

# 1 More on Flows and Lie Derivatives (July 7)

## 1.1 The Fundamental Theorem of Flows

Consider  $X \in \mathfrak{X}(M)$ . Let  $(U, \phi) = (U, x^1, \dots, x^n)$  be a coordinate chart on  $M$ . The theory of ODEs in  $\mathbb{R}^n$  applies almost exactly to the integral curves of  $X|_U \in \mathfrak{X}(U)$ , which follows from  $U$  being diffeomorphic to an open subset of  $\mathbb{R}^n$ . In particular,

1. Given  $p \in U$ , there is, by the existence and uniqueness theorems, a unique maximal integral curve of  $X|_U$  starting at  $p$ .
2. By "collecting" all of the maximal integral curves from the previous step, we get flows. More precisely, we define

$$\mathcal{D} = \{(t, p) \in \mathbb{R} \times M : t \text{ lies in the domain of the maximal integral curve of } X|_U \text{ starting at } p\}.$$

Then, for  $p \in U$ , we let

$$\mathcal{D}^{(p)} = \{t \in \mathbb{R} : (t, p) \in \mathcal{D}\};$$

this is just the domain of the maximal integral curve of  $X|_U$  starting at  $p$ . In particular,  $0 \in \mathcal{D}^{(p)}$ .

Now define  $F : \mathcal{D} \rightarrow U$  by  $F(t, p) = \gamma_p(t)$ , where  $\gamma_p : \mathcal{D}^{(p)} \rightarrow U$  is the maximal integral curve of  $X|_U$  starting at  $p$ .

We are led to the following question: does there exist an interval  $(-\varepsilon, \varepsilon)$  such that  $(-\varepsilon, \varepsilon) \subseteq \mathcal{D}^{(p)}$  for all  $p \in U$ ? If the answer is yes, then the vector field  $X|_U$  is *complete*; each of its maximal integral curves exists for all  $t \in \mathbb{R}$ . This is known as the "Uniform Time Lemma".

Moreover, the function  $F$  is smooth, which follows immediately from the theorem of smooth dependence on initial conditions of ODEs in  $\mathbb{R}^n$ .

3. Given  $p \in U$ , there is, by smooth dependence on initial conditions, a neighbourhood  $W \subseteq U$  of  $p$  such that the hypotheses of the uniform time lemma are satisfied on  $W$ . (?)

All of the above was in a coordinate chart, and so we have seen nothing new. Everything followed directly from the theory of ODEs in  $\mathbb{R}^n$ . We must then ask if the above claims hold in general on  $M$ , not restricted to a single coordinate open set. The answer is yes, all of these claims hold when  $U$  is replaced with  $M$ .

**Theorem 1.1. (Fundamental Theorem of Flows)** Suppose  $X \in \mathfrak{X}(M)$ . Then there exists a unique smooth maximal flow  $F : \mathcal{D} \rightarrow M$ , where  $\mathcal{D} \subseteq \mathbb{R} \times M$ , generated by  $X$ , satisfying

- (a) For each  $p \in M$ , the curve  $\gamma_p(t) = F(t, p)$  is the unique maximal integral curve of  $X$  starting at  $p$ .
- (b) If  $s \in \mathcal{D}^{(p)}$ , then  $\mathcal{D}^{(F(s, p))} = \mathcal{D}^{(p)} - s$ .

- (c) For each  $t \in \mathbb{R}$ , the set  $M_t = \{p \in M : (t, p) \in \mathcal{D}\}$  is open in  $M$ , and  $F_t : M_t \rightarrow M_{-t}$  is a diffeomorphism.

