adaptive expectation, $\widetilde{b_{jk}}^{k}$ is written as $\mathbb{E}_{k,i^*}(b_j)$ which equals to b_{i^*}. If the supplier k's information set at the time of receiving bid i^* I_{i^*k} is the singleton $\{b_{i-1,k}\}$, the such estimate can be seen as supplier k's rational expectation given the information set $I_{i^*k} = \{b_{i-1,k}\}$. Based on this estimated bid price, supplier can form their acceptance rule as described in SAC1 or SAC2. We label this case as SAC3.

[39]Note here D_i as a function of price, which will be defined later in the text, is different from the initial demand the procurement requirement D_i^* to fulfill at the beginning of each trading session. Also note that here S_k^* does not change over price.

his products [40]. On the other hand, supplier k does not have incentive to choose a threshold price b^e that is lower than b^e since at b^e, all his products can been sold (at b^e, he does not need to lower price to compete for demand). As a result, using e^b to estimate future bid price in the acceptance rule is a reasonable choice for suppliers.

On the other hand, we note in the absence of waiting cost, such setting is essentially a Bertrand-Edgeworth oligopoly game. In such game, there is no pure strategy Nash equilibrium [Allen and Hellwig, 1986]. In particular, given all other suppliers use e^b in their acceptance rule, supplier k may want to choose a threshold price that is higher than b^e [41]; in this case, every supplier uses b^e as thresholds in acceptance rule is not a Nash Equilibrium.

Since we assume buyers know about suppliers' acceptance rule and they will not bid higher than the per unit penalty adjusted by per unit search cost (otherwise, they will not search for the product). In this case, buyers' demand D_i can be written as

$$D_i = \begin{cases} 0 \text{ if } b^e > c_i \\ D_i^* \text{ if } b^e \leq c_i \end{cases} \tag{45}$$

Then $Z(b^e) = \sum_i D_i - \sum_k S_k^* = \sum_i D_i^* \mathbb{1}(b^e \leq c_i) - \sum_k S_k^*$. Since b^e is the equilibrium price, we must have

$$Z(b^e) = 0 \tag{46}$$

The equation (46) is easily solvable numerically. Then with the solved b^e, the seller k's acceptance probability is

$$\begin{aligned} \mathbb{P}(B_{ik} = 1|Z) =& \mathbb{P}(b_{ik} > b^e - WC_{ik}|Z) \\ =& e^{-f(Z)(b_{ik} - b^e)} \end{aligned} \tag{47}$$

The content in this section can be summarized in the following acceptance rule.

***Acceptance Rule 3*:** In order to make decision, each supplier k needs to estimate future bid prices. In the settings where there is only a small group of relatively well informed suppliers, the suppliers can use market equilibrium price b^e to estimate future bid prices. Then the supplier will accepts bid i if

$$b_{ik} > b^e - WC_k \tag{48}$$

where b^e and WC_k are defined as above. We label such acceptance rule as SAC4.

4.3 Transaction Price

In this model, each bidder i comes to the supplier k and submits a bid (q_{ik}, b_{ik}) to the supplier. The supplier k then determines whether to accept this bid based on

[40] In the absence of search cost, the buyers can compare suppliers with no cost, then the supplier k with higher threshold price can not sell all his products almost surely (with probability 1).

[41] He will not choose a threshold that is lower that e^b since his capacity is reached.

his interpretation of market conditions. We label the accepted bid prices as transaction prices. They reflect the supplier's estimate of market conditions through the probability of acceptance, which depends on the market conditions. For example, if there exist a large excess demand that is perceivable to both sides in the market, then buyers would like to bid high prices to secure (at least partial) products to fulfill his fixed procurement demand. And suppliers who perceive the large excess demand expect such situation (buyers bid high), and become reluctant to accept the lower bids. They would like wait more for higher prices (and there will actually be high prices since buyers who learn about supply shortage through their searching process will bid high to secure their products). As a result, the transaction prices, as those in the accepted bids, will increase. Thus the market conditions plays an essential role in determining transaction price. This motivates a closer look at the market conditions.

4.4 Market conditions

Case 1: $\sum_k S_k^* \gg \sum_i D_i^*$ $(Z \ll 0)$

In this case, supply is sufficient (much larger than demand). With such large excess supply that is perceivable to both buyers and suppliers, buyers who know this will bid low prices and suppliers may have to accept lower price to clear supply.

Case 2: $\sum_k S_k^* \sim \sum_i D_i^*$ $(Z \sim 0)$

In this case, demand and supply are about the same. The market has no noticeable excess demand or supply to both sides in the market. Since there is no large excess supply, suppliers are confident about selling their product out and the this is no need to sell at a lower price. And no noticeable excess demand means most buyers will not be panic about (not) being able to fulfill their procurement requirement.

Case 3: $\sum_k S_k^* \ll \sum_i D_i^*$ $(Z \gg 0)$

In this case, there is a large noticeable shortage in supply. Buyers who learn about this would like to pay more to guarantee supply. Thus limited supply force buyer to bid high to guarantee fulfillment. And suppliers are confident about selling their supply at high prices and would like to wait until there is one. As a result, the transaction price will increase. This is an essential driving source of price jumps.

5 Analytical Solutions

In this section, I present two exemplary cases with analytical solutions.

5.1 A First Example (SAC1)

In this first example, I first investigate the analytical solution corresponding to supplier's acceptance rule 1 (SAC1).

5.1.1 Supplier Side

In particular, to derive an analytical solution, I assume e_{ik} is the negative Gumbel random variable, i.e.

$$f(e_{ik}) = e^{e_{ik} - e^{e_{ik}}} = e^{e_{ik}} e^{-e^{e_{ik}}} \tag{49}$$

and the cumulative distribution is

$$F(e_{ik}) = 1 - e^{-e^{e_{ik}}} \tag{50}$$

With such assumption, the probability of the supplier accepting a bid (q_{ik}, b_{ik}) (conditioning on Z) in equation (14) can be written as:

$$\begin{aligned} & \mathbb{P}(\text{supplier } k \text{ accepts bids } i^*|Z) = \mathbb{P}(B_{i^*k} = 1|Z) \\ = & \int \mathbb{1}(e_{i^*k} - e_{ik} < b_{i^*k} - b_{ik} \ \forall j \neq i^*) f_Z(e_k) de_k \\ = & \frac{e^{b_{i^*k}}}{\sum_{j \in I} e^{b_j k}} \end{aligned} \tag{51}$$

Note here the market dependent variate δ_k is cancelled. Thus the acceptance rule depends all buyers' bid prices, but not on market condition.

5.1.2 Buyer Side

Now let us consider the buyer side. Recall the buyer's optimal bidding price satisfies:

$$\frac{\partial \mathbb{E}_{\mathbb{Q}_i} q^a_{ik}}{\partial b_{ik}} (b_{ik} - c_i + \frac{\partial \mathbb{E}_{\mathbb{Q}_i} SC_{ik}}{\partial \mathbb{E}_{\mathbb{Q}_i} q^a_{ik}}) + q^a_{ik} = 0 \tag{52}$$

In the case when $SC = \frac{Z^2(D^{*k}_i - q^a_{ik})}{\beta s^*_k}$ and $q^a_{ik} = B_{ik} q_{ik}$, the equation above can be written as:

$$\begin{aligned} (lhs) &= \frac{\partial \mathbb{Q}_i(B_{ik} = 1) s^*_k}{\partial b_{ik}} (b_{ik} - c_i - \frac{\mathbb{E}_{\mathbb{Q}_i} Z^2}{\beta s^*_k}) + \mathbb{Q}_i(B_{ik} = 1) q_{ik} \\ &= q_{ik} \Big\{ \frac{\partial \mathbb{Q}_i(B_{ik} = 1)}{\partial b_{ik}} (b_{ik} - c_i - \frac{\mathbb{E}_{\mathbb{Q}_i} Z^2}{\beta s^*_k}) + \mathbb{Q}_i(B_{ik} = 1) \Big\} = 0 \end{aligned} \tag{53}$$

where *(lhs)* is the left hand side of equation (41). As specified above, the supplier's probability of acceptance is

$$\mathbb{P}(B_{ik} = 1) = \frac{e^{b_{i^*}}}{\sum_{j \in I} e^{b_j}} \tag{54}$$

Since a buyer does not know other buyers' bid prices, he can only estimate it. Denote his estimate of buyer j's bid price b_{jk} is $\widetilde{b_{jk}}^i$, then his estimate of supplier k's probability of acceptance is:

$$\mathbb{Q}_i(B_{ik} = 1) = \frac{e^{b_{i^*k}}}{\sum_{j \in I} e^{\widetilde{b_j k}^i}} \tag{55}$$

In this case, his personal optimal bid price $\widetilde{b^*_{ik}}^i$ must satisfies

$$e^{\widetilde{b^*_{ik}}^i} + \widetilde{b^*_{ik}}^i \widetilde{K}^i - \widetilde{K}^i(c_i + \frac{\mathbb{E}_{\mathbb{Q}_i} Z^2}{\beta s^*_k} - 1) = 0 \tag{56}$$

where $\widetilde{K}^i = \sum_{j \neq i} e^{b_j}$. Solve equation (56) for b_{ik}, and we have :

$$\widetilde{b^*_{ik}}^i = c_i + \frac{\mathbb{E}_{\mathbb{Q}_i} Z^2}{\beta s^*_k} - 1 - \mathscr{W}(\frac{e^{c_i + \frac{\mathbb{E}_{\mathbb{Q}_i} Z^2}{\beta s^*_k} - 1}}{\widetilde{K}^i}) \tag{57}$$

as the buyer i's best personal bid price to supplier k, where $\mathscr{W}$ is Lambert W function: for $f(z) = ze^z$, we have $z = f^{-1}(ze^z) = \mathscr{W}(ze^z)$.

Recall in a simple case when buyer i estimates others' bids using his own bid price $\widetilde{b_j}^i = b_i$, then his best personal bid price is

$$\begin{aligned} \widetilde{b^*_{ik}}^i =& \frac{(N-1)(c_i + \frac{\mathbb{E}_{\mathbb{Q}_i} Z^2}{\beta s^*_k} - 1) - 1}{N-1} \\ =& c_i + \frac{\mathbb{E}_{\mathbb{Q}_i} Z^2}{\beta s^*_k} + \frac{N-2}{N-1} \end{aligned} \tag{58}$$

There are some good analytical properties of this formulation. In particular, we are interested in how a buyer's personal optimal bid price reacts to market condition and other buyers' bid. To see this, by implicit function theorem, we note [42]:

$$\frac{\partial \widetilde{b^*_{ik}}^i}{\partial \mathbb{E}_{\mathbb{Q}_i} Z^2} = -\frac{\partial (lhs)/\partial \mathbb{E}_{\mathbb{Q}_i} Z^2}{\partial (lhs)/\partial \widetilde{b^*_{ik}}^i} = \frac{\widetilde{K}^i}{\widetilde{K}^i + e^{\widetilde{b^*_{ik}}^i}} > 0 \tag{59}$$

$$\frac{\partial \widetilde{b^*_{ik}}^i}{\partial \widetilde{K}^i} == -\frac{\partial (lhs)/\partial \widetilde{K}^i}{\partial (lhs)/\partial \widetilde{b^*_{ik}}^i} = -\frac{\widetilde{b^*_{ik}}^i - c_i - \frac{\mathbb{E}_{\mathbb{Q}_i} Z^2}{\beta s^*_k} + 1}{\widetilde{K}^i + e^{\widetilde{b^*_{ik}}^i}} > 0 \tag{60}$$

where *(lhs)* is the left hand side of equation (52). Equation (59) and (60) show that buyer i's personal optimal bid price increases as excess demand increases, and as his estimate of other buyers' bid prices increase.

5.2 A Second Example (SAC4)

In this section, I will present an analytical solution corresponding to supplier acceptance case 4 (SAC4).

[42] *Proof* of (60): Let $N = c_i + \frac{\mathbb{E}_{\mathbb{Q}_i} Z^2}{\beta s^*_k} - 1$. It suffices to $N > \widetilde{b_{ik}}^i$. Plug N in (56), we have $e^N + \widetilde{K}^i N - \widetilde{K}^i N = e^N > 0$. By $f(x) = e^x + \widetilde{K}^i x - \widetilde{K}^i N$ strictly monotonically increases in x, we know $N > \widetilde{b_{ik}}^i$.

5.2.1 Supplier's Side

Recall in SAC4, we have suppliers accepts an offer if i^* if $b_{i^*k} > b^e - WC_k$, where b^e is the equilibrium price and WC_k the waiting cost. Following the specification of SAC4, we assume

$$WC_k \sim exp(f(Z)) \tag{61}$$

where $f(\cdot)$ is an increasing function. Note this specification means as Z the excess demand increases, the average waiting time $\frac{1}{f(Z)}$ will decrease. Then the probability of the supplier k accepting a bid (q_{ik}, b_{ik}) (conditioning on Z) can be written as:

$$\begin{aligned} & \mathbb{P}(\text{supplier } k \text{ accepts bids } i^*|Z) = \mathbb{P}(B_{i^*k} = 1|Z) \\ = & \ \mathbb{P}(WC_k > b^e - b_{i^*k}) \\ = & \ e^{f(Z)(b_{i^*k} - b^e)} \end{aligned} \tag{62}$$

Note $\frac{\partial e^{f(Z)(b_{i^*k}-b^e)}}{\partial b_{ik}} = e^{f(Z)} > 0$ agrees with the fact that as bid price increases it has a higher probability to be accepted.

5.2.2 Buyer Side

Now let us consider the buyer side. Recall each buyer i's optimization problem can be written as:

$$\begin{aligned} \max_{b_{ik}} \ & \mathbb{E}_{\mathbb{Q}_i}\Big\{(c_i q_{ik} + SC(q_{ik}) - b_{ik}q_{ik})\mathbb{1}(B_{ik} = 1)\Big\} \\ =& \mathbb{E}_Z\Big\{\mathbb{E}_{\mathbb{Q}_i}\Big\{(c_i q_{ik} + SC(q_{ik}) - b_{ik}q_{ik})\mathbb{1}(B_{ik} = 1)\Big\}|Z\Big\} \end{aligned} \tag{63}$$

If we further assume search cost and acceptance rule are conditional independent (conditioning on Z) and $SC(q_{ik}) = Z^2 q_{ik}/\beta s^*_{ik}$ as in the first example, then we have:

$$\begin{aligned} (lhs) =& \mathbb{E}_Z\Big\{\mathbb{E}_{\mathbb{Q}_i}\Big\{(c_i q_{ik} + SC(q_{ik}) - b_{ik}q_{ik})|Z\Big\}\mathbb{Q}_i(B_{ik} = 1|Z)\Big\} \\ =& \mathbb{E}_Z\Big\{\mathbb{Q}_i(B_{ik} = 1|Z)(c_i q_{ik} + \frac{Z^2 q_{ik}}{\beta s^*_k} - b_{ik}q_{ik})\Big\} \end{aligned} \tag{64}$$

Since $q_{ik} > 0$ (as the buyer is still in the market), the first order condition can be written as:

$$\begin{aligned} \frac{\partial \mathscr{L}}{\partial b_{ik}} =& \frac{\partial \mathbb{E}_Z\Big\{\mathbb{Q}_i(B_{ik} = 1|Z)(c_i q_{ik} + \frac{Z^2 q_{ik}}{\beta s^*_k} - b_{ik}q_{ik})\Big\}}{\partial b_{ik}} \\ =& q_{ik}\mathbb{E}_Z\{(c_i f(Z) - b_i f(Z) - 1)e^{f(Z)(b_i - b^e)} + f(Z)\frac{Z^2}{\beta s^*_k}e^{f(Z)(b_i - b^e)}\} \end{aligned} \tag{65}$$

Then the buyer's personal optimal bid price satisfies:

$$\widetilde{b^*_{ik}}^i = c_i - \frac{\mathbb{E}_{\mathbb{Q}_i} e^{f(Z)(b_{ik}-b^e)}}{\mathbb{E}_{\mathbb{Q}_i} f(Z) e^{f(Z)(b_{ik}-b^e)}} + \frac{\mathbb{E}_{\mathbb{Q}_i} Z^2 f(Z) e^{f(Z)(b_{ik}-b^e)}}{\beta s^*_k \mathbb{E}_{\mathbb{Q}_i} f(Z) e^{f(Z)(b_{ik}-b^e)}} \tag{66}$$

In a simple case where $WC \sim exp(\lambda)$ independent of Z the market condition, then we have

$$\widetilde{b^*_{ik}}^i = c_i + \frac{\mathbb{E}_{\mathbb{Q}_i} Z^2}{\beta s^*_k} - \frac{1}{\lambda} \tag{67}$$

which only differs from the optimal solution in the first example by a constant. Hence this solution remains analytical as in the first example (equation (59) and (60)).

These two examples show that if seller's acceptance probability is independent of market condition (and known to buyers), each buyer i's optimal bid price is mostly determined by the marginal instantaneous cost of not meeting requirement c_i plus the marginal search cost $\frac{\partial SC_{ik}}{\partial q_{ik}}$. Thus in the case when search cost is a nonlinear function of the excess demand Z, a shock in supply will trigger a nonlinear increase in the optimal bid price. Such effect will be further augmented by the bias of buyer's personal belief (that Z is close to 0).

