Complex Analysis

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These are the (in-progress) complex analysis notes I created while taking the 4th year undergraduate / core graduate course MAT454/MAT1002, Complex Analysis, at the University of Toronto. The course was taught online by Edward Bierstone in the Winter of 2021. These notes are my attempt to capture some of his great pedagogy in a written form, as well as a way for me to help myself study for the course.

In a prototype version of the notes, I have included the problems that were assigned as homework during the semester, and have tried to place them after the appropriate chapters. This part may be subject to change, as well as the addition of more problems in the future. (For example, past complex analysis qualifying exam problems.)

The first chapter is a (somewhat incompete) review of the material learned in a first course in complex analysis. Missing from it are some basic results, including but not limited to: the argument principle, Rouche's theorem, and material on fractional linear transformations (which, as of the time of this writing, may be covered later in the course). (Also, the material on Cauchy's integral formula could be condensed a little; sections 1.3 and 1.4 could probably be combined.)

Contents

Chapter 1

Basic Complex Analysis

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1.1 Review of Basic Complex Analysis (1-11-2021)

1.1.1 Review of Holomorphic Functions

We recall some basic notions of the theory of complex-valued functions of one complex variable.

Definition 1.1.1. *$f(z)$ is holomorphic at $z \in \mathbb{C}$ if*

$$\lim_{h \to 0} \frac{f(z+h) - f(z)}{h}$$

exists, i.e. if

$$f(z+h) - f(z) = c \cdot h + \varphi(h) \cdot h,$$

where $\lim_{h \to 0} \varphi(h) = 0$*, for some c. We write $c = f'(z)$.*

How is this different from the usual derivative? Write $c = a + ib$, and $h = \xi + i\eta$. Then, the derivative is the linear transformation

$$\begin{pmatrix} \xi \\ \eta \end{pmatrix} \mapsto \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \begin{pmatrix} \xi \\ \eta \end{pmatrix}.$$

This is different from the usual derivative of a function $\mathbb{R}^2 \to \mathbb{R}^2$, as the matrix has a special form. The first column is $\partial f / \partial x$, and the second column is $\partial f / \partial y$. Thus,

$$\frac{\partial f}{\partial x} + i \frac{\partial f}{\partial y} = 0.$$

Writing $f = u + iv$, we obtain the equivalent form

$$\begin{aligned} \frac{\partial u}{\partial x} &= \frac{\partial v}{\partial y}, \\ \frac{\partial u}{\partial y} &= -\frac{\partial v}{\partial x}, \end{aligned}$$

the *Cauchy-Riemann equations*. We see that f is holomorphic at z, if and only if it is differentiable at z as a function of x and y, and it satisfies the Cauchy-Riemann equations.

Suppose, now, that $f(x, y)$ is a differentiable, complex-valued function. One has the *differential*

$$df = \frac{\partial f}{\partial x}\,dx + \frac{\partial f}{\partial y}\,dy.$$

If $z = x + iy$, then $dz = dx + idy$, and $d\bar{z} = dx - idy$. One obtains

$$\begin{aligned} dx &= \frac{1}{2}(dz + d\bar{z}), \\ dy &= \frac{1}{2i}(dz - d\bar{z}). \end{aligned}$$

Thus,

$$df = \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right)dz + \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right)d\bar{z}.$$

Because of this, we define

$$\begin{aligned} \frac{\partial}{\partial z} &= \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right), \\ \frac{\partial}{\partial \bar{z}} &= \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right), \end{aligned}$$

so that we may write

$$df = \frac{\partial f}{\partial z}\,dz + \frac{\partial f}{\partial \bar{z}}\,d\bar{z}.$$

The Cauchy-Riemann equations then take on the particularly simple form

$$\frac{\partial f}{\partial \bar{z}} = 0.$$

1.1.2 Review of Harmonic Functions

Definition 1.1.2. *We say that a function $f(x, y)$ is harmonic if it is C^2, and satisfies the PDE*

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0,$$

Laplace's equation. *The differential operator*

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

is known as the Laplacian*.*

Laplace's equation is equivalent to

$$\frac{\partial^2 f}{\partial z \partial \bar{z}} = 0,$$

which can be seen by multiplying $\partial f/\partial z$ and $\partial f/\partial \bar{z}$. We note two things:

1. A complex-valued function is harmonic, if and only if its real and imaginary parts are harmonic. (Because Δ is a real differential operator.)

2. A holomorphic function is harmonic; thus, its real and imaginary parts are also harmonic.

A real-valued harmonic function $g(x, y)$ is *locally* the real part of a holomorphic functin f, and f is uniquely determined up to an additive constant. Why? One has

$$\frac{\partial}{\partial \bar{z}} \left(\frac{\partial g}{\partial z} \right) = 0,$$

so $\partial g/\partial z$ is holomorphic, thus it admits local primitives. However, it need not admit a *global* primitive: consider $\log |z|$ on $\mathbb{C} \setminus \{0\}$. This function is not the real part of any holomorphic function, because if it were, that holomorphic function would have to be the complex logarithm, which has no single-valued branch in $\mathbb{C} \setminus \{0\}$.

1.1.3 Extending the Complex Plane

We sometimes wish to extend the complex plane to include a point at infinity. We say that a function $f(z)$ is *holomorphic at* ∞ if $f(1/z)$ is holomorphic at 0. In order to think about this globally and geometrically, we will introduce the *Riemann sphere*.

Consider the sphere S^2, defined by $x^2 + y^2 + t^2 = 1$. We will think of the complex plane as $\mathbb{R}^2$, defined by $t = 0$. We use stereographic projection from the north pole $N = (0, 0, 1)$ to identify $S^2 \setminus \{N\}$ with the complex plane.

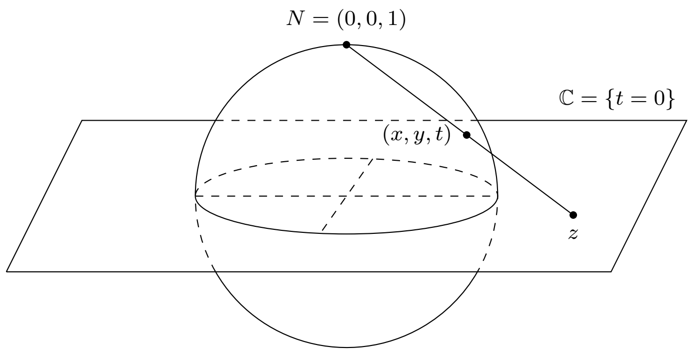

Figure 1.1: (Diagram found on MSE and edited slightly, not mine.)

The stereographic projection of $(x, y, t) \in S^2 \setminus \{N\}$ from N is the point

$$z = \frac{x + iy}{1 - t}.$$

To see this, one needs only verify that $(0, 0, 1)$, (x, y, t), and $(\frac{x}{1-t}, \frac{y}{1-t}, 0)$ are colinear.

This stereographic projection is a homeomorphism of $S^2 \setminus \{N\}$ onto $\mathbb{C}$. We then say that this gives a complex structure on the sphere. In order to think of the Riemann sphere as the complex plane with a point at infinity, we will use the complex conjugate of stereographic projection from the south pole. We will work out the details next time.

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1.2 The Riemann Sphere, Integration (1-13-2021)

We will complete the description of the complex structure on the Riemann sphere, introduce the notion of projective space, and then begin to review the theory of complex integration, with an eye towards Cauchy's theorem.

1.2.1 The Complex Structure on the Riemann Sphere

The complex conjugate of stereographic projection from the south pole $S = (0, 0, -1)$ is given by

$$z' = \frac{x - iy}{1 + t}.$$

This provides a homemorphism of $S^2 \setminus \{S\}$ onto $\mathbb{C}$. For any point $(x, y, t) \in S^2$, other than S or N, we have

$$zz' = \frac{x^2 + y^2}{1 - t^2} = 1;$$

in other words, $z' = 1/z$, a holomorphic transformation. Thus, we have covered the Riemann sphere with two coordinate charts, whose transition mapping is holomorphic. This is the sense in which the Riemann sphere obtains a complex structure.

1.2.2 One-Dimensional Complex Projective Space

We write $P^1(\mathbb{C})$ for the one-dimensional complex space, consisting of all of the lines in $\mathbb{C}^2$ through the origin. That is, $P^1(\mathbb{C}) = (\mathbb{C}^2 \setminus \{0\})/\sim$, where $(x_0, x_1) \sim (x_0', x_1')$ means that there exists a non-zero $\lambda \in \mathbb{C}$ such that $(x_0', x_1') = (\lambda x_0, \lambda x_1)$. We will write $[x_0, x_1]$ for the equivalence class of (x_0, x_1), and we call these classes *homogeneous coordinates.*

We may equip $P^1(\mathbb{C})$ with the structure of a complex manifold. Let

$$U_i = \{[x_0, x_1] \in P^1(\mathbb{C}) : x_i \neq 0\}, \qquad i = 0, 1.$$

We define two mappings

$$U_0 \to \mathbb{C}, \qquad [x_0, x_1] \mapsto \frac{x_1}{x_0} = z,$$

and

$$U_1 \to \mathbb{C}, \qquad [x_0, x_1] \mapsto \frac{x_0}{x_1} = z'.$$

Evidently, $zz' = 1$. Thus, $P^1(\mathbb{C})$ is obtained by gluing together two copies of $\mathbb{C}$ along the complements of $\{0\}$ by the formula $z' = 1/z$. Moreover, $P^1(\mathbb{C}) \cong S^2$, the Riemann sphere.

1.2.3 Integrating Forms along Curves

Let Ω be a (connected) open subset of $\mathbb{R}^2$. By a *differential form*, we mean an expression of the form $\omega = P\,dx + Q\,dy$, where P, Q are continuous (real or complex)-valued functions on Ω. Let $\gamma\colon [a,b] \to \Omega$ be a piecewise-C^1 curve in Ω, $\gamma(t) = (x(t), y(t))$. Then, we define

$$\int_\gamma \omega = \int_a^b f(t)\,dt,$$

where $f(t) = P(x(t), y(t))x'(t) + Q(x(t), y(t))y'(t)$. (That is, $f(t)\,dt = \gamma^*\omega$.)

The integral of ω over γ is independent of the curve's parametrization. Consider a reparametrization $t\colon [c,d] \to [a,b]$, with $t(c) = a$, $t(d) = b$, and $t'(s) > 0$, and set $\delta(s) = \gamma(t(s))$. Then

$$\int_\gamma \omega = \int_\delta \omega$$

by integration by substitution.

An important example of a differential form is the differential of a C^1 function F:

$$\omega = dF = \frac{\partial F}{\partial x}\,dx + \frac{\partial F}{\partial y}\,dy.$$

In this case, we call F a *primitive* of ω. Here,

$$\int_\gamma dF = \int_a^b (F \circ \gamma)'(t)\,dt = F(\gamma(b)) - F(\gamma(a)).$$

In particular, if γ is closed, the integral of dF over γ is zero.

Proposition 1.2.1. *ω has a primitive in Ω, if and only if $\int_\gamma \omega = 0$ for every piecewise-C^1 closed curve γ in Ω.*

Proof. We just saw the forward direction. Conversely, fix a point $(x_0, y_0) \in \Omega$. Given ω satisfying the hypotheses, define F by

$$F(x,y) = \int_\gamma \omega,$$

where γ is a piecewise-C^1 curve in Ω starting at (x_0, y_0) and ending at (x, y). This is independent of γ precisely by the hypothesis on γ.

We check that $dF = \omega$. First, let δ be a straight line in Ω from (x, y) to $(x + h, y)$, for sufficiently small h. Then,

$$F(x+h, y) - F(x,y) = \int_\delta \omega = \int_x^{x+h} P(\xi, y)\,d\xi,$$

so

$$\lim_{h\to 0} \frac{F(x+h,y)-F(x,y)}{h} = P(x,y).$$

Thus, $F(x,y)$ is differentiable in x, and $\frac{\partial F}{\partial x} = P$. The proof for the other variable is identical. Therefore $dF = \omega$. $\square$

In the case that Ω is an open disk, we can simplify the statement, and say that ω has a primitive in Ω, if and only if $\int_\gamma \omega = 0$ whenever γ is the boundary of a rectangle in Ω.

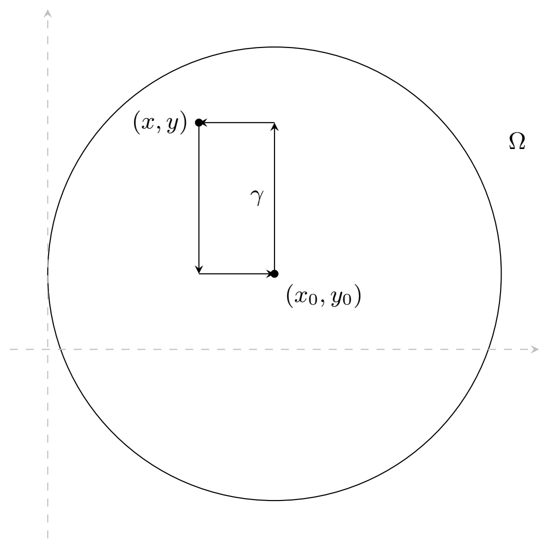

Figure 1.2: The center of Ω, (x_0, y_0), is the basepoint, as in the previous proof. As Ω is a disk, to any (x,y) in Ω one can find a rectangle, with sides parallel to the axes, as pictured. The proof then proceeds unchanged.

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1.3 Cauchy's Theorem (1-15-2021)

We state and prove Cauchy's theorem, and then begin the study of its corollaries.

1.3.1 Closed Forms

We continue with the setting of last time. Let $\omega = P\,dx + Q\,dy$ be a differential form on an open, connected set $\Omega \subseteq \mathbb{R}^2$, P and Q continuous.

Definition 1.3.1. *We say that a form ω is* closed *if any point has a neighbourhood in which ω has a primitive.*

A closed form need not have a global primitive. Take, for example, take $\Omega = \mathbb{C} \setminus \{0\}$ and $\omega = z^{-1}\,dz$. This is closed because local primitives are given by branches of log. It does not admit a global primitive, since its integral over the unit circle is $2\pi i \neq 0$.

Since we can test for the existence of local primitives of ω by looking at the integral of ω along rectangles, we can say that ω is closed, if and only if $\int_\gamma \omega = 0$ whenever γ is the boundary of a small rectangle in Ω.

This is *not* equivalent to the condition that $d\omega = 0$, since we are assuming merely continuity of P,Q. However, *if* ω is C^1, then ω is closed, if and only if $d\omega = 0$ (i.e. $\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$, by Green's formula, which we now recall).

Let γ be the boundary of a rectangle A in Ω, positively oriented. Then

$$\int_\gamma P\,dx + Q\,dy = \iint_A \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dxdy;$$

this formula holds whenever the statement makes sense (i.e. when these derivatives are continuous).

