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0.1 Cauchy's Integral Formula (1-18-2021)

We briefly introduce Cauchy's integral formula.

0.1.1 Remarks

Let us note that in Cauchy's theorem, from last time, it is enough to assume that the function is continuous on the domain, and holomorphic outside a line (or outside even a finite number of lines and points).

We need the following theorem to prove Cauchy's integral formula. The proof will not be given.

Theorem 0.1.1. *A closed differential form ω on a simply-connected open set $\Omega \subseteq \mathbb{R}^2$ has a (globally defined) primitve.*

0.1.2 Cauchy's Integral Formula

Let γ be a closed curve in Ω, and let $a \in \Omega$ be a point not on the curve. We define the *winding number* of γ with respect to a as

$$w(\gamma, a) = \frac{1}{2\pi i} \int_\gamma \frac{dz}{z-a}.$$

The winding number is always an integer, since the integral is given by the difference between two branches of $\log(z-a)$. It is meant to measure how many times the curve γ loops around a. For example, if γ is a circle centered at a oriented counter-clockwise, then $w(\gamma, a) = 1$.

The winding number is invariant with respect to a homotopy of γ not passing through a, since the integrand $(z-a)^{-1}\,dz$ is a closed form on $\mathbb{C} \setminus \{a\}$. Furthermore, as a function of a, $w(\gamma, a)$ is constant on the connected components of $\mathbb{C} \setminus \gamma$.

Theorem 0.1.2. *(Cauchy's integral formula) Let $f(z)$ be holomorphic in the open set Ω, and let $a \in \Omega$. Suppose γ is a closed curve in Ω, not containing a, which is homotopic to a point in Ω. Then,*

$$\frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z-a}\,dz = w(\gamma, a) f(a).$$

We will prove this next time.