MAT454/1002 lecture notes by Kain Dineen. Taught by Edward Bierstone.

0.1 More on Elliptic Functions (2-5-2021)

We show that the Weierstrass $\wp$-function satisfies a certain differential equation, which will be indispensible to us when we study its algebraic properties. We then prove two fundamental results concerning the nature of the zeroes and poles of elliptic functions.

0.1.1 The Differential Equation

Let Γ be a discrete subgroup of $\mathbb{C}$, generated by $e_1, e_2 \in \mathbb{C}$, linearly independent over $\mathbb{R}$. We're going to use the Laurent series expansion of the Weierstrass $\wp$-function to see that it satisfies a certain differential equation. The Laurent series expansion of $\wp(z)$ at 0 is

$$\wp(z) = \frac{1}{z^2} + a_2 z^2 + a_4 z^4 + \cdots ,$$

where we found the coefficients last time. Differentiating,

$$\wp'(z) = -\frac{2}{z^3} + 2a_2 z + 4a_4 z^3 + \cdots .$$

If we square both sides, we obtain

$$\wp'(z)^2 = \frac{4}{z^6} - \frac{8a_2}{z^2} - 16a_4 + \cdots .$$

On the other hand,

$$\wp(z)^3 = \frac{1}{z^6} + \frac{3a_2}{z^2} + 3a_4 + \cdots .$$

Then

$$\wp'(z)^2 - 4\wp(z)^3 = -\frac{20a_2}{z^2} - 28a_4 + z^2(\cdots).$$

To eliminate the $1/z^2$ term, we add $20a_2\wp(z)$:

$$\wp'(z)^2 - 4\wp(z)^2 + 20a_2\wp(z) + 28a_4 = z^2(\cdots).$$

It follows that

$$(\wp')^2 - 4\wp^3 + 20a_2\wp + 28a_4$$

is holomorphic near zero, 0 at 0. It is periodic with group of periods Γ, so it's holomorphic near all points of Γ. So it's holomorphic in $\mathbb{C}$, since it has no poles elsewhere. By periodicity it's bounded, so by Liouville's theorem it's constant. The constant is zero since it vanishes at the origin. Therefore $\wp$ satisfies the differential equation

$$(\wp')^2 = 4\wp^3 - 20a_2\wp - 28a_4.$$

That is, $x = \wp(z)$ and $y = \wp'(z)$ give a parametrization of the algebraic curve

$$y^2 = 4x^3 - 20a_2x - 28a_4.$$

We'll see later that any point (x, y) of this curve is the image of a point $z \in \mathbb{C}$, uniquely determined up to addition of an element of Γ. Analogously to how trigonometric functions and their derivatives parametrize quadratic curves, the Weierstrass $\wp$-function and its derivative parametrize cubic curves.

0.1.2 Doubly Periodic Functions

We require some results about doubly periodic functions before we further study the $\wp$-function. Let Γ be as in the previous part.

Proposition 0.1.1. *Let f be a non-constant meromorphic function on $\mathbb{C}$ with Γ as its group of periods. Then, provided f has no zeroes or poles on the boundary, the number of zeroes of f in a period parallelogram is equal to the number of poles in the same parallelogram, each counted with multiplicity.*

Proof. If γ is the boundary of the period parallelogram pictured below,

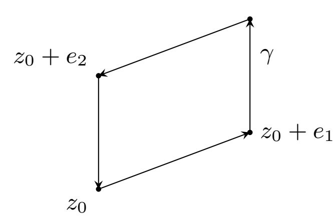

Figure 1: The boundary of the period parallelogram, γ, is oriented counter-clockwise.

then by the argument principle,

$$\frac{1}{2\pi i} \int_\gamma \frac{f'(z)}{f(z)}\, dz = \#\text{ zeroes} - \#\text{ poles},$$

counted with multiplicity. The left-hand side vanishes by periodicity. □

Proposition 0.1.2. *Let f be a non-constant meromorphic function in $\mathbb{C}$ with Γ as its group of periods. For a fixed $a \in \mathbb{C}$, let α_i be the roots of $f(z) = a$, and let β_i be the poles of $f(z)$, each counted with multiplicity, within a period parallelogram. Then $\sum \alpha_i$ is congruent to $\sum \beta_i$, modulo Γ. (In particular, $\sum \alpha_i \bmod \Gamma$ is independent of a.)*

Proof. By the residue theorem, if γ is the boundary of the period parallelogram,

$$\frac{1}{2\pi i} \int_\gamma \frac{zf'(z)}{f(z) - a}\, dz = \text{sum of the residues of } \frac{zf'(z)}{f(z) - a}. \qquad (*)$$

At a root $z = \alpha_i$ of multiplicity k,

$$\begin{aligned} z &= \alpha_i + (z - \alpha_i), \\ f(z) - a &= c(z - \alpha_i)^k + \text{higher order} \\ f'(z) &= kc(z - \alpha_i)^{k-1} + \cdots , \end{aligned}$$

so it follows that

$$\frac{zf'(z)}{f(z) - a} = \frac{k\alpha_i}{z - \alpha_i} + \text{ higher order},$$

so the residue is $k\alpha_i$. Similarly, at a pole β_i, the residue is $-k\beta_i$. It follows that the right-hand side of (*) is simply $\sum \alpha_i - \sum \beta_i$, which we want. Unlike in the proof of the previous proposition, the integrand in the left-hand side of (*) is not periodic. However, the left-hand side is

$$-\frac{e_2}{2\pi i} \int_{\gamma_1} \frac{f'(z)}{f(z) - a}\, dz + \frac{e_1}{2\pi i} \int_{\gamma_2} \frac{f'(z)}{f(z) - a}\, dz,$$

where γ_1, γ_2 are two sides of the period parallelogram:

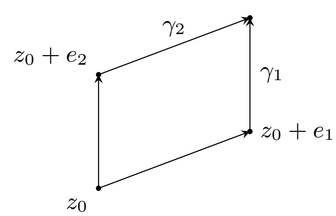

Figure 2: γ_1 and γ_2 are oriented so that they both start at z_0 and end at the opposite vertex, $z_0 + e_1 + e_2$. (The orientation is not a big deal, so long as we end up with *something* which is zero mod Γ.)

Here,

$$\frac{1}{2\pi i} \int_{\gamma_1} \frac{f'(z)}{f(z) - a}\, dz, \int_{\gamma_2} \frac{f'(z)}{f(z) - a}\, dz$$

are integers, because they each equal to the difference between two determinations of $\log(f(z_0) - a)$. $\square$