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1 Review of Basic Complex Analysis (1-11-2021)

1.1 Review of Holomorphic Functions

We recall some basic notions of the theory of complex-valued functions of one complex variable.

Definition 1.1. *$f(z)$ is holomorphic at $z \in \mathbb{C}$ if*

$$\lim_{h \to 0} \frac{f(z+h) - f(z)}{h}$$

exists, i.e. if

$$f(z+h) - f(z) = c \cdot h + \varphi(h) \cdot h,$$

where $\lim_{h \to 0} \varphi(h) = 0$, for some c. We write $c = f'(z)$.

How is this different from the usual derivative? Write $c = a + ib$, and $h = \xi + i\eta$. Then, the derivative is the linear transformation

$$\begin{pmatrix} \xi \\ \eta \end{pmatrix} \mapsto \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \begin{pmatrix} \xi \\ \eta \end{pmatrix}.$$

This is different from the usual derivative of a function $\mathbb{R}^2 \to \mathbb{R}^2$, as the matrix has a special form. The first column is $\partial f/\partial x$, and the second column is $\partial f/\partial y$. Thus,

$$\frac{\partial f}{\partial x} + i \frac{\partial f}{\partial y} = 0.$$

Writing $f = u + iv$, we obtain the equivalent form

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y},$$
$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x},$$

the *Cauchy-Riemann equations*. We see that f is holomorphic at z, if and only if it is differentiable at z as a function of x and y, and it satisfies the Cauchy-Riemann equations.

Suppose, now, that $f(x, y)$ is a differentiable, complex-valued function. One has the *differential*

$$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy.$$

If $z = x + iy$, then $dz = dx + idy$, and $d\bar{z} = dx - idy$. One obtains

$$dx = \frac{1}{2}(dz + d\bar{z}),$$
$$dy = \frac{1}{2i}(dz - d\bar{z}).$$

Thus,

$$df = \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right) dz + \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right) d\bar{z}.$$

Because of this, we define

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right),$$
$$\frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right),$$

so that we may write

$$df = \frac{\partial f}{\partial z}\, dz + \frac{\partial f}{\partial \bar{z}}\, d\bar{z}.$$

The Cauchy-Riemann equations then take on the particularly simple form

$$\frac{\partial f}{\partial \bar{z}} = 0.$$

1.2 Review of Harmonic Functions

Definition 1.2. *We say that a function $f(x, y)$ is harmonic if it is C^2, and satisfies the PDE*

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0,$$

Laplace's equation*. The differential operator*

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

is known as the Laplacian*.*

Laplace's equation is equivalent to

$$\frac{\partial^2 f}{\partial z \partial \bar{z}} = 0,$$

which can be seen by multiplying $\partial f/\partial z$ and $\partial f/\partial \bar{z}$. We note two things:

1. A complex-valued function is harmonic, if and only if its real and imaginary parts are harmonic. (Because Δ is a real differential operator.)

2. A holomorphic function is harmonic; thus, its real and imaginary parts are also harmonic.

A real-valued harmonic function $g(x, y)$ is *locally* the real part of a holomorphic functin f, and f is uniquely determined up to an additive constant. Why? One has

$$\frac{\partial}{\partial \bar{z}}\left(\frac{\partial g}{\partial z}\right) = 0,$$

so $\partial g/\partial z$ is holomorphic, thus it admits local primitives. However, it need not admit a *global* primitive: consider $\log|z|$ on $\mathbb{C} \setminus \{0\}$. This function is not the real part of any holomorphic function, because if it were, that holomorphic function would have to be the complex logarithm, which has no single-valued branch in $\mathbb{C} \setminus \{0\}$.

1.3 Extending the Complex Plane

We sometimes wish to extend the complex plane to include a point at infinity. We say that a function $f(z)$ is *holomorphic at* ∞ if $f(1/z)$ is holomorphic at 0. In order to think about this globally and geometrically, we will introduce the *Riemann sphere*.

Consider the sphere S^2, defined by $x^2 + y^2 + t^2 = 1$. We will think of the complex plane as $\mathbb{R}^2$, defined by $t = 0$. We use stereographic projection from the north pole $N = (0, 0, 1)$ to identify $S^2 \setminus \{N\}$ with the complex plane.

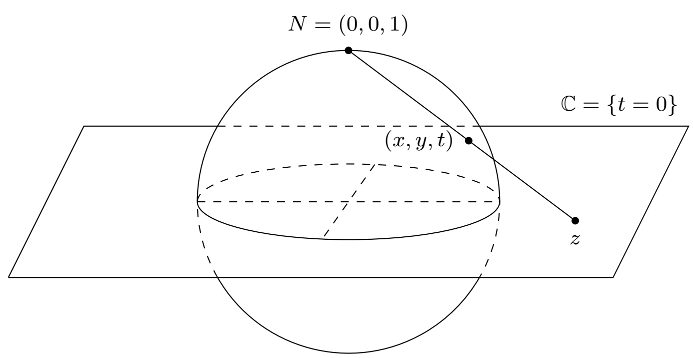

Figure 1: (Diagram found on MSE and edited slightly, not mine.)

The stereographic projection of $(x, y, t) \in S^2 \setminus \{N\}$ from N is the point

$$z = \frac{x + iy}{1 - t}.$$

To see this, one needs only verify that $(0,0,1)$, (x,y,t), and $(\frac{x}{1-t}, \frac{y}{1-t}, 0)$ are colinear.

This stereographic projection is a homeomorphism of $S^2 \setminus \{N\}$ onto $\mathbb{C}$. We then say that this gives a complex structure on the sphere. In order to think of the Riemann sphere as the complex plane with a point at infinity, we will use the complex conjugate of stereographic projection from the south pole. We will work out the details next time.