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1 Harmonic Functions (1-22-2021)

1.1 Further Consequences of Cauchy's Integral Formula

Theorem 1.1. *(Maximum modulus principle) Let f be a holomorphic function on an open subset $\Omega \subset \mathbb{C}$ with the mean value property. If $|f|$ has a local maximum at a point $a \in \Omega$, then f is constant in a neighbourhood of a.*

Theorem 1.2. *(Schwarz's lemma) (include the statement)*

1.2 Harmonic Functions

The real and imaginary parts of a function with the mean value property also have the mean value property. We'll see that the functions with the mean value property are precisely the harmonic functions.

Proposition 1.1. *A real-valued harmonic function $g(x,y)$ is locally the real part of a holomorphic function, uniquely determined up to the addition of a constant.*

Proof. We have

$$\frac{\partial^2 g}{\partial z \partial \bar{z}} = 0,$$

so we know $\frac{\partial g}{\partial z}$ is holomorphic. It therefore has a local primitive f uniquely determined up to an additive constant:

$$df = \frac{\partial g}{\partial z}\, dz.$$

We can write this as

$$d\bar{f} = \frac{\partial g}{\partial \bar{z}}\, d\bar{z}.$$

Since g is real-valued, $\frac{\partial g}{\partial \bar{z}}$ is the conjugate of $\frac{\partial g}{\partial z}$. Therefore

$$d(f + \bar{f}) = dg,$$

which proves that $g = 2 \cdot \mathrm{Re}(f) + (\text{real const.})$. □

Therefore harmonic functions satisfy the mean value property, as well as the maximum modulus principle.

Now, let $g(x,y)$ be a real-valued harmonic function, equal to the real part of some holomorphic function $f(z) = \sum a_n z^n$, converging in some open disk of radius R. Consider inside this disk a smaller one of radius $r < R$. We can assume $a_0 \in \mathbb{R}$. On the boundary of the smaller disk,

$$g(r\cos\theta, r\sin\theta) = \mathrm{Re}(f(re^{i\theta})) = a_0 + \frac{1}{2}\sum_{n=1}^{\infty} r^n \left(a_n e^{in\theta} + \bar{a}_n e^{-in\theta}\right),$$

which is uniformly and absolutely convergent with respect to $\theta \in [0, 2\pi]$. Let's compute the constants.

$$a_0 = \frac{1}{2\pi}\int_0^{2\pi} g(r\cos\theta, r\sin\theta)\, d\theta,$$

and

$$a_n = \frac{1}{\pi}\int_0^{2\pi} \frac{g(r\cos\theta, r\sin\theta)}{(re^{i\theta})^n}\, d\theta.$$

We get, for $|z| < r$,

$$f(z) = \frac{1}{2\pi}\int_0^{2\pi} g(r\cos\theta, r\sin\theta) \underbrace{\left\{1 + 2\sum_{n=1}^{\infty}\left(\frac{z}{re^{i\theta}}\right)^n\right\}}_{=\frac{re^{i\theta}+z}{re^{i\theta}-z}} d\theta$$

This expresses the holomorphic function $f(z)$ in $|z| < r$ in terms of its real part on the boundary. Equating real parts of both sides of the above equation,

$$g(x, y) = \frac{1}{2\pi}\int_0^{2\pi} g(r\cos\theta, r\sin\theta) \cdot \frac{r^2 - |z|^2}{|re^{i\theta} - z|^2}\, d\theta.$$

This formula is valid in $|z| < r$, for any real-valued harmonic function in a neighbourhood of $\{|z| \le r\}$. We call the term

$$\frac{r^2 - |z|^2}{|re^{i\theta} - z|^2}$$

the *Poisson kernel*. Thus, a harmonic function is determined by its values on the boundary of the disk.

1.3 Dirichlet Problem for a Disk

We now study the converse problem. Given a continuous function $f(\theta)$ on the circle with center 0 and radius r, can we find $F(z)$ continuous in $|z| \le r$ and harmonic in $|z| < r$, such that $F(re^{i\theta}) = f(\theta)$?

Theorem 1.3. *We can, and the solution is unique.*

Proof. We can assume that the functions are real-valued. Uniqueness follows from the maximum modulus principle. All that's left is getting the solution. For $|z| < r$, define $F(z)$ by

$$F(z) = \frac{1}{2\pi} \int_0^{2\pi} f(\theta) \cdot \frac{r^2 - |z|^2}{|re^{i\theta} - z|^2} \, d\theta.$$

The Poisson kernel in the above expression is just the real part of $(re^{i\theta}+z)/(re^{i\theta}-z)$, so $F(z)$ is the real part of a holomorphic fuction (you can differentiate under the integral). And we already saw that real parts of holomorphic functions are harmonic. Now, we need only check that

$$\lim_{z \to re^{i\theta_0}} F(z) = f(\theta_0);$$

this is a direct calculation and we will not do it in class. □

Corollary 1.1. *Any continuous function in an open subset $\Omega \subseteq \mathbb{R}^2$ with the mean value property is harmonic.*

Proof. Pick $a \in \Omega$ and $r > 0$ so that $D = \{|z - a| \leq r\} \subset \Omega$. Then $f|_{\partial D}$ extends to a harmonic function in the interior. Then $f - F$ is zero on ∂D. It satisfies the maximum modulus principle, so it's zero. □