1 Smooth Structures, Examples (May 12)

1.1 More on Maximal Atlases

Consider the two atlases $\mathcal{A}_1 = \{(\mathbb{R}^n, Id)\}$ and $\mathcal{A}_2 = \{(B_1(x), Id) : x \in \mathbb{R}^n\}$ on $\mathbb{R}^n$. These two atlases determine the same maximal atlas, or the same smooth structure. Why? We have three equivalent reasons

- for any $(U, \phi) \in \mathcal{A}_1$ and $(V, \psi) \in \mathcal{A}_2$, the charts (U, ϕ) and (V, ψ) are C^∞ compatible.
- $\mathcal{A}_1 \cup \mathcal{A}_2$ is a C^∞ atlas.
- $\mathcal{A}_1$ and $\mathcal{A}_2$ belong to the same maximal atlas.

Define a relation $\sim$ on the atlases by $\mathcal{A}_1 \sim \mathcal{A}_2$ if and only if $\mathcal{A}_1 \cup \mathcal{A}_2$ is another C^∞ atlas. Symmetry and reflexivity are immediate. For transitivity, suppose $\mathcal{A}_1 \cup \mathcal{A}_2$ and $\mathcal{A}_2 \cup \mathcal{A}_3$ are C^∞ atlases. Choose $(U_1, \phi_1) \in \mathcal{A}_1$ and $(U_3, \phi_3) \in \mathcal{A}_3$. We obtain a diffeomorphism

$$\phi_1 \circ \phi_3^{-1} = \phi_1 \circ \phi_2^{-1} \circ \phi_2 \circ \phi_3^1$$

defined on $\phi_3(U_{13} \cap U_2)$. Since $\{U_2 : (U_2, \phi_2) \in \mathcal{A}_2$ covers M, the map $\phi_1 \circ \phi_3^{-1}$ is smooth at every point of $\phi_3(U_{13})$. Therefore $\sim$ is an equivalence relation.

Now given an atlas $\mathcal{A}$ on M, we can talk about the equivalence class $[\mathcal{A}]$. Define

$$\mathcal{M} = \bigcup_{\mathcal{A}' \in [\mathcal{A}]} \mathcal{A}'.$$

Then $\mathcal{M}$ is a new atlas on M; it is the unique maximal atlas containing $\mathcal{A}$. (Exercise.)

So we can make the

Definition 1. *A smooth n-manifold M is a topological n-manifold with a maximal atlas. The choice of maximal atlas is called a smooth structure on M.*

Considering the previous remarks, we arrive at a sufficient condition for a space to be a smooth manifold: If M is a topological space for which

1. M is Hausdorff, second-countable, and
2. M admits a C^∞ atlas $\mathcal{A}$

then M is a smooth manifold with smooth structure $\mathcal{M} = \bigcup_{\mathcal{A}' \in [\mathcal{A}]} \mathcal{A}'$.

1.2 Examples

1. (Open subsets) Let M be a smooth n-manifold with a smooth atlas $\mathcal{A} = \{(U_\alpha, \phi_\alpha)\}$. Let $A \subset M$ be an open set. Then $\mathcal{A}_A = \{(U_\alpha \cap A, \phi_\alpha|_{U_\alpha \cap A})\}$ is a smooth atlas on A, so A is a smooth n-manifold.

2. (Finite dimensional vector spaces) Let V be a finite dimensional real vector space. Choose a basis $\beta = \{v_1, \ldots, v_n\}$ of V, and consider the isomorphism $\Phi : V \to \mathbb{R}^n$ given by $\Phi(v_i) = e_i$.

   Define a norm on V by $\|\sum a_i v_i\| := \|\sum a_i e_i\|$, where the norm on the left is the standard Euclidean norm. With this norm we may define an open ball in V as $B_r(v_0) = \{v \in V : \|v - v_0\| < r\}$. This gives a topology on V. Since all norms on finite dimensional vector spaces are equivalent, this topology does not depend on our choice of basis.

   Then Φ is an isometry (it does not change distances), so it takes balls to balls and so does its inverse. That is, Φ is a homeomorphism, so we have a C^∞ atlas $\{(V, \Phi)\}$ on V, making V a smooth n-manifold.

   This atlas determines a maximal atlas on V. Does this maximal atlas depend on the choice of basis? No. Choose another basis β' of V and define $\Phi' : V \to \mathbb{R}^n$ similarly. Then we'll get another C^∞ atlas $\{(V, \Phi')\}$ on V. The charts (U, Φ) and (V, Φ') are C^∞-compatible, for the transition map $\Phi' \circ \Phi^{-1}$ is a linear isomorphism of $\mathbb{R}^n$ with itself (certainly C^∞).

   Remark: We also could have talked about complex vector spaces, since $\mathbb{C} \cong \mathbb{R}^2$.

3. (Matrices, general linear group) $\mathrm{Mat}_{m\times n}(\mathbb{R}) \cong \mathbb{R}^{mn}$, so $\mathrm{Mat}_{m\times n}(\mathbb{R})$ is a smooth manifold of dimension mn.

   The general linear group is $GL(n, \mathbb{R}) = \{A \in \mathrm{Mat}_{m\times n}(\mathbb{R}) : \det(A) \neq 0\}$. By continuity of det it is an open subset of $\mathrm{Mat}_{m\times n}(\mathbb{R})$, so by the first example we know it's a smooth n^2-dimensional manifold.