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1 Cauchy's Theorem (1-15-2021)

1.1 Closed Forms

We continue with the setting of last time. Let $\omega = P\,dx + Q\,dy$ be a differential form on an open, connected set $\Omega \subseteq \mathbb{R}^2$, P and Q continuous.

Definition 1.1. *We say that a form ω is* closed *if any point has a neighbourhood in which ω has a primitive.*

A closed form need not have a global primitive. Take, for example, take $\Omega = \mathbb{C} \setminus \{0\}$ and $\omega = z^{-1}\,dz$. This is closed because local primitives are given by branches of log. It does not admit a global primitive, since its integral over the unit circle is $2\pi i \neq 0$.

Since we can test for the existence of local primitives of ω by looking at the integral of ω along rectangles, we can say that ω is closed, if and only if $\int_\gamma \omega = 0$ whenever γ is the boundary of a small rectangle in Ω.

This is *not* equivalent to the condition that $d\omega = 0$, since we are assuming merely continuity of P,Q. However, *if* ω is C^1, then ω is closed, if and only if $d\omega = 0$ (i.e. $\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$, by Green's formula, which we now recall).

Let γ be the boundary of a rectangle A in Ω, positively oriented. Then

$$\int_\gamma P\,dx + Q\,dy = \iint_A \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dxdy;$$

this formula holds whenever the statement makes sense (i.e. when these derivatives are continuous).

1.2 Cauchy's Theorem

What follows is one version of Cauchy's theorem, from which we will deduce Cauchy's integral formula later. Let Ω be any open set in $\mathbb{C}$.

Theorem 1.1. *If $f(z)$ is holomorphic in Ω, then the differential form $f(z)\,dz$ is closed.*

If we assume that $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ are continuous, this follows from Green's theorem and the Cauchy-Riemann equations. (Take d of $f(z)\,dz$.) This statement of continuity is actually true, but we are going to use Cauchy's theorem to prove it.

Proof. It's enough to show that the integral $\int_\gamma f(z)\,dz = 0$ for any γ which is the boundary of a rectangle $R \subset \Omega$ whose interior is contained in Ω. Divide R into four equal subrectangles $R_i, i = 1, 2, 3, 4$, each with boundary γ_i. Then,

$$\mu(R) = \int_\gamma f(z)\,dz = \sum_{i=1}^{4} \int_{\gamma_i} f(z)\,dz = \sum_{i=1}^{4} \mu(R_i),$$

since the subrectangles have common edges inside R with opposing orientations. Thus, $|\mu(R_i)| \geq \frac{1}{4}|\mu(R)|$ for some i, and call $R_i = R^{(1)}$, $\gamma_i = \gamma^{(1)}$.

Repeat this process to obtain a decreasing sequence $R \supset R^{(1)} \supset R^{(2)} \supset \cdots$, with

$$\left|\int_{\gamma^{(k)}} f(z)\,dz\right| \geq \frac{1}{4^k}|\mu(R)|.$$

Let z_0 be the single point in the intersection of all of the $R^{(k)}$'s. Since $f(z)$ is holomorphic at z_0,

$$\int_{\gamma^{(k)}} f(z)\,dz = f(z_0)\int_{\gamma^{(k)}} dz + f'(z_0)\int_{\gamma^{(k)}} (z - z_0)\,dz + \int_{\gamma^{(k)}} \varphi(z)(z - z_0)\,dz,$$

where $\lim_{z\to z_0}\varphi(z) = 0$. The first two integrals vanish because they are integrals of forms with (local) primitives. Thus, we need only evaluate the last. Given $\varepsilon > 0$, if k is sufficiently large, then the absolute value of the last integral is

$$\begin{aligned}\left|\int_{\gamma^{(k)}} \varphi(z)(z - z_0)\,dz\right| &\leq \varepsilon \cdot \operatorname{diag}(R^{(k)})\operatorname{perim}(R^{(k)}) \\ &= \frac{\varepsilon}{4^k}\cdot \operatorname{diag}(R)\operatorname{perim}(R).\end{aligned}$$

Now,

$$|\mu(R)| \leq 4^k\left|\int_{\gamma^{(k)}} \varphi(z)(z - z_0)\,dz\right| \leq \varepsilon \cdot \operatorname{diag}(R)\operatorname{perim}(R),$$

so since $\varepsilon > 0$ is arbitrary, $\mu(R) = 0$. □

As noted by someone in class, the estimates done in the proof when we used the fact that $f(z)$ is holomorphic may break down in the case that f is merely a smooth function, or real-analytic function, of two variables. (Check this.)

Corollary 1.1. *A holomorphic function $f(z)$ locally has a primitive, which is itself holomorphic.*

Proof. Locally,

$$f(z)\,dz = dF = \frac{\partial F}{\partial z}\,dz + \frac{\partial F}{\partial \bar{z}}\,d\bar{z},$$

so $\frac{\partial F}{\partial \bar{z}} = 0$. That is, the Cauchy-Riemann equations hold for F. Since F is also differentiable as a function of two variables, it is holomorphic. □