

MAT367 Course Notes

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The following are course notes for the course MAT367 (Differential Geometry) offered in the Summer of 2020, taught by Ahmed Ellithy. The course notes are based off of handwritten notes created during lectures. They may contain errors or other false statements. The dates for each lecture are included, and any additional sections are supplementary material. (Exercises to very important/relevant problems, etc.)

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1 Introduction (May 5)

1.1 Trying to Define Things

The straightforward approach is

Definition 1.1.1. *A set $S \subseteq \mathbb{R}^3$ is a surface if there is an open set $U \subseteq \mathbb{R}^2$ and a smooth function $f : U \rightarrow \mathbb{R}$ for which $S = \Gamma_f$ is the graph of f .*

This isn't a great definition though. Its problem is that it's way too specific. The sphere $S^2 = \{x^2 + y^2 + z^2 = 1\}$ fails to be a surface under this definition, as it's not the graph of a function. We can remedy this by thinking about the following question:

If we were standing on a surface, what should our surroundings look like?

Here's another attempt at defining a surface, albeit in an imprecise way.

Definition 1.1.2. *A set $S \subseteq \mathbb{R}^3$ is a surface if for every $p \in S$ there is a neighbourhood of p in S that "looks like a piece of the plane".*

In more precise (but still not formal) wording, we are "locally diffeomorphic to pieces of \mathbb{R}^2 ". It turns out that this condition is equivalent to S being locally a graph; that follows from the implicit function theorem.

We'd like to generalize the above notions to define a k -dimensional "surface" in \mathbb{R}^n . Following in the footsteps of the previous definition, we obtain a new

Definition 1.1.3. *A set $S \subseteq \mathbb{R}^n$ is a k -dimensional manifold if it "locally looks like \mathbb{R}^k ".*

Equivalently, if for each $p \in S$ there is an open $U \subseteq \mathbb{R}^n$ containing p such that $S \cap U$ is the graph of a smooth function from an open subset of \mathbb{R}^k to \mathbb{R}^{n-k} .

The key idea with the last two definitions is that they are *local* - they are concerned with describing "pieces" of the surface or manifold, as opposed to the first definition being "global" by describing the entire surface.

1.2 Leaving \mathbb{R}^n for the Intrinsic View

Almost all of the geometry that is done on manifolds depends only on the manifold itself, and not on the space in which the manifold lies. (An example of Riemannian geometry is curvature.) Moreover, there are many sets we'd like to call manifolds whose points do not lie in Euclidean space. An example is *real projective space* $\mathbb{R}P^n$, which is defined as the quotient $(\mathbb{R}^{n+1} \setminus \{0\})/(x \sim \lambda x)$, where $\lambda \neq 0$. The real projective space contains equivalence classes of points of Euclidean space, so it is not a subset of Euclidean space. Therefore we'd like to define manifolds so that $\mathbb{R}P^n$ is an n -dimensional manifold.

Concisely, we would like to study manifolds *intrinsically*: we would like to drop all of the unnecessary data around our manifold and consider only the key properties of what a manifold should be.

2 Defining Manifolds (May 7)

2.1 Submanifolds of \mathbb{R}^n

We'll formally write out the definition of a k -manifold M in \mathbb{R}^n now.

Definition 2.1.1. *A subset $M \subseteq \mathbb{R}^n$ is a k -dimensional manifold if for every $p \in M$ there is an open neighbourhood U of p in \mathbb{R}^n , an open $V \subseteq \mathbb{R}^k$, and a function $f : V \rightarrow U \cap M$ such that*

1. *f is a homeomorphism,*
2. *f is smooth,*
3. *$Df(x)$ has rank k at every $x \in V$.*

The first two conditions are natural. Why the third? We'd like the *tangent space to M at p* to be a k -dimensional subspace of \mathbb{R}^n . If $Df(x)$ has rank k , then $Df(x)(\mathbb{R}^k)$ is a k -dimensional subspace of \mathbb{R}^n , which is what we would like $T_p M$ to be (roughly).

We have an equivalent definition, stated here as a theorem:

Theorem 2.1.1. *$M \subseteq \mathbb{R}^n$ is a k -manifold if and only if for each $p \in M$ there is an open neighbourhood U of p in \mathbb{R}^n , an open $V \subseteq \mathbb{R}^k$, and a smooth $f : V \rightarrow \mathbb{R}^{n-k}$ such that $U \cap M = \Gamma_f$ (up to a permutation of the coordinates in U).*

That last condition is a little odd, but what it means is that we can consider graphs of the form $(x, f(x))$ and $(f(y), y)$. This is essential in ensuring that, say, $S^1 = \{x^2 + y^2 = 1\}$ is a manifold. The definition may be shown to be equivalent to the old one using the implicit function theorem.

2.2 Topological Manifolds

By way of the subspace topology, every manifold in \mathbb{R}^n is Hausdorff and second countable. It turns out that these are the conditions we would like our abstract manifolds to have in order to exclude some pathological cases.

Definition 2.2.1. *A topological space M is locally Euclidean of dimension m if for each $p \in U$ there is an open neighbourhood U of p in M and a map $\phi : U \rightarrow \mathbb{R}^m$ which is a homeomorphism onto its image. The pair (U, ϕ) is called a coordinate chart, U is called a coordinate neighbourhood, and ϕ is called a coordinate system.*

Definition 2.2.2. *M is a topological manifold of dimension m if it is Hausdorff, second countable, and locally Euclidean of dimension m .*

Is the dimension of a topological manifold well-defined? That is, if (U, ϕ) and (V, ψ) are two coordinate charts with $U \cap V \neq \emptyset$ and $\phi(U) \subseteq \mathbb{R}^n$, $\psi(V) \subseteq \mathbb{R}^m$, is $n = m$? Consider the *transition mapping*

$$\psi \circ \phi^{-1} : \phi(U \cap V) \rightarrow \psi(U \cap V).$$

This is a homeomorphism from an open subset of \mathbb{R}^n to an open subset of \mathbb{R}^m . If $n \neq m$, this contradicts a non-trivial theorem called *Invariance of Domain*. We will not prove it here.

If we drop the Hausdorff condition, then the "line with two origins" becomes a topological manifold. If we drop the countable basis condition, then the "long line" becomes a topological manifold. These are both topological spaces that intuitively should not be manifolds - the extra conditions excludes them from being so.

2.3 Defining a Smooth Manifold

How should we define a smooth function on a manifold, say, $f : M \rightarrow \mathbb{R}$? The reasonable thing to do is to say that f is smooth if $f \circ \phi^{-1}$ is smooth, for some coordinate system ϕ . Then we run into a problem - this isn't independent of the choice of coordinate system, so long as M is only a topological manifold. We will define a *smooth structure* on M which allows us to make this natural definition.

Definition 2.3.1. *Two coordinate charts (U, ϕ) and (V, ψ) are said to be smoothly compatible (or C^∞ -compatible) if the transition mappings are diffeomorphisms, i.e.*

$$\begin{aligned}\psi \circ \phi^{-1} : \phi(U \cap V) &\rightarrow \psi(U \cap V) \\ \phi \circ \psi^{-1} : \psi(U \cap V) &\rightarrow \phi(U \cap V)\end{aligned}$$

are C^∞ maps of open subsets of Euclidean space.

Smooth compatibility is clearly a reflexive and symmetric relation. Is it transitive? Unfortunately, the answer is no. Suppose (U_1, ϕ_1) is smoothly compatible with (U_2, ϕ_2) and similarly for (U_2, ϕ_2) with (U_3, ϕ_3) . The natural thing to do is write

$$\phi_3 \circ \phi_1^{-1} = (\phi_3 \circ \phi_2^{-1}) \circ (\phi_2 \circ \phi_1^{-1}).$$

But this only makes sense on $\phi_1(U_1 \cap U_2 \cap U_3)$, which may be empty!

Definition 2.3.2. *A smooth atlas (or C^∞ atlas) on M is a collection of pairwise smoothly compatible coordinate charts covering M .*

We can now properly define a smooth function on a manifold. For unsaid technical reasons, however, it's beneficial to consider a little more structure. (Unfortunately the rest of the lecture went a little fast, as we ran out of time.)

Definition 2.3.3. *A smooth maximal atlas is a smooth atlas not contained in any other smooth atlas.*

Definition 2.3.4. *A smooth manifold of dimension n is a Hausdorff, second countable topological manifold of dimension n equipped with a smooth maximal atlas \mathcal{A} . The smooth maximal atlas \mathcal{A} is called a smooth structure on M .*

Lemma 2.3.1. *Any smooth atlas for M is contained in a unique maximal smooth atlas.*

The proof for this lemma proceeds roughly as follows: first one proves that if a two coordinate charts are smoothly compatible with a given atlas (meaning they are compatible with every chart in the atlas), then they are themselves compatible. Then one picks a smooth atlas and adjoins (by union) all of the charts with which the smooth atlas is compatible. It is then shown that this larger atlas is the desired unique maximal atlas.

