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1 Cauchy's Integral Formula (1-18-2021)

1.1 Remarks

Let us note that in Cauchy's theorem, from last time, it is enough to assume that the function is continuous on the domain, and holomorphic outside a line (or outside even a finite number of lines and points).

We need the following theorem to prove Cauchy's integral formula. The proof will not be given.

Theorem 1.1. *A closed differential form ω on a simply-connected open set $\Omega \subseteq \mathbb{R}^2$ has a (globally defined) primitve.*

1.2 Cauchy's Integral Formula

Let γ be a closed curve in Ω, and let $a \in \Omega$ be a point not on the curve. We define the *winding number* of γ with respect to a as

$$w(\gamma, a) = \frac{1}{2\pi i}\int_\gamma \frac{dz}{z-a}.$$

The winding number is always an integer, since the integral is given by the difference between two branches of $\log(z-a)$. It is meant to measure how many times the curve γ loops around a. For example, if γ is a circle centered at a oriented counter-clockwise, then $w(\gamma, a) = 1$.

The winding number is invariant with respect to a homotopy of γ not passing through a, since the integrand $(z-a)^{-1}\,dz$ is a closed form on $\mathbb{C}\setminus\{a\}$. Furthermore, as a function of a, $w(\gamma, a)$ is constant on the connected components of $\mathbb{C}\setminus\gamma$.

Theorem 1.2. *(Cauchy's integral formula) Let $f(z)$ be holomorphic in the open set Ω, and let $a \in \Omega$. Suppose γ is a closed curve in Ω, not containing a, which is homotopic to a point in Ω. Then,*

$$\frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z-a}\,dz = w(\gamma, a) f(a).$$

We will prove this next time.