

1 Orientation, Manifolds With Boundary (August 7)

We will develop the notions of orientation for manifolds, and develop manifolds with boundary. These will allow us a more general theory of integration on manifolds. One develops orientation in order to make sense of integration on manifolds, and one develops manifolds with boundary in order to have a suitable theory of integration on manifolds. One may also develop the notion of a manifold-with-corners, but we shall not do so.

1.1 Orientation on Vector Spaces

Let us take care of notation, first. A basis for a vector space without any ordering is either written as v_1, \dots, v_n or $\{v_1, \dots, v_n\}$. An ordered basis is written as (v_1, \dots, v_n) .

As we did with forms, we will develop things on a vector space first, and then generalize them pointwise to manifolds by doing things on each tangent space.

On \mathbb{R} , we have either the "left" orientation or the "right" orientation. If we take the "right" orientation, then

$$\int_{[a,b]} f = \int_a^b f(x) dx,$$

and if we take the "left" orientation, then

$$\int_{[a,b]} f = \int_b^a f(x) dx.$$

More precisely, we have positive and negative orientations on \mathbb{R} .

On \mathbb{R}^2 , we have the "clockwise" and "counterclockwise" orientations. The latter is the "usual orientation" of \mathbb{R}^2 . Similarly, on \mathbb{R}^3 , we have orientations specified by clockwise and counterclockwise twirls around the z -axis. These orientations can be thought of as picking the standard basis vectors in a certain order.

Let us make these notions precise. Let V be an n -dimensional vector space. We no longer have a standard basis to work with, so we will develop a notion of orientation by using multiple bases; the orientation on a vector space will be a set of bases, each related to another by an orientation-preserving (i.e. of positive determinant) change of basis matrix.

Definition 1. Let $\alpha = (v_1, \dots, v_n)$ and $\beta = (w_1, \dots, w_n)$ be ordered bases of V . We say that α and β specify the same orientation if the change of basis matrix from α to β has positive determinant. This is obviously an equivalence relation, so it partitions the set of ordered bases of V into two equivalence classes. Each class is called an orientation on V .

Let α, β be as in the previous definition, and let Q denote the change of basis matrix from α to β . Then we have $v_i = Q_i^j w_j$ (Einstein notation), so if $\gamma \in \bigwedge^n(V^*)$,

$$\gamma(v_1, \dots, v_n) = \det(Q) \gamma(w_1, \dots, w_n).$$

Thus $\gamma(v_1, \dots, v_n)$ and $\gamma(w_1, \dots, w_n)$ have the same sign if and only if (v_1, \dots, v_n) and (w_1, \dots, w_n) specify the same orientation. We say that the n -covector γ *specifies the orientation* $[(v_1, \dots, v_n)]$ if $\gamma(v_1, \dots, v_n) > 0$. As we just saw, this is a well-defined notion. Thus an n -covector specifies an orientation on V . Since $\bigwedge^n(V^*)$ is one-dimensional, two n -covectors γ and γ' specify the same orientation if and only if there is a positive $a \in \mathbb{R}$ with $\gamma = a\gamma'$. Taking this to be an equivalence relation on the set of non-zero n -covectors on V , we see that an orientation of V is also given by an equivalence class of n -covectors.

Note that we can also think of an orientation as a choice of component of $\bigwedge^n(V^*)$.

1.2 Orientation on Manifolds

We would like to make a "smooth choice" of orientation on each tangent space to M . We will call a choice of orientation at each tangent space of M a *pointwise orientation*. We do not want to haphazardly choose orientations of each tangent space, since then a manifold would have too many ($2^{|M|}$, to be precise) orientations.

The most straightforward way to develop the notion of a smooth choice of orientations is to consider a simple case, and then generalize. Consider an embedded 1-dimensional submanifold S of \mathbb{R}^n (a curve). If we have a non-zero vector field X on S , then each X_p , for $p \in S$, is a (n ordered) basis of $T_p S$, so it determines an orientation of $T_p S$. Suppose that we have chosen a pointwise orientation on S . If, for each $p \in S$, the non-zero tangent vector X_p determines the orientation we chose on $T_p M$, then it would be reasonable to call our pointwise orientation smooth if the vector field X is smooth. This turns out to give us the first notion of orientation of a manifold (note that this did not depend on the fact that S was in \mathbb{R}^n : that simply makes it easier to visualize this scenario).

