0.1 Examples of Conformal Mappings (3-8-2021)

We're going to study the properties of conformal mappings, and work towards proving the Riemann mapping theorem as a first application of the theory of normal families of complex functions.

0.1.1 Automorphisms

Let Ω, Ω' be open sets in $\mathbb{C}$ or in the Riemann sphere. A *conformal* (or *biholomorphic*) mapping $\Omega \to \Omega'$ is a holomorphic mapping with a holomorphic inverse. Some problems related to conformal mappings are:

(1) Given two domains Ω, Ω', are they biholomorphic?

(2) If so, can we find all of the biholomorphisms $\Omega \to \Omega'$?

A necessary condition for (1) is, of course, that the sets are homemorphic (even diffeomorphic). The conditions are not sufficient: the complex plane $\mathbb{C}$ and the unit disk D are diffeomorphic, but by Liouville's theorem there's no non-constant entire mapping $\mathbb{C} \to D$.

If $f, g \colon \Omega \to \Omega'$ are both biholomorphisms, then $S = g^{-1} \circ f$ is an *automorphism* of Ω, a biholomorphism of Ω to itself. The group of automorphisms of Ω is denoted by $\mathrm{Aut}(\Omega)$. A biholomorphism $f \colon \Omega \to \Omega'$ induces a group isomorphism $\mathrm{Aut}(\Omega) \to \mathrm{Aut}(\Omega')$ given by $S \mapsto f \circ S \circ f^{-1}$. Moreover, if we know *one* biholomorphism $f \colon \Omega \to \Omega'$, we obtain a bijection between the set of biholomorphisms $\Omega \to \Omega'$ and the automorphisms $\Omega' \to \Omega'$ given by $g \mapsto f \circ g \circ f^{-1}$. Therefore we ought to study automorphism groups to answer (2).

0.1.2 Examples of Automorphisms

We're interested in computing the automorphism groups of some of the common domains we work with in the complex plane and in the Riemann sphere.

1. What is $\mathrm{Aut}(\mathbb{C})$? It is the set of all affine transformations $z \mapsto az + b$, where $a \neq 0$. Indeed, if $w = f(z)$ is an automorphism of $\mathbb{C}$, then there are two cases:

   (a) f has an essential singularity at ∞;

   (b) f has a pole at ∞, i.e. is a polynomial.

   (a) is not possible. Consider $f(\{|z| > 1\})$ and $f(\{|z| < 1\})$. The latter set is non-empty and open, so the former cannot be dense. By Casorati-Weierstrass, ∞ cannot be an essential singularity. So f is a polynomial, say, of degree n. Since f is one-to-one, $n = 1$, i.e. $w = f(z) = az + b$ for some $a \neq 0$. In terms of familiar groups, $\mathrm{Aut}(\mathbb{C}) \cong \mathbb{C} \rtimes GL(1, \mathbb{C}) = \mathbb{C} \rtimes \mathbb{C}^{\times}$.

2. What about $\mathrm{Aut}(S^2)$? It is the set of all fractional linear transformations
$$w = \frac{az+b}{cz+d}, \qquad ad-bc \neq 0.$$
Note that the coefficients are uniquely determined up to multiplication by a constant, so the inverse mapping is given by
$$z = \frac{dw-b}{-cw+a}.$$
Such a fractional linear transformation acts on ∞ by
$$\infty \mapsto \begin{cases} \frac{a}{c}, & c \neq 0, \\ \infty, & c = 0. \end{cases}$$
The fractional linear transformations form a subgroup G of $\mathrm{Aut}(S^2)$. Consider the isotropy subgroup $H \leq G$ of ∞. Then $c = 0$, so $d \neq 0$; we may assume $d = 1$. Then $w = az + b$, which proves that $H = \mathrm{Aut}(\mathbb{C})$. So then this is also the subgroup of $\mathrm{Aut}(S^2)$ that fixes ∞. Therefore $G = \mathrm{Aut}(S^2)$, by the following lemma:

Lemma 0.1.1. *Let $\Omega \subseteq S^2$ be open, and let G be a subgroup of $\mathrm{Aut}(\Omega)$. Assume that G acts transitively on Ω, and that there is a $z_0 \in \Omega$ such that the isotropy subgroup $(\mathrm{Aut}(\Omega))_{z_0} \subseteq G$. Then $G = \mathrm{Aut}(\Omega)$.*

Proof. Suppose $S \in \mathrm{Aut}(\Omega)$. Then there is a $T \in G$ such that $T(z_0) = S(z_0)$, so that $T^{-1} \circ S \in (\mathrm{Aut}(\Omega))_{z_0} \subseteq G$. So then $S = T \circ (T^{-1} \circ S) \in G$. $\square$

In terms of familiar groups, $\mathrm{Aut}(S^2) \cong PGL(2,\mathbb{C}) := GL(2,\mathbb{C})/\mathbb{C}^\times\{I\}$.

3. What about $\mathrm{Aut}(D)$, where D is the open unit disk? This is the set of all fractional linear transformations
$$w = e^{i\theta}\frac{z-z_0}{1-\overline{z_0}z}, \qquad \theta \in \mathbb{R}, \quad |z_0| < 1.$$
We'll prove this using Schwarz's lemma. Let $T \in \mathrm{Aut}(D)$, and let z_0 be the point with $T(z_0) = 0$, and let $\theta = \arg T'(z_0)$. Consider
$$S(z) = e^{i\theta}\frac{z-z_0}{1-\overline{z_0}z}.$$
Let $f = S \circ T^{-1}$. Then $f(0) = 0$ and $|f(z)| < 1$ if $|z| < 1$. By Schwarz's lemma, $|f(z)| \leq |z|$ for $|z| < 1$. If we also apply Schwarz's lemma to f^{-1}, then we obtain that $|f(z)| = |z|$ for all $|z| < 1$, i.e. f is a rotation $f(z) = e^{i\alpha}z$, for some $\alpha \in \mathbb{R}$. Thus $S(z) = e^{i\alpha}T(z)$, so $S'(z_0) = e^{i\alpha}T'(z_0)$. Comparing arguments gives $\alpha = 0$, i.e. $S = T$.

4. Our last example will be $\mathrm{Aut}(\mathbb{H}^+)$, the group of automorphisms of the open upper half plane. If we consider the biholomorphism $\mathbb{H}^+ \to D$ given by

$$w = \frac{z-i}{z+i}.$$

then we see that $\mathrm{Aut}(\mathbb{H}^+)$ consists entirely of fractional linear transformations $w = (az+b)/(cz+d)$, since all of its elements are conjugates of elements of $\mathrm{Aut}(D)$ by this one. Which ones? Such a fractional linear transformation must take the real line to itself, so we can assume that a, b, c, d are all real, and that $ad - bc = \pm 1$. Since $\mathbb{H}^+$ is mapped to itself, i must go to a point of $\mathbb{H}^+$, so

$$\mathrm{Im}\,\frac{ai+b}{ci+d} = \frac{ad-bc}{c^2+d^2} > 0,$$

i.e. $ad - bc = 1$. Therefore $\mathrm{Aut}(\mathbb{H}^+)$ consists of all of the fractional linear transformations of the form $w = (az+b)/(cz+d)$, where $a, b, c, d \in \mathbb{R}$, and $ad - bc = 1$. In terms of familiar groups, $\mathrm{Aut}(\mathbb{H}^+) \cong PSL(2, \mathbb{R}) := SL(2, \mathbb{R})/\{\pm I\}$.