

1 More on Flows and Lie Derivatives (July 7)

1.1 The Fundamental Theorem of Flows

Consider $X \in \mathfrak{X}(M)$. Let $(U, \phi) = (U, x^1, \dots, x^n)$ be a coordinate chart on M . The theory of ODEs in \mathbb{R}^n applies almost exactly to the integral curves of $X|_U \in \mathfrak{X}(U)$, which follows from U being diffeomorphic to an open subset of \mathbb{R}^n . In particular,

1. Given $p \in U$, there is, by the existence and uniqueness theorems, a unique maximal integral curve of $X|_U$ starting at p .
2. By "collecting" all of the maximal integral curves from the previous step, we get flows. More precisely, we define

$$\mathcal{D} = \{(t, p) \in \mathbb{R} \times M : t \text{ lies in the domain of the maximal integral curve of } X|_U \text{ starting at } p\}.$$

Then, for $p \in U$, we let

$$\mathcal{D}^{(p)} = \{t \in \mathbb{R} : (t, p) \in \mathcal{D}\};$$

this is just the domain of the maximal integral curve of $X|_U$ starting at p . In particular, $0 \in \mathcal{D}^{(p)}$.

Now define $F : \mathcal{D} \rightarrow U$ by $F(t, p) = \gamma_p(t)$, where $\gamma_p : \mathcal{D}^{(p)} \rightarrow U$ is the maximal integral curve of $X|_U$ starting at p .

We are led to the following question: does there exist an interval $(-\varepsilon, \varepsilon)$ such that $(-\varepsilon, \varepsilon) \subseteq \mathcal{D}^{(p)}$ for all $p \in U$? If the answer is yes, then the vector field $X|_U$ is *complete*; each of its maximal integral curves exists for all $t \in \mathbb{R}$. This is known as the "Uniform Time Lemma".

Moreover, the function F is smooth, which follows immediately from the theorem of smooth dependence on initial conditions of ODEs in \mathbb{R}^n .

3. Given $p \in U$, there is, by smooth dependence on initial conditions, a neighbourhood $W \subseteq U$ of p such that the hypotheses of the uniform time lemma are satisfied on W . (?)

All of the above was in a coordinate chart, and so we have seen nothing new. Everything followed directly from the theory of ODEs in \mathbb{R}^n . We must then ask if the above claims hold in general on M , not restricted to a single coordinate open set. The answer is yes, all of these claims hold when U is replaced with M .

Theorem 1.1. (*Fundamental Theorem of Flows*) Suppose $X \in \mathfrak{X}(M)$. Then there exists a unique smooth maximal flow $F : \mathcal{D} \rightarrow M$, where $\mathcal{D} \subseteq \mathbb{R} \times M$, generated by X , satisfying

- (a) For each $p \in M$, the curve $\gamma_p(t) = F(t, p)$ is the unique maximal integral curve of X starting at p .
- (b) If $s \in \mathcal{D}^{(p)}$, then $\mathcal{D}^{(F(s, p))} = \mathcal{D}^{(p)} - s$.

(c) For each $t \in \mathbb{R}$, the set $M_t = \{p \in M : (t, p) \in \mathcal{D}\}$ is open in M , and $F_t : M_t \rightarrow M_{-t}$ is a diffeomorphism.

Proof. The proof is left as an exercise (likely as one of the homework problems). It is also in Lee. \square

1.2 More on Lie Derivatives

Suppose $X, Y \in \mathfrak{X}(M)$, and let F be the flow of X . If $M = \mathbb{R}^n$, then we can compare $Y_{F_t(p)}$ and Y_p , since $T_p\mathbb{R}^n = \mathbb{R}^n$ for every $p \in \mathbb{R}^n$. We cannot, however, do this for abstract manifolds, since the elements of the distinct tangent spaces of M cannot even be compared. We remedy this using the pushforward of F_{-t} .

Definition 1. The Lie derivative of Y with respect to X is the vector field $\mathcal{L}_X Y$ defined by

$$(\mathcal{L}_X Y)_p = \lim_{t \rightarrow 0} \frac{(F_{-t})_{*, F_t(p)}(Y_{F_t(p)}) - Y_p}{t},$$

when the limit exists.

Proposition 1.1. The above limit exists for all $p \in M$, and $\mathcal{L}_X Y \in \mathfrak{X}(M)$.

Proof. Exercise. \square

Consider the question of whether or not the flows of X and Y commute. Let G be the flow of Y . We then may ask ourselves: given $p \in M$, do we have $F_s(G_t(p)) = G_t(F_s(p))$? Define a function A mapping (s, t) to $F_s(G_t(p)) - G_t(F_s(p))$. We have the following proposition:

Proposition 1.2.

$$\frac{\partial^2 A}{\partial s \partial t} = \frac{\partial^2 A}{\partial t \partial s} = \mathcal{L}_X Y,$$

and if $\mathcal{L}_X Y = 0$, then $A \equiv 0$.

Proof. Exercise. \square

So $\mathcal{L}_X Y$ measures "how much the flows commute".

For another perspective, consider a smooth function $f \in C^\infty(M)$ and a vector field $X \in \mathfrak{X}(M)$. We can ask ourselves the question "how does f change along the integral curve of X starting at p ?". If F is the flow of X , then we would like to discuss

$$\left. \frac{d}{dt} \right|_{t=0} f \circ F_t(p)$$

for $p \in M$. A straightforward application of the chain rule gives

$$\left. \frac{d}{dt} \right|_{t=0} f \circ F_t(p) = f_{*,p}(\gamma'_p(0)) = f_{*,p}(X_p) = (Xf)(p).$$

We can also define $\frac{d}{dt}\big|_{t=0} f \circ F_t(p)$ as the first order term in the Taylor expansion of $f \circ F_t(p)$ at 0:

$$f \circ F_t(p) = f(p) + t(Xf)(p) + o(t).$$

Let us apply the same idea in order to get the Lie derivative. Define G to be the map

$$G : t \mapsto (F_{-t})_{*, F_t(p)}(Y_{F_t(p)}) \in T_p M.$$

The Taylor series expansion of G at 0 is

$$G(t) = G(0) + tG'(0) + o(t) = Y_p + t(\mathcal{L}_X Y)_p + o(t),$$

as we will see later. Therefore we may actually define $(\mathcal{L}_X Y)_p$ to be the first order term of the Taylor expansion of the map G defined above.

We end up having three equivalent definitions of the Lie derivative $\mathcal{L}_X Y$: given $p \in M$, define $(\mathcal{L}_X Y)_p$ as

1. the limit

$$(\mathcal{L}_X Y)_p := \lim_{t \rightarrow 0} \frac{(F_{-t})_{*, F_t(p)}(Y_{F_t(p)}) - Y_p}{t}.$$

2. the first-order term of the Taylor expansion of the map G defined by

$$G : t \mapsto (F_{-t})_{*, F_t(p)}(Y_{F_t(p)}).$$

3. the Lie bracket

$$[X, Y] = X_p Y - Y_p X.$$