1.3.2 Cauchy's Theorem

What follows is one version of Cauchy's theorem, from which we will deduce Cauchy's integral formula later. Let Ω be any open set in $\mathbb{C}$.

Theorem 1.3.1. *If $f(z)$ is holomorphic in Ω, then the differential form $f(z)\,dz$ is closed.*

If we assume that $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ are continuous, this follows from Green's theorem and the Cauchy-Riemann equations. (Take d of $f(z)\,dz$.) This statement of continuity is actually true, but we are going to use Cauchy's theorem to prove it.

Proof. It's enough to show that the integral $\int_\gamma f(z)\,dz = 0$ for any γ which is the boundary of a rectangle $R \subset \Omega$ whose interior is contained in Ω. Divide R into four equal subrectangles $R_i, i = 1, 2, 3, 4$, each with boundary γ_i. Then,

$$\mu(R) = \int_\gamma f(z)\,dz = \sum_{i=1}^4 \int_{\gamma_i} f(z)\,dz = \sum_{i=1}^4 \mu(R_i),$$

since the subrectangles have common edges inside R with opposing orientations. Thus, $|\mu(R_i)| \geq \frac{1}{4}|\mu(R)|$ for some i, and call $R_i = R^{(1)}$, $\gamma_i = \gamma^{(1)}$.

Repeat this process to obtain a decreasing sequence $R \supset R^{(1)} \supset R^{(2)} \supset \cdots$, with

$$\left|\int_{\gamma^{(k)}} f(z)\,dz\right| \geq \frac{1}{4^k}|\mu(R)|.$$

Let z_0 be the single point in the intersection of all of the $R^{(k)}$'s. Since $f(z)$ is holomorphic at z_0,

$$\int_{\gamma^{(k)}} f(z)\,dz = f(z_0)\int_{\gamma^{(k)}} dz + f'(z_0)\int_{\gamma^{(k)}} (z - z_0)\,dz + \int_{\gamma^{(k)}} \varphi(z)(z - z_0)\,dz,$$

where $\lim_{z\to z_0}\varphi(z) = 0$. The first two integrals vanish because they are integrals of forms with (local) primitives. Thus, we need only evaluate the last. Given $\varepsilon > 0$, if k is sufficiently large, then the absolute value of the last integral is

$$\begin{aligned}\left|\int_{\gamma^{(k)}} \varphi(z)(z - z_0)\,dz\right| &\leq \varepsilon \cdot \operatorname{diag}(R^{(k)})\operatorname{perim}(R^{(k)}) \\ &= \frac{\varepsilon}{4^k} \cdot \operatorname{diag}(R)\operatorname{perim}(R).\end{aligned}$$

Now,

$$|\mu(R)| \leq 4^k\left|\int_{\gamma^{(k)}} \varphi(z)(z - z_0)\,dz\right| \leq \varepsilon \cdot \operatorname{diag}(R)\operatorname{perim}(R),$$

so since $\varepsilon > 0$ is arbitrary, $\mu(R) = 0$. □

As noted by someone in class, the estimates done in the proof when we used the fact that $f(z)$ is holomorphic may break down in the case that f is merely a smooth function, or real-analytic function, of two variables. (Check this.)

Corollary 1.3.1. *A holomorphic function $f(z)$ locally has a primitive, which is itself holomorphic.*

Proof. Locally,

$$f(z)\,dz = dF = \frac{\partial F}{\partial z}\,dz + \frac{\partial F}{\partial \bar{z}}\,d\bar{z},$$

so $\frac{\partial F}{\partial \bar{z}} = 0$. That is, the Cauchy-Riemann equations hold for F. Since F is also differentiable as a function of two variables, it is holomorphic. □

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1.4 Cauchy's Integral Formula (1-18-2021)

We briefly introduce Cauchy's integral formula.

1.4.1 Remarks

Let us note that in Cauchy's theorem, from last time, it is enough to assume that the function is continuous on the domain, and holomorphic outside a line (or outside even a finite number of lines and points).

We need the following theorem to prove Cauchy's integral formula. The proof will not be given.

Theorem 1.4.1. *A closed differential form ω on a simply-connected open set $\Omega \subseteq \mathbb{R}^2$ has a (globally defined) primitve.*

1.4.2 Cauchy's Integral Formula

Let γ be a closed curve in Ω, and let $a \in \Omega$ be a point not on the curve. We define the *winding number* of γ with respect to a as

$$w(\gamma, a) = \frac{1}{2\pi i} \int_\gamma \frac{dz}{z-a}.$$

The winding number is always an integer, since the integral is given by the difference between two branches of $\log(z-a)$. It is meant to measure how many times the curve γ loops around a. For example, if γ is a circle centered at a oriented counter-clockwise, then $w(\gamma, a) = 1$.

The winding number is invariant with respect to a homotopy of γ not passing through a, since the integrand $(z-a)^{-1}\, dz$ is a closed form on $\mathbb{C} \setminus \{a\}$. Furthermore, as a function of a, $w(\gamma, a)$ is constant on the connected components of $\mathbb{C} \setminus \gamma$.

Theorem 1.4.2. *(Cauchy's integral formula) Let $f(z)$ be holomorphic in the open set Ω, and let $a \in \Omega$. Suppose γ is a closed curve in Ω, not containing a, which is homotopic to a point in Ω. Then,*

$$\frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z-a}\, dz = w(\gamma, a) f(a).$$

We will prove this next time.

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1.5 Cauchy's Integral Formula, Continued (1-20-2021)

We prove Cauchy's integral theorem. We then deduce several important facts about holomorphic functions, including their smoothness, Morera's theorem, Liouville's theorem, the fundamental theorem of algebra, and the mean value property.

1.5.1 Proof of Cauchy's Integral Formula

Let us recall and prove Cauchy's integral formula.

Theorem 1.5.1. *(Cauchy's integral formula) Let $f(z)$ be holomorphic in the open set Ω, and let $a \in \Omega$. Suppose γ is a closed curve in Ω, not containing a, which is homotopic to a point in Ω. Then,*

$$\frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z-a}\,dz = w(\gamma, a)f(a).$$

Proof. Let

$$g(z) = \begin{cases} \frac{f(z)-f(a)}{z-a}, & z \neq a, \\ f'(a), & z = a. \end{cases}$$

Then, $g(z)$ is holomorphic in $\Omega \setminus \{a\}$ and continuous in Ω, so

$$\int_\gamma \frac{f(z)-f(a)}{z-a}\,dz = 0,$$

since $g(z)\,dz$ is closed, by Cauchy's theorem. But then

$$\int_\gamma \frac{f(z)}{z-a}\,dz = \int_\gamma \frac{f(a)}{z-a}\,dz = 2\pi i \cdot f(a)w(\gamma, a).$$

□

The most important case of Cauchy's integral formula to us will be the case where γ is the boundary of a disk, oriented counter-clockwise. If $f(z)$ is holomorphic in a neighbourhood of this disk, then

$$\frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z-a}\,dz = \begin{cases} f(a), & a \text{ inside the circle}, \\ 0, & a \text{ outside the circle}. \end{cases}$$

1.5.2 Applications of Cauchy's Integral Formula

Proposition 1.5.1. *A holomorphic function is infinitely differentiable (and all of its derivatives are holomorphic).*

Proof. Suppose $f(z)$ is holomorphic in some disk $D = \{z : |z| < R\}$, and let γ be the boundary of a smaller circle $\{|z| < r\}$, $r < R$, oriented counter-clockwise. Then, $\gamma(\theta) = re^{i\theta}$, $\theta \in [0, 2\pi]$. If $|z| < r$, then

$$f(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)}{\zeta - z}\, d\zeta.$$

We can differentiate under the integral to obtain.

$$f'(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)\, d\zeta}{(\zeta - z)^2}.$$

Furthermore, for any n,

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_\gamma \frac{f(\zeta)\, d\zeta}{(\zeta - z)^{n+1}}.$$

□

Let us summarize what we have learned thus far concerning Cauchy's theorem.

Proposition 1.5.2. *If $f(z)$ is continuous in Ω, then the following statements are equivalent.*

1. *$f(z)$ is holomorphic.*
2. *$f(z)\, dz$ is a closed form.*
3. *One has*
$$f(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)}{\zeta - z}\, d\zeta$$
for z in an open disk with boundary γ.

Proof. It remains to show (2) implies (1); this statement is known as Morera's theorem. Locally, $f(z)\, dz$ has a primitive $g(z)$. That is,

$$f(z)\, dz = dg = \frac{\partial g}{\partial z}\, dz + \frac{\partial g}{\partial \bar{z}}\, d\bar{z}.$$

Since dz and $d\bar{z}$ are linearly independent, $\frac{\partial g}{\partial \bar{z}} = 0$, i.e. the Cauchy-Riemann equations are satisfied. Thus, g is holomorphic. Therefore $f(z) = \frac{\partial g}{\partial z}(z)$ is also holomorphic. □

We can use the integral formula to compute the Taylor series of a holomorphic function $f(z)$ at 0:

$$f(z) = \sum_{n=0}^{\infty} a_n z^n, \qquad a_n = \frac{f^{(n)}(0)}{n!} = \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)\,d\zeta}{\zeta^{n+1}}.$$

Why is this convergent?

$$\frac{1}{\zeta - z} = \frac{1}{\zeta}\left(1 - \frac{z}{\zeta}\right)^{-1} = \frac{1}{\zeta}\left(1 + \frac{z}{\zeta} + \frac{z^2}{\zeta^2} + \cdots\right).$$

Thus,

$$f(z) = \frac{1}{2\pi i}\int_\gamma \sum_{n=0}^{\infty} z^n \cdot \frac{f(\zeta)}{\zeta^{n+1}}\,d\zeta.$$

We can interchange the sum and the integral, as for a fixed z, $|z| < r$, the series

$$\sum_{n=0}^{\infty} z^n \cdot \frac{f(\zeta)}{\zeta^{n+1}}$$

is uniformly and absolutely convergent on $|\zeta| = r$. Thus, the Taylor series converges when $|z| < r$. Moreover, if we substitute $z = re^{i\theta}$, it becomes

$$f(re^{i\theta}) = \sum_{n=0}^{\infty} a_n r^n e^{in\theta},$$

which proves that

$$a_n r^n = \frac{1}{2\pi}\int_0^{2\pi} e^{-in\theta} f(re^{i\theta})\,d\theta.$$

These are known as the *Fourier coefficients*. This gives us the *Cauchy inequalities*, which give us an upper bound for the coefficients a_n (or their modulus). Let $M(r) = \sup_\theta |f(re^{i\theta})|$, the upper bound of $|f|$ on the circle of radius r. By the integral formula, $|a_n r^n| \leq M(r)$, or

$$|a_n| \leq \frac{M(r)}{r^n}.$$

Now we deduce

Theorem 1.5.2. *(Liouville's theorem) A bounded holomorphic function defined on all of $\mathbb{C}$ is constant.*

Proof. $M(r) \leq M$, for some constant M, so $|a_n| \leq Mr^{-n}$ *for all* r. So $a_n = 0$ for $n > 1$. ☐

Corollary 1.5.1. *(Fundamental theorem of algebra) Every non-constant polynomial has a root in* $\mathbb{C}$*.*

Proof. If $p(z)$ is a polynomial without a root, then $1/p(z)$ is holomorphic in $\mathbb{C}$ and bounded, hence constant. □

Theorem 1.5.3. *(Mean value property) If* $f(z)$ *is holomorphic in* Ω *and* $D \subset \Omega$ *is a compact disk centered at* a*, then*

$$f(a) = \textit{mean value of } f \textit{ on the boundary of } D.$$

Proof. We may assume $a = 0$. Then

$$f(0) = \frac{1}{2\pi} \int_0^{2\pi} f(re^{i\theta})\, d\theta.$$

□

At the beginning of next lecture, we will go over more consequences of Cauchy's integral formula.

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1.6 Harmonic Functions (1-22-2021)

Armed with the mean value property, we take a deeper look at harmonic functions. We show that harmonic functions satisfy three equivalent definitions: those functions whose Laplacians vanish, those functions expressible locally as the real parts of holomorphic functions, and those continuous functions with the mean value property.

1.6.1 The Mean Value Property

Before proceeding, let us make a definition that will be useful to us.

Definition 1.6.1. *Let f be a continuous real-or-complex valued function defined on an open set Ω in $\mathbb{C}$. We say that f has the* mean value property *if for every $a \in \Omega$, there is an $R > 0$ such that $\{|z-a| \leq R\} \subset \Omega$, and for $0 < r < R$,*

$$f(a) = \frac{1}{2\pi} \int_0^{2\pi} f(a + re^{i\theta})\, d\theta.$$

We saw last time that holomorphic functions enjoy the mean value property. By splitting f up into its real and imaginary parts, we see that f has the mean value property if and only if $\mathrm{Re}(f)$ and $\mathrm{Im}(f)$ both have the mean value property.

Theorem 1.6.1. *(Maximum modulus principle) Let f be a complex-valued function on an open subset $\Omega \subset \mathbb{C}$ with the mean value property. If $|f|$ has a local maximum at a point $a \in \Omega$, then f is constant in a neighbourhood of a.*

Proof. Write $f = u + iv$. Since $|f|$ has a local maximum at a, we can find an $R > 0$ such that $D = \{|z-a| \leq R\} \subset \Omega$ and $|f(z)| \leq |f(a)|$ for $z \in D$; furthermore, by choosing R small enough, we can assume that $f(z)$ equals its mean value along the boundary of D. Clearly u is maximized at a in D, so since u also has the mean value property,

$$u(a) = \frac{1}{2\pi} \int_0^{2\pi} u(a + Re^{i\theta})\, d\theta \leq \max_{|z-a| \leq R} u(z) \leq u(a).$$

It follows that $u(z) = u(a)$ for $z \in D$. Similarly, v has the mean value property, so it follows that $f(z) = f(a)$ for $z \in D$. $\square$

The following result is a consequence of the maximum modulus principle. It will be very important to us later in the course. (See Ahlfors, pp. 135, Theorem 13.)

Theorem 1.6.2. *(Schwarz's lemma) Let $f \colon D \to \mathbb{C}$ be a holomorphic function on the open unit disk $D \subset \mathbb{C}$ such that $f(0) = 0$, $|f(z)| \leq 1$ on D, and $|f'(0)| \leq 1$. Then $|f(z)| \leq |z|$ for all $z \in D$. Moreover, if $|f(z_0)| = |z_0|$ for some non-zero $z_0 \in D$, or if $|f'(0)| = 1$, then there is an $a \in \mathbb{C}$, $|a| = 1$, such that $f(z) = az$ for all $z \in D$.*

Proof. Define $g\colon D \to \mathbb{C}$ by

$$g(z) = \begin{cases} \frac{f(z)}{z}, & z \neq 0, \\ f'(0), & z = 0. \end{cases}$$

Then g is holomorphic on D since f is holomorphic and $f(0) = 0$. If $0 < R < 1$, then the maximum modulus principle applied on the disk $\{|z| \leq R\}$ gives

$$|g(z)| \leq \max_{|z|=R} |g(z)| = \max_{|z|=R} \left| \frac{f(z)}{z} \right| \leq \frac{1}{R},$$

so if we let R approach 1 from below, we get $|g(z)| \leq 1$, or $|f(z)| \leq |z|$, for each $z \in D$. Furthermore, if for some non-zero $z_0 \in D$ one has $|f(z_0)| = |z_0|$, then $|g(z_0)| = 1$, so by the maximum modulus principle, g is constant. But if g is constant, it follows that there is an a, $|a| = 1$, such that $f(z) = az$ for all $z \in D$. □

1.6.2 Harmonic Functions

The real and imaginary parts of a function with the mean value property also have the mean value property. We'll see that the functions with the mean value property are precisely the harmonic functions. The following proposition immediately implies one direction: harmonic functions have the mean value property, and in particular, satisfy the maximum modulus principle.

