

MAT454/1002 lecture notes by Kain Dineen. Taught by Edward Bierstone.

# 1 Spaces of Holomorphic Functions (1-29-2021)

## 1.1 Topology of $\mathcal{C}(\Omega)$

Let  $\Omega$  be an open subset of  $\mathbb{C}$ . We write  $\mathcal{C}(\Omega)$  for the ring of continuous complex-valued functions on  $\Omega$ , and  $\mathcal{H}(\Omega)$  for the subring of  $\mathcal{C}(\Omega)$  consisting of holomorphic functions. We are interested in topologizing  $\mathcal{C}(\Omega)$ , with what we call the *compact-open topology*.

A sequence  $\{f_n\}$  in  $\mathcal{C}(\Omega)$  is said to *converge uniformly on compact subsets* if for all compact  $K \subset \Omega$ ,  $\{f_n|_K\}$  converges uniformly. A notion of convergence defines a topology; we need to define the open sets. We start with a system of neighbourhoods of 0. For a compact  $K \subset \Omega$  and an  $\varepsilon > 0$ , define

$$V(K, \varepsilon) = \{f \in \mathcal{C}(\Omega) : |f(z)| < \varepsilon, z \in K\}.$$

Then  $f_n \rightarrow f$  uniformly on compact subsets if and only if for all  $K, \varepsilon$ ,  $f - f_n \in V(K, \varepsilon)$  for sufficiently large  $n$ . Then, a system of neighbourhoods of any point is obtained by translating these neighbourhoods of 0, giving a basis for a topology on  $\mathcal{C}(\Omega)$ .

Actually, this topology on  $\mathcal{C}(\Omega)$  is metrizable, and it can be defined by a translation-invariant metric. Cover  $\Omega$  by the interiors of countably many closed disks  $D_i$  (take all closed disks in  $\Omega$  with rational center, radius). Define

$$d(f) = \sum_{i=1}^{\infty} \frac{1}{2^i} \min\{1, M_i(f)\},$$

where  $M_i(f)$  is the maximum of  $|f|$  on  $D_i$ . It is clear that

- (i)  $d(f) \geq 0$ , and  $d(f) = 0$  if and only if  $f = 0$ , and
- (ii)  $d(f + g) \leq d(f) + d(g)$  (it's certainly true for each term in the sum).

So  $d(f, g) := d(f - g)$  is a translation-invariant metric.

## 1.2 Convergence of Holomorphic Functions

$\mathcal{C}(\Omega)$  is complete: the limit of a sequence of functions that converges uniformly continuous on compact sets is continuous. We'll mostly be concerned with the holomorphic functions  $\mathcal{H}(\Omega)$ ; give it the subspace topology.

**Theorem 1.1.** (*Weierstrass*)

- (1)  $\mathcal{H}(\Omega)$  is a closed subspace of  $\mathcal{C}(\Omega)$ .

(2) The mapping  $\mathcal{H}(\Omega) \rightarrow \mathcal{H}(\Omega)$ ,  $f \mapsto f'$ , is continuous.

(1) means that if  $\{f_n\} \subset \mathcal{H}(\Omega)$  converges uniformly on compact sets, then  $f = \lim f_n$  is holomorphic. (2) means that if  $\{f_n\} \subset \mathcal{H}(\Omega)$  converges to  $f$  uniformly on compact sets, then  $\{f'_n\}$  converges uniformly to  $f'$  on compact sets.

*Proof.* (1) It's enough to show that  $f(z)dz$  is a closed form. Consider a disk with center  $a$  and radius  $r$  contained in  $\Omega$ . We want to show that if  $\gamma$  is any closed curve in  $|z - a| < r$ , then the integral of  $f(z)dz$  over  $\gamma$  vanishes. Since  $\gamma$  is compact,

$$\int_{\gamma} f(z) dz = \lim_{n \rightarrow \infty} \int_{\gamma} f_n(z) dz = 0,$$

since all of the integrals in the limit vanish by holomorphicity of each  $f_n(z)$ .

(2) Suppose  $f_n \rightarrow f$  uniformly on compact sets. It's enough to show that  $f'_n \rightarrow f'$  uniformly on a closed disk  $D \subset \Omega$ . Let  $\gamma$  be the counterclockwise boundary of a larger concentric disk in  $\Omega$ . If  $z \in D$ , then

$$f(z) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(\zeta)}{\zeta - z} d\zeta,$$

so differentiating under the integral sign gives

$$f'(z) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(\zeta)}{(\zeta - z)^2} d\zeta.$$

By uniform convergence on compact sets,

$$f'(z) = \lim_{n \rightarrow \infty} \frac{1}{2\pi i} \int_{\gamma} \frac{f_n(\zeta)}{(\zeta - z)^2} d\zeta = \lim_{n \rightarrow \infty} f'_n(z).$$

The convergence is uniform with respect to  $z \in D$  because  $(\zeta - z)^{-2}$  is bounded away from 0 for  $z \in D$ ,  $\zeta \in \gamma$ . □

Any result about sequences also applies to series.

**Corollary 1.1.** *If a series of holomorphic functions  $\sum f_n$  on  $\Omega$  converges uniformly on compact sets, then the sum  $f = \sum f_n$  is holomorphic, and the series can be differentiated term-by-term.*

Recall that a set  $\Omega$  in  $\mathbb{C}$  is said to be a *domain* if it is open and connected.

**Proposition 1.1.** (Hurwitz) *Suppose that  $\Omega$  is a domain in  $\mathbb{C}$ . If  $\{f_n\} \subset \mathcal{H}(\Omega)$  converges uniformly on compact sets, and each  $f_n$  is nowhere-vanishing on  $\Omega$ , then the limit is either never zero or identically zero.*

*Proof.* Suppose that  $f$  is not identically zero. Since  $\Omega$  is connected, the zeroes of  $f$  are isolated. Suppose  $f(z_0) = 0$ . Let  $\gamma$  be the boundary of a circle in  $\Omega$  with center  $z_0$ . The multiplicity of  $z_0$  is

$$\frac{1}{2\pi i} \int_{\gamma} \frac{f'(z)}{f(z)} dz = \lim_{n \rightarrow \infty} \frac{1}{2\pi i} \int_{\gamma} \frac{f'_n(z)}{f_n(z)} dz.$$

This is zero since  $f_n$  is never zero, i.e. the integrands  $f'_n/f_n$  are holomorphic, a contradiction.  $\square$

**Corollary 1.2.** *If  $\Omega$  is a domain and  $\{f_n\} \subset \mathcal{H}(\Omega)$  converges uniformly on compact sets, and each  $f_n$  is one-to-one, then  $f = \lim f_n$  is either one-to-one or constant.*

*Proof.* Assume that  $f$  is not constant and not one-to-one. Then  $f(z_1) = f(z_2) = a$  for some  $z_1 \neq z_2$  in  $\Omega$ . Let  $U, V$  be disjoint open neighbourhoods of  $z_1, z_2$  in  $\Omega$ . Then  $f(z) - a$  vanishes at a point of  $U$ , so some whole subsequence  $\{f_{n_i}\}$  of  $\{f_n\}$  vanishes at a point of  $U$ . The same argument provides a subsequence  $\{f_{n_{i_j}}\}$  such that  $f_{n_{i_j}}(z) - a$  vanishes at some point of  $V$ , implying that the  $f_{n_{i_j}}$ 's are not one-to-one, contradiction.  $\square$

We'd like to apply these notions to sequences and series of meromorphic functions, however we will have to manage the existence of poles. We will do this next time.