

MAT367 Course Notes

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The following are course notes for the course MAT367 (Differential Geometry) offered in the Summer of 2020, taught by Ahmed Ellithy. The course notes are based off of handwritten notes created during lectures. They may contain errors or other false statements. The dates for each lecture are included, and any additional sections are supplementary material. (Exercises to very important/relevant problems, etc.)

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1 Introduction (May 5)

1.1 Trying to Define Things

The straightforward approach is

Definition 1.1.1. A set $S \subset \mathbb{R}^3$ is a surface if there is an open set $U \subset \mathbb{R}^2$ and a smooth function $f : U \rightarrow \mathbb{R}$ for which $S = \Gamma_f$ is the graph of f .

This isn't a great definition though. Its problem is that it's way too specific. The sphere $S^2 = \{x^2 + y^2 + z^2 = 1\}$ fails to be a surface under this definition, as it's not the graph of a function. We can remedy this by thinking about the following question:

If we were standing on a surface, what should our surroundings look like?

Here's another attempt at defining a surface, albeit in an imprecise way.

Definition 1.1.2. A set $S \subset \mathbb{R}^3$ is a surface if for every $p \in S$ there is a neighbourhood of p in S that "looks like a piece of the plane".

In more precise (but still not formal) wording, we are "locally diffeomorphic to pieces of \mathbb{R}^2 ". It turns out that this condition is equivalent to S being locally a graph; that follows from the implicit function theorem.

We'd like to generalize the above notions to define a k -dimensional "surface" in \mathbb{R}^n . Following in the footsteps of the previous definition, we obtain a new

Definition 1.1.3. A set $S \subset \mathbb{R}^n$ is a k -dimensional manifold if it "locally looks like \mathbb{R}^k ".

Equivalently, if for each $p \in S$ there is an open $U \subset \mathbb{R}^n$ containing p such that $S \cap U$ is the graph of a smooth function from an open subset of \mathbb{R}^k to \mathbb{R}^{n-k} .

The key idea with the last two definitions is that they are *local* - they are concerned with describing "pieces" of the surface or manifold, as opposed to the first definition being "global" by describing the entire surface.

1.2 Leaving \mathbb{R}^n for the Intrinsic View

Almost all of the geometry that is done on manifolds depends only on the manifold itself, and not on the space in which the manifold lies. (An example of Riemannian geometry is curvature.) Moreover, there are many sets we'd like to call manifolds whose points do not lie in Euclidean space. An example is *real projective space* \mathbb{RP}^n , which is defined as the quotient $(\mathbb{R}^{n+1} \setminus \{0\})/(x \sim \lambda x)$, where $\lambda \neq 0$. The real projective space contains equivalence classes of points of Euclidean space, so it is not a subset of Euclidean space. Therefore we'd like to define manifolds so that \mathbb{RP}^n is an n -dimensional manifold.

Concisely, we would like to study manifolds *intrinsically*: we would like to drop all of the unnecessary data around our manifold and consider only the key properties of what a manifold should be.

2 Defining Manifolds (May 7)

2.1 Submanifolds of \mathbb{R}^n

We'll formally write out the definition of a k -manifold M in \mathbb{R}^n now.

Definition 2.1.1. A subset $M \subset \mathbb{R}^n$ is a k -dimensional manifold if for every $p \in M$ there is an open neighbourhood U of p in \mathbb{R}^n , an open $V \subset \mathbb{R}^k$, and a function $f : V \rightarrow U \cap M$ such that

1. f is a homeomorphism,
2. f is smooth,
3. $Df(x)$ has rank k at every $x \in V$.

The first two conditions are natural. Why the third? We'd like the *tangent space to M at p* to be a k -dimensional subspace of \mathbb{R}^n . If $Df(x)$ has rank k , then $Df(x)(\mathbb{R}^k)$ is a k -dimensional subspace of \mathbb{R}^n , which is what we would like $T_p M$ to be (roughly).

We have an equivalent definition, stated here as a theorem:

Theorem 2.1.1. $M \subset \mathbb{R}^n$ is a k -manifold if and only if for each $p \in M$ there is an open neighbourhood U of p in \mathbb{R}^n , an open $V \subset \mathbb{R}^k$, and a smooth $f : V \rightarrow \mathbb{R}^{n-k}$ such that $U \cap M = \Gamma_f$ (up to a permutation of the coordinates in U).

That last condition is a little odd, but what it means is that we can consider graphs of the form $(x, f(x))$ and $(f(y), y)$. This is essential in ensuring that, say, $S^1 = \{x^2 + y^2 = 1\}$ is a manifold. The definition may be shown to be equivalent to the old one using the implicit function theorem.

2.2 Topological Manifolds

By way of the subspace topology, every manifold in \mathbb{R}^n is Hausdorff and second countable. It turns out that these are the conditions we would like our abstract manifolds to have in order to exclude some pathological cases.