Proposition 1.6.1. *A real-valued harmonic function $g(x, y)$ is locally the real part of a holomorphic function, uniquely determined up to the addition of a constant.*

Proof. We have

$$\frac{\partial^2 g}{\partial z \partial \bar{z}} = 0,$$

so we know $\frac{\partial g}{\partial z}$ is holomorphic. It therefore has a local primitive f uniquely determined up to an additive constant:

$$df = \frac{\partial g}{\partial z}\, dz.$$

We can write this as

$$d\bar{f} = \frac{\partial g}{\partial \bar{z}}\, d\bar{z}.$$

Since g is real-valued, $\frac{\partial g}{\partial \bar{z}}$ is the conjugate of $\frac{\partial g}{\partial z}$. Therefore

$$d(f + \bar{f}) = dg,$$

which proves that $g = 2 \cdot \mathrm{Re}(f) + (\text{real const.})$. □

Now, let $g(x, y)$ be a real-valued harmonic function, equal to the real part of some holomorphic function $f(z) = \sum a_n z^n$, converging in some open disk of radius R. Consider inside this disk a smaller one of radius $r < R$. We can assume $a_0 \in \mathbb{R}$. On the boundary of the smaller disk,

$$g(r\cos\theta, r\sin\theta) = \mathrm{Re}(f(re^{i\theta})) = a_0 + \frac{1}{2}\sum_{n=1}^{\infty} r^n \left(a_n e^{in\theta} + \bar{a}_n e^{-in\theta}\right),$$

which is uniformly and absolutely convergent with respect to $\theta \in [0, 2\pi]$. Let's compute the constants.

$$a_0 = \frac{1}{2\pi}\int_0^{2\pi} g(r\cos\theta, r\sin\theta)\, d\theta,$$

and

$$a_n = \frac{1}{\pi}\int_0^{2\pi} \frac{g(r\cos\theta, r\sin\theta)}{(re^{i\theta})^n}\, d\theta.$$

We get, for $|z| < r$,

$$f(z) = \frac{1}{2\pi}\int_0^{2\pi} g(r\cos\theta, r\sin\theta) \underbrace{\left\{1 + 2\sum_{n=1}^{\infty}\left(\frac{z}{re^{i\theta}}\right)^n\right\}}_{=\frac{re^{i\theta}+z}{re^{i\theta}-z}} d\theta$$

This expresses the holomorphic function $f(z)$ in $|z| < r$ in terms of its real part on the boundary. Equating real parts of both sides of the above equation,

$$g(x, y) = \frac{1}{2\pi}\int_0^{2\pi} g(r\cos\theta, r\sin\theta) \cdot \frac{r^2 - |z|^2}{|re^{i\theta} - z|^2}\, d\theta.$$

This formula is valid in $|z| < r$, for any real-valued harmonic function in a neighbourhood of $\{|z| \leq r\}$. We call the term

$$\frac{r^2 - |z|^2}{|re^{i\theta} - z|^2}$$

the *Poisson kernel*. Thus, a harmonic function is determined by its values on the boundary of the disk.

1.6.3 Dirichlet Problem for a Disk

We are now ready to study the converse problem. Given a continuous function $f(\theta)$ on the circle with center 0 and radius r, can we find $F(z)$ continuous in $|z| \leq r$ and harmonic in $|z| < r$, such that $F(re^{i\theta}) = f(\theta)$?

Theorem 1.6.3. *We can, and the solution is unique.*

Proof. We can assume that the functions are real-valued. Uniqueness follows from the maximum modulus principle as follows: if F_1, F_2 are two solutions, then $F_1 - F_2$ is a harmonic function in $|z| < r$, continuous in $|z| \le r$, taking its maximum and minimum values on the boundary. However, it is identically zero on the boundary, so it holds that $F_1 \equiv F_2$.

All that's left is existence. For $|z| < r$, define $F(z)$ by

$$F(z) = \frac{1}{2\pi} \int_0^{2\pi} f(\theta) \cdot \frac{r^2 - |z|^2}{|re^{i\theta} - z|^2} \, d\theta.$$

The Poisson kernel in the above expression is just the real part of $(re^{i\theta} + z)/(re^{i\theta} - z)$, so $F(z)$ is the real part of a holomorphic fuction (differentiate under the integral). And we already saw that real parts of holomorphic functions are harmonic. Now, we need only check that

$$\lim_{z \to re^{i\theta_0}} F(z) = f(\theta_0);$$

this is a direct calculation and we will not do it in class. □

Corollary 1.6.1. *Any continuous function in an open subset $\Omega \subseteq \mathbb{R}^2$ with the mean value property is harmonic.*

Proof. Pick $a \in \Omega$ and $r > 0$ so that $D = \{|z - a| \le r\} \subset \Omega$. Then $f|_{\partial D}$ extends to a harmonic function in the interior. Then $f - F$ is zero on ∂D and has the mean value property, so by the maximum modulus principle it's zero on D. □

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1.7 Meromorphic Functions (1-25-2021)

We study meromorphic functions on the complex plane and on the Riemann sphere, and we develop the Laurent series expansion for meromorphic functions. We also prove the partial fraction decomposition theorem.

1.7.1 Zeroes and Poles

Let Ω be an open subset of $\mathbb{C}$, and let $f(z)$ be holomorphic in Ω. Suppose $f(z_0) = 0$. Then $f(z) = (z - z_0)^k f_1(z)$ for some holomorphic f_1 in Ω with $f_1(z_0) \neq 0$. (To see this, write the Taylor expansion at z_0.) We call k the *order* or *multiplicity* of the zero z_0. It follows that the zeroes of a holomorphic function that doesn't vanish identically are isolated, at least if Ω is connected.

A *meromorphic* function in Ω is a holomorphic function defined in the complement of a discrete set in Ω, which is in a (perhaps punctured) neighbourhood of any point expressible as the quotient of holomorphic functions $f(z)/g(z)$, where $g \not\equiv 0$. The set of meromorphic functions on a *domain* Ω (open, connected) form a field.

At a point z_0, write $f(z) = (z - z_0)^k f_1(z)$ and $g(z) = (z - z_0)^l g_1(z)$, where f_1, g_1 are holomorphic functions with $f_1(z_0), g_1(z_0) \neq 0$. So

$$\frac{f(z)}{g(z)} = (z - z_0)^{k-l} \frac{f_1(z)}{g_1(z)}.$$

If $k \geq l$, then f/g extends to be holomorphic at z_0. If $k < l$, then we say that z_0 is a *pole* of f/g of *order* or *multiplicity* $l - k$. Then, we will say that

$$\lim_{z \to z_0} \frac{f(z)}{g(z)} = \infty,$$

i.e. meromorphic functions take values in the Riemann sphere. Pursuing this, we get a nicer definition of meromorphic functions: a meromorphic function on Ω is a holomorphic function $\Omega \to S^2$.

1.7.2 Laurent Series, Partial Fraction Decomposition

We're going to see that a holomorphic function $f(z)$ in an annulus $r < |z| < R$ has a *Laurent expansion*:

$$\sum_{n=-\infty}^{\infty} a_n z^n = \underbrace{\sum_{n<0} a_n z^n}_{\text{holom. in } |z| > r} + \underbrace{\sum_{n \geq 0} a_n z^n}_{\text{holom. in } |z| < R}.$$

We can rephrase the condition on the first sum as follows: if $z = 1/\zeta$, then

$$\sum_{n<0} a_n z^n = \sum_{n<0} a_n \zeta^{-n} = \sum_{n=1}^{\infty} a_{-n}\zeta^n$$

is holomorphic in $|\zeta| < 1/r$.

We are going to get this from Cauchy's integral formula. Let γ_1 be a circle of radius $r_1 \in (r, R)$, and γ_2 a circle of radius $r_2 \in (r, R)$, with $r < r_2 < r_1 < R$. If $r_2 < |z| < r_1$, then Cauchy's integral gives

$$f(z) = \frac{1}{2\pi i}\int_{\gamma_1} \frac{f(\zeta)}{\zeta - z}\, d\zeta - \frac{1}{2\pi i}\int_{\gamma_2} \frac{f(\zeta)}{\zeta - z}\, d\zeta,$$

which can be seen by connecting γ_1 and γ_2 by an arbitrarily small line segments avoiding z, making a closed curve. (Draw a picture.) In the second integral,

$$\frac{1}{\zeta - z} = -\frac{1}{z}\cdot\frac{1}{1-\frac{\zeta}{z}} = -\sum_{n=0}^{\infty}\frac{\zeta^n}{z^{n+1}} = -\sum_{n<0}\frac{z^n}{\zeta^{n+1}}.$$

This power series is uniformly and absolutely convergent on $|\zeta| = r_2$, so we get

$$f(z) = \sum_{n=-\infty}^{\infty} a_n z^n, \qquad a_n = \frac{1}{2\pi i}\int_{\gamma_2}\frac{f(\zeta)}{\zeta^{n+1}}\, d\zeta, \qquad n < 0.$$

The Laurent series is uniformly and absolutely convergent when $r_2 \le |z| \le r_1$. We can think of this as like a version of Cauchy's theorem, but at a pole or for a function in an annulus.

Holomorphic functions on the Riemann sphere with values in $\mathbb{C}$ are constant by Liouville's theorem. What about meromorphic functions on the Riemann sphere?

Theorem 1.7.1. *Any meromorphic function on S^2 is rational.*

Proof. Let's say $f(z)$ has poles $b_1, \ldots, b_n$, and possibly ∞, with corresponding *principal parts* (negative parts of the Laurent expansions)

$$P_k\left(\frac{1}{z - b_k}\right),$$

polynomials in $1/(z - b_k)$, and possibly

$$P_\infty\left(\frac{1}{\zeta}\right) = P_\infty(z),$$

where $\zeta = 1/z$ is the "coordinate at infinty." Then

$$f(z) - \sum_{k=1}^{n} P_k\left(\frac{1}{z-b_k}\right) - P_\infty(z)$$

is holomorphic on S^2, hence a constant a. Then

$$f(z) = a + P_\infty(z) + \sum_{k=1}^{n} P_k\left(\frac{1}{z-b_k}\right),$$

the *partial fraction decomposition* of a rational function. $\square$

If we write this last expression as $\frac{P(z)}{Q(z)}$, $\deg P(z) = p$, $\deg Q(z) = q$, then $a + P_\infty(z)$ is present only if $p > q$. (Quotient by long division!) As an exercise, deduce the theorem on real partial fraction decomposition.

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1.8 Isolated Singularities and Residues (1-27-2021)

We initiate the study of the zeroes and singularities of holomorphic and meromorphic functions in the complex plane and on the Riemann sphere. We discuss the behaviour of functions near their poles and essential singularities, and then we introduce the notion of residue of a differential form. We state the residue theorem for integration.

1.8.1 Singularities

We say that a holomorphic function $f(z)$ in the punctured disk $0 < |z| < R$ has an *isolated singularity* at 0 if $f(z)$ can't be extended to a holomorphic function on all of $|z| < R$. There are two cases: 0 is either a pole, else we call it an *essential singularity*. What is the difference? For a pole, the principal part of the Laurent expansion of $f(z)$,

$$f(z) = \sum_{n=-\infty}^{\infty} a_n z^n,$$

is a finite sum. So an essential singularity means that the principal part is an infinite sum.

Extension to a holomorphic function in $|z| < R$ is possible if and only if f is bounded in a neighbourhood of 0. Why? Take $r \in (0, R)$. Then

$$f(re^{i\theta}) = \sum_{n=-\infty}^{\infty} a_n r^n e^{in\theta}.$$

Integrate $e^{-in\theta} f(re^{i\theta})$ with respect to θ to get

$$a_n r^n = \frac{1}{2\pi} \int_0^{2\pi} e^{-in\theta} f(re^{i\theta})\, d\theta, \qquad n \in \mathbb{Z}.$$

Therefore $|a_n| \leq M(r) r^{-n}$, where $M(r)$ is an upper bound for $|f(z)|$, $|z| = r$. If f is bounded in the punctured disk, then there is an $M > 0$ such that every $M(r) \leq M$. Then, when $n < 0$, $|a_n| \leq M r^{-n}$; since $-n > 0$, taking $r \to 0$ shows that $a_n = 0$. Therefore f can be extended holomorphically to 0.

Thus, there are three options for a holomorphic function $f(z)$ in a punctured disk centered at 0:

(i) A *removable singularity*: bounded in a neighbourhood of 0, in which case it extends holomorphically to the entire disk.

(ii) A pole, in which case $\lim_{z \to 0} f(z) = \infty$.

(iii) An essential singularity. What can we say about the limit here? It's not even well-defined, as the following theorem shows.

The following theorem is often called the *Casorati-Weierstrass theorem*. It displays, in some sense, just how unmanageable essential singularities are, compared to poles.

Theorem 1.8.1. *If* 0 *is an essential singularity, then for all* $\varepsilon > 0$*,* $f(\{0 < |z| < \varepsilon\})$ *is dense in* $\mathbb{C}$*.*

Proof. If not, we can find an $a \in \mathbb{C}$ and $\delta > 0$, such that $|f(z)-a| \geq \delta$, for all $0 < |z| < \varepsilon$. Consider

$$g(z) = \frac{1}{f(z) - a}.$$

Then $g(z)$ is holomorphic in $0 < |z| < \varepsilon$, and bounded, since $|g(z)| \leq 1/\delta$. So $g(z)$ extends holomorphically to $|z| < \varepsilon$. But then $f(z) = a + 1/g(z)$ is meromorphic, and has either a pole or a removable singularity at 0, contradicting that 0 is an essential singularity of $f(z)$. □

1.8.2 Residues

Let $f(z)$ be a holomorphic function in a punctured neighbourhood of a. Let γ be a curve around a with winding number $w(\gamma, a) = 1$. We call the integral

$$\frac{1}{2\pi i} \int_\gamma f(z)\, dz$$

the *residue* of the differential form $f(z)\, dz$ at a. What does this mean in terms of the Laurent series expansion

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z-a)^n?$$

For every $n \neq -1$, $(z-a)^n\, dz$ has a primitive, so the integral vanishes. For $n = 1$, the integral is $2\pi i a_{-1}$, so the residue is simply the coefficient a_{-1}.

