MAT454/1002 lecture notes by Kain Dineen. Taught by Edward Bierstone.

0.1 Prescribing Poles and Zeroes (2-22-2021)

We continue the general theme of constructing functions with prescribed poles and zeroes. We've already solved this in the case of a lattice Γ in $\mathbb{C}$ using the Weierstrass $\wp$-function. We now turn to the more general problem.

0.1.1 Prescribed Poles

It'll be easier to take on the problem of prescribed poles, since here we ought to add together functions we know have certain poles. Should we do this on the Riemann sphere or on the complex plane? Any meromorphic function on the Riemann sphere is rational, so it can't possibly have infinitely many poles. In $\mathbb{C}$ the situation is quite different, e.g. $\tan z, \sec z$ are meromorphic in $\mathbb{C}$, but ∞ is a limit of poles.

Theorem 0.1.1. *(Mittag-Leffler) Let (b_k) be a sequence in $\mathbb{C}$ with $\lim_{k\to\infty} b_k = \infty$. Let $P_k(z)$ be polynomials without constant terms. There exists a meromorphic function with poles b_k and principal parts $P_k(1/(z-b_k))$; the most general such function can be written*

$$f(z) = \sum_k \left(P_k\left(\frac{1}{z-b_k}\right) - p_k(z) \right) + g(z),$$

where $g(z)$ is entire, and the polynomials $p_k(z)$ are chosen[1] so that the series is uniformly and absolutely convergent on subsets of $\mathbb{C}$.

Proof. We can assume no $b_k = 0$. $P_k(1/(z-b_k))$ is holomorphic in $|z| < |b_k|$, so we can expand it as a Taylor series at 0. Let $p_k(z)$ be the sum of the terms of degree less than or equal to n_k, where n_k is chosen so that

$$\left| P_k\left(\frac{1}{z-b_k}\right) - p_k(z) \right| \le \frac{1}{2^k}, \qquad |z| \le \frac{1}{2}|b_k|.$$

We'll show that the series converges uniformly and absolutely in every disk $|z| \le r$. Choose m such that $|b_k| > 2r$ for all $k > m$. Then

$$\sum_{k=m+1}^{\infty} \left(P_k\left(\frac{1}{z-b_k}\right) - p_k(z) \right)$$

is uniformly and absolutely convergent in $|z| \le r$ by comparison with $\sum 1/2^k$. Any meromorphic function with the same poles and principal parts has to differ from this by an entire function. $\square$

[1] E.g. the $1/\omega^2$ term in the definition of $\wp(z)$.

0.1.2 Infinite Products

To get a function with prescribed zeroes, we ought to multiply together terms rather than add them. Thus, we study the notion of an infinite product of complex numbers or functions.

Let's just try to copy the definition of convergence of an infinite series. Let (b_n) be a sequence in $\mathbb{C}$. We say that the infinite product $\prod_{n=1}^{\infty} b_n$ converges to $p = \lim_{n\to\infty} p_n$, where $p_n = p_1 \cdots p_n$, and the limit is non-zero (to avoid trivial cases like $b_1 = 0$).

This is too restrictive. We'll thus modify our definition as follows: we say that $\prod b_n$ converges if all but finitely many factors are non-zero, and the partial products formed by the non-zero factors have non-zero limit.

We'd like to develop some tests for the convergence of infinite products. An immediate necessary condition is the analogue of the vanishing test: if $\prod b_n$ converges, then $b_n \to 1$. This can be seen by noting that $b_n = p_n/p_{n-1}$. With this in mind, we'll write our product as $\prod(1+a_n)$, so that a necessary condition for convergence is that $a_n \to 0$.

Naturally, we'd like to compare the infinite product $\prod(1 + a_n)$ with the infinite series $\sum \log(1 + a_n)$, where we use the principal branch of log. (It is defined for n large enough.) Let s_n be the nth partial term, so that $p_n = e^{s_n}$. So if the s_n's converge, the p_n's converge. Conversely? Suppose $p_n \to p$. Fix a branch $\log p = \ln |p| + i \arg p$. Take also $\log p_n = \ln |p_n| + i \arg p_n$, where $\arg p_n \in (\arg p - \pi, \arg p + \pi)$. Then

$$s_n = \log p_n + 2\pi i k_n$$

for some $k_n \in \mathbb{Z}$. Write

$$\log(1 + a_{n+1}) = s_{n+1} - s_n = \log p_{n+1} - \log p_n + 2\pi i(k_{n+1} - k_n).$$

For large enough n,

$$\begin{aligned} |\arg(1 + a_{n+1})| &< 2\pi/3, \\ |\arg p_n - \arg p| &< 2\pi/3, \\ |\arg p_{n+1} - \arg p| &< 2\pi/3, \end{aligned}$$

so $|k_{n+1}-k_n| < 1$, i.e. $k_{n+1} = k_n$, a common value k, for large n. Then $s_n \to \log p+2\pi ik$.

Thus convergence of $\prod(1 + a_n)$ is equivalent to convergence of $\sum \log(1 + a_n)$. This will be our main tool in studying infinite products. Next time, we'll discuss the notion of the convergence of an infinite product of funtions in order to develop the theory of functions with prescribed zeroes.