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# 1 Isolated Singularities and Residues (1-27-2021)

## 1.1 Singularities

We say that a holomorphic function $f(z)$ in the punctured disk $0 < |z| < R$ has an *isolated singularity* at 0 if $f(z)$ can't be extended to a holomorphic function on all of $|z| < R$. There are two cases: 0 is either a pole, else we call it an *essential singularity*. What is the difference? For a pole, the principal part of the Laurent expansion of $f(z)$,

$$f(z) = \sum_{n=-\infty}^{\infty} a_n z^n,$$

is a finite sum. So an essential singularity means that the principal part is an infinite sum.

Extension to a holomorphic function in $|z| < R$ is possible if and only if $f$ is bounded in a neighbourhood of 0. Why? Take $r \in (0, R)$. Then

$$f(re^{i\theta}) = \sum_{n=-\infty}^{\infty} a_n r^n e^{in\theta}.$$

Integrate $e^{-in\theta} f(re^{i\theta})$ with respect to $\theta$ to get

$$a_n r^n = \frac{1}{2\pi} \int_0^{2\pi} e^{-in\theta} f(re^{i\theta})\, d\theta, \qquad n \in \mathbb{Z}.$$

Therefore $|a_n| \le M(r) r^{-n}$, where $M(r)$ is an upper bound for $|f(z)|$, $|z| = r$. If $f$ is bounded in the punctured disk, then there is an $M > 0$ such that every $M(r) \le M$. Then, when $n < 0$, $|a_n| \le M r^{-n}$; since $-n > 0$, taking $r \to 0$ shows that $a_n = 0$. Therefore $f$ can be extended holomorphically to 0.

Thus, there are three options for a holomorphic function $f(z)$ in a punctured disk centered at 0:

(i) A *removable singularity*: bounded in a neighbourhood of 0, in which case it extends holomorphically to the entire disk.

(ii) A pole, in which case $\lim_{z\to 0} f(z) = \infty$.

(iii) An essential singularity. What can we say about the limit here? It's not even well-defined, as the following theorem shows.

The following theorem is often called the *Casorati-Weierstrass theorem*. It displays, in some sense, just how unmanageable essential singularities are, compared to poles.

**Theorem 1.1.** *If 0 is an essential singularity, then for all $\varepsilon > 0$, $f(\{0 < |z| < \varepsilon\})$ is dense in $\mathbb{C}$.*

*Proof.* If not, we can find an $a \in \mathbb{C}$ and $\delta > 0$, such that $|f(z)-a| \geq \delta$, for all $0 < |z| < \varepsilon$. Consider

$$g(z) = \frac{1}{f(z) - a}.$$

Then $g(z)$ is holomorphic in $0 < |z| < \varepsilon$, and bounded, since $|g(z)| \leq 1/\delta$. So $g(z)$ extends holomorphically to $|z| < \varepsilon$. But then $f(z) = a + 1/g(z)$ is meromorphic, and has either a pole or a removable singularity at 0, contradicting that 0 is an essential singularity of $f(z)$. □

### 1.2 Residues

Let $f(z)$ be a holomorphic function in a punctured neighbourhood of $a$. Let $\gamma$ be a curve around $a$ with winding number $w(\gamma, a) = 1$. We call the integral

$$\frac{1}{2\pi i} \int_\gamma f(z)\, dz$$

the *residue* of the differential form $f(z)\, dz$ at $a$. What does this mean in terms of the Laurent series expansion

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z-a)^n?$$

For every $n \neq -1$, $(z-a)^n\, dz$ has a primitive, so the integral vanishes. For $n = 1$, the integral is $2\pi i a_{-1}$, so the residue is simply the coefficient $a_{-1}$.

Why is it beneficial to consider the residue of the *form* $f(z)\, dz$ instead of that of the function $f(z)$? What is the residue at $\infty$? Consider coordinates $z'$ at infinity, $z = 1/z'$. Then

$$f(z)\, dz = -\frac{1}{(z')^2} f\left(\frac{1}{z'}\right)\, dz'.$$

We integrate over a curve with winding number 1 around $\infty$ (equivalently, a circle around 0 with winding number $-1$):

$$\operatorname{res}(f, \infty) = -\frac{1}{2\pi i} \int_\gamma \frac{1}{(z')^2} f\left(\frac{1}{z'}\right)\, dz' = -a_{-1},$$

where $\gamma$ is positively oriented with respect to $z' = 0$ (i.e. $z = \infty$), with the Laurent series expansion taken in some annulus $|z| > R$. If $\omega = f(z)\, dz$, then this is just the integral $\frac{1}{2\pi i} \int_\gamma \omega$, where $\gamma$ is positively oriented with respect to $\infty$ (and we computed it using coordinates at $\infty$).

**Theorem 1.2.** *(Residue theorem) Let $\Omega$ be an open subset of the Riemann sphere. Let $K \subset \Omega$ be a compact set with a piecewise-$C^1$ oriented boundary $\Gamma$. Given a function $f(z)$ holomorphic in $\Omega$, except perhaps at isolated points, and with $\Gamma$ not containing any singular points or $\infty$. Then*

$$\int_\Gamma f(z)\,dz = 2\pi i \sum \operatorname{res}(f, z_k),$$

*where the $z_k$'s are the singularities in $K$ (possibly including $\infty$).*

*Remark*: To say that $K$ has a piecewise-$C^1$ oriented boundary $\Gamma$ means that $\Gamma$ is a union of piecewise-$C^1$ closed curves $\gamma(t)$ that are positively oriented with respect to $K$. Informally, if you're walking along $\Gamma$, then $K$ is always to your left. (To state this formally requires a bit of messy second year calculus.)