Why is it beneficial to consider the residue of the *form* $f(z)\, dz$ instead of that of the function $f(z)$? What is the residue at ∞? Consider coordinates z' at infinity, $z = 1/z'$. Then

$$f(z)\, dz = -\frac{1}{(z')^2} f\left(\frac{1}{z'}\right) dz'.$$

We integrate over a curve with winding number 1 around ∞:

$$\operatorname{res}(f, \infty) = -\frac{1}{2\pi i} \int_\gamma \frac{1}{(z')^2} f\left(\frac{1}{z'}\right) dz' = -a_{-1},$$

where γ is positively oriented with respect to $z' = 0$ (i.e. $z = \infty$), with the Laurent series expansion taken in some annulus $|z| > R$. If $\omega = f(z)\,dz$, then this is just the integral $\frac{1}{2\pi i}\int_\gamma \omega$, where γ is positively oriented with respect to ∞ (and we computed it using coordinates at ∞). Thus, the residue of the form $f(z)\,dz$ at ∞ has the exact same formula as the residue at any other point, which is one reason it is beneficial to think of residues of forms instead of residues of functions.

Theorem 1.8.2. *(Residue theorem) Let Ω be an open subset of the Riemann sphere. Let $K \subset \Omega$ be a compact set with a piecewise-C^1 oriented boundary Γ. Given a function $f(z)$ holomorphic in Ω, except perhaps at isolated points, and with Γ not containing any singular points or ∞. Then*

$$\int_\Gamma f(z)\,dz = 2\pi i \sum \operatorname{res}(f, z_k),$$

where the z_k's are the singularities in K (possibly including ∞).

Remark: To say that K has a piecewise-C^1 oriented boundary Γ means that Γ is a union of piecewise-C^1 closed curves $\gamma(t)$ that are positively oriented with respect to K. Informally, if you're walking along Γ, then K is always to your left. (To state this formally requires a bit of messy second year calculus.)

The homework problems assigned for Edward Bierstone's MAT454/1002 in Winter 2021 for the material in Chapter 1 of my notes.

1.9 Problems for Chapter 1

1. The *chordal distance* between two points on the Riemann sphere (considered as the unit sphere in $\mathbb{R}^3$) means the length of the line segment in $\mathbb{R}^3$ joining the points. The chordal distance induces a metric $d(z, w)$ on $\mathbb{C}$; i.e., if $z, w \in \mathbb{C}$, then $d(z, w)$ is defined as the chordal distance between the points of the Riemann sphere corresponding to z, w by stereographic projection. Show that

$$d(z, w) = \frac{2|z - w|}{\sqrt{(1 + |z|^2)(1 + |w|^2)}}.$$

Also show that, for $w = \infty$, the corresponding formula is

$$d(z, \infty) = \frac{2}{\sqrt{(1 + |z|^2)}}.$$

2. Find the image of the upper half-plane by the mapping

$$w = \frac{1 - z^\alpha}{1 + z^\alpha},$$

where $0 < \alpha < 1$ and z^α has its principal value.

3. Let f be a holomorphic function on a simply connected open set Ω. Assume that $f(z) \neq 0$, for all $z \in \Omega$.

   (a) Show how to define $\log f(z)$ as a holomorphic function on Ω, by means of an integral.

   (b) Show that, for any $n \in \mathbb{N}$, there is a holomorphic function g on Ω such that $f = g^n$.

4. Suppose that $f(z)$ is holomorphic. Let γ be a large positively oriented circle enclosing the points $\zeta = 0$ and $\zeta = z$. Show that

$$f(z) - \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)}{\zeta - z} \cdot \frac{z^n}{\zeta^n} \, d\zeta,$$

is a polynomial $g(z)$ of degree $n - 1$, such that

$$g^{(k)}(0) = f^{(k)}(0), \qquad k = 0, \ldots, n - 1.$$

5. How many zeros does

$$P(z) = z^4 + 8z^3 + 3z^2 + 8z + 3$$

have in the right half-plane $x \geq 0$?

6. (a) Show that, if $n \geq 2$, then

$$\int_0^\infty \frac{dx}{1+x^n} = \frac{\pi/n}{\sin(\pi/n)},$$

by integrating along the boundary of a circular sector with angle $2\pi/n$.

(b) Use the residue theorem to evaluate $\displaystyle\int_0^{2\pi} \cos^{2n}\theta\, d\theta$.

7. Formulate and prove a version of Schwarz's reflection principle for harmonic functions.

8. Prove tha all zeros of a polynomial

$$P(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$$

lie in the disk with centre $\{0\}$ and radius

$$\sqrt{1 + |a_{n-1}|^2 + \cdots + |a_1|^2 + |a_0|^2}.$$

9. (a) Let $f(z)$ denote a holomorphic function in $|z| \leq R$ such that $|f(z)| \leq M$. Suppose that $f(z_0) = w_0$, where $|z_0| < R$. Show that

$$\left|\frac{M(f(z) - w_0)}{M^2 - \overline{w_0}f(z)}\right| \leq \left|\frac{R(z - z_0)}{R^2 - \overline{z_0}z}\right|.$$

(Hint. First consider the case $f(0) = 0$.)

(b) Show that if $|f(z)| \leq 1$ for $|z| < 1$, then

$$\frac{|f'(z)|}{1 - |f(z)|^2} \leq \frac{1}{1 - |z|^2}.$$

10. (a) Let $f(z)$ be a holomorphic function. Set $u(x, y) = |f(x + iy)|$ and $F = u^2$. Show that

$$\frac{\partial u}{\partial x} = \frac{\mathrm{Re}(\overline{f}f')}{|f|}, \qquad \frac{\partial u}{\partial y} = -\frac{\mathrm{Im}(\overline{f}f')}{|f|},$$

and

$$\frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2} = 4|f'(z)|^2.$$

(b) Deduce that if f and g are holomorphic functions such that

$$\operatorname{Re} g(z) = |f(z)|,$$

then f and g are constant.

11. Let a be a complex constant lying outside the real interval $[-1, 1]$. Using residues, prove that

$$\int_{-1}^{1} \frac{dx}{(x-a)\sqrt{1-x^2}} = \frac{\pi}{\sqrt{a^2-1}},$$

with the appropriate determination of $\sqrt{a^2-1}$.

12. (a) Prove that a nonconstant holomorphic mapping is *open* (i.e., the image of every open set is open).

    (b) Let U, V denote domains (i.e., connected open sets) in $\mathbb{C}$ and let $f : U \to V$ be a holomorphic mapping. Suppose that f is *proper* (i.e., $f^{-1}(K)$ is compact, for every compact subset K of V). Prove that $f(U) = V$.

    (c) Is the assertion in (a) true if "holomorphic" is replaced by "continuous?" Explain.

Chapter 2

Series and Product Developments

MAT454/1002 lecture notes by Kain Dineen. Taught by Edward Bierstone.

2.1 Spaces of Holomorphic Functions (1-29-2021)

We study the notion of uniform convergence of families and series of holomorphic functions by topologizing the space of continuous and holomorphic functions on open subsets of the complex plane. We then prove two results about the behaviour of uniform limits of sequences of holomorphic functions.

2.1.1 Topology of $\mathcal{C}(\Omega)$

Let Ω be an open subset of $\mathbb{C}$. We write $\mathcal{C}(\Omega)$ for the ring of continuous complex-valued functions on Ω, and $\mathcal{H}(\Omega)$ for the subring of $\mathcal{C}(\Omega)$ consisting of holomorphic functions. We are interested in topologizing $\mathcal{C}(\Omega)$, with what we call the *compact-open topology.*

A sequence $\{f_n\}$ in $C(\Omega)$ is said to *converge uniformly on compact subsets* if for all compact $K \subset \Omega$, $\{f_n|_K\}$ converges uniformly. A notion of convergence defines a topology; we need to define the open sets. We start with a system of neighbourhoods of 0. For a compact $K \subset \Omega$ and an $\varepsilon > 0$, define

$$V(K, \varepsilon) = \{f \in \mathcal{C}(\Omega) : |f(z)| < \varepsilon, z \in K\}.$$

Then $f_n \to f$ uniformly on compact subsets if and only if for all K, ε, $f - f_n \in V(k, \varepsilon)$ for sufficiently large n. Then, a system of neighbourhoods of any point is obtained by translating these neighbourhoods of 0, giving a basis for a topology on $\mathcal{C}(\Omega)$.

Actually, this topology on $\mathcal{C}(\Omega)$ is metrizable, and it can be defined by a translation-invariant metric. Cover Ω by the interiors of countably many closed disks D_i (take all closed disks in Ω with rational center, radius). Define

$$d(f) = \sum_{i=1}^{\infty} \frac{1}{2^i} \min\{1, M_i(f)\},$$

where $M_i(f)$ is the maximum of $|f|$ on D_i. It is clear that

(i) $d(f) \geq 0$, and $d(f) = 0$ if and only if $f = 0$, and

(ii) $d(f+g) \leq d(f) + d(g)$ (it's certainly true for each term in the sum).

So $d(f,g) \coloneqq d(f-g)$ is a translation-invariant metric.

2.1.2 Convergence of Holomorphic Functions

$\mathcal{C}(\Omega)$ is complete; the limit of a sequence of continuous functions that converges uniformly on compact sets is continuous. We'll mostly be concerned with the holomorphic functions $\mathcal{H}(\Omega)$. We give it the subspace topology from the compact-open topology on $\mathcal{C}(\Omega)$. The following result is a fundamental fact about the topology on $\mathcal{H}(\Omega)$ that we will use often.

Theorem 2.1.1. *(Weierstrass) Let Ω be open in $\mathbb{C}$.*

(1) $\mathcal{H}(\Omega)$ is a closed subspace of $\mathcal{C}(\Omega)$.

(2) The mapping $\mathcal{H}(\Omega) \to \mathcal{H}(\Omega)$, $f \mapsto f'$, is continuous.

(1) means that if $\{f_n\} \subset \mathcal{H}(\Omega)$ converges uniformly on compact sets, then $f = \lim f_n$ is holomorphic. (2) means that if $\{f_n\} \subset \mathcal{H}(\Omega)$ converges to f uniformly on compact sets, then $\{f_n'\}$ converges uniformly to f' on compact sets.

Proof. (1) It's enough to show that $f(z)\,dz$ is a closed form. Consider a disk with center a and radius r contained in Ω. We want to show that if γ is any closed curve in $|z-a| < r$, then the integral of $f(z)\,dz$ over γ vanishes. Since γ is compact,

$$\int_\gamma f(z)\,dz = \lim_{n\to\infty} \int_\gamma f_n(z)\,dz = 0,$$

since all of the integrals in the limit vanish by holomorphicity of each $f_n(z)$.

(2) Suppose $f_n \to f$ uniformly on compact sets. It's enough to show that $f_n' \to f'$ uniformly on a closed disk $D \subset \Omega$. Let γ be the counterclockwise boundary of a larger concentric disk in Ω. If $z \in D$, then

$$f(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)}{\zeta - z}\,d\zeta,$$

so differentiating under the integral sign gives

$$f'(z) = \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{(\zeta - z)^2}\, d\zeta.$$

By uniform convergence on compact sets,

$$f'(z) = \lim_{n\to\infty} \frac{1}{2\pi i}\int_\gamma \frac{f_n(\zeta)}{(\zeta - z)^2}\, d\zeta = \lim_{n\to\infty} f_n'(z).$$

The convergence is uniform with respect to $z \in D$ because $(\zeta - z)^{-2}$ is bounded away from 0 for $z \in D$, $\zeta \in \gamma$.

□

Any result about sequences also applies to series.

Corollary 2.1.1. *If a series of holomorphic functions $\sum f_n$ on Ω converges uniformly on compact sets, then the sum $f = \sum f_n$ is holomorphic, and the series can be differentiated term-by-term.*

Recall that a set Ω in $\mathbb{C}$ is said to be a *domain* if it is open and connected.

Proposition 2.1.1. *(Hurwitz) Suppose that Ω is a domain in $\mathbb{C}$. If $\{f_n\} \subset \mathcal{H}(\Omega)$ converges uniformly on compact sets, and each f_n is nowhere-vanishing on Ω, then the limit is either never zero or identically zero.*

Proof. Suppose that f is not identically zero. Since Ω is connected, the zeroes of f are isolated. Suppose $f(z_0) = 0$. Let γ be the boundary of a circle in Ω with center z_0. The multiplicity of z_0 is

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}\, dz = \lim_{n\to\infty} \frac{1}{2\pi i}\int_\gamma \frac{f_n'(z)}{f_n(z)}\, dz.$$

This is zero since f_n is never zero, i.e. the integrands f_n'/f_n are holomorphic, a contradiction. □

Corollary 2.1.2. *If Ω is a domain and $\{f_n\} \subset \mathcal{H}(\Omega)$ converges uniformly on compact sets, and each f_n is one-to-one, then $f = \lim f_n$ is either one-to-one or constant.*

Proof. Assume that f is not constant and not one-to-one. Then $f(z_1) = f(z_2) = a$ for some $z_1 \neq z_2$ in Ω. Let U, V be disjoint open neighbourhoods of z_1, z_2 in Ω. Then $f(z) - a$ vanishes at a point of U, so some whole subsequence $\{f_{n_i}\}$of $\{f_n\}$ vanishes at a point of U. The same argument provides a subsequence $\{f_{n_{i_j}}\}$ such that $f_{n_{i_j}}(z) - a$ vanishes at some point of V, implying that the $f_{n_{i_j}}$'s are not one-to-one, contradiction. □

We'd like to apply these notions to sequences and series of meromorphic functions, however we will have to manage the existence of poles. We will do this next time.

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2.2 Series of Meromorphic Functions (2-1-2021)

Having studied the notion of uniform convergence for sequences and series of holomorphic functions, we turn to the analogous concepts of uniform and absolute convergence for series of meromorphic functions. We then study in detail an example.

2.2.1 Series of Meromorphic Functions

Let $\{f_n\}$ be a sequence of meromorphic functions on an open subset $\Omega \subseteq \mathbb{C}$. We will say that $\sum_{n=1}^{\infty} f_n$ *converges uniformly* (and *absolutely*) on $X \subseteq \Omega$ if all but finitely many f_n's have no pole in X, and form a uniformly (or uniformly and absolutely) convergent series. We wll study series of meromorphic functions that converge uniformly on compact subsets of Ω. We can define the sum on a relatively compact open subset $U \subseteq \Omega$ as

$$\underbrace{\sum_{n \le n_0} f_n}_{\text{meromorphic}} + \underbrace{\sum_{n > n_0} f_n}_{f_n\text{'s no pole in } \overline{U}} .$$

The second sum is uniformly convergent on $\overline{U}$, since none of the summands have poles there. This is independent of the choice of n_0.

The following theorem is the meromorphic analogue of the theorem on series of holomorphic functions from last time. The proof is similar.

