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The Spectrum of Turbulence

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echoes of short delay noted by previous workers are due to atmospheric scattering patches of very low effective reflexion coefficient. It is found that, after the normally recognized regions of the ionosphere, the radio reflecting centres in the atmosphere next in order of importance as regards wireless echo production are transitory bursts of ionization which occur at equivalent heights of from 80 to 160 km. Such bursts of ionization are due to some cosmic agency which is usually effective throughout the whole of the day and night.

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The spectrum of turbulence

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When a prism is set up in the path of a beam of white light it analyses the time variation of electric intensity at a point into its harmonic components and separates them into a spectrum. Since the velocity of light for all wave-lengths is the same, the time variation analysis is exactly equivalent to a harmonic analysis of the space variation of electric intensity along the beam. In a recent paper Mr Simmons (Simmons and Salter 1938) has shown how the time variation in velocity at a field point in a turbulent air stream can be analysed into a spectrum. In the present paper it is proposed to discuss the connexion between the spectrum of turbulence, measured at a fixed point, and the correlation between *simultaneous* values of velocity measured at two points.

If u, the component at a fixed point of turbulent motion in the direction of the main stream in a wind tunnel, is resolved into harmonic components the mean value of u^2 may be regarded as being the sum of contributions from all frequencies. If $\overline{u^2}\,F(n)\,dn$ is the contribution from frequencies between n and $n+dn$, then

$$\int_0^\infty F(n)\,dn = 1. \tag{1}$$

If $F(n)$ is plotted against n, the diagram so produced is a form of the spectrum curve.

The proof that $\overline{u^2}$ may be regarded as being the sum of contributions from all the harmonic components has been given by Rayleigh, using a form of Parseval's theorem. If $u = \phi(t)$ and

$$\left.\begin{aligned} I_1 &= \frac{1}{\pi}\int_{-\infty}^{+\infty} \phi(t)\cos\kappa t\,dt, \\ I_2 &= \frac{1}{\pi}\int_{-\infty}^{+\infty} \phi(t)\sin\kappa t\,dt. \end{aligned}\right\} \tag{2}$$

Rayleigh showed that

$$\int_{-\infty}^{+\infty} [\phi(t)]^2\,dt = \pi\int_0^\infty (I_1^2+I_2^2)\,d\kappa, \tag{3}$$

or if $\kappa = 2\pi n$, so that n represents the number of cycles per second,

$$\int_{-\infty}^{+\infty} [\phi(t)]^2\,dt = 2\pi^2\int_0^\infty (I_1^2+I_2^2)\,dn. \tag{4}$$

If the integrals on the right-hand side of (2) and the left-hand side of (3) are taken over a long time T instead of infinity the left-hand side of (3) is $T\overline{u^2}$, so that

$$\overline{u^2} = 2\pi^2\int_0^\infty \operatorname*{Lt}_{T\to\infty}\left(\frac{I_1^2+I_2^2}{T}\right)dn. \tag{5}$$

The quantity $2\pi^2 \operatorname*{Lt}_{T\to\infty}\left(\frac{I_1^2+I_2^2}{T}\right)$ is therefore the contribution to $\overline{u^2}$ which arises from the components of frequency between n and $n+dn$, i.e.

$$2\pi^2 \operatorname*{Lt}_{T\to\infty}\left(\frac{I_1^2+I_2^2}{T}\right) = F(n). \tag{6}$$

Connexion between spectrum curve and correlation curve

Now consider two cases: (a) where the variation in u is due to eddies of small extent which are carried by a wind stream of velocity U past the fixed point; (b) the variation is due to large eddies carried in the wind stream. In case (a) the fluctuations at the fixed point will be much more rapid than they are in case (*b*). The spectrum analysis in case (*b*) will therefore show greater values of $F(n)$ for small values of n than in case (*a*).

It is clear when the eddies are large the correlation R_x between simultaneous values of u at distance x apart must fall away with increasing x more slowly than when the eddies are small. One may therefore anticipate that when the (R_x, x) curve has a small spread in the x co-ordinate the $F(n)$ curve will extend to large values of n and vice versa.

