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0.1 Normal Families and Equicontinuity (3-1-2021)

We study in more depth the topology of the space of continuous and holomorphic functions on subsets of $\mathbb{C}$. We introduce the notion of a normal family, and begin to discuss some of the basic topological results concerning spaces of complex functions.

0.1.1 Normal Families

Recall that given an open subset $\Omega \subseteq \mathbb{C}$, the space $\mathcal{C}(\Omega)$ is metrizable. For example, if $\Omega = \cup E_i$ is written as a union of closed disks, then a metric is given by

$$d(f,g) = \sum_{i=1}^{\infty} \frac{1}{2^i} \min(1, \max_{z \in E_i} |f(z) - g(z)|).$$

As a (closed) subspace of $\mathcal{C}(\Omega)$, the space $\mathcal{H}(\Omega)$ of holomorphic functions on Ω is also metrizable. We are interested in the notion of compactness of subsets of $\mathcal{C}(\Omega)$. We call $\mathcal{S} \subseteq \mathcal{C}(\Omega)$ a *normal family* if every sequence in $\mathcal{S}$ has a subsequence that converges in $\mathcal{C}(\Omega)$; of course, the limit need not lie in $\mathcal{S}$.

For example, consider $\mathcal{S} = \{z^n\} \subset \mathcal{C}(D)$, where D is the open unit disk. This is a normal family since z^n converges uniformly on compact subsets of D to the zero function, but not on D. As another example, let g_n be 1 for n even and 0 for n odd. This is a normal family, but it doesn't converge.

We see that $\mathcal{S}$ is compact if and only if $\mathcal{S}$ is normal, and the limit functions are themselves in $\mathcal{S}$. We state here some useful facts about normal families.

Lemma 0.1.1. *$\mathcal{S}$ is normal if and only if $\overline{\mathcal{S}}$ is compact.*

Proof. If $\mathcal{S}$ is normal, let $\{f_n\}$ be a sequence in $\overline{\mathcal{S}}$. For each n there is a sequence $\{f_n^{(k)}\}$ in $\mathcal{S}$ converging to f_n. The diagonal $\{f_n^{(n)}\}$ lies in $\mathcal{S}$, so it has a convergent subsequence $\{f_{n_k}^{(n_k)}\} \to f \in \overline{\mathcal{S}}$. Writing $d(f_{n_k}, f) \leq d(f_{n_k}, f_{n_k}^{(n_k)}) + d(f_{n_k}^{(n_k)}, f)$, we see that $f_{n_k} \to f$, which proves that $\overline{\mathcal{S}}$ is compact. The converse is clear. $\square$

Lemma 0.1.2. *$\mathcal{S} \subset \mathcal{C}(\Omega)$ is normal if and only if, for every covering $\Omega = \cup E_i$ by closed disks, and every i, every sequence in $\mathcal{S}$ contains a subsequence that converges uniformly on E_i.*

Proof. The forward direction is clear. Conversely, take a sequence $\{f_n\}$ in $\mathcal{S}$. We can find a subsequence $\{f_n^{(1)}\} \subset \{f_n\}$ which converges uniformly on E_1. Recursively choose subsequences $\{f_n^{(k)}\} \subset \{f_n^{(k-1)}\}$ which converge uniformly on E_k. Then the diagonal sequence $\{f_n^{(n)}\}$ converges uniformly on *every* E_k, so it converges uniformly on every compact subset K of Ω. So the sequence $\{f_n\}$ has a subsequence convergent in $\mathcal{C}(\Omega)$. $\square$

In the conclusion of the proof, we implicitly used the following fact: given a countable covering of an open set by closed disks, their interiors also cover the set.

0.1.2 Equicontinuity

We will relate the ideas of normal families and equicontinuous families. Given $X \subseteq \mathbb{C}$, a family $\mathcal{S} \subseteq \mathcal{C}(X)$ is said to be *equicontinuous* at $a \in X$ if for all $\varepsilon > 0$, there is a $\delta > 0$ such that for all $z \in X$ with $|z - a| < \delta$, one has $|f(z) - f(a)| < \varepsilon$ for all $f \in \mathcal{S}$. We say $\mathcal{S}$ is *equicontinuous* on X if it is equicontinuous at every point of X. Furthermore, we say $\mathcal{S}$ is *uniformly equicontinuous* if for every $\varepsilon > 0$, there is a $\delta > 0$ such that if $z, w \in X$, $|z - w| < \delta$, then $|f(z) - f(w)| < \varepsilon$ for all $f \in \mathcal{S}$.

For example, suppose that $\mathcal{S}$ consists of all of the holomorphic functions f on an open disk D, such that there is an $M < \infty$ for which $|f'| \leq M$, for all $f \in \mathcal{S}$. By integrating, we have $|f(z) - f(w)| \leq M|z - w|$ for $z, w \in D$, which makes it clear that $\mathcal{S}$ is a uniformly equicontinuous family.

Theorem 0.1.1. *(Arzela-Ascoli) Given a domain $\Omega \subseteq \mathbb{C}$, a family $\mathcal{S} \subseteq \mathcal{C}(\Omega)$ is normal if and only if*

(1) $\mathcal{S}$ is equicontinuous on Ω, and

(2) There is a $z_0 \in \Omega$ such that $\{f(z_0) : f \in \mathcal{S}\}$ is bounded in $\mathbb{C}$.

Moreover, if $\mathcal{S}$ is normal, then (2) holds for every $z_0 \in \Omega$.

Note that Arzela-Ascoli holds for families of continuous functions with values in an arbitrary complete metric space. An example of such a space is the Riemann sphere with the chordal metric (since it's compact).

The Arzela-Ascoli theorem has nothing to do with holomorphic functions. However, we will be interested in the consequences it has for holomorphic functions specifically. For families of holomorphic functions, Arzela-Ascoli gives an important criterion for normality (using Cauchy inequalities).

We'll eventually see that in the space of holomorphic functions, "compact" is equivalent to "closed and bonuded." Bounded in what sense? $\mathcal{S} \subset \mathcal{C}(\Omega)$ is *locally bounded* on Ω if it is bounded in a neighbourhood of any given point: for all $z_0 \in \Omega$, there is a $\delta > 0$ and an $M = M(z_0) < \infty$ such that $|f(z)| \leq M$ whenever $|z - z_0| < \delta$, for all $f \in \mathcal{S}$. This is equivalent to the condition that $\int$ is *uniformly bounded on compact sets* in Ω, meaning that for every compact $K \subset \Omega$, there is an $M = M(K)$ such that $|f(z)| \leq M$ when $z \in K$, for all $f \in \mathcal{S}$.

Next time, we will state and prove the consequence of Arzela-Ascoli for holomorphic functions and work towards the promised characterization of compactness in $\mathcal{H}(\Omega)$.