Because of this lemma, we have a simple "test" for a smooth manifold.

Corollary 2.3.1. *A topological space M is a smooth manifold if and only if*

- 1. It is Hausdorff and second countable,*
- 2. It admits a smooth atlas.*

3 Smooth Structures, Examples (May 12)

3.1 More on Maximal Atlases

Consider the two atlases $\mathcal{A}_1 = \{(\mathbb{R}^n, Id)\}$ and $\mathcal{A}_2 = \{(B_1(x), Id) : x \in \mathbb{R}^n\}$ on \mathbb{R}^n . These two atlases determine the same maximal atlas, or the same smooth structure. Why? We have three equivalent reasons

- for any $(U, \phi) \in \mathcal{A}_1$ and $(V, \psi) \in \mathcal{A}_2$, the charts (U, ϕ) and (V, ψ) are C^∞ compatible.
- $\mathcal{A}_1 \cup \mathcal{A}_2$ is a C^∞ atlas.
- \mathcal{A}_1 and \mathcal{A}_2 belong to the same maximal atlas.

Define a relation \sim on the atlases by $\mathcal{A}_1 \sim \mathcal{A}_2$ if and only if $\mathcal{A}_1 \cup \mathcal{A}_2$ is another C^∞ atlas. Symmetry and reflexivity are immediate. For transitivity, suppose $\mathcal{A}_1 \cup \mathcal{A}_2$ and $\mathcal{A}_2 \cup \mathcal{A}_3$ are C^∞ atlases. Choose $(U_1, \phi_1) \in \mathcal{A}_1$ and $(U_3, \phi_3) \in \mathcal{A}_3$. We obtain a diffeomorphism

$$\phi_1 \circ \phi_3^{-1} = \phi_1 \circ \phi_2^{-1} \circ \phi_2 \circ \phi_3^{-1}$$

defined on $\phi_3(U_{13} \cap U_2)$. Since $\{U_2 : (U_2, \phi_2) \in \mathcal{A}_2 \text{ covers } M\}$, the map $\phi_1 \circ \phi_3^{-1}$ is smooth at every point of $\phi_3(U_{13})$. Therefore \sim is an equivalence relation.

Now given an atlas \mathcal{A} on M , we can talk about the equivalence class $[\mathcal{A}]$. Define

$$\mathcal{M} = \bigcup_{\mathcal{A}' \in [\mathcal{A}]} \mathcal{A}'.$$

Then \mathcal{M} is a new atlas on M ; it is the unique maximal atlas containing \mathcal{A} . (Exercise.)

So we can make the

Definition 3.1.1. *A smooth n -manifold M is a topological n -manifold with a maximal atlas. The choice of maximal atlas is called a smooth structure on M .*

Considering the previous remarks, we arrive at a sufficient condition for a space to be a smooth manifold: If M is a topological space for which

1. M is Hausdorff, second-countable, and
2. M admits a C^∞ atlas \mathcal{A}

then M is a smooth manifold with smooth structure $\mathcal{M} = \bigcup_{\mathcal{A}' \in [\mathcal{A}]} \mathcal{A}'$.

3.2 Examples

1. (Open subsets) Let M be a smooth n -manifold with a smooth atlas $\mathcal{A} = \{(U_\alpha, \phi_\alpha)\}$. Let $A \subseteq M$ be an open set. Then $\mathcal{A}_A = \{(U_\alpha \cap A, \phi_\alpha|_{U_\alpha \cap A})\}$ is a smooth atlas on A , so A is a smooth n -manifold.

2. (Finite dimensional vector spaces) Let V be a finite dimensional real vector space. Choose a basis $\beta = \{v_1, \dots, v_n\}$ of V , and consider the isomorphism $\Phi : V \rightarrow \mathbb{R}^n$ given by $\Phi(v_i) = e_i$.

Define a norm on V by $\|\sum a_i v_i\| := \|\sum a_i e_i\|$, where the norm on the right is the standard Euclidean norm. With this norm we may define an open ball in V as $B_r(v_0) = \{v \in V : \|v - v_0\| < r\}$. This gives a topology on V . Since all norms on finite dimensional vector spaces are equivalent, this topology does not depend on our choice of basis.

Then Φ is an isometry (it does not change distances), so it takes balls to balls and so does its inverse. That is, Φ is a homeomorphism, so we have a C^∞ atlas $\{(V, \Phi)\}$ on V , making V a smooth n -manifold.

This atlas determines a maximal atlas on V . Does this maximal atlas depend on the choice of basis? No. Choose another basis β' of V and define $\Phi' : V \rightarrow \mathbb{R}^n$ similarly. Then we'll get another C^∞ atlas $\{(V, \Phi')\}$ on V . The charts (U, Φ) and (V, Φ') are C^∞ -compatible, for the transition map $\Phi' \circ \Phi^{-1}$ is a linear isomorphism of \mathbb{R}^n with itself (certainly C^∞).

Remark: We also could have talked about complex vector spaces, since $\mathbb{C} \cong \mathbb{R}^2$.

3. (Matrices, general linear group) $\text{Mat}_{m \times n}(\mathbb{R}) \cong \mathbb{R}^{mn}$, so $\text{Mat}_{m \times n}(\mathbb{R})$ is a smooth manifold of dimension mn .

The general linear group is $GL(n, \mathbb{R}) = \{A \in \text{Mat}_{n \times n}(\mathbb{R}) : \det(A) \neq 0\}$. By continuity of \det it is an open subset of $\text{Mat}_{n \times n}(\mathbb{R})$, so by the first example we know it's a smooth n^2 -dimensional manifold.

4 More Examples, Quotients (May 14)

4.1 More Examples

1. (The circle) Define $S^1 = \{x^2 + y^2 = 1\} \subseteq \mathbb{R}^2$. We can define four functions on open sets of \mathbb{R} , the collection of which form a set of functions of which S^1 is locally the graph. Define an open cover $\{V_1, V_2, V_3, V_4\}$ of S^1 by

$$\begin{aligned} V_1 &= S^1 \cap ((0, \infty) \times (-1, 1)) && \text{"open right half"} \\ V_2 &= S^1 \cap ((-\infty, 0) \times (-1, 1)) && \text{"open left half"} \\ V_3 &= S^1 \cap ((-1, 1) \times (0, \infty)) && \text{"open top half"} \\ V_4 &= S^1 \cap ((-1, 1) \times (-\infty, 0)) && \text{"open bottom half"} \end{aligned}$$

Define $f_1, f_2, f_3, f_4 : (-1, 1) \rightarrow \mathbb{R}$ by

$$\begin{aligned} f_1(y) &= \sqrt{1 - y^2} && \text{so that } \Gamma_{f_1} = V_1 \\ f_2(y) &= -\sqrt{1 - y^2} && \text{so that } \Gamma_{f_2} = V_2 \\ f_3(x) &= \sqrt{1 - x^2} && \text{so that } \Gamma_{f_3} = V_3 \\ f_4(x) &= -\sqrt{1 - x^2} && \text{so that } \Gamma_{f_4} = V_4 \end{aligned}$$

What are the charts? Define $\phi_1 : V_1 \rightarrow (-1, 1)$ by $\phi_1(x, y) = y$. This is continuous with continuous inverse $\phi_1^{-1}(y) = (\sqrt{1 - y^2}, y)$. The other coordinate systems ϕ_2, ϕ_3, ϕ_4 are defined similarly. Consider

$$\mathcal{A} = \{(V_1, \phi_1), (V_2, \phi_2), (V_3, \phi_3), (V_4, \phi_4)\}.$$

We claim that \mathcal{A} is a smooth atlas on S^1 . For example, one transition map is $\phi_1 \circ \phi_3^{-1} : \phi_3(V_{13}) \rightarrow \phi_1(V_{13})$, which is a map from $(0, 1)$ to itself. It is given by

$$(\phi_1 \circ \phi_3^{-1})(t) = \phi_1(t, \sqrt{1 - t^2}) = \sqrt{1 - t^2},$$

which is a diffeomorphism of $(0, 1)$ with itself. As a similar proposition holds for the other transition maps, we conclude that (S^1, \mathcal{A}) is a smooth manifold of dimension 1.

Let $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ be $f(x, y) = x^2 + y^2$. Then $S^1 = f^{-1}(1)$ (preimage). We get a collection of 1-dimensional manifolds covering $\mathbb{R}^2 \setminus \{0\}$; we say that $\{f^{-1}(r) : r > 0\}$ is a *one-dimensional foliation* of $\mathbb{R}^2 \setminus \{0\}$. (More on that in a later lecture.)

