

# 1 Orientation, Manifolds With Boundary (August 7)

We will develop the notions of orientation for manifolds, and develop manifolds with boundary. These will allow us a more general theory of integration on manifolds. One develops orientation in order to make sense of integration on manifolds, and one develops manifolds with boundary in order to have a suitable theory of integration on manifolds. One may also develop the notion of a manifold-with-corners, but we shall not do so.

## 1.1 Orientation on Vector Spaces

Let us take care of notation, first. A basis for a vector space without any ordering is either written as  $v_1, \dots, v_n$  or  $\{v_1, \dots, v_n\}$ . An ordered basis is written as  $(v_1, \dots, v_n)$ .

As we did with forms, we will develop things on a vector space first, and then generalize them pointwise to manifolds by doing things on each tangent space.

On  $\mathbb{R}$ , we have either the "left" orientation or the "right" orientation. If we take the "right" orientation, then

$$\int_{[a,b]} f = \int_a^b f(x) dx,$$

and if we take the "left" orientation, then

$$\int_{[a,b]} f = \int_b^a f(x) dx.$$

More precisely, we have positive and negative orientations on  $\mathbb{R}$ .

On  $\mathbb{R}^2$ , we have the "clockwise" and "counterclockwise" orientations. The latter is the "usual orientation" of  $\mathbb{R}^2$ . Similarly, on  $\mathbb{R}^3$ , we have orientations specified by clockwise and counterclockwise twirls around the  $z$ -axis. These orientations can be thought of as picking the standard basis vectors in a certain order.

Let us make these notions precise. Let  $V$  be an  $n$ -dimensional vector space. We no longer have a standard basis to work with, so we will develop a notion of orientation by using multiple bases; the orientation on a vector space will be a set of bases, each related to another by an orientation-preserving (i.e. of positive determinant) change of basis matrix.

**Definition 1.** Let  $\alpha = (v_1, \dots, v_n)$  and  $\beta = (w_1, \dots, w_n)$  be ordered bases of  $V$ . We say that  $\alpha$  and  $\beta$  specify the same orientation if the change of basis matrix from  $\alpha$  to  $\beta$  has positive determinant. This is obviously an equivalence relation, so it partitions the set of ordered bases of  $V$  into two equivalence classes. Each class is called an orientation on  $V$ .

Let  $\alpha, \beta$  be as in the previous definition, and let  $Q$  denote the change of basis matrix from  $\alpha$  to  $\beta$ . Then we have  $v_i = Q_i^j w_j$  (Einstein notation), so if  $\gamma \in \bigwedge^n(V^*)$ ,

$$\gamma(v_1, \dots, v_n) = \det(Q) \gamma(w_1, \dots, w_n).$$

Thus  $\gamma(v_1, \dots, v_n)$  and  $\gamma(w_1, \dots, w_n)$  have the same sign if and only if  $(v_1, \dots, v_n)$  and  $(w_1, \dots, w_n)$  specify the same orientation. We say that the  $n$ -covector  $\gamma$  *specifies the orientation*  $[(v_1, \dots, v_n)]$  if  $\gamma(v_1, \dots, v_n) > 0$ . As we just saw, this is a well-defined notion. Thus an  $n$ -covector specifies an orientation on  $V$ . Since  $\bigwedge^n(V^*)$  is one-dimensional, two  $n$ -covectors  $\gamma$  and  $\gamma'$  specify the same orientation if and only if there is a positive  $a \in \mathbb{R}$  with  $\gamma = a\gamma'$ . Taking this to be an equivalence relation on the set of non-zero  $n$ -covectors on  $V$ , we see that an orientation of  $V$  is also given by an equivalence class of  $n$ -covectors.

Note that we can also think of an orientation as a choice of component of  $\bigwedge^n(V^*)$ .

## 1.2 Orientation on Manifolds

We would like to make a "smooth choice" of orientation on each tangent space to  $M$ . We will call a choice of orientation at each tangent space of  $M$  a *pointwise orientation*. We do not want to haphazardly choose orientations of each tangent space, since then a manifold would have too many ( $2^{|M|}$ , to be precise) orientations.

