1 Lie Groups and Algebras (Additional Reading, Incomplete)

We will write the group operation for an arbitrary Lie group as multiplication, but in some cases (e.g. $\mathbb{R}^n$) we will use addition when necessary.

1.1 Motivation and Definitions

A Lie group is a manifold with a group structure such that the group operations of multiplication and inversion are smooth. We can think of Lie groups as the smooth analogue of topological groups; topological spaces with continuous group multiplication and inversion. A Lie group is a "homogeneous" space, in the sense that left-multiplication by a fixed element is a diffeomorphism of the group with itself and so, in some sense, the group is locally everywhere the same. More precisely, if G is a Lie group, $g \in G$, and $\ell_g : G \to G$ is left-multiplication by g, then ℓ_g is a diffeomorphism of G with itself taking the identity element e to g. Therefore, in the study of Lie groups, it suffices to study the properties of the group at the identity. We now make these notions precise.

Definition 1. *A Lie group is a smooth manifold G equipped with a group structure such that multiplication*

$$\mu : G \times G \to G \qquad (g,h) \mapsto gh$$

and inversion

$$\imath : G \to G \qquad g \mapsto g^{-1}$$

are smooth.

Since $\ell_g^{-1} = \ell_{g^{-1}}$, left-multiplication is a diffeomorphism of G with itself, and similarly for right-multiplication. We have the following equivalent condition for being a Lie group.

Proposition 1.1. *G is a Lie group if and only if the map $k : G \times G \to G$ defined by $k(g,h) = gh^{-1}$ is smooth.*

Proof. If G is a Lie group, then $k = \mu \circ (\mathrm{id}_G, \imath)$ is smooth. Conversely, if k is smooth, then $\imath = k \circ (e, \mathrm{id}_G)$ is smooth and $\mu = k \circ (\mathrm{id}_G, \imath)$ is smooth, where e denotes the identity element of G. $\square$

Definition 2. *A Lie subgroup of the Lie group G is an immersed submanifold H that is also a subgroup such that the group operations on H are smooth.*

We must impose the condition that the group operations on H are smooth; H is merely an immersed submanifold, so it does not follow that the group operations on H are smooth. If H is embedded then this is the case, which is the content of the following theorem. First, recall the following technical lemma.

Lemma 1.1. *Let $F : N \to M$ be a smooth map and let S be a subset of M containing $F(N)$. If S is a regular submanifold of M, then the restriction of the codomain of F to S is smooth.*

Proof. For convenience, denote by $\widetilde{F}: N \to S$ the map obtained by restricting the codomain of F. Let $p \in N$, and suppose the dimensions of N, M, S are n, m, s, respectively. Choose a slice chart $(V, \psi) = (V, y^1, \dots, y^m)$ on M at $F(p)$ such that $V \cap S$ is defined by the vanishing of $y^{s+1}, \dots, y^m$. Let $\psi_S = (y^1, \dots, y^s)$ be the induced coordinate chart on S at $F(p)$. Choose an open neighbourhood U of p with $F(U) \subseteq V$. Then $F(U) \subseteq V \cap S$, so if $q \in U$ we have

$$(\psi \circ F)(q) = (y^1(F(q)), \dots, y^s(F(q)), 0, \dots, 0),$$

so on U we have

$$\psi_S \circ \widetilde{F} = (y^1 \circ F, \dots, y^s \circ F).$$

Thus $\widetilde{F}$ is smooth on U. Since p was arbitrary, $\widetilde{F}$ is smooth. □

For a counterexample of the preceding lemma in the case of an immersed submanifold, consider a parametrization of the figure eight in $\mathbb{R}^2$.

Theorem 1.2. *If H is an abstract subgroup of the Lie group G and is also an embedded submanifold of G, then H is a Lie subgroup of G.*

Proof. Embedded submanifolds are immersed submanifolds, so all we have to check is the smoothness of the operations on H. Let $i: H \hookrightarrow G$ be the inclusion map. Then multiplication on H is given by $\mu \circ (i, i)$, where μ is the multiplication on G. The image of this map lies in the regular submanifold H, so restricting the codomain leaves us with the multiplication on H. This is smooth by the preceding lemma, since H is an embedded submanifold. Similarly for the inversion map on H. □

We have the following important theorem, due to Cartan, which provides us with many examples of Lie groups.

