

MAT454/1002 lecture notes by Kain Dineen. Taught by Edward Bierstone.

1 Cauchy's Integral Formula (1-18-2021)

1.1 Remarks

Let us note that in Cauchy's theorem, from last time, it is enough to assume that the function is continuous on the domain, and holomorphic outside a line (or outside a finite number of lines and points).

We need the following theorem to prove Cauchy's integral formula. The proof will not be given.

Theorem 1.1. *A closed differential form ω on a simply-connected open set $\Omega \subseteq \mathbb{R}^2$ has a (globally defined) primitive.*

1.2 Cauchy's Integral Formula

Let γ be a closed curve in Ω , and let $a \in \Omega$ be a point not on the curve. We define the *winding number* of γ with respect to a as

$$w(\gamma, a) = \frac{1}{2\pi i} \int_{\gamma} \frac{dz}{z - a}.$$

The winding number is always an integer, since it is given by the difference between two branches of $\log(z - a)$. It is meant to measure how many times the curve γ loops around a . For example, if γ is a circle centered at a oriented counter-clockwise, then $w(\gamma, a) = 1$.

The winding number is invariant with respect to a homotopy of γ not passing through a , since the integrand $(z - a)^{-1} dz$ is a closed form on $\mathbb{C} \setminus \{a\}$. Furthermore, as a function of a , $w(\gamma, a)$ is constant on the connected components of $\mathbb{C} \setminus \gamma$.

Theorem 1.2. *(Cauchy's integral formula) Let $f(z)$ be holomorphic in the open set Ω , and let $a \in \Omega$. Suppose γ is a closed curve in Ω , not containing a , which is homotopic to a point in Ω . Then,*

$$\frac{1}{2\pi i} \int_{\gamma} \frac{f(z)}{z - a} dz = w(\gamma, a) f(a).$$

We will prove this next time.