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1 Isolated Singularities and Residues (1-27-2021)

1.1 Singularities

We say that a holomorphic function $f(z)$ in the punctured disk $0 < |z| < R$ has an *isolated singularity* at 0 if $f(z)$ can't be extended to a holomorphic function on all of $|z| < R$. There are two cases: 0 is either a pole, else we call it an *essential singularity*. What is the difference? For a pole, the principal part of the Laurent expansion of $f(z)$,

$$f(z) = \sum_{n=-\infty}^{\infty} a_n z^n,$$

is a finite sum. So an essential singularity means that the principal part is an infinite sum.

Extension to a holomorphic function in $|z| < R$ is possible if and only if f is bounded in a neighbourhood of 0. Why? Take $r \in (0, R)$. Then

$$f(re^{i\theta}) = \sum_{n=-\infty}^{\infty} a_n r^n e^{in\theta}.$$

Integrate $e^{-in\theta} f(re^{i\theta})$ with respect to θ to get

$$a_n r^n = \frac{1}{2\pi} \int_0^{2\pi} e^{-in\theta} f(re^{i\theta})\, d\theta, \qquad n \in \mathbb{Z}.$$

Therefore $|a_n| \leq M(r) r^{-n}$, where $M(r)$ is an upper bound for $|f(z)|$, $|z| = r$. If f is bounded in the punctured disk, then there is an $M > 0$ such that every $M(r) \leq M$. Then, when $n < 0$, $|a_n| \leq M r^{-n}$; since $-n > 0$, taking $r \to 0$ shows that $a_n = 0$. Therefore f can be extended holomorphically to 0.

Thus, there are three options for a holomorphic function $f(z)$ in a punctured disk centered at 0:

(i) A *removable singularity*: bounded in a neighbourhood of 0, in which case it extends holomorphically to the entire disk.

(ii) A pole, in which case $\lim_{z \to 0} f(z) = \infty$.

(iii) An essential singularity. What can we say about the limit here? It's not even well-defined, as the following theorem shows.

The following theorem is often called the *Casorati-Weierstrass theorem*. It displays, in some sense, just how unmanageable essential singularities are, compared to poles.

Theorem 1.1. *If* 0 *is an essential singularity, then for all* $\varepsilon > 0$, $f(\{0 < |z| < \varepsilon\})$ *is dense in* $\mathbb{C}$.

Proof. If not, we can find an $a \in \mathbb{C}$ and $\delta > 0$, such that $|f(z)-a| \geq \delta$, for all $0 < |z| < \varepsilon$. Consider

$$g(z) = \frac{1}{f(z) - a}.$$

Then $g(z)$ is holomorphic in $0 < |z| < \varepsilon$, and bounded, since $|g(z)| \leq 1/\delta$. So $g(z)$ extends holomorphically to $|z| < \varepsilon$. But then $f(z) = a + 1/g(z)$ is meromorphic, and has either a pole or a removable singularity at 0, contradicting that 0 is an essential singularity of $f(z)$. □

1.2 Residues

Let $f(z)$ be a holomorphic function in a punctured neighbourhood of a. Let γ be a curve around a with winding number $w(\gamma, a) = 1$. We call the integral

$$\frac{1}{2\pi i} \int_\gamma f(z)\, dz$$

the *residue* of the differential form $f(z)\, dz$ at a. What does this mean in terms of the Laurent series expansion

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z-a)^n?$$

For every $n \neq -1$, $(z-a)^n\, dz$ has a primitive, so the integral vanishes. For $n = 1$, the integral is $2\pi i a_{-1}$, so the residue is simply the coefficient a_{-1}.

Why is it beneficial to consider the residue of the *form* $f(z)\, dz$ instead of that of the function $f(z)$? What is the residue at ∞? Consider coordinates z' at infinity, $z = 1/z'$. Then

$$f(z)\, dz = -\frac{1}{(z')^2} f\left(\frac{1}{z'}\right) dz'.$$

We integrate over a curve with winding number 1 around ∞:

$$\operatorname{res}(f, \infty) = -\frac{1}{2\pi i} \int_\gamma \frac{1}{(z')^2} f\left(\frac{1}{z'}\right) dz' = -a_{-1},$$

where γ is positively oriented with respect to $z' = 0$ (i.e. $z = \infty$), with the Laurent series expansion taken in some annulus $|z| > R$. If $\omega = f(z)\, dz$, then this is just the integral $\frac{1}{2\pi i} \int_\gamma \omega$, where γ is positively oriented with respect to ∞ (and we computed it using coordinates at ∞). Thus, the residue of the form $f(z)\, dz$ at ∞ has the exact same formula as the residue at any other point, which is one reason it is beneficial to think of residues of forms instead of residues of functions.

Theorem 1.2. *(Residue theorem) Let Ω be an open subset of the Riemann sphere. Let $K \subset \Omega$ be a compact set with a piecewise-C^1 oriented boundary Γ. Given a function $f(z)$ holomorphic in Ω, except perhaps at isolated points, and with Γ not containing any singular points or ∞. Then*

$$\int_\Gamma f(z)\,dz = 2\pi i \sum \operatorname{res}(f, z_k),$$

where the z_k's are the singularities in K (possibly including ∞).

Remark: To say that K has a piecewise-C^1 oriented boundary Γ means that Γ is a union of piecewise-C^1 closed curves $\gamma(t)$ that are positively oriented with respect to K. Informally, if you're walking along Γ, then K is always to your left. (To state this formally requires a bit of messy second year calculus.)