Theorem 2.2.1. *Consider a series $\sum f_n$ of meromorphic functions on Ω. If the series converges uniformly on compact subsets of Ω, then the sum is a meromorphic function f on Ω, and it can be differentiated term-by-term; $\sum f_n'$ converges uniformly on compact subsets of Ω to f'.*

Since the sum f is meromorphic, its poles are isolated. Its poles form a subset of the poles of the f_n's, but some of the poles of the f_n's might cancel out.

2.2.2 A Meromorphic Series

Consider $\sum_{n=-\infty}^{\infty}(z-n)^{-2}$. We want to show that this series is uniformly and absolutely convergent on compact subsets of $\mathbb{C}$, and then we want to find a closed form for the sum. It's enough to show this on any vertical strip $a_1 \le x \le a_2$, since any compact subset of $\mathbb{C}$ can be covered by finitely many of these.

We are going to remove the terms where n lies inside this strip. First, consider

$$\sum_{n<a_1} \frac{1}{(z-n)^2},$$

for z inside the vertical strip. This is uniformly and absolutely convergent in the strip since each summand is bounded above in modulus by $(a_1 - n)^{-2}$, and this converges since each term is comparable to n^{-2}. The argument for the sum over $n > a_2$ is similar.

With the theorem in mind, consider the meromorphic function defined by $f(z) = \sum_{n=-\infty}^{\infty}(z-n)^{-2}$. The function f has period 1 ($f(z+1) = f(z)$), the poles are precisely the integers, and they are all double poles with principal parts $(z-n)^{-2}$ (and residues 0). We claim that

$$f(z) = \left(\frac{\pi}{\sin \pi z}\right)^2;$$

call this function $g(z)$. It's enough to show that $g(z)$ is meromorphic, with the same poles and corresponding principal parts as $f(z)$, and that $f - g$ is bounded (in fact, we want it to be zero).

Note that $f(z) \to 0$ uniformly with respect to x as $|y| \to \infty$; that is, for every $\varepsilon > 0$, there is a b such that $|f(z)| < \varepsilon$ when $|y| > b$. By periodicity, it's enough to show this in a strip $a_1 \leq x \leq a_2$. This clearly holds for each summand, in the strip; it follows that it holds for the sum as well. (The proof of this is left as an exercise, which might be given next time.)

Now, $g(z)$ has the same properties as f:

(i) Meromorphic in $\mathbb{C}$, with period 1.

(ii) The poles are precisely the integers, and each pole is a double pole with principal part $(z-n)^{-2}$.

(iii) $g(z) \to 0$ as $|y| \to \infty$, uniformly with respect to x; to see this, use the fact that

$$|\sin \pi z|^2 = \sin^2 \pi x + \sinh^2 \pi y.$$

Why is $f - g$ bounded? Consider again a strip $a_1 \leq x \leq a_2$. In some part $|y| \geq b$ of this strip, $f - g$ goes to 0 as $|y| \to \infty$, uniformly with respect to x. The rest of the strip is compact, so $f - g$ is bounded on it. Therefore $f - g$ is bounded on the strip, and bounded on all of $\mathbb{C}$. (Why?)

$f - g$ is holomorphic in $\mathbb{C}$, since f, g have the same poles and principal parts. $f - g$ is constant by Liouville's theorem. The constant is zero by property (iii). As an exercise, show that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

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2.3 The Weierstrass ℘-function (2-3-2021)

We introduce an important example of a meromorphic function defined by a series of meromorphic functions known as the *Weierstrass* ℘-function. This is the prototypical example of an *elliptic*, or doubly-periodic, function, in the sense that all other elliptic functions can be written as rational functions of $\wp$ and $\wp'$.

2.3.1 Another Example

Let us consider

$$\frac{1}{z} + \sum_{n \neq 0} \left(\frac{1}{z-n} + \frac{1}{n} \right).$$

This series is uniformly and absolutely convergent on compact subsets of $\mathbb{C}$, for we may write each term as

$$\frac{z}{n(z-n)},$$

which is comparable to $1/n^2$ for z in a compact set. It follows that the sum, $f(z)$, is a meromorphic function in $\mathbb{C}$. The poles are precisely the integers, and they are all simple poles of residue 1. The series can be differentiated term-by-term:

$$f'(z) = -\frac{1}{z^2} - \sum_{n \neq 0} \frac{1}{(z-n)^2} = -\frac{\pi^2}{\sin^2 \pi z} = \frac{d}{dz}(\pi \cot \pi z).$$

It follows that $f(z) - \pi \cot \pi z$ is a constant; the constant is zero since $f(z)$ and $\pi \cot \pi z$ are both odd functions. Grouping the positive and negative terms, we obtain

$$\frac{1}{z} + \sum_{n=1}^{\infty} \frac{2z}{z^2 - n^2} = \pi \cot \pi z.$$

In particular, this function is periodic.

2.3.2 The Weierstrass ℘-function

Let $e_1, e_2 \in \mathbb{C}$ be two complex numbers which are linearly independent over the reals. We want to look at a *doubly periodic* function over the lattice generated by e_1 and e_2. The set

$$\Gamma = \{n_1 e_1 + n_2 e_2 : n_1, n_2 \in \mathbb{Z}\}$$

is a discrete subgroup (lattice) of $\mathbb{C}$. We will say that $f(z)$ *has* Γ *as a group of periods* (or is *doubly periodic* for short) if $f(z+n_1e_1+n_2e_2)=f(z)$, for all $n_1,n_2\in\mathbb{Z}$, i.e.

$$f(z+e_1)=f(z),$$
$$f(z+e_2)=f(z),$$

for all z. We call e_1,e_2 a *basis* of Γ.

Given a point $z_0\in\mathbb{C}$, we will call the parallelogram with vertices $z_0, z_0+e_1, z_0+e_2, z_0+e_1+e_2$ the *period parallelogram with first vertex* z_0.

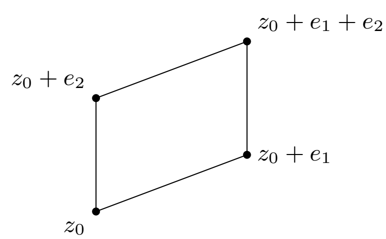

Figure 2.1: A period parallelogram with first vertex z_0 and vertices z_0, z_0+e_1, z_0+e_2,and $z_0+e_1+e_2$.

Note that e_1', e_2' is a basis of Γ if and only if e_1', e_2' is a linear combination of e_1,e_2 with integer coefficients, and the determinant of the matrix of coefficients is ±1 (i.e. a unit in $\mathbb{Z}$).

We define[1] the *Weierstrass* $\wp$*-function* by

$$\wp(z) := \frac{1}{z^2}+\sum_{\omega\in\Gamma\setminus\{0\}}\left(\frac{1}{(z-\omega)^2}-\frac{1}{\omega^2}\right).$$

We're going to see that this series is uniformly and absolutely convergent on compact subsets of $\mathbb{C}$.

Lemma 2.3.1. *The series*

$$\sum_{\omega\in\Gamma\setminus\{0\}}\frac{1}{|\omega|^3}$$

converges.

Proof. Think of the lattice as an expanding series of parallelograms. For $n\geq1$, let

$$P_n=\{t_1e_1+t_2e_2 : t_1,t_2\in\mathbb{Z}, \max\{|t_1|,|t_2|\}=n\}.$$

[1]Why do we add the $-1/\omega^2$ term? To quote Professor Bierstone, it's "a device to make the series converge." We'll see that this term is required to make the series' terms comparable to $1/\omega^3$, when z lies in a compact set. Compare with the example from the start of this class.

There are $8n$ points on P_n, each of distance at least kn from 0, where k is the shortest distance from the origin to P_1. We have

$$\sum_{\omega\in P_n}\frac{1}{|\omega|^3}\leq\frac{8n}{k^3n^3}=\frac{8}{k^3n^2},$$

so

$$\sum_{\omega\in\Gamma\setminus\{0\}}\frac{1}{|\omega|^3}\leq\sum_{n=1}^{\infty}\frac{8}{k^3n^2}<\infty.$$

$\square$

To show that the series defining $\wp$ converges uniformly and absolutely on compact subsets of $\mathbb{C}$, it's enough to check it on disks $|z|\leq r$, for any r. For $|z|\leq r$ and $|\omega|\geq 2r$,

$$\left|\frac{1}{(z-\omega)^2}-\frac{1}{\omega^2}\right|=\left|\frac{2\omega z-z^2}{\omega^2(z-\omega)^2}\right|=\frac{|z||2-\frac{z}{\omega}|}{|\omega|^3|1-\frac{z}{\omega}|^2}\leq\frac{r\cdot\frac{5}{2}}{|\omega|^3\cdot\frac{1}{4}}=\frac{10r}{|\omega|^3}.$$

So the series converges uniformly and absolutely in $|z|\leq r$ by comparison with $\sum 1/|\omega|^3$. It follows that the Weierstrass $\wp$-function is well-defined as a meromorphic function on $\mathbb{C}$.

Its poles are just the points of the lattice Γ, each of multiplicity 2, with principal part $(z-\omega)^{-2}$ to the pole $\omega\in\Gamma$. Moreover, $\wp$ is an even function. We would like to see that it is doubly periodic.

$$\wp'(z)=-2\sum_{\omega\in\Gamma}\frac{1}{(z-\omega)^3},$$

which is uniformly and absolutely convergent on compact subsets of $\mathbb{C}$, and again periodic: $\wp'(z+\omega)=\wp'(z)$ for any $\omega\in\Gamma$. It's also an odd function. We want to check that

$$\wp(z+e_i)=\wp(z),\qquad i=1,2,$$

for all z. By periodicity of $\wp'$, $\wp(z+e_i)-\wp(z)$ is a constant. Setting $z=-e_i/2$, we see that the constant is

$$\text{const.}=\wp(e_i/2)-\wp(-e_i/2)=0,$$

since $\wp$ is even. So $\wp$ is doubly periodic with group of periods Γ.

What is the Laurent expansion of $\wp(z)$ at 0?

$$\wp(z)=\frac{1}{z^2}+a_2z^2+a_4z^4+\cdots,$$

because $\wp$ is even, and $\wp(z)-z^{-2}$ vanishes at 0. The coefficients are sums of things of the form

$$\frac{1}{(2k)!}\cdot\left(\frac{(2k+1)!}{(z-\omega)^{2k+2}}\right)\bigg|_{z=0},$$

i.e. $(2k!)^{-1}$ times the $2k$th derivative of $(z-\omega)^{-2}$. This leads to

$$a_{2k} = (2k+1) \sum_{\omega \neq 0} \frac{1}{\omega^{2(k+1)}}.$$

In particular,

$$a_2 = 3 \sum_{\omega \neq 0} \frac{1}{\omega^4}$$

and

$$a_4 = 5 \sum_{\omega \neq 0} \frac{1}{\omega^6},$$

and so on.

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2.4 More on Elliptic Functions (2-5-2021)

We show that the Weierstrass $\wp$-function satisfies a certain differential equation, which will be indispensible to us when we study its algebraic properties. We then prove two fundamental results concerning the nature of the zeroes and poles of elliptic functions.

2.4.1 The Differential Equation

Let Γ be a discrete subgroup of $\mathbb{C}$, generated by $e_1, e_2 \in \mathbb{C}$, linearly independent over $\mathbb{R}$. We're going to use the Laurent series expansion of the Weierstrass $\wp$-function to see that it satisfies a certain differential equation. The Laurent series expansion of $\wp(z)$ at 0 is

$$\wp(z) = \frac{1}{z^2} + a_2 z^2 + a_4 z^4 + \cdots,$$

where we found the coefficients last time. Differentiating,

$$\wp'(z) = -\frac{2}{z^3} + 2a_2 z + 4a_4 z^3 + \cdots.$$

If we square both sides, we obtain

$$\wp'(z)^2 = \frac{4}{z^6} - \frac{8a_2}{z^2} - 16a_4 + \cdots.$$

On the other hand,

$$\wp(z)^3 = \frac{1}{z^6} + \frac{3a_2}{z^2} + 3a_4 + \cdots.$$

Then

$$\wp'(z)^2 - 4\wp(z)^3 = -\frac{20a_2}{z^2} - 28a_4 + z^2(\cdots).$$

To eliminate the $1/z^2$ term, we add $20a_2\wp(z)$:

$$\wp'(z)^2 - 4\wp(z)^2 + 20a_2\wp(z) + 28a_4 = z^2(\cdots).$$

It follows that

$$(\wp')^2 - 4\wp^3 + 20a_2\wp + 28a_4$$

is holomorphic near zero, 0 at 0. It is periodic with group of periods Γ, so it's holomorphic near all points of Γ. So it's holomorphic in $\mathbb{C}$, since it has no poles elsewhere. By periodicity it's bounded, so by Liouville's theorem it's constant. The constant is zero since it vanishes at the origin. Therefore $\wp$ satisfies the differential equation

$$(\wp')^2 = 4\wp^3 - 20a_2\wp - 28a_4.$$

That is, $x = \wp(z)$ and $y = \wp'(z)$ give a parametrization of the algebraic curve

$$y^2 = 4x^3 - 20a_2x - 28a_4.$$

We'll see later that any point (x, y) of this curve is the image of a point $z \in \mathbb{C}$, uniquely determined up to addition of an element of Γ. Analogously to how trigonometric functions and their derivatives parametrize quadratic curves, the Weierstrass $\wp$-function and its derivative parametrize cubic curves.

2.4.2 Doubly Periodic Functions

We require some results about doubly periodic functions before we further study the $\wp$-function. Let Γ be as in the previous part.

