1 k-forms (July 24)

1.1 Multilinear Algebra

Definition 1. *An (ℓ, k)-tensor on a real vector space V is a multilinear map*

$$T : \underbrace{V^* \times \cdots \times V^*}_{\ell \text{ times}} \times \underbrace{V \times \cdots \times V}_{k \text{ times}} \to \mathbb{R}.$$

In this course we will mainly be concerned with $(0, k)$-tensors, and we'll mainly refer to them as k-tensors. Why are these important? We have some reasons:

1. The set of tensors have a rich algebraic structure. (They will form a "graded algebra.")
2. They give us the objects we can integrate over. (It turns that the multilinear algebraic properties of forms allow us to define their integrals in a coordinate independent way.)
3. They provide the framework needed to generalize vector calculus to manifolds.

Definition 2. *A k-tensor (hereafter this refers to a $(0, k)$-tensor, as defined above) is alternating if*

$$f(v_1, \ldots, v_i, \ldots, v_j, \ldots, v_k) = -f(v_1, \ldots, v_j, \ldots, v_i, \ldots, v_k)$$

for all $v_1, \ldots, v_k \in V$.

We have multiple characterizations of algebraic tensors that will make working with them easier.

Proposition 1.1. *Let f be a k-tensor on V. The following are equivalent:*

1. *f is alternating.*
2. *$f(v_1, \ldots, v_k) = 0$ whenever $v_i = v_k$ for some $i \neq j$.*
3. *$f(v_1, \ldots, v_k) = 0$ whenever $\{v_1, \ldots, v_k\}$ is linearly independent.*
4. *For all $\sigma \in S_k$, $\sigma f = \operatorname{sgn}(\sigma) f$, where σf is defined as the k-tensor $(\sigma f)(v_1, \ldots, v_k) = f(v_{\sigma(1)}, \ldots, v_{\sigma(k)})$.*

Let us introduce some notation for the spaces of different kinds of tensors.

- $T_k(V)$ for the vector space of k-tensors.
- $A_k(V)$ for the vector space of alternating k-tensors.
- $S_k(V)$ for the vector space of symmetric k-tensors; those k-tensors f satisfying $\sigma f = f$ for any $\sigma \in S_k$.

Now we define projection operators:

$$\begin{aligned} \mathrm{Sym}: & T_k(V) \to S_k(V) \\ & f \mapsto \sum_{\sigma \in S_k} \sigma f \end{aligned}$$

and

$$\begin{aligned} \mathrm{Alt}: & T_k(V) \to A_k(V) \\ & f \mapsto \frac{1}{k!} \sum_{\sigma} (\mathrm{sgn}(\sigma)) \sigma f. \end{aligned}$$

The reason for the mysterious $1/k!$ in the definition of the operator Alt is a technical one: it makes a lot of results come out nicer. In particular,

- f is symmetric if and only if $f = \mathrm{Sym}(f)$,
- f is alternating if and only if $f = \mathrm{Alt}(f)$.

Definition 3. *For $f \in T_k(V)$ and $g \in T_\ell(V)$, define the tensor product $f \otimes g \in T_{k+\ell}(V)$ by $(f \otimes g)(v_1, \ldots, v_k, v_{k+1}, \ldots, v_{k+\ell}) := f(v_1, \ldots, v_k) g(v_{k+1}, \ldots, v_{k+\ell})$.*

We want a product operation of the form $A_k(V) \times A_\ell(V) \to A_{k+\ell(V)}$. The tensor product does not satisfy this property, unfortunately. Our projection operators will help us define it, however.

Definition 4. *For $f \in A_k(V)$ and $g \in A_\ell(V)$, define the wedge product $f \wedge g \in A_{k+\ell}(V)$ by*

$$f \wedge g = \frac{(k+\ell)!}{k!\ell!} \mathrm{Alt}(f \otimes g).$$

The mysterious scalar multiple is, again, there for technical reasons. We also have

$$f \wedge g = \frac{1}{k!\ell!} \sum_{\sigma \in S_{k+\ell}} \mathrm{sgn}(\sigma) \sigma (f \otimes g).$$

Here are some properties of the wedge product $\wedge : A_k(V) \times A_\ell(V) \to A_{k+\ell}(V)$:

1. Bilinearity.
2. Associativity.
3. Anticommutatitivty: $f \wedge g = (-1)^{k\ell} g \wedge f$. (This is the reason we always sum over increasing indices!)

