PROJECT-2
MECH 5390/6390/6396, Fundamentals of Finite Element Method, Spring 2016

Due Date: April 29, 2016

(Professor Lall)

INSTRUCTIONS

1. Develop the software code in steps indicated below. Document your software code with comment statements where necessary.

2. Use of MATLAB™ is suggested but not required. Use in-built function in MATLAB™ where convenient. The project can be completed in any programming language.

3. Write a report with documentation and results for each step. Submit your report as a WORD (.doc) file. Include your (a) software code (with extensive and detailed documentation in the WORD file) and (b) computation results.

4. Submit your work via CANVAS. Turn-in a hard-copy of your report in class.

QUESTION-1

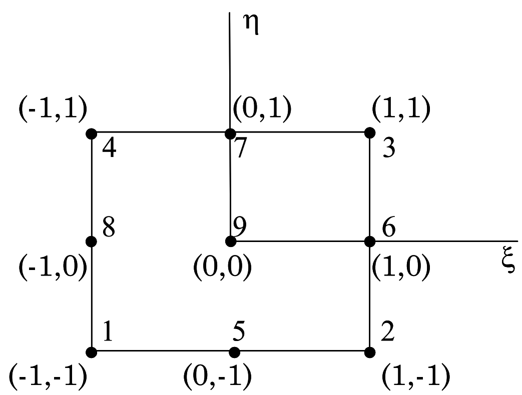
Write a computer program that evaluates the element stiffness matrix for a 9 node quadrilateral plane stress element (Q9 Element).

Submit the final program and the results required in step 11 (see below).

PHYSICAL AND NATURAL COORDINATES

1. Write software code that prompts you to enter the x and y coordinates of the nine nodes in physical space. See the discussion on Q9 isoparametric quadratic quadrilateral element on Section 6.4, page 213 of Cook, et.al. [2002] for reference.

   Store the x-coordinates in a one dimensional array, $\{d_x\}$ containing 9 numbers.
   Store the y-coordinates in a one dimensional array, $\{d_y\}$ containing 9 numbers.
   Write code that displays the 9 x-coordinates on the computer screen.
   Write code that displays the 9 y-coordinates on the computer screen.
   Verify that your code operates correctly for a variety of sets of test data.



$$\{d_x\} = \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \\ x_8 \\ x_9 \end{Bmatrix} \qquad \{d_y\} = \begin{Bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \\ y_6 \\ y_7 \\ y_8 \\ y_9 \end{Bmatrix} \qquad \{d_\xi\} = \begin{Bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \\ \xi_4 \\ \xi_5 \\ \xi_6 \\ \xi_7 \\ \xi_8 \\ \xi_9 \end{Bmatrix} \qquad \{d_\eta\} = \begin{Bmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \\ \eta_4 \\ \eta_5 \\ \eta_6 \\ \eta_7 \\ \eta_8 \\ \eta_9 \end{Bmatrix}$$

SHAPE FUNCTIONS

2. Write code that given the values of any two natural coordinates, ξ and η, returns a one dimensional array of shape functions, $\{N\}$, containing 9 numbers. The shape functions for the Q9 element are given below,

$$N_1 = \frac{1}{4}\xi(1-\xi)\eta(1-\eta) \qquad N_2 = \frac{-1}{4}\xi(1+\xi)\eta(1-\eta)$$

$$N_3 = \frac{1}{4}\xi(1+\xi)\eta(1+\eta) \qquad N_4 = \frac{-1}{4}\xi(1-\xi)\eta(1+\eta)$$

$$N_5 = \frac{-1}{2}(1-\xi^2)\eta(1-\eta) \qquad N_6 = \frac{1}{2}\xi(1+\xi)(1-\eta^2)$$

$$N_7 = \frac{1}{2}(1-\xi^2)\eta(1+\eta) \qquad N_8 = \frac{-1}{2}\xi(1-\xi)(1-\eta^2)$$

$$N_9 = (1-\xi^2)(1-\eta^2)$$

$$\{N\} = \lfloor N_1 \quad N_2 \quad N_3 \quad N_4 \quad N_5 \quad N_6 \quad N_7 \quad N_8 \quad N_9 \rfloor^T$$

3. Write code that given two one-dimensional arrays, each containing 9 numbers, returns the "dot product" product of the two arrays, that is the sum of the products of the corresponding elements of the arrays.

$$\text{dotprdt} = \sum_{k=1}^{9} v_k w_k$$

Verify that your code operates correctly by evaluating $\{N\}$ for each of the natural coordinate pairs (ξ_i, η_i) associated with the nodal points, then taking the "dot product" of the corresponding $\{N\}$ vector

$$\{N\}\Big|_{\substack{\xi=\xi_i \\ \eta=\eta_i}} = \lfloor N_1 \quad N_2 \quad N_3 \quad N_4 \quad N_5 \quad N_6 \quad N_7 \quad N_8 \quad N_9 \rfloor^T$$

$$\{d_\xi\} = \lfloor \xi_1 \quad \xi_2 \quad \xi_3 \quad \xi_4 \quad \xi_5 \quad \xi_6 \quad \xi_7 \quad \xi_8 \quad \xi_9 \rfloor^T$$

with each of the ξ and η coordinate arrays and verifying that it returns the expected coordinate value. For example evaluating N(0,1) and multiplying it by the ξ array should yield ξ_7.

