1 Cartan Calculus Continued (August 5)

Our plan is to continue developing the Cartan calculus.

1.1 Interior Multiplication

We will first define *interior multiplication* on a vector space, and then define it on a manifold pointwise.

Definition 1. *Given a vector space V and $\beta \in \bigwedge^k(V^*)$ with $k \geq 2$, define, for $v \in V$, $i_v\beta \in \bigwedge^{k-1}(V^*)$ by*

$$i_v\beta(v_1, \ldots, v_{k-1}) = \beta(v, v_1, \ldots, v_{k-1}).$$

The map $i_v\beta$ is called the interior multiplication (or contraction) of β with v. If $k = 1$, we define $i_v\beta$ as the scalar $\beta(v)$, and if $k = 0$, we define $i_v\beta$ to be 0.

It is obvious that $i : V \times \bigwedge^k(V^*) \to \bigwedge^{k-1}(V^*)$ is linear with respect to the vector space structures in both arguments. We list some properties of interior multiplication.

Proposition 1.1.

1. *If $\alpha^1, \ldots, \alpha^k \in V^* = \bigwedge^1(V^*)$ and $v \in V$, then*

$$i_v(\alpha^1 \wedge \cdots \wedge \alpha^k) = \sum_{i=1}^{k-1}(-1)^{i-1}\alpha_1 \wedge \cdots \wedge \widehat{\alpha^i} \wedge \cdots \wedge \alpha^k.$$

2. *$i_v^2 = 0$. (Compare with $d^2 = 0$.)*
3. *For any $\beta \in \bigwedge^k(V^*)$ and $\gamma \in \bigwedge^\ell(V^*)$, one has $i_v(\beta \wedge \gamma) = i_v\beta \wedge \gamma + (-1)^k\beta \wedge i_v\gamma$.*

Proof.

1. Expand along the first column in

$$i_v(\alpha^1 \wedge \cdots \wedge \alpha^k)(v_1, \ldots, v_{k-1}) = \det \begin{pmatrix} \alpha^1(v) & \alpha^1(v_1) & \cdots & \alpha^k(v_{k-1}) \\ \vdots & \vdots & \ddots & \vdots \\ \alpha^k(v) & \alpha^k(v_1) & \cdots & \alpha^k(v_{k-1}) \end{pmatrix}$$

2. If $k \geq 2$, then

$$i_v^2\beta(v_1, \ldots, v_{k-2}) = \beta(v, v, v_1, \ldots, v_{k-2}) = 0,$$

since β is alternating. If $k = 1$ or $k = 0$, then this is obvious.

3. Reduce to the case where β and γ are of the form $\alpha^1 \wedge \cdots \wedge \alpha^k$ and $\alpha^{k+1} \wedge \cdots \wedge \alpha^{k+\ell}$ by linearity.

$\square$

So $i_v : \bigwedge^*(V^*) \to \bigwedge^*(V^*)$ is an antiderivation of degree -1 whose square is zero. Compare with d!

Having defined interior multiplication on a vector space, we make the obvious generalization to manifolds by defining interior multiplication pointwise.

Definition 2. *Given $X \in \mathfrak{X}(M)$ and $\omega \in \Omega^k(M)$, define $i_X\omega$ as the $(k-1)$-form given by $(i_X\omega)_p \coloneqq i_{X_p}\omega_p$.*

For $X_1, \dots, X_{k-1} \in \mathfrak{X}(M)$, $i_X\omega(X_1, \dots, X_{k-1}) = \omega(X, X_1, \dots, X_{k-1})$ is a smooth function on M, so we conclude that $i_X\omega \in \Omega^{k-1}(M)$, since any form that takes smooth vector fields to smooth functions must be a smooth form.

By the way we defined the edge cases $k = 0, 1$ for the interior multiplication of tensors on vector spaces, we have the following matching edge cases for the interior multiplication of forms on manifolds: if $k = 1$, then $i_X\omega = \omega(X)$, and if $k = 0$, then $i_X\omega = 0$.