Proposition 2.4.1. *Let f be a non-constant meromorphic function on $\mathbb{C}$ with Γ as its group of periods. Then, provided f has no zeroes or poles on the boundary, the number of zeroes of f in a period parallelogram is equal to the number of poles in the same parallelogram, each counted with multiplicity.*

Proof. If γ is the boundary of the period parallelogram pictured below,

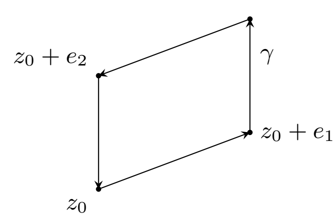

Figure 2.2: The boundary of the period parallelogram, γ, is oriented counter-clockwise.

then by the argument principle,

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}\,dz = \#\text{ zeroes} - \#\text{ poles},$$

counted with multiplicity. The left-hand side vanishes by periodicity. ☐

Proposition 2.4.2. *Let f be a non-constant meromorphic function in $\mathbb{C}$ with Γ as its group of periods. For a fixed $a \in \mathbb{C}$, let α_i be the roots of $f(z) = a$, and let β_i be the poles of $f(z)$, each counted with multiplicity, within a period parallelogram. Then $\sum \alpha_i$ is congruent to $\sum \beta_i$, modulo Γ. (In particular, $\sum \alpha_i \bmod \Gamma$ is independent of a.)*

Proof. By the residue theorem, if γ is the boundary of the period parallelogram,

$$\frac{1}{2\pi i}\int_\gamma \frac{zf'(z)}{f(z)-a}\,dz = \text{sum of the residues of } \frac{zf'(z)}{f(z)-a}. \qquad (*)$$

At a root $z = \alpha_i$ of multiplicity k,

$$\begin{aligned} z &= \alpha_i + (z - \alpha_i), \\ f(z) - a &= c(z - \alpha_i)^k + \text{higher order} \\ f'(z) &= kc(z - \alpha_i)^{k-1} + \cdots , \end{aligned}$$

so it follows that

$$\frac{zf'(z)}{f(z) - a} = \frac{k\alpha_i}{z - \alpha_i} + \text{ higher order},$$

so the residue is $k\alpha_i$. Similarly, at a pole β_i, the residue is $-k\beta_i$. It follows that the right-hand side of (*) is simply $\sum \alpha_i - \sum \beta_i$, which we want. Unlike in the proof of the previous proposition, the integrand in the left-hand side of (*) is not periodic. However, the left-hand side is

$$-\frac{e_2}{2\pi i}\int_{\gamma_1} \frac{f'(z)}{f(z) - a}\, dz + \frac{e_1}{2\pi i}\int_{\gamma_2} \frac{f'(z)}{f(z) - a}\, dz,$$

where γ_1, γ_2 are two sides of the period parallelogram:

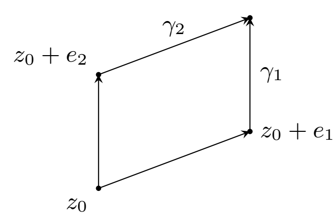

Figure 2.3: γ_1 and γ_2 are oriented so that they both start at z_0 and end at the opposite vertex, $z_0 + e_1 + e_2$. (The orientation is not a big deal, so long as we end up with *something* which is zero mod Γ.)

Here,

$$\frac{1}{2\pi i}\int_{\gamma_1} \frac{f'(z)}{f(z) - a}\, dz, \int_{\gamma_2} \frac{f'(z)}{f(z) - a}\, dz$$

are integers, because they each equal to the difference between two determinations of $\log(f(z_0) - a)$. $\square$

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2.5 Algebraic Curves and the $\wp$-function (2-8-2021)

We begin to study the algebraic properties of the Weierstrass $\wp$-function. Namely, we prove that it always gives a parametrization, up to addition of an element of Γ, of the elliptic curve in $\mathbb{C}^2$ given by the differential equation of $\wp$.

2.5.1 Parametrization by $\wp$

Let Γ be a discrete subgroup of $\mathbb{C}$, and let $\wp(z)$ be the associated Weierstrass $\wp$-function. We saw last time that $(x, y) = (\wp(z), \wp'(z))$ satisfies the algebraic equation $y^2 = 4x^3 - 20a_2x - 28a_4$, where

$$a_2 = 3 \sum_{\omega \in \Gamma \setminus \{0\}} \frac{1}{\omega^4}, \qquad a_4 = 5 \sum_{\omega \in \Gamma \setminus \{0\}} \frac{1}{\omega^6}.$$

Theorem 2.5.1. *$P(x) = 4x^3 - 20a_2x - 28a_4$ has three distinct zeroes. Moreover, for all $(x, y) \in \mathbb{C}^2$ on the curve $y^2 = 4x^3 - 20a_2x - 28a_4$, there is a unique $z \in \mathbb{C}$, mod Γ, such that $x = \wp(z)$, $y = \wp'(z)$.*

Later in the course, we'll discuss the following theorem, due to Abel.

Theorem 2.5.2. *(Abel) Conversely, given an equation $y^2 = 4x^3 - 20a_2x - 28a_4$, where the right-hand side has three distinct zeroes, then one can find a discrete group Γ such that a_2, a_4 are given as above. (So by the previous theorem, if $\wp$ is the corresponding Weierstrass $\wp$-function, then $x = \wp(z), y = \wp'(z)$ parametrizes the curve.)*

The significance of the "three distinct zeroes" condition is that any smooth cubic curve has the given form. This is not too difficult to prove, but we may not get to it in this course.

Proof. (Of the first theorem.) Each value of $\wp(z)$ is taken twice in a period parallelogram, and three times for $\wp'(z)$. Consider

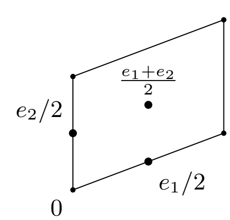

Figure 2.4:

Consider the points $z \in \mathbb{C}$ such that $z \notin \Gamma$, and $2z \in \Gamma$. Any such point is congruent mod Γ to one of $e_1/2$, $e_2/2$, or $(e_1+e_2)/2$; moreover, the classes mod Γ of these three points are distinct. At such z,

$$\begin{aligned} \wp'(z) &= \wp'(-z) && \text{by periodicity,} \\ \wp'(z) &= -\wp'(-z) && \text{since } \wp' \text{ is odd,} \end{aligned}$$

so $\wp'(z) = 0$. This means that each value $\wp(e_1/2)$, $\wp(e_2/2)$, and $\wp((e_1+e_2)/2)$ is taken exactly once in a period parallelogram (with multiplicity 2), and these three values are distinct. These values are all distinct because $\wp$ takes on each value exactly twice in a period parallelogram. Since $(\wp')^2 = 4\wp^3 - 20a_2\wp - 28a_4$, it follows that these three values are three distinct zeroes of $P(x)$.

If $2z_0 \notin \Gamma$, then the value $\wp(z_0)$ is taken exactly twice but at different points, since $\wp'(z_0) \neq 0$. The other point not congruent to z_0 mod Γ at which $\wp$ takes on this value is $-z_0$, since $\wp$ is even. (In the period parallelogram as shown above, it's z_0 flipped over the midpoint, which is congruent mod Γ to $-z_0$.)

Consider (x, y) on the curve, with $y \neq 0$. Let z_0 be a point of the period parallelogram such that $x = \wp(z_0)$. Then $\wp'(z_0) \neq 0$, so $2z_0 \notin \Gamma$, so then $x = \wp(z_0) = \wp(-z_0)$, but $y = \wp'(z_0)$ and $-y = \wp'(-z_0)$ by odd-ness, so the two distinct points $z_0, -z_0$ in the period parallogram map to different points on the curve. Therefore *every* point on the curve $y^2 = 4x^3 - 20a_2x - 28a_4$ is the image under $(\wp, \wp')$ of some point, uniquely determined mod Γ. □

2.5.2 Implicit Function Theorem

Consider $X \subset \mathbb{C}^2$ given by the equation $y^2 = 4x^3 - 20a_2x - 28a_4$. The right-hand side, $P(x)$, has three distinct zeroes. Then X is a smooth curve[2] (i.e. X is locally the graph of a *holomorphic* function $y = f(x)$ or $x = g(y)$). Write $F(x, y) = y^2 - P(x)$. If $(x_0, y_0) \in X$ has $y_0 \neq 0$, then

$$\frac{\partial F}{\partial y}(x_0, y_0) = 2y_0 \neq 0,$$

so the curve is of the form $y = f(x)$ near (x_0, y_0). On the other hand, if $(x_0, y_0) \in X$ has $y_0 = 0$, then $P'(x_0) \neq 0$, since there are three distinct zeroes. (We have $P(x_0) = 0$, so if $P'(x_0) = 0$, then x_0 is a zero of multiplicity 2, contradiction.) This gives $\frac{\partial F}{\partial x}(x_0, y_0) \neq 0$, so near (x_0, y_0) we can write the curve as $x = g(y)$.

Why will these functions f, g be holomorphic? The classic implicit function theorem only ensures that the functions f, g will be differentiable, but we don't yet know if they will be holomorphic. To this end, we will prove a "weak" implicit function theorem for two complex variables that will give us holomorphicity of f, g in the previous discussion.

[2] A 2-dimensional real manifold, or 1-dimensional complex manifold.

Theorem 2.5.3. *Consider the equation* $f(x,y) = 0$, *for* $(x,y) \in \mathbb{C}^2$. *Assume that* f *is* C^1, *and that* f *is separately holomorphic (i.e. holomorphic in* y *for fixed* x, *and holomorphic in* x *for fixed* y*). If* $\frac{\partial f}{\partial y}(x_0, y_0) \neq 0$, *where* $f(x_0, y_0) = 0$, *then we can locally solve for* y *as a holomorphic function of* x.

Proof. Write $z = f(x,y)$, where

$$\begin{aligned} x &= x_1 + ix_2, \\ y &= y_1 + iy_2, \\ z &= z_1 + iz_2, \\ f &= f_1 + if_2. \end{aligned}$$

For fixed x,

$$dz = \frac{\partial f}{\partial y}\, dy,$$

so taking conjugates gives

$$d\bar{z} = \frac{\partial \bar{f}}{\partial \bar{y}}\, d\bar{y}.$$

Then

$$dz \wedge d\bar{z} = \left|\frac{\partial f}{\partial y}\right|^2 dy \wedge d\bar{y}.$$

Note that $dz \wedge d\bar{z} = -2i\, dz_1 \wedge dz_2$ and $dy \wedge d\bar{y} = -2i\, dy_1 \wedge dy_2$. It follows that

$$\det \frac{\partial(z_1, z_2)}{\partial(y_1, y_2)} = \left|\frac{\partial f}{\partial y}\right|^2 \neq 0$$

at (x_0, y_0). By the real implicit function theorem, we can solve $f(x,y) = 0$ for $y = y_1 + iy_2$ as a C^1 function $y = y(x)$ of $x = x_1 + ix_2$.

We have to check that y is holomorphic. Taking the differential of $f(x, y(x)) = 0$ gives

$$\frac{\partial f}{\partial x}\, dx + \frac{\partial f}{\partial y}\left(\frac{\partial y}{\partial x}\, dx + \frac{\partial y}{\partial \bar{x}}\, d\bar{x}\right) = 0.$$

So $\frac{\partial y}{\partial \bar{x}} = 0$, i.e. $y = y(x)$ is holomorphic. $\square$

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2.6 Projective Space (2-10-2021)

We detour into a brief study of higher-dimensional complex projective space for the sake of understanding the geometric properties of the elliptic curve in $\mathbb{C}^2$ given by the differential equation of the $\wp$-function.

2.6.1 Higher-Dimensional Complex Projective Space

Consider, as before, the curve $X = \{y^2 = 4x^3 - 20a_2x - 28a_4\}$. The curve sits inside $\mathbb{C}^2$ via the inclusion $X \hookrightarrow \mathbb{C}^2$. If we project $\mathbb{C}^2$ onto $\mathbb{C}$ via $(x, y) \mapsto x$, we obtain the commutative diagram

$$\begin{tikzcd} & \mathbb{C}^2 \arrow[d] & (x,y) \arrow[d, mapsto] \\ X \arrow[ur, hook] \arrow[r, "\varphi"] & \mathbb{C} & x \end{tikzcd}$$

We call this a *Riemann surface*[3] over $\mathbb{C}$. Note that, except for the three roots of the right-hand side, each $x \in \mathbb{C}$ corresponds to two points of X. Geometrically speaking, the curve X lies over $\mathbb{C}$ in two sheets that come together over these three points, the roots of the right-hand side (*branch points*). We will return to the notion of a Riemann surface later in the course.

We can compactify $\mathbb{C}$ to get the Riemann sphere S^2. Can we compactify X to get a Riemann surface over $S^2 = P^1(\mathbb{C})$? Define $P^n(\mathbb{C})$ to be the set of complex lines through the origin 0 in $\mathbb{C}^{n+1}$. That is,

$$P^n(\mathbb{C}) = (\mathbb{C}^{n+1} \setminus \{0\})/\sim,$$

where $(x'_0, \dots, x'_n) \sim (x_0, \dots, x_n)$ if there is a $\lambda \in \mathbb{C} \setminus \{0\}$ such that $(x'_0, \dots, x'_n) = \lambda(x_0, \dots, x_n)$. We write $[x_0, \dots, x_n]$ for the equivalence class of $(x_0, \dots, x_n)$; these coordinates are called *homgeneous coordinates*. Our coordinate charts are

$$U_i = \{[x_0, \dots, x_n] \in P^n(\mathbb{C}) : x_i \neq 0\}, \qquad i = 0, \dots, n.$$

For each i, there is a homeomorphism $U_i \to \mathbb{C}^n$ given by

$$[x_0, \dots, x_n] \mapsto \left(\frac{x_0}{x_i}, \dots, \widehat{\frac{x_i}{x_i}}, \dots, \frac{x_n}{x_i}\right),$$

[3] To be precise with the terminology, the Riemann surface is actually X *along with* φ, i.e. the pair (X, φ).

where the hat denotes omission. The inverse is given by

$$(z_1, \ldots, z_n) \mapsto [z_1, \ldots, z_i, 1, z_{i+1}, \ldots, z_n].$$

This gives $P^n(\mathbb{C})$ the structure of a complex n-manifold, i.e. the transition mappings are holomorphic. (Actually, in this case, the transition mappings are rational; as an exercise, write them down.) Note that $P^n(\mathbb{C})$ is compact.

We may think of $P^n(\mathbb{C})$ as $U_0 \cong \mathbb{C}^n$, together with the points not in U_0,

$$\{[x_0, \ldots, x_n] \in P^n(\mathbb{C}) : x_0 = 0\};$$

we call this latter set the *points at infinity* or the *hyperplane at infinity*. (That is, we compactified $\mathbb{C}^n$ by adding some points at infinity.) This hyperplane at infinity is (isomorphic to) $P^{n-1}(\mathbb{C})$. For example, $P^1(\mathbb{C}) \cong S^2$, the Riemann sphere.

For $P^2(\mathbb{C})$, we're going to consider $\mathbb{C}^2$ sitting inside of $P^2(\mathbb{C})$ as U_3. Write the coordinates as $[x, y, t]$. Now, $\mathbb{C}^2$ is isomorphic to the set $\{[x, y, t] : t \neq 0\}$, so we can think of $P^2(\mathbb{C})$ as $\mathbb{C}^2$ together with the points at infinity, $P^1(\mathbb{C})$. Recall that our curve X sits inside of $\mathbb{C}^2$. What is the closure of $X \subset \mathbb{C}^2$ in $P^2(\mathbb{C})$? We add a third variable t and *homogenize*: X' is given by

$$y^2 t = 4x^3 - 20a_2 x t^2 - 28a_4 t^3 \tag{*}$$

in $P^2(\mathbb{C})$. Why? The homeomorphism $U_3 \to \mathbb{C}^2$ is given by $[x, y, t] \mapsto (x/t, y/t)$, so under this homeomorphism, the equation of the curve X becomes

$$\left(\frac{y}{t}\right)^2 = 4\left(\frac{x}{t}\right)^2 - 20a_2\left(\frac{x}{t}\right) - 28a_4;$$

clearing denominators gives the homogeneous equation for $X' \subset P^2(\mathbb{C})$. That is,

$$X' = \{[x, y, t] \in P^2(\mathbb{C}) : y^2 t = 4x^3 - 20a_2 x t^2 - 28a_4 t^3\}.$$

X' is a compactification of X; X' consists of X together with the points at infinity, i.e. the points of X' where $t = 0$. Setting $t = 0$ in (*), we obtain $4x^3 = 0$, leaving just the single point at infinity, $[0, 1, 0]$.

