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1 Series of Meromorphic Functions (2-1-2021)

1.1 Series of Meromorphic Functions

Let $\{f_n\}$ be a sequence of meromorphic functions on an open subset $\Omega \subseteq \mathbb{C}$. We will say that $\sum_{n=1}^{\infty} f_n$ *converges uniformly* (and *absolutely*) on $X \subseteq \Omega$ if all but finitely many f_n's have no pole in X, and form a uniformly (or uniformly and absolutely) convergent series. We wlll study series of meromorphic functions that converge uniformly on compact subsets of Ω. We can define the sum on a relatively compact open subset $U \subseteq \Omega$ as

$$\underbrace{\sum_{n \leq n_0} f_n}_{\text{meromorphic}} + \underbrace{\sum_{n > n_0} f_n}_{f_n\text{'s no pole in } \overline{U}} .$$

The second sum is uniformly convergent on $\overline{U}$, since none of the summands have poles there. This is independent of the choice of n_0.

The following theorem is the meromorphic analogue of the theorem on series of holomorphic functions from last time. The proof is similar.

Theorem 1.1. *Consider a series $\sum f_n$ of meromorphic functions on Ω. If the series converges uniformly on compact subsets of Ω, then the sum is a meromorphic function f on Ω, and it can be differentiated term-by-term; $\sum f_n'$ converges uniformly on compact subsets of Ω to f'.*

Since the sum f is meromorphic, its poles are isolated. Its poles form a subset of the poles of the f_n's, but some of the poles of the f_n's might cancel out.

1.2 A Meromorphic Series

Consider $\sum_{n=-\infty}^{\infty} (z-n)^{-2}$. We want to show that this series is uniformly and absolutely convergent on compact subsets of $\mathbb{C}$, and then we want to find a closed form for the sum. It's enough to show this on any vertical strip $a_1 \leq x \leq a_2$, since any compact subset of $\mathbb{C}$ can be covered by finitely many of these.

We are going to remove the terms where n lies inside this strip. First, consider

$$\sum_{n<a_1} \frac{1}{(z-n)^2},$$

for z inside the vertical strip. This is uniformly and absolutely convergent in the strip since each summand is bounded above in modulus by $(a_1 - n)^{-2}$, and this converges since each term is comparable to n^{-2}. The argument for the sum over $n > a_2$ is similar.

With the theorem in mind, consider the meromorphic function defined by $f(z) = \sum_{n=-\infty}^{\infty}(z-n)^{-2}$. The function f has period 1 ($f(z+1) = f(z)$), the poles are precisely the integers, and they are all double poles with principal parts $(z-n)^{-2}$ (and residues 0). We claim that

$$f(z) = \left(\frac{\pi}{\sin \pi z}\right)^2;$$

call this function $g(z)$. It's enough to show that $g(z)$ is meromorphic, with the same poles and corresponding principal parts as $f(z)$, and that $f - g$ is bounded (in fact, we want it to be zero).

Note that $f(z) \to 0$ uniformly with respect to x as $|y| \to \infty$; that is, for every $\varepsilon > 0$, there is a b such that $|f(z)| < \varepsilon$ when $|y| > b$. By periodicity, it's enough to show this in a strip $a_1 \leq x \leq a_2$. This clearly holds for each summand, in the strip; it follows that it holds for the sum as well. (The proof of this is left as an exercise, which might be given next time.)

Now, $g(z)$ has the same properties as f:

(i) Meromorphic in $\mathbb{C}$, with period 1.

(ii) The poles are precisely the integers, and each pole is a double pole with principal part $(z-n)^{-2}$.

(iii) $g(z) \to 0$ as $|y| \to \infty$, uniformly with respect to x; to see this, use the fact that

$$|\sin \pi z|^2 = \sin^2 \pi x + \sinh^2 \pi y.$$

Why is $f - g$ bounded? Consider again a strip $a_1 \leq x \leq a_2$. In some part $|y| \geq b$ of this strip, $f - g$ goes to 0 as $|y| \to \infty$, uniformly with respect to x. The rest of the strip is compact, so $f - g$ is bounded on it. Therefore $f - g$ is bounded on the strip, and bounded on all of $\mathbb{C}$. (Why?)

$f - g$ is holomorphic in $\mathbb{C}$, since f, g have the same poles and principal parts. $f - g$ is constant by Liouville's theorem. The constant is zero by property (iii). As an exercise, show that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$