

MAT454/1002 lecture notes by Kain Dineen. Taught by Edward Bierstone.

## 1 Cauchy's Integral Formula (1-18-2021)

### 1.1 Remarks

Let us note that in Cauchy's theorem, from last time, it is enough to assume that the function is continuous on the domain, and holomorphic outside a line (or outside a finite number of lines and points).

We need the following theorem to prove Cauchy's integral formula. The proof will not be given.

**Theorem 1.1.** *A closed differential form  $\omega$  on a simply-connected open set  $\Omega \subseteq \mathbb{R}^2$  has a (globally defined) primitive.*

### 1.2 Cauchy's Integral Formula

Let  $\gamma$  be a closed curve in  $\Omega$ , and let  $a \in \Omega$  be a point not on the curve. We define the *winding number* of  $\gamma$  with respect to  $a$  as

$$w(\gamma, a) = \frac{1}{2\pi i} \int_{\gamma} \frac{dz}{z - a}.$$

The winding number is always an integer, since it is given by the difference between two branches of  $\log(z - a)$ . It is meant to measure how many times the curve  $\gamma$  loops around  $a$ . For example, if  $\gamma$  is a circle centered at  $a$  oriented counter-clockwise, then  $w(\gamma, a) = 1$ .

The winding number is invariant with respect to a homotopy of  $\gamma$  not passing through  $a$ , since the integrand  $(z - a)^{-1} dz$  is a closed form on  $\mathbb{C} \setminus \{a\}$ . Furthermore, as a function of  $a$ ,  $w(\gamma, a)$  is constant on the connected components of  $\mathbb{C} \setminus \gamma$ .

**Theorem 1.2.** *(Cauchy's integral formula) Let  $f(z)$  be holomorphic in the open set  $\Omega$ , and let  $a \in \Omega$ . Suppose  $\gamma$  is a closed curve in  $\Omega$ , not containing  $a$ , which is homotopic to a point in  $\Omega$ . Then,*

$$\frac{1}{2\pi i} \int_{\gamma} \frac{f(z)}{z - a} dz = w(\gamma, a)f(a).$$

We will prove this next time.