Definition 2.2.1. A topological space M is locally Euclidean of dimension m if for each $p \in U$ there is an open neighbourhood U of p in M and a map $\phi : U \rightarrow \mathbb{R}^m$ which is a homeomorphism onto its image. The pair (U, ϕ) is called a coordinate chart, U is called a coordinate neighbourhood, and ϕ is called a coordinate system.

Definition 2.2.2. M is a topological manifold of dimension m if it is Hausdorff, second countable, and locally Euclidean of dimension m .

Is the dimension of a topological manifold well-defined? That is, if (U, ϕ) and (V, ψ) are two coordinate charts with $U \cap V \neq \emptyset$ and $\phi(U) \subset \mathbb{R}^n$, $\psi(V) \subset \mathbb{R}^m$, is $n = m$? Consider the *transition mapping*

$$\psi \circ \phi^{-1} : \phi(U \cap V) \rightarrow \psi(U \cap V).$$

This is a homeomorphism from an open subset of \mathbb{R}^n to an open subset of \mathbb{R}^m . If $n \neq m$, this contradicts a non-trivial theorem called *Invariance of Domain*. We will not prove it here.

If we drop the Hausdorff condition, then the "line with two origins" becomes a topological manifold. If we drop the countable basis condition, then the "long line" becomes a topological manifold. These are both topological spaces that intuitively should not be manifolds - the extra conditions excludes them from being so.

2.3 Defining a Smooth Manifold

How should we define a smooth function on a manifold, say, $f : M \rightarrow \mathbb{R}$? The reasonable thing to do is to say that f is smooth if $f \circ \phi^{-1}$ is smooth, for some coordinate system ϕ . Then we run into a problem - this isn't independent of the choice of coordinate system, so long as M is only a topological manifold. We will define a *smooth structure* on M which allows us to make this natural definition.

Definition 2.3.1. Two coordinate charts (U, ϕ) and (V, ψ) are said to be smoothly compatible (or C^∞ -compatible) if the transition mappings are diffeomorphisms, i.e.

$$\begin{aligned}\psi \circ \phi^{-1} &: \phi(U \cap V) \rightarrow \psi(U \cap V) \\ \phi \circ \psi^{-1} &: \psi(U \cap V) \rightarrow \phi(U \cap V)\end{aligned}$$

are C^∞ maps of open subsets of Euclidean space.

Smooth compatibility is clearly a reflexive and symmetric relation. Is it transitive? Unfortunately, the answer is no. Suppose (U_1, ϕ_1) is smoothly compatible with (U_2, ϕ_2) and similarly for (U_2, ϕ_2) with (U_3, ϕ_3) . The natural thing to do is write

$$\phi_3 \circ \phi_1^{-1} = (\phi_3 \circ \phi_2^{-1}) \circ (\phi_2 \circ \phi_1^{-1}).$$

But this only makes sense on $\phi_1(U_1 \cap U_2 \cap U_3)$, which may be empty!

Definition 2.3.2. A smooth atlas (or C^∞ atlas) on M is a collection of pairwise smoothly compatible coordinate charts covering M .

We can now properly define a smooth function on a manifold. For unsaid technical reasons, however, it's beneficial to consider a little more structure. (Unfortunately the rest of the lecture went a little fast, as we ran out of time.)

Definition 2.3.3. A smooth maximal atlas is a smooth atlas not contained in any other smooth atlas.

Definition 2.3.4. A smooth manifold of dimension n is a Hausdorff, second countable topological manifold of dimension n equipped with a smooth maximal atlas \mathcal{A} . The smooth maximal atlas \mathcal{A} is called a smooth structure on M .

Lemma 2.3.1. *Any smooth atlas for M is contained in a unique maximal smooth atlas.*

The proof for this lemma proceeds roughly as follows: first one proves that if two coordinate charts are smoothly compatible with a given atlas (meaning they are compatible with every chart in the atlas), then they are themselves compatible. Then one picks a smooth atlas and adjoins (by union) all of the charts with which the smooth atlas is compatible. It is then shown that this larger atlas is the desired unique maximal atlas.

Because of this lemma, we have a simple "test" for a smooth manifold.

Corollary 2.3.1. *A topological space M is a smooth manifold if and only if*

1. *It is Hausdorff and second countable,*
2. *It admits a smooth atlas.*

3 Smooth Structures, Examples (May 12)

3.1 More on Maximal Atlases

Consider the two atlases $\mathcal{A}_1 = \{(\mathbb{R}^n, Id)\}$ and $\mathcal{A}_2 = \{(B_1(x), Id) : x \in \mathbb{R}^n\}$ on \mathbb{R}^n . These two atlases determine the same maximal atlas, or the same smooth structure. Why? We have three equivalent reasons

- for any $(U, \phi) \in \mathcal{A}_1$ and $(V, \psi) \in \mathcal{A}_2$, the charts (U, ϕ) and (V, ψ) are C^∞ compatible.
- $\mathcal{A}_1 \cup \mathcal{A}_2$ is a C^∞ atlas.
- \mathcal{A}_1 and \mathcal{A}_2 belong to the same maximal atlas.

