1 The Exterior Derivative of a k-form (July 29)

1.1 Motivating the Local Definition

Definition 1. *An antiderivation on a graded algebra $A = \bigoplus_k A^k$ is an $\mathbb{R}$-linear map $D : A \to A$ such that*

$$D(\omega \cdot \tau) = D(\omega) \cdot \tau + (-1)^k \omega \cdot D(\tau) \qquad \textit{whenever } \omega \in A^k.$$

An element $\omega \in A^k$ is said to be (homogeneous) of degree k. The antiderivation D is said to be of degree m if $\deg(D(\omega)) = \deg(\omega) + m$ for all $\omega \in A$.

Recall that $\Omega^*(M) = \bigoplus_k \Omega^k(M)$. The exterior derivative d on $\Omega^*(M)$ that we wish to define will be an antiderivation of degree 1.

On 0-forms, we defined d in a coordinate-independent way as $d : f \mapsto (X \mapsto X(f))$. Alternatively, we could have defined it in each coordinate chart and showed that the definition is independent of the chart. Specifically, if (U, x^i) is a coordinate chart on M, we can define df on U as the 1-form $df = \frac{\partial f}{\partial x^i} dx^i$. To show that this is independent of the coordinate chart, let (V, y^i) be another chart with $U \cap V \neq \emptyset$. Then, on $U \cap V$,

$$\frac{\partial f}{\partial x^i} dx^i = \frac{\partial f}{\partial x^i} \frac{\partial x^i}{\partial y^j} dy^j = \frac{\partial f}{\partial y^j} dy^j,$$

which shows that the local definition of df is coordinate-independent. We can actually define the exterior derivative d locally in the same manner for forms of higher degree, and show that our definition is independent of the chart we defined it in.

1.2 Defining d Locally For 1-forms

Some time ago, we asked when a 1-form ω was expressible as df for some 0-form f. A sufficient condition for *local exactness* is that $\frac{\partial \omega_i}{\partial x^j} - \frac{\partial \omega_j}{\partial x^i} = 0$ on every chart. This condition is also expressible in a coordinate independent manner:

$$\frac{\partial \omega_i}{\partial x^j} - \frac{\partial \omega_j}{\partial x^i} = 0 \iff X(\omega(Y)) - Y(\omega(X)) = \omega([X, Y]) \qquad \text{for all } X, Y \in \mathfrak{X}(U).$$

The left side in the above equivalence is antisymmetric in i and j, so we might hope that if we define

$$d\omega := \sum_{i<j} \left(\frac{\partial \omega_i}{\partial x^j} - \frac{\partial \omega_j}{\partial x^i} \right) dx^i \wedge dx^j,$$

then this 2-form would be independent. Note that this is equivalent to saying that $d\omega = d\omega_i \wedge dx^i$. The following lemma shows that this is indeed a coordinate-independent definition.

Lemma 1.1. *Let (U, x^i) and (V, y^i) be coordinate charts on M such that $\omega = a_i dx^i = b_i dy^i$ on $U \cap V$. Then $da_i \wedge dx^i = db_i \wedge dy^i$ on $U \cap V$.*

Proof. On $U \cap V$,

$$
\begin{aligned}
da_i \wedge dx^i &= \frac{\partial a_i}{\partial x^j} dx^j \wedge dx^i \\
&= \frac{\partial}{\partial x^j}\left(\omega\left(\frac{\partial}{\partial x^i}\right)\right) dx^j \wedge dx^i \\
&= \frac{\partial}{\partial x^j}\left(b_k dy^k\left(\frac{\partial}{\partial x^i}\right)\right) dx^j \wedge dx^i \\
&= \frac{\partial}{\partial x^j}\left(b_k \frac{\partial y^k}{\partial x^i}\right) dx^j \wedge dx^i \\
&= \left(\frac{\partial b_k}{\partial x^j}\frac{\partial y^k}{\partial x^i} + b_k \frac{\partial^2 y^k}{\partial x^j \partial x^i}\right) dx^j \wedge dx^i \\
&= \frac{\partial b_k}{\partial x^j}\frac{\partial y^k}{\partial x^i} dx^j \wedge dx^i \\
&= \frac{\partial b_k}{\partial x^j}\frac{\partial y^k}{\partial x^i}\left(\frac{\partial x^j}{\partial y^\ell} dy^\ell\right) \wedge \left(\frac{\partial x^i}{\partial y^m} dy^m\right) \\
&= \left(\frac{\partial b_k}{\partial x^j}\frac{\partial x^j}{\partial y^\ell}\right)\left(\frac{\partial y^k}{\partial x^i}\frac{\partial x^i}{\partial y^m}\right) dy^\ell \wedge dy^m \\
&= \frac{\partial b_k}{\partial y^\ell}\delta^k_m dy^\ell \wedge dy^m \\
&= \frac{\partial b_k}{\partial y^\ell} dy^\ell \wedge dy^k \\
&= db_k \wedge dy^k
\end{aligned}
$$

□

Therefore the exterior derivative of a 1-form is a well-defined 2-form.

1.3 Higher-Degree Exterior Derivatives

Definition 2. *Let $\omega \in \Omega^k(M)$. Define $d\omega$ locally as follows: if (U, x^i) is a coordinate chart, then define $d\omega$ on U by $d\omega \coloneqq d\omega_I \wedge dx_I$. One shows that this definition is coordinate-independent by a calculation similar to (but more tedious than) the previous one, giving rise to a $(k+1)$-form $d\omega \in \Omega^k(M)$.*

Proposition 1.1. *$d : \Omega^k(M) \to \Omega^{k+1}(M)$ is an $\mathbb{R}$-linear map satisfying*

(i) d is an antiderivation of degree 1 on $\Omega^*(M)$.

(ii) $d^2 = 0$.

(iii) As just defined, df agrees with the differential of a 0-form f as defined way before this lecture.

Proof. Exercise! □

1.4 Outlining Uniqueness

It turns out that the properties above actually characterize d.

Theorem 1.2. *There is a unique $\mathbb{R}$-linear map $d : \Omega^k(M) \to \Omega^{k+1}(M)$ satisfying properties (i)-(iii) of the previous proposition.*

Proof. We provide an outline of the proof here. A full proof is given in e.g. Tu or Lee. We have shown existence already. For uniqueness, suppose that $D : \Omega^k(M) \to \Omega^{k+1}(M)$ also satisfies those three properties and is $\mathbb{R}$-linear. We proceed in three main steps.

1. Show that D is a *local operator*: for any $\omega \in \Omega^k(M)$, $(D\omega)_p$ depends only on ω in a neighbourhood of p.

2. Given $\omega \in \Omega^k(M)$ and a chart (U, x^i), write $\omega = a_I dx^I$ on U. Since D is a local operator, $D|_U : \Omega^k(U) \to \Omega^{k+1}(U)$ defined by $D|_U \eta \coloneqq (D\widetilde{\eta})|_U$, where $\widetilde{\eta}$ is an extension of η to M, is well-defined.

3. One then shows that $D\omega = da_I \wedge dx^I = d\omega$ on U, proving uniqueness.

$\square$

Theorem 1.3. *If $F : N \to M$ is smooth, then $F^*(d\omega) = d(F^*\omega)$ for all $\omega \in \Omega^k(M)$.*