The most straightforward way to develop the notion of a smooth choice of orientations is to consider a simple case, and then generalize. Consider an embedded 1-dimensional submanifold  $S$  of  $\mathbb{R}^n$  (a curve). If we have a non-zero vector field  $X$  on  $S$ , then each  $X_p$ , for  $p \in S$ , is a (n ordered) basis of  $T_p S$ , so it determines an orientation of  $T_p S$ . Suppose that we have chosen a pointwise orientation on  $S$ . If, for each  $p \in S$ , the non-zero tangent vector  $X_p$  determines the orientation we chose on  $T_p M$ , then it would be reasonable to call our pointwise orientation smooth if the vector field  $X$  is smooth. This turns out to give us the first notion of orientation of a manifold (note that this did not depend on the fact that  $S$  was in  $\mathbb{R}^n$ : that simply makes it easier to visualize this scenario).

**Definition 2.** *An orientation on a manifold  $M$  is a pointwise orientation on  $M$  such that for all  $p \in M$ , there is a neighbourhood  $U$  of  $p$  and a smooth local frame  $X_1, \dots, X_n \in \mathfrak{X}(U)$  such that for each  $q \in U$ , the orientation specified by  $(X_{1q}, \dots, X_{nq})$  on  $T_q M$  is consistent with the choice of orientation on  $T_q M$ .*

Equivalently, an orientation on  $M$  is a pointwise orientation on  $M$  such that for all  $p \in M$ , there is a chart  $(U, x^1, \dots, x^n)$  near  $p$  such that for each  $q \in U$ , the orientation specified by  $\left(\frac{\partial}{\partial x^1}\big|_q, \dots, \frac{\partial}{\partial x^n}\big|_q\right)$  on  $T_q M$  is consistent with the choice of orientation on  $T_q M$ . The proof of this fact will be a homework problem.

This gives rise to the notion of an oriented atlas. An *oriented atlas* on  $M$  is an atlas with the property that  $\det(D(\psi \circ \phi^{-1})) > 0$  for all transition maps  $\psi \circ \phi^{-1}$ . So equivalently, an orientation on  $M$  is a pointwise orientation admitting an oriented atlas.

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**Definition 3.** *A manifold  $M$  is said to be orientable if it admits an orientation. An oriented manifold is an orientable manifold together with a choice of orientation.*

**Proposition 1.1.** *An orientable manifold  $M$  admits precisely  $2^c$  orientations, where  $c$  is the number of components of  $M$ .*

*Proof.* It suffices to show that a connected manifold  $M$  admits precisely two orientations. This is a standard topological argument: construct a locally constant function and argue by connectedness that the function must be constant on the entire topological space. A full proof is given in the book.  $\square$

In particular, a connected orientable manifold admits precisely two orientations.

Let  $\omega$  be an  $n$ -form on  $M$ . If  $\omega_p \neq 0$ , then  $\omega_p$ , being a non-zero  $n$ -covector on  $T_pM$ , determines an orientation on  $T_pM$ . Thus a non-vanishing  $n$ -form  $\omega$  on  $M$  determines (uniquely) a pointwise orientation on  $M$ . One would expect that a smooth non-vanishing  $n$ -form on a manifold determines an orientation of that manifold. This is, in fact, true, and we provide a proof.

**Theorem 1.1.** *A manifold  $M$  is orientable if and only if it admits a non-vanishing smooth top-degree form.*

*Proof.* Suppose that  $\omega$  is a non-vanishing smooth top degree form on  $M$ . For each  $p \in M$ , give  $T_pM$  the orientation specified by  $\omega_p \in \bigwedge^n(T_p^*M)$ . This gives a pointwise orientation on  $M$ . Given  $p \in M$ , let  $(U, \phi) = (U, x^1, \dots, x^n)$  be a connected coordinate chart at  $p$ . By connectedness, we have that  $\omega\left(\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^n}\right)$  is either strictly positive or strictly negative on  $U$ . Assume without loss of generality that it is a strictly positive function on  $U$ . Then, for each  $q \in U$ ,  $\omega\left(\frac{\partial}{\partial x^1}|_q, \dots, \frac{\partial}{\partial x^n}|_q\right) > 0$ , meaning that the ordered basis  $\left(\frac{\partial}{\partial x^1}|_q, \dots, \frac{\partial}{\partial x^n}|_q\right)$  of  $T_qM$  specifies the same orientation as  $\omega_q$  did. Therefore  $M$  is orientable.

