1 Introducing Cartan's Calculus (August 4)

1.1 Plan

We will develop *Cartan's calculus*, which could be described as the calculus of differential forms. We will, in particular, do four things:

1. Develop a global intrinsic formula for the exterior derivative.
2. Develop interior multiplication of forms, a certain antiderivation $\imath_X$ of degree -1.
3. Develop the notion of the Lie derivative of a k-form.
4. Discuss how the previous three concepts interact with each other. In particular, we will prove *Cartan's homotopy formula:* $\mathcal{L}_X = d\imath_X + \imath_X d$.

The focus of today's lecture is (1).

1.2 A Global Intrinsic Formula For The Exterior Derivative of a 1-form

Suppose $\omega \in \Omega^1(M)$. Recall that we said that ω is closed if

$$\frac{\partial \omega_j}{\partial x^i} - \frac{\partial \omega_i}{\partial x^j} = 0 \qquad \text{on every chart } (U, x^1, \ldots, x^n).$$

Notice two things:

(i) $\frac{\partial \omega_j}{\partial x^i} - \frac{\partial \omega_i}{\partial x^j}$ is antisymmetric in i, j, and it is the i, j-th component of $d\omega$:

$$d\omega = \sum_{1 \le i < j \le n} \left(\frac{\partial \omega_j}{\partial x^i} - \frac{\partial \omega_i}{\partial x^j} \right) dx^i \wedge dx^j.$$

This component is not coordinate-independent, but the property that it is zero is. Thus ω is closed if and only if $d\omega = 0$. We use this to generalize the notions of closedness and exactness to higher degree forms.

Definition 1. *$\omega \in \Omega^k(M)$ is closed if $d\omega = 0$, and exact if $\omega = d\eta$ for some $\eta \in \Omega^{k-1}(M)$.*

(ii) We showed that $\frac{\partial \omega_j}{\partial x^i} - \frac{\partial \omega_i}{\partial x^j} = 0$ on a coordinate open set U if and only if for all $X, Y \in \mathfrak{X}(U)$, $X(\omega(Y)) - Y(\omega(X)) - \omega([X, Y]) = 0$. One shows with an easy bump function argument that if this holds on every chart, then this holds for vector fields on M as well.

We therefore have that $\omega \in \Omega^1(M)$ is closed if and only if $d\omega = 0$, if and only if $X(\omega(Y)) - Y(\omega(X)) - \omega([X, Y]) = 0$ for all $X, Y \in \mathfrak{X}(M)$. This might lead one to thing that $d\omega$ and $X(\omega(Y)) - Y(\omega(X)) - \omega([X, Y])$ are related. In fact, they're the same thing.

Theorem 1.1. *For every $\omega \in \Omega^1(M)$, $d\omega$ is, by its action on vector fields, given by*

$$d\omega(X, Y) = X(\omega(Y)) - Y(\omega(X)) - \omega([X, Y]).$$

Proof. I will give a different proof than the one given in class. By linearity of both sides in ω, we may assume that $\omega = f dg$, where $f, g \in C^\infty(U)$ for some (coordinate) open set U. If X and Y are smooth vector fields, then the left side is

$$d(fdg)(X, Y) = df \wedge dg(X, Y) = df(X)dg(Y) - dg(X)df(Y) = (Xf)(Yg) - (Xg)(Yf),$$

and the right side is

$$X(fYg) - Y(fXg) - fdg([X, Y]) = ((Xf)(Yg) + fXYg) - ((Yf)(Xg) + fYXg) - f(XYg - YXg),$$

which simplifies to equal the left side. □

1.3 A Global Intrinsic Formula For The Exterior Derivative of a k-form

The general case is a not-so-straightforward of the case for 1-forms.

Theorem 1.2. *For every $\omega \in \Omega^k(M)$, $d\omega$ is, by its action on vector fields, given by*

$$d\omega(X_0, \dots, X_k) = \sum_{i=0}^{k} (-1)^i X_i \omega\left(X_0, \dots, \widehat{X_i}, \dots, X_k\right) + \sum_{0 \le i < j \le k} (-1)^{i+j} \omega\left([X_i, X_j], \dots, \widehat{X_i}, \dots, \widehat{X_j}, \dots, X_k\right).$$

Proof. We will be able to give a very short proof of this once we develop Cartan's homotopy formula. For now, we outline a proof that does not use this. Define

$$D\omega : \mathfrak{X}(M) \times \cdots \times \mathfrak{X}(M) \to C^\infty(M)$$

by the right-hand side of the formula. We can use uniqueness of the exterior derivative to prove that $d\omega = D\omega$. This proceeds in two main steps.

1. Show that $D\omega$ is a $(k+1)$-form by showing that $D\omega$ is $C^\infty(M)$-multilinear and alternating.
2. Show that D satisfies the characterizing properties of the exterior derivative to show that $d\omega = D\omega$, and conclude the result.

□