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0.1 Normal Families of Meromorphic Functions (3-5-2021)

We explore the idea of a normal family of *meromorphic* functions, and prove an analogue of Montel's theorem for such families, which is due to Marty.

0.1.1 Meromorphic Functions

Recall from the end of last time that a family of continuous functions on a domain $\Omega \subseteq \mathbb{C}$, with values in the Riemann sphere (i.e. extended complex plane with the chordal metric) is normal if and only if it is equicontinuous. This is a simplified version of Arzela-Ascoli which follows because every continuous function is bounded by 2 in the chordal metric. We're interested in families of meromorphic functions, meaning holomorphic functions into the Riemann sphere.

Lemma 0.1.1. *Let $\{f_n\}$ be a sequence of meromorphic functions on a domain Ω which converges uniformly on compact subsets of Ω in the chordal metric. Then the limit function f is meromorphic or identically ∞.*

Proof. If $|f(z_0)| < \infty$, then f is bounded in a neighbourhood of z_0, implying that $f_n \to f$ uniformly (in the Euclidean metric) in a neighbourhood of z_0. So f is holomorphic in a neighbourhood of z_0 (meaning $1/f$ has no zero near z_0).

If $f(z_0) = \infty$, then $1/f_n$ is bounded in a neighbourhood of z_0 for large enough n, so $1/f$ is holomorphic in a neighbourhood of z_0. So $1/f$ has an isolated zero at z_0, i.e. a pole of f, or $1/f$ is identically zero near z_0. In the latter case, the set of non-isolated zeroes of $1/f$ is open and closed and non-empty in Ω, and is therefore equal to Ω. $\square$

Does Montel's theorem have a sensible analogue for families of meromorphic functions in the chordal metric? Montel's criteria were

(1) Normality,

(2) Local-boundedness,

(3) Local-boundedness of derivatives, plus boundedness at a point.

Since all continuous functions are bounded in the chordal metric, the equivalence of (1) and (2) has no sensible generalization. However, the equivalence of (1) and (3) does, using the spherical derivative. (Of course, we can drop the "boundedness at a point" in (3).)

0.1.2 The Spherical Derivative

The *spherical derivative* of a meromorphic function f on a domain $\Omega \subseteq \mathbb{C}$ is

$$f^\sharp(z) = \lim_{w \to z} \frac{d(f(z), f(w))}{|z-w|}.$$

If $\Omega \subseteq S^2$, we use the chordal metric in the denominator instead of the Euclidean metric.

What properties does the spherical derivative have? If z is not a pole, then

$$f^\sharp(z) = \lim_{w \to z} \frac{2|f(z)-f(w)|}{|z-w|\sqrt{(1+|f(z)|^2)(1+|f(w)|^2)}} = \frac{2|f'(z)|}{1+|f(z)|^2}.$$

Near poles? Since the chordal metric is invariant under inversion, $(1/f)^\sharp = f^\sharp$. So $f^\sharp(z)$ is finite and continuous at all $z \in \Omega$. Also, note that (by the above formula) $f^\sharp(z) > 0$ if and only if f is one-to-one in a neighbourhood of z.

Theorem 0.1.1. *(Marty) Let $\mathcal{S}$ be a family of meromorphic functions on a domain Ω. Then $\mathcal{S}$ is normal in the chordal metric if and only if $\mathcal{S}^\sharp = \{f^\sharp : f \in \mathcal{S}\}$ is locally bounded.*

Proof. Suppose that $\mathcal{S}$ is normal in the chordal metric, but that the spherical derivatives are not bounded in any neighbourhood of z_0, i.e. there exist $f_n \in \mathcal{S}$ and $z_n \to z_0$ such that $f_n^\sharp(z_n) \to \infty$. By normality, we can assume that f_n converges uniformly on compact subsets of Ω in the chordal metric. By the lemma, the limit f is either meromorphic or identically ∞.

- If $f(z_0) \neq \infty$: f is bounded in the Euclidean metric in a neighbourhood U of z_0. Since $f_n \to f$ in the chordal metric, f_n is also bounded on U for large enough n. So $f_n \to f$ uniformly on compact subsets of U in the Euclidean metric. Then $f_n' \to f'$ uniformly on compact subsets of U, and so $f_n^\sharp \to f^\sharp$, a contradiction.

- If $f(z_0) = \infty$, apply the same argument to $1/f_n$.

Conversely, suppose that the set of spherical derivatives of $f \in \mathcal{S}$ is bounded, say by M, in some disk $D \subset \Omega$. For any n, let $z_j = z + (w-z) \cdot j/n$, for $0 \leq j \leq n$. Then

$$d(f(z), f(w)) \leq \sum_{j=1}^{n} d(f(z_{j-1}), f(z_j)) \approx \sum_{j=1}^{n} f^\sharp(z_j)|z_j - z_{j-1}| \leq M|z-w|.$$

So $\mathcal{S}$ is equicontinuous on D with respect to the chordal metric. By Arzela-Ascoli, $\mathcal{S}$ is normal on D. Therefore $\mathcal{S}$ is normal on Ω. □

One can make the last estimate in the proof more precise by writing

$$d(f(z), f(w)) \leq \int_\gamma f^\sharp(\zeta)\,|d\zeta| \leq M|z-w|,$$

where γ is the straight line in D joining z and w in D. This is because $d(f(z), f(w))$ is no greater than the spherical distance between $f(z), f(w)$ measured along $f \circ \gamma$ as points on the Riemann sphere, which is measured by the integral

$$\int_{f\circ\gamma} \frac{2}{1+|\zeta|^2}\,|d\zeta| = \int_\gamma \frac{2|f'(\zeta)|}{1+|f(\zeta)|^2}\,|d\zeta| = \int_\gamma f^\sharp(\zeta)\,|d\zeta|,$$

where γ is the straight-line curve in D joining z to w. See e.g. Gamelin's book on complex analysis for more details on the geometric interpretation of the spherical derivative.

We're going to use the results developed up until now about the topology of spaces of holomorphic and meromorphic functions to deduce some remarkable geometric results. We'll first study conformal mappings and work on proving the Riemann mapping theorem. Then, we'll work towards Picard's great theorem.