2. (Level sets) Consider a smooth map $F : \mathbb{R}^{n+1} \rightarrow \mathbb{R}$. Let $c \in \mathbb{R}$ be such that $F^{-1}(c) \neq \emptyset$ and $\nabla F(a) \neq 0$ for each $a \in F^{-1}(c)$.

For example, if $F(x) = \|x\|^2$, then $S^n = F^{-1}(1)$ and $\nabla F|_{F^{-1}(c)} \neq 0$. (We say $\{F^{-1}(r) : r > 0\}$ is an *n-dimensional foliation* of $\mathbb{R}^{n+1} \setminus \{0\}$.)

Choose $a \in F^{-1}(c)$. Then $DF(a) \neq 0$, so there is an i such that $\frac{\partial F}{\partial x_i}(a) \neq 0$. Then the equation $F(x_1, \dots, x_i, \dots, x_{n+1}) = c$ can be solved locally for x_i in terms of the other coordinates, i.e. $F^{-1}(c)$ is the graph of a smooth function near a .

Making this precise, the implicit function theorem provides us with a neighbourhood U of $(a_1, \dots, \hat{a}_i, \dots, a_{n+1})$ in \mathbb{R}^n and a smooth function $g : U \rightarrow \mathbb{R}$ satisfying

- $g(a_1, \dots, \hat{a}_i, \dots, a_{n+1}) = a_i$,
- $F(x_1, \dots, g(x_1, \dots, \hat{x}_i, \dots, x_{n+1}), \dots, x_{n+1}) = c$ for all $(x_1, \dots, \hat{x}_i, \dots, x_{n+1}) \in U$,

i.e.

$$\Gamma_g = \{(x_1, \dots, g(x_1, \dots, \hat{x}_i, \dots, x_{n+1}), \dots, x_{n+1}) \in \mathbb{R}^{n+1} : (x_1, \dots, \hat{x}_i, \dots, x_{n+1}) \in U\} = V \cap F^{-1}(c)$$

for some neighbourhood V of a in \mathbb{R}^{n+1} .

So we conclude that if $\nabla F(a) \neq 0$ for all $a \in F^{-1}(c) \neq \emptyset$, then $F^{-1}(c)$ is locally the graph of a function. What are the charts? $(V \cap F^{-1}(c), \phi)$, where $\phi : V \cap F^{-1}(c) \rightarrow U$ is given by $\phi(x_1, \dots, x_{n+1}) = (x_1, \dots, \hat{x}_i, \dots, x_{n+1})$ with the inverse $\phi^{-1}(x_1, \dots, \hat{x}_i, \dots, x_{n+1}) = (x_1, \dots, g(x_1, \dots, \hat{x}_i, \dots, x_{n+1}), \dots, x_{n+1})$. This is clearly a chart.

Now consider the collection of such charts $\mathcal{A} = \{(V_a \cap F^{-1}(c), \phi_a)\}$. Consider a transition mapping $\phi_a \circ \phi_b^{-1} : \phi_b(V_{ab}) \rightarrow \phi_a(V_{ab})$. This is

$$\begin{aligned} (\phi_a \circ \phi_b^{-1})(x_1, \dots, x_i, \dots, \hat{x}_j, \dots, x_{n+1}) &= \phi_a(x_1, \dots, x_i, \dots, g_b(x_1, \dots, \hat{x}_j, \dots, x_{n+1}), \dots, x_{n+1}) \\ &= (x_1, \dots, \hat{x}_i, \dots, g_b(x_1, \dots, \hat{x}_j, \dots, x_{n+1}), \dots, x_{n+1}) \end{aligned}$$

which is C^∞ , and similarly for its inverse. So \mathcal{A} is a C^∞ atlas on $F^{-1}(c)$, making $F^{-1}(c)$ a smooth manifold of dimension n .

3. (Products) Consider two smooth manifolds M and N of dimensions m and n , respectively. Equip them with smooth atlases \mathcal{A}_M and \mathcal{A}_N , respectively. Define

$$\mathcal{A}_{M \times N} = \{(U \times V, \phi \times \psi) : (U, \phi) \in \mathcal{A}_M \text{ and } (V, \psi) \in \mathcal{A}_N\}.$$

$\mathcal{A}_{M \times N}$ is a smooth atlas on $M \times N$, making $M \times N$ a smooth manifold of dimension $m + n$. To see this, note that the sets $U \times V$ certainly cover $M \times N$, and that the products of homeomorphisms are homeomorphisms. If $(U_1 \times V_1, \phi_1 \times \psi_1), (U_2 \times V_2, \phi_2 \times \psi_2) \in \mathcal{A}_{M \times N}$, then the transition map

$$(\phi_1 \times \psi_1) \circ (\phi_2 \times \psi_2)^{-1} : (\phi_2 \times \psi_2)((U_1 \times V_1) \cap (U_2 \times V_2)) \rightarrow (\phi_1 \times \psi_1)((U_1 \times V_1) \cap (U_2 \times V_2))$$

is, by set theory, equal to

$$(\phi_1 \circ \phi_2^{-1}) \times (\psi_1 \circ \psi_2^{-1}) : \phi_2(U_{12}) \times \psi_2(V_{12}) \rightarrow \phi_1(U_{12}) \times \psi_1(V_{12}),$$

which is clearly a diffeomorphism.

For example, the cylinder $S^1 \times \mathbb{R}$ is a smooth manifold of dimension 2, and the torus $S^1 \times S^1$ is a smooth manifold of dimension 2. We also have the higher tori $T^n = S^1 \times \cdots \times S^1$, a smooth manifold of dimension n .

(Algebraic topology remark: $T^n \not\cong S^n$, as the former has first fundamental group \mathbb{Z}^n , whereas the latter is simply connected for $n \geq 2$.)

4.2 Gluing Manifolds

Due to the informal visual nature of this part of the lecture, the examples can only be described in words.

1. Glue the endpoints of $[0, 1]$ to get the circle. They aren't homeomorphic however, since removing an interior point from $[0, 1]$ disconnects it, whereas the circle will remain connected if a point is removed.
2. Glue the two vertical sides of $[0, 1]^2$ to get a cylinder. (Note: in order to visualize this, we need to go up one dimension.)
3. Glue the two vertical sides of $[0, 1]^2$, but with points identified "by reflecting through the centre $(1/2, 1/2)$ ". This produces a Mobius strip.
4. Glue the opposite sides of $[0, 1]^2$ together as in example 2, but with each opposite side glued. This produces a torus.
5. Glue the opposite vertical sides of $[0, 1]^2$ together as in example 2, and the opposite horizontal sides together as in example 3. This produces a "Klein bottle", an example of a manifold which cannot be embedded in \mathbb{R}^3 .

4.3 The Quotient Topology

Let S be a topological space and \sim an equivalence relation on S . Let $\pi : S \rightarrow S/\sim$ be the projection map $\pi(x) = [x]$. Topologize S/\sim by declaring $U \subseteq S/\sim$ to be open if and only if $\pi^{-1}(U)$ is open in S . This topology on S/\sim is called the *quotient topology* - it is the finest topology on S/\sim with respect to which π is continuous, as is easily seen.

Now consider a function $f : S \rightarrow Y$, where Y is a set. Suppose f is constant on the fibres of π (i.e. f is constant on every equivalence class of \sim). Then f induces a map $\tilde{f} : S/\sim \rightarrow Y$ for which the following diagram is commutative:

$$\begin{array}{ccc} S & & \\ \downarrow \pi & \searrow f & \\ S/\sim & \xrightarrow{\tilde{f}} & Y \end{array}$$

The function \tilde{f} is defined in the obvious way: $\tilde{f}([x]) = f(x)$. The new function \tilde{f} is well-defined since we assumed f was constant on equivalence classes. We say that f *descends to the quotient*. If Y is a topological space, we have a very useful lemma.

Lemma 4.3.1. *Suppose $f : S \rightarrow Y$ is a function of topological spaces, and that \sim is an equivalence relation on S on whose equivalence classes f is constant. Then the induced map $\tilde{f} : S/\sim \rightarrow Y$ is continuous if and only if f is continuous.*

Proof. If \tilde{f} is continuous, then $f = \tilde{f} \circ \pi$ is continuous as a composition of continuous maps. If f is continuous, then given U open in Y , $f^{-1}(U)$ is open in S . But $f^{-1}(U) = \pi^{-1}(\tilde{f}^{-1}(U))$, so by the definition of the quotient topology, $\tilde{f}^{-1}(U)$ is open in S/\sim , proving continuity of \tilde{f} . \square

Let's discuss the example of gluing the endpoints of the interval. Define \sim on $I = [0, 1]$ by $x \sim x$ for $x \in (0, 1)$ and $x \sim y$ for $x, y \in \{0, 1\}$. We claim that $I/\sim \cong S^1$. An explicit homeomorphism can be found by descending to the quotient.