6 Extensions to Other Types of Trading Sessions

In previous sections we presented the model of a single type 1 trading session (TS1). In this section we generalize the model to other types of trading sessions as shown in table 1. We extend the model to accommodate buyers' coincidental arrival (as in TS2 and TS4) and intertemporal dependence (as in TS3 and TS4).

6.1 Extensions to TS2

In a type 2 trading session, we introduce the possibility that multiple buyers arrive coincidentally. This setting could allow for only for one-time, simultaneous bids, or repeated bidding, i.e. buyers submit multiple bids at the same encounter. Formally, depending on whether buyers can submit repeated bids and what information they are specified as having, there are three sub-cases in a type 2 trading session.

- TS2-1: Buyers can submit only one bid at the encounter. They submit sealed bids simultaneously and hence can not observe each other's bid. In this case, the only information available to buyers is the number of people showing up.
- TS2-2: Buyers can submit one bid at the encounter as in TS2-1, however they can observe bids of buyers that coincidentally arrive at the supplier.
- TS2-3: Buyers can submit multiple bids at the same encounter which results in repeated gaming. They can observe others' bids as in TS2-2 and are allowed to bid against each other to secure supply. Buyers are *fast reactors* in a sense that they can react to other buyers and submit multiple bids within the same encounter.

These assumptions lead to richer information sets of buyers (than in TS1). In particular, in the first case (TS2-1), the buyer's information set becomes $\Phi^i_k \cup \{n_k\}$, where Φ_k^i is the buyer information set at supplier k as in TS1 and n_k is the number of buyers that coincidentally come to the supplier. In the other cases (TS2-2 and TS2-3), the buyer's information includes that in TS2-1 plus the set of other buyers' bids.

In each of these cases, the buyers' optimal bids are determined by whether and how they evaluate the new information. If a buyer ignores the new information, his estimate of market condition and hence his optimal bid price will be exactly the same as those in TS1. As a result, given supplier's acceptance rule and waiting cost, the transaction prices will be the same as those in TS1. If the buyer uses the additional information to revise his estimate of market condition under some rule, his optimal bid price will generally deviate from that in TS1. The deviation depends on how buyers integrate the information observed at the encounter into his learning rule to form his estimate of market condition.

A possible approach to model the interaction between buyers (as in TS2-3) is to consider the herd behaviour among buyers. In literature buyers (traders) are usually categorized into two types: *rational traders* and *noise traders* (see [De Long et al., 1990], [Shleifer and Summers, 1990]). Rational traders base their expectation on economic fundamentals while noise traders are subject to erroneous stochastic beliefs and have irrational expectations. Within rational traders, we label those well-informed ones as *informed traders* and those less-informed ones as *follower*. Followers can be viewed as rational as they mimic decisions of informed,rational traders. Herd behaviour refers to that of either follower behaviour (i.e. followers follow informed traders) or momentum trading, through the process information is transmitted from informed traders to followers and/or noise traders.

Shiller (1980) employed volatility tests and found stock market volatility to be far greater than described in efficient market hypothesis (see [Shiller, 1980]). Pindyck and Rotemberg (1988) argued that extreme movements of commodity prices are partially induced by "herd behaviour" of traders ([Pindyck and Rotemberg, 1988]). That is, noise traders are either bullish or bearish across all commodity markets without justification from economic fundamentals. Bouchaud and Cont (2000) discussed the relationship between herd behaviour and the fluctuation in financial markets and showed that herd behaviour tend to induce extreme (heavy tail distributed) stock market returns (see [Bouchaud and Cont, 2000]). Thus we suspect the transaction price tends to be higher through herd behaviour.

The settings of this model are not compatible with common herd effect models for two reasons. First, since early models of herd behaviour ([Banerjee, 1992], [Bikhchandani et al., 1992]), most herd behaviour literature concerns with discrete action and signal spaces; while in our model, a buyer's action space (the space of his bid) and signal space (the space of others' bids) and the state space (the space of true market conditions) are continuous. Although it is not impossible to model herd effect and information cascades with continuous action space (see [Huck and Oechssler, 1998], [Eyster and Rabin, 2009], [Eyster and Rabin, 2010]), it is hard to capture the continuous state space in this context.

More importantly, agents' information acquiring process in our model is rather different from the information cascade described in the herd effect models. In particular, in our model buyers' learning rule is based on nonparametric Bayesian learning on the market condition through the searching process, rather than from the signals of some other pioneer buyers. That there is another buyer i' bidding at the same supplier and his bid price are signals to the current buyer i, since the other buyer i' bid price based on his estimate of market condition. Such signals are hard to be integrated into buyer i's own learning process through observations of suppliers' capacity. In addition, buyer i could hardly assess the quality of the signal since a buyer's estimate of market condition is not revealed from his optimal bid (e.g. in equation (57) and (67), a buyer i can not infer another buyer i' estimate of market condition from buyer i''s optimal bid $\widetilde{b_{i'k}}^{i}$ since buyer i does not know buyer i''s penalty cost $c_{i'}$.). It is of great interest to integrate herd behaviour into the context of this model. On the other hand, the construction could be rather complicated. We do not have precise idea about how to integrate herd behaviour into our model yet.

6.2 Extensions to TS3 and TS4

In a type 3 or type 4 trading session, we allow for "*far-sighted*" suppliers. That is, a supplier is allowed to consider selling in the current session versus selling in the future. If a supplier k thinks the equilibrium in the current trading session is relatively "low" compared to normal (the products are currently undervalued), he can choose to put some of his supply quantity into storage and sell in the future.

To model such storage behaviour, we add ab expectation of the future equilibrium price (in future trading sessions) to the current acceptance rule [43]. Under acceptance rule 3 (SAC4), for a bid b_{ik}, the supplier accepts the bid if $b_{ik} > b^e_t - WC_k$ where b^e_t is the market equilibrium price in current session. With consideration of storage, the supplier will only accept the bid if it is higher than current equilibrium price and less than future equilibrium price adjusted by storage cost. These results can be summarized as the following rule:

Acceptance Rule in a Type 3 Trading Session: In a type 3 trading session, each supplier k can make tradeoff between selling in the current session and selling in the future. The supplier's acceptance rule is as follows:

- For a bid $b_{ik} > b^e_t - WC_k$, a *myopic* supplier k will accept the bid and ignore future sessions.
- For a bid $b_{ik} > \widetilde{b^e_{t+1}} - stc_k - WC_k > b^e_t$, where we define $\widetilde{b^e_{t+1}}^{k}$ as the supplier k's estimated future equilibrium price in the next trading session and str_k is supplier k's storage cost, a *far-sighted* supplier k will accept the bid.

[43]There are actually two possible ways to model such storage behaviour. Another way is to view storage as a special type of buyer (inventory speculator). The estimated bid price of the storage buyer is the estimated equilibrium price of the next session. The waiting cost of the buyer is storage cost (here we assume the inter-trading session waiting costs are much larger than the inter-trading session ones). Then if the storage price is accepted based on some acceptance rule (for example, acceptance rule 4 in TS3), then the supplier chooses to store his supply to sell in the future.

- For a bid $\widetilde{b^e_{t+1}} - stc_k - WC_k < b_{ik} < b^e_t - WC_k$, a *far-sighted* supplier k will reject the bid and choose to sell in the future (equivalently, he puts q^a_{ik} of his supply into storage).

We label such acceptance rule as SAC5.

7 Simulation

In this section we present numerical simulation to show that the model proposed in this model can generate price jump that can not be characterized by competitive market model and resemble those in reality. As shown in the previous sections, the fundamental driving forces to price jumps are:

1. Nonlinear effect of market condition in buyers' search cost;
2. Buyer's personal belief bias in market condition;

We also suspect given the same total supply, different allocation/concentration of the total supply across suppliers give rise to different transaction price distributions. In this section we will simulate all these three effects and show how these effects can drive up the transaction when there is shortage in supply.

In the simulation, we assume there are 100 buyers and 10 suppliers in the market. We simulate 100 trading sessions. In each trading session, the supply and demand are generated from normal distribution with parameter $(n_s, 1)$ and $(10, 9)$ respectively. Across trading session, the parameter n_s are generated from Weibull distribution $H(10, 10)$ (with scale and location parameters equalling to 10) to characterize extreme events (e.g. severe shortage in supply). By using the hierarchical method to generate supplies, we are able to capture the circumstances that supplies across all suppliers comove with presence of extreme events (e.g. all farmer's harvests of a grain product decrease with extreme weather). As a result of this setting, in most cases, the total demand is about the same as total supply; while under under extreme events, there could be rather large excess demand in the market. The generated excess demand across trading sessions are plotted in figure 4. The parameter β in search costs is chosen to be 100 so that search cost and penalty cost are comparable (note we also simulate the case $\beta = 300$ in section 7.1). We generate penalty costs based on normal distribution $N(100, 25)$ to capture heterogeneity in buyers' penalty cost. The sequence of encounters as in figure 1 is determined by the order of the time of the encounters generated from a Weibull distribution $H(1, 1)$. The buyers' learning rule, offer rule, and suppliers' acceptance rules are as specified in previous sections.

7.1 Single Trading Session Simulation

In this section we simulate the market in single session. We first simulate the nonlinear effect of market condition in buyers' search cost and hence their optimal bid price using simulation. Recall the buyer i's *personal* optimal bid price based on acceptance rule 1 in equation (58) is

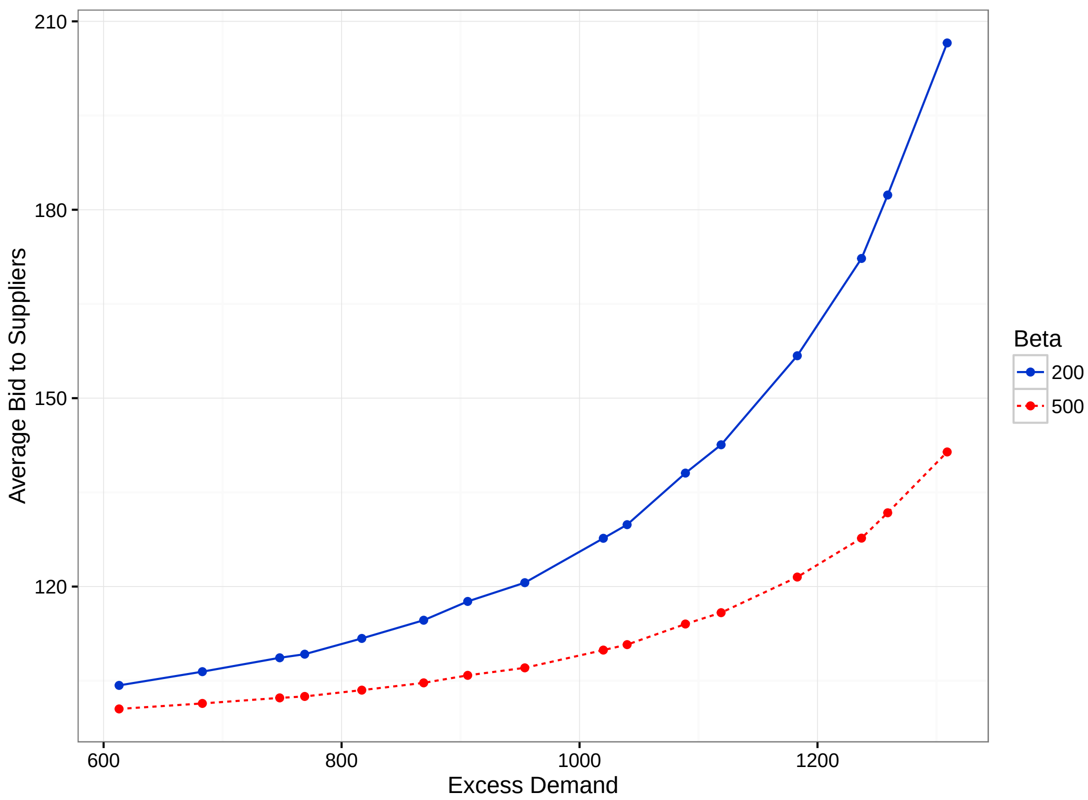
$$\widetilde{b_{ik}^*}^i = c_i + \frac{\mathbb{E}_{\mathbb{Q}_i} Z^2}{\beta s_k^*} - 1 - \mathscr{W}\left(\frac{e^{c_i + \frac{\mathbb{E}_{\mathbb{Q}_i} Z^2}{\beta s_k^*} - 1}}{\widetilde{K}^i}\right) \tag{68}$$

Similarly based on acceptance rule 4, the buyer i's *personal* optimal bid price is

$$\widetilde{b_{ik}^*}^i = c_i - \frac{\mathbb{E}_{\mathbb{Q}_i} e^{f(Z)(b_{ik} - b^e)}}{\mathbb{E}_{\mathbb{Q}_i} f(Z) e^{f(Z)(b_{ik} - b^e)}} + \frac{\mathbb{E}_{\mathbb{Q}_i} Z^2 f(Z) e^{f(Z)(b_{ik} - b^e)}}{\beta s_k^* \mathbb{E}_{\mathbb{Q}_i} f(Z) e^{f(Z)(b_{ik} - b^e)}} \tag{69}$$

In either case, the buyer i's *personal* optimal bid price is a nonlinear function of market condition Z. Figure 3 shows that the buyer's optimal bids reacts to excess demand nonlinearly. The solid line the optimal bid price versus different market condition under the case $\beta = 200$. This is the parameter value we later use. In comparison, we also simulate the case when $\beta = 500$. c_i are generated from $N(100, 10)$. The figure shows that higher β gives lower optimal bid price. This is consistent with our model since search cost decreases with β. We also note, in both case, the optimal bid price increase nonlinearly in excess demand.

Figure 3: Single Buyer Average Optimal Bids to Suppliers Versus Excess Demand



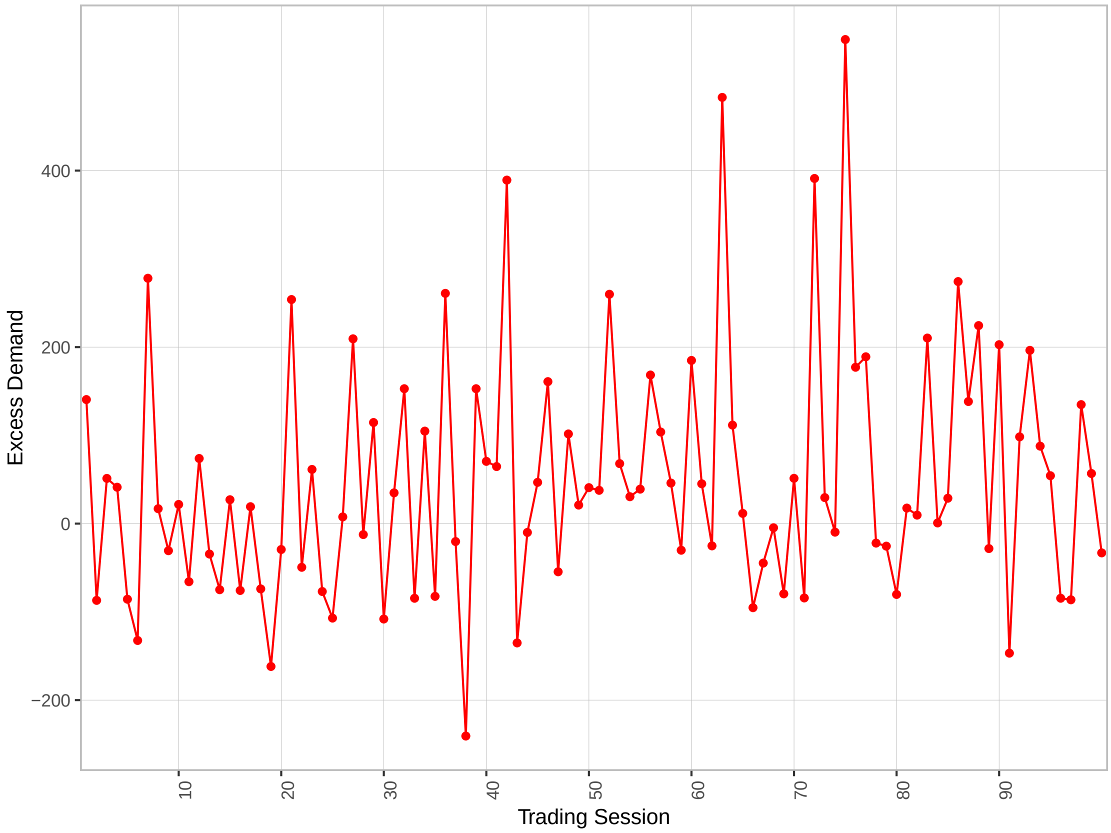
7.2 Multiple Trading Session Simulation

In this section, we simulation 100 trading session under different market condition. We simulate different trading session cases under different assumptions on buyers and suppliers. We first simulate a type 1 trading session (TS1) with assumption buyer's optimal bid case 2 (BOB2 thereafter) and supplier's waiting cost and acceptance rule case 4 (SAC4 thereafter). We label this case as TS1BOB2SAC4. We also simulate prices in a type 3 trading session (TS3) with the same assumption on buyers and suppliers. We label this second case as TS3BOB2SAC4.

