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## 1 Cauchy's Integral Formula, Continued (1-20-2021)

### 1.1 Proof of Cauchy's Integral Formula

Let us recall and prove Cauchy's integral formula.

**Theorem 1.1.** (*Cauchy's integral formula*) Let  $f(z)$  be holomorphic in the open set  $\Omega$ , and let  $a \in \Omega$ . Suppose  $\gamma$  is a closed curve in  $\Omega$ , not containing  $a$ , which is homotopic to a point in  $\Omega$ . Then,

$$\frac{1}{2\pi i} \int_{\gamma} \frac{f(z)}{z-a} dz = w(\gamma, a) f(a).$$

*Proof.* Let

$$g(z) = \begin{cases} \frac{f(z)-f(a)}{z-a}, & z \neq a, \\ f'(a), & z = a. \end{cases}$$

Then,  $g(z)$  is holomorphic in  $\Omega \setminus \{a\}$  and continuous in  $\Omega$ , so

$$\int_{\gamma} \frac{f(z) - f(a)}{z - a} dz = 0,$$

since  $g(z) dz$  is closed, by Cauchy's theorem. But then

$$\int_{\gamma} \frac{f(z)}{z-a} dz = \int_{\gamma} \frac{f(a)}{z-a} dz = 2\pi i \cdot f(a) w(\gamma, a).$$

□

The most important case of Cauchy's integral formula to us will be the case where  $\gamma$  is the boundary of a disk, oriented counter-clockwise. If  $f(z)$  is holomorphic in a neighbourhood of this disk, then

$$\frac{1}{2\pi i} \int_{\gamma} \frac{f(z)}{z-a} dz = \begin{cases} f(a), & a \text{ inside the circle,} \\ 0, & a \text{ outside the circle.} \end{cases}$$

### 1.2 Applications of Cauchy's Integral Formula

**Proposition 1.1.** *A holomorphic function is infinitely differentiable (and all of its derivatives are holomorphic).*

*Proof.* Suppose  $f(z)$  is holomorphic in some disk  $D = \{z : |z| < R\}$ , and let  $\gamma$  be the boundary of a smaller circle  $\{|z| < r\}$ ,  $r < R$ , oriented counter-clockwise. Then,  $\gamma(\theta) = re^{i\theta}$ ,  $\theta \in [0, 2\pi]$ . If  $|z| < r$ , then

$$f(z) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(\zeta)}{\zeta - z} d\zeta.$$

We can differentiate under the integral to obtain.

$$f'(z) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(\zeta) d\zeta}{(\zeta - z)^2}.$$

Furthermore, for any  $n$ ,

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_{\gamma} \frac{f(\zeta) d\zeta}{(\zeta - z)^{n+1}}.$$

□

Let us summarize what we have learned thus far concerning Cauchy's theorem.

**Proposition 1.2.** *If  $f(z)$  is continuous in  $\Omega$ , then the following statements are equivalent.*

1.  $f(z)$  is holomorphic.
2.  $f(z) dz$  is a closed form.
3. One has

$$f(z) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(\zeta)}{\zeta - z} d\zeta$$

for  $z$  in an open disk with boundary  $\gamma$ .

*Proof.* It remains to show (2) implies (1); this statement is known as Morera's theorem. Locally,  $f(z) dz$  has a primitive  $g(z)$ . That is,

$$f(z) dz = dg = \frac{\partial g}{\partial z} dz + \frac{\partial g}{\partial \bar{z}} d\bar{z}.$$

Since  $dz$  and  $d\bar{z}$  are linearly independent,  $\frac{\partial g}{\partial \bar{z}} = 0$ , i.e. the Cauchy-Riemann equations are satisfied. Thus,  $g$  is holomorphic. Therefore  $f(z) = \frac{\partial g}{\partial z}(z)$  is also holomorphic. □

We can use the integral formula to compute the Taylor series of a holomorphic function  $f(z)$  at 0:

$$f(z) = \sum_{n=0}^{\infty} a_n z^n, \quad a_n = \frac{f^{(n)}(0)}{n!} = \frac{1}{2\pi i} \int_{\gamma} \frac{f(\zeta) d\zeta}{\zeta^{n+1}}.$$

Why is this convergent?

$$\frac{1}{\zeta - z} = \frac{1}{\zeta} \left(1 - \frac{z}{\zeta}\right)^{-1} = \frac{1}{\zeta} \left(1 + \frac{z}{\zeta} + \frac{z^2}{\zeta^2} + \cdots\right).$$

Thus,

$$f(z) = \frac{1}{2\pi i} \int_{\gamma} \sum_{n=0}^{\infty} z^n \cdot \frac{f(\zeta)}{\zeta^{n+1}} d\zeta.$$

We can interchange the sum and the integral, as for a fixed  $z$ ,  $|z| < r$ , the series

$$\sum_{n=0}^{\infty} z^n \cdot \frac{f(\zeta)}{\zeta^{n+1}}$$

is uniformly and absolutely convergent on  $|\zeta| = r$ . Thus, the Taylor series converges when  $|z| < r$ . Moreover, if we substitute  $z = re^{i\theta}$ , it becomes

$$f(re^{i\theta}) = \sum_{n=0}^{\infty} a_n r^n e^{in\theta},$$

which proves that

$$a_n r^n = \frac{1}{2\pi} \int_0^{2\pi} e^{-in\theta} f(re^{i\theta}) d\theta.$$

These are known as the *Fourier coefficients*. This gives us the *Cauchy inequalities*, which give us an upper bound for the coefficients  $a_n$  (or their modulus). Let  $M(r) = \sup_{\theta} |f(re^{i\theta})|$ , the upper bound of  $|f|$  on the circle of radius  $r$ . By the integral formula,  $|a_n r^n| \leq M(r)$ , or

$$|a_n| \leq \frac{M(r)}{r^n}.$$

Now we deduce

**Theorem 1.2.** (*Liouville's theorem*) *A bounded holomorphic function defined on all of  $\mathbb{C}$  is constant.*

*Proof.*  $M(r) \leq M$ , for some constant  $M$ , so  $|a_n| \leq Mr^{-n}$  for all  $r$ . So  $a_n = 0$  for  $n > 1$ .  $\square$

**Corollary 1.1.** (*Fundamental theorem of algebra*) *Every non-constant polynomial has a root in  $\mathbb{C}$ .*

*Proof.* If  $p(z)$  is a polynomial without a root, then  $1/p(z)$  is holomorphic in  $\mathbb{C}$  and bounded, hence constant.  $\square$

**Theorem 1.3.** (*Mean value property*) If  $f(z)$  is holomorphic in  $\Omega$  and  $D \subset \Omega$  is a compact disk centered at  $a$ , then

$$f(a) = \text{mean value of } f \text{ on the boundary of } D.$$

*Proof.* We may assume  $a = 0$ . Then

$$f(0) = \frac{1}{2\pi} \int_0^{2\pi} f(re^{i\theta}) d\theta.$$

□

At the beginning of next lecture, we will go over more consequences of Cauchy's integral formula.