1 Examples of Using The Euler-Lagrange Equation (July 30)

1.1 DuBois-Raymond Lemma

The next lemma allows us to relax the restrictions for the Euler-Lagrange equation to hold from twice continuously differentiable to only once continuously differentiable.

Lemma 1.1. *(DuBois-Raymond lemma) Suppose α, β are continuous functions on $[a,b]$ such that*

$$\int_a^b (\alpha(x)v(x) + \beta(x)v'(x))\,dx = 0$$

for all test functions v on $[a,b]$. Then β is C^1, and $\beta' = \alpha$ on $[a,b]$.

Proof. Let $A(x) = \int_a^x \alpha(t)\,dt$ be an antiderivative of α. Since α is continuous, A is C^1. Then

$$\int_a^b \alpha(x)v(x)\,dx = \int_a^b A'(x)v(x)\,dx = -\int_a^b A(x)v'(x)\,dx.$$

By the original assumption,

$$0 = \int_a^b (\alpha(x)v(x) + \beta(x)v'(x))\,dx = \int_a^b (-A(x) + \beta(x))v'(x)\,dx.$$

We are done if we are able to show that $-A(x) + \beta(x)$ is constant on $[a,b]$. Let $\gamma = -A + \beta$. Define C to be the constant

$$C := \frac{1}{b-a}\int_a^b \gamma(t)\,dt,$$

so that $\int_a^b (\gamma(t) - C)\,dt = 0$. Define $v(x) := \int_a^x (\gamma(t) - C)\,dt$. The function v is C^1 since $\gamma(t) - C$ is continuous, and $v(a) = v(b) = 0$; so v is a test function on $[a,b]$. By some algebra,

$$\int_a^b (\gamma(x) - C)^2\,dx = \int_a^b (\gamma(x) - C)v'(x)\,dx = 0.$$

Since $(\gamma(x) - C)^2 \geq 0$ on $[a,b]$, we must have $\gamma(x) = C$. Therefore γ is constant, which proves the lemma. $\square$

1.2 Examples

1. Consider two points $(a,A), (b,B)$ in $\mathbb{R}^2$ with $a < b$. We seek a function u on $[a,b]$ with $u(a) = A$, $u(b) = B$, and with

$$F[u] := \int_a^b \sqrt{1 + u'(x)^2}\,dx$$

minimized. Denote by $\mathcal{A}$ the set of C^1 functions u with $u(a) = A$ and $u(b) = B$. Suppose that u_* is a minimizer. Then, by the previous lemma applied to the last result of the previous lecture, u_* satisfies the Euler-Lagrange equation. Let $L(x, z, p) \coloneqq \sqrt{1 + p^2}$. Then $L_z = 0$, and

$$L_p = \frac{p}{\sqrt{1+p^2}}.$$

The Euler-Lagrange equation is, in this case,

$$0 = -\frac{d}{dx} L_p + L_z = -\frac{d}{dx} \frac{u_*'(x)}{\sqrt{1 + u_*'(x))^2}}. \tag{*}$$

This implies that

$$u_*'(x) = C\sqrt{1 + u_*'(x)^2}$$

for some constant C, implying

$$u_*'(x)^2 = C(1 + u_*'(x)^2) = C + Cu_*'(x)^2,$$

giving

$$u_*'(x)^2(1 - C) = C,$$

hence u_*' is constant, or $u_*(x) = \alpha x + \beta$ for some constants α, β. As expected, the minimizer is a line. This answer is expected, since the shortest path joining two points is the line joining them.

2. Suppose u is a C^1 function on an interval $[a, b]$. Consider the surface of revolution obtained by rotating the graph of u on $[a, b]$ about the x-axis. Consider the functional

   $$F[u] \coloneqq \text{area of the surface of revolution obtained by rotating } \Gamma_u \text{ about the } x\text{-axis},$$

   which is, by some calculus,

   $$F[u] = \int_a^b 2\pi u(x)\sqrt{1 + u'(x)^2}\, dx.$$

   With the set $\mathcal{A}$ of functions defined as in the previous example, we seek to find a function $u_* \in \mathcal{A}$ minimizing F on $\mathcal{A}$.

   In this example, the Lagrangian is $L(x, z, p) = 2\pi z\sqrt{1 + p^2}$, which gives

   $$L_z = 2\pi\sqrt{1+p^2}$$

   and

   $$L_p = \frac{2\pi z p}{\sqrt{1+p^2}}$$

The Euler-Lagrange equation is, in this case,

$$0 = -\frac{d}{dx}L_p + L_z = -\frac{d}{dx}\left[\frac{2\pi u(x)u'(x)}{\sqrt{1+u'(x)^2}} + 2\pi\sqrt{1+u'(x)^2}\right].$$

Cancel the 2π's to get the ODE

$$\left[\frac{u(x)u'(x)}{\sqrt{1+u'(x)^2}} + \sqrt{1+u'(x)^2}\right] = 0. \tag{**}$$

By magic, the general solution to this differential equation has the form

$$u(x) = \beta\cosh\left(\frac{x-\alpha}{\beta}\right)$$

for some constants α, β. We won't argue why we got this solution, but we can differentiate it and check that it solves the ODE; uniqueness theorems give us what we want.

$$u'(x) = \beta\sinh\left(\frac{x-\alpha}{\beta}\right)\frac{1}{\beta} = \sinh\left(\frac{x-\alpha}{\beta}\right),$$

implying

$$\sqrt{1+u'(x)^2} = \cosh\left(\frac{x-\alpha}{\beta}\right).$$

It is now obvious that u solves (**). Therefore a minimizer u_* must be of the form

$$u_*(x) = \beta\cosh\left(\frac{x-\alpha}{\beta}\right).$$

We may use the boundary conditions to find α, β.

Consider the special case $(a, A) = (0, 1)$ and $(b, B) = (1, 0)$. The boundary conditions on u_* give us the system

$$\beta\cosh\left(\frac{x-\alpha}{\beta}\right) = 1$$
$$\beta\cosh\left(\frac{1-\alpha}{\beta}\right) = 0.$$

Since cosh is strictly positive, the second equation gives us $\beta = 0$, a contradiction. We conclude that there is no C^1 minimizer in this special case.