MAT454/1002 lecture notes by Kain Dineen. Taught by Edward Bierstone.

0.1 Projective Space (2-10-2021)

We detour into a brief study of higher-dimensional complex projective space for the sake of understanding the geometric properties of the elliptic curve in $\mathbb{C}^2$ given by the differential equation of the $\wp$-function.

0.1.1 Higher-Dimensional Complex Projective Space

Consider, as before, the curve $X = \{y^2 = 4x^3 - 20a_2x - 28a_4\}$. The curve sits inside $\mathbb{C}^2$ via the inclusion $X \hookrightarrow \mathbb{C}^2$. If we project $\mathbb{C}^2$ onto $\mathbb{C}$ via $(x, y) \mapsto x$, we obtain the commutative diagram

$$\begin{tikzcd} & \mathbb{C}^2 \arrow[d] & (x,y) \arrow[d, mapsto] \\ X \arrow[ur, hook] \arrow[r, "\varphi"] & \mathbb{C} & x \end{tikzcd}$$

We call this a *Riemann surface*[1] over $\mathbb{C}$. Note that, except for the three roots of the right-hand side, each $x \in \mathbb{C}$ corresponds to two points of X. Geometrically speaking, the curve X lies over $\mathbb{C}$ in two sheets that come together over these three points, the roots of the right-hand side (*branch points*). We will return to the notion of a Riemann surface later in the course.

We can compactify $\mathbb{C}$ to get the Riemann sphere S^2. Can we compactify X to get a Riemann surface over $S^2 = P^1(\mathbb{C})$? Define $P^n(\mathbb{C})$ to be the set of complex lines through the origin 0 in $\mathbb{C}^{n+1}$. That is,

$$P^n(\mathbb{C}) = (\mathbb{C}^{n+1} \setminus \{0\})/\sim,$$

where $(x_0', \ldots, x_n') \sim (x_0, \ldots, x_n)$ if there is a $\lambda \in \mathbb{C} \setminus \{0\}$ such that $(x_0', \ldots, x_n') = \lambda(x_0, \ldots, x_n)$. We write $[x_0, \ldots, x_n]$ for the equivalence class of $(x_0, \ldots, x_n)$; these coordinates are called *homgeneous coordinates.* Our coordinate charts are

$$U_i = \{[x_0, \ldots, x_n] \in P^n(\mathbb{C}) : x_i \neq 0\}, \qquad i = 0, \ldots, n.$$

For each i, there is a homeomorphism $U_i \to \mathbb{C}^n$ given by

$$[x_0, \ldots, x_n] \mapsto \left(\frac{x_0}{x_i}, \ldots, \widehat{\frac{x_i}{x_i}}, \ldots, \frac{x_n}{x_i}\right),$$

[1]To be precise with the terminology, the Riemann surface is actually X *along with* φ, i.e. the pair (X, φ).

where the hat denotes omission. The inverse is given by

$$(z_1, \dots, z_n) \mapsto [z_1, \dots, z_i, 1, z_{i+1}, \dots, z_n].$$

This gives $P^n(\mathbb{C})$ the structure of a complex n-manifold, i.e. the transition mappings are holomorphic. (Actually, in this case, the transition mappings are rational; as an exercise, write them down.) Note that $P^n(\mathbb{C})$ is compact.

We may think of $P^n(\mathbb{C})$ as $U_0 \cong \mathbb{C}^n$, together with the points not in U_0,

$$\{[x_0, \dots, x_n] \in P^n(\mathbb{C}) : x_0 = 0\};$$

we call this latter set the *points at infinity* or the *hyperplane at infinity*. (That is, we compactified $\mathbb{C}^n$ by adding some points at infinity.) This hyperplane at infinity is (isomorphic to) $P^{n-1}(\mathbb{C})$. For example, $P^1(\mathbb{C}) \cong S^2$, the Riemann sphere.

For $P^2(\mathbb{C})$, we're going to consider $\mathbb{C}^2$ sitting inside of $P^2(\mathbb{C})$ as U_3. Write the coordinates as $[x, y, t]$. Now, $\mathbb{C}^2$ is isomorphic to the set $\{[x, y, t] : t \neq 0\}$, so we can think of $P^2(\mathbb{C})$ as $\mathbb{C}^2$ together with the points at infinity, $P^1(\mathbb{C})$. Recall that our curve X sits inside of $\mathbb{C}^2$. What is the closure of $X \subset \mathbb{C}^2$ in $P^2(\mathbb{C})$? We add a third variable t and *homogenize*: X' is given by

$$y^2 t = 4x^3 - 20a_2 x t^2 - 28a_4 t^3 \tag{*}$$

in $P^2(\mathbb{C})$. Why? The homeomorphism $U_3 \to \mathbb{C}^2$ is given by $[x, y, t] \mapsto (x/t, y/t)$, so under this homeomorphism, the equation of the curve X becomes

$$\left(\frac{y}{t}\right)^2 = 4\left(\frac{x}{t}\right)^2 - 20a_2\left(\frac{x}{t}\right) - 28a_4;$$

clearing denominators gives the homogeneous equation for $X' \subset P^2(\mathbb{C})$. That is,

$$X' = \{[x, y, t] \in P^2(\mathbb{C}) : y^2 t = 4x^3 - 20a_2 x t^2 - 28a_4 t^3\}.$$

X' is a compactification of X; X' consists of X together with the points at infinity, i.e. the points of X' where $t = 0$. Setting $t = 0$ in (*), we obtain $4x^3 = 0$, leaving just the single point at infinity, $[0, 1, 0]$.

We would like to learn what X' looks like near this point at infinity. In particular, we would like to know whether or not X' is smooth, even near the point at infinity. We will do this next time.