Theorem 1.3. *(Closed subgroup theorem) A closed subgroup of a Lie group is an embedded Lie subgroup.*

1.2 Examples of Lie Groups

1. $\mathbb{R}^n$ with addition.
2. $\mathbb{R}^*$, the non-zero real numbers, with multiplication.
3. The (direct) product of Lie groups is another Lie group.
4. $\mathbb{C} \setminus \{0\}$ with complex multiplication, and its embedded Lie subgroup S^1. $\mathbb{C} \setminus \{0\}$ is a Lie group for obvious reasons, and S^1 is an embedded Lie subgroup by either the closed subgroup theorem, or the theorem above.
5. $\mathrm{Mat}_{n\times n}(\mathbb{R})$ with matrix multiplication, and similarly $\mathrm{Mat}_{n\times n}(\mathbb{C})$. This can be proven by identifying with $\mathbb{R}^{n^2}$ and $\mathbb{C}^{n^2}$, respectively.

6. $GL(n, \mathbb{R})$ is an abstract subgroup of $\mathrm{Mat}_{n\times n}(\mathbb{R})$ and an embedded submanifold, since it is an open set, so $GL(n, \mathbb{R})$ is an embedded Lie subgroup of G. Similarly with $\mathbb{R}$ replaced by $\mathbb{C}$.

7. The orthogonal group $O(n)$ and the special linear group $SL(n, \mathbb{R})$ are embedded Lie subgroups of $GL(n, \mathbb{R})$, since they are embedded submanifolds and abstract subgroups. The unitary group $U(n)$ and the complex special linear group $SL(n, \mathbb{C})$ are embedded Lie subgroups of $GL(n, \mathbb{C})$ for similar reasons.

1.3 The Differential of det at I

To compute the differential of a map on a subgroup of $GL(n, \mathbb{R})$, we need a curve of invertible matrices. Since $\det(e^X) = e^{\mathrm{Tr}(X)}$, the matrix exponential is useful for this purpose.

Consider the determinant $\det : GL(n, \mathbb{R}) \to \mathbb{R}$. It is a smooth map because it is given by a polynomial. After the usual identifications, its differential at the identity is a map $\det_{*,I} : \mathbb{R}^{n\times n} \to \mathbb{R}$.

Theorem 1.4. $\det_{*,I}(X) = \mathrm{Tr}(X)$.

Proof. Let $c(t) = e^{tX}$. Then c is a smooth curve starting at I with $c'(0) = X$. Therefore

$$\det_{*,I}(X) = \left.\frac{d}{dt}\right|_{t=0} \det(e^{tX}) = \left.\frac{d}{dt}\right|_{t=0} e^{t\mathrm{Tr}(X)} = \mathrm{Tr}(X).$$

□

1.4 Some Properties of Lie Groups

It is possible to do the next exercise without invoking the closed subgroup theorem, but I would like to give an example of its application (due to the relative lack of content thus far).

Theorem 1.5. *(Exercise 15.3, slight variation) The connected component of a Lie group G containing the identity is an embedded Lie subgroup.*

Proof. Let G_0 be the connected component of G which contains e. Let $\mu : G \times G \to G$ be multiplication and $\imath : G \to G$ be inversion. These are both continuous maps. Fix $x \in G_0$. The image $\mu(\{x\} \times G_0)$ is connected, and it intersects G_0 because $\mu(x, e) = x \in G_0$. (This is where $e \in G_0$ is used.) Therefore $\mu(\{x\} \times G_0) \subseteq G_0$ and similarly $\imath(G_0) \subseteq G_0$. It follows that G_0 is a subgroup of G.

Since G is a manifold, it is locally connected and thus the connected components of G are open. Then $G \setminus G_0$ is either empty or the union of the other connected components of G, so G_0 is closed. Then G_0 is a closed subgroup of G, so by the closed subgroup theorem G_0 is an embedded Lie subgroup of G. (To prove this without invoking the closed subgroup theorem, note that G_0 is open and so it is an embedded submanifold of G, so by the theorem directly following the statement of the closed subgroup theorem, it is an embedded Lie subgroup.) □

It is not hard to see that every connected component of a Lie group G is of the form gG_0 for some $g \in G$.

The following property holds for the more general topological group, whose space is merely a topological space and whose group operations are continuous with respect to the group's topology.

Theorem 1.6. *(Exercise 15.4) An open subgroup H of a connected Lie group G is equal to G.*

Proof. H is a subgroup, so it contains the identity e of G and is thus non-empty, so to show $H = G$ it suffices to show that H is closed. If $g \in G \setminus H$, then gH is an open neighbourhood of g disjoint from H (because the cosets partition the group). Therefore G is nonempty, open, and closed, so we must have $H = G$. □

1.5 The Tangent Space at I

(Finish this and the following sections.)

1.6 Left-Invariant Vector Fields

1.7 Lie Algebras

1.8 The Differential as a Lie Algebra Homomorphism