Define $f : I \rightarrow S^1$ by $f(t) = (\cos 2\pi t, \sin 2\pi t)$. Then $f(0) = f(1) = (1, 0)$, so f is constant on the equivalence classes of \sim . Then f descends to a continuous map $\tilde{f} : I/\sim \rightarrow S^1$, given by

$$\tilde{f}([t]) = \begin{cases} (\cos 2\pi t, \sin 2\pi t), & [t] \neq [0] \\ (1, 0), & t = [0] = [1] \end{cases}$$

which is bijective. Since $I/\sim = \pi(I)$ is compact and S^1 is Hausdorff, the map \tilde{f} is a homeomorphism of topological spaces. So indeed, $I/\sim \cong S^1$.

In order to tackle the question of "when is a quotient a manifold", we need to derive some conditions for when the quotient of a space is Hausdorff or second countable. Here's a simple necessary condition.

Lemma 4.3.2. *If S/\sim is Hausdorff, then equivalence classes are closed in S .*

Proof. Each $\{[x]\} = \{\pi(x)\}$ is closed in S/\sim by Hausdorffness, so by continuity $\pi^{-1}(\{\pi(x)\}) = [x]$ is closed in S . \square

For a simple application of this necessary condition, consider $\mathbb{R}/(0, \infty)$ - the quotient space obtained by identifying all points of $(0, \infty)$. The lemma dictates that $\mathbb{R}/(0, \infty)$ is not Hausdorff because the equivalence class $(0, \infty)$ is not closed in \mathbb{R} .

4.4 Open Equivalence Relations

Definition 4.4.1. *An equivalence relation \sim on a space S is said to be open if the projection $\pi : S \rightarrow S/\sim$ is an open mapping. Equivalently, \sim is open if and only if*

$$\pi^{-1}(\pi(U)) = \bigcup_{x \in U} [x]$$

is open in S , for each U open in S .

This definition is worth making, as the projections need not be open in general. Consider $\mathbb{R}/\{-1, 1\}$. The interval $(-2, 0)$ is open, but

$$\pi^{-1}(\pi((-2, 0))) = \bigcup_{-2 < x < 0} [x] = (-2, 0) \cup \{1\}$$

is not open in \mathbb{R} . Therefore \sim identifying -1 and 1 on \mathbb{R} is not an open equivalence relation. (Note that $\mathbb{R}/\{-1, 1\}$ is not a topological manifold, as it is homeomorphic to the symbol \propto with the ends extending infinitely.)

Definition 4.4.2. *The graph of an equivalence relation \sim on S is the set $R = \{(x, y) \in S \times S : x \sim y\}$.*

Theorem 4.4.1. *Suppose \sim is an open equivalence relation on S . Then S/\sim is Hausdorff if and only if the graph R of \sim is closed in $S \times S$.*

Proof. Was left as an exercise in class, so here's a solution. We have a sequence of equivalent statements

$$\begin{aligned} R \text{ is closed} &\iff S \times S \setminus R \text{ is open} \\ &\iff \text{for all } (x, y) \in S \times S \setminus R \text{ there are open sets } U, V \text{ such that } (x, y) \in U \times V \subseteq S \times S \setminus R \\ &\iff \text{for all } x \not\sim y \text{ in } S \text{ there are open sets } U \ni x, V \ni y \text{ such that } (U \times V) \cap R = \emptyset \\ &\iff \text{for all } [x] \neq [y] \text{ in } S/\sim \text{ there are open sets } U \ni x, V \ni y \text{ such that } \pi(U) \cap \pi(V) = \emptyset \end{aligned}$$

This last statement is equivalent to S/\sim being Hausdorff, which we now prove. If this statement is true, then $\pi(U)$ and $\pi(V)$ are disjoint open (because \sim is open) sets of S/\sim separating $[x]$ and $[y]$, which shows that S/\sim is Hausdorff. Conversely, suppose S/\sim is Hausdorff. Given $[x] \neq [y]$ in S/\sim , we can find disjoint open sets $U \ni [x]$, $V \ni [y]$ of S/\sim . By surjectivity, $U = \pi(\pi^{-1}(U))$ and $V = \pi(\pi^{-1}(V))$, so $\pi^{-1}(U)$ and $\pi^{-1}(V)$ are open sets of S containing x and y , respectively, satisfying the condition of the last statement. So the last statement is equivalent to S/\sim being Hausdorff. \square

With it is a corollary - a classic exercise in point-set topology.

Corollary 4.4.1. *S is Hausdorff if and only if $\Delta = \{(x, x) \in S \times S : x \in S\}$ is closed.*

Proof. Let \sim be the equivalence relation identifying every point only with itself. Then \sim is an open equivalence relation and $R = \Delta$. The spaces S and S/\sim are homeomorphic, so the statement follows from the theorem immediately. \square

It turns out that the above theorem and its corollary are equivalent. It's not too hard to see that the corollary implies the theorem by using the fact that π is continuous and open.

What about second countability?

Theorem 4.4.2. *If \sim is an open equivalence relation on S and $\{B_n\}$ is a countable basis of S , then $\{\pi(B_n)\}$ is a countable basis of S/\sim .*

Proof. Was left as an exercise in class, so here's a solution. Note that the collection $\{\pi(B_n)\}$ is a collection of open sets because π is an open mapping. Let $U \subseteq S/\sim$ be open and consider $[x] \in U$. Then $x \in \pi^{-1}(U)$, so we can find a B_n with $x \in B_n \subseteq \pi^{-1}(U)$. Then $[x] = \pi(x) \subseteq \pi(B_n) \subseteq \pi(\pi^{-1}(U)) = U$, proving that $\{\pi(B_n)\}$ is a basis of S/\sim . \square

To summarize,

- quotient spaces of Hausdorff spaces under open equivalence relations are Hausdorff if and only if the graph of the relation is closed
- quotient spaces of second-countable spaces under open equivalence relations are second-countable, and bases for the quotient are obtained in the obvious way.

4.5 Real Projective Space

Define \sim on $\mathbb{R}^{n+1} \setminus \{0\}$ by $x \sim \lambda x$ for $\lambda \neq 0$. The quotient space $(\mathbb{R}^{n+1} \setminus \{0\})/\sim$ is denoted $\mathbb{R}P^n$ and is called *real projective space*. It may be thought of as the set of lines passing through the origin.

Each element of $\mathbb{R}P^n$ can be thought of as a pair of antipodal points on S^n , which motivates the following

Theorem 4.5.1. *Define \sim on S^n by identifying antipodal points, i.e. $x \sim \pm x$. Define $f : \mathbb{R}^{n+1} \setminus \{0\} \rightarrow S^n$ by $f(x) = \frac{x}{\|x\|}$. Then f induces a homeomorphism $\mathbb{R}P^n \xrightarrow{\sim} S^n/\sim$.*

The proof will be essentially the proof given in class, but much more complete and explicit about how maps induce other maps.

Proof. Consider the following diagram:

$$\begin{array}{ccc} \mathbb{R}^{n+1} \setminus \{0\} & \xrightarrow{f} & S^n \\ \downarrow \pi_1 & & \downarrow \pi_2 \\ \mathbb{R}P^n & & S^n/\sim \end{array}$$

where π_1 and π_2 are the projections to each quotient space as shown in the diagram. The map $\pi_2 \circ f : \mathbb{R}^{n+1} \setminus \{0\} \rightarrow S^n/\sim$ is given by

$$(\pi_2 \circ f)(x) = \pi_2 \left(\frac{x}{\|x\|} \right) = \left\{ -\frac{x}{\|x\|}, \frac{x}{\|x\|} \right\} = [x]_2,$$

which is continuous and constant on the fibres of π_1 ; the lines through the origin. It thus induces a continuous map $\tilde{f} : \mathbb{R}P^n \rightarrow S^n/\sim$ for which the following diagram is commutative:

$$\begin{array}{ccc} \mathbb{R}^{n+1} \setminus \{0\} & \xrightarrow{f} & S^n \\ \downarrow \pi_1 & \searrow \pi_2 \circ f & \downarrow \pi_2 \\ \mathbb{R}P^n & \xrightarrow{\tilde{f}} & S^n/\sim \end{array}$$

We define a continuous inverse of \tilde{f} . Consider $g : S^n \rightarrow \mathbb{R}^{n+1} \setminus \{0\}$ given by $g(x) = x$. As before, consider the diagram

$$\begin{array}{ccc} \mathbb{R}^{n+1} \setminus \{0\} & \xleftarrow{g} & S^n \\ \downarrow \pi_1 & & \downarrow \pi_2 \\ \mathbb{R}P^n & & S^n/\sim \end{array}$$

