

1 Lie Derivatives, Lie Algebras, and Frobenius' Theorem (July 10)

(This lecture was pushed a day forward as the instructor could not make the usual lecture time on Thursday.)

1.1 Equivalent Conditions for Zero Lie Derivative

Suppose $X, Y \in \mathfrak{X}(M)$, and let F denote the flow of X . Suppose $\mathcal{L}_X Y$ is identically zero on M . Define a C^∞ curve $H : \mathcal{D}^{(p)} \rightarrow T_p M$ by

$$H : t \mapsto (F_{-t})_{*, F_t(p)}(Y_{F_t(p)}).$$

Since H is a smooth curve in a vector space, we can identify its derivative with an element of the vector space $T_p M$. We have

$$\begin{aligned} H'(t_0) &= \left. \frac{d}{dt} \right|_{t=t_0} (F_{-t})_{*, F_t(p)}(Y_{F_t(p)}) \\ &= \left. \frac{d}{dt} \right|_{t=t_0} (F_{-(t-t_0)-t_0})_{*, F_{t-t_0+t_0}(p)}(Y_{F_{t-t_0+t_0}(p)}) \\ &= \left. \frac{d}{dt} \right|_{t=0} (F_{-t-t_0})_{*, F_{t+t_0}(p)}(Y_{F_{t+t_0}(p)}) \\ &= \left. \frac{d}{dt} \right|_{t=0} (F_{t_0} \circ F_{-t})_{*, F_t(F_{t_0}(p))}(Y_{F_t(F_{t_0}(p))}) \\ &= \left. \frac{d}{dt} \right|_{t=0} (F_{t_0})_{*, F_{t_0}(p)} \left((F_{-t})_{*, F_t(F_{t_0}(p))}(Y_{F_t(F_{t_0}(p))}) \right) \\ &= (F_{t_0})_{*, F_{t_0}(p)} \left. \frac{d}{dt} \right|_{t=0} (F_{-t})_{*, F_t(F_{t_0}(p))}(Y_{F_t(F_{t_0}(p))}) \\ &= (F_{t_0})_{*, F_{t_0}(p)} (\mathcal{L}_X Y)_{F_{t_0}(p)} \\ &= 0. \end{aligned}$$

So H is constant. Therefore $H(t) = Y_p$ for all $t \in \mathcal{D}^{(p)}$. Applying $(F_t)_{*, p}$ to both sides gives

$$Y_{F_t(p)} = (F_t)_{*, p}(Y_p) \text{ for all } t \in \mathcal{D}^{(p)}.$$

When this condition is satisfied, we say that Y is *invariant under the flow of X* . We have just shown the following proposition.

Proposition 1.1. *If $\mathcal{L}_X Y$ is identically zero on M , then Y is invariant under the flow of X .*

We will eventually show that the hypothesis $\mathcal{L}_X Y \equiv 0$ also implies that X is invariant under the flow of Y , which will be a corollary of the identity $\mathcal{L}_X Y = -\mathcal{L}_Y X$.

One way to imagine this situation is through curves. Suppose $\gamma(s)$ is a C^∞ curve in M starting at p with $\gamma'(0) = Y_p$. Suppose the conclusion of the above proposition is satisfied. We have

$$(F_t \circ \gamma)'(0) = (F_t)_{*,p}(\gamma'(0)) = (F_t)_{*,p}(Y_p) = Y_{F_t(p)},$$

the last step following from the previous proposition. In particular, choose γ to be the unique integral curve of Y starting at p . Then, for all s at which γ is defined,

$$(F_t \circ \gamma)'(s) = (F_t)_{*,\gamma(s)}(\gamma'(s)) = (F_t)_{*,p}(Y_{\gamma(s)}) = Y_{(F_t \circ \gamma)(s)},$$

the last step again following from the previous proposition. We have proven the following proposition.

Proposition 1.2. *If Y is invariant under the flow F of X , then for each t , F_t takes integral curves of Y to integral curves of Y .*

Now suppose the conclusion of this proposition holds. Let F be the flow of $X \in \mathfrak{X}(M)$ and G the flow of $Y \in \mathfrak{X}(M)$. Assume for simplicity that the vector fields are complete. Fix $p \in M$. For a fixed t_0 , this conclusion implies that $F_{t_0}(G_s(p))$ is the integral curve of Y starting at $F_{t_0}(p)$. But then $F_{t_0}(G_s(p)) = G_s(F_{t_0}(p))$ for all s for which it makes sense. Since t_0 and p were arbitrary, we have $F_t \circ G_s = G_s \circ F_t$ for all s, t for which it makes sense. We say, in this case, that the flows of X and Y *commute*. We have proven the following proposition.

