0.1 The Riemann Mapping Theorem (3-10-2021)

We state and prove the Riemann mapping theorem as an application of the theory of normal families of functions.

0.1.1 The Riemann Mapping Theorem

The Riemann mapping theorem is as follows.

Theorem 0.1.1. *Any simply-connected open subset Ω of $\mathbb{C}$, except for $\mathbb{C}$ itself, is biholomorphic to the open unit disk $D = \{|z| < 1\}$.*

To prove the theorem, we're going to find the biholomorphism by solving a sort of extremum problem. This will be done using normal families. The proof proceeds in three steps, which we state as a series of lemmas.

Lemma 0.1.1. *There is a biholomorphism of Ω onto a bounded open subset of $\mathbb{C}$.*

Proof. Suppose $a \notin \Omega$. By simple-connectedness, there is a $g \in \mathcal{H}(\Omega)$ such that $z - a = \exp g(z)$. In particular, g is one-to-one. Since $g(\Omega)$ is open, given any $z_0 \in \Omega$, we can find a disk $E \subseteq g(\Omega)$ centered at $g(z_0)$. Since $\exp g(z)$ is one-to-one, the disk $E + 2\pi i$ and $g(\Omega)$ are disjoint. Consider

$$\frac{1}{g(z) - (g(z_0) + 2\pi i)}.$$

This is holomorphic, one-to-one, and bounded in Ω. □

So we can assume that $0 \in \Omega$ and $\Omega \subseteq D$ by translation and scaling. Let

$$\mathcal{A} = \{f \in \mathcal{H}(\Omega) : f \text{ is one-to-one, } f(0) = 0, \text{ and } |f(z)| < 1 \text{ for } z \in \Omega\}.$$

We want to find an element of $\mathcal{A}$ with the largest possible derivative at 0.

Lemma 0.1.2. *For every $g \in \mathcal{A}$, $g(\Omega) = D$ if and only if $|g'(0)| = \sup_{f\in\mathcal{A}} |f'(0)|$.*

This lemma reduces the problem of finding a biholomorphism $\Omega \to D$ to an extremum problem; if there is a solution, then there is a biholomorphism.

Proof. For the "only if" direction, suppose $g(\Omega) = D$ for some $g \in \mathcal{A}$. Given $f \in \mathcal{A}$, we want to show that $|f'(0)| \le |g'(0)|$. Let $h = f \circ g^{-1}$, which takes D biholomorphically onto $f(\Omega) \subset D$. Then $h(0) = 0$, so $|h'(0)| \le 1$ by Schwarz's lemma. Since $f = h \circ g$, $|f'(0)| \le |g'(0)|$.

For the "if" direction, suppose that $g \in \mathcal{A}$, and that $a \in D \setminus g(\Omega)$. We have to show that there is an $f \in \mathcal{A}$ with $|g'(0)| < |f'(0)|$. Consider the fractional linear transformation

$$\varphi(\zeta) = \frac{\zeta - a}{1 - \bar{a}\zeta}$$

in $\mathrm{Aut}(D)$. Then

$$(\varphi \circ g)(z) = \frac{g(z) - a}{1 - \overline{a} g(z)}$$

is non-vanishing, so it has a single-valued holomorphic square root $F(z)$, since Ω is simply connected. That is, $g = \varphi^{-1} \circ \theta \circ F$, where θ is the squaring function. Consider the fractional linear transformation

$$\psi(\eta) = \frac{\eta - F(0)}{1 - \overline{F(0)}\eta},$$

again an automorphism of D. Write

$$g = \underbrace{(\varphi^{-1} \circ \theta \circ \psi^{-1})}_{h} \circ \underbrace{(\psi \circ F)}_{f}.$$

Then $h \colon D \to D$ takes 0 to 0, so by Schwarz's lemma, $|h'(0)| \leq 1$, and it equals 1 only if h is a rotation, i.e. a biholomorphism. h cannot be a biholomorphism since θ is not a biholomorphism, so $|h'(0)| < 1$. By the chain rule, $|g'(0)| < |f'(0)|$. $\square$

Finally, we show that the extremum problem may be solved, i.e. that there is a biholomorphism $\Omega \to D$.

Lemma 0.1.3. $\sup_{f \in \mathcal{A}} |f'(0)|$ *is attained.*

Proof. Consider the continuous mapping $\mathcal{H}(\Omega) \to \mathbb{R}$ given by $f \mapsto |f'(0)|$. Let

$$\mathcal{B} = \{f \in \mathcal{A} : |f'(0)| \geq 1\}.$$

$\mathcal{B} \neq \emptyset$ because it contains $f(z) = z$. To prove the lemma, it's enough to show that $\mathcal{B}$ is compact. It is bounded uniformly on compact sets since, by definition, $|f(z)| < 1$ for all $z \in \Omega$ and $f \in \mathcal{B}$. To check that it is closed in $\mathcal{H}(\Omega)$, consider $f \in \mathcal{H}(\Omega)$ given by

$$f = \lim_{n \to \infty} f_n, \qquad f_n \in \mathcal{B}.$$

Then $f(0) = \lim f_n(0) = 0$ and

$$|f'(0)| = \lim_{n \to \infty} |f_n'(0)| \geq 1.$$

Then f is not constant, so by Hurwitz's theorem, f is one-to-one. Since $|f_n(z)| < 1$ for $z \in \Omega$, so $|f(z)| \leq 1$; it can't be equal to 1 by the maximum modulus principle. Therefore $f \in \mathcal{B}$, showing that $\mathcal{B}$ is closed. Since $\mathcal{B}$ is normal, it is thus compact. $\square$

0.1.2 Next Steps

As a follow-up to the Riemann mapping theorem, we would like to address the following two questions:

1. Can we explicitly write down a biholomorphisms?
2. How does it behave on the boundary of the sets?

It turns out that for the open disk and for a polygonal domain, one can explicitly write down a biholomorphism, and moreover, that it extends to a homeomorphism of the sets with their boundaries included.