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0.1 Harmonic Functions (1-22-2021)

Armed with the mean value property, we take a deeper look at harmonic functions. We show that harmonic functions satisfy three equivalent definitions: those functions whose Laplacians vanish, those functions expressible locally as the real parts of holomorphic functions, and those continuous functions with the mean value property.

0.1.1 The Mean Value Property

Before proceeding, let us make a definition that will be useful to us.

Definition 0.1.1. *Let f be a continuous real-or-complex valued function defined on an open set Ω in $\mathbb{C}$. We say that f has the* mean value property *if for every $a \in \Omega$, there is an $R > 0$ such that $\{|z-a| \leq R\} \subset \Omega$, and for $0 < r < R$,*

$$f(a) = \frac{1}{2\pi} \int_0^{2\pi} f(a + re^{i\theta})\, d\theta.$$

We saw last time that holomorphic functions enjoy the mean value property. By splitting f up into its real and imaginary parts, we see that f has the mean value property if and only if $\mathrm{Re}(f)$ and $\mathrm{Im}(f)$ both have the mean value property.

Theorem 0.1.1. *(Maximum modulus principle) Let f be a complex-valued function on an open subset $\Omega \subset \mathbb{C}$ with the mean value property. If $|f|$ has a local maximum at a point $a \in \Omega$, then f is constant in a neighbourhood of a.*

Proof. Write $f = u + iv$. Since $|f|$ has a local maximum at a, we can find an $R > 0$ such that $D = \{|z-a| \leq R\} \subset \Omega$ and $|f(z)| \leq |f(a)|$ for $z \in D$; furthermore, by choosing R small enough, we can assume that $f(z)$ equals its mean value along the boundary of D. Clearly u is maximized at a in D, so since u also has the mean value property,

$$u(a) = \frac{1}{2\pi} \int_0^{2\pi} u(a + Re^{i\theta})\, d\theta \leq \max_{|z-a| \leq R} u(z) \leq u(a).$$

It follows that $u(z) = u(a)$ for $z \in D$. Similarly, v has the mean value property, so it follows that $f(z) = f(a)$ for $z \in D$. $\square$

The following result is a consequence of the maximum modulus principle. It will be very important to us later in the course. (See Ahlfors, pp. 135, Theorem 13.)

Theorem 0.1.2. *(Schwarz's lemma) Let $f\colon D \to \mathbb{C}$ be a holomorphic function on the open unit disk $D \subset \mathbb{C}$ such that $f(0) = 0$, $|f(z)| \leq 1$ on D, and $|f'(0)| \leq 1$. Then $|f(z)| \leq |z|$ for all $z \in D$. Moreover, if $|f(z_0)| = |z_0|$ for some non-zero $z_0 \in D$, or if $|f'(0)| = 1$, then there is an $a \in \mathbb{C}$, $|a| = 1$, such that $f(z) = az$ for all $z \in D$.*

Proof. Define $g\colon D \to \mathbb{C}$ by

$$g(z) = \begin{cases} \frac{f(z)}{z}, & z \neq 0, \\ f'(0), & z = 0. \end{cases}$$

Then g is holomorphic on D since f is holomorphic and $f(0) = 0$. If $0 < R < 1$, then the maximum modulus principle applied on the disk $\{|z| \leq R\}$ gives

$$|g(z)| \leq \max_{|z|=R} |g(z)| = \max_{|z|=R} \left| \frac{f(z)}{z} \right| \leq \frac{1}{R},$$

so if we let R approach 1 from below, we get $|g(z)| \leq 1$, or $|f(z)| \leq |z|$, for each $z \in D$. Furthermore, if for some non-zero $z_0 \in D$ one has $|f(z_0)| = |z_0|$, then $|g(z_0)| = 1$, so by the maximum modulus principle, g is constant. But if g is constant, it follows that there is an a, $|a| = 1$, such that $f(z) = az$ for all $z \in D$. □

0.1.2 Harmonic Functions

The real and imaginary parts of a function with the mean value property also have the mean value property. We'll see that the functions with the mean value property are precisely the harmonic functions. The following proposition immediately implies one direction: harmonic functions have the mean value property, and in particular, satisfy the maximum modulus principle.