Conversely, suppose that  $M$  is orientable. Given  $p \in M$ , we may find a coordinate chart  $(U, x^1, \dots, x^n)$  at  $p$  such that for each  $q \in U$ , the orientation on  $T_qM$  coincides with the orientation specified by the  $n$ -covector  $(dx^1 \wedge \dots \wedge dx^n)_q$ . We therefore have a smooth non-vanishing top-degree form defined on an open set of  $M$  which gives the orientation there. This is vulnerable to a standard partition of unity argument. Let  $\{(U_\alpha, \phi_\alpha)\}$  be an oriented atlas for the oriented manifold  $M$ . Let  $\{\rho_\alpha\}$  be a partition of unity subordinated to this atlas. Define

$$\omega := \sum_{\alpha} \rho_{\alpha} dx_{\alpha}^1 \wedge \dots \wedge dx_{\alpha}^n.$$

It is easy to check that  $\omega$  is the desired non-vanishing smooth top-degree form, and moreover, that the orientation  $\omega$  specifies on  $M$  is the same as the orientation we gave  $M$ .  $\square$

The following corollary of this characterization of orientability provides a wealth of oriented manifolds for us to work with. The proof was left as an exercise in class. The proof we will give provides a technique for finding *orientation forms* for such regular hypersurfaces, and once Riemannian manifolds are developed, the result may be generalized. (That said, the proof already shows something a lot more general than the statement. An *orientation form* is a non-vanishing smooth top degree form giving the orientation on  $M$  as in the preceding theorem.)

**Corollary 1.1.1.** *Any regular hypersurface in  $\mathbb{R}^n$  is orientable.*

*Proof.* Suppose  $S = f^{-1}(\{0\})$ , where  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  is a smooth function with 0 as a regular value. Denote by  $\nabla f$  the gradient vector field of  $f$  on  $\mathbb{R}^n$ . This vector field is, by assumption, non-zero on  $S$ . Let  $\omega$  be an orientation form on  $\mathbb{R}^n$  (for example, we can take  $\omega = dx^1 \wedge \cdots \wedge dx^n$ ). Let  $\eta = i_S^*(i_{\nabla f}(\omega))$ , where  $i_{\nabla f}$  is interior multiplication by  $\nabla f$  and  $i_S : S \hookrightarrow \mathbb{R}^n$  is the inclusion map. Then  $\eta$  is a smooth top degree form on  $S$ . Given  $p \in S$ ,  $\nabla f|_p \in ((i_S)_{*,p}(T_p S))^\perp$  is non-zero, so if  $\{v_1, \dots, v_{n-1}\}$  is a basis of  $T_p S$ , we have

$$\eta_p(v_1, \dots, v_{n-1}) = \omega_p(\nabla f|_p, (i_S)_{*,p}(v_1), \dots, (i_S)_{*,p}(v_{n-1})) \neq 0,$$

for  $\{\nabla f|_p, (i_S)_{*,p}(v_1), \dots, (i_S)_{*,p}(v_{n-1})\}$  is a basis of  $T_p \mathbb{R}^n$  and  $\omega_p \neq 0$ . Then  $\eta$  is a non-vanishing smooth top degree form on  $S$ , so  $S$  is orientable by the previous theorem.  $\square$

As a corollary,  $S^{n-1}$  is orientable (as one would hope), and the Mobius band, being non-orientable, is not a regular hypersurface in  $\mathbb{R}^3$ .

Given an  $n$ -manifold  $M$ , define an equivalence relation on the set of smooth non-vanishing  $k$ -forms as follows:  $\omega \sim \omega'$  if and only if  $\omega = f\omega'$  for some strictly positive continuous function  $f : M \rightarrow \mathbb{R}$ . This partitions the non-vanishing smooth top degree forms on  $M$  into two equivalence classes, each specifying an orientation on  $M$ . We have the following correspondences:

$$\begin{aligned} \text{orientations} &\iff \text{equivalence classes of non-vanishing smooth } k\text{-forms on } M \\ &\iff \text{equivalence classes of oriented atlases on } M \end{aligned}$$

### 1.3 Manifolds With Boundary

As a manifold is locally modelled by  $\mathbb{R}^n$ , a manifold with boundary is locally modelled by the *half-space*  $\mathbb{H}^n = \{(x^1, \dots, x^n) \in \mathbb{R}^n : x^n \geq 0\}$  with the subspace topology. Many notions from our original definition of a manifold carry over almost word-for-word, but there is a particularly important notion of interior and boundary for a manifold with boundary.

