# 1 Constrained Optimization (May 26)

Consider the following minimization problem:

$$\text{minimize } f(x,y) = xy$$
$$\text{subject to } x^2 + y^2 \leq 1$$

Let $\Omega$ be the feasible set. The feasible directions at a point $(x_0, y_0) \in \Omega$ are the $(v, w) \in \mathbb{R}^2$ such that $(v, w) \cdot (x_0, y_0) < 0$, or $vx_0 + wy_0 < 0$. By the FONC for a minimizer, $\nabla f(x_0, y_0) \cdot (v, w) \geq 0$, so $wx_0 + vy_0 \geq 0$. Note that a local minimum must occur on the boundary. (Why?) We have three cases, depending on the sign of $x_0 + y_0$.

(i) $x_0 + y_0 < 0$: can't occur

(ii) $x_0 + y_0 > 0$: can't occur

(iii) $x_0 + y_0 = 0$: good!

(This part could not be finished as attention had to be diverted from the lecture.)

## 1.1 Second Order Necessary Condition for a Local Minimizer

**Theorem 1.1.** *(Second order sufficient condition for a local minimizer) Let $f$ be $C^2$ on $\Omega \subseteq \mathbb{R}^n$ and suppose $x_0 \in \Omega$ satisfies*

*(i) $\nabla f(x_0) \cdot v \geq 0$ for all feasible directions $v$ at $x_0$,*

*(ii) if $\nabla f(x_0) \cdot v = 0$ for some such $v$, then $v^T \nabla^2 f(x_0) v > 0$.*

*Then $x_0$ is a local minimizer of $f$ on $\Omega$.*

## 1.2 Optimization with Equality Constraints

Consider the minimization problem

$$\text{minimize } f(x,y)$$
$$\text{subject to } h(x,y) = x^2 + y^2 - 1 = 0$$

Suppose $(x_0, y_0)$ is a local minimizer. Two cases:

1. $\nabla f(x_0, y_0) \neq 0$: we claim that $\nabla f(x_0, y_0)$ is perpendicular to the tangent space to the unit circle $h^{-1}(\{0\})$ at $(x_0, y_0)$. If this is not the case, then we obtain a contradiction by looking at the level sets of $f$, to which $\nabla f$ is perpendicular. Therefore $\nabla f(x_0, y_0) = \lambda \nabla h(x_0, y_0)$ for some $\lambda$.

2. $\nabla f(x_0, y_0) = 0$: as in the previous case, $\lambda = 0$.

In either case, at a local minimizer, the gradient of the function to be minimized is parallel to the gradient of the constraints.

We now recall some elementary differential geometry.

**Definition 1.** *For us, a surface is the set of common zeroes of a finite set of $C^1$ functions.*

**Definition 2.** *For us, a differentiable curve on the surface $M \subseteq \mathbb{R}^n$ is the image of a $C^1$ function $x : (a, b) \to M$.*

**Definition 3.** *Let $x(s)$ be a differentiable curve on $M$ that passes through $x_0 \in M$ at time $x(0) = x_0$. The velocity vector $v = \frac{d}{ds}\big|_{s=0} x(s)$ of $x(s)$ at $x_0$ is, for us, said to be a tangent vector to the surface $M$ at $x_0$. The set of all tangent vectors to $M$ at $x_0$ is called the tangent space to $M$ at $x_0$ and is denoted by $T_{x_0}M$.*

**Definition 4.** *Let $M = \{x \in \mathbb{R}^n : h_1(x) = \cdots = h_k(x) = 0\}$ be a surface. If $\nabla h_1(x_0), \ldots, \nabla h_k(x_0)$ are all linearly independent, then $x_0$ is said to be a regular point of $M$.*

**Theorem 1.2.** *At a regular point $x_0 \in M$, the tangent space $T_{x_0}M$ is given by*

$$T_{x_0}M = \{y \in \mathbb{R}^n : \nabla \mathbf{h}(x_0)y = 0\}.$$

*Proof.* It's in the book. Use the implicit function theorem. □

**Lemma 1.3.** *Let $f, h_1, \ldots, h_k$ be $C^1$ functions on the open set $\Omega \subseteq \mathbb{R}^n$. Let $x_0 \in M = \{x \in \Omega : h_1(x) = \cdots = h_k(x) = 0\}$. Suppose $x_0$ is a local minimizer of $f$ subject to the constraints $h_i(x) = 0$. Then $\nabla f(x_0)$ is perpendicular to $T_{x_0}M$.*

*Proof.* Without loss of generality, suppose $\Omega = \mathbb{R}^n$. Let $v \in T_{x_0}M$. Then $v = \frac{d}{ds}\big|_{s=0} x(s)$ for some differentiable curve $x(s)$ in $M$ with $x(0) = x_0$. Since $x_0$ is a local minimizer of $f$, 0 is a local minimizer of $f \circ x$, so $\nabla f(x_0) \cdot x'(0) = \nabla f(x_0) \cdot v = 0$. □