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0.1 Series of Meromorphic Functions (2-1-2021)

Having studied the notion of uniform convergence for sequences and series of holomorphic functions, we turn to the analogous concepts of uniform and absolute convergence for series of meromorphic functions. We then study in detail an example.

0.1.1 Series of Meromorphic Functions

Let $\{f_n\}$ be a sequence of meromorphic functions on an open subset $\Omega \subseteq \mathbb{C}$. We will say that $\sum_{n=1}^{\infty} f_n$ *converges uniformly* (and *absolutely*) on $X \subseteq \Omega$ if all but finitely many f_n's have no pole in X, and form a uniformly (or uniformly and absolutely) convergent series. We wll study series of meromorphic functions that converge uniformly on compact subsets of Ω. We can define the sum on a relatively compact open subset $U \subseteq \Omega$ as

$$\underbrace{\sum_{n \leq n_0} f_n}_{\text{meromorphic}} + \underbrace{\sum_{n > n_0} f_n}_{f_n\text{'s no pole in } \overline{U}} .$$

The second sum is uniformly convergent on $\overline{U}$, since none of the summands have poles there. This is independent of the choice of n_0.

The following theorem is the meromorphic analogue of the theorem on series of holomorphic functions from last time. The proof is similar.

Theorem 0.1.1. *Consider a series $\sum f_n$ of meromorphic functions on Ω. If the series converges uniformly on compact subsets of Ω, then the sum is a meromorphic function f on Ω, and it can be differentiated term-by-term; $\sum f_n'$ converges uniformly on compact subsets of Ω to f'.*

Since the sum f is meromorphic, its poles are isolated. Its poles form a subset of the poles of the f_n's, but some of the poles of the f_n's might cancel out.

0.1.2 A Meromorphic Series

Consider $\sum_{n=-\infty}^{\infty}(z-n)^{-2}$. We want to show that this series is uniformly and absolutely convergent on compact subsets of $\mathbb{C}$, and then we want to find a closed form for the sum. It's enough to show this on any vertical strip $a_1 \leq x \leq a_2$, since any compact subset of $\mathbb{C}$ can be covered by finitely many of these.

We are going to remove the terms where n lies inside this strip. First, consider

$$\sum_{n<a_1} \frac{1}{(z-n)^2},$$

for z inside the vertical strip. This is uniformly and absolutely convergent in the strip since each summand is bounded above in modulus by $(a_1 - n)^{-2}$, and this converges since each term is comparable to n^{-2}. The argument for the sum over $n > a_2$ is similar.

With the theorem in mind, consider the meromorphic function defined by $f(z) = \sum_{n=-\infty}^{\infty}(z-n)^{-2}$. The function f has period 1 ($f(z+1) = f(z)$), the poles are precisely the integers, and they are all double poles with principal parts $(z-n)^{-2}$ (and residues 0). We claim that

$$f(z) = \left(\frac{\pi}{\sin \pi z}\right)^2;$$

call this function $g(z)$. It's enough to show that $g(z)$ is meromorphic, with the same poles and corresponding principal parts as $f(z)$, and that $f - g$ is bounded (in fact, we want it to be zero).

Note that $f(z) \to 0$ uniformly with respect to x as $|y| \to \infty$; that is, for every $\varepsilon > 0$, there is a b such that $|f(z)| < \varepsilon$ when $|y| > b$. By periodicity, it's enough to show this in a strip $a_1 \leq x \leq a_2$. This clearly holds for each summand, in the strip; it follows that it holds for the sum as well. (The proof of this is left as an exercise, which might be given next time.)

Now, $g(z)$ has the same properties as f:

(i) Meromorphic in $\mathbb{C}$, with period 1.

(ii) The poles are precisely the integers, and each pole is a double pole with principal part $(z-n)^{-2}$.

(iii) $g(z) \to 0$ as $|y| \to \infty$, uniformly with respect to x; to see this, use the fact that

$$|\sin \pi z|^2 = \sin^2 \pi x + \sinh^2 \pi y.$$

Why is $f - g$ bounded? Consider again a strip $a_1 \leq x \leq a_2$. In some part $|y| \geq b$ of this strip, $f - g$ goes to 0 as $|y| \to \infty$, uniformly with respect to x. The rest of the strip is compact, so $f - g$ is bounded on it. Therefore $f - g$ is bounded on the strip, and bounded on all of $\mathbb{C}$. (Why?)

$f - g$ is holomorphic in $\mathbb{C}$, since f, g have the same poles and principal parts. $f - g$ is constant by Liouville's theorem. The constant is zero by property (iii). As an exercise, show that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$