7.2.1 Simulation of TS1BOB2SAC4

In this section, we simulate a case under the assumption trading session case 1, buyer's optimal bid case 2 and supplier's waiting cost and acceptance rule case 4 (or, acceptance rule 3). We start with simulating a sequence of supply capacity across trading sessions. The suppliers supply are generated in the procedure specified at the beginning of section 6. Figure 4 reports the excess demand in each session.

Figure 4: Excess Demand across Trading Sessions



Under the market condition shows in Figure 4, in each single session, the buyers make optimal bid prices and the supplier accept or reject bids based on acceptance rule 4 as specified in the model. Figure 5 reports accepted bid prices (transaction prices) and quantities in each session using candlestick chart. In each trading session, the top of the vertical line reports the maximum transactional price, and the bottom of the vertical line reports the minimum transactional price. The opening price and the closing price in one trading session are plotted as the top and bottom of the candle bar. If the opening price is lower than the closing price, then we mark the trading session as a price rise session. Otherwise, it is a price fall session. The equilibrium prices across trading sessions are plotted as blue dots and connected with dashed line. The trading volume of each trading session is represented as the width of candle stick.

As seen in Figure 5, when there is a large excess demand in a trading session, the transaction prices are significantly higher than the equilibrium price. Such difference reflects that buyers' search cost increase nonlinearly as market becomes tight and suppliers becomes more reluctant to accept bids in tight market. This figure shows that the model proposed in this paper can generate price jumps that resemble those in reality but are not characterized by the market equilibrium price.

Figure 5 shows in the session with large excess demand, the transaction prices are significantly higher than the market equilibrium price. Such price jumps reflect that buyers are difficult to search and that suppliers are reluctant to acceptance offers in tight market. These effects are characterized as buyers' search cost nonlinear increases and suppliers' acceptance probability nonlinear decreases in tight market by our model.

We also note there is relatively large variation in transaction prices in each trading session. We suspect that this is due to the specification of penalty cost c_i's and search cost. This motivates a closer look into these factors.

In order to measure the jump size across trading sessions in figure 5, we define some jump metric as percentage difference between some representation of the sequence of transaction prices in a trading session (e.g. weighted average price, closing price, or minimum pirce of that session) and market equilibrium price in that trading session. Here we choose closing price[44] [45] as such representation in that closing price characterizes the most up-to-date status of the market within a trading session. Formally, we define jump size as the difference between contemporaneous closing bid price and equilibrium price normalized by equilibrium price. We denote contemporaneous jump size by J_t, i.e.

$$J_t \equiv \frac{\text{(Closing Bid Price - Equilibrium Price)}}{\text{Equilibrium Price}} \times 100\% \tag{70}$$

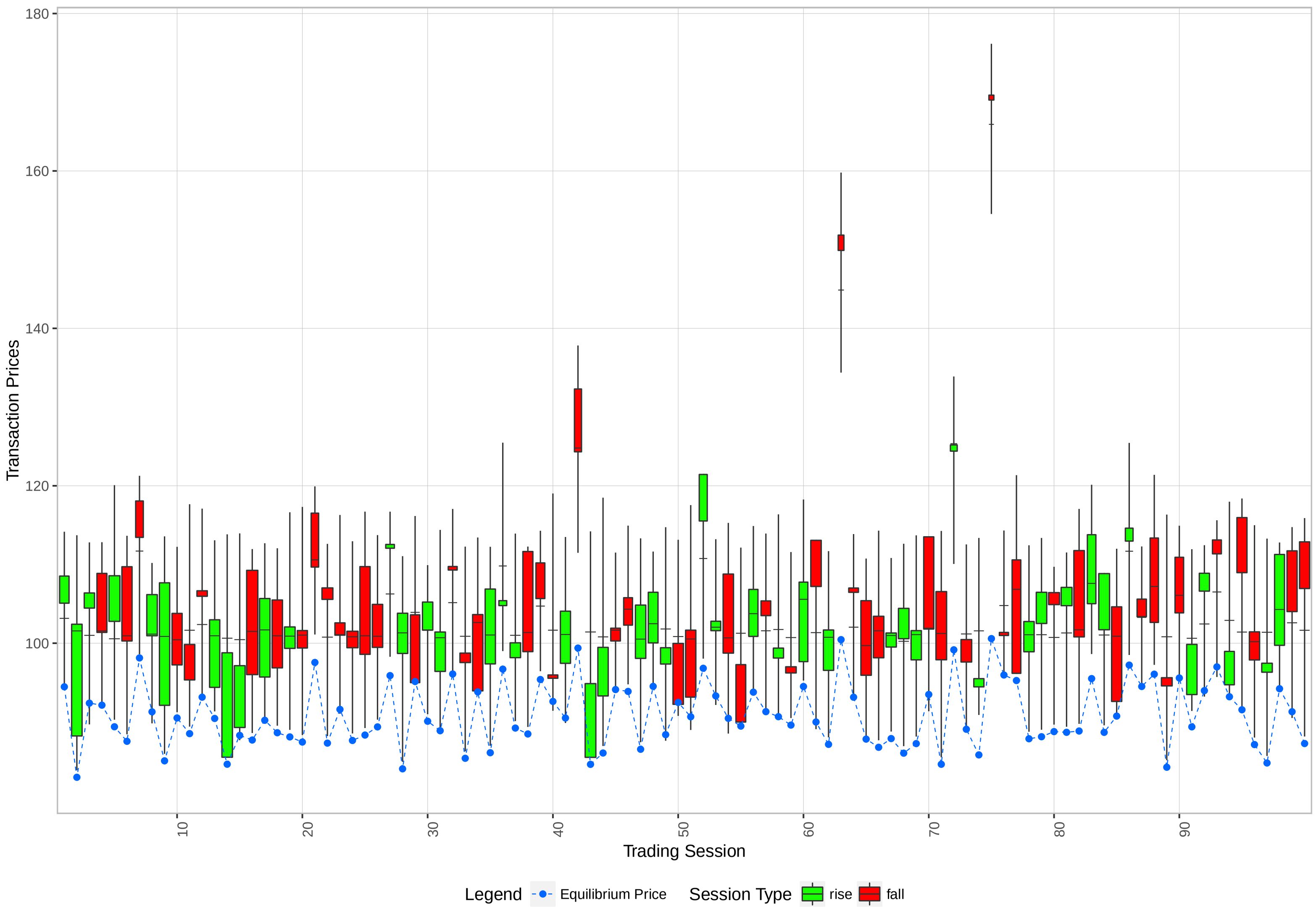
[44]We assume there is no transaction or encounter outside a trading session (no *after-hours trading*). Thus the closing price is defined as the last transaction price in a trading session.

[45]Some other jump metrics can be used here, for example:

- *Jump Metric 1* $\equiv \frac{\text{(Weighted Average Bid Price - Equilibrium Price)}}{\text{Equilibrium Price}} \times 100\%$
- *Jump Metric 2* $\equiv \frac{\text{(Minimum Bid Price - Equilibrium Price)}}{\text{Equilibrium Price}} \times 100\%$

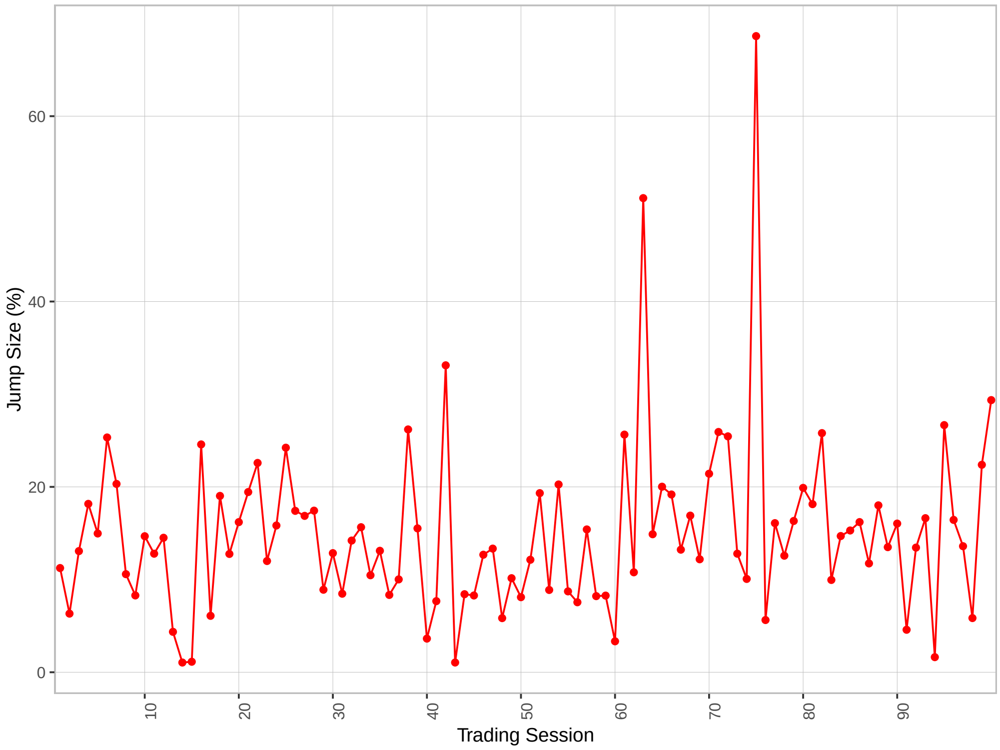
The plots of jump size using these metrics are in appendix. On the other, the jump size measured by these jump metrics basically shows the similar pattern and reflects jumps shown in figure 5 as that measured by closing price. Thus we will stick to use the jump metric by closing price.

Figure 5: Bid Price across Trading Sessions ($c_i \sim N(100, 25)$ and $\beta = 100$)



We plot price jump size measured such jump metric in figure 6. We see that in most trading sessions, jump size fluctuates around 10%. Such difference reflects the search cost under normal market condition (demand and supply are approximately the same). In cases when supply is in severe shortage (for example, trading session 63, trading session 75), the jump size exceeds 50%.

Figure 6: Jump Size across Trading Sessions Measured by Closing Price



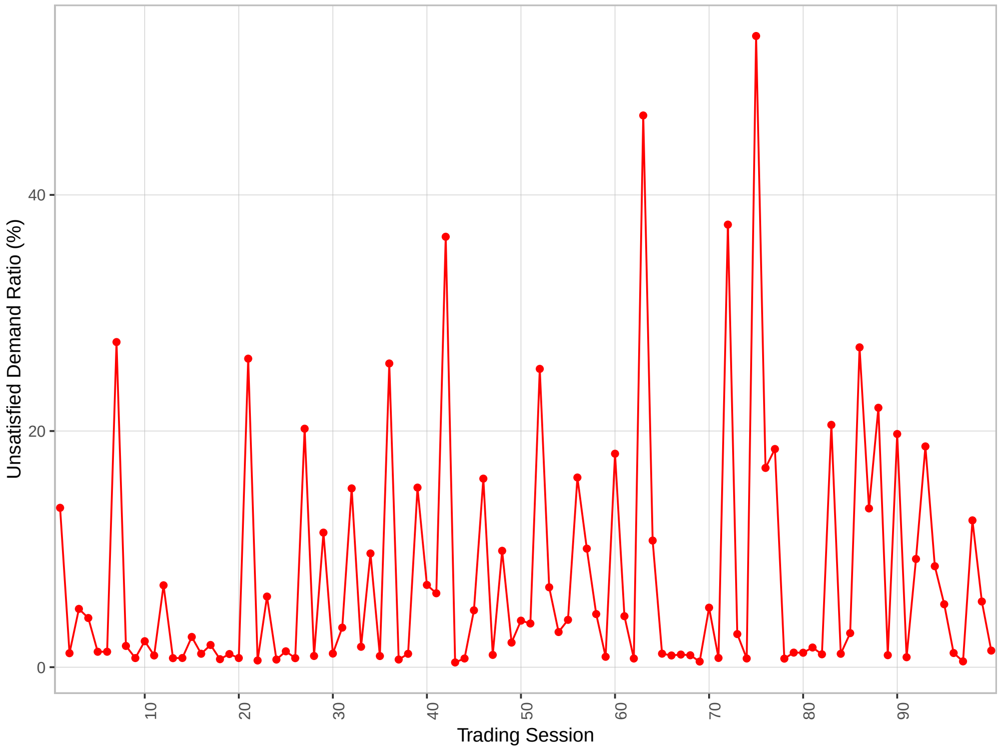
In this type of trading session (TS1), buyers' unfulfilled demand in current trading session can not be carried over to the next session. Thus it is of importance to know the unfulfilled demand at the end of each trading session. We define *unfulfilled demand ratio* as the ratio of unfulfilled demand at the end of each trading session and excess demand at the beginning of each trading session. We denote unfulfilled demand ratio as U_t, i.e.

$$U_t \equiv \frac{\text{Unfulfilled Demand at the End of Each Trading Session}}{\text{Excess Demand at the Beginning of Each Trading Session}} \times 100\%$$

The unfulfilled demand ratio's across trading sessions are plott in figure 7. We see from the figure that the peaks unfulfilled demand ratio directly corresponds to the peaks in the excess demand plotted in figure 4. This reflects the fact that the unfulfilled demand ratio is the positive part of excess demand scaled by itself (i.e. unfulfilled demand ratio $\approx \frac{max(0,Z)}{Z}$).

To compare the frequency of jumps in closing prices (as in our model) and equilibrium prices (as in competitive model), we use the empirical outliers test proposed by in [Chen and Liu, 1993] (CL test thereafter). [Charles and Darné, 2005] extended the method to GARCH model and applied it to examine the effects of outliers in three daily stock market indexes. The outlier identification procedure

Figure 7: Unfulfilled Demand Ratio across Trading Sessions



can be described as follows. Suppose the simulated closing prices can be thought as a univariate time series indexed by trading sessions t and hence can be written as:

$$P_t = \frac{\theta(B)}{\alpha(B)\phi(B)} e_t + \sum_{j=1}^{T_o} \omega_j L_j(B) \mathbf{1}_O(t_j) \tag{71}$$

where $e_t \sim WN(0, \sigma^2)$; B is the backward shifter: $BP_t = P_{t-1}$. $\theta(B)$ is the moving average polynomial, $\phi(B)$ is the autoregressive polynomial, and $\alpha(B)$ is the integration polynomial. All roots of $\theta(B)$ and $\phi(B)$ are outside unit circle; and all roots of $\alpha(B)$ are on the unit circle. Thus, $\frac{\theta(B)}{\alpha(B)\phi(B)} e_t$ represents the smooth trend of the price and follows a ARIMA process. $L(B)$ is a polynomial of B representing the dynamic pattern of the outlier effects. $\mathbf{1}_O(t)$ is an indicator variable that equals to 1 when there is outliers effect at time t ($t \in O$)and 0 ($t \notin O$) elsewhere; O is the set of possible latent time locations of outliers; and ω_j represents the magnitude of the outlier effects; and T_o is the total number of outliers.

If there are no outlier effects, the second term in equation (72) is 0 and hence

$$\widetilde{e_t} \equiv \frac{\alpha(B)\phi(B)}{\theta(B)} P_t \tag{72}$$

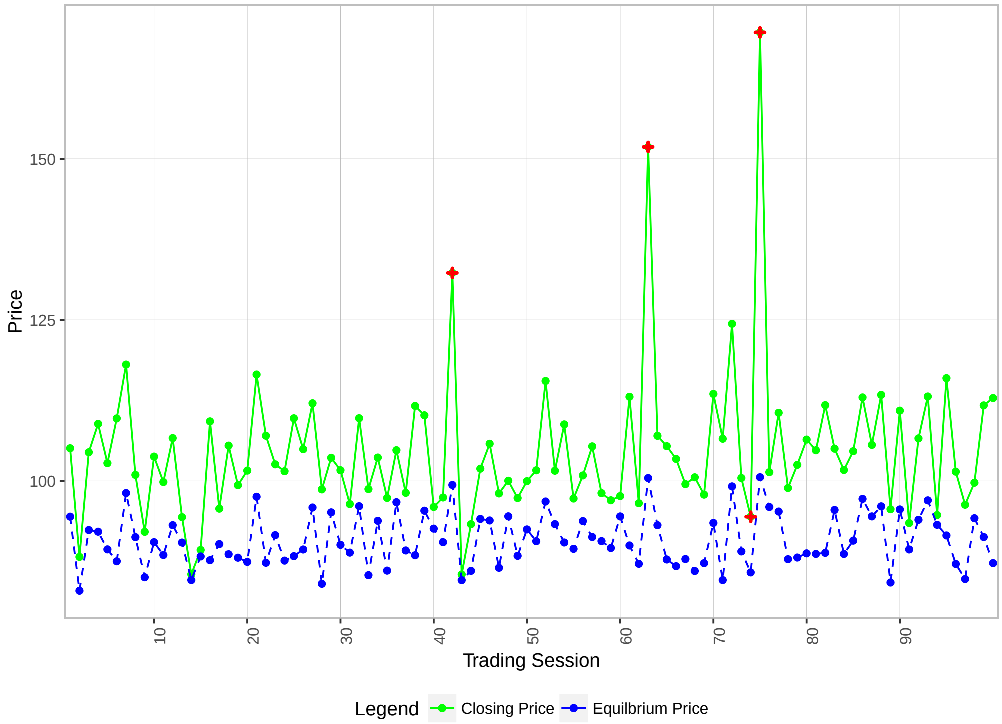
should be sufficiently close to the white noise e_t. Thus to test the presence of outliers, CL test examines to what extent $\widetilde{e_t}$ is not zero. The test statistic is:

$$\hat{\tau}_t = \frac{\widetilde{e_t}}{\hat{\sigma_e}} \tag{73}$$

where $\hat{\sigma_e} = 1.483 \times \text{median}\{|\hat{e}_t - \bar{e}_t|\}$ and $\bar{e}_t$ is the median of the estimate residuals.