The map $\pi_1 \circ g : S^n \rightarrow \mathbb{R}P^n$ is given by

$$(\pi_1 \circ g)(x) = \pi_1(x) = \{\lambda x : \lambda \neq 0\} = [x]_1,$$

which is continuous and constant on the fibres of π_2 ; antipodal points on the n -sphere. It thus induces a continuous map $\tilde{g} : S^n/\sim \rightarrow \mathbb{R}P^n$ for which the following diagram is commutative:

$$\begin{array}{ccc} \mathbb{R}^{n+1} \setminus \{0\} & \xleftarrow{g} & S^n \\ \downarrow \pi_1 & \swarrow \pi_1 \circ g & \downarrow \pi_2 \\ \mathbb{R}P^n & \xleftarrow{\tilde{g}} & S^n/\sim \end{array}$$

We claim that \tilde{f} and \tilde{g} are inverses to each other, which will show that \tilde{f} is a homeomorphism $\mathbb{R}P^n \xrightarrow{\sim} S^n/\sim$. We have

$$(\tilde{g} \circ \tilde{f})([x]_1) = (\tilde{g} \circ \tilde{f} \circ \pi_1)(x) = (\tilde{g} \circ \pi_2 \circ f)(x) = (\pi_1 \circ g \circ f)(x) = \pi_1 \left(g \left(\frac{x}{\|x\|} \right) \right) = \pi_1 \left(\frac{x}{\|x\|} \right) = [x]_1$$

$$(\tilde{f} \circ \tilde{g})([x]_2) = (\tilde{f} \circ \tilde{g} \circ \pi_2)(x) = (\tilde{f} \circ \pi_1 \circ g)(x) = (\pi_2 \circ f \circ g)(x) = \pi_2(f(x)) = \pi_2 \left(\frac{x}{\|x\|} \right) = [x]_2$$

So \tilde{f} is a homeomorphism $\mathbb{R}P^n \xrightarrow{\sim} S^n/\sim$. □

In particular, $\mathbb{R}P^n$ is compact! Note that we could have just explicitly defined

$$\begin{aligned} \tilde{f} : \mathbb{R}P^n &\rightarrow S^n/\sim & \tilde{f}([x]_1) &:= \pi_2(f(x)) = \left[\frac{x}{\|x\|} \right]_2 \\ \tilde{g} : S^n/\sim &\rightarrow \mathbb{R}P^n & \tilde{g}([x]_2) &:= \pi_1(g(x)) = [x]_1 \end{aligned}$$

checked for well-definedness and continuity, and then we'd have been done. That's how the proof on page 362 of Tu goes. However, the abuse of tikz diagrams makes it very clear where the homeomorphism and its inverse come from, and that they're continuous (which is basically what Tu is doing anyway).

4.6 Visualizing $\mathbb{R}P^2$

In order to visualize $\mathbb{R}P^2$ we will consider some homeomorphisms. Define

$$\begin{aligned} H^2 &= \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1, z \geq 0\} \\ D^2 &= \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}. \end{aligned}$$

Consider the maps

$$\begin{aligned} \phi : H^2 &\rightarrow D^2 & \phi(x, y, z) &= (x, y) \\ \psi : D^2 &\rightarrow H^2 & \psi(x, y) &= (x, y, \sqrt{1 - x^2 - y^2}) \end{aligned}$$

which are continuous inverses of each other. Define equivalence relations on H^2 and D^2 as follows:

- On H^2 : identify antipodal points on the equator, call the projection π_3
- On D^2 : identify antipodal points on the boundary, call the projection π_4

Considering diagrams similar to those in the previous proof, the map $\pi_4 \circ \phi$ induces a continuous map $\tilde{\phi} : H^2/\sim \rightarrow D^2/\sim$ with $\tilde{\phi} \circ \pi_3 = \pi_4 \circ \phi$, and the map $\pi_3 \circ \psi$ induces a continuous map $\tilde{\psi} : D^2/\sim \rightarrow H^2/\sim$ with $\tilde{\psi} \circ \pi_4 = \pi_3 \circ \psi$. The maps $\tilde{\phi}$ and $\tilde{\psi}$ are continuous inverses of each other (which can be seen using just these given compositions), which shows that we have a homeomorphism $H^2/\sim \xrightarrow{\sim} D^2/\sim$.

If we accept on faith that there is a homeomorphism $S^2/\sim \xrightarrow{\sim} H^2/\sim$, then we have a sequence of homeomorphisms

$$\mathbb{R}P^2 \xrightarrow{\sim} S^2/\sim \xrightarrow{\sim} H^2/\sim \xrightarrow{\sim} D^2/\sim.$$

Therefore we can visualize the real projective plane $\mathbb{R}P^2$ as a disk with the antipodal boundary points identified. Such a homeomorphism $S^2/\sim \xrightarrow{\sim} H^2/\sim$ can be shown by a proof similar to the previous quotient space homeomorphisms that we did, by considering the inclusion map $i : H^2 \rightarrow S^2$ and its obvious inverse, and working through steps similar to the proofs of the previous homeomorphisms.

5 Smooth Maps and Differentiable Structures (May 19)

5.1 Smooth Maps on a Manifold

The notion of the pullback of a function on a manifold (which by MAT257 we know is a 0-form on a manifold - not that that's important right now) is the following:

Definition 5.1.1. *Let $F : M \rightarrow N$ and $f : N \rightarrow \mathbb{R}$ be functions. The pullback of f by F is the function $F^*f : M \rightarrow \mathbb{R}$ defined by $F^*f = f \circ F$. That is, the pullback of f by F is the unique function for which the following diagram commutes:*

$$\begin{array}{ccc} M & & \\ \downarrow F & \searrow F^*f & \\ N & \xrightarrow{f} & \mathbb{R} \end{array}$$

Now for the main definitions.

Definition 5.1.2. *Fix a smooth manifold M . A function $f : M \rightarrow \mathbb{R}$ is C^∞ at $p \in M$ if there is a chart (U, ϕ) about p such that $f \circ \phi^{-1} : \phi(U) \rightarrow \mathbb{R}$ is C^∞ at $\phi(p)$, in the usual sense. Alternatively, f is C^∞ at p if the pullback $(\phi^{-1})^*f$ of f by the inverse of some coordinate system ϕ about p is C^∞ at $\phi(p)$.*

We'd like to show that this does not depend on the choice of chart about p . If (V, ψ) is another chart about p , then

$$f \circ \psi^{-1} = (f \circ \phi^{-1}) \circ (\phi \circ \psi^{-1}),$$

is C^∞ at $\psi(p)$ on the open set $\psi(U \cap V)$, since $\phi \circ \psi^{-1}$ is C^∞ at $\psi(p)$ and $f \circ \phi^{-1}$ is C^∞ at $\phi(p)$. Therefore smoothness of a function on a manifold at a point doesn't depend on the choice of chart about that point. We will say that f is C^∞ on M if it is C^∞ at every point of M . Note that if $f : M \rightarrow \mathbb{R}$ is C^∞ at p , then $f = (f \circ \phi^{-1}) \circ \phi$ is continuous at p .

These considerations give us a

Proposition 1. *Let $f : M \rightarrow \mathbb{R}$ be a continuous function on a smooth manifold M . The following are equivalent:*

- (i) $f : M \rightarrow \mathbb{R}$ is C^∞ .
- (ii) There is an atlas \mathcal{A} of M such that for any $(U, \phi) \in \mathcal{A}$, the function $f \circ \phi^{-1} : \phi(U) \rightarrow \mathbb{R}$ is C^∞ .

Note that we implicitly assume \mathcal{A} in the above is a subset of our choice of maximal atlas for M . When we say M is a smooth manifold, we also assume a choice of maximal atlas has been made.

What about maps between manifolds? The definition is a natural extension of the one we just made.

Definition 5.1.3. Let N and M be smooth manifolds and let $F : N \rightarrow M$ be continuous. We say F is C^∞ at $p \in N$ if there is a chart (V, ψ) about $F(p)$ and a chart (U, ϕ) about p such that

$$\psi \circ F \circ \phi^{-1} : \phi(U \cap F^{-1}(V)) \rightarrow \mathbb{R}^m$$

is C^∞ at $\phi(p)$.

Note that continuity of F was essential, for if that were not the case, the set $\phi(U \cap F^{-1}(V))$ may not be open, in which case we may not be able to talk about smoothness at p .