4. Fix a basis $e_1, \ldots, e_n$ of V and let $\alpha^1, \ldots, \alpha^n$ be the dual basis for $V^* = A_1(V)$. For any increasing multi-index $I \subseteq \{1, \ldots, n\}$ of length k, define α^I as the unique element of $A_k(V)$ sending $e_J = (e_{j_1}, \ldots, e_{j_k})$ to δ^I_J, where J is another increasing multi-index of length k from $\{1, \ldots, n\}$.

   Then
   $$\{\alpha^I : I \text{ an increasing multi-index of length } k \text{ from } \{1, \ldots, n\}\}$$
   forms a basis of $A_k(V)$. In particular, $\dim(A_k(V)) = \binom{n}{k}$. Also, $a^I = a^{i_1} \wedge \cdots \wedge a^{i_k}$.

5. For any $\omega^1, \ldots, \omega^k \in V^*$ and $v_1, \ldots, v_k \in V$, $\omega^1 \wedge \cdots \wedge \omega^k(v_1, \ldots, v_k) = \det(\omega^i(v_j))$.

6. The wedge product is actually characterized by the above properties.

Because of these properties, we will hereafter denote by $\bigwedge^k(V)$ the space of alternating k-tensors on a vector space V.

Definition 5. *An $\mathbb{R}$-algebra A is said to be graded if $A = \bigoplus_{k=0}^{\infty} A^k$, where each A^k is an $\mathbb{R}$-vector space, such that the multiplication $A^k \times A^\ell$ maps into $A^{k+\ell}$. A graded algebra A is said to be anticommutative if $ab = (-1)^{k\ell} ba$ for $a \in A^k$ and $b \in A^\ell$.*

Define $\bigwedge(V^*) \coloneqq \bigoplus_{k=0}^{\infty} \bigwedge^k(V^*) = \bigoplus_{k=0}^{n} \bigwedge^k(V^*)$. The properties of the wedge product make $\bigwedge(V^*)$ an associative anticommutatitve graded algebra over $\mathbb{R}$ of dimension $\sum_{k=0}^{n} \binom{n}{k} = 2^n$.

1.2 k-forms On Manifolds

We developed a notion of smoothness for 1-forms on manifolds. We defined a 1-form ω on M to be smooth if it was smooth as a section of the cotangent bundle. We will follow a similar approach by giving the union of all of the spaces $A_k(T_pM)$, over $p \in M$, a smooth structure, which will allow us to talk about a smooth k-form. (Along the way, our notation will change a little.)

Let $(U, x^1, \ldots, x^n)$ be a coordinate chart on M containing p. Then we have a basis $\{ \frac{\partial}{\partial x^1}\big|_p, \ldots, \frac{\partial}{\partial x^n}\big|_p\}$ of T_pM with the dual basis $\{dx^1_p, \ldots, dx^n_p\}$. Therefore
$$\{dx^{i_1}_p \wedge \cdots \wedge dx^{i_k}_p : 1 \leq i_1 < \cdots < i_k \leq n\}$$
is a basis of $\bigwedge^k(T^*_pM)$.

Define $\bigwedge^k(T^*M) \coloneqq \bigcup_{p \in M} \bigwedge^k(T^*_pM)$. We call $\bigwedge^k(T^*M)$ the *bundle of alternating k-tensors.* This comes with a projection map
$$\pi : \bigwedge^k(T^*M) \to M$$
$$\omega \mapsto p \qquad \text{whenever } \omega \in \bigwedge^k(T^*_pM)$$

We can equip $\bigwedge^k(T^*M)$ with a topology and smooth structure making it into a rank $\binom{n}{k}$ vector bundle. In fact, there is a unique topology and smooth structure for which this is the case. The construction is very similar to that for TM and for T^*M. The idea is that for a chart (U, ϕ), define $\widetilde{\phi} : \bigwedge^k(T^*U) \to \phi(U) \times \mathbb{R}^n$ by

$$\widetilde{\phi} : \omega \mapsto (\phi(p), \{c_I\}_I) \qquad \text{whenever } \omega = \sum_I c_I dx^I \in \bigwedge^k(T_p^*M).$$

(The sum is over increasing multi-indices I.) A detailed proof that $\bigwedge^k(T^*M)$ is a smooth rank-$\binom{n}{k}$ vector bundle is left as an exercise. We will sometimes call $\bigwedge^k(T^*M)$ the *kth exterior power of the cotangent bundle.*

With a smooth structure on $\bigwedge^k(T^*M)$, we can talk begin to talk about smooth forms of higher degree.