With CL test presented, we can empirically examine jump frequency of a set of simulated prices. We first examine the number jumps in closing and equilibrium prices simulated in the first model (with $\beta = 100$). The closing and equilibrium prices are plotted in figure 8 and jumps are marked by cross. As shown in figure 8, there are four jumps in closing prices. These jumps correspond to the positive peaks of the excess demand. In contrast, there are no jumps in equilibrium prices under the same market conditions. This shows our model captures more price jumps than the competitive market model.

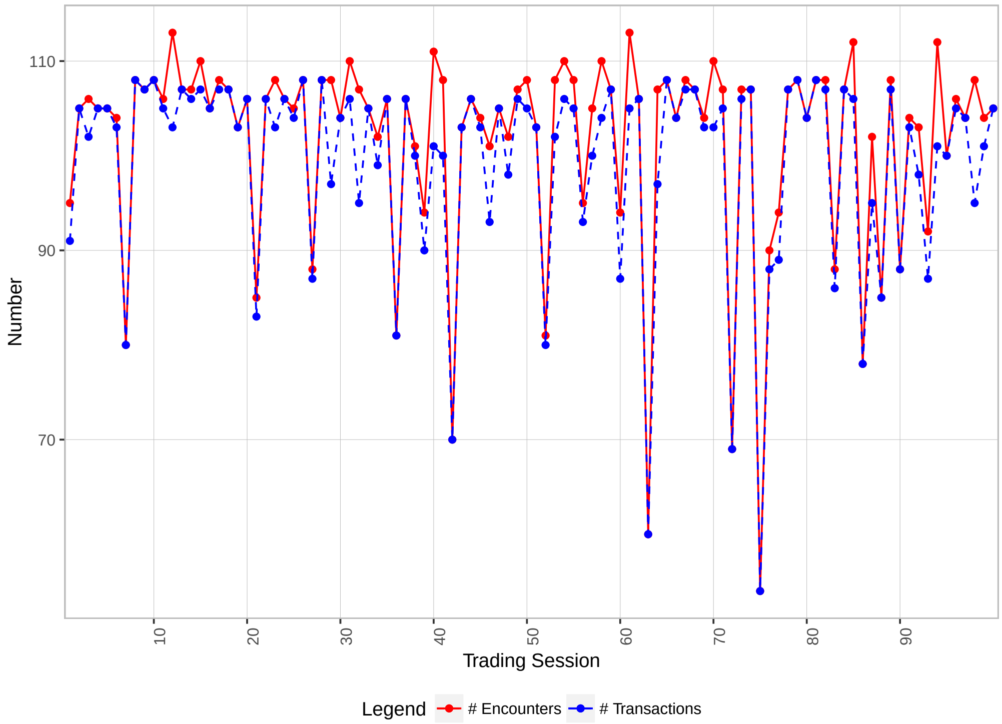
Figure 8: Jumps in Closing and Equilibrium Prices



- **The Effect of Search Cost**

In figure 5 and figure 6, we have seen that the search cost is a key driver in forming prices and inducing price jumps. From another perspective, search cost can be thought as a type of transaction cost inducing market *frictions* which by definition is a type of market incompleteness that prevents a trade from being executed smoothly (see [Cohen et al., 1983], [DeGennaro and Robotti, 2007]). Here we suspect search cost introduces frictions in the buyers' search process: buyers are reluctant to search under high search cost. To examine this, we simulate the number of *actual* encounters and transactions across trading sessions. We consider the difference between them as a measure of frictions. That is, a small difference means buyers are reluctant to search due to high frictions while a big difference shows that the market encourages buyers to search. We see in Figure 9, the number of actual encounters and transactions moves in a similar way across trading sessions. Thus, we conclude that in general (*actual*) encounters lead to transactions. We suspect the small difference between them is due to the specification of relative high search cost, since under high search cost ($\beta = 100$), buyers are less willing to search and forced to bid high (otherwise they need to face high search cost). Then, suppliers tend to accept such high bids (given the same waiting cost) and as a result, encounters are more likely to lead to transactions. In comparison, low search cost ($\beta = 300$) as plotted in figure 10 encourages buyers to search more. As a result, the difference between the number of actual encounters and that of transaction becomes bigger, showing small frictions in the search process.

Figure 9: Number of Encounters and Transactions: High Search Cost ($\beta = 100$)

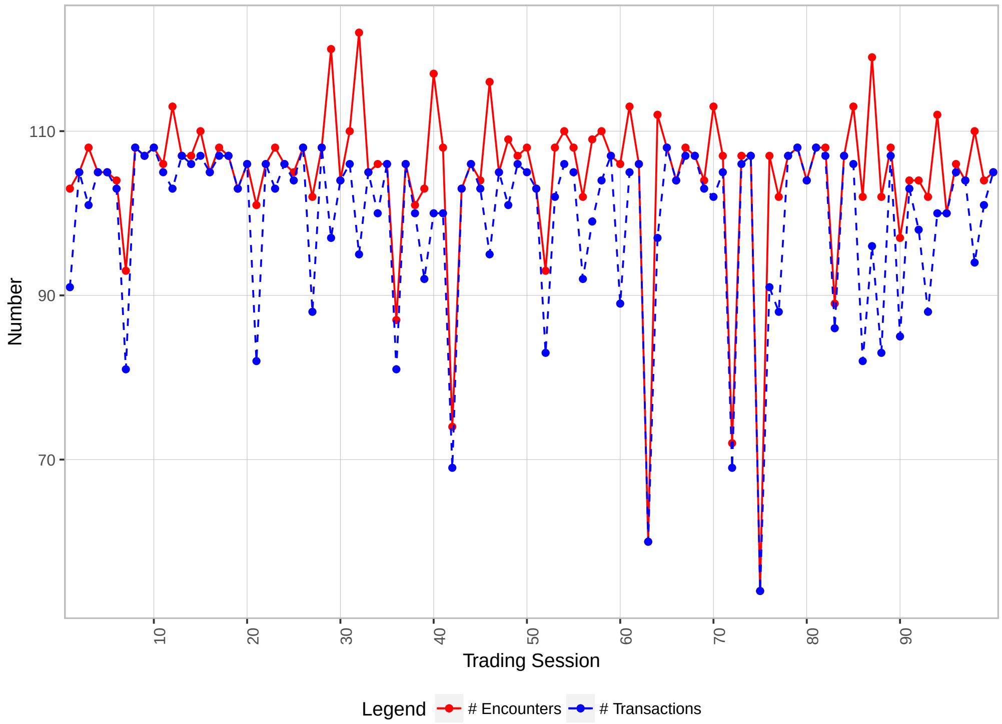


Since search cost is a key driver to price jumps, we more closely examine the effect of search cost. We examine jump frequencies under different search cost specifications in the same way as we did for the first model. We are also interested in the volatility of the closing prices under different cases. We suspect that with high search cost (low β) the prices become more volatile since a low β will exaggerate the difference in buyers' belief of the market condition. Formally, we use *realized volatility* (see [Andersen and Bollerslev, 1998]) to measure the variation in the closing prices, denoted as RV:

$$RV \equiv \sqrt{\sum_{t=2}^{T} (\log P_t - \log P_{t-1})^2} \tag{74}$$

where T is the number of trading sessions (here $T = 100$) and P_t is the closing

Figure 10: Number of Encounters and Transactions: Low Search Cost ($\beta = 300$)



price. The jump frequency, size and realized volatility under difference search cost specifications are reported in table 2.

Table 2: Jump Frequency, Jump Size and Realized Volatility

β	Jumps (Closing Price)	Jumps (b^e)	Max Jump Size	Realized Volatility
50	10	0	133.7769 %	2.086653
75	5	0	90.35319 %	1.665363
100	4	0	68.64135 %	1.440626
150	2	0	46.92952 %	1.217818
200	2	0	36.07360 %	1.102902
250	2	0	29.56005 %	1.028019
300	1	0	29.3648 %	0.9517208

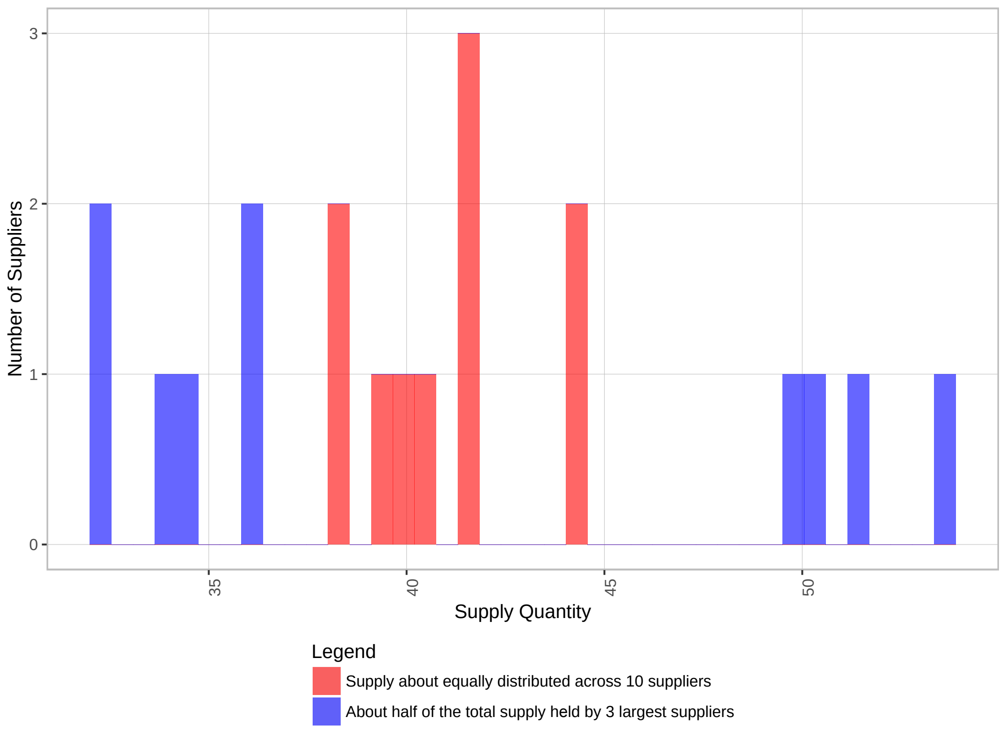
Note: In table 2, the second and third columns are the number of jumps in closing prices and equilibrium prices (b^e) in 100 trading sessions. The fourth column reports the maximum of jump size using jump metric defined as above. The last column reports the realized volatility of closing prices defined in equation 74.

As seen in table 2, as search cost increases (β decreases), there are more jumps in closing price. In contrast, the number of jumps in equilibrium price stays the same (all 0) since they do not have the search cost component. We also note, as search cost increases, closing prices becomes more volatile. This proves our previous conjecture. A small β will exaggerate buyers' belief in market condition and generate more extreme bid prices.

- **The Effect of Concentration of Supply**

We next examine the role of concentration of *total* supply in a trading session when supply varies substantially across suppliers. That is, observed supply at any encounter with a supplier provides the buyer with one observation that may be useful for predicting total supply in the market. For example, if only a few suppliers hold most supply available in the market, a buyer is more likely to observe supply availability that is not representative of market level total supply condition. As a result, he will think the market is very tight (tighter than it will already be in supply shortage), i.e., his $\widetilde{Z}^i$ could be very low. In contrast, with a more dispersed distribution of supply across suppliers, a buyer is more likely to observe available supply that reflects market conditions. As a result, his personal estimate of the market is more accurate and the accuracy (unbiasedness) increases more rapidly across encounters. Figure 11 plots two supply allocations. The red allocation corresponds to the case when every supplier holds roughly the same supply. The blue histogram shows the case with high concentration of supply.

Figure 11: Histogram of Two Difference Supply Allocations



To measure market concentration, we use Herfindahl–Hirschman Index (see [Hirschman, 1964], [Jacquemin and Berry, 1979]), which is denoted by HHI. Formally HHI can be calculated as follows:

$$HHI \equiv \sqrt{\sum_{k=1}^{|K|} \left(\frac{S_k^*}{\sum_{j=1}^{|K|} S_j^*} \right)^2} \tag{75}$$

where $\sum_{j=1}^{|K|} S_j^*$ is the total market supply and hence $\frac{S_k^*}{\sum_{j=1}^{|K|} S_j^*}$ is the market share of supplier k in the market. A higher HHI means that the market is more concentrated. Note HHI is bounded from below by $\frac{1}{|K|}$ when every supplier holds exactly the same market share. And HHI is bounded (from above) by 1 when one supplier is the only one with stock and holds all supply in the market.

Note in the first simulation of TS1BOB2SAC4, each supplier's initial allocation of supply is drawn from a normal distribution. In this case the average score of the concentration index equals to 0.3162. In this section, we simulate a higher market concentration case when the three largest supplier hold about 50% of the total supply. The result is reported in the third row of table 4. As seen from the table, with high concentration of the supply with $HHI = 0.3204$, there are 6 price jumps compared to 4. We also note maximal jump size is significantly higher than that in the first case. These results show that (even slight) higher concentration of supply can induce price jumps. We conclude that the distribution of supply is a key determinant in price jumps. This result is not so trivial from analytical solutions (in contrast, the effect of search cost in price jumps is much easier to identify from analytical solutions). Transaction prices and jump size in the case of high concentration are plotted in figure 23 and 29 respectively.

To better estimate jump frequency and examine its change due to higher concentration, we simulate 10000 trading sessions under the two cases. The results are reported in table 3. As shown in the table, the number of detected jumps increases from 469 to 546 with higher concentration. This is consistent with our previous result (with 100 trading sessions) and suggests that higher concentration of supply can induce price jumps.

Table 3: Jump Frequency (10K Trading Sessions)

Case	*HHI*	Jump Frequency
Low Concentration	0.3162	469
High Concentration	0.3228	546

- **The Effect of Penalty Cost Variation**

Intuitively, if penalty cost has larger variation, the transaction prices should be more volatile since high variance encourages the presence of extreme values. As a result, there could be more jumps in the transaction prices. In this part, we simulate the case when $\sigma_c^2 = 10$ (recall originally we have $\sigma_c = 5$). The result is reported in the fourth row of table 4. We see although prices becomes significantly more volatile (realized volatility is rather high compared to other cases), the numbers of jumps in both closing and market equilibrium prices do not increase. At the first glance, the result is rather surprising since price volatility and the presence of price jumps are closely related. A possible explanation is that the price fluctuations in this case are better characterized by the conditional variance model (for example, GARCH) than by the jump models.