As before, we check that this is independent of the charts. Choose charts $(\tilde{U}, \tilde{\phi})$ about p and $(\tilde{V}, \tilde{\psi})$ about $F(p)$. Then

$$\tilde{\psi} \circ F \circ \tilde{\phi}^{-1} = (\tilde{\psi} \circ \psi^{-1}) \circ (\psi \circ F \circ \phi^{-1}) \circ (\phi \circ \tilde{\phi}^{-1})$$

is C^∞ at $\tilde{\phi}(p)$ by similar reasoning as before. We say that $F : N \rightarrow M$ is C^∞ if it is so at every point of N .

We have a similar proposition coming from the independence of charts:

Proposition 2. Let $F : N \rightarrow M$ be a continuous function of smooth manifolds N and M . The following are equivalent:

- (i) F is C^∞ on N .
- (ii) There are atlases \mathcal{A} of N and \mathcal{B} of M such that for every $(U, \phi) \in \mathcal{A}$ and $(V, \psi) \in \mathcal{B}$, the map $\psi \circ F \circ \phi^{-1} : \phi(U \cap F^{-1}(V)) \rightarrow \mathbb{R}^m$ is C^∞ .

We need to make sure this is actually a generalization of the notion of smoothness we know from calculus. We will make sure that our definition is the usual notion of smoothness when the manifolds are Euclidean spaces, and we will make sure that smoothness is preserved by compositions.

Proposition 3. If $N = \mathbb{R}^n$ and $M = \mathbb{R}^m$ are given their usual smooth structures, then $F : N \rightarrow M$ is smooth as defined above if and only if it is smooth as a function of Euclidean spaces.

Proof. Choose the atlases $\{(\mathbb{R}^n, \text{Id}_{\mathbb{R}^n})\}$ on \mathbb{R}^n and $\{(\mathbb{R}^m, \text{Id}_{\mathbb{R}^m})\}$ on \mathbb{R}^m . Then $F : N \rightarrow M$ is smooth as defined above if and only if

$$\text{Id}_{\mathbb{R}^m} \circ F \circ \text{Id}_{\mathbb{R}^n}^{-1} : N \rightarrow M$$

is smooth. But this function is just $F : \mathbb{R}^n \rightarrow \mathbb{R}^m$. □

Note that this holds if N and M had merely been open sets of Euclidean spaces, for the usual smooth structure on them (i.e. the one we do ordinary calculus with) is the maximal atlas corresponding to the restrictions of the charts given above to those open sets.

Proposition 4. If $F : N \rightarrow M$ and $G : M \rightarrow P$ are C^∞ maps of manifolds, then $G \circ F : N \rightarrow P$ is C^∞ .

Proof. Suppose $p \in N$. Choose charts (U, ϕ) about p , (V, ψ) about $F(p)$, and (W, σ) about $G(F(p))$. Then

$$\sigma \circ (G \circ F) \circ \phi^{-1} = (\sigma \circ G \circ \psi^{-1}) \circ (\psi \circ F \circ \phi^{-1})$$

is C^∞ at $\phi(p)$, since $\sigma \circ G \circ \psi^{-1}$ is C^∞ at $\psi(F(p))$ and $\psi \circ F \circ \phi^{-1}$ is C^∞ at $\phi(p)$. \square

We have one last property: vector-valued functions behave how we want them to.

Proposition 5. *Let N be a smooth manifold and $F : N \rightarrow \mathbb{R}^m$ be a continuous function. The following are equivalent:*

- (i) F is C^∞ .
- (ii) Each component function $F^i : N \rightarrow \mathbb{R}$ is smooth.

Proof. The proof was left as an exercise, so here's a solution. We have

$$\begin{aligned} F \text{ is } C^\infty &\iff \text{for every chart } (U, \phi) \text{ on } N, \text{ the map } F \circ \phi^{-1} : \phi(U) \rightarrow \mathbb{R}^m \text{ is } C^\infty \\ &\iff \text{for each } i \text{ and for every chart } (U, \phi) \text{ on } N, \text{ the map } F^i \circ \phi^{-1} : \phi(U) \rightarrow \mathbb{R} \text{ is } C^\infty \\ &\iff \text{for each } i, \text{ the map } F^i : N \rightarrow \mathbb{R} \text{ is } C^\infty. \end{aligned}$$

\square

Just as two vector spaces or groups are equivalent if they are isomorphic, or two topological spaces are equivalent if they are homeomorphic, or two sets are equivalent if they are in bijection with each other, we have a notion of "isomorphism" or equivalence of smooth manifolds.

Definition 5.1.4. *A function $F : N \rightarrow M$ of smooth manifolds is said to be a diffeomorphism if it is smooth and has a smooth inverse.*

Then we can state: *differential topology is the study of properties of smooth manifolds invariant under diffeomorphism.*

5.2 Differentiable Structures

We can exhibit two diffeomorphic but unequal smooth structures on \mathbb{R} . Define two atlases

$$\begin{aligned} \mathcal{A}_1 &= \{(\mathbb{R}, \text{Id})\} && \text{(call this one } \mathbb{R}) \\ \mathcal{A}_2 &= \{(\mathbb{R}, \psi(x) := x^3)\} && \text{(call this one } \mathbb{R}') \end{aligned}$$

These charts are not C^∞ compatible, since $\text{Id} \circ \psi^{-1}$ sends x to $\sqrt[3]{x}$; not a diffeomorphism. Therefore the smooth structures corresponding to \mathcal{A}_1 and \mathcal{A}_2 are different.

Nevertheless, define $f : \mathbb{R} \rightarrow \mathbb{R}'$ by $f(x) = \sqrt[3]{x}$. Then

$$\psi \circ f \circ \text{Id}^{-1} : \mathbb{R} \rightarrow \mathbb{R}' \quad x \mapsto x$$

is a diffeomorphism!

We can exhibit non-diffeomorphic smooth structures on manifolds; see the exotic sphere S^7 . Even better, \mathbb{R}^4 has uncountably many smooth structures *up to diffeomorphism*. It is known that every topological manifold of dimension < 4 admits a unique smooth structure, up to diffeomorphism.

6 Inverse Function Theorem, Tangent Spaces (May 21)

6.1 Diffeomorphisms and Coordinate Systems

By convention, any manifold labelled N will have dimension n and any labelled M will have dimension m .

Definition 6.1.1. *A diffeomorphism $F : N \rightarrow M$ of smooth manifolds is a bijective smooth map with smooth inverse.*

Proposition 6. *Coordinate systems are diffeomorphisms.*

Proof. Let M be a smooth manifold and (U, ϕ) a coordinate chart on M . Choose the atlases

$$\begin{aligned} &\{(U, \phi)\} \text{ on } U \\ &\{\phi(U), \text{Id}\} \text{ on } \phi(U). \end{aligned}$$

Then

$$\begin{aligned} &\text{Id} \circ \phi \circ \phi^{-1} : \phi(U) \rightarrow \phi(U) \\ &\phi \circ \phi^{-1} \circ \text{Id}^{-1} : \phi(U) \rightarrow \phi(U) \end{aligned}$$

are both smooth, implying that ϕ and ϕ^{-1} are smooth, respectively. \square

The converse is true; it uses maximality of the smooth structure.

Proposition 7. *Diffeomorphisms from open subsets of manifolds to open subsets of Euclidean space are coordinate systems belonging to the manifold's smooth structure.*

Proof. Was left as an exercise in class, so here's a solution. Let $F : U \rightarrow F(U)$ be a diffeomorphism of the open subset U of the smooth manifold M with an open subset $F(U) \subseteq \mathbb{R}^m$. Then (U, F) is a coordinate chart on M . Choose a coordinate chart (V, ψ) for M . If $U \cap V = \emptyset$ we are done, and otherwise, the transition maps are $F \circ \phi^{-1}$ and $\phi \circ F^{-1}$, both of which are clearly smooth. So the transition map is a diffeomorphism, and so (U, F) is a coordinate chart belonging to the smooth structure by maximality. \square

6.2 Coordinate Derivatives, Inverse Function Theorem

In calculus, we take derivatives. How can we take derivatives of functions on manifolds? The first thing we can try is differentiating with respect to local coordinates. If (U, ϕ) is a coordinate system on a manifold, we will write $(U, \phi) = (U, x^1, \dots, x^m)$ to mean that x^i is the i th component of ϕ . More specifically, if r^1, \dots, r^m are the coordinates on \mathbb{R}^m , then $x^i = r^i \circ \phi$.

Definition 6.2.1. Let $f : M \rightarrow \mathbb{R}$ be a smooth function on the smooth M . Let $p \in M$ and let $(U, \phi) = (U, x^1, \dots, x^m)$ be a coordinate chart around p . Define

$$\left. \frac{\partial f}{\partial x^i} \right|_p := \left. \frac{\partial (f \circ \phi^{-1})}{\partial r^i} \right|_{\phi(p)}$$

as the i th partial derivative of f at p with respect to the coordinates (U, x^1, \dots, x^m) .

