1 Sufficient Condition for an Interior Local Minimizer (May 21)

1.1 A Sufficient Condition

Lemma 1.1. *If A is symmetric and positive-definite, then there is an $a > 0$ such that $v^T Av \geq a\|v\|^2$ for all v.*

Proof. There is an orthogonal matrix Q with $Q^T AQ = \operatorname{diag}(\lambda_1, \ldots, \lambda_n)$. If $v = Qw$,

$$\begin{aligned} v^T Av &= (Qw)^T AQw \\ &= w^T (Q^T AQ)w \\ &= \lambda_1 w_1^2 + \cdots + \lambda_n w_n^2 \\ &\geq \min\{\lambda_1, \ldots, \lambda_n\}\|w\|^2 \\ &= \min\{\lambda_1, \ldots, \lambda_n\}\|v\|^2 \qquad \text{since } Q \text{ is orthogonal} \end{aligned}$$

Since A is positive-definite, every eigenvalue is positive and we are done. □

Theorem 1.2. *(Second order sufficient conditions for interior local minimizers) Let f be C^2 on $\Omega \subseteq \mathbb{R}^n$, and let x_0 be an interior point of Ω such that $\nabla f(x_0) = 0$ and $\nabla^2 f(x_0) \succ 0$. Then x_0 is a strict local minimizer of f.*

Proof. The condition $\nabla^2 f(x_0) \succ 0$ implies there is an $a > 0$ such that $v^T \nabla^2 f(x_0)v \geq a \cdot \|v\|^2$ for all v. By Taylor's theorem we have

$$f(x_0 + v) - f(x_0) = \frac{1}{2}v^T \nabla^2 f(x_0)v + o(\|v\|^2) \geq \frac{1}{2}a\|v\|^2 + o(\|v\|^2) = \|v\|^2 \left(\frac{a}{2} + \frac{o(\|v\|^2)}{\|v\|^2}\right).$$

For sufficiently small v the right hand side is positive, so $f(x_0 + v) > f(x_0)$ for all such v. Therefore x_0 is a strict local minimizer of f on Ω. □

1.2 Examples

(i) Consider $f(x, y) = xy$. The gradient is $\nabla f(x, y) = (y, x)$ and the Hessian is

$$\nabla^2 f(x, y) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Suppose we want to minimize f on all of $\Omega = \mathbb{R}^2$. By the FONC, the only candidate for a local minimizer is $(0, 0)$. The Hessian's eigenvalues are ± 1, so it is not positive definite. We conclude by the SONC that the origin is not a local minimizer of f.

(ii) Consider the same function $f(x, y) = xy$ on $\Omega = \{(x, y) \in \mathbb{R}^2, x, y \geq 0\}$. We claim that every point of the boundary of Ω is a local minimizer of f.

Consider $(x, 0)$ with $x > 0$. The feasible directions here are v with $v_2 \geq 0$. The FONC tells us that $\nabla f(x, 0) \cdot v \geq 0$. This dot product is $xv_2 \geq 0$, so $(x, 0)$ satisfies the FONC. Therefore every point on the positive x-axis is a candidate for a local minimizer. As for the SONC, $\nabla f(x, 0) \cdot v = xv_2 = 0$ if and only if $v_2 = 0$. Then $v^T \nabla^2 f(x, 0)v = 0$. Of course, this tells us nothing; we need a sufficient condition that works for boundary points. That's for next lecture.

Or, you could just say that $f = 0$ on the boundary of Ω and is positive on the interior, so every point of the boundary of Ω is a local minimizer (not strict) of f.