Proposition 1.3. *If X and Y are complete and if the flow of X takes integral curves of Y to integral curves of Y , then the flows of X and Y commute.*

Note that each proposition implies the next one. It turns out that they are actually equivalent, which we now state and prove. The theorem holds without the assumption that X and Y are complete.

Theorem 1.1. *(Equivalent conditions for zero Lie derivative, complete case) Let $X, Y \in \mathfrak{X}(M)$ be complete vector fields. The following are equivalent:*

1. $\mathcal{L}_X Y \equiv 0$.
2. Y is invariant under the flow of X .
3. The flow of X takes integral curves of Y to integral curves of Y .
4. The flows of X and Y commute.

Proof. We have shown that $(1) \implies (2) \implies (3) \implies (4)$. All we need to prove is that (4) implies (1), which is very easy. Let F denote the flow of X and G the flow of Y . Given $p \in M$, we have

$$F_t(G_s(p)) = G_s(F_t(p))$$

for all t, s . Differentiating with respect to s at 0 gives

$$(F_t)_{*,p}(Y_p) = Y_{F_t(p)},$$

implying

$$Y_p = (F_{-t})_{*,F_t(p)}(Y_{F_t(p)}).$$

Since this holds for all t , $(\mathcal{L}_X Y)_p = 0$. Since p was arbitrary, $\mathcal{L}_X Y \equiv 0$. \square

Note that the fourth condition in the theorem is symmetric in X and Y . Therefore all of the conditions of the theorem hold when X and Y are switched: in particular, $\mathcal{L}_X Y \equiv 0$ if and only if $\mathcal{L}_Y X \equiv 0$.

1.2 A Formula for the Lie Derivative

We will now derive a very simple formula for the Lie derivative. Given $X, Y \in \mathfrak{X}(M)$, let F be the flow of X . Fix $p \in M$ and consider $H : \mathcal{D}^{(p)} \rightarrow T_p M$ defined by

$$H : t \mapsto (F_{-t})_{*,F_t(p)}(Y_{F_t(p)}).$$

Then H is a smooth function into the vector space $T_p M$. We may therefore identify $H'(0)$ with an element of $T_p M$; that element is precisely $H'(0) = (\mathcal{L}_X Y)_p$. Since H takes elements in a vector space, we can look at its Taylor series expansion near $t = 0$:

$$(F_{-t})_{*,F_t(p)}(Y_{F_t(p)}) = H(t) = H(0) + tH'(0) + o(t) = Y_p + t(\mathcal{L}_X Y)_p + o(t).$$

Applying $(F_t)_{*,p}$ to both sides and rearranging

$$Y_{F_t(p)} - (F_t)_{*,p}(Y_p) = t(F_t)_{*,p}((\mathcal{L}_X Y)_p) + o(t).$$

Suppose $f \in C^\infty(M)$. Evaluating both sides at f gives

$$(Y(f) \circ F_t)(p) - (Y(f \circ F_t))(p) = t(\mathcal{L}_X Y)((f \circ F_t))(p) + o(t).$$

Note that $(Y(f) \circ F_t)(p)$ and $f \circ F_t$ are functions of t into vector spaces, so we can look at their Taylor series expansions near $t = 0$:

$$\begin{aligned} (Y(f) \circ F_t)(p) &= Y_p(f) + t \left. \frac{d}{dt} \right|_{t=0} (Y(f) \circ F_t)(p) + o(t) = Y_p(f) + tX_p(Y(f)) + o(t) \\ f \circ F_t &= f + tX(f) + o(t). \end{aligned}$$

Substituting these back in gives us

$$\cancel{Y_p(f)} + tX_p(Y(f)) - (Y(\cancel{f} + tX(f) + o(t)))(p) = t(\mathcal{L}_X Y)((f + tX(f) + o(t)))(p) + o(t),$$

