MAT454/1002 lecture notes by Kain Dineen. Taught by Edward Bierstone.

0.1 Spaces of Holomorphic Functions (1-29-2021)

We study the notion of uniform convergence of families and series of holomorphic functions by topologizing the space of continuous and holomorphic functions on open subsets of the complex plane. We then prove two results about the behaviour of uniform limits of sequences of holomorphic functions.

0.1.1 Topology of $\mathcal{C}(\Omega)$

Let Ω be an open subset of $\mathbb{C}$. We write $\mathcal{C}(\Omega)$ for the ring of continuous complex-valued functions on Ω, and $\mathcal{H}(\Omega)$ for the subring of $\mathcal{C}(\Omega)$ consisting of holomorphic functions. We are interested in topologizing $\mathcal{C}(\Omega)$, with what we call the *compact-open topology.*

A sequence $\{f_n\}$ in $C(\Omega)$ is said to *converge uniformly on compact subsets* if for all compact $K \subset \Omega$, $\{f_n|_K\}$ converges uniformly. A notion of convergence defines a topology; we need to define the open sets. We start with a system of neighbourhoods of 0. For a compact $K \subset \Omega$ and an $\varepsilon > 0$, define

$$V(K, \varepsilon) = \{f \in \mathcal{C}(\Omega) : |f(z)| < \varepsilon, z \in K\}.$$

Then $f_n \to f$ uniformly on compact subsets if and only if for all K, ε, $f - f_n \in V(k, \varepsilon)$ for sufficiently large n. Then, a system of neighbourhoods of any point is obtained by translating these neighbourhoods of 0, giving a basis for a topology on $\mathcal{C}(\Omega)$.

Actually, this topology on $\mathcal{C}(\Omega)$ is metrizable, and it can be defined by a translation-invariant metric. Cover Ω by the interiors of countably many closed disks D_i (take all closed disks in Ω with rational center, radius). Define

$$d(f) = \sum_{i=1}^{\infty} \frac{1}{2^i} \min\{1, M_i(f)\},$$

where $M_i(f)$ is the maximum of $|f|$ on D_i. It is clear that

(i) $d(f) \geq 0$, and $d(f) = 0$ if and only if $f = 0$, and

(ii) $d(f + g) \leq d(f) + d(g)$ (it's certainly true for each term in the sum).

So $d(f, g) \coloneqq d(f - g)$ is a translation-invariant metric.

0.1.2 Convergence of Holomorphic Functions

$\mathcal{C}(\Omega)$ is complete; the limit of a sequence of continuous functions that converges uniformly on compact sets is continuous. We'll mostly be concerned with the holomorphic functions $\mathcal{H}(\Omega)$. We give it the subspace topology from the compact-open topology on $\mathcal{C}(\Omega)$. The following result is a fundamental fact about the topology on $\mathcal{H}(\Omega)$ that we will use often.

Theorem 0.1.1. *(Weierstrass) Let Ω be open in $\mathbb{C}$.*

(1) $\mathcal{H}(\Omega)$ is a closed subspace of $\mathcal{C}(\Omega)$.

(2) The mapping $\mathcal{H}(\Omega) \to \mathcal{H}(\Omega)$, $f \mapsto f'$, is continuous.

(1) means that if $\{f_n\} \subset \mathcal{H}(\Omega)$ converges uniformly on compact sets, then $f = \lim f_n$ is holomorphic. (2) means that if $\{f_n\} \subset \mathcal{H}(\Omega)$ converges to f uniformly on compact sets, then $\{f_n'\}$ converges uniformly to f' on compact sets.

Proof. (1) It's enough to show that $f(z)\,dz$ is a closed form. Consider a disk with center a and radius r contained in Ω. We want to show that if γ is any closed curve in $|z-a| < r$, then the integral of $f(z)\,dz$ over γ vanishes. Since γ is compact,

$$\int_\gamma f(z)\,dz = \lim_{n\to\infty} \int_\gamma f_n(z)\,dz = 0,$$

since all of the integrals in the limit vanish by holomorphicity of each $f_n(z)$.

(2) Suppose $f_n \to f$ uniformly on compact sets. It's enough to show that $f_n' \to f'$ uniformly on a closed disk $D \subset \Omega$. Let γ be the counterclockwise boundary of a larger concentric disk in Ω. If $z \in D$, then

$$f(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)}{\zeta - z}\,d\zeta,$$

so differentiating under the integral sign gives

$$f'(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)}{(\zeta - z)^2}\,d\zeta.$$

By uniform convergence on compact sets,

$$f'(z) = \lim_{n\to\infty} \frac{1}{2\pi i} \int_\gamma \frac{f_n(\zeta)}{(\zeta - z)^2}\,d\zeta = \lim_{n\to\infty} f_n'(z).$$

The convergence is uniform with respect to $z \in D$ because $(\zeta - z)^{-2}$ is bounded away from 0 for $z \in D$, $\zeta \in \gamma$.

□

Any result about sequences also applies to series.

Corollary 0.1.1. *If a series of holomorphic functions $\sum f_n$ on Ω converges uniformly on compact sets, then the sum $f = \sum f_n$ is holomorphic, and the series can be differentiated term-by-term.*

Recall that a set Ω in $\mathbb{C}$ is said to be a *domain* if it is open and connected.

Proposition 0.1.1. *(Hurwitz) Suppose that Ω is a domain in $\mathbb{C}$. If $\{f_n\} \subset \mathcal{H}(\Omega)$ converges uniformly on compact sets, and each f_n is nowhere-vanishing on Ω, then the limit is either never zero or identically zero.*

Proof. Suppose that f is not identically zero. Since Ω is connected, the zeroes of f are isolated. Suppose $f(z_0) = 0$. Let γ be the boundary of a circle in Ω with center z_0. The multiplicity of z_0 is

$$\frac{1}{2\pi i} \int_\gamma \frac{f'(z)}{f(z)}\, dz = \lim_{n\to\infty} \frac{1}{2\pi i} \int_\gamma \frac{f_n'(z)}{f_n(z)}\, dz.$$

This is zero since f_n is never zero, i.e. the integrands f_n'/f_n are holomorphic, a contradiction. □

Corollary 0.1.2. *If Ω is a domain and $\{f_n\} \subset \mathcal{H}(\Omega)$ converges uniformly on compact sets, and each f_n is one-to-one, then $f = \lim f_n$ is either one-to-one or constant.*

Proof. Assume that f is not constant and not one-to-one. Then $f(z_1) = f(z_2) = a$ for some $z_1 \neq z_2$ in Ω. Let U, V be disjoint open neighbourhoods of z_1, z_2 in Ω. Then $f(z) - a$ vanishes at a point of U, so some whole subsequence $\{f_{n_i}\}$of $\{f_n\}$ vanishes at a point of U. The same argument provides a subsequence $\{f_{n_{i_j}}\}$ such that $f_{n_{i_j}}(z) - a$ vanishes at some point of V, implying that the $f_{n_{i_j}}$'s are not one-to-one, contradiction. □

We'd like to apply these notions to sequences and series of meromorphic functions, however we will have to manage the existence of poles. We will do this next time.