MAT454/1002 lecture notes by Kain Dineen. Taught by Edward Bierstone.

0.1 Algebraic Curves and the $\wp$-function (2-8-2021)

We begin to study the algebraic properties of the Weierstrass $\wp$-function. Namely, we prove that it always gives a parametrization, up to addition of an element of Γ, of the elliptic curve in $\mathbb{C}^2$ given by the differential equation of $\wp$.

0.1.1 Parametrization by $\wp$

Let Γ be a discrete subgroup of $\mathbb{C}$, and let $\wp(z)$ be the associated Weierstrass $\wp$-function. We saw last time that $(x, y) = (\wp(z), \wp'(z))$ satisfies the algebraic equation $y^2 = 4x^3 - 20a_2x - 28a_4$, where

$$a_2 = 3 \sum_{\omega \in \Gamma \setminus \{0\}} \frac{1}{\omega^4}, \qquad a_4 = 5 \sum_{\omega \in \Gamma \setminus \{0\}} \frac{1}{\omega^6}.$$

Theorem 0.1.1. *$P(x) = 4x^3 - 20a_2x - 28a_4$ has three distinct zeroes. Moreover, for all $(x, y) \in \mathbb{C}^2$ on the curve $y^2 = 4x^3 - 20a_2x - 28a_4$, there is a unique $z \in \mathbb{C}$, mod Γ, such that $x = \wp(z)$, $y = \wp'(z)$.*

Later in the course, we'll discuss the following theorem, due to Abel.

Theorem 0.1.2. *(Abel) Conversely, given an equation $y^2 = 4x^3 - 20a_2x - 28a_4$, where the right-hand side has three distinct zeroes, then one can find a discrete group Γ such that a_2, a_4 are given as above. (So by the previous theorem, if $\wp$ is the corresponding Weierstrass $\wp$-function, then $x = \wp(z), y = \wp'(z)$ parametrizes the curve.)*

The significance of the "three distinct zeroes" condition is that any smooth cubic curve has the given form. This is not too difficult to prove, but we may not get to it in this course.

Proof. (Of the first theorem.) Each value of $\wp(z)$ is taken twice in a period parallelogram, and three times for $\wp'(z)$. Consider

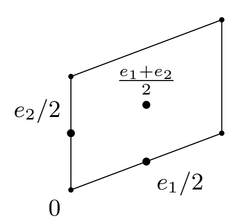

Figure 1:

Consider the points $z \in \mathbb{C}$ such that $z \notin \Gamma$, and $2z \in \Gamma$. Any such point is congruent mod Γ to one of $e_1/2$, $e_2/2$, or $(e_1+e_2)/2$; moreover, the classes mod Γ of these three points are distinct. At such z,

$$\begin{aligned} \wp'(z) &= \wp'(-z) && \text{by periodicity,} \\ \wp'(z) &= -\wp'(-z) && \text{since } \wp' \text{ is odd,} \end{aligned}$$

so $\wp'(z) = 0$. This means that each value $\wp(e_1/2)$, $\wp(e_2/2)$, and $\wp((e_1+e_2)/2)$ is taken exactly once in a period parallelogram (with multiplicity 2), and these three values are distinct. These values are all distinct because $\wp$ takes on each value exactly twice in a period parallelogram. Since $(\wp')^2 = 4\wp^3 - 20a_2\wp - 28a_4$, it follows that these three values are three distinct zeroes of $P(x)$.

If $2z_0 \notin \Gamma$, then the value $\wp(z_0)$ is taken exactly twice but at different points, since $\wp'(z_0) \neq 0$. The other point not congruent to z_0 mod Γ at which $\wp$ takes on this value is $-z_0$, since $\wp$ is even. (In the period parallelogram as shown above, it's z_0 flipped over the midpoint, which is congruent mod Γ to $-z_0$.)