Definition 6. *A (differential) k-form on M is a section of the kth exterior power of the cotangent bundle $\pi : \bigwedge^k(T^*M) \to M$.*

For example, if $(U, x^1, \dots, x^n)$ is a chart on M, we can define $dx^I : U \to \bigwedge^k(T^*M)$ by $d^I = dx^{i_1} \wedge \cdots \wedge dx^{i_k}$, where the wedge product of two forms is defined pointwise. Thus dx^I is a k-form on U.

Just as 1-forms act on vector fields, k-forms act on k-tuples of vector fields. Let ω be a k-form on M. For $X_1, \dots, X_k \in \mathfrak{X}(M)$, define $\omega(X_1, \dots, X_k) : M \to \mathbb{R}$ pointwise: $\omega(X_1, \dots, X_k)(p) := \omega_p(X_{1p}, \dots, X_{kp})$. We note the following important property: if $h : M \to \mathbb{R}$ is a function, then

$$\omega(X_1, \dots, hX_i, \dots, X_k) = h\omega(X_1, \dots, X_k).$$

We now begin to discuss smooth k-forms. The definition is exactly what one would expect.

Definition 7. *A k-form ω on M is smooth if it is smooth as a section of $\bigwedge^k(T^*M)$. The set of all smooth k-forms on M is denoted $\Omega^k(M)$. We have $\Omega^k(M) = \Gamma(\bigwedge^k(T^*M))$, using vector bundle notation. We also define $\Omega^0(M) = C^\infty(M)$.*

The space $\Omega^k(M)$ is an $\mathbb{R}$-vector space and a $C^\infty(M)$-module, as we should expect by now. We have some equivalent conditions for smoothness of a k-form. The proofs are left as easy exercises.

Proposition 1.2. *Let ω be a k-form on M. The following are equivalent:*

1. *ω is smooth as a section of $\bigwedge^k(T^*M)$.*
2. *For any chart $(U, x^1, \dots, x^n)$, $\omega = \sum_I c_I dx^I$ for some $c_I \in C^\infty(U)$, where the sum is over all increasing multi-indices I.*
3. *By its action on vector fields, $\omega : \mathfrak{X}(M) \times \cdots \times \mathfrak{X}(M) \to C^\infty(M)$, and is $C^\infty(M)$-multilinear.*

The next proposition is a higher degree form of a surprising result that we saw for 1-forms. The proof is identical.

Proposition 1.3. *Every $C^\infty(M)$-multilinear map $\omega : \mathfrak{X}(M) \times \cdots \times \mathfrak{X}(M) \to C^\infty(M)$ is a k-form.*

Let's see some examples. Let $f^1, \ldots, f^k \in C^\infty(M)$. Then we have $df^1, \ldots, df^k \in \Omega^1(M)$. If $(U, x^1, \ldots, x^n)$ is a chart, then $df^1 \wedge \cdots \wedge df^k = \sum c_I dx^I$, where the sum is over increasing multi-indices I. If $p \in M$, then evaluating at p gives

$$df^1_p \wedge \cdots \wedge df^k_p = \sum c_I(p) dx^{i_1}_p \wedge \cdots \wedge dx^{i_k}_p.$$

Evaluation at $\frac{\partial}{\partial x^I}\big|_p$ (which means exactly what you think it means) gives

$$c_I(p) = df^1_p \wedge \cdots \wedge dx^k_p \left(\left.\frac{\partial}{\partial x^I}\right|_p \right) = \det \left(df^i_p \left(\left.\frac{\partial}{\partial x^{i_j}}\right|_p \right) \right) = \frac{\partial(f^1, \ldots, f^k)}{\partial(x^{i_1}, \ldots, x^{i_k})}(p),$$

which is the determinant of the Jacobian evaluated at p. Therefore

$$df^1 \wedge \cdots \wedge df^k = \sum_{1 \le i_1 < \cdots < i_k \le n} \frac{\partial(f^1, \ldots, f^k)}{\partial(x^{i_1}, \ldots, x^{i_k})} dx^{i_1} \wedge \cdots \wedge dx^{i_k}.$$

So $df^1 \wedge \cdots \wedge df^k \in \Omega^k(M)$. This leads us to ask the question: is it true in general that wedges of smooth forms on M are also smooth forms on M? The answer is yes.