JACOBIAN MATRIX, [J]

4. Write code that given the values of two natural coordinates, ξ and η, returns a one dimensional array, $\{DNDS\}$, containing 9 numbers defined as follows:

$$DNDS_i(\xi, \eta) = \frac{d}{d\xi} N_i(\xi, \eta)$$

$$\{DNDS\} = \frac{d}{d\xi} \lfloor N_1(\xi,\eta) \quad N_2(\xi,\eta) \quad N_3(\xi,\eta) \quad N_4(\xi,\eta) \quad N_5(\xi,\eta) \quad N_6(\xi,\eta) \quad N_7(\xi,\eta) \quad N_8(\xi,\eta) \quad N_9(\xi,\eta) \rfloor^T$$

Write code that given the values of two natural coordinates, ξ and η, returns a one dimensional array, $\{DNDT\}$, containing 9 numbers defined as follows

$$DNDT_i(\xi, \eta) = \frac{d}{d\eta} N_i(\xi, \eta)$$

$$\{DNDT\} = \frac{d}{d\eta}\lfloor N_1(\xi,\eta) \quad N_2(\xi,\eta) \quad N_3(\xi,\eta) \quad N_4(\xi,\eta) \quad N_5(\xi,\eta) \quad N_6(\xi,\eta) \quad N_7(\xi,\eta) \quad N_8(\xi,\eta) \quad N_9(\xi,\eta)\rfloor^T$$

Verify that your code operates correctly by comparing the results of your code to hand calculations for a number of simple values of ξ and η.

5. Write code that given the values of two natural coordinates, ξ and η, and values of two one-dimensional arrays of x and y physical coordinates, $\{d_x\}$ and $\{d_y\}$, each containing 9 numbers, uses the code written above and returns the determinant of the Jacobian, J, defined as,

$$J = \frac{\partial x}{\partial \xi}\cdot\frac{\partial y}{\partial \eta} - \frac{\partial x}{\partial \eta}\cdot\frac{\partial y}{\partial \xi}$$

$$\frac{\partial x}{\partial \xi} = \text{dotprdt}[\{DNDS\},\{d_x\}]$$

$$\frac{\partial x}{\partial \eta} = \text{dotprdt}[\{DNDT\},\{d_x\}]$$

$$\frac{\partial y}{\partial \xi} = \text{dotprdt}[\{DNDS\},\{d_y\}]$$

$$\frac{\partial y}{\partial \eta} = \text{dotprdt}[\{DNDT\},\{d_y\}]$$

Verify your code operates correctly by comparing the results of your code to hand calculations for a number of simple values of ξ and η.

INVERSE OF JACOBIAN MATRIX, $[J]^{-1}$

6. Write code that given the values of two natural coordinates, ξ and η, and values of two one-dimensional arrays of x and y physical coordinates, $\{d_x\}$ and $\{d_y\}$, each containing 9 numbers, uses the code written above and returns the following four derivatives.

$$\frac{\partial \xi}{\partial x} = \frac{1}{|J|}\cdot\frac{\partial y}{\partial \eta} = \frac{1}{|J|}.\text{dotprdt}[\{DNDT\},\{d_y\}]$$

$$\frac{\partial \eta}{\partial x} = -\frac{1}{|J|}\cdot\frac{\partial y}{\partial \xi} = -\frac{1}{|J|}.\text{dotprdt}[\{DNDS\},\{d_y\}]$$

$$\frac{\partial \xi}{\partial y} = -\frac{1}{|J|}\cdot\frac{\partial x}{\partial \eta} = -\frac{1}{|J|}.\text{dotprdt}[\{DNDT\},\{d_x\}]$$

$$\frac{\partial \eta}{\partial y} = \frac{1}{|J|}\cdot\frac{\partial x}{\partial \xi} = \frac{1}{|J|}.\text{dotprdt}[\{DNDS\},\{d_x\}]$$