Table 4: Jump Frequency, Jump Size, Realized Volatility and Trading Volume: 100 Trading Sessions

Case	Jumps (Closing Price)	Jumps (b^e)	Max Jump Size	Realized Volatility	Mean Volume	Max Volume
TS1BOB2SAC4 $\beta = 100, \sigma_c = 5,$ $HHI = 0.3162$	4	0	68.64135%	1.440626	940.2868	1072.494
TS1BOB2SAC4 $\beta = 300, \sigma_c = 5,$ $HHI = 0.3162$	1	0	29.3648%	0.951721	940.2868	1072.494
TS1BOB2SAC4 $\beta = 100, \sigma_c = 5,$ $HHI = 0.3204$	6	0	86.16272%	1.463181	942.3243	1094.446
TS1BOB2SAC4 $\beta = 100, \sigma_c = 10,$ $HHI = 0.3162$	2	0	70.84985%	2.008205	940.1837	1072.494
TS3BOB2SAC5 $\beta = 100, \sigma_c = 5,$ $HHI = 0.3162$	3	0	63.91316%	1.354189	970.6786	1081.269

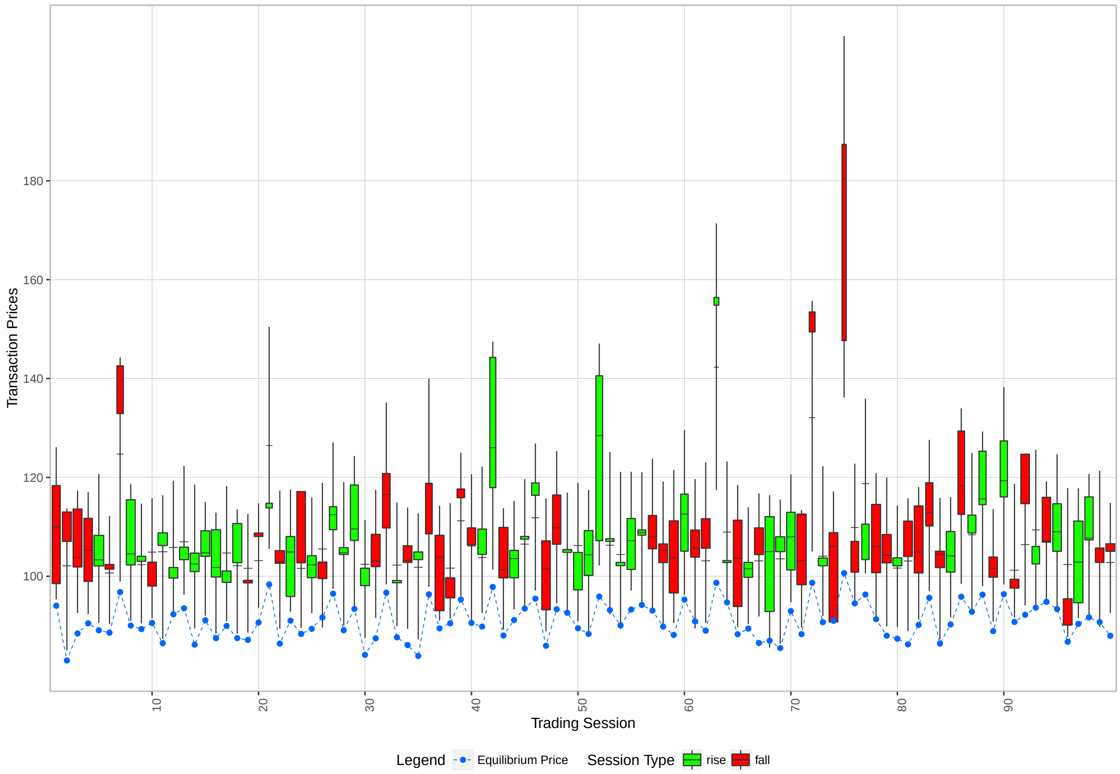
Note: In table 4, the first column indicates the case and parameter we simulate. The second and third columns are the number of jumps in closing prices and equilibrium prices (b^e) in 100 trading sessions. The fourth column reports the maximum of jump size using jump metric defined as above. The fifth column reports the realized volatility of closing prices defined in equation 74. The last two columns report mean trading volume and maximal trading volume respectively.

Table 5: Jump Frequency, Jump Size, Realized Volatility and Trading Volume: 10000 Trading Sessions

Case	Jumps (Closing Price)	Jumps (b^e)	Max Jump Size	Realized Volatility	Mean Volume	Max Volume
TS1BOB2SAC4 $\beta = 100, \sigma_c = 5,$ $HHI = 0.1000105$	469	0	691.6276%	12.7131	937.611	1105.919
TS1BOB2SAC4 $\beta = 300, \sigma_c = 5,$ $HHI = 0.1000105$	126	0	222.9008%	8.212194	937.611	1105.919
TS1BOB2SAC4 $\beta = 100, \sigma_c = 5,$ $HHI = 0.1042273$	546	0	902.1253%	15.42406	937.2741	1101.769
TS1BOB2SAC4 $\beta = 100, \sigma_c = 10,$ $HHI = 0.1000105$	223	0	644.4719%	18.22781	937.41	1105.919
TS3BOB2SAC5 $\beta = 100, \sigma_c = 5,$ $HHI = 0.102851$	399	39	459.9445%	11.40037	967.3564	1116.094

Note: In table 5, the first column indicates the case and parameter we simulate. The second and third columns are the number of jumps in closing prices and equilibrium prices (b^e) in 10000 trading sessions. The fourth column reports the maximum of jump size using jump metric defined as above. The fifth column reports the realized volatility of closing prices defined in equation 74. The last two columns report mean trading volume and maximal trading volume respectively.

Figure 12: Bid Price in TS1: High Penalty Cost Variation ($c_i \sim N(100, 100)$ and $\beta = 100$)



We take a closer look into this case. Figure 12 plots the transaction prices across 100 trading sessions. We notice that transaction prices are more volatile than previous case. The presence of extreme prices are also high. A possible reason that our statistics do not capture these jumps is that we use closing prices to represent the transaction prices within a whole trading session. For example, in trading session 75, the large jump in transaction prices does not appear in the closing price. This shows although the closing price is a good representation of the prices in a trading session, it might lead to some misinformation. Combining the results from figure 12 and table 4, we conclude that high penalty cost variation do induce price jumps and higher volatility in transaction prices.

7.2.2 The Effect of Storage: Simulation of TS3BOB2SAC5

In the section, we present the simulation results of type 3 trading. The new allocated demand and supply will be the same as in TS1BOB2SAC4 and hence the excess demand is the same as in Figure 4. We assume suppliers use acceptance rule 5 to determine whether to accept or not. Storage cost is drawn from $N(10, 25)$ to characterize suppliers' heterogeneity.

For the purpose of comparison, we first simulate transaction prices and trading volume as in type 1 trading sessions. Figure 13 reports the result in the same fashion of Figure 5.

We note that in Figure 5, the two biggest price jumps (at trading session 63 and 75) directly correspond to the large excess demand in Figure 4. This is consistent with that each trading session is assumed to be independent. In contrast, Figure 13 shows that in type 3 trading sessions the transaction price jumps do not directly to supply shortage. This can be caused by the supply storage. That is, price jumps can be mitigated by storage (for example, trading session 75). On the other hand, the carryover of unfulfilled demand can generate price jumps that are not induced by new generated excess demand (for example, trading session 90).

To further compare the transaction price in type 1 trading session and type 3 trading session, we plot the weighted average price and close price in each type of trading sessions in Figure 14 and Figure 15. We see from the figures that both close price and weighted average price in type 1 trading sessions are more volatile than those in type 3 trading sessions. This is consistent with previous results that suppliers' storage can mitigate the price oscillations. We further plot jump size (using jump size 1 and jump size 3) in figure 16 and figure 17. We see from the figures that jump sizes in type 1 trading sessions and type 3 trading sessions have roughly the same magnitude. On the other hand, we see type 3 trading sessions do no produce extreme price jumps as type 1 trading sessions do when there is large supply shortage (for example, trading session 75 in figure 17). This says suppliers' storage behaviour helps mitigating extreme price jumps.

Figure 13: Bid Price in a Type 3 Trading Sessions ($c_i \sim N(100, 25)$ and $\beta = 100$)

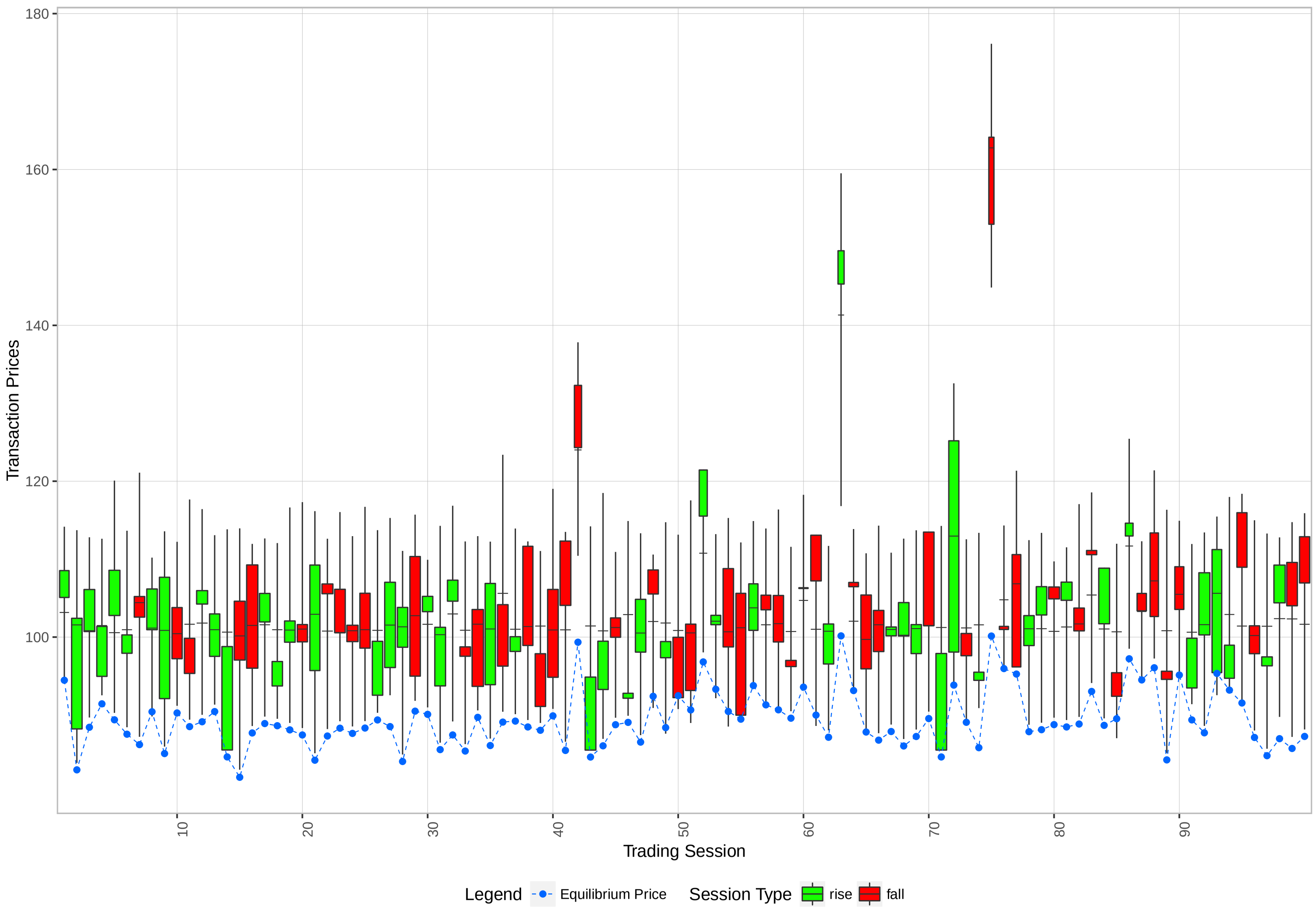


Figure 14: Comparison of Average Bid Price in TS1 VS TS3

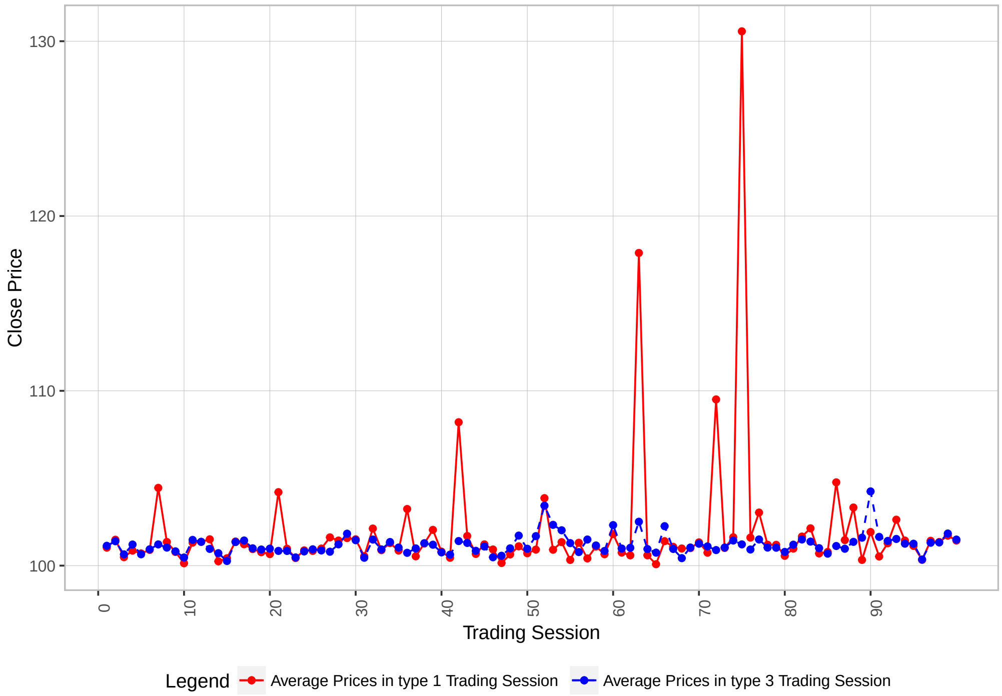


Figure 15: Comparison of Close Price in TS1 VS TS3

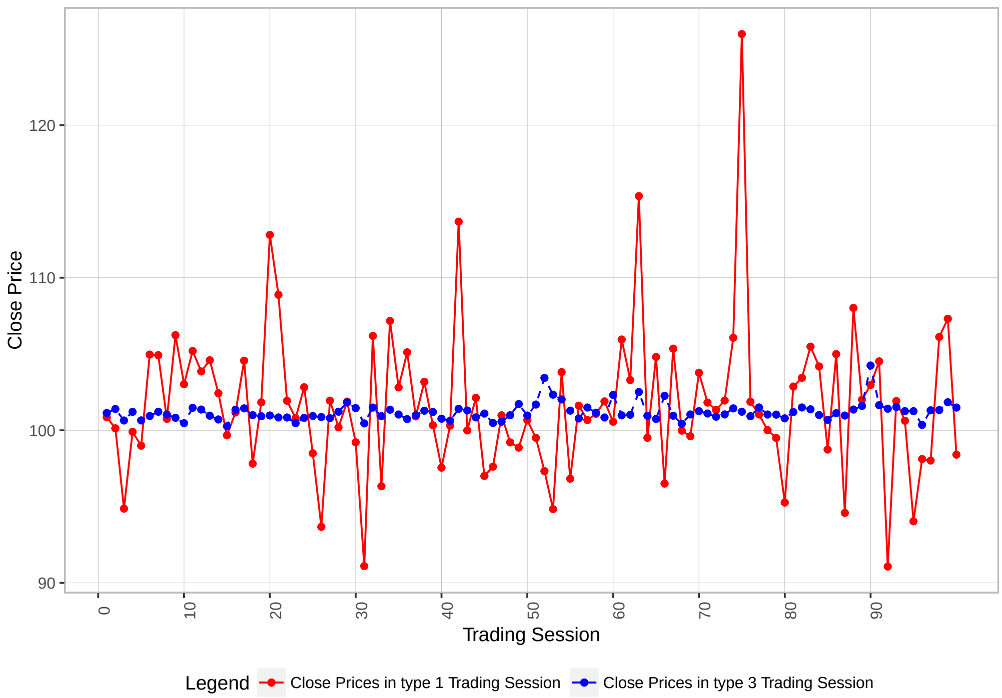


Figure 16: Comparison of Jump Size in TS1 VS TS3

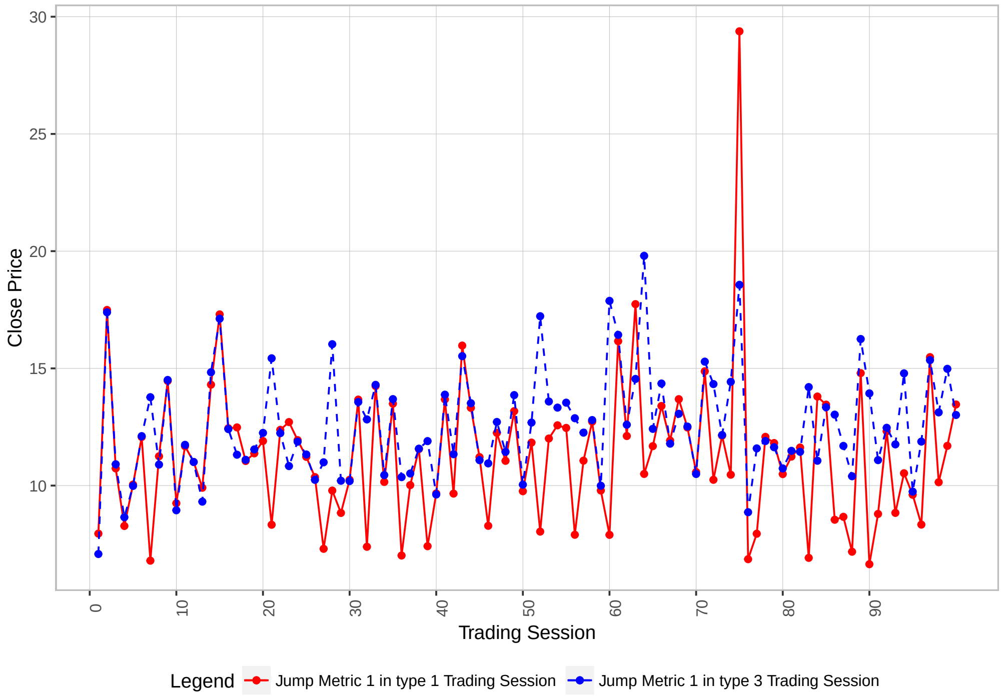
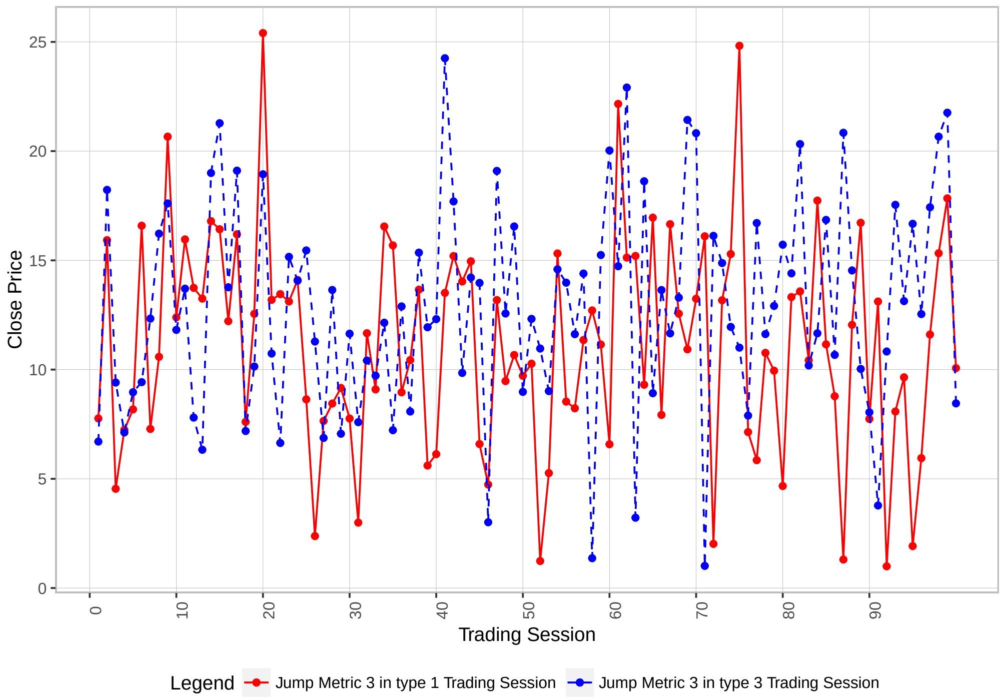