What about maps between manifolds? We can do something similar. Let $F : N \rightarrow M$ be a smooth map between smooth manifolds. Let $(U, \phi) = (U, x^1, \dots, x^n)$ and $(V, \psi) = (V, y^1, \dots, y^m)$ be coordinate charts on N and M , respectively. Define the i th component of F with respect to the coordinates (V, y^1, \dots, y^m) by $F^i := y^i \circ F = r^i \circ \psi \circ F$. Then $F^i : N \rightarrow \mathbb{R}$, so by our previous definition we can look at

$$\left. \frac{\partial F^i}{\partial x^j} \right|_p = \left. \frac{\partial (F^i \circ \phi^{-1})}{\partial r^j} \right|_{\phi(p)} = \left. \frac{\partial (r^i \circ \psi \circ F \circ \phi^{-1})}{\partial r^j} \right|_{\phi(p)} = \left. \frac{\partial (\psi \circ F \circ \phi^{-1})^i}{\partial r^j} \right|_{\phi(p)}.$$

We will call the $m \times n$ matrix $\left[\left. \frac{\partial F^i}{\partial x^j} \right|_p \right]$ the *Jacobian of F at p (relative to the coordinates (U, x^1, \dots, x^n) and (V, y^1, \dots, y^m))*.

The Jacobian itself is not independent of the coordinate systems, but since transition maps are diffeomorphisms, its rank is independent of the coordinate systems chosen. Precisely, if $(\tilde{U}, \tilde{\phi})$ and $(\tilde{V}, \tilde{\psi})$ are alternate coordinate charts around p and $F(p)$, respectively, then we have

$$\tilde{\psi} \circ F \circ \tilde{\phi}^{-1} = (\tilde{\psi} \circ \psi^{-1}) \circ (\psi \circ F \circ \phi^{-1}) \circ (\phi \circ \tilde{\phi}^{-1}),$$

implying

$$D(\tilde{\psi} \circ F \circ \tilde{\phi}^{-1})(\tilde{\phi}(p)) = \underbrace{D(\tilde{\psi} \circ \psi^{-1})(\psi(F(p)))}_{\in GL(m, \mathbb{R})} \cdot D(\psi \circ F \circ \phi^{-1})(\phi(p)) \cdot \underbrace{D(\phi \circ \tilde{\phi}^{-1})(\tilde{\phi}(p))}_{\in GL(n, \mathbb{R})},$$

and so linear algebra tells us that

$$\text{rank}(D(\psi \circ F \circ \phi^{-1})(\phi(p))) = \text{rank}(D(\tilde{\psi} \circ F \circ \tilde{\phi}^{-1})(\tilde{\phi}(p))).$$

Therefore "the rank of the Jacobian of F at p " is a well-defined quantity, independent of the coordinate charts. We state this as a proposition.

Proposition 8. If $F : N \rightarrow M$ is C^∞ at p , then the rank of the Jacobian of F at p is the same no matter what coordinate charts around p and $F(p)$ are used to calculate it.

In particular, if $m = n$, then we are led to a generalization of the inverse function theorem, as we can then speak of invertibility of the Jacobian.

Theorem 6.2.1. (*Inverse function theorem for manifolds*) Let $F : N \rightarrow M$ be a smooth map of smooth manifolds of the same dimension. If the Jacobian of F at $p \in N$ is invertible, then there is an open neighbourhood U of p in N and an open neighbourhood V of $F(p)$ in M such that $F|_U : U \rightarrow V$ is a diffeomorphism.

Proof. Was left as an exercise in class, so here's a solution. Choose coordinate charts $(U, \phi) = (U, x^1, \dots, x^n)$ at p and $(V, \psi) = (V, y^1, \dots, y^n)$ at $F(p)$. Then $\left[\frac{\partial F^i}{\partial x^j} \right]_p$ is invertible, but as we saw above, this is equivalent to saying $\left[\frac{\partial(\psi \circ F \circ \phi^{-1})^i}{\partial r^j} \right]_{\phi(p)}$ is invertible. By the inverse function theorem in \mathbb{R}^n , the map $\psi \circ F \circ \phi^{-1}$ is a diffeomorphism on a small neighbourhood of $\phi(p)$ in $\phi(U \cap F^{-1}(V))$. Since coordinate systems are diffeomorphisms, F is a diffeomorphism on a small neighbourhood of p . \square

(The following was not part of the lecture.) Note that the converse of the above theorem is true; if F restricts to a diffeomorphism in a neighbourhood of p , then the Jacobian with respect to any choices of coordinates is invertible. This can be seen by taking two coordinate charts (U, ϕ) around p and (V, ψ) around $F(p)$ and noting that since $\psi \circ F \circ \phi^{-1}$ is then a diffeomorphism of open sets of \mathbb{R}^n , the Jacobian of F with respect to these coordinate systems is invertible (and hence with respect to any coordinate systems). Therefore we have the following slightly stronger theorem:

Theorem 6.2.2. (*Stronger inverse function theorem for manifolds*) Let $F : N \rightarrow M$ be a smooth map of smooth manifolds of the same dimension. Then F is a local diffeomorphism at $p \in N$ if and only if the Jacobian of F at p is invertible.

Of course, *local diffeomorphism at p* means that F restricts to a diffeomorphism on an open neighbourhood of p .

We would like a "coordinate-free" derivative. In MAT257, the derivative of a map $F : \mathbb{R}^n \rightarrow \mathbb{R}^m$ at p was thought of as the map $F_* : T_p \mathbb{R}^n \rightarrow T_{F(p)} \mathbb{R}^m$ defined by $F_*(v_p) = (DF(p)v)_{F(p)}$, where the subscript indicates the tangent space in which the vector lies. The difficulty in generalizing this to manifolds lies in defining the tangent space of an abstract manifold.

6.3 Abstracting the Tangent Space

If M is a submanifold in \mathbb{R}^n in the MAT257 sense, then we can define the tangent space as follows. Suppose $p \in M$. Then there is an open neighbourhood V of p in \mathbb{R}^n , an open set $U \subseteq \mathbb{R}^k$, and a C^∞ homeomorphism $\phi : U \rightarrow V \cap M$ with $\text{rank}(D\phi(q)) = k$ for each $q \in U$. If $q = \phi^{-1}(p)$, then let $T_p U$ be the "set of all vectors in \mathbb{R}^k thought of as pointing from q ". Then we define $T_p M := D\phi(q)(T_p U)$. Since the derivative has rank k at q , the space $T_p M$ will be a vector subspace of $T_p \mathbb{R}^n$ of dimension k . It is not hard to see that this is independent of the "parametrization" chosen near p .

The problem with this is that it doesn't generalize to abstract manifolds. We'd like to modify the definition so that it abstracts.

We will first attempt to do so by using curves. Let $p \in \mathbb{R}^n$ and $v \in T_p \mathbb{R}^n$. If $F : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is smooth, then we can speak of its Jacobian at a point $p \in \mathbb{R}^n$. If $\gamma : (-\varepsilon, \varepsilon) \rightarrow \mathbb{R}^n$ is a smooth curve with $\gamma(0) = p$ and $\gamma'(0) = v$, then

$$\left. \frac{d}{dt} \right|_{t=0} F \circ \gamma = DF(p) \cdot v,$$

so we can think of $DF(p)$ as a map sending tangent vectors to tangent vectors. (A picture would really help here.)

Define $A = \{\text{smooth curves with } \gamma(0) = p\}$. Define \sim on A by $\gamma \sim \tilde{\gamma}$ if and only if $\gamma'(0) = \tilde{\gamma}'(0)$. Then we can think of a vector $v \in T_p \mathbb{R}^n$ as the equivalence class $[\gamma]$ of a curve γ with $\gamma(0) = p$, and we can think of $T_{F(p)} \mathbb{R}^m$ as A/\sim .

This generalizes to manifolds, since we know what a smooth curve is on a manifold. But who wants to work with equivalence classes? We don't.

6.4 Germs and Derivations

Introduce a smooth function $f : \mathbb{R}^n \rightarrow \mathbb{R}$. We can speak of the directional derivative of f . We have

$$D_v f = \left. \frac{d}{dt} \right|_{t=0} f \circ \gamma = \nabla f(p) \cdot v,$$

where the quantity $\nabla f(p) \cdot v$ is independent of γ . We can therefore choose to identify $v \in T_p \mathbb{R}^n$ with the map $D_v : C^\infty(\mathbb{R}^n) \rightarrow \mathbb{R}$. But the value of $D_v f$ only depends on "the local behaviour of f at p ", and so we would like to consider two inputs of D_v to be equivalent if they are equal on a smaller neighbourhood of p . For this, we develop germs.