Define a relation \sim on the atlases by $\mathcal{A}_1 \sim \mathcal{A}_2$ if and only if $\mathcal{A}_1 \cup \mathcal{A}_2$ is another C^∞ atlas. Symmetry and reflexivity are immediate. For transitivity, suppose $\mathcal{A}_1 \cup \mathcal{A}_2$ and $\mathcal{A}_2 \cup \mathcal{A}_3$ are C^∞ atlases. Choose $(U_1, \phi_1) \in \mathcal{A}_1$ and $(U_3, \phi_3) \in \mathcal{A}_3$. We obtain a diffeomorphism

$$\phi_1 \circ \phi_3^{-1} = \phi_1 \circ \phi_2^{-1} \circ \phi_2 \circ \phi_3^1$$

defined on $\phi_3(U_{13} \cap U_2)$. Since $\{U_2 : (U_2, \phi_2) \in \mathcal{A}_2\}$ covers M , the map $\phi_1 \circ \phi_3^{-1}$ is smooth at every point of $\phi_3(U_{13})$. Therefore \sim is an equivalence relation.

Now given an atlas \mathcal{A} on M , we can talk about the equivalence class $[\mathcal{A}]$. Define

$$\mathcal{M} = \bigcup_{\mathcal{A}' \in [\mathcal{A}]} \mathcal{A}'.$$

Then \mathcal{M} is a new atlas on M ; it is the unique maximal atlas containing \mathcal{A} . (Exercise.)

So we can make the

Definition 3.1.1. A smooth n -manifold M is a topological n -manifold with a maximal atlas. The choice of maximal atlas is called a smooth structure on M .

Considering the previous remarks, we arrive at a sufficient condition for a space to be a smooth manifold: If M is a topological space for which

1. M is Hausdorff, second-countable, and
2. M admits a C^∞ atlas \mathcal{A}

then M is a smooth manifold with smooth structure $\mathcal{M} = \bigcup_{\mathcal{A}' \in [\mathcal{A}]} \mathcal{A}'$.

3.2 Examples

1. (Open subsets) Let M be a smooth n -manifold with a smooth atlas $\mathcal{A} = \{(U_\alpha, \phi_\alpha)\}$. Let $A \subset M$ be an open set. Then $\mathcal{A}_A = \{(U_\alpha \cap A, \phi_\alpha|_{U_\alpha \cap A})\}$ is a smooth atlas on A , so A is a smooth n -manifold.
2. (Finite dimensional vector spaces) Let V be a finite dimensional real vector space. Choose a basis $\beta = \{v_1, \dots, v_n\}$ of V , and consider the isomorphism $\Phi : V \rightarrow \mathbb{R}^n$ given by $\Phi(v_i) = e_i$.

Define a norm on V by $\|\sum a_i v_i\| := \|\sum a_i e_i\|$, where the norm on the left is the standard Euclidean norm. With this norm we may define an open ball in V as $B_r(v_0) = \{v \in V : \|v - v_0\| < r\}$. This gives a topology on V . Since all norms on finite dimensional vector spaces are equivalent, this topology does not depend on our choice of basis.

Then Φ is an isometry (it does not change distances), so it takes balls to balls and so does its inverse. That is, Φ is a homeomorphism, so we have a C^∞ atlas $\{(V, \Phi)\}$ on V , making V a smooth n -manifold.

This atlas determines a maximal atlas on V . Does this maximal atlas depend on the choice of basis? No. Choose another basis β' of V and define $\Phi' : V \rightarrow \mathbb{R}^n$ similarly. Then we'll get another C^∞ atlas $\{(V, \Phi')\}$ on V . The charts (U, Φ) and (V, Φ') are C^∞ -compatible, for the transition map $\Phi' \circ \Phi^{-1}$ is a linear isomorphism of \mathbb{R}^n with itself (certainly C^∞).

Remark: We also could have talked about complex vector spaces, since $\mathbb{C} \cong \mathbb{R}^2$.

3. (Matrices, general linear group) $\text{Mat}_{m \times n}(\mathbb{R}) \cong \mathbb{R}^{mn}$, so $\text{Mat}_{m \times n}(\mathbb{R})$ is a smooth manifold of dimension mn .

The general linear group is $GL(n, \mathbb{R}) = \{A \in \text{Mat}_{m \times n}(\mathbb{R}) : \det(A) \neq 0\}$. By continuity of \det it is an open subset of $\text{Mat}_{m \times n}(\mathbb{R})$, so by the first example we know it's a smooth n^2 -dimensional manifold.