The interior multiplication $i_X : \Omega^k(M) \to \Omega^{k-1}(M)$ has the following properties:

1. $\mathbb{R}$-linearity. (We now have to specify the type of linearity, because simply "linearity" could refer to the $C^\infty(M)$-module structure or to the $\mathbb{R}$-vector space structure.)

2. For $\omega \in \Omega^k(M)$ and $\eta \in \Omega^\ell(M)$,

$$i_X(\omega \wedge \eta) = i_X\omega \wedge \eta + (-1)^k \omega \wedge i_X\eta.$$

3. $i_X^2 = 0$.

As before, i_X is an antiderivation on the graded algebra $\Omega^*(M)$ of degree -1 whose square is zero. Again, compare with d! Since $i_X\omega$ is defined pointwise, the map $i : \mathfrak{X}(M) \times \Omega^k(M) \to \Omega^{k-1}(M)$ is $C^\infty(M)$-linear in both arguments.

We note an important property of i_X whose proof is obvious.

Proposition 1.2. *$i_X \circ i_Y + i_Y \circ i_X = 0$.*

1.2 Lie Derivative of Forms

Fix a smooth vector field $X \in \mathfrak{X}(M)$ with flow F. We generally want to study how things change along the flow of X at a point. We defined the Lie derivative on $\Omega^0(M)$ as

$$(\mathcal{L}_X f)_p \coloneqq \lim_{t \to 0} \frac{f(F_t(p)) - f(p)}{t} = \frac{d}{dt}\bigg|_{t=0} f(F_t(p)) = X_p(f).$$

We also defined the Lie derivative on $\mathfrak{X}(M)$ using pushforwards to compare the tangent vectors:

$$(\mathcal{L}_X Y)_p \coloneqq \lim_{t \to 0} \frac{(F_{-t})_{*,F_t(p)}(Y_{F_t(p)}) - Y_p}{t} = \frac{d}{dt}\bigg|_{t=0} (F_{-t})_{*,F_t(p)}(Y_{F_t(p)}).$$

Just as vector fields push forwards and differential forms pull back, we now define the Lie derivative of a k-form by using pullbacks to compare the forms:

$$(\mathcal{L}_X\omega)_p \coloneqq \lim_{t\to 0} \frac{F_t^*(\omega_{F_t(p)}) - \omega_p}{t} = \left.\frac{d}{dt}\right|_{t=0} F_t^*(\omega_{F_t(p)}).$$

Proposition 1.3. *The limit always exists, and $\mathcal{L}_X\omega \in \Omega^k(M)$ whenever $\omega \in \Omega^k(M)$.*

Proof. Write $\mathcal{L}_X\omega$ in local coordinates. □

We could also have defined $(\mathcal{L}_X\omega)_p$ as the first order term in the Taylor expansion of $F_t^*(\omega_{F_t(p)})$:

$$F_t^*(\omega_{F_t(p)}) = \omega_p + t(\mathcal{L}_X\omega)_p + o(t).$$

1.3 Properties of These Operations

We state and prove many properties of this Lie derivative, including how it interacts with the exterior derivative and with interior multiplication.

Theorem 1.1. *1. $\mathcal{L}_X : \Omega^*(M) \to \Omega^*(M)$ is a derivation: it is $\mathbb{R}$-linear and satisfies*

$$\mathcal{L}_X(\omega \wedge \eta) = (\mathcal{L}_X\omega) \wedge \eta + \omega \wedge \mathcal{L}_X\eta.$$

2. *The Lie derivative commutes with the exterior derivative: $\mathcal{L}_X \circ d = d \circ \mathcal{L}_X$.*

3. *We have a "global intrinsic formula" for the Lie derivative:*

$$\mathcal{L}_X(\omega(X_1, \ldots, X_k)) = (\mathcal{L}_X\omega)(X_1, \ldots, X_k) + \sum_{i=1}^{k} \omega(X_1, \ldots, \mathcal{L}_X X_i, \ldots, X_k).$$

4. *(Cartan's magic formula) $\mathcal{L}_X = d \circ i_X + i_X \circ d$.*

5. *$\mathcal{L}_X \circ i_Y - i_Y \circ \mathcal{L}_X = i_{[X,Y]}$.*

Proof. For most of these properties, we shall merely outline a proof.