Proposition 0.1.1. *A real-valued harmonic function $g(x, y)$ is locally the real part of a holomorphic function, uniquely determined up to the addition of a constant.*

Proof. We have

$$\frac{\partial^2 g}{\partial z \partial \bar{z}} = 0,$$

so we know $\frac{\partial g}{\partial z}$ is holomorphic. It therefore has a local primitive f uniquely determined up to an additive constant:

$$df = \frac{\partial g}{\partial z} \, dz.$$

We can write this as

$$d\bar{f} = \frac{\partial g}{\partial \bar{z}} \, d\bar{z}.$$

Since g is real-valued, $\frac{\partial g}{\partial \bar{z}}$ is the conjugate of $\frac{\partial g}{\partial z}$. Therefore

$$d(f + \bar{f}) = dg,$$

which proves that $g = 2 \cdot \mathrm{Re}(f) + (\text{real const.})$. □

Now, let $g(x,y)$ be a real-valued harmonic function, equal to the real part of some holomorphic function $f(z) = \sum a_n z^n$, converging in some open disk of radius R. Consider inside this disk a smaller one of radius $r < R$. We can assume $a_0 \in \mathbb{R}$. On the boundary of the smaller disk,

$$g(r\cos\theta, r\sin\theta) = \mathrm{Re}(f(re^{i\theta})) = a_0 + \frac{1}{2}\sum_{n=1}^{\infty} r^n \left(a_n e^{in\theta} + \bar{a}_n e^{-in\theta}\right),$$

which is uniformly and absolutely convergent with respect to $\theta \in [0, 2\pi]$. Let's compute the constants.

$$a_0 = \frac{1}{2\pi}\int_0^{2\pi} g(r\cos\theta, r\sin\theta)\, d\theta,$$

and

$$a_n = \frac{1}{\pi}\int_0^{2\pi} \frac{g(r\cos\theta, r\sin\theta)}{(re^{i\theta})^n}\, d\theta.$$

We get, for $|z| < r$,

$$f(z) = \frac{1}{2\pi}\int_0^{2\pi} g(r\cos\theta, r\sin\theta) \underbrace{\left\{1 + 2\sum_{n=1}^{\infty}\left(\frac{z}{re^{i\theta}}\right)^n\right\}}_{=\frac{re^{i\theta}+z}{re^{i\theta}-z}} d\theta$$

This expresses the holomorphic function $f(z)$ in $|z| < r$ in terms of its real part on the boundary. Equating real parts of both sides of the above equation,

$$g(x,y) = \frac{1}{2\pi}\int_0^{2\pi} g(r\cos\theta, r\sin\theta) \cdot \frac{r^2 - |z|^2}{|re^{i\theta} - z|^2}\, d\theta.$$

This formula is valid in $|z| < r$, for any real-valued harmonic function in a neighbourhood of $\{|z| \le r\}$. We call the term

$$\frac{r^2 - |z|^2}{|re^{i\theta} - z|^2}$$

the *Poisson kernel*. Thus, a harmonic function is determined by its values on the boundary of the disk.

0.1.3 Dirichlet Problem for a Disk

We are now ready to study the converse problem. Given a continuous function $f(\theta)$ on the circle with center 0 and radius r, can we find $F(z)$ continuous in $|z| \le r$ and harmonic in $|z| < r$, such that $F(re^{i\theta}) = f(\theta)$?

Theorem 0.1.3. *We can, and the solution is unique.*

Proof. We can assume that the functions are real-valued. Uniqueness follows from the maximum modulus principle as follows: if F_1, F_2 are two solutions, then $F_1 - F_2$ is a harmonic function in $|z| < r$, continuous in $|z| \leq r$, taking its maximum and minimum values on the boundary. However, it is identically zero on the boundary, so it holds that $F_1 \equiv F_2$.

All that's left is existence. For $|z| < r$, define $F(z)$ by

$$F(z) = \frac{1}{2\pi} \int_0^{2\pi} f(\theta) \cdot \frac{r^2 - |z|^2}{|re^{i\theta} - z|^2} \, d\theta.$$

The Poisson kernel in the above expression is just the real part of $(re^{i\theta} + z)/(re^{i\theta} - z)$, so $F(z)$ is the real part of a holomorphic fuction (differentiate under the integral). And we already saw that real parts of holomorphic functions are harmonic. Now, we need only check that

$$\lim_{z \to re^{i\theta_0}} F(z) = f(\theta_0);$$

this is a direct calculation and we will not do it in class. $\square$

Corollary 0.1.1. *Any continuous function in an open subset $\Omega \subseteq \mathbb{R}^2$ with the mean value property is harmonic.*

Proof. Pick $a \in \Omega$ and $r > 0$ so that $D = \{|z - a| \leq r\} \subset \Omega$. Then $f|_{\partial D}$ extends to a harmonic function in the interior. Then $f - F$ is zero on ∂D and has the mean value property, so by the maximum modulus principle it's zero on D. $\square$