*Proof.* The proof is left as an exercise (likely as one of the homework problems). It is also in Lee.  $\square$

## 1.2 More on Lie Derivatives

Suppose  $X, Y \in \mathfrak{X}(M)$ , and let  $F$  be the flow of  $X$ . If  $M = \mathbb{R}^n$ , then we can compare  $Y_{F_t(p)}$  and  $Y_p$ , since  $T_p \mathbb{R}^n = \mathbb{R}^n$  for every  $p \in \mathbb{R}^n$ . We cannot, however, do this for abstract manifolds, since the elements of the distinct tangent spaces of  $M$  cannot even be compared. We remedy this using the pushforward of  $F_{-t}$ .

**Definition 1.** The Lie derivative of  $Y$  with respect to  $X$  is the vector field  $\mathcal{L}_X Y$  defined by

$$(\mathcal{L}_X Y)_p = \lim_{t \rightarrow 0} \frac{(F_{-t})_{*, F_t(p)}(Y_{F_t(p)}) - Y_p}{t},$$

when the limit exists.

**Proposition 1.1.** The above limit exists for all  $p \in M$ , and  $\mathcal{L}_X Y \in \mathfrak{X}(M)$ .

*Proof.* Exercise.  $\square$

Consider the question of whether or not the flows of  $X$  and  $Y$  commute. Let  $G$  be the flow of  $Y$ . We then may ask ourselves: given  $p \in M$ , do we have  $F_s(G_t(p)) = G_t(F_s(p))$ ? Define a function  $A$  mapping  $(s, t)$  to  $F_s(G_t(p)) - G_t(F_s(p))$ . We have the following proposition:

**Proposition 1.2.**

$$\frac{\partial^2 A}{\partial s \partial t} = \frac{\partial^2 A}{\partial t \partial s} = \mathcal{L}_X Y,$$

and if  $\mathcal{L}_X Y = 0$ , then  $A \equiv 0$ .

*Proof.* Exercise.  $\square$

So  $\mathcal{L}_X Y$  measures "how much the flows commute".

For another perspective, consider a smooth function  $f \in C^\infty(M)$  and a vector field  $X \in \mathfrak{X}(M)$ . We can ask ourselves the question "how does  $f$  change along the integral curve of  $X$  starting at  $p$ ?". If  $F$  is the flow of  $X$ , then we would like to discuss

$$\left. \frac{d}{dt} \right|_{t=0} f \circ F_t(p)$$

for  $p \in M$ . A straightforward application of the chain rule gives

$$\left. \frac{d}{dt} \right|_{t=0} f \circ F_t(p) = f_{*,p}(\gamma'_p(0)) = f_{*,p}(X_p) = (Xf)(p).$$

We can also define  $\frac{d}{dt}|_{t=0} f \circ F_t(p)$  as the first order term in the Taylor expansion of  $f \circ F_t(p)$  at 0:

$$f \circ F_t(p) = f(p) + t(Xf)(p) + o(t).$$

Let us apply the same idea in order to get the Lie derivative. Define  $G$  to be the map

$$G : t \mapsto (F_{-t})_{*,F_t(p)}(Y_{F_t(p)}) \in T_p M.$$

The Taylor series expansion of  $G$  at 0 is

$$G(t) = G(0) + tG'(0) + o(t) = Y_p + t(\mathcal{L}_X Y)_p + o(t),$$

as we will see later. Therefore we may actually define  $(\mathcal{L}_X Y)_p$  to be the first order term of the Taylor expansion of the map  $G$  defined above.

We end up having three equivalent definitions of the Lie derivative  $\mathcal{L}_X Y$ : given  $p \in M$ , define  $(\mathcal{L}_X Y)_p$  as

1. the limit

$$(\mathcal{L}_X Y)_p := \lim_{t \rightarrow 0} \frac{(F_{-t})_{*,F_t(p)}(Y_{F_t(p)}) - Y_p}{t}.$$

2. the first-order term of the Taylor expansion of the map  $G$  defined by

$$G : t \mapsto (F_{-t})_{*,F_t(p)}(Y_{F_t(p)}).$$

3. the Lie bracket

$$[X, Y] = X_p Y - Y_p X.$$