1 Sufficient Conditions and Convexity (August 13)

1.1 Sufficient Convexity Condition on The Lagrangian, No Constraints

Recall that if $u_* \in \mathcal{A} = \{u \in C^1([a,b]) : u(a) = A, u(b) = B\}$ is a minimizer of the functional

$$F[u(\cdot)] \coloneqq \int_a^b L(x, u(x), u'(x))\, dx,$$

Then it satisfies the "primitive" Euler-Lagrange equation

$$\int_a^b \left[L_z(x, u_*(x), u_*'(x))v(x) + L_p(x, u_*(x), u_*'(x))v'(x) \right] dx = 0 \tag{*}$$

for all test functions v on $[a,b]$.

Lemma 1.1. *Assume the conditions above. Suppose $L = L(x,z,p)$ is a C^1 function, and that for each $x \in [a,b]$, $L(x,\cdot,\cdot)$ is convex. If u_* satisfies (*) above, then $F[u_*(\cdot)] \le F[u_*(\cdot) + v(\cdot)]$ for all test functions v on $[a,b]$.*

The lemma states, roughly, that if u_* is a minimizer, then it must be a global minimizer. We will look at an example of the lemma and then prove it.

(See the "model example" in the professor's notes.) Consider $L(x,z,p) = \frac{1}{2}(z^2 + p^2)$. This satisfies the hypotheses of the lemma. Suppose u_* is a minimizer (of the functional defined by integrating $L(x, u(x), u'(x))$). Then u_* satisfies (*). The lemma then states that $F[u_*(\cdot)] \le F[u_*(\cdot) + v(\cdot)]$ for all test functions v on $[a,b]$. But every function $u \in \mathcal{A}$ is of the form $u_* + v$ for some test function v on $[a,b]$, since $u - u_*$ is a test function on $[a,b]$. Thus u_* is a global minimizer.

We now prove the lemma. We can think of the lemma as generalizing this model example.

Proof. Recall the C^1 criterion for convexity of a function $f : \mathbb{R}^n \to \mathbb{R}$: f is convex if and only if $f(a+b) \ge f(a) + \nabla f(a) \cdot b$ for all $a, b \in \mathbb{R}^n$. We will apply this criterion to the convex function $L(x,\cdot,\cdot) : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$. Directly applying the criterion gives

$$L(x, z + \widetilde{z}, p + \widetilde{p}) \ge L(x,z,p) + \underbrace{\nabla_{(z,p)} L(z,z,p) \begin{pmatrix} \widetilde{z} \\ \widetilde{p} \end{pmatrix}}_{= L_z(x,z,p)\widetilde{z} + L_p(x,z,p)\widetilde{p}}.$$

Then

$$\begin{aligned}
F[u_*(\cdot) + v(\cdot)] &= \int_a^b L(x, u_*(x) + v(x), u_*'(x) + v'(x))\,dx \\
&\geq \int_a^b L(x, u_*(x), u_*'(x))\,dx + \underbrace{\int_a^b \Big[L_z(\cdots)v(x) + L_p(\cdots)v'(x) \Big]\,dx}_{=0} \\
&= \int_a^b L(x, u_*(x), u_*'(x))\,dx \\
&= L[u_*(\cdot)].
\end{aligned}$$

□

1.2 Convex Domains

Recall that we were originally looking at convex functions $f : \Omega \to \mathbb{R}$, where f is C^1 and convex and $\Omega \subseteq \mathbb{R}^n$ is convex. We had a theorem:

Theorem 1.2. *Assume the above conditions. x_* is a minimizer of f on Ω if and only if $\nabla f(x_*)(x - x_*) \geq 0$ for all $x \in \Omega$.*

To this theorem there is a variational analogue. (Are we assuming the conditions of the previous subsection? This is unclear.)

Theorem 1.3. *Consider a functional*

$$F[u(\cdot)] \coloneqq \int_a^b L(x, u(x), u'(x))\,dx.$$

Let $\mathcal{A} = \{u \in C^1([a,b]) : u(a) = A, u(b) = B\}$. Let Ω be a convex subset of $\mathcal{A}$, and suppose that u_* is a minimizer of F on Ω. Then $F[u_*(\cdot) + sv(\cdot)] \geq F[u_*(\cdot)]$ for all s and all test functions v on $[a,b]$, and

$$\int_a^b \frac{\delta F}{\delta u}(u_*)(x)(u(x) - u_*(x))\,dx \int_a^b \frac{\delta F}{\delta u}(u_*)(x)v(x)\,dx = \frac{d}{ds}\bigg|_{s=0} F[u_*(\cdot) + sv(\cdot)] \geq 0$$

for all $u \in \Omega$. (See how the above condition parallels to the finite-dimensional case.)

We will not prove this, but we will look at an example. Suppose

$$F[u(\cdot)] = \int_0^1 \frac{1}{2}\left(u'(t)^2 + u(t)\right)\,dt,$$

where $\mathcal{A}$ is the set of C^1 functions on $[0,1]$ that are zero at the endpoints, and $\Omega = \{u \in \mathcal{A} : u(t) \geq \sin^2(\pi t)\}$. One can check that Ω is convex. The Lagrangian function is $L(x,z,p) = \frac{1}{2}(p^2 + z)$. This is a convex function of (z,p), so it satisfies the conditions of the first lemma. Then if u_* minimizes F on Ω, let $f(s) = F[(1-s)u_*(\cdot) + su(\cdot)]$. We are asked to show that $f'(0) \geq 0$. (Author could not write this part.)