4 More examples of manifolds, Quotients (May 14)

4.1 More Examples

1. (The circle) Define $S^1 = \{x^2 + y^2 = 1\} \subset \mathbb{R}^2$. We can define four functions on open sets of \mathbb{R} , the collection of which form a set of functions of which S^1 is locally the graph. Define an open cover $\{V_1, V_2, V_3, V_4\}$ of S^1 by

$$\begin{aligned} V_1 &= S^1 \cap ((0, \infty) \times (-1, 1)) && \text{"open right half"} \\ V_2 &= S^1 \cap ((-\infty, 0) \times (-1, 1)) && \text{"open left half"} \\ V_3 &= S^1 \cap ((-1, 1) \times (0, \infty)) && \text{"open top half"} \\ V_4 &= S^1 \cap ((-1, 1) \times (-\infty, 0)) && \text{"open bottom half"} \end{aligned}$$

Define $f_1, f_2, f_3, f_4 : (-1, 1) \rightarrow \mathbb{R}$ by

$$\begin{aligned} f_1(y) &= \sqrt{1 - y^2} && \text{so that } \Gamma_{f_1} = V_1 \\ f_2(y) &= -\sqrt{1 - y^2} && \text{so that } \Gamma_{f_2} = V_2 \\ f_3(x) &= \sqrt{1 - x^2} && \text{so that } \Gamma_{f_3} = V_3 \\ f_4(x) &= -\sqrt{1 - x^2} && \text{so that } \Gamma_{f_4} = V_4 \end{aligned}$$

What are the charts? Define $\phi_1 : V_1 \rightarrow (-1, 1)$ by $\phi_1(x, y) = y$. This is continuous with continuous inverse $\phi_1^{-1}(y) = (\sqrt{1 - y^2}, y)$. The other coordinate systems ϕ_2, ϕ_3, ϕ_4 are defined similarly. Consider

$$\mathcal{A} = \{(V_1, \phi_1), (V_2, \phi_2), (V_3, \phi_3), (V_4, \phi_4)\}.$$

We claim that \mathcal{A} is a smooth atlas on S^1 . For example, one transition map is $\phi_1 \circ \phi_3^{-1} : \phi_3(V_{13}) \rightarrow \phi_1(V_{13})$, which is a map from $(0, 1)$ to itself. It is given by

$$(\phi_1 \circ \phi_3^{-1})(t) = \phi_1(t, \sqrt{1 - t^2}) = \sqrt{1 - t^2},$$

which is a diffeomorphism of $(0, 1)$ with itself. As a similar proposition holds for the other transition maps, we conclude that (S^1, \mathcal{A}) is a smooth manifold of dimension 1.

Let $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ be $f(x, y) = x^2 + y^2$. Then $S^1 = f^{-1}(1)$ (preimage). We get a collection of 1-dimensional manifolds covering $\mathbb{R}^2 \setminus \{0\}$; we say that $\{f^{-1}(r) : r > 0\}$ is a *one-dimensional foliation* of $\mathbb{R}^2 \setminus \{0\}$. (More on that in a later lecture.)

2. (Level sets) Consider a smooth map $F : \mathbb{R}^{n+1} \rightarrow \mathbb{R}$. Let $c \in \mathbb{R}$ be such that $F^{-1}(c) \neq \emptyset$ and $\nabla F(a) \neq 0$ for each $a \in F^{-1}(c)$.

For example, if $F(x) = \|x\|^2$, then $S^n = F^{-1}(1)$ and $\nabla F|_{F^{-1}(c)} \neq 0$. (We say $\{F^{-1}(r) : r > 0\}$ is an *n-dimensional foliation* of $\mathbb{R}^{n+1} \setminus \{0\}$.)

Choose $a \in F^{-1}(c)$. Then $DF(a) \neq 0$, so there is an i such that $\frac{\partial F}{\partial x_i}(a) \neq 0$. Then the equation $F(x_1, \dots, x_i, \dots, x_{n+1}) = c$ can be solved locally for x_i in terms of the other coordinates, i.e. $F^{-1}(c)$ is the graph of a smooth function near a .

Making this precise, the implicit function theorem provides us with a neighbourhood U of $(a_1, \dots, \hat{a}_i, \dots, a_{n+1})$ in \mathbb{R}^n and a smooth function $g : U \rightarrow \mathbb{R}$ satisfying

- $g(a_1, \dots, \hat{a}_i, \dots, a_{n+1}) = a_i$,
- $F(x_1, \dots, g(x_1, \dots, \hat{x}_i, \dots, x_{n+1}), \dots, x_{n+1}) = c$ for all $(x_1, \dots, \hat{x}_i, \dots, x_{n+1}) \in U$,

i.e.

$$\Gamma_g = \{(x_1, \dots, g(x_1, \dots, \hat{x}_i, \dots, x_{n+1}), \dots, x_{n+1}) \in \mathbb{R}^{n+1} : (x_1, \dots, \hat{x}_i, \dots, x_{n+1}) \in U\} = V \cap F^{-1}(c)$$

for some neighbourhood V of a in \mathbb{R}^{n+1} .

So we conclude that if $\nabla F(a) \neq 0$ for all $a \in F^{-1}(c) \neq \emptyset$, then $F^{-1}(c)$ is locally the graph of a function. What are the charts? $(V \cap F^{-1}(c), \phi)$, where $\phi : V \cap F^{-1}(c) \rightarrow U$ is given by $\phi(x_1, \dots, x_{n+1}) = (x_1, \dots, \hat{x}_i, \dots, x_{n+1})$ with the inverse $\phi^{-1}(x_1, \dots, \hat{x}_i, \dots, x_{n+1}) = (x_1, \dots, g(x_1, \dots, \hat{x}_i, \dots, x_{n+1}), \dots, x_{n+1})$. This is clearly a chart.