We would like to learn what X' looks like near this point at infinity. In particular, we would like to know whether or not X' is smooth, even near the point at infinity. We will do this next time.

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2.7 The Elliptic Integral (2-12-2021)

We examine the behaviour of the homogenization of the $\wp$-function's parametrized curve in complex projective space near the point at infinity. We mention that this curve is a torus. Then, we uncover the relationship between the $\wp$-function and elliptic integrals.

2.7.1 Recap

Consider the setup from last time: Γ is a lattice in $\mathbb{C}$, and X is the smooth curve in $\mathbb{C}^2$ given by $y^2 = 4x^3 - 20a_2x - 28a_4$, where

$$a_2 = 3\sum_{\omega\in\Gamma\setminus\{0\}} \frac{1}{\omega^4}, \qquad a_4 = 5\sum_{\omega\in\Gamma\setminus\{0\}} \frac{1}{\omega^6}.$$

Its compactification in $P^2(\mathbb{C})$ is given in homogeneous coordinates $[x, y, t]$ by the equation

$$y^2t = 4x^3 - 20a_2xt^2 - 28a_4t^3. \tag{*}$$

X' consists of X together with the point at infinity, given by $t = 0$, i.e. $[0, 1, 0]$. We'd like to try to understand this compactified curve as a Riemann surface over the Riemann sphere, i.e. $P^1(\mathbb{C})$. Specifically, is there a mapping $\varphi\colon X' \to P^1(\mathbb{C})$ that extends the mapping $\varphi\colon X \to \mathbb{C}, (x, y) \mapsto x$ from before?

2.7.2 The Point at Infinity

We examine the point at infinity in coordinates. $[0, 1, 0]$ lies in the chart

$$\{[x, y, t] \in P^2(\mathbb{C}) : y \neq 0\}.$$

In this chart, we have affine coordinates $(x', t') = (x/y, t/y)$. The equation (*) in these coordinates is obtained by dehomogenizing with respect to y:

$$t' = 4x'^3 - 20a_2x't'^2 - 28a_4t'^3.$$

The point $[0, 1, 0]$ at infinity has coordinates $(x', t') = (0, 0)$. In some neighbourhood of this point, the implicit function theorem gives t' as a holomorphic function of x':

$$t' = 4x'^3 - 320a_2x'^7 + \cdots .$$

That is, in a neighbourhood of the point at infinity in X', we can take x' as a local coordinate. In this way, we get a complex 1-manifold structure on X'.

We return to the problem of extending $\varphi\colon X \to \mathbb{C}$ to a mapping $\varphi'\colon X' \to S^2$ sending $[0,1,0]$ to the point at infinity in the Riemann sphere. Write $P^1(\mathbb{C})$ with coordinates $[x,t]$. $S^2 \cong P^1(\mathbb{C})$ consists of $\mathbb{C}$ together with the point at infinity, $[1,0]$:

$$\mathbb{C} \cong \{[x,t] : t \neq 0\}.$$

The coordinate $z = x/t$ of $\mathbb{C}$ corresponds to $1/z = t/x$ in coordinates at infinity. In the chart above where $y \neq 0$, X' consists of the points

$$[x', 1, t'] = [x', 1, 4x'^3 - 320a_2x'^7 + \cdots].$$

If $t' \neq 0$, then φ takes this to $[x', t']$ in $P^1(\mathbb{C}) = S^2$, i.e. x'/t' in $\mathbb{C}$. In coordinates at infinity, t'/x':

$$\frac{t'}{x'} = \frac{4x'^3 - 320a_2x'^7 + \cdots}{x'},$$

which goes to the point at infinity in $S^2 = P^1(\mathbb{C})$ as $x' \to 0$. Summarizing, we have the diagram

$$\begin{tikzcd}
(x,y) \arrow[dr, mapsto] & & \mathbb{C}^2 \arrow[dr] & X \arrow[l, hook'] \arrow[r, hook] \arrow[d, "\varphi"] & X' \arrow[r, hook] \arrow[d, "\varphi'"] & P^2(\mathbb{C}) \\
& x & & \mathbb{C} \arrow[r, hook] & P^1(\mathbb{C}) = S^2,
\end{tikzcd}$$

where the hooked arrows represent inclusions, and φ' is an extension of φ taking the point at infinity of X', $[0,1,0]$, to the point at infinity of the Riemann sphere, which is in the case of $P^1(\mathbb{C})$ the point $[1,0]$. We can therefore say that X' forms a Riemann surface over the Riemann sphere. (We will return to the notion of Riemann surfaces later in the course.)

The meromorphic transformation $x = \wp(z), y = \wp'(z)$, defines a mapping

$$\mathbb{C}/\Gamma \xrightarrow{\cong} X',$$

a homeomorphism, where $\mathbb{C}/\Gamma$ has the quotient topology. This is because it's a continuous bijection from a compact space to a Hausdorff space. Since $\mathbb{C}/\Gamma$ is a torus, our curve X' is topologically a torus.

2.7.3 Elliptic Integrals

The inverse map $X' \to \mathbb{C}/\Gamma$ defines z as a holomorphic *multi-valued* function of X', whose branches differ by constants belonging to Γ. We have

$$\frac{dx}{dz} = \frac{d\wp(z)}{dz} = \wp'(z) = y,$$

so $dx = y\,dz$. Since

$$y^2 = P(x) = 4x^3 - 20a_2x - 28a_4,$$

$2y\,dy = P'(x)\,dx$. Wherever x is a local coordinate, $dz = \frac{dx}{y}$. Wherever y is a local coordinate, $dz = \frac{2\,dy}{P'(x)}$. Thus, we can think of dz as an extension to X' of the holomorphic differential form

$$\frac{dx}{\sqrt{4x^3 - 20a_2x - 28a_4}}.$$

So

$$z = \wp^{-1}(x) = \int_{[0,1,0]}^{[\wp(z),\wp'(z),1]} \frac{dx}{\sqrt{4x^3 - 20a_2x - 28a_4}}. \qquad (**)$$

Thus, the Weierstrass $\wp$-function is given by "inversion" of an elliptic integral.

Compare with the circle, $x^2 + y^2 = 1$. We have $x\,dx + y\,dy = 0$, so whenever each term makes sense,

$$-\frac{dx}{y} = \frac{dy}{x} = \frac{dy}{\sqrt{1-y^2}} = d\theta,$$

or $dy = x\,d\theta$. The function θ is not well-defined on the circle, but it is well-defined as a multi-valued function on S^1 whose branches differ by constants belonging to a group isomorphic to $\mathbb{Z}$. In first year calculus, we invert

$$\int \frac{dy}{x} = \int \frac{dy}{\sqrt{1-y^2}}$$

in a neighbourhood of $(1,0)$ by defining the trig functions by the relation

$$\theta = \int_{(1,0)}^{(\cos\theta,\sin\theta)} \frac{dy}{x} = \int_0^{\sin\theta} \frac{dy}{\sqrt{1-y^2}}.$$

This is exactly the same relationship between the Weierstrass $\wp$-function and the integral in (**); here, we obtained the trigonometric functions by the inversion of a trigonometric integral.

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2.8 Prescribing Poles and Zeroes (2-22-2021)

We continue the general theme of constructing functions with prescribed poles and zeroes. We've already solved this in the case of a lattice Γ in $\mathbb{C}$ using the Weierstrass $\wp$-function. We now turn to the more general problem.

2.8.1 Prescribed Poles

It'll be easier to take on the problem of prescribed poles, since here we ought to add together functions we know have certain poles. Should we do this on the Riemann sphere or on the complex plane? Any meromorphic function on the Riemann sphere is rational, so it can't possibly have infinitely many poles. In $\mathbb{C}$ the situation is quite different, e.g. $\tan z, \sec z$ are meromorphic in $\mathbb{C}$, but ∞ is a limit of poles.

Theorem 2.8.1. *(Mittag-Leffler) Let (b_k) be a sequence in $\mathbb{C}$ with $\lim_{k\to\infty} b_k = \infty$. Let $P_k(z)$ be polynomials without constant terms. There exists a meromorphic function with poles b_k and principal parts $P_k(1/(z-b_k))$; the most general such function can be written*

$$f(z) = \sum_k \left(P_k\left(\frac{1}{z-b_k}\right) - p_k(z) \right) + g(z),$$

where $g(z)$ is entire, and the polynomials $p_k(z)$ are chosen[4] so that the series is uniformly and absolutely convergent on subsets of $\mathbb{C}$.

Proof. We can assume no $b_k = 0$. $P_k(1/(z-b_k))$ is holomorphic in $|z| < |b_k|$, so we can expand it as a Taylor series at 0. Let $p_k(z)$ be the sum of the terms of degree less than or equal to n_k, where n_k is chosen so that

$$\left| P_k\left(\frac{1}{z-b_k}\right) - p_k(z) \right| \le \frac{1}{2^k}, \qquad |z| \le \frac{1}{2}|b_k|.$$

We'll show that the series converges uniformly and absolutely in every disk $|z| \le r$. Choose m such that $|b_k| > 2r$ for all $k > m$. Then

$$\sum_{k=m+1}^{\infty} \left(P_k\left(\frac{1}{z-b_k}\right) - p_k(z) \right)$$

is uniformly and absolutely convergent in $|z| \le r$ by comparison with $\sum 1/2^k$. Any meromorphic function with the same poles and principal parts has to differ from this by an entire function. $\square$

[4] E.g. the $1/\omega^2$ term in the definition of $\wp(z)$.

2.8.2 Infinite Products

To get a function with prescribed zeroes, we ought to multiply together terms rather than add them. Thus, we study the notion of an infinite product of complex numbers or functions.

Let's just try to copy the definition of convergence of an infinite series. Let (b_n) be a sequence in $\mathbb{C}$. We say that the infinite product $\prod_{n=1}^{\infty} b_n$ converges to $p = \lim_{n\to\infty} p_n$, where $p_n = p_1 \cdots p_n$, and the limit is non-zero (to avoid trivial cases like $b_1 = 0$).

This is too restrictive. We'll thus modify our definition as follows: we say that $\prod b_n$ converges if all but finitely many factors are non-zero, and the partial products formed by the non-zero factors have non-zero limit.

We'd like to develop some tests for the convergence of infinite products. An immediate necessary condition is the analogue of the vanishing test: if $\prod b_n$ converges, then $b_n \to 1$. This can be seen by noting that $b_n = p_n/p_{n-1}$. With this in mind, we'll write our product as $\prod(1+a_n)$, so that a necessary condition for convergence is that $a_n \to 0$.

Naturally, we'd like to compare the infinite product $\prod(1 + a_n)$ with the infinite series $\sum \log(1 + a_n)$, where we use the principal branch of log. (It is defined for n large enough.) Let s_n be the nth partial term, so that $p_n = e^{s_n}$. So if the s_n's converge, the p_n's converge. Conversely? Suppose $p_n \to p$. Fix a branch $\log p = \ln |p| + i \arg p$. Take also $\log p_n = \ln |p_n| + i \arg p_n$, where $\arg p_n \in (\arg p - \pi, \arg p + \pi)$. Then

$$s_n = \log p_n + 2\pi i k_n$$

for some $k_n \in \mathbb{Z}$. Write

$$\log(1 + a_{n+1}) = s_{n+1} - s_n = \log p_{n+1} - \log p_n + 2\pi i(k_{n+1} - k_n).$$

For large enough n,

$$\begin{aligned} |\arg(1 + a_{n+1})| &< 2\pi/3, \\ |\arg p_n - \arg p| &< 2\pi/3, \\ |\arg p_{n+1} - \arg p| &< 2\pi/3, \end{aligned}$$

so $|k_{n+1}-k_n| < 1$, i.e. $k_{n+1} = k_n$, a common value k, for large n. Then $s_n \to \log p+2\pi ik$.

Thus convergence of $\prod(1 + a_n)$ is equivalent to convergence of $\sum \log(1 + a_n)$. This will be our main tool in studying infinite products. Next time, we'll discuss the notion of the convergence of an infinite product of funtions in order to develop the theory of functions with prescribed zeroes.

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2.9 Infinite Products of Functions (2-24-2021)

Having defined the notion of an infinite product of complex numbers, we'd like to now define the notion of an infinite product of complex functions.

2.9.1 Infinite Products of Functions

First, we should discuss the notions of absolute and uniform convergence of infinite products. We say that an infinite product $\prod_{n=1}^{\infty}(1+a_n)$ *converges absolutely* if $\sum_{n=1}^{\infty}\log(1+a_n)$ converges absolutely. This is equivalent to absolute convergence of $\sum_{n=1}^{\infty} a_n$. Why? Recall that

$$\lim_{z\to 0}\frac{\log(1+z)}{z}=1,$$

so for any $\varepsilon>0$,

$$\left|\frac{|\log(1+a_n)|}{|a_n|}-1\right|<\varepsilon$$

for large enough n, i.e.

$$(1-\varepsilon)|a_n|<|\log(1+a_n)|<(1+\varepsilon)|a_n|.$$

Now, consider an infinite product $\prod_{n=1}^{\infty} f_n(z)$, where the f_n's are continuous and complex-valued functions on an open set $\Omega\subseteq\mathbb{C}$. We say that the infinite product *converges uniformly and absolutely* on a subset $K\subseteq\Omega$ if

(1) $f_n(z)\to 1$ uniformly on K, and

(2) $\sum\log f_n$ is uniformly and absolutely convergent on K.

Note that condition (1) ensures that the principal branch of $\log f_n$ is defined when n is large enough, so that condition (2) makes sense. If $\prod f_n$ conveges uniformly and absolutely on compact subsets of Ω, then the partial products converge uniformly on compact subsets to a limit function $f(z)$, which is therefore continuous. We're mainly interested in the case when these functions are holomorphic.

Theorem 2.9.1. *Suppose that the f_n's are holomorphic in Ω, and that $\prod f_n$ converges uniformly and absolutely on compact subsets of Ω. Then*

(1) $f=\prod f_n$ is holomorphic in Ω, and for any p, we have $f=f_1\cdots f_p\prod_{n>p} f_n$.

(2) The set of zeroes of f is the union of the zero sets of all of the f_n's. Moreover, the multiplicity of a zero of f is the sum of the multiplicities for each f_n.

