1 Equivalence of Regular and Embedded Submanifolds (June 2)

1.1 Regular Submanifolds

Recall the definition of a regular submanifold.

Definition 1. *Let M be a smooth manifold. $S \subseteq M$ is a regular submanifold of dimension k if for each $p \in S$ there is a chart $(U, \phi) = (U, x^1, \ldots, x^n)$ for M at p such that $U \cap S$ is defined by the vanishing of exactly $n - k$ of the coordinates (we will usually take these to be the last such coordinates). Such a chart is called an adapted chart relative to S.*

If $\{(U, \phi)\}$ is an atlas for M of adapted charts relative to S, then it is not hard to see that $\{(U \cap S, \phi_S)\}$ is an atlas for S in the subspace topology, where $\phi_S := \pi \circ \phi|_S$. Therefore S is a smooth manifold of dimension k.

A regular submanifold "inherits" the smooth structure from M in the following sense:

Proposition 1.1. *If $f : M \to \mathbb{R}$ is C^∞ and $S \subseteq M$ is a regular submanifold, then $f|_S : S \to \mathbb{R}$ is C^∞.*

Proof. For any adapted chart (U, ϕ) relative to S, $f \circ \phi^{-1}$ is C^∞. Then $f \circ \phi_S^{-1}$ is C^∞, since it is the composition $f \circ \phi^{-1} \circ g$, where $g : (x^1, \ldots, x^k) \mapsto (x^1, \ldots, x^k, 0, \ldots, 0)$ is the "canonical immersion". $\qquad\square$

For example, consider a C^∞ function $f : \mathbb{R} \to \mathbb{R}$. Then Γ_f becomes a smooth manifold with the atlas $\{(\Gamma_f, \pi)\}$, where $\pi : (x, f(x)) \mapsto x$. For an open set $U \subseteq \mathbb{R}^2$ intersecting Γ_f, define $\psi : U \to \mathbb{R}^2$ by $\psi(x, y) = (x, y - f(x))$. Then ψ is a local diffeomorphism, which implies that, after shrinking U, the pair (U, ψ) is a coordinate chart belonging to the standard smooth structure on $\mathbb{R}^2$. Moreover, $\Gamma_f \cap U$ is defined by the vanishing of the last coordinate of ψ, so (U, ψ) is an adapted chart relative to Γ_f. We can do this at any point of Γ_f, so we can conclude that Γ_f is a regular submanifold of $\mathbb{R}^2$ of dimension 1.

What is the tangent space to a regular submanifold $S \subseteq M$? Note that we cannot write $T_pS \subseteq T_pM$, since the elements are not even the same. However, if $v \in T_pS$, there is a unique $\widetilde{v} \in T_pM$ such that for any $f \in C_p^\infty(M)$, $\widetilde{v}(f) = v(f|_S)$. (Uniqueness is immediate, and existence follows by defining $\widetilde{v}$ by that formula.) Let Φ be the map $v \mapsto \widetilde{v}$. Linearity is obvious, and for injectivity, suppose $\Phi(v) = \widetilde{v} = 0$. Fix an adapted chart $(U, x^1, \ldots, x^n)$ at p, so that if $y^i = x^i|_S$, then $(U \cap S, y^1, \ldots, y^k)$ is a chart on S at p. Then $\{\frac{\partial}{\partial y^i}\big|_p\}$ is a basis of T_pS, so

$$v = \sum v(y^i)\, \frac{\partial}{\partial y^i}\bigg|_p = \sum v(x^i|_S)\, \frac{\partial}{\partial y^i}\bigg|_p = \sum \widetilde{v}(x^i)\, \frac{\partial}{\partial y^i}\bigg|_p = 0,$$

so Φ is injective. Therefore we may think of the k-dimensional subspace $\Phi(T_pS) \subseteq T_pM$ as "T_pS living inside T_pM".

1.2 Embedded Submanifolds

Recall the definition of an embedded submanifold.

Definition 2. *Let M be a smooth manifold. $S \subseteq M$ is an embedded submanifold of dimension k if it is a smooth manifold of dimension k such that the inclusion map $i : S \hookrightarrow M$ is an embedding (topological embedding and an immersion).*

Let M be a smooth manifold and $S \subseteq M$ a subset which is also a smooth manifold. Is it true that the inclusion $i : S \hookrightarrow M$ is C^∞? Not always. Consider the case Γ_f for $f(x) = |x|$. Then Γ_f is a smooth manifold and a subset of the smooth manifold $\mathbb{R}^2$, but the inclusion $\Gamma_f \hookrightarrow \mathbb{R}^2$ is not smooth.

Give S the subspace topology, so that $i : S \hookrightarrow M$ is a topological embedding. Suppose S is equipped with a smooth structure such that i is C^∞. We claim that i is then an embedding, in the sense that, in addition to being a topological embedding, it is an immersion. (The proof will be a homework exercise.)

