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1 More on Elliptic Functions (2-5-2021)

1.1 The Differential Equation

Let Γ be some discrete subgroup of $\mathbb{C}$ generated by $e_1, e_2 \in \mathbb{C}$, linearly independent over $\mathbb{R}$. We're going to use the Laurent series expansion of the Weierstrass $\wp$-function to see that it satisfies a certain differential equation. The Laurent series expansion of $\wp(z)$ at 0 is

$$\wp(z) = \frac{1}{z^2} + a_2 z^2 + a_4 z^4 + \cdots ,$$

where we found the coefficients last time. Differentiating,

$$\wp'(z) = -\frac{2}{z^3} + 2a_2 z + 4a_4 z^3 + \cdots .$$

If we square both sides, we obtain

$$\wp'(z)^2 = \frac{4}{z^6} - \frac{8a_2}{z^2} - 16a_4 + \cdots .$$

On the other hand,

$$\wp(z)^3 = \frac{1}{z^6} + \frac{3a_2}{z^2} + 3a_4 + \cdots .$$

Then

$$\wp'(z)^2 - 4\wp(z)^3 = -\frac{20a_2}{z^2} - 28a_4 + z^2(\cdots).$$

To eliminate the $1/z^2$ term, we add $20a_2\wp(z)$:

$$\wp'(z)^2 - 4\wp(z)^2 + 20a_2\wp(z) + 28a_4 = z^2(\cdots).$$

It follows that

$$(\wp')^2 - 4\wp^3 + 20a_2\wp + 28a_4$$

is holomorphic near zero, 0 at 0. However, it is periodic with group of periods Γ, so it's holomorphic near all points of Γ. So it's holomorphic in $\mathbb{C}$. By periodicity it's bounded, so by Liouville's theorem it's constant. The constant is zero since it vanishes at the origin. Therefore $\wp$ satisfies the differential equation

$$(\wp')^2 = 4\wp^3 - 20a_2\wp - 28a_4.$$

That is, $x = \wp(z)$ and $y = \wp'(z)$ give a parametrization of the algebraic curve

$$y^2 = 4x^3 - 20a_2 x - 28a_4.$$

We'll see later that any point (x, y) of this curve is the image of a point $z \in \mathbb{C}$, uniquely determined up to addition of an element of Γ. Analogously to how sin and cos parametrize quadratic curves, the Weierstrass $\wp$-function parametrizes cubic curves.

1.2 Doubly Periodic Functions

We require some results about doubly periodic functions before we further study the $\wp$-function. Let Γ be as in the previous part.

Proposition 1.1. *Let f be a non-constant meromorphic function on $\mathbb{C}$ with Γ as its group of periods. Then, provided f has no zeroes or poles on the boundary, the number of zeroes of f in a period parallelogram is equal to the number of poles in the same parallelogram, each counted with multiplicity.*

Proof. By the argument principle, if γ is the boundary of the period parallelogram,

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}\,dz = \#\text{ zeroes} - \#\text{ poles},$$

counted with multiplicity. The left-hand side vanishes by periodicity. $\square$

Proposition 1.2. *Let f be a non-constant meromorphic function in $\mathbb{C}$ with Γ as its group of periods. For a fixed $a \in \mathbb{C}$, let α_i be the roots of $f(z) = a$, and let β_i be the poles of $f(z)$, each counted with multiplicity, within a period parallelogram. Then $\sum \alpha_i$ is congruent to $\sum \beta_i$, modulo Γ. (In particular, $\sum \alpha_i$ mod Γ is independent of a.)*

Proof. By the residue theorem, if γ is the boundary of the period parallelogram,

$$\frac{1}{2\pi i}\int_\gamma \frac{zf'(z)}{f(z)-a}\,dz = \text{sum of residues of } \frac{zf'(z)}{f(z)-a}. \tag{*}$$

At a root $z = \alpha_i$ of multiplicity k,

$$\begin{aligned} z &= \alpha_i + (z-\alpha_i), \\ f(z) - a &= c(z-\alpha_i)^k + \text{higher order} \\ f'(z) &= kc(z-\alpha_i)^{k-1} + \cdots, \end{aligned}$$

so it follows that

$$\frac{zf'(z)}{f(z)-a} = \frac{k\alpha_i}{z-\alpha_i} + \text{ higher order},$$

so the residue is $k\alpha_i$. Similarly, at a pole β_i, the residue is $-k\beta_i$. It follows that the right-hand side of (*) is simply $\sum \alpha_i - \sum \beta_i$, which we want. Unlike in the proof of the previous proposition, the integrand in the left-hand side of (*) is not periodic. However, the left-hand side is

$$-\frac{e_2}{2\pi i}\int_{\gamma_1} \frac{f'(z)}{f(z)-a}\,dz + \frac{e_1}{2\pi i}\int_{\gamma_2} \frac{f'(z)}{f(z)-a}\,dz,$$

where γ_1, γ_2 are certain sides of the period parallelogram. The integrals

$$\frac{1}{2\pi i}\int_{\gamma_1} \frac{f'(z)}{f(z)-a}\,dz, \int_{\gamma_2} \frac{f'(z)}{f(z)-a}\,dz$$

are integers, because they each equal to the difference between two determinations of $\log(f(z_0)-a)$. $\square$