Figure 17: Comparison of Jump Size in TS1 VS TS3



We further plot the number of transactions and encounters in type 3 trading sessions in Figure 18. We see that in general *actual* encounters lead to transactions. Similar to trading session 1, this is due to the relative high search costs. Since in the case of TS1BOB2SAC4 we already examine the effect of search cost on number of transactions. We do not re-simulate the effect here again.

In figure 19, we plot buyers' unfulfilled demand ratio in current trading session. Such demand can be carried over to the next trading session. In figure 19, we see that in the most cases when there is no large excess demand, the unfulfilled demand ratio is smaller than that in TS1. Such observation is consistent with the result that total realized volatility in TS3 is smaller than that in TS1 (as reported in table 4). On the other hand, when there is large excess demand (stock-out), the unfilled demand ratio is much higher than that in TS1. This creates more price fluctuations in the session with large excess demand. This result is consistent with [Helmberger and Weaver, 1977].

Figure 18: Number of Encounters in TS3

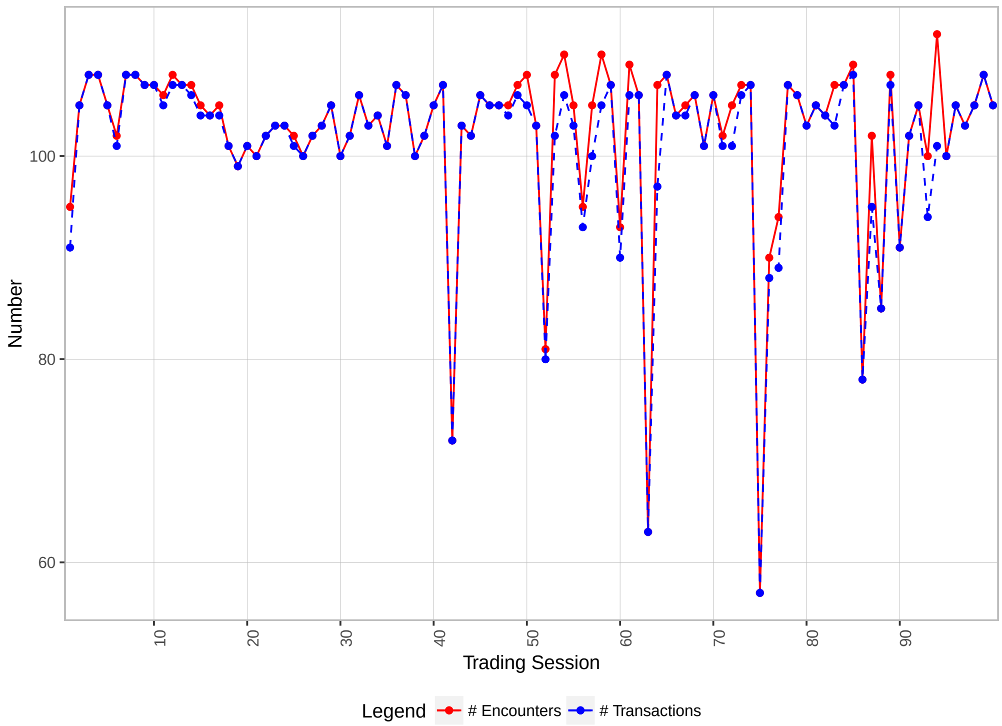
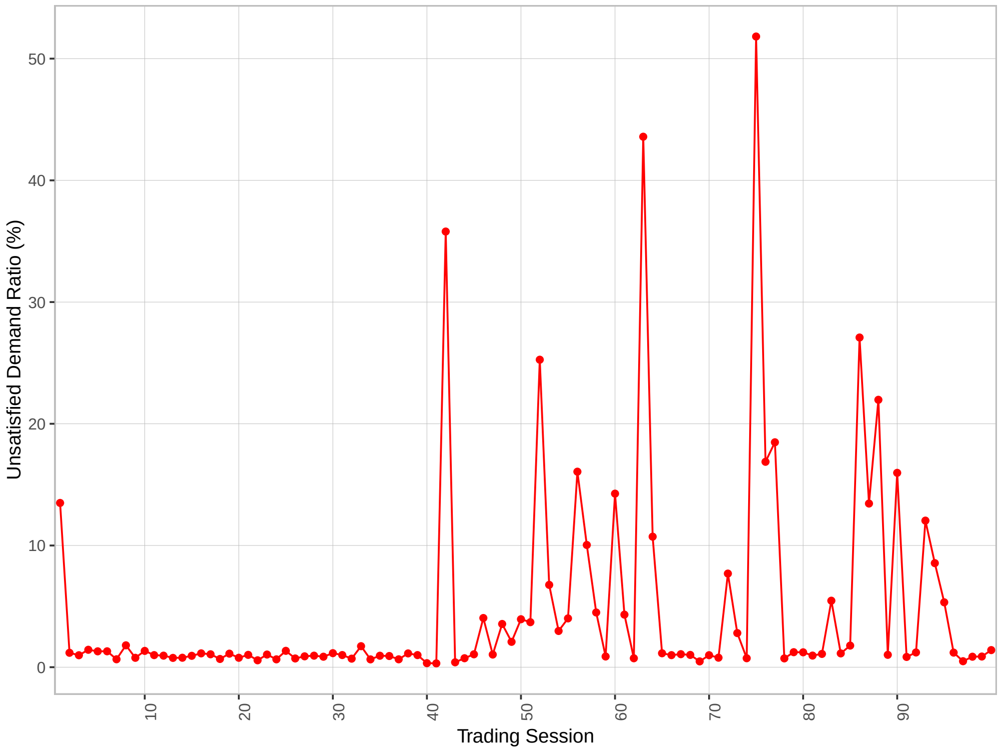


Figure 19: Unfulfilled Demand Ratio in TS3



In summary, we see from the simulation of case TS1BOB2SAC4 that the model proposed in this paper can generate price jumps that resemble those in reality. We show jump sizes are sensitive to the search cost specification. We also show that high search costs induce high market frictions.

In the simulation of case TS3BOB2SAC4, we show in this section that the storage behaviour (or inventory speculation) in TS3 can mitigate shocks in supply. They also induce intertemporal correlation in price (of trading sessions). We show that if storage behaviour is allowed, the extreme price jumps can be smoothed to what extent the carry over storage exceeds zero. This result is consistent with our intuition that carry over demand or supply storage generate serial (auto)correlatiion of prices. It also agrees with the results from previous literature (for example, [Deaton and Laroque, 1996], [Muth, 1961]).

8 Concluding Remarks

In this paper we proposed a rather general model of procurement with a set of different assumptions of the market. We discussed different cases of buyers, suppliers, and trading sessions. In particular, we present two exemplary analytical solutions to the model in two cases (labelled as TS1BOB2SAC1 and TS3BOB2SAC5). Using simulation, we show that buyers' search cost and their personal belief, suppliers' acceptance rule and the allocation of total supply (across suppliers) are the key driving forces of price jumps. As a result, our model generates jumps of similar magnitude and frequency as those in observed economic price series that generally can not be characterized by the perfect competitive market models.

Although the linkage between agents' risk aversion and price volitity has been noted in literature (see [Newbery and Stiglitz, 1979], [Deaton and Laroque, 1992]), most literature investigating commodity price dynamics assume risk-neutral agents (see [Helmberger and Weaver, 1977], [Wright and Williams, 1982], [Deaton and Laroque, 1992], [Deaton and Laroque, 1996], [Chambers and Bailey, 1996]). A key feature of our model is that we integrate buyers' risk aversion in market conditions (though they are risk neutral in payoffs) through their personal belief and search costs. That is, buyers' risk aversion in market condition affects their *estimated* search costs through their personal belief, which leads to large price movements. Buyers' risk aversion increases their sensitivity to market conditions, which exaggerates price movements with presence of supply shocks.

With discussions of type 3 trading sessions, we extend the work of [Helmberger and Weaver, 1977] in examining the effect of storage in commodity price dynamics by allowing for extreme events and risk aversion. Simulation results show although storage behaviour might introduce some fluctuation in general cases (this agrees with [Helmberger and Weaver, 1977]), it to some extent mitigates severe price movements with presence of extreme events.

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9 Appendix

9.1 Comparison of CandleStick charts

Figure 20: Bid Price in TS1 ($c_i \sim N(100, 25)$ and $\beta = 100$)

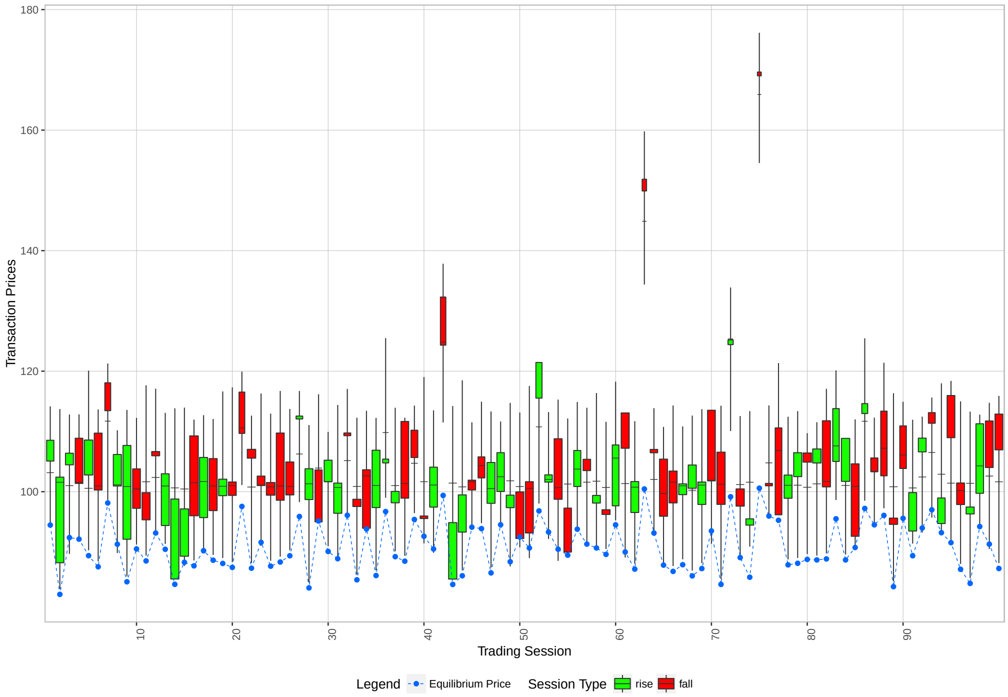


Figure 21: Bid Price in TS1: Low Seach Cost ($c_i \sim N(100, 25)$ and $\beta = 300$)

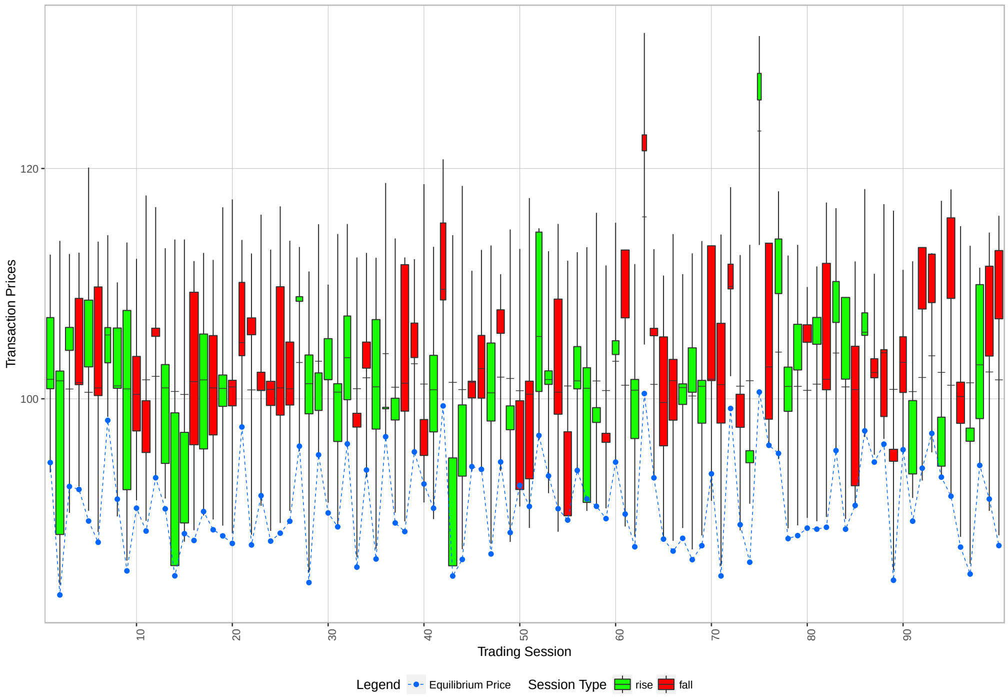


Figure 22: Bid Price in TS1: High Concentration ($c_i \sim N(100, 25)$ and $\beta = 100$)

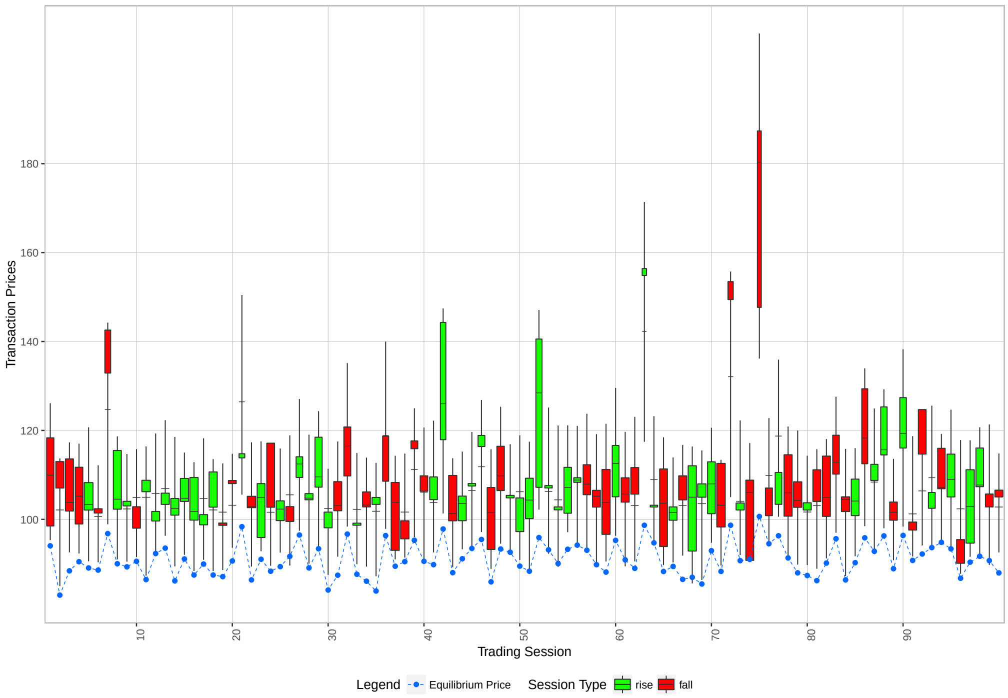


Figure 23: Bid Price in TS1: High Waiting Cost ($c_i \sim N(100, 25)$ and $\beta = 100$)