(the backslashes indicate that those terms cancel out) simplifying to

$$t[X_p(Y(f)) - Y_p(X(f))] = t(\mathcal{L}_X Y)((f + tX(f) + o(t))(p) + o(t)).$$

By linearity on the right side, we can take out the term $tX(f) + o(t)$ from the argument of $\mathcal{L}_X Y$. This term will be absorbed into the rightmost $o(t)$. Thus

$$t[X_p(Y(f)) - Y_p(X(f))] = t(\mathcal{L}_X Y)_p(f) + o(t).$$

Using the Lie bracket $[X, Y] := XY - YX$, this may be written as

$$t[X, Y]_p(f) = t(\mathcal{L}_X Y)_p(f) + o(t).$$

Dividing by t and taking the limit $t \rightarrow 0$ gives us the following

Theorem 1.2. $\mathcal{L}_X Y = [X, Y]$.

With this formula in hand, we may very easily compute Lie derivatives. Note that this implies $\mathcal{L}_X Y = -\mathcal{L}_Y X$. Also, with this, we can say that X and Y commute if $[X, Y] = 0$.

Note the following properties of the Lie bracket:

1. Bilinearity.
2. Anticommutativity.
3. The *Jacobi identity*

$$\sum_{\text{cyclic}} [X, [Y, Z]] = 0,$$

or equivalently

$$[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0.$$

All of these are relatively obvious from the definition. Note that every property of the Lie bracket implies a property of the Lie derivative (after all, they are the same). Note that the Lie bracket is *not* in general associative; the Jacobi identity ruins any hope of associativity in general.

1.3 Lie Algebras

Let us abstract away from vector fields on manifolds and consider only the structure a vector space is given when we define on it a "product" holding properties similar to the Lie bracket.

Definition 1. A *Lie algebra* is a vector space V over a field F together with a bilinear, anticommutative operation $[\cdot, \cdot] : V \times V \rightarrow V$ which satisfies the Jacobi identity.

Note that a Lie algebra is, in general, not an algebra. Recall that an *algebra* is a vector space V over a field equipped with a "product" $\cdot : V \times V \rightarrow V$ making V into a ring (with or without identity) which satisfies a homogeneity condition with respect to the vector space's scalar multiplication.

However, if (V, \cdot) is an algebra, we can consider the ring commutator on V with respect to \cdot :

$$[v, w] := v \cdot w - w \cdot v.$$

One easily checks that the operation defined above gives the algebra V a Lie algebra structure.

The most familiar example of a Lie algebra to us is $\mathfrak{X}(M)$ with the Lie bracket $[X, Y] = XY - YX$. (Note that the "multiplication" XY here is a vector field defined by $(XY)(f) = X(Y(f))$, viewing vector fields as derivations of $C^\infty(M)$.) There are three main algebraic structures at play here:

- A real vector space structure.
- A $C^\infty(M)$ -module structure
- A real Lie algebra structure.

Definition 2. A *derivation* on a Lie algebra V is a linear map $D : V \rightarrow V$ with respect to the vector space structure satisfying the Leibnitz rule

$$D([X, Y]) = [D(X), Y] + [X, D(Y)].$$

The next proposition provides an important class of derivations, one special member of which we are very familiar with.

Proposition 1.4. For X in a Lie algebra V , define $\text{ad}_X : V \rightarrow V$ by $\text{ad}_X(Y) = [X, Y]$. Then ad_X is a derivation on V .

Proof. Linearity of ad_X follows from bilinearity of $[\cdot, \cdot]$. The Jacobi identity may be written as

$$[X, [Y, Z]] = [[X, Y], Z] + [Y, [X, Z]],$$

or alternatively,

$$\text{ad}_X([Y, Z]) = [\text{ad}_X(Y), Z] + [Y, \text{ad}_X(Z)].$$

□

Corollary 1.2.1. For a fixed $X \in \mathfrak{X}(M)$, the map $\mathcal{L}_X : \mathfrak{X}(M) \rightarrow \mathfrak{X}(M)$ defined by $\mathcal{L}_X(Y) = \mathcal{L}_X Y$ is a derivation of the Lie algebra $\mathfrak{X}(M)$.

Let us note some important properties of the Lie derivative, whose proofs are relatively straightforward computations.

