# Problem 1

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 4 \\ 1 \end{bmatrix} u \tag{1}$$

$$y = \begin{bmatrix} 1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{2}$$

Equations 1 and 2 can be represented as the following block diagram.

# Problem 2

Equations initially given with the problem (manipulated for use later):

$$\begin{aligned} ml^2\ddot{\theta} &= mgl\sin\theta - b\dot{\theta} + T \\ \ddot{\theta} &= \frac{g\sin\theta}{l} - \frac{b\dot{\theta}}{ml^2} + \frac{T}{ml^2} \end{aligned} \tag{3}$$

$$T = sat(u) \tag{4}$$

$$y = \theta \tag{5}$$

**Part A**–when $\theta = 0$

$$y = \delta\theta_1 \tag{6}$$

$$\delta\dot{\theta} = \frac{g\cos\theta}{l}\Big|_{\theta=0} * \delta\theta_1 - \frac{b\delta\theta_2}{ml^2} + \frac{T}{ml^2} \tag{7}$$

$$\delta\dot{\theta} = \frac{g\delta\theta_1}{l} - \frac{b\delta\theta_2}{ml^2} + \frac{T}{ml^2} \tag{8}$$

Converting equation 8 to linear state space form allows us to observe the system as a whole easily.

$$\begin{aligned} \delta\dot{\theta} &= \begin{bmatrix} 0 & 1 \\ \frac{g}{l} & -\frac{b}{ml^2} \end{bmatrix} \delta\theta + \begin{bmatrix} 0 \\ \frac{1}{ml^2} \end{bmatrix} T \\ y &= \begin{bmatrix} 1 & 0 \end{bmatrix} \delta\theta \end{aligned} \tag{9}$$

From equation 9 you can define the matrices $A, B, C$ as:

$$A = \begin{bmatrix} 0 & 1 \\ \frac{g}{l} & -\frac{b}{ml^2} \end{bmatrix},\ B = \begin{bmatrix} 0 \\ \frac{1}{ml^2} \end{bmatrix},\ C = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

**Part B**–when $\theta = \pi$

$$\frac{g}{l}\cos\theta\Big|_{\theta=\pi} = -\frac{g}{l} \tag{10}$$

$$\delta\dot{\theta} = -\frac{g}{l}\delta\theta_1 - \frac{b\delta\theta_2}{ml^2} + \frac{T}{ml^2} \tag{11}$$

$$\begin{aligned} \delta\dot{\theta} &= \begin{bmatrix} 0 & 1 \\ -\frac{g}{l} & -\frac{b}{ml^2} \end{bmatrix} \delta\theta + \begin{bmatrix} 0 \\ \frac{1}{ml^2} \end{bmatrix} T \\ y &= \begin{bmatrix} 1 & 0 \end{bmatrix} \delta\theta \end{aligned} \tag{12}$$

From equation 12 you can define the matrices $A, B, C$ as:

$$A = \begin{bmatrix} 0 & 1 \\ -\frac{g}{l} & -\frac{b}{ml^2} \end{bmatrix},\ B = \begin{bmatrix} 0 \\ \frac{1}{ml^2} \end{bmatrix},\ C = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

# Problem 3

$$A_1 = \begin{bmatrix} 3 & 6 \\ 1 & 4 \end{bmatrix},\ A_2 = \begin{bmatrix} 0 & 1 & -1 \\ 1 & 0 & -1 \\ 1 & 1 & -2 \end{bmatrix}$$

The definition of the eigenvalues are:

$$\det(A - \lambda I) = 0 \tag{13}$$

Using equation 13 to solve Matrix $A_1$ we get:

$$\begin{gathered} \det\left(\begin{bmatrix} 3 & 6 \\ 1 & 4 \end{bmatrix} - \lambda I\right) = 0 \\ \begin{vmatrix} 3-\lambda & 6 \\ 1 & 4-\lambda \end{vmatrix} = 0 \end{gathered} \tag{14}$$