(3) *The series* $\sum f_n'/f_n$ *converges uniformly and absolutely on compact subsets of* Ω*, and its sum is* f'/f*.*

Proof. The proofs of (1) and (2) are things we've essentially seen before, so we'll only prove (3). Consider a relatively compact set $U \subseteq \Omega$, and write $f = f_1 \cdots f_p \cdot g_p$, where $g_p = \prod_{n>p} f_n$. Then

$$\frac{f'}{f} = \sum_{n=1}^{p} \frac{f_n'}{f_n} + \frac{g_p'}{g_p},$$

where $g_p = \exp(\sum_{n>p} \log f_n)$ (well-defined in U when p is large enough). We have

$$\frac{g_p'}{g_p} = \sum_{n>p} \frac{f_n'}{f_n},$$

since $\sum_{n>p} \log f_n$ converges uniformly and absolutely on compact subsets of U to a branch of $\log g_p$. Therefore $f'/f = \sum_{n=1}^{\infty} f_n'/f$ converges uniformly and absolutely on compact subsets of U, hence on compact subsets of Ω. □

When can we express a function as an infinite product? Let's try to write $\sin \pi z$ as an infinite product. Since $\sin \pi z$ has zeroes at exactly the integers, all simple, we should write down an infinite product with simple zeroes exactly at the integers; for example

$$f(z) = z \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right).$$

This converges uniformly and absolutely on compact subsets of $\mathbb{C}$ by comparison with $\sum 1/n^2$, implying that $f(z)$ is holomorphic and has zeroes precisely at the integers, all simple. Differentiating logarithmically term-by-term, we have

$$\frac{f'(z)}{f(z)} = \frac{1}{z} + \sum_{n=1}^{\infty} \frac{2z}{z^2 - n^2} = \pi \cot \pi z = \frac{g'(z)}{g(z)},$$

where $g(z) = \sin \pi z$. So $f(z) = Cg(z)$, since $(f/g)' = 0$. What is the constant? As $z \to 0$, $f(z)/z \to 1$ and $(\sin \pi z)/z \to \pi$, so $C = 1/\pi$. Therefore

$$\frac{\sin \pi z}{\pi z} = \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right).$$

2.9.2 Holomorphic Functions with Prescribed Zeroes

Any entire function that is *never* zero has the form $f(z) = e^{g(z)}$. Why? f is never zero, so f'/f is holomorphic in $\mathbb{C}$, and since $\mathbb{C}$ is simply connected, it's the derivative of an entire function $g(z)$. Then

$$\frac{d}{dz}\left(\frac{f(z)}{e^{g(z)}}\right) = \frac{f'(z)e^{g(z)} - f(z)e^{g(z)}g'(z)}{e^{2g(z)}} = \frac{f'(z)e^{g(z)} - f'(z)e^{g(z)}}{e^{2g(z)}} = 0,$$

so after absorbing the constant into $g(z)$ we have $f(z) = e^{g(z)}$.

Now, what's the most general entire function $f(z)$ with finitely many zeroes? Let's say that 0 is a zero of multiplicity $m \geq 0$, and let $a_1, \ldots, a_n$ be the non-zero zeroes of f, repeated according to multiplicity. Then

$$f(z) = z^m e^{g(z)} \prod_{k=1}^{n} \left(1 - \frac{z}{a_k}\right)$$

for some entire function $g(z)$. (Divide $f(z)$ by the non-exponential factors on the right-hand side to get an entire function with no zeroes, and then apply the previous considerations.)

We'd like to play the same game but for entire functions with *infinitely* many zeroes. We have to take the same care that we did for poles in the theorem of Mittag-Leffler and multiply by "convergence factors" that make the obvious infinite product converge. We'll do this next time.

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2.10 Weierstrass's Factorization Theorem (2-26-2021)

We finish off the story of finding functions with prescribed poles and zeroes by stating and proving Weierstrass's factorization theorem, and then we discuss a couple of its corollaries.

2.10.1 Weierstrass's Factorization Theorem

Here is the promised result.

Theorem 2.10.1. *(Weierstrass) Given $a_k \in \mathbb{C}$ (not necessarily distinct) with $a_k \to \infty$, there is an entire function with zeroes a_k. The most general such entire function is of the form*

$$f(z) = z^m e^{g(z)} \prod_{k=1}^{\infty} \left[\left(1 - \frac{z}{a_k}\right) e^{\frac{z}{a_k} + \frac{1}{2}\left(\frac{z}{a_k}\right)^2 + \cdots + \frac{1}{m_k}\left(\frac{z}{a_k}\right)^{m_k}} \right],$$

where the product is taken over all of the non-zero a_k's, $g(z)$ is entire, and the m_k's are integers so that the product converges uniformly and absolutely on compact sets.

We remark that this is very similar to the construction of a meromorphic function with prescribed poles. In that construction, we had to subtract off a certain number of terms from the Taylor expansion of the prescribed principal part to make the sum converge. Here, since we take logarithms to test convergence of the product, we ought to multiply by the exponential of a certain number of terms of the Taylor expansion of the principal branch of $\log(1 - z/a_k)$.

Proof. The infinite product $\prod(1 - z/a_k)e^{p_k(z)}$ converges together with the series with general term $g_k(z) = \log(1 - z/a_k) + p_k(z)$, where the branch of log is to be chosen so that $g_k(z)$ is the principal branch of the logarithm of the kth factor in the product. To do this, we'll choose the branch of log so that $\operatorname{Im} g_k(z) \in (-\pi, \pi)$. Given r, consider the factors with $|a_k| > r$. In $|z| \leq r$, let

$$p_k(z) = \frac{z}{a_k} + \frac{1}{2}\left(\frac{z}{a_k}\right)^2 + \cdots + \frac{1}{m_k}\left(\frac{z}{a_k}\right)^{m_k}.$$

Using the principal branch of log as above,

$$g_k(z) = -\frac{1}{m_k + 2}\left(\frac{z}{a_k}\right)^{m_k+1} - \frac{1}{m_k + 2}\left(\frac{z}{a_k}\right)^{m_k+2} - \cdots;$$

we want to estimate $|g_k(z)|$ and choose m_k accordingly. We have

$$|g_k(z)| \leq \frac{1}{m_k+1}\left(\frac{r}{|a_k|}\right)^{m_k+1} \cdot \left(1-\frac{r}{|a_k|}\right)^{-1}.$$

So choose m_k so that

$$\sum_{k=1}^{\infty} \frac{1}{m_k+1}\left(\frac{r}{|a_k|}\right)^{m_k+1}$$

converges; for example, $m_k = k$ works! Then $g_k(z) \to 0$ uniformly in $|z| \leq r$, so its imaginary part lies in $(-\pi, \pi)$ for k large enough. Therefore $\sum g_k(z)$ is uniformly and absolutely convergent in $|z| \leq r$, and it follows that the product represents a holomorphic function. □

We remark that the theorem we just proved, and the theorem of Mittag-Leffler, are very similar. In fact, one can prove the theorem of Mittag-Leffler using the Weierstrass factorization theorem, simply by taking reciprocals. (What about the converse? Could one combine the two theorems?)

2.10.2 Corollaries to Weierstrass's Factorization Theorem

We explore some consequences of the theorem we just proved.

Corollary 2.10.1. *Every meromorphic function on $\mathbb{C}$ is the quotient of two entire functions.*

Proof. Let $h(z)$ be a meromorphic function on $\mathbb{C}$. Let $g(z)$ be an entire function whose zeroes are precisely the poles of $h(z)$, counted with multiplicity. Then $f(z) = g(z)h(z)$ is an entire function, and $h(z) = f(z)/g(z)$. □

Corollary 2.10.2. *Let $a_k, b_k \in \mathbb{C}$ be sequences with $a_k \to \infty$, and choose multiplicities $n_k \in \mathbb{N}$. There is an entire function $f(z)$ such that each a_k is a root of order n_k of the equation $f(z) = b_k$.*

Proof. Near a_k, f should have the form $f(z) = b_k + (z-a_k)^{n_k}\widetilde{f}(z)$ for some holomorphic function $\widetilde{f}(z)$ with $\widetilde{f}(a_k) \neq 0$. By the theorem, there is an entire function $g(z)$ with a zero of order n_k at a_k. Write

$$f(z) = g(z)h(z) = b_k + g(z)\left(h(z) - \frac{b_k}{g(z)}\right),$$

where $h(z)$ is a meromorphic function with poles a_k having principal part equal to that of $b_k/g(z)$ at a_k. $g(z)$ has a zero of order n_k at a_k, and $h(z) - b_k/g(z)$ is holomorphic, so $f(z) = b_k$ has a_k as a root of order *at least* a_k. To remedy this, change the order of the zero of $g(z)$ at a_k from n_k to $n_k + 1$, and change the principal part of $h(z)$ at a_k to be $b_k/g(z) + 1/(z - a_k)$. □

The homework problems assigned for Edward Bierstone's MAT454/1002 in Winter 2021 for the material in Chapter 2 of my notes.

2.11 Problems for Chapter 2

1. Let Ω denote the union of an increasing sequence $\{\Omega_n\}$ of connected open subsets of $\mathbb{C}$, and let f_n denote a holomorphic function on Ω_n, for each n. Prove that, if f_n converges uniformly on compact subsets of Ω to a function f, then f is holomorphic on Ω (even though perhaps none of the f_n is defined on all of Ω).

2. Prove that
$$\sum_{n=-\infty}^{\infty} \frac{1}{(z+n)^2 + a^2} = \frac{\pi}{a} \cdot \frac{\sinh 2\pi a}{\cosh 2\pi a - \cos 2\pi z}.$$

3. Show

   (a) $\displaystyle\sum_{n=-\infty}^{\infty} \frac{(-1)^n}{(z-n)^2} = \frac{\pi^2}{(\sin \pi z)(\tan \pi z)}.$

   (b) $\displaystyle\frac{1}{z} + \sum_{n=1}^{\infty} (-1)^n \frac{2z}{z^2 - n^2} = \frac{\pi}{\sin \pi z}.$

4. Let $f(z)$ denote a doubly periodic meromorphic function with group of periods Γ, where Γ is generated by $e_1, e_2 \in \mathbb{C}$ (linearly independent over $\mathbb{R}$). Suppose that

   (i) $f(z)$ has zeroes (all simple) precisely at the points $(me_1 + ne_2)/2$, where $m, n \in \mathbb{Z}$ and $m + n$ is odd;

   (ii) $f(z)$ has poles (all simple) precisely at the points $(pe_1 + qe_2)/2$, where $p, q \in \mathbb{Z}$ and $p + q$ is even.

   Show that $f(z)$ is a constant multiple of the function
$$\frac{\wp'(z)}{\wp(z) - a_3}, \qquad \text{where } a_3 = \wp((e_1 + e_2)/2).$$

5. Show that, for all $a, b \in \mathbb{C}$,the function
$$\wp'(z) - a\wp(z) - b$$
has 3 zeroes in a period parallelogram, and their sum equals a period. Show that if $u, v \in \mathbb{C}$, $u \pm v \not\equiv 0 \mod \Gamma$, then we can find a, b such that the function above has zeros at $u, v, -u - v$. Deduce the *addition theorem*: if $u + v + w = 0$, then
$$\det \begin{pmatrix} \wp(u) & \wp'(u) & 1 \\ \wp(v) & \wp'(v) & 1 \\ \wp(w) & \wp'(w) & 1 \end{pmatrix} = 0.$$

6. Prove the following variant of the addition theorem:

$$\wp(u+v) = -\wp(u) - \wp(v) + \frac{1}{4}\left(\frac{\wp'(u)-\wp'(v)}{\wp(u)-\wp(v)}\right)^2.$$

7. (a) An *elliptic function* on $\mathbb{C}$ means a doubly-period meromorphic function. Show that any *even* elliptic function $f(z)$ can be written in the form

$$f(z) = c\prod_{k=1}^{n}\frac{\wp(z)-\wp(a_k)}{\wp(z)-\wp(b_k)}$$

(where c is a constant and $\wp(z)$ denotes the Weierstrass $\wp$-function with the same periods), provided that 0 is neither a zero nor a pole of f. Conclude that every even elliptic function f can be written $f = R(\wp)$, where R is a rational function.

   (b) Show that every odd elliptic function f can be written $f = \wp' R(\wp)$, where R is a rational function.

   (c) Show that every elliptic function f can be written $f = R(\wp, \wp')$, where R is rational.

8. Let $\{a_k\}$ and $\{b_k\}$ be sequences of complex numbers such that $b_k \to \infty$, and $|a_k/b_k| \leq M < \infty$ for all $k \geq k_0$. Prove that

$$\sum_{k=1}^{\infty}\left(\frac{a_k}{z-b_k} + \frac{a_k}{b_k}\sum_{j=0}^{k}\left(\frac{z}{b_k}\right)^j\right)$$

defines a meromorphic function on $\mathbb{C}$ with poles b_k and principal parts $a_k/(z-b_k)$, and no other poles in $\mathbb{C}$.

9. Show that the infinite product $\prod_{n=0}^{\infty}(1+z^{2^n})$ converges uniformly on compact subsets of the disk $|z| < 1$.

10. (a) Show that the infinite product

$$\prod_{n=1}^{\infty}\left(1-\frac{z}{n}\right)e^{z/n}$$

represents an entire function $g(z)$ with simple zeros at the positive integers.

   (b) Prove that $g(z+1) = ze^c g(z), z \in \mathbb{C}$, where c is a constant.

   (c) Find a nonzero entire function $f(z)$ such that $f(z+1) = zf(z), z \in \mathbb{C}$.

(d) Let $P(z)$ be a polynomial. Show there is a nonzero entire function $f(z)$ such that $f(z+1) = P(z)f(z)$.

11. Let Γ denote a discrete subgroup of $\mathbb{C}$ generated by two complex numbers e_1, e_2 that are linearly independent over $\mathbb{R}$.

    (a) Show that

$$\sigma(z) = z \prod_{\omega \in \Gamma \setminus \{0\}} \left(1 - \frac{z}{\omega}\right) e^{\frac{z}{\omega} + \frac{1}{2}\frac{z^2}{\omega^2}}$$

    converges to an entire function.

    (b) Prove that

$$\left(\frac{\sigma'}{\sigma}\right)'(z) = -\wp(z),$$

    where $\wp(z)$ is the Weierstrass $\wp$-function associated to Γ.

12. (a) Show that there are constants η_1, η_2 such that

$$\sigma(z + e_j) = -\sigma(z) e^{\eta_j (z + e_j/2)}, \qquad j = 1, 2.$$

    (b) Show that any elliptic function with periods e_1, e_2 can be written

$$C \prod_{k=1}^{n} \frac{\sigma(z - a_k)}{\sigma(z - b_k)}, \qquad \text{where } C \text{ is a constant.}$$

13. Let f denote an entire function (not identically zero). Let n be a natural number. Prove that there exists an entire function g such that $f = g^n$ if and only if every zero of f has order divisible by n.

14. Denote by $Z(f)$ the set of zeroes of a function f. Prove that if f_1 and f_2 are entire functions such that $Z(f_1) \cap Z(f_2) = \emptyset$, then there exist entire functions g_1, g_2 such that $f_1 g_1 + f_2 g_2 = 1$.