1. One has
$$\mathcal{L}_X(\omega \wedge \eta) = \left.\frac{d}{dt}\right|_{t=0} (F_t^*\omega \wedge F_t^*\eta).$$
Assume without loss of generality that $\omega = f dg$ and $\eta = u dv$ for some smooth functions f, g, u, v. Then work out the computation.

2. One has
$$d(F_t^*\omega) = d\omega + td(\mathcal{L}_X\omega) + o(t).$$
Since pullback and exterior differentiation commute, this is also equal to
$$F_t^*(d\omega) = d\omega + t\mathcal{L}_X(d\omega) + o(t).$$
As they are equal, we may cancel the $d\omega$ terms, divide by t, and take the limit $t \to 0$. This gives the desired $\mathcal{L}_X \circ d = d \circ \mathcal{L}_X$.

3. By the definition of $\mathcal{L}_X\omega$, we obtain
$$F_t^*\omega(X_1, \ldots, X_k) = \omega(X_1, \ldots, X_k) + t(\mathcal{L}_X\omega)(X_1, \ldots, X_k) + o(t).$$
Note that
$$\begin{aligned}\omega((F_t)_*(X_1), \ldots, (F_t)_*(X_k)) &= \omega(X_1 - t\mathcal{L}_X X_1 + o(t), \ldots, X_k - t\mathcal{L}_X X_k + o(t)) \\ &= \omega(X_1, \ldots, X_k) - t\sum_{i=1}^{k} \omega(X_1, \ldots, \mathcal{L}_X X_i, \ldots, X_k) + o(t).\end{aligned}$$
Substituting this into the first equation and moving the sum, as well as the $o(t)$ terms, to the right gives
$$\omega(X_1, \ldots, X_k) \circ F_t = \omega(X_1, \ldots, X_k) + t\left((\mathcal{L}_X\omega)(X_1, \ldots, X_k) + \sum_{i=1}^{k} \omega(X_1, \ldots, \mathcal{L}_X X_i, \ldots, X_k)\right) + o(t).$$
Taylor expansion of the left hand side at $t = 0$ gives
$$\omega(X_1, \ldots, X_k) \circ F_t = \omega(X_1, \ldots, X_k) + t\mathcal{L}_X(\omega(X_1, \ldots, X_k)) + o(t),$$
so the constant terms on both sides cancel, leaving us with
$$t\mathcal{L}_X(\omega(X_1, \ldots, X_k)) + o(t) = t\left((\mathcal{L}_X\omega)(X_1, \ldots, X_k) + \sum_{i=1}^{k} \omega(X_1, \ldots, \mathcal{L}_X X_i, \ldots, X_k)\right) + o(t).$$
Dividing by t and taking the limit $t \to 0$ gives the desired formula.

4. This is another proof in which we will abuse uniqueness properties. First, a lemma;

   Lemma 1.2. *(Uniqueness of the Lie derivative)* $\mathcal{L}_X : \Omega^k(M) \to \Omega^k(M)$ *is the unique* $\mathbb{R}$*-linear map satisfying*

   (i) $\mathcal{L}_X f = X(f)$ *for* $f \in \Omega^0(M)$,

   (ii) $\mathcal{L}_X \circ d = d \circ \mathcal{L}_X$,

(iii) $\mathcal{L}_X(\omega \wedge \eta) = (\mathcal{L}_X\omega) \wedge \eta + \omega \wedge \mathcal{L}_X\eta$.

Proof. Let $D : \Omega^k(M) \to \Omega^k(M)$ be an $\mathbb{R}$-linear map satisfying properites (i)-(iii). Let $\omega \in \Omega^k(M)$ and let $(U, x^1, \ldots, x^n)$ be a chart on M. Since property (iii) is satisfied, D is a local operator; $D\omega$ on U depends only on ω on U, roughly. (The notion of a local operator was precisely defined a few lectures ago, and is in the textbook.)

