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1 Spaces of Holomorphic Functions (1-29-2021)

1.1 Topology of $\mathcal{C}(\Omega)$

Let Ω be an open subset of $\mathbb{C}$. We write $\mathcal{C}(\Omega)$ for the ring of continuous complex-valued functions on Ω, and $\mathcal{H}(\Omega)$ for the subring of $\mathcal{C}(\Omega)$ consisting of holomorphic functions. We are interested in topologizing $\mathcal{C}(\Omega)$, with what we call the *compact-open topology*.

A sequence $\{f_n\}$ in $C(\Omega)$ is said to *converge uniformly on compact subsets* if for all compact $K \subset \Omega$, $\{f_n|_K\}$ converges uniformly. A notion of convergence defines a topology; we need to define the open sets. We start with a system of neighbourhoods of 0. For a compact $K \subset \Omega$ and an $\varepsilon > 0$, define

$$V(K, \varepsilon) = \{f \in \mathcal{C}(\Omega) : |f(z)| < \varepsilon, z \in K\}.$$

Then $f_n \to f$ uniformly on compact subsets if and only if for all K, ε, $f - f_n \in V(k, \varepsilon)$ for sufficiently large n. Then, a system of neighbourhoods of any point is obtained by translating these neighbourhoods of 0, giving a basis for a topology on $\mathcal{C}(\Omega)$.

Actually, this topology on $\mathcal{C}(\Omega)$ is metrizable, and it can be defined by a translation-invariant metric. Cover Ω by the interiors of countably many closed disks D_i (take all closed disks in Ω with rational center, radius). Define

$$d(f) = \sum_{i=1}^{\infty} \frac{1}{2^i} \min\{1, M_i(f)\},$$

where $M_i(f)$ is the maximum of $|f|$ on D_i. It is clear that

(i) $d(f) \geq 0$, and $d(f) = 0$ if and only if $f = 0$, and

(ii) $d(f + g) \leq d(f) + d(g)$ (it's certainly true for each term in the sum).

So $d(f, g) \coloneqq d(f - g)$ is a translation-invariant metric.

1.2 Convergence of Holomorphic Functions

$\mathcal{C}(\Omega)$ is complete; the limit of a sequence of continuous functions that converges uniformly on compact sets is continuous. We'll mostly be concerned with the holomorphic functions $\mathcal{H}(\Omega)$. We give it the subspace topology from the compact-open topology on $\mathcal{C}(\Omega)$. The following result is a fundamental fact about the topology on $\mathcal{H}(\Omega)$ that we will use often.

Theorem 1.1. *(Weierstrass) Let Ω be open in $\mathbb{C}$.*

(1) $\mathcal{H}(\Omega)$ is a closed subspace of $\mathcal{C}(\Omega)$.

(2) The mapping $\mathcal{H}(\Omega) \to \mathcal{H}(\Omega)$, $f \mapsto f'$, is continuous.

(1) means that if $\{f_n\} \subset \mathcal{H}(\Omega)$ converges uniformly on compact sets, then $f = \lim f_n$ is holomorphic. (2) means that if $\{f_n\} \subset \mathcal{H}(\Omega)$ converges to f uniformly on compact sets, then $\{f_n'\}$ converges uniformly to f' on compact sets.

Proof. (1) It's enough to show that $f(z)\,dz$ is a closed form. Consider a disk with center a and radius r contained in Ω. We want to show that if γ is any closed curve in $|z-a| < r$, then the integral of $f(z)\,dz$ over γ vanishes. Since γ is compact,

$$\int_\gamma f(z)\,dz = \lim_{n\to\infty} \int_\gamma f_n(z)\,dz = 0,$$

since all of the integrals in the limit vanish by holomorphicity of each $f_n(z)$.

(2) Suppose $f_n \to f$ uniformly on compact sets. It's enough to show that $f_n' \to f'$ uniformly on a closed disk $D \subset \Omega$. Let γ be the counterclockwise boundary of a larger concentric disk in Ω. If $z \in D$, then

$$f(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)}{\zeta - z}\,d\zeta,$$

so differentiating under the integral sign gives

$$f'(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)}{(\zeta - z)^2}\,d\zeta.$$

By uniform convergence on compact sets,

$$f'(z) = \lim_{n\to\infty} \frac{1}{2\pi i} \int_\gamma \frac{f_n(\zeta)}{(\zeta - z)^2}\,d\zeta = \lim_{n\to\infty} f_n'(z).$$

The convergence is uniform with respect to $z \in D$ because $(\zeta - z)^{-2}$ is bounded away from 0 for $z \in D$, $\zeta \in \gamma$.

□

Any result about sequences also applies to series.

Corollary 1.1. *If a series of holomorphic functions $\sum f_n$ on Ω converges uniformly on compact sets, then the sum $f = \sum f_n$ is holomorphic, and the series can be differentiated term-by-term.*

Recall that a set Ω in $\mathbb{C}$ is said to be a *domain* if it is open and connected.

Proposition 1.1. *(Hurwitz) Suppose that Ω is a domain in $\mathbb{C}$. If $\{f_n\} \subset \mathcal{H}(\Omega)$ converges uniformly on compact sets, and each f_n is nowhere-vanishing on Ω, then the limit is either never zero or identically zero.*

Proof. Suppose that f is not identically zero. Since Ω is connected, the zeroes of f are isolated. Suppose $f(z_0) = 0$. Let γ be the boundary of a circle in Ω with center z_0. The multiplicity of z_0 is

$$\frac{1}{2\pi i} \int_\gamma \frac{f'(z)}{f(z)}\, dz = \lim_{n\to\infty} \frac{1}{2\pi i} \int_\gamma \frac{f_n'(z)}{f_n(z)}\, dz.$$

This is zero since f_n is never zero, i.e. the integrands f_n'/f_n are holomorphic, a contradiction. □

Corollary 1.2. *If Ω is a domain and $\{f_n\} \subset \mathcal{H}(\Omega)$ converges uniformly on compact sets, and each f_n is one-to-one, then $f = \lim f_n$ is either one-to-one or constant.*

Proof. Assume that f is not constant and not one-to-one. Then $f(z_1) = f(z_2) = a$ for some $z_1 \neq z_2$ in Ω. Let U, V be disjoint open neighbourhoods of z_1, z_2 in Ω. Then $f(z)-a$ vanishes at a point of U, so some whole subsequence $\{f_{n_i}\}$of $\{f_n\}$ vanishes at a point of U. The same argument provides a subsequence $\{f_{n_{i_j}}\}$ such that $f_{n_{i_j}}(z) - a$ vanishes at some point of V, implying that the $f_{n_{i_j}}$'s are not one-to-one, contradiction. □

We'd like to apply these notions to sequences and series of meromorphic functions, however we will have to manage the existence of poles. We will do this next time.