0.1 Montel's Theorem (3-3-2021)

We state and prove Montel's theorem, a fundamental result about normal families of holomorphic functions with values in the (finite) complex plane.

0.1.1 A Remark on Equicontinuity

Given the formulation of the Arzela-Ascoli theorem from last time, it's worth making a remark about the second condition; in particular, to link the formulation of the theorem last time with any version of Arzela-Ascoli with which the student is familiar.

Proposition 0.1.1. *If Ω is a domain and $\mathcal{S} \subseteq \mathcal{C}(\Omega)$ is equicontinuous, then the following are equivalent:*

(a) There is a $z_0 \in \Omega$ for which $\{f(z_0) : f \in \mathcal{S}\}$ is bounded (the second condition in Arzela-Ascoli).

(b) For all $z \in \Omega$, the set $\{f(z) : f \in \mathcal{S}\}$ is bounded.

(c) $\mathcal{S}$ is locally bounded.

Proof. Equicontinuity implies that each $w \in \Omega$ is the center of a disk $D_w \subset \Omega$ such that $|f(z) - f(w)| < 1$, for all $z \in D_w$ and $f \in \mathcal{S}$. To prove (a) implies (b), let

$$U = \{z \in \Omega : \text{the set } \{f(z) : f \in \mathcal{S}\} \text{ is bounded}\}.$$

$U \neq \emptyset$ by part (a). U is open by equicontinuity, as above. To see that it is closed, note that if $w \in \Omega \setminus U$, then D_w as above is also in $\Omega \setminus U$. By connectedness, $U = \Omega$. That (b) implies (c) is immediate from equicontinuity, and it's clear that (c) implies (a). $\square$

0.1.2 Montel's Theorem

Of course, we are mainly interested in the consequences of Arzela-Ascoli for holomorphic functions. The following is the main such result.

Theorem 0.1.1. *(Montel) Let $\Omega \subseteq \mathbb{C}$ be a domain, and let $\mathcal{S} \subseteq \mathcal{H}(\Omega)$. The following are equivalent:*

(1) $\mathcal{S}$ is normal.

(2) $\mathcal{S}$ is locally bounded.

(3) $\mathcal{S}' = \{f' : f \in \mathcal{S}\}$ is locally bounded, and there is a $z_0 \in \Omega$ such that $\{f(z_0) : f \in \mathcal{S}\}$ is bounded in $\mathbb{C}$.

Proof. (1) implies (2) is by Arzela-Ascoli. To see that (2) implies (3), local boundedness gives an $r > 0$ and an $M < \infty$ such that $|f(z)| \leq M$ on a disk with center z_0 and radius r, for all $f \in \mathcal{S}$. Then

$$|f'(z)| \leq \frac{M}{\frac{r}{2}} = \frac{2M}{r}$$

on $B(z_0, r/2)$, by Cauchy inequalities.

To show that (3) implies (1), it suffices (by Arzela-Ascoli) to show that if $\mathcal{S}'$ is locally bounded, then $\mathcal{S}$ is equicontinuous. Given $z_0 \in \Omega$, $|f'(z)| \leq M < \infty$ in a disk D, center z_0, for all $f \in \mathcal{S}$. For $z \in D$, it follows that

$$|f(z) - f(z_0)| \leq M|z - z_0|, \qquad f \in \mathcal{S},$$

which proves equicontinuity at z_0. □

Corollary 0.1.1. *$\mathcal{S} \subseteq \mathcal{H}(\Omega)$ is compact if and only if $\mathcal{S}$ is closed and locally bounded.*

Proof. Local-boundedness of $\mathcal{S}$ is equivalent to precompactness by Montel's theorem, which together with closedness is equivalent to compactness. □

0.1.3 Functions into the Riemann Sphere

It is worth remarking that Arzela-Ascoli holds for families of continuous functions with valued in any complete metric space. We'll mainly be interested in families of functions with values in the Riemann sphere, or with values in $\mathbb{C}^* = \mathbb{C} \cup \{\infty\}$ with the chordal metric

$$d(z, w) = \frac{2|z - w|}{\sqrt{1 + |z|^2}\sqrt{1 + |w|^2}}.$$

Note that if neither of z or w are zero or infinity, then $d(z, w) = d(1/z, 1/w)$.

We remark that for f, with $f(z_0) = \infty$, to be continuous at $z_0 = \infty$, means that for all $R < \infty$, there is a $\delta > 0$ such that if $d(z, z_0) < \delta$, then $|f(z)| > R$. Also, the topology of $\mathbb{C}$ induced by the chordal metric is the usual Euclidean topology. Using the chordal metric, we can simplify the statement of Arzela-Ascoli.

Proposition 0.1.2. *(Arzela-Ascoli) A family of continuous functions on a domain Ω is normal in the chordal metric if and only if it is equicontinuous in the chordal metric.*

That is, the second condition in Arzela-Ascoli from before is not needed, because the Riemann sphere is compact. We will explore this in more depth next time by studying families of holomorphic functions into the Riemann sphere i.e. families of meromorphic functions.