Now consider the collection of such charts $\mathcal{A} = \{(V_a \cap F^{-1}(c), \phi_a)\}$. Consider a transition mapping $\phi_a \circ \phi_b^{-1} : \phi_b(V_{ab}) \rightarrow \phi_a(V_{ab})$. This is

$$\begin{aligned} (\phi_a \circ \phi_b^{-1})(x_1, \dots, x_i, \dots, \hat{x}_j, \dots, x_{n+1}) &= \phi_a(x_1, \dots, x_i, \dots, g_b(x_1, \dots, \hat{x}_j, \dots, x_{n+1}), \dots, x_{n+1}) \\ &= (x_1, \dots, \hat{x}_i, \dots, g_b(x_1, \dots, \hat{x}_j, \dots, x_{n+1}), \dots, x_{n+1}) \end{aligned}$$

which is C^∞ , and similarly for its inverse. So \mathcal{A} is a C^∞ atlas on $F^{-1}(c)$, making $F^{-1}(c)$ a smooth manifold of dimension n .

3. (Products) Consider two smooth manifolds M and N of dimensions m and n , respectively. Equip them with smooth atlases \mathcal{A}_M and \mathcal{A}_N , respectively. Define

$$\mathcal{A}_{M \times N} = \{(U \times V, \phi \times \psi) : (U, \phi) \in \mathcal{A}_M \text{ and } (V, \psi) \in \mathcal{A}_N\}.$$

$\mathcal{A}_{M \times N}$ is a smooth atlas on $M \times N$, making $M \times N$ a smooth manifold of dimension $m+n$. To see this, note that the sets $U \times V$ certainly cover $M \times N$, and that the products of homeomorphisms are homeomorphisms. If $(U_1 \times V_1, \phi_1 \times \psi_1), (U_2 \times V_2, \phi_2 \times \psi_2) \in \mathcal{A}_{M \times N}$, then the transition map

$$(\phi_1 \times \psi_1) \circ (\phi_2 \times \psi_2)^{-1} : (\phi_2 \times \psi_2)((U_1 \times V_1) \cap (U_2 \times V_2)) \rightarrow (\phi_1 \times \psi_1)((U_1 \times V_1) \cap (U_2 \times V_2))$$

is, by set theory, equal to

$$(\phi_1 \circ \phi_2^{-1}) \times (\psi_1 \circ \psi_2^{-1}) : \phi_2(U_{12}) \times \psi_2(V_{12}) \rightarrow \phi_1(U_{12}) \times \psi_1(V_{12}),$$

which is clearly a diffeomorphism.

For example, the cylinder $S^1 \times \mathbb{R}$ is a smooth manifold of dimension 2, and the torus $S^1 \times S^1$ is a smooth manifold of dimension 2. We also have the higher tori $T^n = S^1 \times \cdots \times S^n$, a smooth manifold of dimension n .

(Algebraic topology remark: $T^n \not\cong S^n$, as the former has first fundamental group \mathbb{Z}^n , whereas the latter is simply connected for $n \geq 2$.)

4.2 Gluing Manifolds

Due to the informal visual nature of this part of the lecture, the examples can only be described in words.

1. Glue the endpoints of $[0, 1]$ to get the circle. They aren't homeomorphic however, since removing an interior point from $[0, 1]$ disconnects it, whereas the circle will remain connected if a point is removed.
2. Glue the two vertical sides of $[0, 1]^2$ to get a cylinder. (Note: in order to visualize this, we need to go up one dimension.)
3. Glue the two vertical sides of $[0, 1]^2$, but with points identified "by reflecting through the centre $(1/2, 1/2)$ ". This produces a Möbius strip.
4. Glue the opposite sides of $[0, 1]^2$ together as in example 2, but with each opposite side glued. This produces a torus.
5. Glue the opposite vertical sides of $[0, 1]^2$ together as in example 2, and the opposite horizontal sides together as in example 3. This produces a "Klein bottle", an example of a manifold which cannot be embedded in \mathbb{R}^3 .

4.3 The Quotient Topology

Let S be a topological space and \sim an equivalence relation on S . Let $\pi : S \rightarrow S/\sim$ be the projection map $\pi(x) = [x]$. Topologize S/\sim by declaring $U \subset S/\sim$ to be open if and only if $\pi^{-1}(U)$ is open in S . This topology on S/\sim is called the *quotient topology* - it is the finest topology on S/\sim with respect to which π is continuous, as is easily seen.

Now consider a function $f : S \rightarrow Y$, where Y is a set. Suppose f is constant on the fibres of π (i.e. f is constant on every equivalence class of \sim). Then f induces a map $\tilde{f} : S/\sim \rightarrow Y$ for which the following diagram is commutative:

$$\begin{array}{ccc} S & & \\ \downarrow \pi & \searrow f & \\ S/\sim & \dashrightarrow \tilde{f} & \rightarrow Y \end{array}$$

The function \tilde{f} is defined in the obvious way: $\tilde{f}([x]) = f(x)$. The new function \tilde{f} is well-defined since we assumed f was constant on equivalence classes. We say that f descends to the quotient. If Y is a topological space, we have a very useful lemma.

