1 Lagrange Multipliers (May 28)

1.1 First Order Necessary Condition for a Local Minimizer Under Equality Constraints

Here is the first order necessary condition for a local minimizer under equality constraints.

Theorem 1.1. *(Lagrange multipliers) Let $f, h_1, \ldots, h_k$ be C^1 functions on some open $\Omega \subseteq \mathbb{R}^n$. Suppose x_0 is a local minimizer of f subject to the constraints $h_1(x), \ldots, h_k(x) = 0$, which is also a regular point of these constraints. Then there are $\lambda_1, \ldots, \lambda_k \in \mathbb{R}$ ("Lagrange multipliers") such that*

$$\nabla f(x_0) + \lambda_1 \nabla h_1(x_0) + \cdots + \lambda_k \nabla h_k(x_0) = 0.$$

Proof. Since x_0 is regular, $T_{x_0}M = \operatorname{span}(\{\nabla h_1(x_0), \ldots, \nabla h_k(x_0)\})^\perp$. By a lemma from last class, $\nabla f(x_0) \in (T_{x_0}M)^\perp$. Therefore $\nabla f(x_0) \in \operatorname{span}(\{\nabla h_1(x_0), \ldots, \nabla h_k(x_0)\})$, since we are dealing with a finite dimensional vector space. We are done. □

1.2 The Box Example

Given a fixed area $A > 0$, how do we construct a box of maximum volume with surface area A? Suppose the volume is $V(x, y, z) = xyz$ and the area is $A(x, y, z) = 2(xy + xz + yz)$. Our problem is stated as a maximization problem, so we have to convert it to a minimization problem. Let $f = -V$. We are therefore dealing with the problem

$$\begin{aligned} \text{minimize } & f(x, y, z) = -xyz \\ \text{subject to } & h(x, y, z) = A(x, y, z) - A = 0, x, y, z \geq 0 \end{aligned}$$

But we don't know how to deal with inequality constraints right now, so we have to make some changes. Note that if any one of x, y, z is zero, then the volume is zero. Therefore the problem we want to consider is really the problem

$$\begin{aligned} \text{minimize } & f(x, y, z) \\ \text{subject to } & h(x, y, z) = 0, x, y, z > 0 \end{aligned}$$

Now, if $\Omega = \{(x, y, z) \in \mathbb{R}^3 : x, y, z > 0\}$, then the above minimization problem may be solved using the first order necessary condition we gave above, for the set Ω is open.

Suppose (x_0, y_0, z_0) is a local minimizer of f subject to the constraint $h(x, y, z) = 0$. This point is regular because we are only considering points whose coordinates are all positive. Then there is a $\lambda \in \mathbb{R}$ such that $\nabla f(x_0, y_0, z_0) + \lambda \nabla h(x_0, y_0, z_0) = 0$. Therefore

$$(-y_0 z_0, -x_0 z_0, -x_0 y_0) + \lambda(2y_0 + 2z_0, 2x_0 + 2z_0, 2x_0 + 2y_0) = (0, 0, 0).$$

Equivalently,

$$
\begin{aligned}
2\lambda(y_0 + z_0) &= y_0 z_0 \\
2\lambda(x_0 + z_0) &= x_0 z_0 \\
2\lambda(x_0 + y_0) &= x_0 y_0
\end{aligned}
$$

Add all of these equations together:

$$
2\lambda(2x_0 + 2y_0 + 2z_0) = x_0 z_0 + x_0 y_0 + y_0 z_0 = \frac{A}{2} > 0
$$

implying that $\lambda > 0$. The first two equations tell us that

$$
\begin{aligned}
2\lambda x_0(y_0 + z_0) &= x_0 y_0 z_0 \\
2\lambda y_0(x_0 + z_0) &= x_0 y_0 z_0.
\end{aligned}
$$

Subtracting these two equations gives $2\lambda(x_0 z_0 - y_0 z_0) = 0$. Cancelling the z_0's gives $2\lambda(x_0 - y_0) = 0$, and since $\lambda > 0$, we have $x_0 = y_0$. Since we could have done the same thing with the other pairs of equations, we get $x_0 = y_0 = z_0$.

Physically, this tells us that in order to maximize the volume of a rectangular solid of fixed area, we must make a cube. Note that we haven't actually solved the maximization problem; we've only figured out what form its solutions must take.

1.3 Second Order Necessary Conditions for a Local Minimizer Under Equality Constraints

Theorem 1.2. *Let $f, h_1, \ldots, h_k$ be C^2 on some open set $\Omega \subseteq \mathbb{R}^n$. Suppose x_0 is a regular point which is a local minimizer of f subject to the constraints. Then*

(i) There are $\lambda_1, \ldots, \lambda_k \in \mathbb{R}$ such that

$$
\nabla f(x_0) + \lambda_1 \nabla h_1(x_0) + \cdots + \lambda_k \nabla h_k(x_0) = 0.
$$

(ii) The "Lagrangian"

$$
L(x_0) = \nabla^2 f(x_0) + \sum \lambda_i \nabla^2 h_i(x_0)
$$

is positive semi-definite on the tangent space $T_{x_0}M$, where $M = h_1^{-1}(\{0\}) \cap \cdots \cap h_k^{-1}(\{0\})$.