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0.1 Cauchy's Integral Formula, Continued (1-20-2021)

We prove Cauchy's integral theorem. We then deduce several important facts about holomorphic functions, including their smoothness, Morera's theorem, Liouville's theorem, the fundamental theorem of algebra, and the mean value property.

0.1.1 Proof of Cauchy's Integral Formula

Let us recall and prove Cauchy's integral formula.

Theorem 0.1.1. *(Cauchy's integral formula) Let $f(z)$ be holomorphic in the open set Ω, and let $a \in \Omega$. Suppose γ is a closed curve in Ω, not containing a, which is homotopic to a point in Ω. Then,*

$$\frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z-a}\,dz = w(\gamma,a)f(a).$$

Proof. Let

$$g(z) = \begin{cases} \frac{f(z)-f(a)}{z-a}, & z \neq a, \\ f'(a), & z = a. \end{cases}$$

Then, $g(z)$ is holomorphic in $\Omega \setminus \{a\}$ and continuous in Ω, so

$$\int_\gamma \frac{f(z)-f(a)}{z-a}\,dz = 0,$$

since $g(z)\,dz$ is closed, by Cauchy's theorem. But then

$$\int_\gamma \frac{f(z)}{z-a}\,dz = \int_\gamma \frac{f(a)}{z-a}\,dz = 2\pi i \cdot f(a)w(\gamma,a).$$

□

The most important case of Cauchy's integral formula to us will be the case where γ is the boundary of a disk, oriented counter-clockwise. If $f(z)$ is holomorphic in a neighbourhood of this disk, then

$$\frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z-a}\,dz = \begin{cases} f(a), & a \text{ inside the circle}, \\ 0, & a \text{ outside the circle}. \end{cases}$$

0.1.2 Applications of Cauchy's Integral Formula

Proposition 0.1.1. *A holomorphic function is infinitely differentiable (and all of its derivatives are holomorphic).*

Proof. Suppose $f(z)$ is holomorphic in some disk $D = \{z : |z| < R\}$, and let γ be the boundary of a smaller circle $\{|z| < r\}$, $r < R$, oriented counter-clockwise. Then, $\gamma(\theta) = re^{i\theta}$, $\theta \in [0, 2\pi]$. If $|z| < r$, then

$$f(z) = \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{\zeta - z}\, d\zeta.$$

We can differentiate under the integral to obtain.

$$f'(z) = \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)\, d\zeta}{(\zeta - z)^2}.$$

Furthermore, for any n,

$$f^{(n)}(z) = \frac{n!}{2\pi i}\int_\gamma \frac{f(\zeta)\, d\zeta}{(\zeta - z)^{n+1}}.$$

$\square$

Let us summarize what we have learned thus far concerning Cauchy's theorem.

Proposition 0.1.2. *If $f(z)$ is continuous in Ω, then the following statements are equivalent.*

1. *$f(z)$ is holomorphic.*
2. *$f(z)\, dz$ is a closed form.*
3. *One has*
$$f(z) = \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{\zeta - z}\, d\zeta$$
for z in an open disk with boundary γ.

Proof. It remains to show (2) implies (1); this statement is known as Morera's theorem. Locally, $f(z)\, dz$ has a primitive $g(z)$. That is,

$$f(z)\, dz = dg = \frac{\partial g}{\partial z}\, dz + \frac{\partial g}{\partial \bar{z}}\, d\bar{z}.$$

Since dz and $d\bar{z}$ are linearly independent, $\frac{\partial g}{\partial \bar{z}} = 0$, i.e. the Cauchy-Riemann equations are satisfied. Thus, g is holomorphic. Therefore $f(z) = \frac{\partial g}{\partial z}(z)$ is also holomorphic. $\square$

We can use the integral formula to compute the Taylor series of a holomorphic function $f(z)$ at 0:

$$f(z) = \sum_{n=0}^{\infty} a_n z^n, \qquad a_n = \frac{f^{(n)}(0)}{n!} = \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)\, d\zeta}{\zeta^{n+1}}.$$

Why is this convergent?

$$\frac{1}{\zeta - z} = \frac{1}{\zeta}\left(1 - \frac{z}{\zeta}\right)^{-1} = \frac{1}{\zeta}\left(1 + \frac{z}{\zeta} + \frac{z^2}{\zeta^2} + \cdots\right).$$

Thus,

$$f(z) = \frac{1}{2\pi i} \int_\gamma \sum_{n=0}^{\infty} z^n \cdot \frac{f(\zeta)}{\zeta^{n+1}}\, d\zeta.$$

We can interchange the sum and the integral, as for a fixed z, $|z| < r$, the series

$$\sum_{n=0}^{\infty} z^n \cdot \frac{f(\zeta)}{\zeta^{n+1}}$$

is uniformly and absolutely convergent on $|\zeta| = r$. Thus, the Taylor series converges when $|z| < r$. Moreover, if we substitute $z = re^{i\theta}$, it becomes

$$f(re^{i\theta}) = \sum_{n=0}^{\infty} a_n r^n e^{in\theta},$$

which proves that

$$a_n r^n = \frac{1}{2\pi} \int_0^{2\pi} e^{-in\theta} f(re^{i\theta})\, d\theta.$$

These are known as the *Fourier coefficients*. This gives us the *Cauchy inequalities*, which give us an upper bound for the coefficients a_n (or their modulus). Let $M(r) = \sup_\theta |f(re^{i\theta})|$, the upper bound of $|f|$ on the circle of radius r. By the integral formula, $|a_n r^n| \leq M(r)$, or

$$|a_n| \leq \frac{M(r)}{r^n}.$$

Now we deduce

Theorem 0.1.2. *(Liouville's theorem) A bounded holomorphic function defined on all of $\mathbb{C}$ is constant.*

Proof. $M(r) \leq M$, for some constant M, so $|a_n| \leq Mr^{-n}$ *for all* r. So $a_n = 0$ for $n > 1$. ☐

Corollary 0.1.1. *(Fundamental theorem of algebra) Every non-constant polynomial has a root in $\mathbb{C}$.*

Proof. If $p(z)$ is a polynomial without a root, then $1/p(z)$ is holomorphic in $\mathbb{C}$ and bounded, hence constant. □

Theorem 0.1.3. *(Mean value property) If $f(z)$ is holomorphic in Ω and $D \subset \Omega$ is a compact disk centered at a, then*

$$f(a) = \textit{mean value of } f \textit{ on the boundary of } D.$$

Proof. We may assume $a = 0$. Then

$$f(0) = \frac{1}{2\pi} \int_0^{2\pi} f(re^{i\theta})\, d\theta.$$

□

At the beginning of next lecture, we will go over more consequences of Cauchy's integral formula.