Definition 2. *An orientation on a manifold M is a pointwise orientation on M such that for all $p \in M$, there is a neighbourhood U of p and a smooth local frame $X_1, \dots, X_n \in \mathfrak{X}(U)$ such that for each $q \in U$, the orientation specified by (X_{1q}, \dots, X_{nq}) on $T_q M$ is consistent with the choice of orientation on $T_q M$.*

Equivalently, an orientation on M is a pointwise orientation on M such that for all $p \in M$, there is a chart (U, x^1, \dots, x^n) near p such that for each $q \in U$, the orientation specified by $\left(\frac{\partial}{\partial x^1}\big|_q, \dots, \frac{\partial}{\partial x^n}\big|_q\right)$ on $T_q M$ is consistent with the choice of orientation on $T_q M$. The proof of this fact will be a homework problem.

This gives rise to the notion of an oriented atlas. An *oriented atlas* on M is an atlas with the property that $\det(D(\psi \circ \phi^{-1})) > 0$ for all transition maps $\psi \circ \phi^{-1}$. So equivalently, an orientation on M is a pointwise orientation admitting an oriented atlas.

Equivalently, an orientation on M is a pointwise orientation on M such that for all $p \in M$, there is a chart (U, x^1, \dots, x^n) near p such that for each $q \in U$, the orientation specified by $(dx^1 \wedge \dots \wedge dx^n)_q$ on $T_q M$ is consistent with the choice of orientation on $T_q M$.

Definition 3. *A manifold M is said to be orientable if it admits an orientation. An oriented manifold is an orientable manifold together with a choice of orientation.*

Proposition 1.1. *An orientable manifold M admits precisely 2^c orientations, where c is the number of components of M .*

Proof. It suffices to show that a connected manifold M admits precisely two orientations. This is a standard topological argument: construct a locally constant function and argue by connectedness that the function must be constant on the entire topological space. A full proof is given in the book. \square

In particular, a connected orientable manifold admits precisely two orientations.

Let ω be an n -form on M . If $\omega_p \neq 0$, then ω_p , being a non-zero n -covector on $T_p M$, determines an orientation on $T_p M$. Thus a non-vanishing n -form ω on M determines (uniquely) a pointwise orientation on M . One would expect that a smooth non-vanishing n -form on a manifold determines an orientation of that manifold. This is, in fact, true, and we provide a proof.

Theorem 1.1. *A manifold M is orientable if and only if it admits a non-vanishing smooth top-degree form.*

Proof. Suppose that ω is a non-vanishing smooth top degree form on M . For each $p \in M$, give $T_p M$ the orientation specified by $\omega_p \in \bigwedge^n(T_p^* M)$. This gives a pointwise orientation on M . Given $p \in M$, let $(U, \phi) = (U, x^1, \dots, x^n)$ be a connected coordinate chart at p . By connectedness, we have that $\omega\left(\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^n}\right)$ is either strictly positive or strictly negative on U . Assume without loss of generality that it is a strictly positive function on U . Then, for each $q \in U$, $\omega\left(\frac{\partial}{\partial x^1}|_q, \dots, \frac{\partial}{\partial x^n}|_q\right) > 0$, meaning that the ordered basis $\left(\frac{\partial}{\partial x^1}|_q, \dots, \frac{\partial}{\partial x^n}|_q\right)$ of $T_q M$ specifies the same orientation as ω_q did. Therefore M is orientable.

Conversely, suppose that M is orientable. Given $p \in M$, we may find a coordinate chart (U, x^1, \dots, x^n) at p such that for each $q \in U$, the orientation on $T_q M$ coincides with the orientation specified by the n -covector $(dx^1 \wedge \dots \wedge dx^n)_q$. We therefore have a smooth non-vanishing top-degree form defined on an open set of M which gives the orientation there. This is vulnerable to a standard partition of unity argument. Let $\{(U_\alpha, \phi_\alpha)\}$ be an oriented atlas for the oriented manifold M . Let $\{\rho_\alpha\}$ be a partition of unity subordinate to this atlas. Define

$$\omega := \sum_{\alpha} \rho_{\alpha} dx_{\alpha}^1 \wedge \dots \wedge dx_{\alpha}^n.$$

It is easy to check that ω is the desired non-vanishing smooth top-degree form, and moreover, that the orientation ω specifies on M is the same as the orientation we gave M . \square

The following corollary of this characterization of orientability provides a wealth of oriented manifolds for us to work with.

Corollary 1.1.1. *Any regular hypersurface in \mathbb{R}^n is orientable.*

Proof. Exercise. (Author will think on it and fill it in, maybe. See exercise 19.11 in Tu.) \square

As a corollary, S^{n-1} is orientable (as one would hope), and the Mobius band, being non-orientable, is not a regular hypersurface in \mathbb{R}^3 .