Lemma 4.3.1. *Suppose $f : S \rightarrow Y$ is a function of topological spaces, and that \sim is an equivalence relation on S on whose equivalence classes f is constant. Then the induced map $\tilde{f} : S/\sim \rightarrow Y$ is continuous if and only if f is continuous.*

Proof. If \tilde{f} is continuous, then $f = \tilde{f} \circ \pi$ is continuous as a composition of continuous maps. If f is continuous, then given U open in Y , $f^{-1}(U)$ is open in S . But $f^{-1}(U) = \pi^{-1}(\tilde{f}^{-1}(U))$, so by the definition of the quotient topology, $\tilde{f}^{-1}(U)$ is open in S/\sim , proving continuity of \tilde{f} . \square

Let's discuss the example of gluing the endpoints of the interval. Define \sim on $I = [0, 1]$ by $x \sim x$ for $x \in (0, 1)$ and $x \sim y$ for $x, y \in \{0, 1\}$. We claim that $I/\sim \cong S^1$. An explicit homeomorphism can be found by descending to the quotient.

Define $f : I \rightarrow S^1$ by $f(t) = (\cos 2\pi t, \sin 2\pi t)$. Then $f(0) = f(1) = (1, 0)$, so f is constant on the equivalence classes of \sim . Then f descends to a continuous map $\tilde{f} : I/\sim \rightarrow S^1$, given by

$$\tilde{f}([t]) = \begin{cases} (\cos 2\pi t, \sin 2\pi t), & [t] \neq [0] \\ (1, 0), & t = [0] = [1] \end{cases}$$

which is bijective. Since $I/\sim = \pi(I)$ is compact and S^1 is Hausdorff, the map \tilde{f} is a homeomorphism of topological spaces. So indeed, $I/\sim \cong S^1$.

In order to tackle the question "when is a quotient a manifold", we need to derive some conditions for when the quotient of a space is Hausdorff or second countable. Here's a simple necessary condition.

Lemma 4.3.2. *If S/\sim is Hausdorff, then equivalence classes are closed in S .*

Proof. Each $\{[x]\} = \{\pi(x)\}$ is closed in S/\sim by Hausdorffness, so by continuity $\pi^{-1}(\{\pi(x)\}) = [x]$ is closed in S . \square

For a simple application of this necessary condition, consider $\mathbb{R}/(0, \infty)$ - the quotient space obtained by identifying all points of $(0, \infty)$. The lemma dictates that $\mathbb{R}/(0, \infty)$ is not Hausdorff because the equivalence class $(0, \infty)$ is not closed in \mathbb{R} .

4.4 Open Equivalence Relations

Definition 4.4.1. *An equivalence relation \sim on a space S is said to be open if the projection $\pi : S \rightarrow S/\sim$ is an open mapping. Equivalently, \sim is open if and only if*

$$\pi^{-1}(\pi(U)) = \bigcup_{x \in U} [x]$$

is open in S , for each U open in S .

This definition is worth making, as the projections need not be open in general. Consider $\mathbb{R}/\{-1, 1\}$. The interval $(-2, 0)$ is open, but

$$\pi^{-1}(\pi((-2, 0))) = \bigcup_{-2 < x < 0} [x] = (-2, 0) \cup \{1\}$$

is not open in \mathbb{R} . Therefore \sim identifying -1 and 1 on \mathbb{R} is not an open equivalence relation. (Note that $\mathbb{R}/\{-1, 1\}$ is not a topological manifold, as it is homeomorphic to the symbol ∞ with the ends extending infinitely.)

Definition 4.4.2. *The graph of an equivalence relation \sim on S is the set $R = \{(x, y) \in S \times S : x \sim y\}$.*

Theorem 4.4.1. *Suppose \sim is an open equivalence relation on S . Then S/\sim is Hausdorff if and only if the graph R of \sim is closed in $S \times S$.*

Proof. Was left as an exercise in class, so here's a solution. We have a sequence of equivalent statements

$$\begin{aligned} R \text{ is closed} &\iff S \times S \setminus R \text{ is open} \\ &\iff \text{for all } (x, y) \in S \times S \setminus R \text{ there are open sets } U, V \text{ such that } (x, y) \in U \times V \subset S \times S \setminus R \\ &\iff \text{for all } x \not\sim y \text{ in } S \text{ there are open sets } U \ni x, V \ni y \text{ such that } (U \times V) \cap R = \emptyset \\ &\iff \text{for all } [x] \neq [y] \text{ in } S/\sim \text{ there are open sets } U \ni x, V \ni y \text{ such that } \pi(U) \cap \pi(V) = \emptyset \end{aligned}$$