On U we thus have

$$\begin{aligned} D\omega &= D\left(\sum a_I dx^I\right) & \\ &= \sum \left(Da_I dx^I + a_I D(dx^I)\right) & \mathbb{R}\text{-linearity + property (iii)} \\ &= \sum \left(Da_I dx^I + a_I \left[d(Dx^{i_1}) \wedge \cdots \wedge d(Dx^{i_k})\right]\right) & \text{property (ii)} \\ &= \sum \left(X(a_I)dx^I + a_I \left[dX^{i_1} \wedge \cdots \wedge dX^{i_k}\right]\right) & \text{property (i).} \end{aligned}$$

Since this does not depend on D, we have shown that D is unique. Therefore $\mathcal{L}_X$, satisfying these properties, must be unique. □

With the lemma in mind, to prove Cartan's magic formula we may prove that $i_X \circ d + d \circ i_X$ is an $\mathbb{R}$-linear map satisfying properties (i)-(iii), and conclude by the uniqueness lemma that $\mathcal{L}_X = i_X \circ d + d \circ i_X$. These properties are all relatively straightforward to check and will be omitted.

5. Show this for a 1-form, and then use that to prove it for a k-form.

□

1.4 An Easy Proof of The Global Formula For $d\omega$

Suppose $\omega \in \Omega^1(M)$ and $X, Y \in \mathfrak{X}(M)$. We compute:

$$\begin{aligned} d\omega(X,Y) &= i_Y(i_X(d\omega)) & \\ &= i_Y\mathcal{L}_X\omega - i_Y d i_X\omega & \text{Cartan's magic formula} \\ &= \mathcal{L}_X i_Y\omega - i_{[X,Y]}\omega - i_Y d i_X \omega & \mathcal{L}_X i_Y - i_Y\mathcal{L}_X = i_{[X,Y]} \\ &= X(\omega(Y)) - \omega([X,Y]) - Y(\omega(X)) & i_Y d i_X\omega = Y(\omega(X)). \end{aligned}$$

This gives a straightforward proof for the $k = 1$ case. To prove the more general formula

$$d\omega(X_0, \ldots, X_k) = \sum_{i=0}^{k}(-1)^{i-1}X_i\omega(X_0, \ldots, \widehat{X_i}, \ldots, X_k) + \sum_{0 \le i < j \le k}(-1)^{i+j}\omega([X_i, X_j], X_0, \ldots, \widehat{X_i}, \ldots, \widehat{X_j}, \ldots, X_k),$$

one uses induction as well as the formulas we have developed here. We shall not give the proof.

1.5 Bringing it All Together (Cartan Calculus)

Let us summarize what happened over the last two lectures.

For $X \in \mathfrak{X}(M)$, we introduced three "operators":

- $d : \Omega^k(M) \to \Omega^{k+1}(M)$, an antiderivation of degree 1 - exterior differentiation.
- $i_X : \Omega^k(M) \to \Omega^{k-1}(M)$, an antiderivation of degree -1 - interior multiplication.
- $\mathcal{L}_X : \Omega^k(M) \to \Omega^k(M)$, a derivation - the Lie derivative.

They each interact with the wedge product $\wedge$:

- $d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^k \omega \wedge d\eta$.
- $i_X(\omega \wedge \eta) = i_X\omega \wedge \eta + (-1)^k \omega \wedge i_X\eta$.
- $\mathcal{L}_X(\omega \wedge \eta) = \mathcal{L}_X\omega \wedge \eta + \omega \wedge \mathcal{L}_X\eta$.

And finally, they interact with each other:

$$
\begin{aligned}
d^2, i_X^2 &= 0 \\
\mathcal{L}_X\mathcal{L}_Y - \mathcal{L}_Y\mathcal{L}_X &= \mathcal{L}_{[X,Y]} \\
i_X i_Y + i_Y i_X &= 0 \\
d\mathcal{L}_X - \mathcal{L}_X d &= 0 \\
\mathcal{L}_X i_Y - i_Y \mathcal{L}_X &= i_{[X,Y]} \\
d i_X + i_X d &= \mathcal{L}_X .
\end{aligned}
$$

This completes our study of the Cartan calculus.