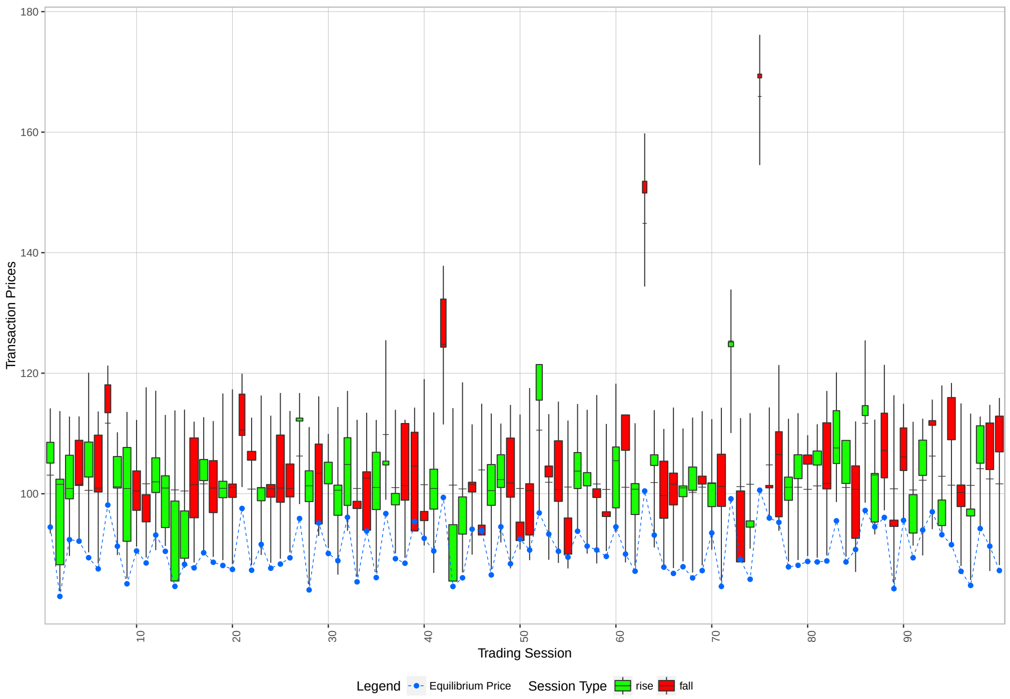


Figure 24: Bid Price in TS1: High Penalty Cost Variation ($c_i \sim N(100, 100)$ and $\beta = 100$)

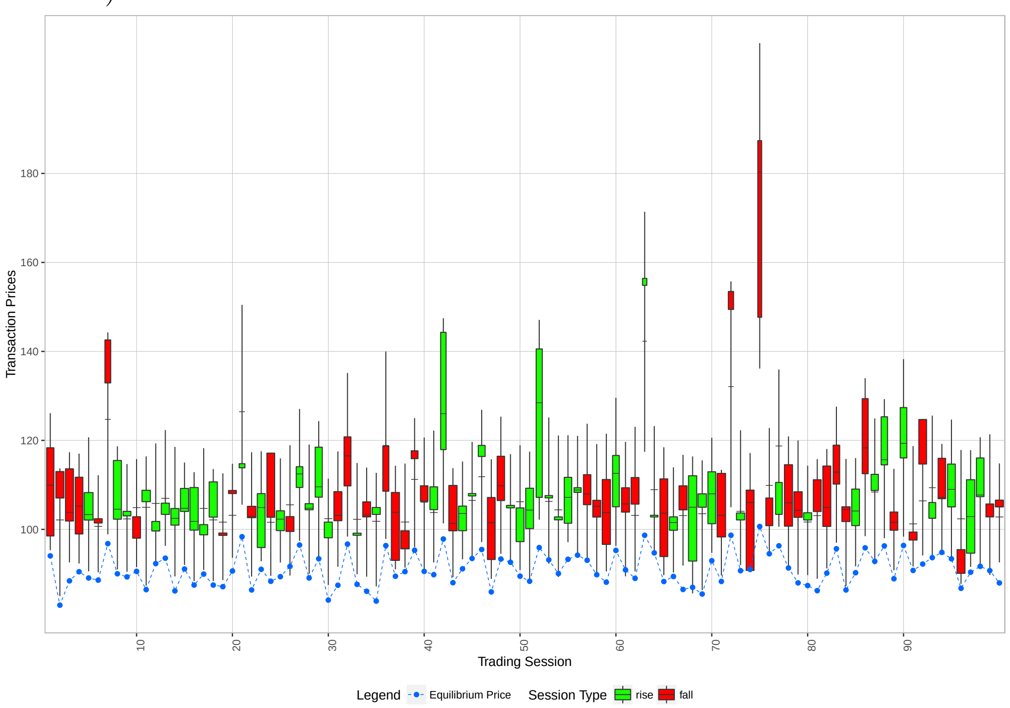
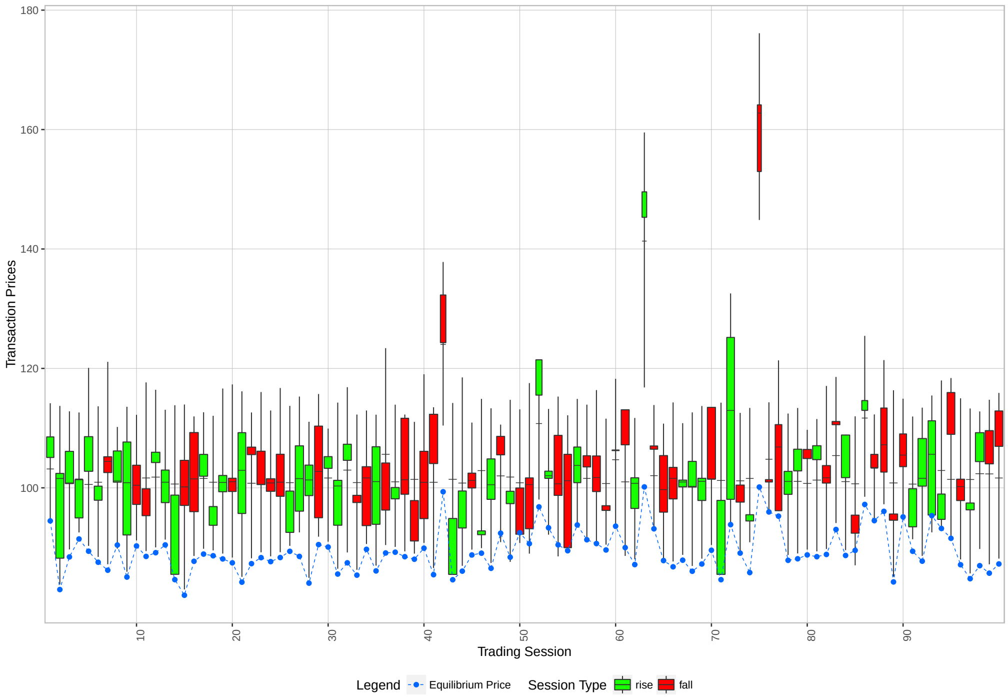


Figure 25: Bid Price in TS3 ($c_i \sim N(100, 25)$ and $\beta = 100$)



9.2 Comparison of Jump Size

Figure 26: Jump Size in TS1 (Base Case)