An embedded submanifold "inherits" the smooth structure from M in the following sense:

Proposition 1.2. *If $f : M \to \mathbb{R}$ is C^∞ and $S \subseteq M$ is an embedded submanifold, then $f|_S : S \to \mathbb{R}$ is C^∞.*

Proof. $f|_S = f \circ i$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

What is the tangent space to an embedded submanifold $S \subseteq M$? The inclusion $i : S \hookrightarrow M$ has injective differential $i_{*,p} : T_p S \to T_p M$, and so we can think of the k-dimensional subspace $i_{*,p}(T_p S) \subseteq T_p M$ as "$T_p S$ living inside $T_p M$". Moreover, in reference to the tangent space of a regular submanifold, we have $i_{*,p} = \Phi$, since

$$i_{*,p}(v)(f) = v(f \circ i) = v(f|_S) = \widetilde{v}(f)$$

for every $f \in C_p^\infty(M)$ and $v \in T_p S$.

1.3 Equivalence of the Two

After noticing the similarities between regular and embedded submanifolds, one might ask whether or not they are the same. The answer is yes.

Theorem 1.1. *Let M be a smooth manifold and $S \subseteq M$. S is a regular submanifold of dimension k if and only if S is an embedded submanifold of dimension k.*

Proof. Suppose S is a regular submanifold of dimension k. It is given the subspace topology, so $i : S \hookrightarrow M$ is a topological embedding. Let (U, ϕ) be an adapted chart relative to S. Then $(U \cap S, \phi_S)$ is a coordinate chart on S. The coordinate representation of i in these two charts is

$$\phi \circ i \circ \phi_S^{-1} : (x^1, \ldots, x^k) \mapsto (x^1, \ldots, x^k, 0, \ldots, 0),$$

since $U \cap S$ is defined by the vanishing of the last $n - k$ coordinates. In this form it is clear that $i : S \hookrightarrow M$ is an immersion, so S is an embedded submanifold.

The converse follows from the following slightly more general proposition. $\qquad\square$

Proposition 1.3. *If $f : N \to M$ is an embedding, then $f(N)$ is a regular submanifold of M.*

Proof. Let $p \in N$. By the immersion theorem, we can find coordinate charts $(U, \phi) = (U, x^1, \ldots, x^n)$ at p and $(V, \psi) = (V, y^1, \ldots, y^m)$ at $f(p)$ with respect to which f, in coordinates, takes on the form

$$\psi \circ f \circ \phi^{-1} : \phi(U \cap f^{-1}(V)) \to \mathbb{R}^m, \qquad (x^1, \ldots, x^n) \mapsto (x^1, \ldots, x^n, 0, \ldots, 0).$$

By possibly shrinking U, assume that $f(U) \subseteq V$. We may do this by replacing U with $U \cap f^{-1}(V)$, which is open in N; we will still have a coordinate chart at p and the above identity will still hold.

We show that $f(U)$ is defined by the vanishing of $y^{n+1}, \ldots, y^m$. More precisely, that

$$f(U) = \{z \in V : y^{n+1}(z) = \cdots = y^m(z) = 0\}.$$

Suppose $q \in U$. Then $f(q)$ satisfies $\psi(f(q)) = (\psi \circ f \circ \phi^{-1})(\phi(q))$, of which the last $m - n$ coordinates vanish. This proves the $\subseteq$ inclusion. Conversely, suppose $z \in V$ satisfies $y^{n+1}(z) = \cdots = y^m(z) = 0$. Then $\psi(z)$ is in the image of $\psi \circ f \circ \phi^{-1}$ because of the vanishing of the last $m - n$ coordinates, so there is a $q \in \phi(U)$ such that $(\psi \circ f \circ \phi^{-1})(q) = \psi(z)$, implying $z = f(\phi^{-1}(q)) \in f(U)$. This proves the $\supseteq$ inclusion, and completes the proof that $f(U)$ is defined by the vanishing of $y^{n+1}, \ldots, y^m$.

Since f is a homeomorphism onto its image, $f(U)$ is open in the subspace topology on $f(N)$, so we can find an open set W of M such that $f(U) = W \cap f(N)$. Then

$$(V \cap W) \cap f(N) = V \cap f(U)$$
$$= f(U) \qquad \text{(because we made } f(U) \subseteq V\text{)}$$

is defined by the vanishing of $y^{n+1}, \ldots, y^m$, which implies that $(V \cap W, y^1, \ldots, y^m)$ is an adapted chart at $f(p)$ relative to $f(N)$. Therefore $f(N)$ is a regular submanifold of M, of the same dimension as N. $\qquad\square$

Therefore *embedded submanifolds and regular submanifolds are one and the same thing.*