Definition 6.4.1. Let $f : U \rightarrow \mathbb{R}$ and $g : V \rightarrow \mathbb{R}$ be smooth functions defined on open neighbourhoods of p . We will say that $f \sim g$ if and only if $f|_W = g|_W$ for some open neighbourhood $W \subseteq U \cap V$ of p . Denote by $C_p^\infty(\mathbb{R}^n)$ the set of all such equivalence classes. The equivalence class $[f]$ is called the germ of f at p .

The map D_v is constant on germs, so it induces a map $D_v : C_p^\infty(\mathbb{R}^n) \rightarrow \mathbb{R}$ (note the notational abuse). The set of germs at p has some nice algebraic properties. That it is an "algebra" was not covered in lecture.

Proposition 9. $C_p^\infty(\mathbb{R}^n)$ is a vector space over \mathbb{R} . It can be made into a ring with multiplication of germs, and it can be made into an "algebra over \mathbb{R} "; a ring which is also a vector space over \mathbb{R} with the vector space scalar multiplication satisfying the homogeneity condition $a(vw) = (av) \cdot w = v \cdot (aw)$.

Proof. Was left as an exercise in class, so here's a solution. We define three operations on $C_p^\infty(\mathbb{R}^n)$:

- Vector addition: $[f] + [g] := [f + g]$.
- Vector scaling: $a[f] := [af]$.

- Ring multiplication: $[f] \cdot [g] := [fg]$.

We must first check that these operations are well defined. If $[f] = [\tilde{f}]$ and $[g] = [\tilde{g}]$, then $f = \tilde{f}$ on a neighbourhood of p and $g = \tilde{g}$ on another neighbourhood of p . It then follows that $f + g = \tilde{f} + \tilde{g}$, $af = a\tilde{f}$, and $fg = \tilde{f}\tilde{g}$ on the intersections of these neighbourhoods. By definition we have $[f + g] = [\tilde{f} + \tilde{g}]$, $[af] = [a\tilde{f}]$, and $[fg] = [\tilde{f}\tilde{g}]$, implying that our three operations are well-defined.

$C_p^\infty(\mathbb{R}^n)$ is clearly a vector space over \mathbb{R} under the first two operations, and is also a ring over the first and last operation. Homogeneity of the ring multiplication with respect to vector scaling follows from the corresponding assertion for $C^\infty(\mathbb{R}^n)$. (At this point it is just definition pushing.) \square

Note that the ring $C_p^\infty(\mathbb{R}^n)$ is commutative and has unity - the identity element of the multiplication is the germ $[x \mapsto 1]$. From now on we will abuse notation (even more) and let f denote its germ $[f] \in C_p^\infty(\mathbb{R}^n)$.

We note two properties of our map $D_v : C_p^\infty(\mathbb{R}^n) \rightarrow \mathbb{R}$:

1. D_v is linear.
2. D_v satisfies the "Leibnitz rule"

$$D_v(fg) = f(p)D_v(g) + D_v(f)g(p).$$

Any map $D : C_p^\infty(\mathbb{R}^n) \rightarrow \mathbb{R}$ satisfying the above properties is called a *derivation at p* . The set of all derivations at p is denoted \mathcal{D}_p . It turns out that this view of the tangent space is what generalizes to manifolds. Before we prove the "identification theorem", we need two lemmas.

Lemma 6.4.1. *If f is C^∞ on an open ball U centred at p , then there are smooth $g_i \in C^\infty(U)$ such that $g_i(p) = \frac{\partial f}{\partial x_i}(p)$ and*

$$f(x) = f(p) + \sum_{i=1}^n (x^i - p^i)g_i(x).$$

Proof. Define $\gamma(t) = p + t(x - p)$. Then

$$f(x) - f(p) = \int_0^1 \frac{d}{dt} f(\gamma(t)) dt = \int_0^1 \sum_{i=1}^n \frac{\partial f}{\partial x^i} \Big|_{\gamma(t)} (x^i - p^i) dt = \sum_{i=1}^n (x^i - p^i) \underbrace{\int_0^1 \frac{\partial f}{\partial x^i}(p + t(x - p)) dt}_{g_i(x)}.$$

\square

Lemma 6.4.2. *Derivations of constants are zero.*

Proof. Let $c \in \mathbb{R}$. Then

$$\begin{aligned} D(c) &= c \cdot D(1) = c \cdot D(1 \cdot 1) \\ &= c \cdot 1 \cdot D(1) + c \cdot D(1) \cdot 1 \\ &= 2c \cdot D(1) \\ &= 2D(c), \end{aligned}$$

implying $D(c) = 0$. □

Theorem 6.4.1. *We can identify $T_p\mathbb{R}^n$ with \mathcal{D} . More specifically,*

1. \mathcal{D}_p is a vector space over \mathbb{R} .
2. The map $\Phi : T_p\mathbb{R}^n \rightarrow \mathcal{D}_p$ sending v to D_v is a vector space isomorphism.

Proof. 1. Was left as an exercise in class, so here's a proof. We must check that if $a \in \mathbb{R}$ and $D_1, D_2 \in \mathcal{D}_p$, then the function $aD_1 + D_2$ is a derivation. It is linear as a sum of linear functions. If $f, g \in C_p^\infty(\mathbb{R}^n)$, then

$$\begin{aligned} (aD_1 + D_2)(fg) &= aD_1(fg) + D_2(fg) \\ &= a[f(p)D_1(g) + D_1(f)g(p)] + f(p)D_2(g) + D_2(f)g(p) \\ &= f(p)[aD_1(g) + D_2(g)] + [aD_1(f) + D_2(f)]g(p) \\ &= f(p)(aD_1 + D_2)(g) + (aD_1 + D_2)(f)g(p), \end{aligned}$$

so $aD_1 + D_2$ satisfies the Leibnitz rule and is thus a derivation at p . Therefore \mathcal{D}_p is a vector space over \mathbb{R} .

2. We check linearity, injectivity, and surjectivity.

- Linearity: if $a \in \mathbb{R}$ and $v_1, v_2 \in T_p\mathbb{R}^n$, then for $f \in C_p^\infty(\mathbb{R}^n)$ we have

$$\Phi(av_1 + v_2)(f) = D_{av_1 + v_2}(f) = Df(p)(av_1 + v_2) = aDf(p)v_1 + Df(p)v_2 = (a\Phi(v_1) + \Phi(v_2))(f).$$

- Injectivity: suppose $D_v(f) = 0$ for all $f \in C_p^\infty(\mathbb{R}^n)$. In particular, $D_v x^i = 0$ for the i th coordinate map $x^i \in C_p^\infty(\mathbb{R}^n)$. Expanded, this says

$$0 = D_v x^i = Dx^i(p)v = e_i^T v = v^i,$$

where e_i is the i th standard basis vector of \mathbb{R}^n . So $v = 0$ if $\Phi(v) = 0$.

- Surjectivity: suppose $D \in \mathcal{D}_p$. For any $f \in C_p^\infty(\mathbb{R}^n)$, we have, by the two lemmas,

$$\begin{aligned} Df &= D\left(f(p) + \sum_{i=1}^n (x^i - p^i)g_i\right) \\ &= \sum_{i=1}^n [(p^i - p^i)Dg_i + D(x^i - p^i)g_i(p)] \\ &= \sum_{i=1}^n Dx^i \frac{\partial f}{\partial x^i}(p), \end{aligned}$$

so if we take $v = (Dx^1, \dots, Dx^n)$ then $Df = D_v f$ for all $f \in C_p^\infty(\mathbb{R}^n)$. Therefore $\Phi(v) = D$.

So Φ is a vector space isomorphism $T_p\mathbb{R}^n \xrightarrow{\sim} \mathcal{D}_p$.

□

We can finally define the tangent space to a point on an abstract manifold. The space of germs $C_p^\infty(M)$ for $p \in M$ is defined in the exact same way as in \mathbb{R}^n .

Definition 6.4.2. *Let M be a smooth manifold. We say $v : C_p^\infty(M) \rightarrow \mathbb{R}$ is a derivation at $p \in M$ if v is linear and satisfies the Leibnitz rule. We define the tangent space T_pM to M at p to be the set of all derivations at p .*

(The following was not part of the lecture and is included for completeness.) Now we can finally define the derivative of a map between manifolds.

Definition 6.4.3. *Let $F : N \rightarrow M$ be a smooth map of smooth manifolds and let $p \in N$. The map F induces a linear map $F_* : T_pN \rightarrow T_{F(p)}M$ defined by*

$$(F_*X_p)(f) = X_p(f \circ F),$$

where $X_p \in T_pN$ is a derivation at p and $f \in C_{F(p)}^\infty(M)$.