Proposition 1.5. The Lie derivative satisfies

$$(i) \quad \mathcal{L}_X Y = -\mathcal{L}_Y X.$$

$$(ii) \quad \mathcal{L}_X([Y, Z]) = [\mathcal{L}_X Y, Z] + [Y, \mathcal{L}_X Z].$$

$$(iii) \quad \mathcal{L}_{[X, Y]} Z = \mathcal{L}_X(\mathcal{L}_Y Z) - \mathcal{L}_Y(\mathcal{L}_X Z).$$

$$(iv) \quad \text{If } g \in C^\infty(M), \mathcal{L}_X(gY) = g\mathcal{L}_X Y + X(g) \cdot Y.$$

$$(v) \quad \text{If } F : M \rightarrow N \text{ is a diffeomorphism, } F_*(\mathcal{L}_X Y) = \mathcal{L}_{F_* X}(F_* Y).$$

Proof. (i) and (ii) are immediate. For (iii), the Jacobi identity may be written as

$$[[X, Y], Z] = [X, [Y, Z]] - [Y, [X, Z]],$$

which is the desired identity. For (iv), if $f \in C^\infty(M)$ then

$$\begin{aligned} (\mathcal{L}_X(gY))(f) &= X((gY)(f)) - (gY)(X(f)) \\ &= X(g \cdot Y(f)) - g \cdot Y(X(f)) \\ &= g \cdot X(Y(f)) + Y(f) \cdot X(g) - g \cdot Y(X(f)) \\ &= g(\mathcal{L}_X Y)(f) - (X(g)Y)(f), \end{aligned}$$

so the identity holds. The last is left as an exercise (was not included in lecture). \square

1.4 Mini Frobenius' Theorem

Suppose $X \in \mathfrak{X}(M)$. Given $p \in M$, we can think of finding the integral curve of X starting at p as finding a 1-dimensional (immersed) submanifold of M to which X is everywhere tangent. If we increase to k vector fields, when can we find k -dimensional submanifolds of M to which those vector fields are everywhere tangent? Such questions lead us to Frobenius' theorem.

Let $(U, \phi) = (U, x^1, \dots, x^n)$ be a chart on M . Suppose $f \in C^\infty(M)$. Then the function $\frac{\partial f}{\partial x^i}$ is again a C^∞ function on M , for each i . Equality of second-order mixed partials gives

$$\frac{\partial^2 f}{\partial x^i \partial x^j} = \frac{\partial^2 f}{\partial x^j \partial x^i},$$

implying that the Lie bracket

$$\left[\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \right] = 0.$$

That is, the coordinate vector fields commute. Is this condition also sufficient? That is, is the following statement true?

"Suppose X_1, \dots, X_k is a (smooth) k -frame; a collection of k smooth vector fields such that $(X_1)_p, \dots, (X_k)_p$ are linearly independent for all $p \in M$. Suppose $[X_i, X_j] = 0$ for each $1 \leq i, j \leq k$.

Are X_1, \dots, X_k coordinate vector fields? That is, for $p \in M$, does there exist a chart (U, x^1, \dots, x^n) near p such that

$$\frac{\partial}{\partial x^i} = X_i \text{ for } 1 \leq i \leq k?"$$

(We do not require $k = n$.)

Suppose X, Y is a smooth commuting 2-frame on M . Let $p \in M$ and let U be a neighbourhood of p such that the flows F of X and G of Y are defined on some $(-\varepsilon, \varepsilon) \times U$. Define $A : (-\varepsilon, \varepsilon)^2 \rightarrow U$ by

$$A(s, t) = G_s(F_t(p)) = F_t(G_s(p)).$$

(Equality holds since $[X, Y] = 0$.) A is C^∞ , and

$$\begin{aligned} A_{*,(s_0, t_0)} \left(\frac{\partial}{\partial s} \right) &= \frac{\partial A}{\partial s} \Big|_{(s_0, t_0)} = Y_{A(s_0, t_0)} \\ A_{*,(s_0, t_0)} \left(\frac{\partial}{\partial t} \right) &= \frac{\partial A}{\partial t} \Big|_{(s_0, t_0)} = X_{A(s_0, t_0)}. \end{aligned}$$

