

# 1 Bump Functions and Partitions of Unity (Additional Reading)

## 1.1 Bump Functions

Define  $f : \mathbb{R} \rightarrow \mathbb{R}$  by

$$f(t) = \begin{cases} e^{-1/t} & t > 0 \\ 0 & t \leq 0 \end{cases}.$$

This will be the basic function off of which our bump functions will be modelled. The construction of bump functions on manifolds proceeds in four steps.

**Lemma 1.1.** *The function  $f$  defined above is  $C^\infty$ .*

*Proof.* We claim that for  $t > 0$  and  $k \geq 0$ , there is a polynomial  $p_{2k}$  of degree  $2k$  such that  $f^{(k)}(t) = p_{2k}(1/t)e^{-1/t}$ . For  $k = 0$  this is obvious, so suppose that this holds true for some  $k \geq 0$ . Then we have

$$\begin{aligned} f^{(k+1)}(t) &= \frac{d}{dt} p_{2k}\left(\frac{1}{t}\right) e^{-1/t} \\ &= -\frac{1}{t^2} p'_{2k}\left(\frac{1}{t}\right) e^{-1/t} + \frac{1}{t^2} p_{2k}\left(\frac{1}{t}\right) e^{-1/t} \\ &= \underbrace{\left[ -\frac{1}{t^2} p'_{2k}\left(\frac{1}{t}\right) + \frac{1}{t^2} p_{2k}\left(\frac{1}{t}\right) \right]}_{p_{2(k+1)}(1/t)} e^{-1/t}, \end{aligned}$$

so by induction the claim holds true.

(Finish this. Since  $f$  is  $C^\infty$  on  $\mathbb{R} \setminus 0$ , all we need to do is show that each  $f^{(k)}(0)$  makes sense and is equal to 0, by induction.)  $\square$

**Lemma 1.2.** *Given real numbers  $r_1 < r_2$ , there is a  $C^\infty$  function  $h : \mathbb{R} \rightarrow \mathbb{R}$  such that  $h^{-1}(1) = (-\infty, r_1]$ ,  $h^{-1}(0) = [r_2, \infty)$ , and  $0 < h(t) < 1$  for  $t \in (r_1, r_2)$ .*

*Proof.* Taking  $f$  as defined above, define

$$h(t) = \frac{f(r_2 - t)}{f(r_2 - t) + f(t - r_1)}.$$

This function is well-defined because  $f(r_2 - t) + f(t - r_1) = 0$  if and only if each is zero, which is true if and only if  $t \geq r_2$  and  $t \leq r_1$ , which is clearly impossible. Then  $h$  is  $C^\infty$ . It is clear that  $0 < h(t) < 1$  for  $t \in (r_1, r_2)$ , so we are left with checking the other two conditions.

$h(t) = 0$  if and only if  $f(r_2 - t) = 0$ , which holds if and only if  $r_2 - t \leq 0$ , which holds if and only if  $t \geq r_2$ . So  $h^{-1}(0) = [r_2, \infty)$ .

$h(t) = 1$  if and only if  $f(t - r_1) = 0$ , which holds if and only if  $t - r_1 \leq 0$ , which holds if and only if  $t \leq r_1$ . So  $h^{-1}(1) = (-\infty, r_1]$ .  $\square$

**Lemma 1.3.** *Given real numbers  $0 < r_1 < r_2$ , there is a  $C^\infty$  function  $H : \mathbb{R}^n \rightarrow \mathbb{R}$  such that  $H^{-1}(1) = \overline{B_{r_1}(0)}$ ,  $H^{-1}(0) = \mathbb{R}^n \setminus B_{r_2}(0)$ , and  $0 < H(x) < 1$  for  $x \in B_{r_2}(0) \setminus \overline{B_{r_1}(0)}$ .*

*Proof.* With  $h$  as in the previous lemma, define  $H(x) = h(\|x\|)$ . The function  $H$  is  $C^\infty$  because when  $\|x\| < r_1$ , it is identically 1, and it is a composition of  $C^\infty$  maps away from the origin. The rest of the lemma is clear.  $\square$

Using this, we now work in a coordinate chart to get a bump function on a manifold.

**Theorem 1.4. (Existence of bump functions)** *Given a smooth manifold  $M$ , a point  $q \in M$ , and a neighbourhood  $U$  of  $q$ , there exists a  $\rho \in C^\infty(M)$  such that  $\rho|_V \equiv 1$  on some neighbourhood  $V \subseteq U$  of  $q$ , and  $\text{supp}(\rho) \subseteq U$ .*

*Proof.* Choose a coordinate chart  $(W, \phi)$  at  $q$  such that  $\phi(q) = 0$ . The set  $\phi(W \cap U)$  is an open neighbourhood of the origin, so we can find  $0 < r_1 < r_2$  such that

$$0 = \phi(q) \in B_{r_1}(0) \subset B_{r_2}(0) \subset \phi(W \cap U).$$

This implies that

$$q \in \phi^{-1}(B_{r_1}(0)) \subset \phi^{-1}(B_{r_2}(0)) \subset W \cap U \subseteq U.$$

Let  $H$  be as in the previous lemma. Define

$$\rho(x) = \begin{cases} (H \circ \phi)(x) & x \in U \cap W \\ 0 & x \notin U \cap W \end{cases}.$$

The function  $\rho$  is  $C^\infty$  on  $U \cap W$  because it is a composition of  $C^\infty$  functions on an open set. If  $x \notin U \cap W$ , then  $x \notin \phi^{-1}(\overline{B_{r_2}(0)})$ , so we can find a neighbourhood of  $x$  on which  $\rho$  is identically zero. (Finish this.)  $\square$