

MAT454/1002 lecture notes by Kain Dineen. Taught by Edward Bierstone.

## 1 The Weierstrass $\wp$ -function (2-3-2021)

### 1.1 Another Example

Let us consider

$$\frac{1}{z} + \sum_{n \neq 0} \left( \frac{1}{z-n} + \frac{1}{n} \right).$$

This series is uniformly and absolutely convergent on compact subsets of  $\mathbb{C}$ , for we may write each term as

$$\frac{z}{n(z-n)},$$

which is comparable to  $1/n^2$  for  $z$  in a compact set. It follows that the sum,  $f(z)$ , is a meromorphic function in  $\mathbb{C}$ . The poles are precisely the integers, and they are all simple poles of residue 1. The series can be differentiated term-by-term:

$$f'(z) = -\frac{1}{z^2} - \sum_{n \neq 0} \frac{1}{(z-n)^2} = -\frac{\pi^2}{\sin^2 \pi z} = \frac{d}{dz}(\pi \cot \pi z).$$

It follows that  $f(z) - \pi \cot \pi z$  is a constant; the constant is zero since  $f(z)$  and  $\pi \cot \pi z$  are both odd functions. Grouping the positive and negative terms, we obtain

$$\frac{1}{z} + \sum_{n=1}^{\infty} \frac{2z}{z^2 - n^2} = \pi \cot \pi z.$$

In particular, this function is periodic.

### 1.2 The Weierstrass $\wp$ -function

Let  $e_1, e_2 \in \mathbb{C}$  be two complex numbers which are linearly independent over the reals. We want to look at a *doubly periodic* function over the lattice generated by  $e_1$  and  $e_2$ . The set

$$\Gamma = \{n_1 e_1 + n_2 e_2 : n_1, n_2 \in \mathbb{Z}\}$$

is a discrete subgroup (lattice) of  $\mathbb{C}$ . We will say that  $f(z)$  has  $\Gamma$  as a group of periods (or is *doubly periodic* for short) if  $f(z + n_1 e_1 + n_2 e_2) = f(z)$ , for all  $n_1, n_2 \in \mathbb{Z}$ , i.e.

$$\begin{aligned} f(z + e_1) &= f(z), \\ f(z + e_2) &= f(z), \end{aligned}$$

for all  $z$ . We call  $e_1, e_2$  a *basis* of  $\Gamma$ .

Given a point  $z_0 \in \mathbb{C}$ , we will call the parallelogram with vertices  $z_0, z_0 + e_1, z_0 + e_2, z_0 + e_1 + e_2$  the *period parallelogram with first vertex  $z_0$* .

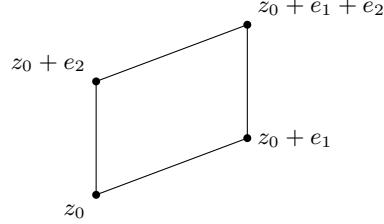


Figure 1: A period parallelogram with first vertex  $z_0$  and vertices  $z_0, z_0 + e_1, z_0 + e_2$ , and  $z_0 + e_1 + e_2$ .

Note that  $e'_1, e'_2$  is a basis of  $\Gamma$  if and only if  $e'_1, e'_2$  is a linear combination of  $e_1, e_2$  with integer coefficients, and the determinant of the matrix of coefficients is  $\pm 1$  (i.e. a unit in  $\mathbb{Z}$ ).

We define<sup>1</sup> the *Weierstrass  $\wp$ -function* by

$$\wp(z) := \frac{1}{z^2} + \sum_{\omega \in \Gamma \setminus \{0\}} \left( \frac{1}{(z - \omega)^2} - \frac{1}{\omega^2} \right).$$

We're going to see that this series is uniformly and absolutely convergent on compact subsets of  $\mathbb{C}$ .

