

1 Bump Functions and Partitions of Unity (Additional Reading)

1.1 Bump Functions

Define $f : \mathbb{R} \rightarrow \mathbb{R}$ by

$$f(t) = \begin{cases} e^{-1/t} & t > 0 \\ 0 & t \leq 0 \end{cases}.$$

This will be the basic function off of which our bump functions will be modelled. The construction of bump functions on manifolds proceeds in four steps.

Lemma 1.1. *The function f defined above is C^∞ .*

Proof. We claim that for $t > 0$ and $k \geq 0$, there is a polynomial p_{2k} of degree $2k$ such that $f^{(k)}(t) = p_{2k}(1/t)e^{-1/t}$. For $k = 0$ this is obvious, so suppose that this holds true for some $k \geq 0$. Then we have

$$\begin{aligned} f^{(k+1)}(t) &= \frac{d}{dt} p_{2k} \left(\frac{1}{t} \right) e^{-1/t} \\ &= -\frac{1}{t^2} p'_{2k} \left(\frac{1}{t} \right) e^{-1/t} + \frac{1}{t^2} p_{2k} \left(\frac{1}{t} \right) e^{-1/t} \\ &= \underbrace{\left[-\frac{1}{t^2} p'_{2k} \left(\frac{1}{t} \right) + \frac{1}{t^2} p_{2k} \left(\frac{1}{t} \right) \right]}_{p_{2(k+1)}(1/t)} e^{-1/t}, \end{aligned}$$

so by induction the claim holds true.

(Finish this. Since f is C^∞ on $\mathbb{R} \setminus 0$, all we need to do is show that each $f^{(k)}(0)$ makes sense and is equal to 0, by induction.) \square

Lemma 1.2. *Given real numbers $r_1 < r_2$, there is a C^∞ function $h : \mathbb{R} \rightarrow \mathbb{R}$ such that $h^{-1}(1) = (-\infty, r_1]$, $h^{-1}(0) = [r_2, \infty)$, and $0 < h(t) < 1$ for $t \in (r_1, r_2)$.*

Proof. Taking f as defined above, define

$$h(t) = \frac{f(r_2 - t)}{f(r_2 - t) + f(t - r_1)}.$$

This function is well-defined because $f(r_2 - t) + f(t - r_1) = 0$ if and only if each is zero, which is true if and only if $t \geq r_2$ and $t \leq r_1$, which is clearly impossible. Then h is C^∞ . It is clear that $0 < h(t) < 1$ for $t \in (r_1, r_2)$, so we are left with checking the other two conditions.

$h(t) = 0$ if and only if $f(r_2 - t) = 0$, which holds if and only if $r_2 - t \leq 0$, which holds if and only if $t \geq r_2$. So $h^{-1}(0) = [r_2, \infty)$.

$h(t) = 1$ if and only if $f(t - r_1) = 0$, which holds if and only if $t - r_1 \leq 0$, which holds if and only if $t \leq r_1$. So $h^{-1}(1) = (-\infty, r_1]$. \square

Lemma 1.3. *Given real numbers $0 < r_1 < r_2$, there is a C^∞ function $H : \mathbb{R}^n \rightarrow \mathbb{R}$ such that $H^{-1}(1) = \overline{B_{r_1}(0)}$, $H^{-1}(0) = \mathbb{R}^n \setminus B_{r_2}(0)$, and $0 < H(x) < 1$ for $x \in B_{r_2}(0) \setminus \overline{B_{r_1}(0)}$.*

Proof. With h as in the previous lemma, define $H(x) = h(\|x\|)$. The function H is C^∞ because when $\|x\| < r_1$, it is identically 1, and it is a composition of C^∞ maps away from the origin. The rest of the lemma is clear. \square

Using this, we now work in a coordinate chart to get a bump function on a manifold.

Theorem 1.4. *(Existence of bump functions) Given a smooth manifold M , a point $q \in M$, and a neighbourhood U of q , there exists a $\rho \in C^\infty(M)$ such that $\rho|_V \equiv 1$ on some neighbourhood $V \subseteq U$ of q , and $\text{supp}(\rho) \subseteq U$.*

Proof. Choose a coordinate chart (W, ϕ) at q such that $\phi(q) = 0$. The set $\phi(W \cap U)$ is an open neighbourhood of the origin, so we can find $0 < r_1 < r_2$ such that

$$0 = \phi(q) \in B_{r_1}(0) \subset B_{r_2}(0) \subset \phi(W \cap U).$$

This implies that

$$q \in \phi^{-1}(B_{r_1}(0)) \subset \phi^{-1}(B_{r_2}(0)) \subset W \cap U \subseteq U.$$

Let H be as in the previous lemma. Define

$$\rho(x) = \begin{cases} (H \circ \phi)(x) & x \in U \cap W \\ 0 & x \notin U \cap W \end{cases}.$$

The function ρ is C^∞ on $U \cap W$ because it is a composition of C^∞ functions on an open set. If $x \notin U \cap W$, then $x \notin \phi^{-1}(\overline{B_{r_2}(0)})$, so we can find a neighbourhood of x on which ρ is identically zero. (Finish this.) \square