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1 Meromorphic Functions (1-25-2021)

1.1 Zeroes and Poles

Let Ω be an open subset of $\mathbb{C}$, and let $f(z)$ be holomorphic in Ω. Suppose $f(z_0) = 0$. Then $f(z) = (z - z_0)^k f_1(z)$ for some holomorphic f_1 in Ω with $f_1(z_0) \neq 0$. (To see this, write the Taylor expansion at z_0.) We call k the *order* or *multiplicity* of the zero z_0. It follows that the zeroes of a holomorphic function that doesn't vanish identically are isolated, at least if Ω is connected.

A *meromorphic* function in Ω is a holomorphic function defined in the complement of a discrete set in Ω, which is in a (perhaps punctured) neighbourhood of any point expressible as the quotient of holomorphic functions $f(z)/g(z)$, where $g \not\equiv 0$. The set of meromorphic functions on a *domain* Ω (open, connected) form a field.

At a point z_0, write $f(z) = (z - z_0)^k f_1(z)$ and $g(z) = (z - z_0)^l g_1(z)$, where f_1, g_1 are holomorphic functions with $f_1(z_0), g_1(z_0) \neq 0$. So

$$\frac{f(z)}{g(z)} = (z - z_0)^{k-l} \frac{f_1(z)}{g_1(z)}.$$

If $k \geq l$, then f/g extends to be holomorphic at z_0. If $k < l$, then we say that z_0 is a *pole* of f/g of *order* or *multiplicity* $l - k$. Then, we will say that

$$\lim_{z \to z_0} \frac{f(z)}{g(z)} = \infty,$$

i.e. meromorphic functions take values in the Riemann sphere. Pursuing this, we get a nicer definition of meromorphic functions: a meromorphic function on Ω is a holomorphic function $\Omega \to S^2$.

1.2 Laurent Series, Partial Fraction Decomposition

We're going to see that a holomorphic function $f(z)$ in an annulus $r < |z| < R$ has a *Laurent expansion*:

$$\sum_{n=-\infty}^{\infty} a_n z^n = \underbrace{\sum_{n<0} a_n z^n}_{\text{holom. in } |z| > r} + \underbrace{\sum_{n \geq 0} a_n z^n}_{\text{holom. in } |z| < R}.$$

We can rephrase the condition on the first sum as follows: if $z = 1/\zeta$, then

$$\sum_{n<0} a_n z^n = \sum_{n<0} a_n \zeta^{-n} = \sum_{n=1}^{\infty} a_{-n} \zeta^n$$

is holomorphic in $|\zeta| < 1/r$.

We are going to get this from Cauchy's integral formula. Let γ_1 be a circle of radius $r_1 \in (r, R)$, and γ_2 a circle of radius $r_2 \in (r, R)$, with $r < r_2 < r_1 < R$. If $r_2 < |z| < r_1$, then Cauchy's integral gives

$$f(z) = \frac{1}{2\pi i}\int_{\gamma_1} \frac{f(\zeta)}{\zeta - z}\, d\zeta - \frac{1}{2\pi i}\int_{\gamma_2} \frac{f(\zeta)}{\zeta - z}\, d\zeta,$$

which can be seen by connecting γ_1 and γ_2 by an arbitrarily small line segments avoiding z, making a closed curve. (Draw a picture.) In the second integral,

$$\frac{1}{\zeta - z} = -\frac{1}{z} \cdot \frac{1}{1 - \frac{\zeta}{z}} = -\sum_{n=0}^{\infty} \frac{\zeta^n}{z^{n+1}} = -\sum_{n<0} \frac{z^n}{\zeta^{n+1}}.$$

This power series is uniformly and absolutely convergent on $|\zeta| = r_2$, so we get

$$f(z) = \sum_{n=-\infty}^{\infty} a_n z^n, \qquad a_n = \frac{1}{2\pi i}\int_{\gamma_2} \frac{f(\zeta)}{\zeta^{n+1}}\, d\zeta, \qquad n < 0.$$

The Laurent series is uniformly and absolutely convergent when $r_2 \leq |z| \leq r_1$.

Holomorphic functions on the Riemann sphere with values in $\mathbb{C}$ are constant by Liouville's theorem. What about meromorphic functions?

Theorem 1.1. *Any meromorphic function on S^2 is rational.*

Proof. Let's say $f(z)$ has poles $b_1, \ldots, b_n$, and possibly ∞, with corresponding *principal parts* (negative parts of the Laurent expansion)

$$P_k\left(\frac{1}{z - b_k}\right),$$

polynomials in $1/(z - b_k)$, and possibly

$$P_\infty\left(\frac{1}{\zeta}\right) = P_\infty(z),$$

where $\zeta = 1/z$ is the "coordinate at infinty." Then

$$f(z) - \sum_{k=1}^{n} P_k\left(\frac{1}{z - b_k}\right) - P_\infty(z)$$

is holomorphic on S^2, hence a constant a. Then

$$f(z) = a + P_\infty(z) + \sum_{k=1}^{\infty} P_k\left(\frac{1}{z - b_k}\right),$$

the *partial fraction decomposition* of a rational function. $\square$

If we write this last expression as $\frac{P(z)}{Q(z)}$, $\deg P(z) = p$, $\deg Q(z) = q$, then $a + P_\infty(z)$ is present if and only if $p > q$. (Quotient by long division!) As an exercise, deduce the theorem on real partial fraction decomposition.