If the velocity of the air stream which carries the eddies is very much greater than the turbulent velocity, one may assume that the sequence of changes in u at the fixed point are simply due to the passage of an unchanging pattern of turbulent motion over the point, i.e. one may assume that

$$u = \phi(t) = \phi\left(\frac{x}{U}\right), \tag{7}$$

where x is measured upstream at time $t = 0$ from the fixed point where u is measured. In the limit when $u/U \to 0$ (7) is certainly true. Assuming that (7) is still true when u/U is small but not zero, R_x is defined as

$$R_x = \frac{\overline{\phi(t)\,\phi\left(t+\frac{x}{U}\right)}}{\overline{u^2}}. \tag{8}$$

We now introduce another expression analogous to (3). It can be shown that*

$$\int_{-\infty}^{+\infty} \phi(t)\,\phi\left(t+\frac{x}{U}\right) dt = 2\pi^2 \int_0^\infty (I_1^2 + I_2^2) \cos\frac{2\pi n x}{U}\, du, \tag{9}$$

where I_1 and I_2 have the same meaning as in (3).

Substituting for $I_1^2 + I_2^2$ from (6), (9) becomes

$$\frac{\overline{\phi(t)\,\phi\left(t+\frac{x}{U}\right)}}{\overline{u^2}} = \int_0^\infty F(n) \cos\frac{2\pi n x}{U}\, dn; \tag{10}$$

hence from (8)

$$R_x = \int_0^\infty F(n) \cos\frac{2\pi n x}{U}\, dn. \tag{11}$$

It will be noticed that the form of (11) is very similar to that of the Fourier

* This formula can be deduced from the theorem 9·09 given on p. 70 of Norbert Wiener's *The Fourier Integral*, Camb. Univ. Press, 1933.

integral. The Fourier integral theorem is usually expressed by the pair of formulae

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} g(\mu) \cos \mu x \, d\mu, \tag{12}$$

$$g(\mu) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(x) \cos \mu x \, dx. \tag{13}$$

When $f(x)$ is symmetrical, so that $f(x) = f(-x)$, (12) and (13) may be written

$$f(x) = \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} g(\mu) \cos \mu x \, d\mu, \tag{14}$$

$$g(\mu) = \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} f(x) \cos \mu x \, dx. \tag{15}$$

Comparing (11) and (14) it will be seen that if

$$\mu = \frac{2\pi n}{U}, \quad f(x) = R_x, \quad g(\mu) = \frac{UF(n)}{2\sqrt{2\pi}},$$

then (11) and (14) are identical.

Making these substitutions in (15), the following expression is found for $F(n)$:

$$F(n) = \frac{4}{U} \int_{0}^{\infty} R_x \cos \frac{2\pi n x}{U} \, dx. \tag{16}$$

It seems therefore that R_x and $\dfrac{UF(n)}{2\sqrt{2\pi}}$ are Fourier transforms of one another.

If $F(n)$ is observed we can calculate R_x using (11), and if R_x is observed we can calculate the spectrum curve $F(n)$ using (16).

Comparison with observation

Measurements have been made by Mr L. F. G. Simmons of R_x and of $F(n)$ at a point 6 ft. 10 in. from a turbulence-producing grid with a mesh 3×3 in. at wind speeds $U = 15$, 20, 25, 30 and 35 ft./sec. It was found that except very close to $x = 0$, R_x is nearly independent of U within that range.* When R_x is independent of U it will be seen from (16) that $UF(n)$ must be a function of n/U.

Accordingly Mr Simmons' measurements of $F(n)$ for all values of U have

* A similar result has been obtained by Dryden, N.A.C.A. report 581, 1937.

been plotted on the same diagram (figs. 1 and 1 a), in which the ordinates are $UF(n)$ and the abscissae are n/U. The fact that the points fall so closely on one curve is very satisfactory evidence that the measurements of $F(n)$ are accurate.