This last statement is equivalent to S/\sim being Hausdorff, which we now prove. If this statement is true, then $\pi(U)$ and $\pi(V)$ are disjoint open (because \sim is open) sets of S/\sim separating $[x]$ and $[y]$, which shows that S/\sim is Hausdorff. Conversely, suppose S/\sim is Hausdorff. Given $[x] \neq [y]$ in S/\sim , we can find disjoint open sets $U \ni x, V \ni y$ of S/\sim . By surjectivity, $U = \pi(\pi^{-1}(U))$ and $V = \pi(\pi^{-1}(V))$, so $\pi^{-1}(U)$ and $\pi^{-1}(V)$ are open sets of S containing x and y , respectively, satisfying the condition of the last statement. So the last statement is equivalent to S/\sim being Hausdorff. \square

With it is a corollary - a classic exercise in point-set topology.

Corollary 4.4.1. *S is Hausdorff if and only if $\Delta = \{(x, x) \in S \times S : x \in S\}$ is closed.*

Proof. Let \sim be the equivalence relation identifying every point only with itself. Then \sim is an open equivalence relation and $R = \Delta$. The spaces S and S/\sim are homeomorphic, so the statement follows from the theorem immediately. \square

What about second countability?

Theorem 4.4.2. *If \sim is an open equivalence relation on S and $\{B_n\}$ is a countable basis of S , then $\{\pi(B_n)\}$ is a countable basis of S/\sim .*

Proof. Was left as an exercise in class, so here's a solution. Note that the collection $\{\pi(B_n)\}$ is a collection of open sets because π is an open mapping. Let $U \subset S/\sim$ be open and consider $[x] \in U$. Then $x \in \pi^{-1}(U)$, so we can find a B_n with $x \in B_n \subset \pi^{-1}(U)$. Then $[x] = \pi(x) \subset \pi(B_n) \subset \pi(\pi^{-1}(U)) = U$, proving that $\{B_n\}$ is a basis of S/\sim . \square

To summarize,

- quotient spaces of Hausdorff spaces under open equivalence relations are Hausdorff if and only if the graph of the relation is closed
- quotient spaces of second-countable spaces under open equivalence relations are second-countable, and bases for the quotient are obtained in the obvious way.

4.5 Real Projective Space

Define \sim on $\mathbb{R}^{n+1} \setminus \{0\}$ by $x \sim \lambda x$ for $\lambda \neq 0$. The quotient space $(\mathbb{R}^{n+1} \setminus \{0\})/\sim$ is denoted $\mathbb{R}P^n$ and is called *real projective space*. It may be thought of as the set of lines passing through the origin.

Each element of $\mathbb{R}P^n$ can be thought of as a pair of antipodal points on S^n , which motivates the following

Theorem 4.5.1. Define \sim on S^n by identifying antipodal points, i.e. $x \sim \pm x$. Define $f : \mathbb{R}^{n+1} \setminus \{0\} \rightarrow S^n$ by $f(x) = \frac{x}{\|x\|}$. Then f induces a homeomorphism $\mathbb{R}P^n \xrightarrow{\sim} S^n/\sim$.

The proof will be essentially the proof given in class, but much more complete and explicit about how maps induce other maps.

Proof. Consider the following diagram:

$$\begin{array}{ccc} \mathbb{R}^{n+1} \setminus \{0\} & \xrightarrow{f} & S^n \\ \downarrow \pi_1 & & \downarrow \pi_2 \\ \mathbb{R}P^n & & S^n/\sim \end{array}$$

where π_1 and π_2 are the projections to each quotient space as shown in the diagram. The map $\pi_2 \circ f : \mathbb{R}^{n+1} \setminus \{0\} \rightarrow S^n/\sim$ is given by

$$(\pi_2 \circ f)(x) = \pi_2 \left(\frac{x}{\|x\|} \right) = \left\{ -\frac{x}{\|x\|}, \frac{x}{\|x\|} \right\} = [x]_2,$$

which is continuous and constant on the fibres of π_1 ; the lines through the origin. It thus induces a continuous map $\tilde{f} : \mathbb{R}P^n \rightarrow S^n/\sim$ for which the following diagram is commutative:

$$\begin{array}{ccc} \mathbb{R}^{n+1} \setminus \{0\} & \xrightarrow{f} & S^n \\ \downarrow \pi_1 & \searrow \pi_2 \circ f & \downarrow \pi_2 \\ \mathbb{R}P^n & \xrightarrow{\tilde{f}} & S^n/\sim \end{array}$$

We define a continuous inverse of \tilde{f} . Consider $g : S^n \rightarrow \mathbb{R}^{n+1} \setminus \{0\}$ given by $g(x) = x$. As before, consider the diagram

$$\begin{array}{ccc} \mathbb{R}^{n+1} \setminus \{0\} & \xleftarrow{g} & S^n \\ \pi_1 \downarrow & & \downarrow \pi_2 \\ \mathbb{R}P^n & & S^n / \sim \end{array}$$

