1 An Example (August 6)

1.1 Recap

Recall from last lecture the "isoperimetric problem"

$$\text{minimize } F[u] := \int_{-a}^{a} u(x)\,dx$$
$$\text{subject to } G[u] := \int_{-a}^{a} \sqrt{1+u'(x)^2}\,dx = \ell$$
$$u \in \mathcal{A} := \{u : [-a,a] \to \mathbb{R} : u \in C^1([a,b]), u(-a) = u(a) = 0\}.$$

We derived the following Euler-Lagrange equation for a minimizer:

$$\lambda^2 \frac{u'(x)^2}{1+u'(x)^2} = (c_1 - x)^2. \tag{*}$$

We claim that any solution to (*) must satisfy

$$(x-c_1)^2 + (u(x)-c_2)^2 = \lambda^2,$$

which follows from implicit differentiation. Therefore the solution is on a circle of the form $(x - c_1)^2 + (y-c_2)^2 = \lambda^2$. Compare with the isoperimetric inequality, which states that of all the simple closed curves of fixed length, the circle of that circumference encloses the most area.

1.2 Surface of Revolution Example

Among all the curves $y = u(x)$ joining $(0, b)$ and $(a, 0)$ which enclose a region of fixed area S, find the one which minimizes the surfacea rea of the surface of revolution about the x-axis. Our functional is

$$F[u] := \int_0^a u(x)\sqrt{1+u'(x)^2}\,dx,$$

which we wish to minimize under the constraint

$$G[u] := \int_0^a u(x)\,dx = S,$$

where $u \in \mathcal{A} = \{u \in C^1([0,a]) : u(0) = b, u(a) = 0\}$. (We cancel the 2π from the definition of F for simplicity.) The Lagrangians are $L^F(x,z,p) = z\sqrt{1+p^2}$ and $L^G(x,z,p) = z$. The Euler-Lagrange equation is

$$-\frac{d}{dx}(L^F_p + \lambda L^G_p) + (L^F_z + \lambda L^G_z) \equiv 0,$$

with the linear combinations of the Lagrangians evaluated at $(x, u(x), u'(x))$. We obtain

$$-\frac{d}{dx}\left(\frac{u(x)u'(x)}{\sqrt{1+u'(x)^2}} + \lambda 0\right) + \sqrt{1+u'(x)^2} + \lambda = 0.$$

A computation shows that a function of the form $u(x) = \alpha x + \beta$ solves these Euler-Lagrange equations. Plugging in the boundary conditions and solving for α, β in terms of a, b gives

$$\frac{x}{a} + \frac{y}{b} = 1.$$

These notes are somewhat incomplete.