Given an n -manifold M , define an equivalence relation on the set of smooth non-vanishing k -forms as follows: $\omega \sim \omega'$ if and only if $\omega = f\omega'$ for some strictly positive continuous function $f : M \rightarrow \mathbb{R}$. This partitions the non-vanishing smooth top degree forms on M into two equivalence classes, each specifying an orientation on M . We have the following correspondences:

$$\begin{aligned} \text{orientations} &\iff \text{equivalence classes of non-vanishing smooth } k\text{-forms on } M \\ &\iff \text{equivalence classes of oriented atlases on } M \end{aligned}$$

1.3 Manifolds With Boundary

As a manifold is locally modelled by \mathbb{R}^n , a manifold with boundary is locally modelled by the *half-space* $\mathbb{H}^n = \{(x^1, \dots, x^n) \in \mathbb{R}^n : x^n \geq 0\}$ with the subspace topology. Many notions from our original definition of a manifold carry over almost word-for-word, but there is a particularly important notion of interior and boundary for a manifold with boundary.

Definition 4. A point $x \in \mathbb{H}^n$ is said to be an *interior point* if $x^n > 0$, and is said to be a *boundary point* if $x^n = 0$. The set of interior points is denoted by $(\mathbb{H}^n)^o$, and the set of boundary points is denoted by $\partial\mathbb{H}^n$.

The set of interior points and boundary points, as just defined, coincide with the topological interior and boundary of \mathbb{H}^n , so there is no harm in simply referring to "interior points" and "boundary points" when referring to the "prototype manifold with boundary" \mathbb{H}^n . Once we develop the more general manifold with boundary, this will not necessarily be the case. It is important to note that any open neighbourhood of a boundary point of \mathbb{H}^n will not be an open set in \mathbb{R}^n .

To establish clear notation, we will denote by $\text{Int}(S)$ the topological interior of a set and $\text{Bd}(S)$ the topological boundary of a set.

We say that a topological space M is *locally- \mathbb{H}^n* if every point has an open neighbourhood homeomorphic to an open subset of \mathbb{H}^n . With this, we define manifolds with boundary in the continuous category. We will then generalize to smooth manifolds with boundary.

Definition 5. A *topological n -manifold with boundary* is a second-countable Hausdorff locally- \mathbb{H}^n space. These homeomorphisms are called (coordinate) *charts*, as one would expect.

The standard terminology which applies to coordinates on a smooth manifold, as defined before, also applies to the coordinates on a manifold with boundary, smooth or not.

Definition 6. A collection $\{(U_\alpha, \phi_\alpha)\}$ of charts on the topological manifold with boundary M is said to be a *smooth atlas* on M if it covers M and if the transition maps (same notion as before) are C^∞ functions on open subsets of \mathbb{H}^n . (Here we mean C^∞ in the extended sense.) Maximal atlases are defined as before.

A *smooth manifold with boundary* is a topological manifold with boundary equipped with a maximal smooth atlas.

Now we must define the notion of the interior point and the boundary point for a smooth manifold with boundary. We will define them in a seemingly coordinate-dependent way, but it will turn out that our definition is actually coordinate-independent.

Definition 7. *A point p in the manifold with boundary M is said to be an interior point of M if there is a chart (U, ϕ) at p such that $\phi(p) \in (\mathbb{H}^n)^o$, and is said to be a boundary point if $\phi(p) \in \partial\mathbb{H}^n$. These notions are well-defined (coordinate-independent) by the following theorem and its corollary:*

Theorem 1.2. *(Smooth invariance of domain) Let $U \subseteq \mathbb{R}^n$ be open and $S \subseteq \mathbb{R}^n$ be arbitrary. Then S is open if there is a diffeomorphism $U \rightarrow S$.*

Corollary 1.2.1. *Let $U, V \subseteq \mathbb{H}^n$ be open and $f : U \rightarrow V$ a diffeomorphism. Then f maps interior points to interior points and boundary points to boundary points.*

Thus we denote by ∂M the boundary points of a manifold with boundary M . (We will not really use the notion of an interior point.)

Proposition 1.2. *If M is a smooth n -manifold with boundary, then ∂M is an embedded codimension 1 submanifold of M with empty boundary.*

Proof. If (U, x^1, \dots, x^n) is a coordinate chart on M with $U \cap \partial M \neq \emptyset$, then $U \cap \partial M = \{x^n = 0\}$. \square

Corollary 1.2.2. $\partial^2 = 0$. (Compare with $d^2 = 0$ for forms.)

We will eventually see that if M is an oriented manifold with boundary, then ∂M has a natural orientation induced by that of M . This will be crucial in developing Stokes' theorem $\int_M d\omega = \int_{\partial M} \omega$.