1 Necessary Conditions (July 28)

1.1 Necessary Conditions, Brachistochrone Problem

For the brachistochrone problem, we derived the following mathematical formulation:

$$\begin{aligned} &\text{minimize } F[u] := \int_0^b \left(\frac{1 + u'(x)^2}{2g(a - u(x))} \right)^{1/2} dx \\ &\text{subject to } u \in \mathcal{A}, \end{aligned}$$

where $\mathcal{A} = \{v \in C^1([0, b]) : v(0) = a, v(b) = 0\}$. We would like to find necessary conditions for a minimizer.

Suppose u_* is the minimizer of F on $\mathcal{A}$. Fix a test function v on $[0, b]$. Consider the function $u_* + sv$, for $s \in \mathbb{R}$. Clearly $u_* + sv \in \mathcal{A}$, which implies that $F[u_*] \leq F[u_* + sv]$ for all $s \in \mathbb{R}$ and all test functions v on $[0, b]$. Define $f(s) := F[u_* + sv]$; then f is minimized at $s = 0$ and it's C^1, so we have $f'(0) = 0$, or

$$\left.\frac{d}{ds}\right|_{s=0} F[u_* + sv] = 0.$$

We'd like to compute this derivative. This is deferred to a homework problem, which will require a result which we give later today. The answer is that u_* satisfies

$$(1 + u_*'(x)^2)(a - u_*(x))c^2 = 1 \qquad c \text{ is some constant.} \tag{*}$$

This is a differential equation. Note that $a > 0$ and $a > u_*(x)$. We claim (guess) that $u_*(x(t)) = a - l(1 - \cos(t))$. Some computations give

$$u_*'(x(t))x'(t) = \frac{d}{dt} u_*(x(t)) = -k \sin(t),$$

or

$$u_*(x(t)) = -\frac{k \sin(t)}{x'(t)}.$$

Substituting these into (*) gives

$$c^2 k \left(1 + \frac{k^2 \sin^2(t)}{x'(t)^2}\right)(1 - \cos(t)) = 1.$$

Choose c so that $c^2 k = 1/2$. Then

$$\left(1 + \frac{k^2 \sin^2(t)}{x'(t)^2}\right)(1 - \cos(t)) = 2.$$

Expansion of the left side results in

$$\frac{k^2 \sin^2(t)}{x'(t)^2}(1 - \cos(t)) + (1 - \cos(t)) = 2.$$

Move the second $1-\cos(t)$ to the right to get

$$\frac{k^2\sin^2(t)}{x'(t)^2}(1-\cos(t)) = 1+\cos(t).$$

Multiply both sides by $x'(t)^2$ to get

$$k^2\sin^2(t)(1-\cos(t)) = x'(t)^2(1+\cos(t)).$$

Multiply both sides by $1-\cos(t)$ to get

$$k^2\sin^2(t)(1-\cos(t)) = x'(t)^2\sin^2(t),$$

and cancelling the $\sin^2(t)$'s gives

$$k^2(1-\cos(t))^2 = x'(t)^2,$$

and taking square roots gives

$$x'(t) = k(1-\cos(t)).$$

Integration gives

$$x(t) = kt - k\sin(t),$$

where the constant of integration is 0 since $x(0)=0$. Therefore

$$c(t) = \begin{pmatrix} x(t) \\ u_*(x(t)) \end{pmatrix} = \begin{pmatrix} k(t-\sin(t)) \\ a-k(1-\cos(t)) \end{pmatrix}$$

is a candidate for a minimizer. By writing $c(T)=(b,0)$, we can solve to T,k.

1.2 First Order Necessary Conditions

Our general space of functions will be

$$\mathcal{A} := \{u \in C^1([a,b]) : u(a)=A, u(b)=B\},$$

and the functionals to be minimized will be of the form

$$F[u] := \int_a^b L(x, u'(x), u'(x))\,dx$$

for some real-valued function $L = L(x,z,p)$ defined on $[a,b]\times\mathbb{R}^2$. We shall use subscripts to denote partial derivatives.

Definition 1. *Given $u \in \mathcal{A}$, suppose there is a function $g : [a,b] \to \mathbb{R}$ such that*

$$\left.\frac{d}{ds}\right|_{s=0} F[u+sv] = \int_a^b g(x)v(x)\,dx$$

for all test functions v on $[a,b]$. Then g is called the variational derivative of F at u. We denote the function g by $\frac{\delta F}{\delta u}(u)$. (Why is this unique?)

We can think of $\frac{\delta F}{\delta u}(u)$ as an analogue of the gradient. We have

$$\left.\frac{d}{ds}\right|_{s=0} F[u+sv] = \int_a^b \frac{\delta F}{\delta u}(u)(x)v(x)\,dx$$

for all test functions v on $[a,b]$. Compare this with the finite-dimensional formula

$$\left.\frac{d}{ds}\right|_{s=0} f(u+sv) = \nabla f(u)\cdot v = \sum_{i=1}^{n} \nabla f(u)_i v_i;$$

if one thinks of the integral as an "infinite sum of infinitesimally small pieces", then we can understand how the functional derivative might be an "infinite-dimensional" version of the gradient.

We now work towards deriving the desired first order necessary conditions for a minimizer.

Proposition 1. *Suppose $u_* \in \mathcal{A}$ satisfies $u_* + v \in \mathcal{A}$ for all test functions v on $[a,b]$. Then if u_* minimizes F on $\mathcal{A}$ and if $\frac{\delta F}{\delta u}(u_*)$ exists and is continuous, then $\frac{\delta F}{\delta u}(u_*) \equiv 0$.*

Proof. This is an easy application of the fundamental lemma of the calculus of variations. $\square$

We would like to find a formula for the variational derivative.

Theorem 1.1. *Suppose L, u are C^2 functions. Then $\frac{\delta F}{\delta u}(u)$ exists, is continuous, and*

$$\frac{\delta F}{\delta u}(u)(x) = -\frac{d}{dx}L_p(x,u(x),u'(x)) + L_z(x,u(x),u'(x)). \tag{**}$$

Equation (**) is known as the *Euler-Lagrange equation.* Recall that $L = L(x,z,p)$.

Proof. Let v be a test function on $[a,b]$. Then

$$\begin{aligned}
\left.\frac{d}{ds}\right|_{s=0} F[u+sv] &= \left.\frac{d}{ds}\right|_{s=0} \int_a^b L(x,u(x)+sv(x),u'(x)+sv'(x))\,dx \\
&= \int_a^b \left.\frac{d}{ds}\right|_{s=0} L(x,u(x)+sv(x),u'(x)+sv'(x))\,dx \\
&= \int_a^b \left(L_x(\cdots)\frac{dx}{ds} + L_z(\cdots)\frac{d}{ds}(u(x)+sv(x)) + L_p(\cdots)\frac{d}{ds}(u'(x)+sv'(x))\right)\,dx \\
&= \int_a^b \big(L_z(x,u(x),u'(x))v(x) + L_p(x,u(x),u'(x))v'(x)\big)\,dx \\
&= \int_a^b L_z(x,u(x),u'(x))v(x)\,dx + \int_a^b L_p(x,u(x),u'(x))v'(x)\,dx \\
&= \int_a^b \left(-\frac{d}{dx}L_p(x,u(x),u'(x)) + L_z(x,u(x),u'(x))\right)v(x)\,dx \qquad \text{integration by parts.}
\end{aligned}$$

Since u and L are C^2, the function in the integrand is continuous. By the definition of the variational derivative we have the desired result. $\square$