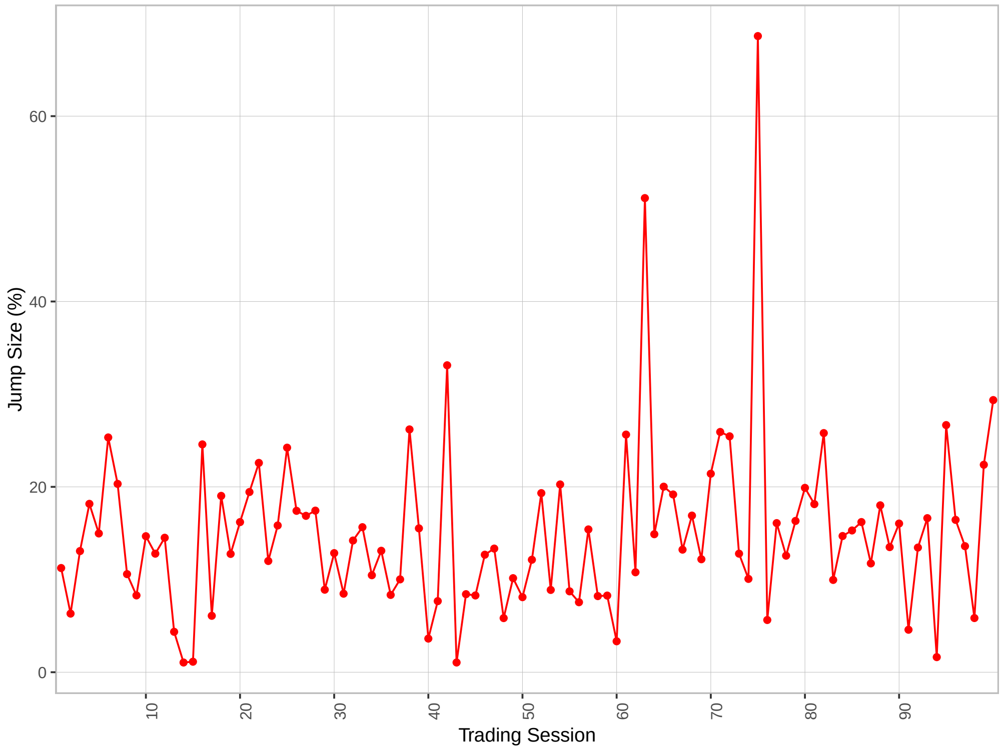


Figure 27: Jump Size in TS1: Low Search Cost

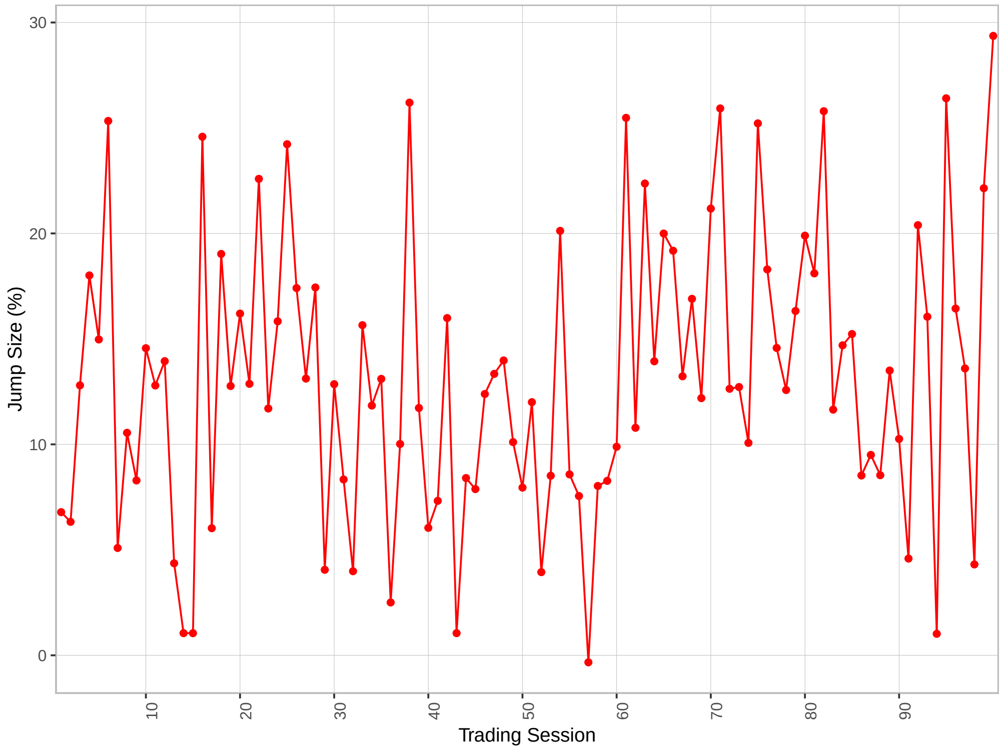


Figure 28: Jump Size in TS1: High Concentration

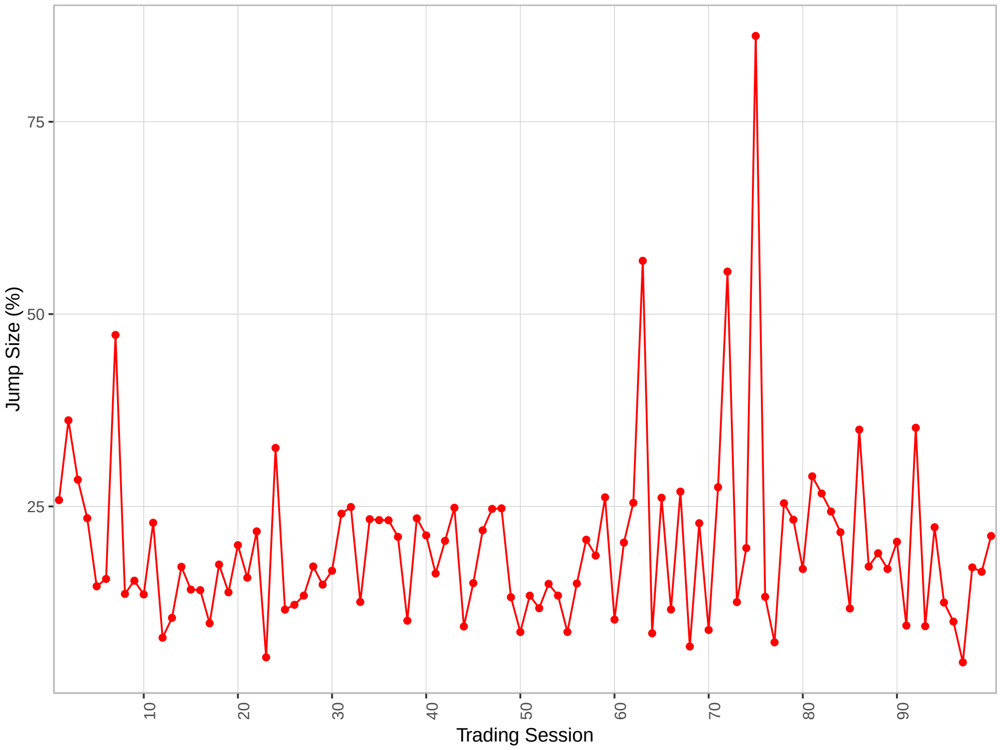


Figure 29: Jump Size in TS1: High Waiting Cost

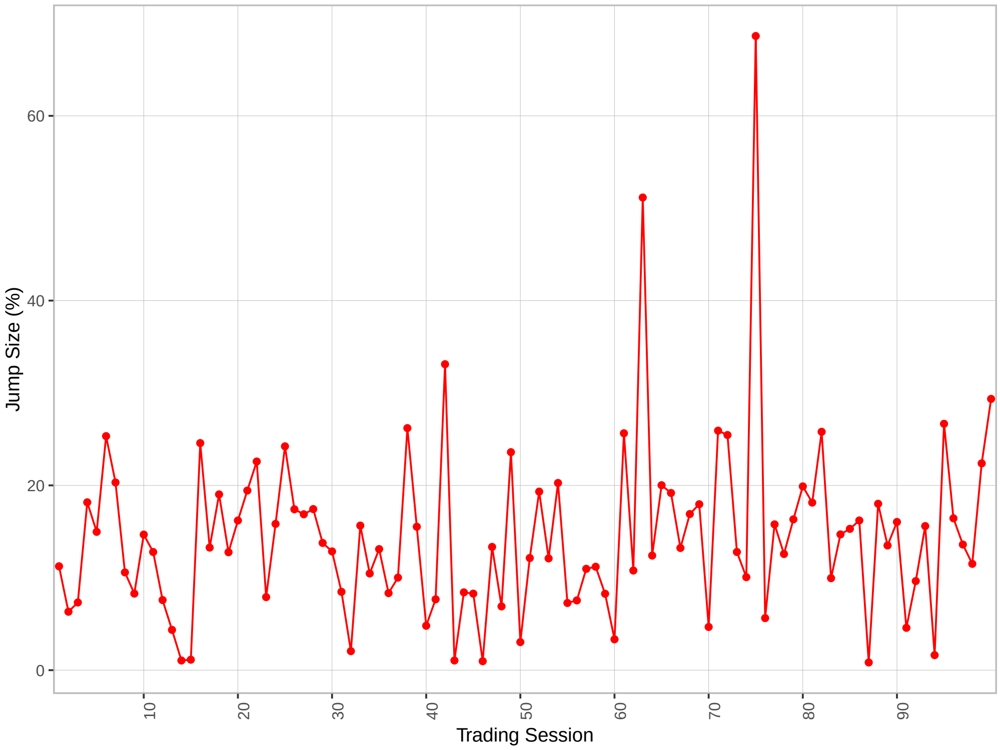


Figure 30: Jump Size in TS1: High Penalty Cost Variation

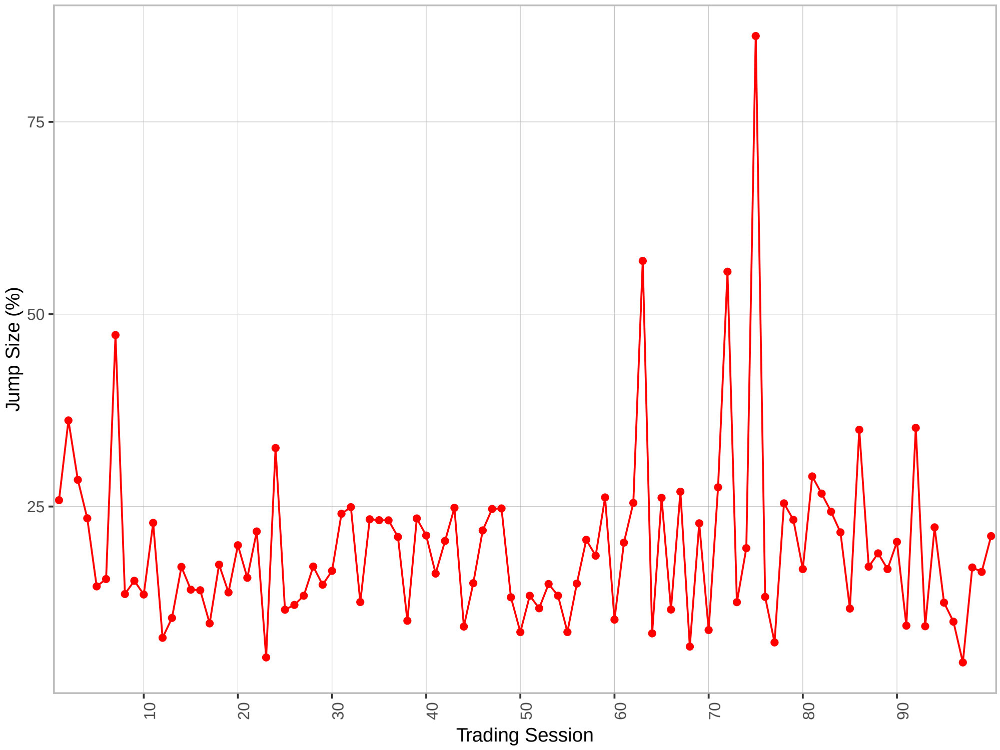
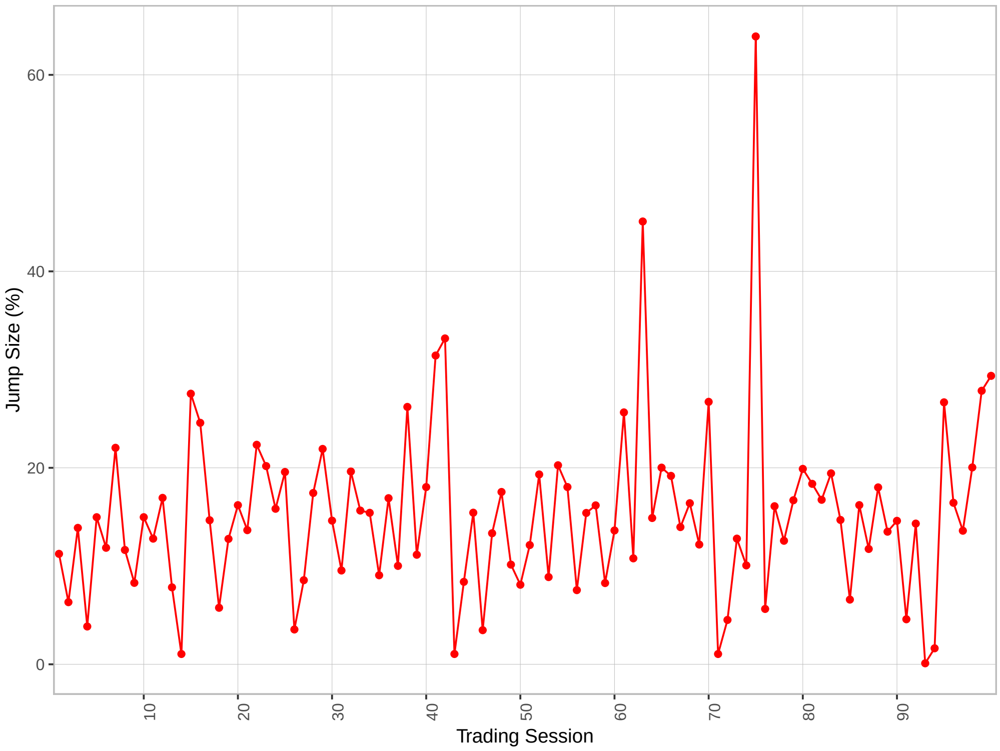


Figure 31: Jump Size in TS3



9.3 Comparison of the Number of Encounters and Transactions

Figure 32: Number of Encounters and Transactions in TS1 (Base Case)

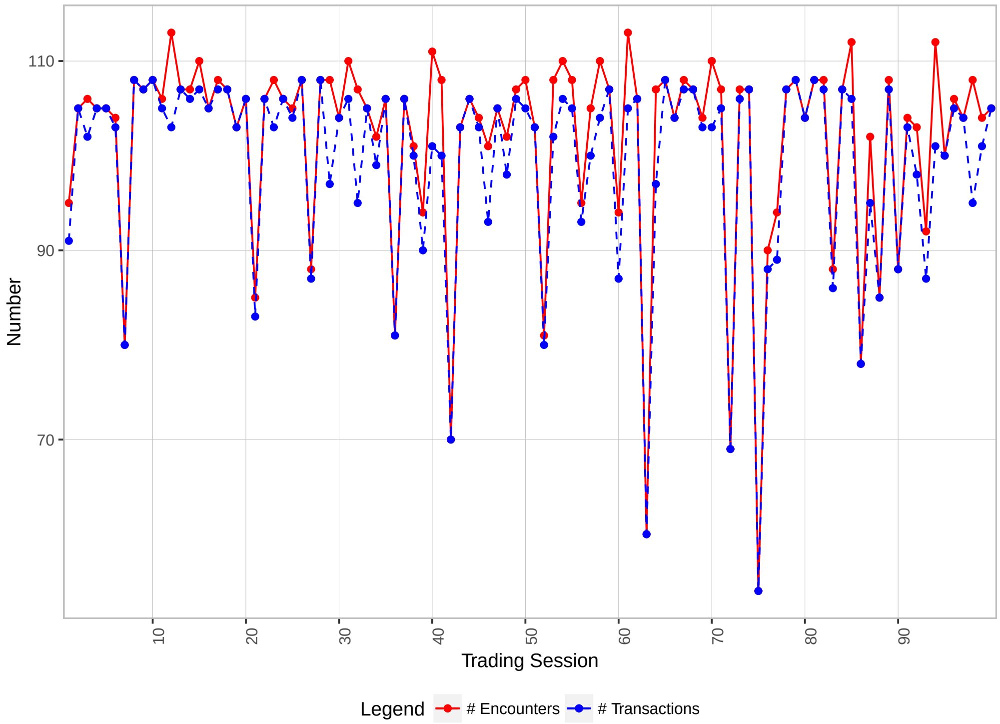


Figure 33: Number of Encounters and Transactions in TS1: Low Search Cost

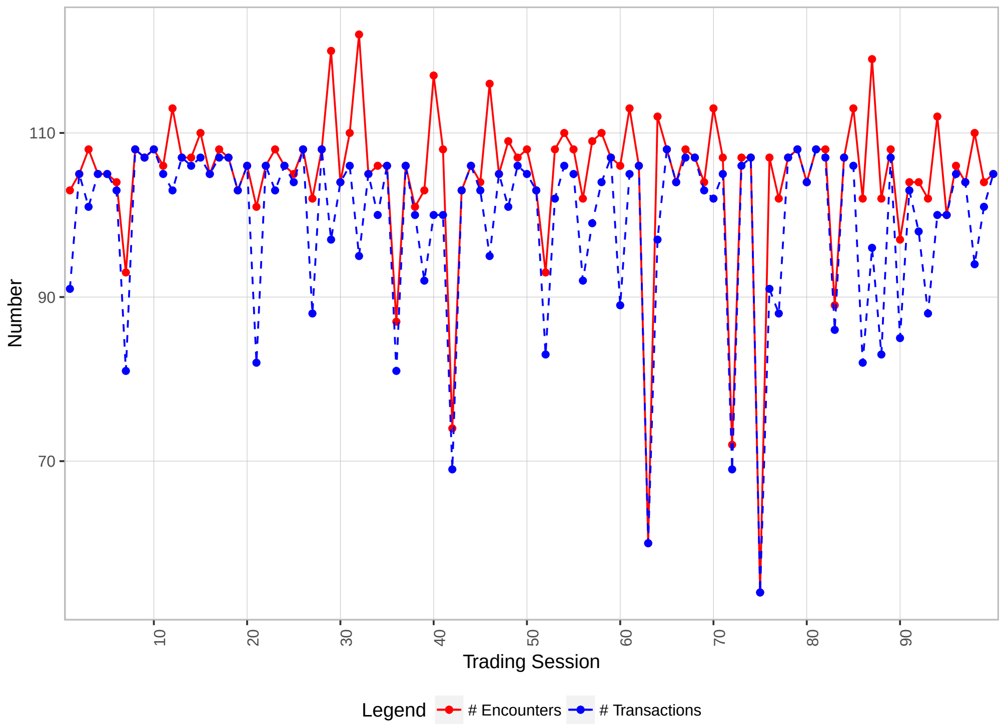


Figure 34: Number of Encounters and Transactions in TS1: High Concentration

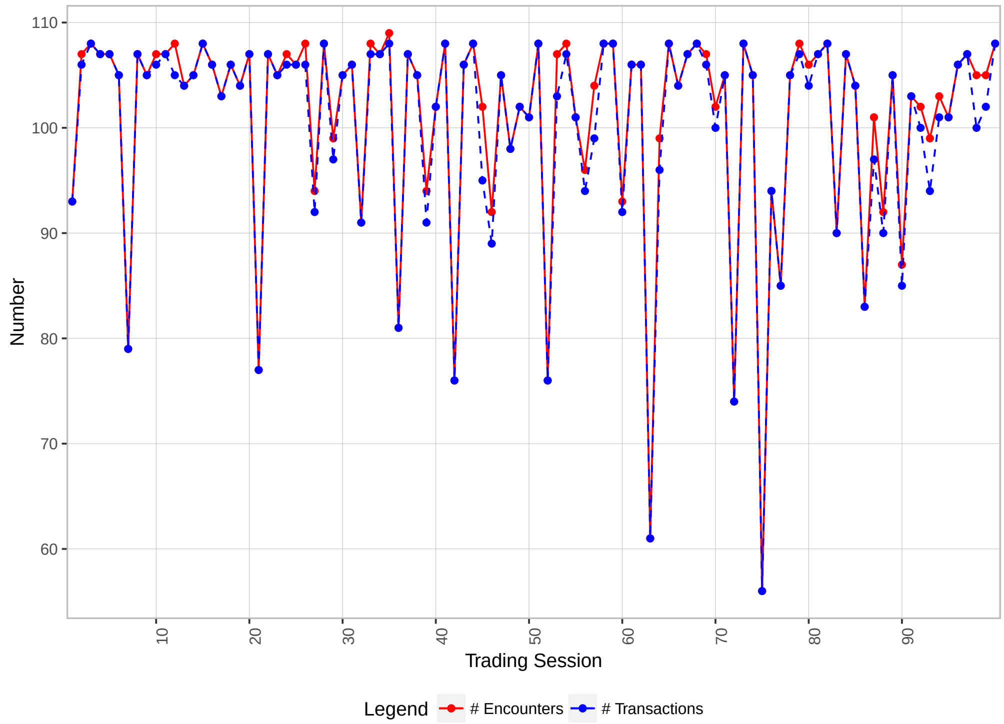


Figure 35: Number of Encounters and Transactions in TS1: High Waiting Cost

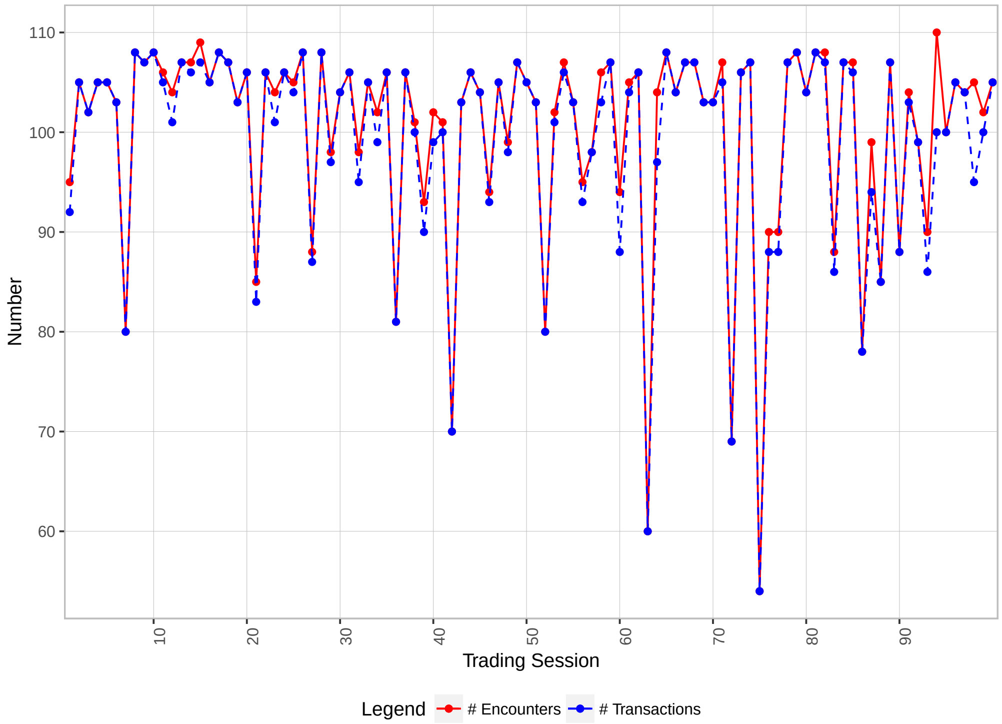


Figure 36: Number of Encounters and Transactions in TS1: High Penalty Cost Variation

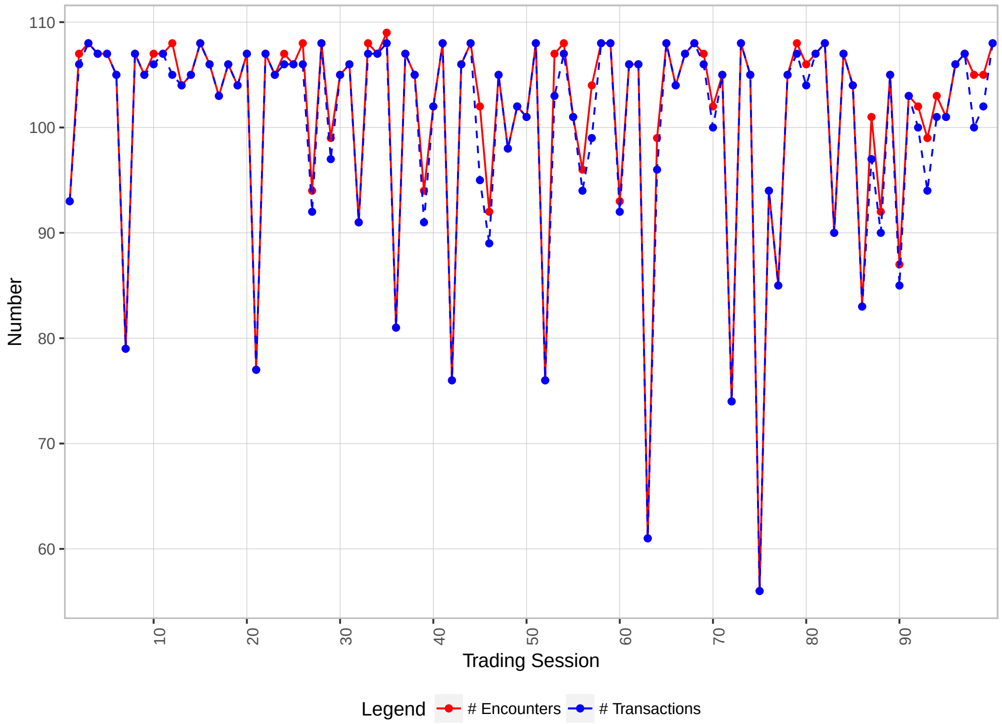


Figure 37: Number of Encounters and Transactions in TS3