**Lemma 1.1.** *The series*

$$\sum_{\omega \in \Gamma \setminus \{0\}} \frac{1}{|\omega|^3}$$

*converges.*

*Proof.* Think of the lattice as an expanding series of parallelograms. For  $n \geq 1$ , let

$$P_n = \{t_1 e_1 + t_2 e_2 : t_1, t_2 \in \mathbb{Z}, \max\{|t_1|, |t_2|\} = n\}.$$

There are  $8n$  points on  $P_n$ , each of distance at least  $kn$  from 0, where  $k$  is the shortest distance from the origin to  $P_1$ . We have

$$\sum_{\omega \in P_n} \frac{1}{|\omega|^3} \leq \frac{8n}{k^3 n^3} = \frac{8}{k^3 n^2},$$

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<sup>1</sup>Why do we add the  $-1/\omega^2$  term? To quote Professor Bierstone, it's "a device to make the series converge." We'll see that this term is required to make the series' terms comparable to  $1/\omega^3$ , when  $z$  lies in a compact set. Compare with the example from the start of this class.

so

$$\sum_{\omega \in \Gamma \setminus \{0\}} \frac{1}{|\omega|^3} \leq \sum_{n=1}^{\infty} \frac{8}{k^3 n^2} < \infty.$$

□

To show that the series defining  $\wp$  converges uniformly and absolutely on compact subsets of  $\mathbb{C}$ , it's enough to check it on disks  $|z| \leq r$ , for any  $r$ . For  $|z| \leq r$  and  $|\omega| \geq 2r$ ,

$$\left| \frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} \right| = \left| \frac{2\omega z - z^2}{\omega^2(z-\omega)^2} \right| = \frac{|z||2 - \frac{z}{\omega}|}{|\omega|^3 |1 - \frac{z}{\omega}|^2} \leq \frac{r \cdot \frac{5}{2}}{|\omega|^3 \cdot \frac{1}{4}} = \frac{10r}{|\omega|^3}.$$

So the series converges uniformly and absolutely in  $|z| \leq r$  by comparison with  $\sum 1/|\omega|^3$ . It follows that the Weierstrass  $\wp$ -function is well-defined as a meromorphic function on  $\mathbb{C}$ .

Its poles are just the points of the lattice  $\Gamma$ , each of multiplicity 2, with principal part  $(z-\omega)^{-2}$  to the pole  $\omega \in \Gamma$ . Moreover,  $\wp$  is an even function. We would like to see that it is doubly periodic.

$$\wp'(z) = -2 \sum_{\omega \in \Gamma} \frac{1}{(z-\omega)^3},$$

which is uniformly and absolutely convergent on compact subsets of  $\mathbb{C}$ , and again periodic:  $\wp'(z+\omega) = \wp'(z)$  for any  $\omega \in \Gamma$ . It's also an odd function. We want to check that

$$\wp(z+e_i) = \wp(z), \quad i = 1, 2,$$

for all  $z$ . By periodicity of  $\wp'$ ,  $\wp(z+e_i) - \wp(z)$  is a constant. Setting  $z = -e_i/2$ , we see that the constant is

$$\text{const.} = \wp(e_i/2) - \wp(-e_i/2) = 0,$$

since  $\wp$  is even. So  $\wp$  is doubly periodic with group of periods  $\Gamma$ .

What is the Laurent expansion of  $\wp(z)$  at 0?

$$\wp(z) = \frac{1}{z^2} + a_2 z^2 + a_4 z^4 + \dots,$$

because  $\wp$  is even, and  $\wp(z) - z^{-2}$  vanishes at 0. The coefficients are sums of things of the form

$$\frac{1}{(2k)!} \cdot \left( \frac{(2k+1)!}{(z-\omega)^{2k+2}} \right) \Big|_{z=0},$$

i.e.  $(2k!)^{-1}$  times the  $2k$ th derivative of  $(z-\omega)^{-2}$ . This leads to

$$a_{2k} = (2k+1) \sum_{\omega \neq 0} \frac{1}{\omega^{2(k+1)}}.$$

In particular,

$$a_2 = 3 \sum_{\omega \neq 0} \frac{1}{\omega^4}$$

and

$$a_4 = 5 \sum_{\omega \neq 0} \frac{1}{\omega^6},$$

and so on.