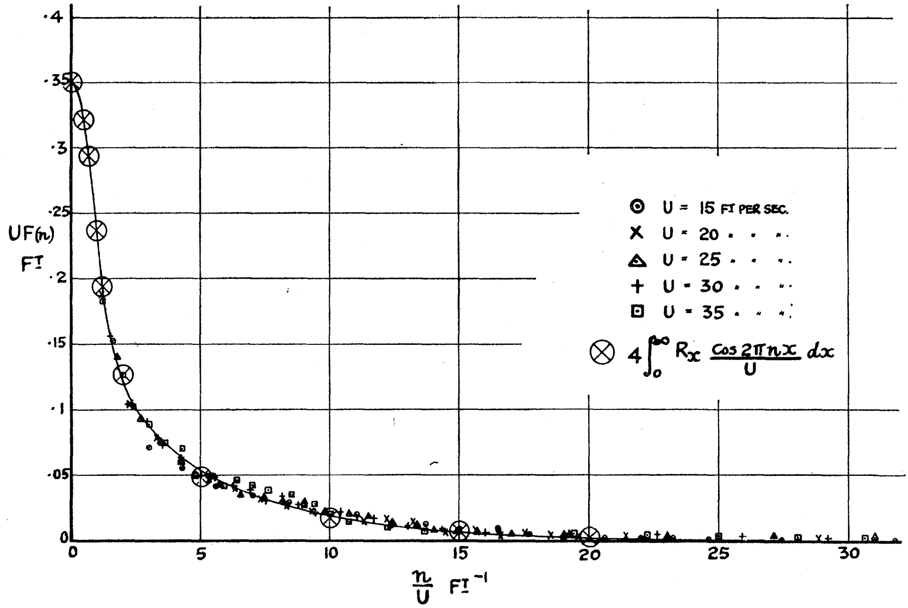

Fig. 1

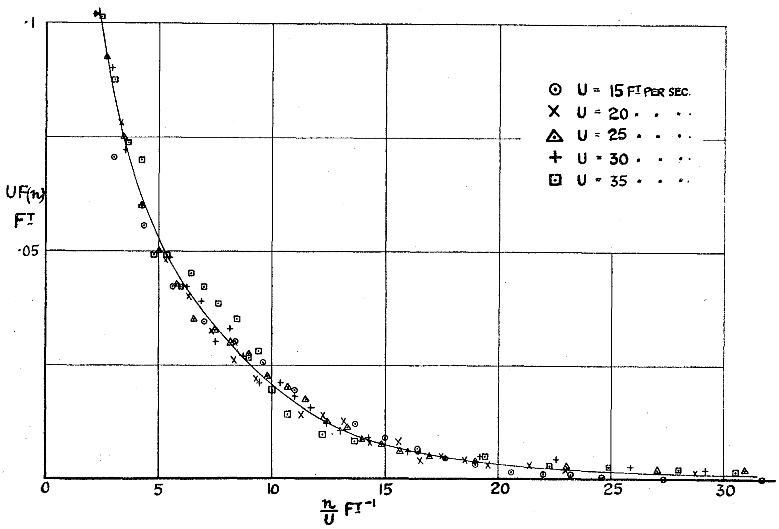

Fig. 1 a

Mr Simmons' measurements of R_x are shown in fig. 2. The values of $F(n)$ calculated using the measured values of R_x in (16) are shown in fig. 1, but as the points corresponding with the lower part of fig. 1 are rather close together an enlarged version is shown in fig. 1*a*. The values of R_x calculated using the measured values of $F(n)$ in (11) are shown in fig. 2.

It will be seen that the agreement in both cases is good.

