MAT454/1002 lecture notes by Kain Dineen. Taught by Edward Bierstone.

1 The Weierstrass $\wp$-function (2-3-2021)

1.1 Another Example

Let us consider

$$\frac{1}{z} + \sum_{n \neq 0} \left(\frac{1}{z-n} + \frac{1}{n} \right).$$

This series is uniformly and absolutely convergent on compact subsets of $\mathbb{C}$, for we may write each term as

$$\frac{z}{n(z-n)},$$

which is comparable to $1/n^2$ for z in a compact set. It follows that the sum, $f(z)$, is a meromorphic function in $\mathbb{C}$. The poles are precisely the integers, and they are all simple poles of residue 1. The series can be differentiated term-by-term:

$$f'(z) = -\frac{1}{z^2} - \sum_{n \neq 0} \frac{1}{(z-n)^2} = -\frac{\pi^2}{\sin^2 \pi z} = \frac{d}{dz}(\pi \cot \pi z).$$

It follows that $f(z) - \pi \cot \pi z$ is a constant; the constant is zero since $f(z)$ and $\pi \cot \pi z$ are both odd functions. Grouping the positive and negative terms, we obtain

$$\frac{1}{z} + \sum_{n=1}^{\infty} \frac{2z}{z^2 - n^2} = \pi \cot \pi z.$$

In particular, this function is periodic.

1.2 The Weierstrass $\wp$-function

Let $e_1, e_2 \in \mathbb{C}$ be two complex numbers which are linearly independent over the reals. We want to look at a *doubly periodic* function over the lattice generated by e_1 and e_2. The set

$$\Gamma = \{n_1 e_1 + n_2 e_2 : n_1, n_2 \in \mathbb{Z}\}$$

is a discrete subgroup (lattice) of $\mathbb{C}$. We will say that $f(z)$ *has* Γ *as a group of periods* (or is *doubly periodic* for short) if $f(z + n_1 e_1 + n_2 e_2) = f(z)$, for all $n_1, n_2 \in \mathbb{Z}$, i.e.

$$f(z + e_1) = f(z),$$
$$f(z + e_2) = f(z),$$

for all z. We call e_1, e_2 a *basis* of Γ.

Given a point $z_0 \in \mathbb{C}$, we will call the parallelogram with vertices $z_0, z_0 + e_1, z_0 + e_2, z_0 + e_1 + e_2$ the *parallelogram with first vertex* z_0. Note that e_1', e_2' is a basis of Γ if and only if e_1', e_2' is a linear combination of e_1, e_2 with integer coefficients, and the determinant of the matrix of coefficients is ± 1 (i.e. a unit in $\mathbb{Z}$).

We define[1] the *Weierstrass* $\wp$*-function* by

$$\wp(z) := \frac{1}{z^2} + \sum_{\omega \in \Gamma \setminus \{0\}} \left(\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} \right).$$

We're going to see that this series is uniformly and absolutely convergent on compact subsets of $\mathbb{C}$.

Lemma 1.1. *The series*

$$\sum_{\omega \in \Gamma \setminus \{0\}} \frac{1}{|\omega|^3}$$

converges.

Proof. Think of the lattice as an expanding series of parallelograms. For $n \geq 1$, let

$$P_n = \{t_1 e_1 + t_2 e_2 : t_1, t_2 \in \mathbb{Z}, \max\{|t_1|, |t_2|\} = n\}.$$

There are $8n$ points on P_n, each of distance at least kn from 0, where k is the shortest distance from the origin to P_1. We have

$$\sum_{\omega \in P_n} \frac{1}{|\omega|^3} \leq \frac{8n}{k^3 n^3} = \frac{8}{k^3 n^2},$$

so

$$\sum_{\omega \in \Gamma \setminus \{0\}} \frac{1}{|\omega|^3} \leq \sum_{n=1}^{\infty} \frac{8}{k^3 n^2} < \infty.$$

$\square$

To show that the series defining $\wp$ converges uniformly and absolutely on compact subsets of $\mathbb{C}$, it's enough to check it on disks $|z| \leq r$, for any r. For $|z| \leq r$ and $|\omega| \geq 2r$,

$$\left| \frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} \right| = \left| \frac{2\omega z - z^2}{\omega^2 (z-\omega)^2} \right| = \frac{|z||2 - \frac{z}{\omega}|}{|\omega|^3 |1 - \frac{z}{\omega}|^2} \leq \frac{r \cdot \frac{5}{2}}{|\omega|^3 \cdot \frac{1}{4}} = \frac{10r}{|\omega|^3}.$$

So the series converges uniformly and absolutely in $|z| \leq r$ by comparison with $\sum 1/|\omega|^3$. It follows that the Weierstrass $\wp$-function is well-defined as a meromorphic function on $\mathbb{C}$.

[1] Why do we add the $-1/\omega^2$ term? To quote Professor Bierstone, it's "a device to make the series converge." We'll see that this term is required to make the series' terms comparable to $1/\omega^3$, when z lies in a compact set. Compare with the example from the start of this class.

Its poles are just the points of the lattice Γ, each of multiplicity 2, with principal part $(z-\omega)^{-2}$ to the pole $\omega \in \Gamma$. Moreover, $\wp$ is an even function. We would like to see that it is doubly periodic.

$$\wp'(z) = -2 \sum_{\omega \in \Gamma} \frac{1}{(z-\omega)^3},$$

which is uniformly and absolutely convergent on compact subsets of $\mathbb{C}$, and again periodic: $\wp'(z+\omega) = \wp'(z)$ for any $\omega \in \Gamma$. It's also an odd function. We want to check that

$$\wp(z+e_i) = \wp(z), \qquad i = 1, 2,$$

for all z. By periodicity of $\wp'$, $\wp(z+e_i) - \wp(z)$ is a constant. Setting $z = -e_i/2$, we see that the constant is

$$\text{const.} = \wp(e_i/2) - \wp(-e_i/2) = 0,$$

since $\wp$ is even. So $\wp$ is doubly periodic with group of periods Γ.

What is the Laurent expansion of $\wp(z)$ at 0?

$$\wp(z) = \frac{1}{z^2} + a_2 z^2 + a_4 z^4 + \cdots,$$

because $\wp$ is even, and $\wp(z) - z^{-2}$ vanishes at 0. The coefficients are sums of things of the form

$$\frac{1}{(2k)!} \cdot \left(\frac{(2k+1)!}{(z-\omega)^{2k+2}} \right) \bigg|_{z=0},$$

i.e. $(2k!)^{-1}$ times the $2k$th derivative of $(z-\omega)^{-2}$. This leads to

$$a_{2k} = (2k+1) \sum_{\omega \neq 0} \frac{1}{\omega^{2(k+1)}}.$$

In particular,

$$a_2 = 3 \sum_{\omega \neq 0} \frac{1}{\omega^4}$$

and

$$a_4 = 5 \sum_{\omega \neq 0} \frac{1}{\omega^6},$$

and so on.