The map $\pi_1 \circ g : S^n \rightarrow \mathbb{R}P^n$ is given by

$$(\pi_1 \circ g)(x) = \pi_1(x) = \{\lambda x : \lambda \neq 0\} = [x]_1,$$

which is continuous and constant on the fibres of π_2 ; antipodal points on the n -sphere. It thus induces a continuous map $\tilde{g} : S^n / \sim \rightarrow \mathbb{R}P^n$ for which the following diagram is commutative:

$$\begin{array}{ccc} \mathbb{R}^{n+1} \setminus \{0\} & \xleftarrow{g} & S^n \\ \pi_1 \downarrow & \swarrow \pi_1 \circ g & \downarrow \pi_2 \\ \mathbb{R}P^n & \xleftarrow{\tilde{g}} & S^n / \sim \end{array}$$

We claim that \tilde{f} and \tilde{g} are inverses to each other, which will show that \tilde{f} is a homeomorphism $\mathbb{R}P^n \xrightarrow{\sim} S^n / \sim$. We have

$$\begin{aligned} (\tilde{g} \circ \tilde{f})([x]_1) &= (\tilde{g} \circ \tilde{f} \circ \pi_1)(x) = (\tilde{g} \circ \pi_2 \circ f)(x) = (\pi_1 \circ g \circ f)(x) = \pi_1 \left(g \left(\frac{x}{\|x\|} \right) \right) = \pi_1 \left(\frac{x}{\|x\|} \right) = [x]_1 \\ (\tilde{f} \circ \tilde{g})([x]_2) &= (\tilde{f} \circ \tilde{g} \circ \pi_2)(x) = (\tilde{f} \circ \pi_1 \circ g)(x) = (\pi_2 \circ f \circ g)(x) = \pi_2(f(x)) = \pi_2 \left(\frac{x}{\|x\|} \right) = [x]_2 \end{aligned}$$

So \tilde{f} is a homeomorphism $\mathbb{R}P^n \xrightarrow{\sim} S^n / \sim$. □

In particular, $\mathbb{R}P^n$ is compact! Note that we could have just explicitly defined

$$\begin{aligned} \tilde{f} : \mathbb{R}P^n &\rightarrow S^n / \sim & \tilde{f}([x]_1) &:= \pi_2(f(x)) = \left[\frac{x}{\|x\|} \right]_2 \\ \tilde{g} : S^n / \sim &\rightarrow \mathbb{R}P^n & \tilde{g}([x]_2) &:= \pi_1(g(x)) = [x]_1 \end{aligned}$$

checked for well-definedness and continuity, and then we'd have been done. That's how the proof on page 362 of Tu goes. However, the abuse of tikz diagrams makes it very clear where the homeomorphism and its inverse come from, and that they're continuous (which is basically what Tu is doing anyway).

4.6 Visualizing $\mathbb{R}P^2$

In order to visualize $\mathbb{R}P^2$ we will consider some homeomorphisms. Define

$$\begin{aligned} H^2 &= \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1, z \geq 0\} \\ D^2 &= \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}. \end{aligned}$$

Consider the maps

$$\begin{aligned} \phi : H^2 &\rightarrow D^2 & \phi(x, y, z) &= (x, y) \\ \psi : D^2 &\rightarrow H^2 & \psi(x, y) &= (x, y, \sqrt{1 - x^2 - y^2}) \end{aligned}$$

which are continuous inverses of each other. Define equivalence relations on H^2 and D^2 as follows:

- On H^2 : identify antipodal points on the equator, call the projection π_3
- On D^2 : identify antipodal points on the boundary, call the projection π_4

Considering diagrams similar to those in the previous proof, the map $\pi_4 \circ \phi$ induces a continuous map $\tilde{\phi} : H^2/\sim \rightarrow D^2/\sim$ with $\tilde{\phi} \circ \pi_3 = \pi_4 \circ \phi$, and the map $\pi_3 \circ \psi$ induces a continuous map $\tilde{\psi} : D^2/\sim \rightarrow H^2/\sim$ with $\tilde{\psi} \circ \pi_4 = \pi_3 \circ \psi$. The maps $\tilde{\phi}$ and $\tilde{\psi}$ are continuous inverses of each other (which can be seen using just these given compositions), which shows that we have a homeomorphism $H^2/\sim \xrightarrow{\sim} D^2/\sim$.

If we accept on faith that there is a homeomorphism $S^2/\sim \xrightarrow{\sim} H^2/\sim$, then we have a sequence of homeomorphisms

$$\mathbb{R}P^2 \xrightarrow{\sim} S^2/\sim \xrightarrow{\sim} H^2/\sim \xrightarrow{\sim} D^2/\sim.$$

Therefore we can visualize the real projective plane $\mathbb{R}P^2$ as a disk with the antipodal boundary points identified. Such a homeomorphism $S^2/\sim \xrightarrow{\sim} H^2/\sim$ can be shown by a proof similar to the previous quotient space homeomorphisms that we did, by considering the inclusion map $i : H^2 \rightarrow S^2$ and its obvious inverse, and working through steps similar to the proofs of the previous homeomorphisms.