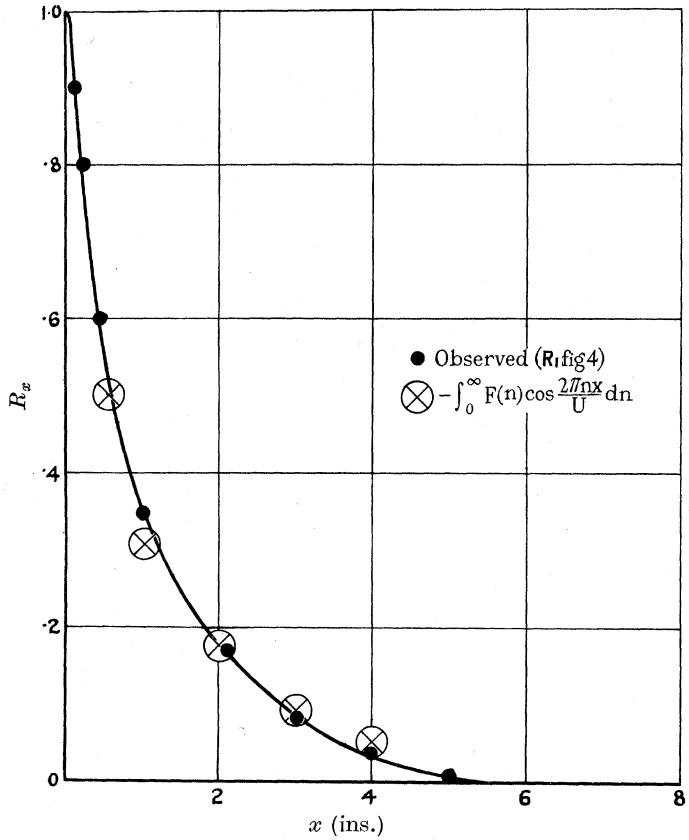

FIG. 2

It has been seen that the points in fig. 1 seem to fall on one curve, as is predicted by (16), when it is assumed that R_x is independent of U. On the other hand it is known that the curvature of the R_x curve at its vertex is not independent of U. This curvature is defined (Taylor 1935) by means of a length λ, where

$$\frac{1}{\lambda^2} = 2 \underset{x \to 0}{\mathrm{Lt}} \left(\frac{1 - R_x}{x^2}\right). \tag{17}$$

If the turbulence is isotropic experiments show that λ is proportional to $U^{-\frac{1}{2}}$.

To find what effect this variation in λ with U may be expected to produce in the $F(n)$ curve, λ may be expressed in terms of $F(n)$. When n is small $\cos\frac{2\pi nx}{U}$ may be replaced in (11) by $1-\frac{2\pi^2x^2n^2}{U^2}$. Hence

$$\frac{1}{\lambda^2}=\frac{4\pi^2}{U^2}\int_0^\infty n^2F(n)\,dn. \tag{18}$$

Since (18) can be written in the form

$$\frac{1}{\lambda^2}=4\pi^2\int_0^\infty \frac{n^2}{U^2}\,UF(n)\,\frac{dn}{U}, \tag{19}$$

λ must be independent of U if the $\{UF(n), n/U\}$ curve is independent of U. This deduction is inconsistent with the observed fact that λ is proportional to $U^{-\frac{1}{2}}$.

The explanation of this apparent discrepancy is that the value of $\int_0^\infty n^2F(n)\,dn$ depends chiefly on the values of $F(n)$ for large values of n, i.e. on the parts of figs. 1 and 1 *a* where the points are so close to the axis that variations in their height above it are hardly visible. In fig. 3 the vertical scale of $UF(n)$ has been enlarged very greatly. It will be seen that above $n/U = 16$ the $UF(n)$ curves separate, that for $U = 15$ ft./sec. falling below those for 20 and 35 ft./sec.

Calculation of λ from the spectrum curve

To determine λ from the spectrum curve the integral (19) must be evaluated using the values of $UF(n)$ taken from figs. 1 and 3. It is instructive to tabulate the contributions to this integral which arise from various ranges of n/U. These are set forth in Table I, where they are expressed in ft.-sec. units. It will be seen that when $U = 35$ about half of the integral is due to components for which $n/U > 30$, in spite of the fact that the highest

Table I. Contributions to $\int_0^\infty \frac{n^2}{U^2}F(n)\,dn$ expressed in ft.-sec. units

n/U	$U=15$	$U=20$	$U=35$
0–16	24·0	24·0	24·0
17–30	7·7	19·0	23·9
31–∞	0	15·5	44·4
Total	31·7	58·5	92·3

value of $F(n)$ in this range is only $\frac{1}{200}$ of its maximum value (namely 0·35 when $n=0$). The value of λ in feet is found by inserting the numbers given at the foot of Table I for the integral in (19). Thus when $U = 35$ ft./sec. $\lambda = (4\pi^2 \times 92{\cdot}3)^{-\frac{1}{2}} = 0{\cdot}00165$ ft. $= 0{\cdot}50$ cm. The values of λ so calculated are given in column 2, Table II.

