MAT454/1002 lecture notes by Kain Dineen. Taught by Edward Bierstone.

0.1 The Riemann Sphere, Integration (1-13-2021)

We will complete the description of the complex structure on the Riemann sphere, introduce the notion of projective space, and then begin to review the theory of complex integration, with an eye towards Cauchy's theorem.

0.1.1 The Complex Structure on the Riemann Sphere

The complex conjugate of stereographic projection from the south pole $S = (0, 0, -1)$ is given by
$$z' = \frac{x - iy}{1 + t}.$$
This provides a homemorphism of $S^2 \setminus \{S\}$ onto $\mathbb{C}$. For any point $(x, y, t) \in S^2$, other than S or N, we have
$$zz' = \frac{x^2 + y^2}{1 - t^2} = 1;$$
in other words, $z' = 1/z$, a holomorphic transformation. Thus, we have covered the Riemann sphere with two coordinate charts, whose transition mapping is holomorphic. This is the sense in which the Riemann sphere obtains a complex structure.

0.1.2 One-Dimensional Complex Projective Space

We write $P^1(\mathbb{C})$ for the one-dimensional complex space, consisting of all of the lines in $\mathbb{C}^2$ through the origin. That is, $P^1(\mathbb{C}) = (\mathbb{C}^2 \setminus \{0\})/\sim$, where $(x_0, x_1) \sim (x_0', x_1')$ means that there exists a non-zero $\lambda \in \mathbb{C}$ such that $(x_0', x_1') = (\lambda x_0, \lambda x_1)$. We will write $[x_0, x_1]$ for the equivalence class of (x_0, x_1), and we call these classes *homogeneous coordinates*.

We may equip $P^1(\mathbb{C})$ with the structure of a complex manifold. Let
$$U_i = \{[x_0, x_1] \in P^1(\mathbb{C}) : x_i \neq 0\}, \qquad i = 0, 1.$$
We define two mappings
$$U_0 \to \mathbb{C}, \qquad [x_0, x_1] \mapsto \frac{x_1}{x_0} = z,$$
and
$$U_1 \to \mathbb{C}, \qquad [x_0, x_1] \mapsto \frac{x_0}{x_1} = z'.$$
Evidently, $zz' = 1$. Thus, $P^1(\mathbb{C})$ is obtained by gluing together two copies of $\mathbb{C}$ along the complements of $\{0\}$ by the formula $z' = 1/z$. Moreover, $P^1(\mathbb{C}) \cong S^2$, the Riemann sphere.

0.1.3 Integrating Forms along Curves

Let Ω be a (connected) open subset of $\mathbb{R}^2$. By a *differential form*, we mean an expression of the form $\omega = P\,dx + Q\,dy$, where P, Q are continuous (real or complex)-valued functions on Ω. Let $\gamma\colon [a,b] \to \Omega$ be a piecewise-C^1 curve in Ω, $\gamma(t) = (x(t), y(t))$. Then, we define

$$\int_\gamma \omega = \int_a^b f(t)\,dt,$$

where $f(t) = P(x(t), y(t))x'(t) + Q(x(t), y(t))y'(t)$. (That is, $f(t)\,dt = \gamma^*\omega$.)

The integral of ω over γ is independent of the curve's parametrization. Consider a reparametrization $t\colon [c,d] \to [a,b]$, with $t(c) = a$, $t(d) = b$, and $t'(s) > 0$, and set $\delta(s) = \gamma(t(s))$. Then

$$\int_\gamma \omega = \int_\delta \omega$$

by integration by substitution.

An important example of a differential form is the differential of a C^1 function F:

$$\omega = dF = \frac{\partial F}{\partial x}\,dx + \frac{\partial F}{\partial y}\,dy.$$

In this case, we call F a *primitive* of ω. Here,

$$\int_\gamma dF = \int_a^b (F \circ \gamma)'(t)\,dt = F(\gamma(b)) - F(\gamma(a)).$$

In particular, if γ is closed, the integral of dF over γ is zero.

Proposition 0.1.1. *ω has a primitive in Ω, if and only if $\int_\gamma \omega = 0$ for every piecewise-C^1 closed curve γ in Ω.*

Proof. We just saw the forward direction. Conversely, fix a point $(x_0, y_0) \in \Omega$. Given ω satisfying the hypotheses, define F by

$$F(x,y) = \int_\gamma \omega,$$

where γ is a piecewise-C^1 curve in Ω starting at (x_0, y_0) and ending at (x, y). This is independent of γ precisely by the hypothesis on γ.

We check that $dF = \omega$. First, let δ be a straight line in Ω from (x, y) to $(x+h, y)$, for sufficiently small h. Then,

$$F(x+h, y) - F(x, y) = \int_\delta \omega = \int_x^{x+h} P(\xi, y)\,d\xi,$$

so

$$\lim_{h \to 0} \frac{F(x+h,y) - F(x,y)}{h} = P(x,y).$$

Thus, $F(x,y)$ is differentiable in x, and $\frac{\partial F}{\partial x} = P$. The proof for the other variable is identical. Therefore $dF = \omega$. $\square$

In the case that Ω is an open disk, we can simplify the statement, and say that ω has a primitive in Ω, if and only if $\int_\gamma \omega = 0$ whenever γ is the boundary of a rectangle in Ω.

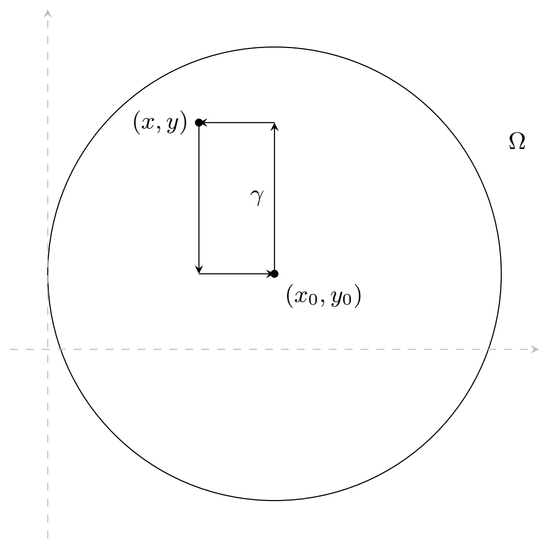

Figure 1: The center of Ω, (x_0, y_0), is the basepoint, as in the previous proof. As Ω is a disk, to any (x, y) in Ω one can find a rectangle, with sides parallel to the axes, as pictured. The proof then proceeds unchanged.