MAT454/1002 lecture notes by Kain Dineen. Taught by Edward Bierstone.

0.1 Meromorphic Functions (1-25-2021)

We study meromorphic functions on the complex plane and on the Riemann sphere, and we develop the Laurent series expansion for meromorphic functions. We also prove the partial fraction decomposition theorem.

0.1.1 Zeroes and Poles

Let Ω be an open subset of $\mathbb{C}$, and let $f(z)$ be holomorphic in Ω. Suppose $f(z_0) = 0$. Then $f(z) = (z - z_0)^k f_1(z)$ for some holomorphic f_1 in Ω with $f_1(z_0) \neq 0$. (To see this, write the Taylor expansion at z_0.) We call k the *order* or *multiplicity* of the zero z_0. It follows that the zeroes of a holomorphic function that doesn't vanish identically are isolated, at least if Ω is connected.

A *meromorphic* function in Ω is a holomorphic function defined in the complement of a discrete set in Ω, which is in a (perhaps punctured) neighbourhood of any point expressible as the quotient of holomorphic functions $f(z)/g(z)$, where $g \not\equiv 0$. The set of meromorphic functions on a *domain* Ω (open, connected) form a field.

At a point z_0, write $f(z) = (z - z_0)^k f_1(z)$ and $g(z) = (z - z_0)^l g_1(z)$, where f_1, g_1 are holomorphic functions with $f_1(z_0), g_1(z_0) \neq 0$. So

$$\frac{f(z)}{g(z)} = (z - z_0)^{k-l} \frac{f_1(z)}{g_1(z)}.$$

If $k \geq l$, then f/g extends to be holomorphic at z_0. If $k < l$, then we say that z_0 is a *pole* of f/g of *order* or *multiplicity* $l - k$. Then, we will say that

$$\lim_{z \to z_0} \frac{f(z)}{g(z)} = \infty,$$

i.e. meromorphic functions take values in the Riemann sphere. Pursuing this, we get a nicer definition of meromorphic functions: a meromorphic function on Ω is a holomorphic function $\Omega \to S^2$.

0.1.2 Laurent Series, Partial Fraction Decomposition

We're going to see that a holomorphic function $f(z)$ in an annulus $r < |z| < R$ has a *Laurent expansion*:

$$\sum_{n=-\infty}^{\infty} a_n z^n = \underbrace{\sum_{n<0} a_n z^n}_{\text{holom. in } |z| > r} + \underbrace{\sum_{n \geq 0} a_n z^n}_{\text{holom. in } |z| < R} .$$

We can rephrase the condition on the first sum as follows: if $z = 1/\zeta$, then

$$\sum_{n<0} a_n z^n = \sum_{n<0} a_n \zeta^{-n} = \sum_{n=1}^{\infty} a_{-n}\zeta^n$$

is holomorphic in $|\zeta| < 1/r$.

We are going to get this from Cauchy's integral formula. Let γ_1 be a circle of radius $r_1 \in (r, R)$, and γ_2 a circle of radius $r_2 \in (r, R)$, with $r < r_2 < r_1 < R$. If $r_2 < |z| < r_1$, then Cauchy's integral gives

$$f(z) = \frac{1}{2\pi i}\int_{\gamma_1} \frac{f(\zeta)}{\zeta - z}\,d\zeta - \frac{1}{2\pi i}\int_{\gamma_2} \frac{f(\zeta)}{\zeta - z}\,d\zeta,$$

which can be seen by connecting γ_1 and γ_2 by an arbitrarily small line segments avoiding z, making a closed curve. (Draw a picture.) In the second integral,

$$\frac{1}{\zeta - z} = -\frac{1}{z}\cdot\frac{1}{1-\frac{\zeta}{z}} = -\sum_{n=0}^{\infty}\frac{\zeta^n}{z^{n+1}} = -\sum_{n<0}\frac{z^n}{\zeta^{n+1}}.$$

This power series is uniformly and absolutely convergent on $|\zeta| = r_2$, so we get

$$f(z) = \sum_{n=-\infty}^{\infty} a_n z^n, \qquad a_n = \frac{1}{2\pi i}\int_{\gamma_2}\frac{f(\zeta)}{\zeta^{n+1}}\,d\zeta, \qquad n < 0.$$

The Laurent series is uniformly and absolutely convergent when $r_2 \le |z| \le r_1$. We can think of this as like a version of Cauchy's theorem, but at a pole or for a function in an annulus.

Holomorphic functions on the Riemann sphere with values in $\mathbb{C}$ are constant by Liouville's theorem. What about meromorphic functions on the Riemann sphere?

Theorem 0.1.1. *Any meromorphic function on S^2 is rational.*

Proof. Let's say $f(z)$ has poles $b_1, \dots, b_n$, and possibly ∞, with corresponding *principal parts* (negative parts of the Laurent expansions)

$$P_k\left(\frac{1}{z - b_k}\right),$$

polynomials in $1/(z - b_k)$, and possibly

$$P_\infty\left(\frac{1}{\zeta}\right) = P_\infty(z),$$

where $\zeta = 1/z$ is the "coordinate at infinty." Then

$$f(z) - \sum_{k=1}^{n} P_k\left(\frac{1}{z-b_k}\right) - P_\infty(z)$$

is holomorphic on S^2, hence a constant a. Then

$$f(z) = a + P_\infty(z) + \sum_{k=1}^{n} P_k\left(\frac{1}{z-b_k}\right),$$

the *partial fraction decomposition* of a rational function. □

If we write this last expression as $\frac{P(z)}{Q(z)}$, $\deg P(z) = p$, $\deg Q(z) = q$, then $a + P_\infty(z)$ is present only if $p > q$. (Quotient by long division!) As an exercise, deduce the theorem on real partial fraction decomposition.