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0.1 The Elliptic Integral (2-12-2021)

We examine the behaviour of the homogenization of the $\wp$-function's parametrized curve in complex projective space near the point at infinity. We mention that this curve is a torus. Then, we uncover the relationship between the $\wp$-function and elliptic integrals.

0.1.1 Recap

Consider the setup from last time: Γ is a lattice in $\mathbb{C}$, and X is the smooth curve in $\mathbb{C}^2$ given by $y^2 = 4x^3 - 20a_2x - 28a_4$, where

$$a_2 = 3 \sum_{\omega \in \Gamma \setminus \{0\}} \frac{1}{\omega^4}, \qquad a_4 = 5 \sum_{\omega \in \Gamma \setminus \{0\}} \frac{1}{\omega^6}.$$

Its compactification in $P^2(\mathbb{C})$ is given in homogeneous coordinates $[x, y, t]$ by the equation

$$y^2 t = 4x^3 - 20a_2 x t^2 - 28a_4 t^3. \tag{*}$$

X' consists of X together with the point at infinity, given by $t = 0$, i.e. $[0, 1, 0]$. We'd like to try to understand this compactified curve as a Riemann surface over the Riemann sphere, i.e. $P^1(\mathbb{C})$. Specifically, is there a mapping $\varphi\colon X' \to P^1(\mathbb{C})$ that extends the mapping $\varphi\colon X \to \mathbb{C}, (x, y) \mapsto x$ from before?

0.1.2 The Point at Infinity

We examine the point at infinity in coordinates. $[0, 1, 0]$ lies in the chart

$$\{[x, y, t] \in P^2(\mathbb{C}) : y \neq 0\}.$$

In this chart, we have affine coordinates $(x', t') = (x/y, t/y)$. The equation (*) in these coordinates is obtained by dehomogenizing with respect to y:

$$t' = 4x'^3 - 20a_2 x' t'^2 - 28a_4 t'^3.$$

The point $[0, 1, 0]$ at infinity has coordinates $(x', t') = (0, 0)$. In some neighbourhood of this point, the implicit function theorem gives t' as a holomorphic function of x':

$$t' = 4x'^3 - 320a_2 x'^7 + \cdots .$$

That is, in a neighbourhood of the point at infinity in X', we can take x' as a local coordinate. In this way, we get a complex 1-manifold structure on X'.

We return to the problem of extending $\varphi\colon X \to \mathbb{C}$ to a mapping $\varphi'\colon X' \to S^2$ sending $[0,1,0]$ to the point at infinity in the Riemann sphere. Write $P^1(\mathbb{C})$ with coordinates $[x,t]$. $S^2 \cong P^1(\mathbb{C})$ consists of $\mathbb{C}$ together with the point at infinity, $[1,0]$:

$$\mathbb{C} \cong \{[x,t] : t \neq 0\}.$$

The coordinate $z = x/t$ of $\mathbb{C}$ corresponds to $1/z = t/x$ in coordinates at infinity. In the chart above where $y \neq 0$, X' consists of the points

$$[x', 1, t'] = [x', 1, 4x'^3 - 320a_2x'^7 + \cdots].$$

If $t' \neq 0$, then φ takes this to $[x', t']$ in $P^1(\mathbb{C}) = S^2$, i.e. x'/t' in $\mathbb{C}$. In coordinates at infinity, t'/x':

$$\frac{t'}{x'} = \frac{4x'^3 - 320a_2x'^7 + \cdots}{x'},$$

which goes to the point at infinity in $S^2 = P^1(\mathbb{C})$ as $x' \to 0$. Summarizing, we have the diagram

$$\begin{tikzcd}
(x,y) \arrow[dr, mapsto] & \mathbb{C}^2 \arrow[dr] & X \arrow[l, hook'] \arrow[r, hook] \arrow[d, "\varphi"] & X' \arrow[r, hook] \arrow[d, "\varphi'"] & P^2(\mathbb{C}) \\
& x & \mathbb{C} \arrow[r, hook] & P^1(\mathbb{C}) = S^2, &
\end{tikzcd}$$

where the hooked arrows represent inclusions, and φ' is an extension of φ taking the point at infinity of X', $[0,1,0]$, to the point at infinity of the Riemann sphere, which is in the case of $P^1(\mathbb{C})$ the point $[1,0]$. We can therefore say that X' forms a Riemann surface over the Riemann sphere. (We will return to the notion of Riemann surfaces later in the course.)

The meromorphic transformation $x = \wp(z), y = \wp'(z)$, defines a mapping

$$\mathbb{C}/\Gamma \xrightarrow{\cong} X',$$

a homeomorphism, where $\mathbb{C}/\Gamma$ has the quotient topology. This is because it's a continuous bijection from a compact space to a Hausdorff space. Since $\mathbb{C}/\Gamma$ is a torus, our curve X' is topologically a torus.

0.1.3 Elliptic Integrals

The inverse map $X' \to \mathbb{C}/\Gamma$ defines z as a holomorphic *multi-valued* function of X', whose branches differ by constants belonging to Γ. We have

$$\frac{dx}{dz} = \frac{d\wp(z)}{dz} = \wp'(z) = y,$$

so $dx = y\,dz$. Since

$$y^2 = P(x) = 4x^3 - 20a_2x - 28a_4,$$

$2y\,dy = P'(x)\,dx$. Wherever x is a local coordinate, $dz = \frac{dx}{y}$. Wherever y is a local coordinate, $dz = \frac{2\,dy}{P'(x)}$. Thus, we can think of dz as an extension to X' of the holomorphic differential form

$$\frac{dx}{\sqrt{4x^3 - 20a_2x - 28a_4}}.$$

So

$$z = \wp^{-1}(x) = \int_{[0,1,0]}^{[\wp(z),\wp'(z),1]} \frac{dx}{\sqrt{4x^3 - 20a_2x - 28a_4}}. \tag{**}$$

Thus, the Weierstrass $\wp$-function is given by "inversion" of an elliptic integral.

Compare with the circle, $x^2 + y^2 = 1$. We have $x\,dx + y\,dy = 0$, so whenever each term makes sense,

$$-\frac{dx}{y} = \frac{dy}{x} = \frac{dy}{\sqrt{1-y^2}} = d\theta,$$

or $dy = x\,d\theta$. The function θ is not well-defined on the circle, but it is well-defined as a multi-valued function on S^1 whose branches differ by constants belonging to a group isomorphic to $\mathbb{Z}$. In first year calculus, we invert

$$\int \frac{dy}{x} = \int \frac{dy}{\sqrt{1-y^2}}$$

in a neighbourhood of $(1,0)$ by defining the trig functions by the relation

$$\theta = \int_{(1,0)}^{(\cos\theta,\sin\theta)} \frac{dy}{x} = \int_0^{\sin\theta} \frac{dy}{\sqrt{1-y^2}}.$$

This is exactly the same relationship between the Weierstrass $\wp$-function and the integral in (**); here, we obtained the trigonometric functions by the inversion of a trigonometric integral.