Consider (x, y) on the curve, with $y \neq 0$. Let z_0 be a point of the period parallelogram such that $x = \wp(z_0)$. Then $\wp'(z_0) \neq 0$, so $2z_0 \notin \Gamma$, so then $x = \wp(z_0) = \wp(-z_0)$, but $y = \wp'(z_0)$ and $-y = \wp'(-z_0)$ by odd-ness, so the two distinct points $z_0, -z_0$ in the period parallogram map to different points on the curve. Therefore *every* point on the curve $y^2 = 4x^3 - 20a_2x - 28a_4$ is the image under $(\wp, \wp')$ of some point, uniquely determined mod Γ. $\square$

0.1.2 Implicit Function Theorem

Consider $X \subset \mathbb{C}^2$ given by the equation $y^2 = 4x^3 - 20a_2x - 28a_4$. The right-hand side, $P(x)$, has three distinct zeroes. Then X is a smooth curve[1] (i.e. X is locally the graph of a *holomorphic* function $y = f(x)$ or $x = g(y)$). Write $F(x, y) = y^2 - P(x)$. If $(x_0, y_0) \in X$ has $y_0 \neq 0$, then

$$\frac{\partial F}{\partial y}(x_0, y_0) = 2y_0 \neq 0,$$

so the curve is of the form $y = f(x)$ near (x_0, y_0). On the other hand, if $(x_0, y_0) \in X$ has $y_0 = 0$, then $P'(x_0) \neq 0$, since there are three distinct zeroes. (We have $P(x_0) = 0$, so if $P'(x_0) = 0$, then x_0 is a zero of multiplicity 2, contradiction.) This gives $\frac{\partial F}{\partial x}(x_0, y_0) \neq 0$, so near (x_0, y_0) we can write the curve as $x = g(y)$.

Why will these functions f, g be holomorphic? The classic implicit function theorem only ensures that the functions f, g will be differentiable, but we don't yet know if they will be holomorphic. To this end, we will prove a "weak" implicit function theorem for two complex variables that will give us holomorphicity of f, g in the previous discussion.

[1] A 2-dimensional real manifold, or 1-dimensional complex manifold.

Theorem 0.1.3. *Consider the equation $f(x, y) = 0$, for $(x, y) \in \mathbb{C}^2$. Assume that f is C^1, and that f is separately holomorphic (i.e. holomorphic in y for fixed x, and holomorphic in x for fixed y). If $\frac{\partial f}{\partial y}(x_0, y_0) \neq 0$, where $f(x_0, y_0) = 0$, then we can locally solve for y as a holomorphic function of x.*

Proof. Write $z = f(x, y)$, where

$$\begin{aligned} x &= x_1 + ix_2, \\ y &= y_1 + iy_2, \\ z &= z_1 + iz_2, \\ f &= f_1 + if_2. \end{aligned}$$

For fixed x,

$$dz = \frac{\partial f}{\partial y}\, dy,$$

so taking conjugates gives

$$d\bar{z} = \frac{\partial \bar{f}}{\partial \bar{y}}\, d\bar{y}.$$

Then

$$dz \wedge d\bar{z} = \left| \frac{\partial f}{\partial y} \right|^2 dy \wedge d\bar{y}.$$

Note that $dz \wedge d\bar{z} = -2i\, dz_1 \wedge dz_2$ and $dy \wedge d\bar{y} = -2i\, dy_1 \wedge dy_2$. It follows that

$$\det \frac{\partial(z_1, z_2)}{\partial(y_1, y_2)} = \left| \frac{\partial f}{\partial y} \right|^2 \neq 0$$

at (x_0, y_0). By the real implicit function theorem, we can solve $f(x, y) = 0$ for $y = y_1 + iy_2$ as a C^1 function $y = y(x)$ of $x = x_1 + ix_2$.

We have to check that y is holomorphic. Taking the differential of $f(x, y(x)) = 0$ gives

$$\frac{\partial f}{\partial x}\, dx + \frac{\partial f}{\partial y} \left(\frac{\partial y}{\partial x}\, dx + \frac{\partial y}{\partial \bar{x}}\, d\bar{x} \right) = 0.$$

So $\frac{\partial y}{\partial \bar{x}} = 0$, i.e. $y = y(x)$ is holomorphic. □