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1 Infinite Products of Functions (2-24-2021)

Having defined the notion of an infinite product of complex numbers, we'd like to now define the notion of an infinite product of complex functions.

1.1 Infinite Products of Functions

First, we should discuss the notions of absolute and uniform convergence of infinite products. We say that an infinite product $\prod_{n=1}^{\infty}(1+a_n)$ *converges absolutely* if $\sum_{n=1}^{\infty}\log(1+a_n)$ converges absolutely. This is equivalent to absolute convergence of $\sum_{n=1}^{\infty} a_n$. Why? Recall that

$$\lim_{z\to 0}\frac{\log(1+z)}{z}=1,$$

so for any $\varepsilon>0$,

$$\left|\frac{|\log(1+a_n)|}{|a_n|}-1\right|<\varepsilon$$

for large enough n, i.e.

$$(1-\varepsilon)|a_n|<|\log(1+a_n)|<(1+\varepsilon)|a_n|.$$

Now, consider an infinite product $\prod_{n=1}^{\infty} f_n(z)$, where the f_n's are continuous and complex-valued functions on an open set $\Omega\subseteq\mathbb{C}$. We say that the infinite product *converges uniformly and absolutely* on a subset $K\subseteq\Omega$ if

(1) $f_n(z)\to 1$ uniformly on K, and

(2) $\sum\log f_n$ is uniformly and absolutely convergent on K.

Note that condition (1) ensures that the principal branch of $\log f_n$ is defined when n is large enough, so that condition (2) makes sense. If $\prod f_n$ conveges uniformly and absolutely on compact subsets of Ω, then the partial products converge uniformly on compact subsets to a limit function $f(z)$, which is therefore continuous. We're mainly interested in the case when these functions are holomorphic.

Theorem 1.1. *Suppose that the f_n's are holomorphic in Ω, and that $\prod f_n$ converges uniformly and absolutely on compact subsets of Ω. Then*

(1) $f=\prod f_n$ is holomorphic in Ω, and for any p, we have $f=f_1\cdots f_p\prod_{n>p} f_n$.

(2) The set of zeroes of f is the union of the zero sets of all of the f_n's. Moreover, the multiplicity of a zero of f is the sum of the multiplicities for each f_n.

(3) *The series* $\sum f_n'/f_n$ *converges uniformly and absolutely on compact subsets of* Ω, *and its sum is* f'/f.

Proof. The proofs of (1) and (2) are things we've essentially seen before, so we'll only prove (3). Consider a relatively compact set $U \subseteq \Omega$, and write $f = f_1 \cdots f_p \cdot g_p$, where $g_p = \prod_{n>p} f_n$. Then

$$\frac{f'}{f} = \sum_{n=1}^{p} \frac{f_n'}{f_n} + \frac{g_p'}{g_p},$$

where $g_p = \exp(\sum_{n>p} \log f_n)$ (well-defined in U when p is large enough). We have

$$\frac{g_p'}{g_p} = \sum_{n>p} \frac{f_n'}{f_n},$$

since $\sum_{n>p} \log f_n$ converges uniformly and absolutely on compact subsets of U to a branch of $\log g_p$. Therefore $f'/f = \sum_{n=1}^{\infty} f_n'/f$ converges uniformly and absolutely on compact subsets of U, hence on compact subsets of Ω. $\square$

When can we express a function as an infinite product? Let's try to write $\sin \pi z$ as an infinite product. Since $\sin \pi z$ has zeroes at exactly the integers, all simple, we should write down an infinite product with simple zeroes exactly at the integers; for example

$$f(z) = z \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right).$$

This converges uniformly and absolutely on compact subsets of $\mathbb{C}$ by comparison with $\sum 1/n^2$, implying that $f(z)$ is holomorphic and has zeroes precisely at the integers, all simple. Differentiating logarithmically term-by-term, we have

$$\frac{f'(z)}{f(z)} = \frac{1}{z} + \sum_{n=1}^{\infty} \frac{2z}{z^2 - n^2} = \pi \cot \pi z = \frac{g'(z)}{g(z)},$$

where $g(z) = \sin \pi z$. So $f(z) = Cg(z)$, since $(f/g)' = 0$. What is the constant? As $z \to 0$, $f(z)/z \to 1$ and $(\sin \pi z)/z \to \pi$, so $C = 1/\pi$. Therefore

$$\frac{\sin \pi z}{\pi z} = \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right).$$

1.2 Holomorphic Functions with Prescribed Zeroes

Any entire function that is *never* zero has the form $f(z) = e^{g(z)}$. Why? f is never zero, so f'/f is holomorphic in $\mathbb{C}$, and since $\mathbb{C}$ is simply connected, it's the derivative of an entire function $g(z)$. Then

$$\frac{d}{dz}\left(\frac{f(z)}{e^{g(z)}}\right) = \frac{f'(z)e^{g(z)} - f(z)e^{g(z)}g'(z)}{e^{2g(z)}} = \frac{f'(z)e^{g(z)} - f'(z)e^{g(z)}}{e^{2g(z)}} = 0,$$

so after absorbing the constant into $g(z)$ we have $f(z) = e^{g(z)}$.

Now, what's the most general entire function $f(z)$ with finitely many zeroes? Let's say that 0 is a zero of multiplicity $m \geq 0$, and let $a_1, \ldots, a_n$ be the non-zero zeroes of f, repeated according to multiplicity. Then

$$f(z) = z^m e^{g(z)} \prod_{k=1}^{n} \left(1 - \frac{z}{a_k}\right)$$

for some entire function $g(z)$. (Divide $f(z)$ by the non-exponential factors on the right-hand side to get an entire function with no zeroes, and then apply the previous considerations.)

We'd like to play the same game but for entire functions with *infinitely* many zeroes. We have to take the same care that we did for poles in the theorem of Mittag-Leffler and multiply by "convergence factors" that make the obvious infinite product converge. We'll do this next time.