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0.1 Weierstrass's Factorization Theorem (2-26-2021)

We finish off the story of finding functions with prescribed poles and zeroes by stating and proving Weierstrass's factorization theorem, and then we discuss a couple of its corollaries.

0.1.1 Weierstrass's Factorization Theorem

Here is the promised result.

Theorem 0.1.1. *(Weierstrass) Given $a_k \in \mathbb{C}$ (not necessarily distinct) with $a_k \to \infty$, there is an entire function with zeroes a_k. The most general such entire function is of the form*

$$f(z) = z^m e^{g(z)} \prod_{k=1}^{\infty} \left[\left(1 - \frac{z}{a_k}\right) e^{\frac{z}{a_k} + \frac{1}{2}\left(\frac{z}{a_k}\right)^2 + \cdots + \frac{1}{m_k}\left(\frac{z}{a_k}\right)^{m_k}} \right],$$

where the product is taken over all of the non-zero a_k's, $g(z)$ is entire, and the m_k's are integers so that the product converges uniformly and absolutely on compact sets.

We remark that this is very similar to the construction of a meromorphic function with prescribed poles. In that construction, we had to subtract off a certain number of terms from the Taylor expansion of the prescribed principal part to make the sum converge. Here, since we take logarithms to test convergence of the product, we ought to multiply by the exponential of a certain number of terms of the Taylor expansion of the principal branch of $\log(1 - z/a_k)$.

Proof. The infinite product $\prod(1 - z/a_k)e^{p_k(z)}$ converges together with the series with general term $g_k(z) = \log(1 - z/a_k) + p_k(z)$, where the branch of log is to be chosen so that $g_k(z)$ is the principal branch of the logarithm of the kth factor in the product. To do this, we'll choose the branch of log so that $\operatorname{Im} g_k(z) \in (-\pi, \pi)$. Given r, consider the factors with $|a_k| > r$. In $|z| \leq r$, let

$$p_k(z) = \frac{z}{a_k} + \frac{1}{2}\left(\frac{z}{a_k}\right)^2 + \cdots + \frac{1}{m_k}\left(\frac{z}{a_k}\right)^{m_k}.$$

Using the principal branch of log as above,

$$g_k(z) = -\frac{1}{m_k + 2}\left(\frac{z}{a_k}\right)^{m_k+1} - \frac{1}{m_k + 2}\left(\frac{z}{a_k}\right)^{m_k+2} - \cdots;$$

we want to estimate $|g_k(z)|$ and choose m_k accordingly. We have

$$|g_k(z)| \leq \frac{1}{m_k+1}\left(\frac{r}{|a_k|}\right)^{m_k+1} \cdot \left(1 - \frac{r}{|a_k|}\right)^{-1}.$$

So choose m_k so that

$$\sum_{k=1}^{\infty} \frac{1}{m_k+1}\left(\frac{r}{|a_k|}\right)^{m_k+1}$$

converges; for example, $m_k = k$ works! Then $g_k(z) \to 0$ uniformly in $|z| \leq r$, so its imaginary part lies in $(-\pi, \pi)$ for k large enough. Therefore $\sum g_k(z)$ is uniformly and absolutely convergent in $|z| \leq r$, and it follows that the product represents a holomorphic function. $\square$

We remark that the theorem we just proved, and the theorem of Mittag-Leffler, are very similar. In fact, one can prove the theorem of Mittag-Leffler using the Weierstrass factorization theorem, simply by taking reciprocals. (What about the converse? Could one combine the two theorems?)

0.1.2 Corollaries to Weierstrass's Factorization Theorem

We explore some consequences of the theorem we just proved.

Corollary 0.1.1. *Every meromorphic function on $\mathbb{C}$ is the quotient of two entire functions.*

Proof. Let $h(z)$ be a meromorphic function on $\mathbb{C}$. Let $g(z)$ be an entire function whose zeroes are precisely the poles of $h(z)$, counted with multiplicity. Then $f(z) = g(z)h(z)$ is an entire function, and $h(z) = f(z)/g(z)$. $\square$

Corollary 0.1.2. *Let $a_k, b_k \in \mathbb{C}$ be sequences with $a_k \to \infty$, and choose multiplicities $n_k \in \mathbb{N}$. There is an entire function $f(z)$ such that each a_k is a root of order n_k of the equation $f(z) = b_k$.*

Proof. Near a_k, f should have the form $f(z) = b_k + (z-a_k)^{n_k}\widetilde{f}(z)$ for some holomorphic function $\widetilde{f}(z)$ with $\widetilde{f}(a_k) \neq 0$. By the theorem, there is an entire function $g(z)$ with a zero of order n_k at a_k. Write

$$f(z) = g(z)h(z) = b_k + g(z)\left(h(z) - \frac{b_k}{g(z)}\right),$$

where $h(z)$ is a meromorphic function with poles a_k having principal part equal to that of $b_k/g(z)$ at a_k. $g(z)$ has a zero of order n_k at a_k, and $h(z) - b_k/g(z)$ is holomorphic, so $f(z) = b_k$ has a_k as a root of order *at least* a_k. To remedy this, change the order of the zero of $g(z)$ at a_k from n_k to $n_k + 1$, and change the principal part of $h(z)$ at a_k to be $b_k/g(z) + 1/(z - a_k)$. $\square$