1 The Brachistochrone Problem (July 23)

1.1 Fundamental Lemma of the Calculus of Variations

Recall that v is said to be a *test function on* $[a,b]$ if it is C^1 and $v(a) = v(b) = 0$.

Theorem 1.1. *(Fundamental Lemma of the Calculus of Variations) If g is a continuous function on $[a,b]$ with the property that*

$$\int_a^b g(x)v(x)\,dx = 0$$

for all test functions v on $[a,b]$, then $g \equiv 0$.

Proof. Suppose $g \not\equiv 0$. Then there is an $x_0 \in (a,b)$ such that $g(x_0) \neq 0$. (We can ensure that x_0 is in the interior of the interval because of continuity.) Assume without loss of generality that $g(x_0) > 0$. There exists an open neighbourhood (c,d) of x_0 inside (a,b) on which g is positive. Let v be a C^1 function on $[a,b]$ such that $v > 0$ on (c,d) and $v = 0$ otherwise. Then v is a test function on $[a,b]$, so by the hypotheses,

$$0 = \int_a^b g(x)v(x)\,dx = \int_c^d g(x)v(x)\,dx > 0,$$

a contradiction. □

The test function v we chose in the proof of the preceding theorem could be, for example,

$$v(x) = \begin{cases} (x-c)^2(x-d)^2 & x \in [c,d] \\ 0 & \text{otherwise} \end{cases}.$$

Then

$$v(x) = \begin{cases} 2(x-c)(x-d)^2 + 2(x-c)^2(x-d) & x \in (c,d) \\ 0 & \text{otherwise} \end{cases},$$

which is easily seen to be continuous. Therefore v is a test function on $[a,b]$ which is positive only on (c,d).

1.2 The Brachistochrone Problem

The brachistrochrone problem is the problem from which the calculus of variations was born. In approximately 1638, Galileo Galilei was studying the problem of a ball rolling along a slope from a point A to a point B. Galileo experimented with multiple kinds of slopes, such as a straight line from A to B, or some non-straight curve from A to B, and so on. He measured the time it takes for the ball to roll. He first noticed that the straight line from A to B did *not* minimize the time. He posed the following problem:

Find the curve connecting A and B on which a point mass moves without friction under the influence of gravity in the least time possible.

Around 1696, Johan Bernoulli posted this problem somewhere as a challenge to the mathematicians of the world.

Let us pose the problem more mathematically. Let $c : [0, T] \to \mathbb{R}^2$ describe a curve (the graph of a function) that starts at A at time $t = 0$ and ends at B at time $t = T$. So if $c(t) = (x(t), y(t))$ satisfies $c(0) = A$ and $c(T) = B$. Assuming $y = u(x)$, we have $c(t) = (x(t), u(x(t)))$. Assume $A = (0, a)$ and $B = (b, 0)$.

Now what is the velocity of the point mass along this curve?

$$v(t) = \frac{d}{dt}c(t) = \begin{pmatrix} x'(t) \\ u'(x(t))x'(t) \end{pmatrix} = x'(t) \begin{pmatrix} 1 \\ u'(x(t)) \end{pmatrix}.$$

The kinetic energy of the point mass is $K(t) = \frac{1}{2}m|v|^2 = \frac{m}{2}x'(t)^2(1 + u'(x(t)))^2$, and the potential energy is $V(t) = mgy = mgu(x(t))$. The total energy is $E = K + P$. There is no friction, so energy is conserved, hence the total energy at any time is equal to the total energy at time $t = 0$: $E(t) = E(0)$ for all t. Written out, this means

$$\frac{m}{2}x'(t)^2(1 + u'(x(t)))^2 + mgu(x(t)) = mga.$$

Some algebra shows that this is equal to

$$\frac{1}{2}x'(t)^2 = \frac{g(a - u(x(t)))}{1 + u'(x(t))^2}.$$

Multiplying by 2 and taking square roots gives

$$x'(t) = \sqrt{\frac{2g(a - u(x(t)))}{1 + u'(x(t))^2}},$$

a differential equation in x! What is the total time it takes the point mass to go from A to B along c? We have

$$T = \int_0^T 1\,dt = \int_0^T \sqrt{\frac{1 + u'(x(t))^2}{2g(a - u(x(t)))}}x'(t)\,dt. \tag{*}$$

How does this give us anything we want? It appears that T is on both sides, so that this reveals nothing about T.

Let f be some function, and consider the integral

$$\int_{t_0}^{t_1} f(x(t))x'(t)\,dt = \int_{t_0}^{t_1} F'(x(t))x'(t)\,dt = F(x(t_1)) - F(x(t_0)) = \int_{x(t_0)}^{x(t_1)} f(x)\,dx, \tag{**}$$

where F is an antiderivative of f.

Now, (**) applied to (*) gives

$$T = \int_{x(0)}^{x(1)} \sqrt{\frac{1 + u'(x)^2}{2g(a - u(x))}}\, dx = \int_0^b \sqrt{\frac{1 + u'(x)^2}{2g(a - u(x))}}\, dx.$$

With this, we may pose Galileo's original problem as a minimization problem: the *brachistochrone problem in the calculus of variations.*

$$\begin{aligned} \text{minimize } F[u] &\coloneqq \int_0^b \sqrt{\frac{1 + u'(x)^2}{2g(a - u(x))}}\, dx \\ u \in \mathcal{A} &\coloneqq \{u \in C^1([0, b], \mathbb{R}) : u(0) = a, u(b) = 0\}. \end{aligned}$$

This is a problem in the calculus of variations. Next time, we will find first order conditions for a minimizer and attempt to find a function u_* satisfying these conditions.