The fact that X, Y is a 2-frame implies that A is an immersion. By the immersion theorem there is a chart $(U, \phi) = (U, x^1, \dots, x^n)$ near p such that

$$\begin{aligned} \phi \circ A : (-\varepsilon', \varepsilon')^2 &\rightarrow \mathbb{R}^n \\ (s, t) &\mapsto (s, t, 0, \dots, 0) \end{aligned}$$

for some $\varepsilon' \in (0, \varepsilon]$. Then

$$\begin{aligned} \frac{\partial}{\partial x^1} &= Y \\ \frac{\partial}{\partial x^2} &= X \end{aligned}$$

on $A((-\varepsilon', \varepsilon')^2)$. Since A is an embedding on $(-\varepsilon', \varepsilon')^2$, $S = A((-\varepsilon', \varepsilon')^2)$ is a 2-dimensional embedded submanifold of M such that for every $q \in S$, $T_q S = \text{span}(\{X_q, Y_q\})$, and for which (U, ϕ) is an adapted chart. Moreover, X and Y are coordinate vector fields with respect to the chart $\phi_S = \pi \circ S$ on S . In this case, we call S an *integral submanifold of the 2-frame X, Y* . We have proven the following theorem:

Theorem 1.3. (*Mini-Frobenius*) *Let X_1, \dots, X_k be a smooth commuting k -frame on M . Let $p \in M$. There exists a k -dimensional integral submanifold S of this frame which contains p . Also, there exists an adapted chart (U, ϕ) near p relative to S such that the coordinate vector fields relative to ϕ_S are*

$$\frac{\partial}{\partial x^i} = X_i \text{ for } 1 \leq i \leq k.$$

(We may state the result for arbitrary k since the only thing that changes in the proof is the notation.)

1.5 Leading up to Frobenius' Theorem

We will be able to weaken the hypotheses of the preceding theorem. We will replace "commuting" with "such that $[X_i, X_j] \in \text{span}(\{X_1, \dots, X_k\})$ ".

Let S be a k -dimensional submanifold of M which is an integral submanifold for the k -frame X_1, \dots, X_k (i.e. $T_q S = \text{span}(\{(X_1)_q, \dots, (X_k)_q\})$ for all $q \in S$). Then

$$[X_i, X_j] = \sum a_\ell X_\ell$$

for some functions a_ℓ . The proof is an exercise. (Hint: write in coordinates the Lie bracket.) This shows that if $X_p, Y_p \in T_p S$, then $[X, Y]_p \in T_p S$. In particular, the condition stated in the previous paragraph is necessary; Frobenius' theorem tells us that it is sufficient.

Now let X, Y be a 2-frame. Let (U, ϕ) be a chart as before. Then

$$\frac{\partial}{\partial x^1} = Y \quad \frac{\partial}{\partial x^2} = X$$

on S . We chose ϕ so that the integral curves of Y on S are precisely the integral curves of $\partial/\partial x^1$, and similarly for X and $\partial/\partial x^2$, since $\phi \circ A : (s, t) \mapsto (s, t, 0, \dots, 0)$. But we do not necessarily have this on all of U ; we have still not shown that X and Y are coordinate vector fields on U . We shall remedy this.

Consider the map

$$H : (s, t, x^3, \dots, x^n) \mapsto G_s(F_t(\phi^{-1}(0, 0, x^3, \dots, x^n))).$$

Proposition 1.6. *Let ψ be the inverse of H , defined above. Then ψ is a coordinate chart whose first two coordinate vector fields are Y and X , respectively.*

We have a

Theorem 1.4. *Let X_1, \dots, X_k be a smooth commuting k -frame on M . Let $p \in M$. Then there exists a chart (U, ϕ) near p such that*

$$\frac{\partial}{\partial x^i} = X_i$$

for $1 \leq i \leq k$. (What does this mean when $k = 1$?)

Lemma 1.5. *(Lemma for Frobenius' theorem) If X_1, \dots, X_k is a smooth k -frame such that $[X_i, X_j] \in \text{span}(\{X_1, \dots, X_k\})$, then there is a smooth commuting k -frame Y_1, \dots, Y_k such that $\text{span}(\{X_1, \dots, X_k\}) = \text{span}(\{Y_1, \dots, Y_k\})$.*

"Commuting" is a necessary and sufficient condition for the vector fields to be coordinate vector fields.