MAT454/1002 lecture notes by Kain Dineen. Taught by Edward Bierstone.

1 The Riemann Sphere, Complex Integration (1-13-2021)

1.1 The Complex Structure on the Riemann Sphere

The complex conjugate of stereographic projection from the south pole $S = (0, 0, -1)$ is given by

$$z' = \frac{x - iy}{1 + t}.$$

This provides a homemorphism of $S^2 \setminus \{S\}$ onto $\mathbb{C}$. For any point $(x, y, t) \in S^2$, other than S or N, we have

$$zz' = \frac{x^2 + y^2}{1 - t^2} = 1;$$

in other words, $z' = 1/z$, a holomorphic transformation. Thus, we have covered the Riemann sphere with two coordinate charts, whose transition mapping is holomorphic. This is the sense in which the Riemann sphere obtains a complex structure.

1.2 One-Dimensional Complex Projective Space

We write $P^1(\mathbb{C})$ for the one-dimensional complex space, consisting of all of the lines in $\mathbb{C}^2$ through the origin. That is, $P^1(\mathbb{C}) = (\mathbb{C}^2 \setminus \{0\})/\sim$, where $(x_0, x_1) \sim (x_0', x_1')$ means that there exists a non-zero $\lambda \in \mathbb{C}$ such that $(x_0', x_1') = (\lambda x_0, \lambda x_1)$. We will write $[x_0, x_1]$ for the equivalence class of (x_0, x_1), and we call these classes *homogeneous coordinates.*

We may equip $P^1(\mathbb{C})$ with the structure of a complex manifold. Let

$$U_i = \{[x_0, x_1] \in P^1(\mathbb{C}) : x_i \neq 0\}, \qquad i = 0, 1.$$

We define two mappings

$$U_0 \to \mathbb{C}, \qquad [x_0, x_1] \mapsto \frac{x_1}{x_0} = z,$$

and

$$U_1 \to \mathbb{C}, \qquad [x_0, x_1] \mapsto \frac{x_0}{x_1} = z'.$$

Evidently, $zz' = 1$. Thus, $P^1(\mathbb{C})$ is obtained by gluing together two copies of $\mathbb{C}$ along the complements of $\{0\}$ by the formula $z' = 1/z$. Moreover, $P^1(\mathbb{C}) \cong S^2$, the Riemann sphere.

1.3 Cauchy's Theorem

Let Ω be a (connected) open subset of $\mathbb{R}^2$. By a *differential form*, we mean an expression of the form $\omega = P\,dx + Q\,dy$, where P, Q are continuous (real or complex)-valued functions on Ω. Let $\gamma\colon [a,b] \to \Omega$ be a piecewise-C^1 curve in Ω, $\gamma(t) = (x(t), y(t))$. Then, we define

$$\int_\gamma \omega = \int_a^b f(t)\,dt,$$

where $f(t) = P(x(t), y(t))x'(t) + Q(x(t), y(t))y'(t)$. (That is, $f(t)\,dt = \gamma^*\omega$.)

The integral of ω over γ is independent of the curve's parametrization. Consider a reparametrization $t\colon [c,d] \to [a,b]$, with $t(c) = a$, $t(d) = b$, and $t'(s) > 0$, and set $\delta(s) = \gamma(t(s))$. Then

$$\int_\gamma \omega = \int_\delta \omega$$

by integration by substitution.

An important example of a differential form is the differential of a C^1 function F:

$$\omega = dF = \frac{\partial F}{\partial x}\,dx + \frac{\partial F}{\partial y}\,dy.$$

In this case, we call F a *primitive* of ω. Here,

$$\int_\gamma dF = \int_a^b (F \circ \gamma)'(t)\,dt = F(\gamma(b)) - F(\gamma(a)).$$

In particular, if γ is closed, the integral of dF over γ is zero.

Proposition 1.1. *ω has a primitive in Ω, if and only if $\int_\gamma \omega = 0$ for every piecewise-C^1 closed curve γ in Ω.*

Proof. We just saw the forward direction. Conversely, fix a point $(x_0, y_0) \in \Omega$. Given ω satisfying the hypotheses, define F by

$$F(x,y) = \int_\gamma \omega,$$

where γ is a piecewise-C^1 curve in Ω starting at (x_0, y_0) and ending at (x, y). This is independent of γ precisely by the hypothesis on γ.

We check that $dF = \omega$. First, let δ be a straight line in Ω from (x,y) to $(x+h, y)$, for sufficiently small h. Then,

$$F(x+h, y) - F(x,y) = \int_\delta \omega = \int_x^{x+h} P(\xi, y)\,d\xi,$$

so

$$\lim_{h \to 0} \frac{F(x+h, y) - F(x, y)}{h} = P(x, y).$$

Thus, $F(x, y)$ is differentiable in x, and $\frac{\partial F}{\partial x} = P$. The proof for the other variable is identical. Therefore $dF = \omega$. □

In the case that Ω is an open disk, we can simplify the statement, and say that ω has a primitive in Ω, if and only if $\int_\gamma \omega = 0$ whenever γ is the boundary of a rectangle in Ω.

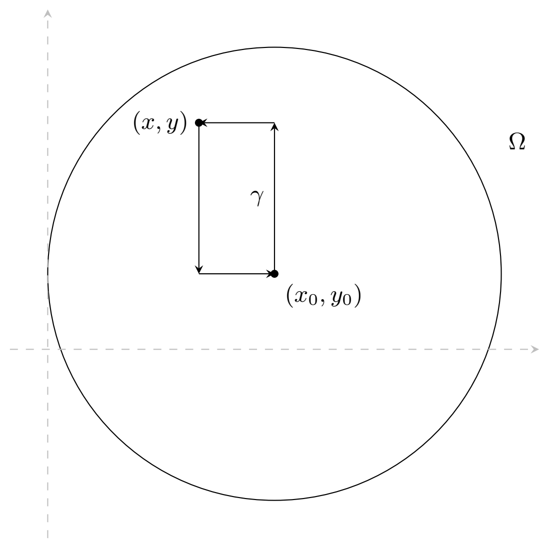

Figure 1: The center of Ω, (x_0, y_0), is the basepoint, as in the previous proof. As Ω is a disk, to any (x, y) in Ω one can find a rectangle, with sides parallel to the axes, as pictured. The proof then proceeds unchanged.