Since

$$[J]=\begin{pmatrix} x_{,\xi} & y_{,\xi} \\ x_{,\eta} & y_{,\eta} \end{pmatrix}=\begin{pmatrix} \sum N_{i,\xi}x_i & \sum N_{i,\xi}y_i \\ \sum N_{i,\eta}x_i & \sum N_{i,\eta}y_i \end{pmatrix}=\begin{pmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{pmatrix}$$

$$[\Gamma]=\begin{pmatrix} \xi_{,x} & \eta_{,x} \\ \xi_{,y} & \eta_{,y} \end{pmatrix}=[J]^{-1}=\frac{1}{|J|}\begin{pmatrix} J_{22} & -J_{12} \\ -J_{21} & J_{11} \end{pmatrix}=\frac{1}{|J|}\begin{pmatrix} y_{,\eta} & -y_{,\xi} \\ -x_{,\eta} & x_{,\xi} \end{pmatrix}$$

Verify your code operates correctly by comparing the results of your code to hand calculations for a number of simple values of ξ and η.

STRAIN-DISPLACEMENT MATRIX, [B]

7. Write code that given the values of two natural coordinates, ξ and η, and values of the two one-dimensional arrays of x and y physical coordinates, $\{d_x\}$ and $\{d_y\}$, each containing 9 numbers, uses the code written above and returns the two-dimensional array, B (i.e. strain-displacement matrix), containing three rows of 18 numbers each defined as follows,

$$\begin{Bmatrix} u_{,x} \\ u_{,y} \end{Bmatrix}=\frac{1}{|J|}\begin{pmatrix} J_{22} & -J_{12} \\ -J_{21} & J_{11} \end{pmatrix}\begin{Bmatrix} u_{,\xi} \\ u_{,\eta} \end{Bmatrix} \qquad \begin{Bmatrix} v_{,x} \\ v_{,y} \end{Bmatrix}=\frac{1}{|J|}\begin{pmatrix} J_{22} & -J_{12} \\ -J_{21} & J_{11} \end{pmatrix}\begin{Bmatrix} v_{,\xi} \\ v_{,\eta} \end{Bmatrix}$$

thus,

$$\{\varepsilon\}=\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix}=\begin{Bmatrix} u_{,x} \\ v_{,y} \\ u_{,y}+v_{,x} \end{Bmatrix}=\frac{1}{|J|}\begin{pmatrix} J_{22} & -J_{12} & 0 & 0 \\ 0 & 0 & -J_{21} & J_{11} \\ -J_{21} & J_{11} & J_{22} & -J_{12} \end{pmatrix}\begin{Bmatrix} u_{,\xi} \\ u_{,\eta} \\ v_{,\xi} \\ v_{,\eta} \end{Bmatrix}$$

$$\{\varepsilon\}=[A]\begin{Bmatrix} u_{,\xi} \\ u_{,\eta} \\ v_{,\xi} \\ v_{,\eta} \end{Bmatrix} \qquad \text{where, } [A]=\frac{1}{|J|}\begin{pmatrix} J_{22} & -J_{12} & 0 & 0 \\ 0 & 0 & -J_{21} & J_{11} \\ -J_{21} & J_{11} & J_{22} & -J_{12} \end{pmatrix}$$

The gradient vector can be expressed as follows,

$$\begin{Bmatrix} u_{,\xi} \\ u_{,\eta} \\ v_{,\xi} \\ v_{,\eta} \end{Bmatrix}=[G]\{D\}$$

where,

$$\{D\}=\lfloor u_1 \quad v_1 \quad u_2 \quad v_2 \quad u_3 \quad v_3 \quad u_4 \quad v_4 \quad u_5 \quad v_5 \quad u_6 \quad v_6 \quad u_7 \quad v_7 \quad u_8 \quad v_8 \quad u_9 \quad v_9 \rfloor^T$$

and

$$[G]=\begin{pmatrix} N_{1,\xi} & 0 & N_{2,\xi} & 0 & N_{3,\xi} & 0 & N_{4,\xi} & 0 & N_{5,\xi} & 0 & N_{6,\xi} & 0 & N_{7,\xi} & 0 & N_{8,\xi} & 0 & N_{9,\xi} & 0 \\ N_{1,\eta} & 0 & N_{2,\eta} & 0 & N_{3,\eta} & 0 & N_{4,\eta} & 0 & N_{5,\eta} & 0 & N_{6,\eta} & 0 & N_{7,\eta} & 0 & N_{8,\eta} & 0 & N_{9,\eta} & 0 \\ 0 & N_{1,\xi} & 0 & N_{2,\xi} & 0 & N_{3,\xi} & 0 & N_{4,\xi} & 0 & N_{5,\xi} & 0 & N_{6,\xi} & 0 & N_{7,\xi} & 0 & N_{8,\xi} & 0 & N_{9,\xi} \\ 0 & N_{1,\eta} & 0 & N_{2,\eta} & 0 & N_{3,\eta} & 0 & N_{4,\eta} & 0 & N_{5,\eta} & 0 & N_{6,\eta} & 0 & N_{7,\eta} & 0 & N_{8,\eta} & 0 & N_{9,\eta} \end{pmatrix}$$

The strain displacement matrix is,

$$[B]=[A][G]$$

Verify your code operates correctly by comparing the results of your code to hand calculations for a number of simple values of s and t.