Suppose $\omega \in \Omega^k(M)$ and $\eta \in \Omega^\ell(M)$. In local coordinates, we have

$$\omega \wedge \eta = \left(\sum_I c_I dx^I \right) \wedge \left(\sum_J b_J dx^J \right) = \sum_{I,J} c_I b_J dx^{IJ} \in \Omega^{k+\ell}(M),$$

where IJ is the multi-index $\{i_1, \ldots, i_k, j_1, \ldots, j_\ell\}$, and all sums are over increasing multi-indices. Therefore the wedge product gives us a map $\wedge : \Omega^k(M) \times \Omega^\ell(M) \to \Omega^{k+\ell}(M)$. (We are not being too careful here, but it doesn't really matter in the end.)

We extend the wedge product to 0-forms in the obvious way: since $\Omega^0(M) = C^0(M)$, $f \wedge \omega = f\omega$ for $f \in \Omega^0(M)$ and $\omega \in \Omega^k(M)$.

1.3 Pullbacks of k-forms

Let $F : N \to M$ be a smooth map. We define

$$\begin{aligned} F^{*,p} : \bigwedge^k (T^*_{F(p)} M) &\to \bigwedge^k (T^*_p N) \\ \theta &\mapsto F^{*,p}(\theta) := \theta \circ (F_{*,p}, \ldots, F_{*,p}). \end{aligned}$$

That is, if $\theta \in \bigwedge^k(T^*_{F(p)}M)$ and $v_1, \ldots, v_k \in T_{F(p)}M$, then $F^{*,p}(\theta)(v_1, \ldots, v_k) = \theta(F_{*,p}(v_1), \ldots, F_{*,p}(v_k))$.

With this, we define the pullback of a k-form as follows: if ω is a k-form on M, define $F^*\omega$ on N by

$$(F^*\omega)_p := F^{*,p}\omega_{F(p)} = \omega_{F(p)} \circ (F_{*,p}, \ldots, F_{*,p}).$$

The pullback has the following properties:

Proposition 1.4. *1. $F^*(a\omega + \eta) = aF^*\omega + F^*\eta$.*

2. For any k-form ω and ℓ-form η, $F^*(\omega \wedge \eta) = F^*\omega \wedge F^*\eta$. (The pullback distributes over the wedge product.)

3. $F^* : \Omega^k(M) \to \Omega^k(N)$.

Proof. We will prove only (3). In local coordinates,

$$\begin{aligned} F^*\omega &= F^*(\sum c_I dx^I) \\ &= \sum (c_I \circ F) F^*(dx^{i_1} \wedge \cdots \wedge dx^{i_k}) \\ &= \sum (c_I \circ F) d(x^{i_1} \circ F) \wedge \cdots \wedge d(x^{i_k} \circ F) \end{aligned}$$

the sum, as always, ranging over increasing multi-indices. Since each $d(x^{i_1} \circ F) \wedge \cdots \wedge d(x^{i_k} \circ F)$ is a smooth k-form, $F^*\omega$ must be a smooth k-form. □

1.4 A Remark About Top Degree Forms

Let M, N be smooth manifolds of common dimension n with charts $(V, y^1, \ldots, y^n)$ and $(U, x^1, \ldots, x^n)$, respectively, and let $F : N \to M$ be a smooth map with $F(U) \subseteq V$, for simplicity. Then

$$\Omega^n(V) = \{f dy^1 \wedge \cdots \wedge dy^n : f \in C^\infty(V)\}$$

is a 1-dimensional $C^\infty(V)$-module. On U,

$$F^*(dy^1 \wedge \cdots \wedge dy^n) = dF^1 \wedge \cdots \wedge dF^n = \det\left(\frac{\partial F^i}{\partial x^j}\right) dx^1 \wedge \cdots \wedge dx^n,$$

giving us the very important identity

$$\boxed{F^*(f dy^1 \wedge \cdots \wedge dy^n) = (f \circ F) \det\left(\frac{\partial F^i}{\partial x^j}\right) dx^1 \wedge \cdots \wedge dx^n}$$

that we will (likely) use extensively.

Define

$$\Omega^*(M) := \bigoplus_{k=0}^{\infty} \Omega^k(M) = \bigoplus k = 0^n \Omega^k(M).$$

Equipped with the wedge product, $\Omega^*(M)$ is an associative anticommutative graded algebra over $\mathbb{R}$. This is what was meant in the first section of this lesson by "tensors have a very rich algebraic structure." As we can see, the algebraic structure of differential forms on a manifold is *extremely* rich. In particular, $\Omega^*(M)$ is studied extensively in algebraic topology. (See, for example, de Rham cohomology.)

Next time, we will develop the exterior derivative

$$
\begin{aligned}
d &: \Omega^k(M) \to \Omega^{k+1}(M) \\
d &: \Omega^*(M) \to \Omega^*(M).
\end{aligned}
$$