The solution to a 2-by-2 determinate is expressed as:

$$ad - bc$$

A polynomial can be formed from the determinate in equation 14.

$$(3-\lambda)(4-\lambda) - 6 = 0 \tag{15}$$

Solving equation 15 will give us our eigenvalues $\lambda_1, \lambda_2$ .

$$\begin{gathered} \lambda^2 - 7\lambda + 6 = 0 \\ (\lambda - 1)(\lambda - 6) = 0 \\ \lambda_1 = 1, \lambda_2 = 6 \end{gathered} \tag{16}$$

The same approach can be used for $A_2$, since that's the case detail for each step will not be given.

$$\begin{aligned} \det\left(\begin{bmatrix} 0 & 1 & -1 \\ 1 & 0 & -1 \\ 1 & 1 & -2 \end{bmatrix} - \lambda I\right) &= 0 \\ \begin{vmatrix} -\lambda & 1 & -1 \\ 1 & -\lambda & -1 \\ 1 & 1 & -2-\lambda \end{vmatrix} &= 0 \\ -\lambda \begin{vmatrix} -\lambda & -1 \\ 1 & -2-\lambda \end{vmatrix} - 1 \begin{vmatrix} 1 & -1 \\ 1 & -2-\lambda \end{vmatrix} - 1 \begin{vmatrix} 1 & -\lambda \\ 1 & 1 \end{vmatrix} &= 0 \\ -\lambda(2\lambda + \lambda^2 + 1) - 1(1+\lambda) - 1(-2-\lambda+1) &= 0 \\ -\lambda^3 - 2\lambda^2 - \lambda &= 0 \\ \lambda(\lambda^2 + 2\lambda &= 0 \\ \lambda(\lambda+1)^2 &= 0 \\ \lambda_1 = -1, \lambda_2 = -1, \lambda_3 &= 0 \end{aligned} \tag{17}$$

The eigenvalues of $A_1$ are $\lambda_1 = 1, \lambda_2 = 6$ and the eigenvalues of $A_2$ are $\lambda_1 = -1, \lambda_2 = -1, \lambda_3 = 0$

# Problem 4

The voltage V can be described as the voltage across each element summed together.

$$V = V_R + V_L \tag{18}$$

$$V_R = IR \tag{19}$$

$$V_L = L\frac{dI}{dt} \tag{20}$$

Substituting equations 19 and 20 into equation 18 results in the differential equation that governs the system.

$$\begin{aligned} V &= IR + L\frac{dI}{dt} \\ \frac{dI}{dt} &= -\frac{IR}{L} + \frac{V}{L} \end{aligned} \tag{21}$$

Utilizing the relationship between current and charge you can use it to augment equation 21; that relationship is:

$$I = \frac{dq}{dt} \tag{22}$$

Choosing proper state variables $z_1 = q, z_2 = \dot{q}$ yields the following first order differential equations

$$\begin{aligned} \dot{z}_1 &= z_2 \\ \dot{z}_2 &= -\frac{Rz_2}{L} + \frac{V}{L} \end{aligned} \tag{23}$$

This set of state space equations can be expressed in matrix form as:

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & -\frac{R}{L} \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{V}{L} \end{bmatrix} \tag{24}$$

Using equation 24 and 22 it is possible to write the resulting system in linear state space form.

$$\begin{aligned} \dot{x}(t) &= \begin{bmatrix} 0 & 1 \\ 0 & -\frac{R}{L} \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ \frac{1}{L} \end{bmatrix} u(t) \\ y(t) &= \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 0 \end{bmatrix} u(t) \end{aligned} \tag{25}$$

From equation 25 you can define the matrices $A, B, C, D$ as:

$$A = \begin{bmatrix} 0 & 1 \\ 0 & -\frac{R}{L} \end{bmatrix},\ B = \begin{bmatrix} 0 \\ \frac{1}{L} \end{bmatrix},\ C = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix},\ D = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$