CONSTITUTIVE MATRIX

8. Write code that prompts the user to enter values for the modulus of elasticity, E, and Poisson's ratio, ν, and returns the two-dimensional constitutive matrix, [E], that relates stress to strain for an isotropic material in a state of plane stress. Verify your code operates correctly by comparing the results of your code to hand calculations for a number of simple values of E and ν.

$$[E]=\frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{(1-\nu)}{2} \end{bmatrix}$$

STIFFNESS MATRIX INTEGRAND

9. Write code that given the values of two natural coordinates, ξ and η, values of two one-dimensional arrays, $\{d_x\}$ and $\{d_y\}$, each containing 9 numbers, and values for the modulus of elasticity, E, and the Poisson's ratio, ν, uses the code written above and returns the two-dimensional stiffness matrix <u>integrand</u> $[k_t]$, containing 18 rows of 18 numbers each, defined as:

$$[k_t]=[B]^T[E][B]\,|J|$$

where

$$[k]=\iint [B]^T[E]\,[B]\,t\;dx\;dy=\int_{-1}^{1}\int_{-1}^{1}[B]^T[E]\,[B]\,t\;|J|\;d\xi\;d\eta=\int_{-1}^{1}\int_{-1}^{1}[k_t]\,t\;\;d\xi\;d\eta$$

STIFFNESS MATRIX BY NUMERICAL INTEGRATION

10. Write code that given the values of two one-dimensional arrays, $\{d_x\}$ and $\{d_y\}$, each containing 9 numbers, and values for the modulus of elasticity, E, a thickness, "t", and the Poisson's ratio, ν, uses the code written above and returns the two-dimensional element stiffness matrix, $[k_e]$, containing 18 rows of 18 numbers each, using a 25-point Gauss quadrature using the point values and weights given below,

$$I=\int_{-1}^{1}\int_{-1}^{1}\phi(\xi,\,\eta)\;d\xi\;d\eta$$

$$\Rightarrow I=\int_{-1}^{1}\left(\sum_i W_i\;\phi(\xi_i,\,\eta)\right)d\eta$$

$$\Rightarrow I=\sum_j\sum_i W_i\;\phi(\xi_i,\,\eta_j)$$

$$[k_e] = t.\sum_{i=1}^{5}\sum_{j=1}^{5} w_i.w_j.k_t(\xi_i, \eta_j)$$

where, the weights are,

$$w_1 = w_2 = 0.2369268851$$
$$w_3 = w_4 = 0.4786286705$$
$$w_5 = 0.5688888889$$

and integration point locations are,

$$\xi_1 = -\xi_2 = 0.9061798459 \qquad \eta_1 = -\eta_2 = 0.9061798459$$
$$\xi_3 = -\xi_4 = 0.5384693101 \qquad \eta_3 = -\eta_4 = 0.5384693101$$
$$\xi_5 = 0 \qquad \eta_5 = 0$$

VERIFICATION OF SOFTWARE CODE

11. Use your program to evaluate the element stiffness matrix for a nine node element plane stress quadrilateral element (Q9) defined by

$$E = 10e6 \text{ psi}$$
$$\nu = 0.33$$
$$t = 1.65 \text{ in}$$

$$(x_1, y_1) = (-3\text{in}, -5 \text{ in})$$
$$(x_2, y_2) = (3\text{in}, -5 \text{ in})$$
$$(x_3, y_3) = (4\text{in}, 5.5 \text{ in})$$
$$(x_4, y_4) = (-4\text{in}, 2.5 \text{ in})$$
$$(x_5, y_5) = (0\text{in}, -4.5 \text{ in})$$
$$(x_6, y_6) = (3.5\text{in}, 0 \text{ in})$$
$$(x_7, y_7) = (0\text{in}, 4 \text{ in})$$
$$(x_8, y_8) = (-3.5\text{in}, 0 \text{ in})$$
$$(x_9, y_9) = (0\text{in}, 0 \text{ in})$$

Attach the results in your report.

<u>QUESTION-2</u>

The following linkage system is used to lift a 3000 lb cylindrical object vertically in the clamp at the bottom. The linkage-system with the dimensions below does not work. Using ANSYS Workbench Finite Element Analysis <u>design and analyze</u> the linkage system with a factor-of-Safety of 3.5.