**Definition 4.** *A point  $x \in \mathbb{H}^n$  is said to be an interior point if  $x^n > 0$ , and is said to be a boundary point if  $x^n = 0$ . The set of interior points is denoted by  $(\mathbb{H}^n)^o$ , and the set of boundary points is denoted by  $\partial\mathbb{H}^n$ .*

The set of interior points and boundary points, as just defined, coincide with the topological interior and boundary of  $\mathbb{H}^n$ , so there is no harm in simply referring to "interior points" and "boundary points" when referring to the "prototype manifold with boundary"  $\mathbb{H}^n$ . Once we develop the more general manifold with boundary, this will not necessarily be the case. It is important to note that any open neighbourhood of a boundary point of  $\mathbb{H}^n$  will not be an open set in  $\mathbb{R}^n$ .

To establish clear notation, we will denote by  $\text{Int}(S)$  the topological interior of a set and  $\text{Bd}(S)$  the topological boundary of a set.

We say that a topological space  $M$  is *locally- $\mathbb{H}^n$*  if every point has an open neighbourhood homeomorphic to an open subset of  $\mathbb{H}^n$ . With this, we define manifolds with boundary in the continuous category. We will then generalize to smooth manifolds with boundary.

**Definition 5.** A topological  $n$ -manifold with boundary is a second-countable Hausdorff locally- $\mathbb{H}^n$  space. These homeomorphisms are called (coordinate) charts, as one would expect.

The standard terminology which applies to coordinates on a smooth manifold, as defined before, also applies to the coordinates on a manifold with boundary, smooth or not.

**Definition 6.** A collection  $\{(U_\alpha, \phi_\alpha)\}$  of charts on the topological manifold with boundary  $M$  is said to be a smooth atlas on  $M$  if it covers  $M$  and if the transition maps (same notion as before) are  $C^\infty$  functions on open subsets of  $\mathbb{H}^n$ . (Here we mean  $C^\infty$  in the extended sense.) Maximal atlases are defined as before.

A smooth manifold with boundary is a topological manifold with boundary equipped with a maximal smooth atlas.

Now we must define the notion of the interior point and the boundary point for a smooth manifold with boundary. We will define them in a seemingly coordinate-dependent way, but it will turn out that our definition is actually coordinate-independent.

**Definition 7.** A point  $p$  in the manifold with boundary  $M$  is said to be an interior point of  $M$  if there is a chart  $(U, \phi)$  at  $p$  such that  $\phi(p) \in (\mathbb{H}^n)^\circ$ , and is said to be a boundary point if  $\phi(p) \in \partial\mathbb{H}^n$ . These notions are well-defined (coordinate-independent) by the following theorem and its corollary:

**Theorem 1.2.** (Smooth invariance of domain) Let  $U \subseteq \mathbb{R}^n$  be open and  $S \subseteq \mathbb{R}^n$  be arbitrary. Then  $S$  is open if there is a diffeomorphism  $U \rightarrow S$ .

**Corollary 1.2.1.** Let  $U, V \subseteq \mathbb{H}^n$  be open and  $f : U \rightarrow V$  a diffeomorphism. Then  $f$  maps interior points to interior points and boundary points to boundary points.

Thus we denote by  $\partial M$  the boundary points of a manifold with boundary  $M$ . (We will not really use the notion of an interior point.)

**Proposition 1.2.** If  $M$  is a smooth  $n$ -manifold with boundary, then  $\partial M$  is an embedded codimension 1 submanifold of  $M$  with empty boundary.

*Proof.* If  $(U, x^1, \dots, x^n)$  is a coordinate chart on  $M$  with  $U \cap \partial M \neq \emptyset$ , then  $U \cap \partial M = \{x^n = 0\}$ .  $\square$

**Corollary 1.2.2.**  $\partial^2 = 0$ . (Compare with  $d^2 = 0$  for forms.)

We will eventually see that if  $M$  is an oriented manifold with boundary, then  $\partial M$  has a natural orientation induced by that of  $M$ . This will be crucial in developing Stokes' theorem  $\int_M d\omega = \int_{\partial M} \omega$ .