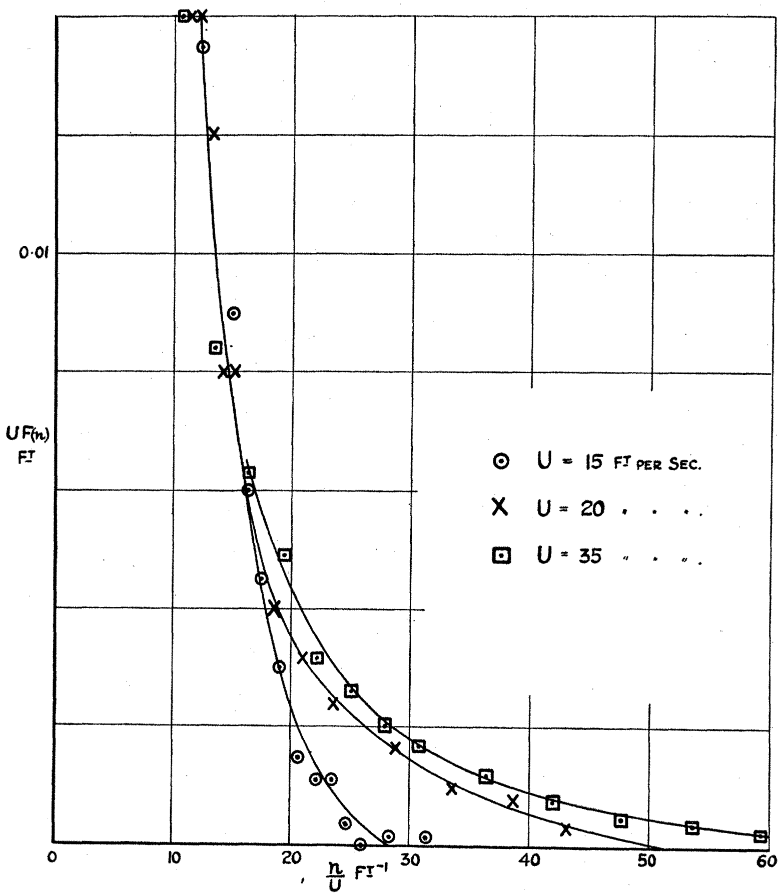

FIG. 3

VALUE OF $F(n)$ AT $n = 0$

Putting $x = 0$ in (16)

$$U[F(n)]_{n=0} = 4\int_0^\infty R_x dx.$$

By numerical integration of the measured R_x curve in fig. 2 it is found that

$$\int_0^\infty R_x dx = 1\cdot07\,\text{in.} = 0\cdot089\,\text{ft.},$$

so that, when $n = 0$,

$$U[F(n)]_{n=0} = 4 \times 0\cdot089 = 0\cdot35\,\text{ft.}$$

This upper limit is marked in fig. 1.

Proof that turbulence is isotropic

Though the theory and measurements so far discussed do not involve any assumption as to whether or not the turbulence is statistically isotropic, yet since the theory of isotropic turbulence has been discussed so completely it is worth while to describe measurements which prove that the turbulence was in fact isotropic.

It has been shown by Kármán (1937) that when turbulence is isotropic there is a definite relationship between the correlation curves R_x and R_y. Kármán defines two correlation functions R_1 and R_2. R_1 is the correlation between components of velocity along the line AB, where A and B are the points at which the velocities are measured, R_2 is the correlation between components at right angles to AB. In isotropic turbulence R_1 and R_2 are functions of r only where r is the length AB. When correlation measurements are made in a wind tunnel by means of a hot wire, only the component parallel to the length of the tunnel produces any appreciable effect on the hot wire. If therefore A and B are situated on a line parallel to the mean wind stream the correlation R_x is identical with Kármán's R_1. If A and B are situated on a line perpendicular to the stream the correlation R_y is identical with Kármán's R_2.

Kármán's relationship between R_1 and R_2, namely

$$R_2 = R_1 + \tfrac{1}{2}r\frac{dR_1}{dr}, \tag{20}$$

is therefore a relationship between the correlations R_x and R_y which have been measured. The measured values of R_x or R_1 are given in fig. 2, and they are repeated in fig. 4. To this curve the (negative) values of $\frac{1}{2}r(dR_1/dr)$ are added and the calculated values of R_y or R_2 thus obtained are shown in fig. 4. The values of R_y measured by Mr Simmons at 6 ft. 10 in. behind a 3×3 in. grid are also shown in fig. 4. It will be seen that Kármán's relationship (20) is very well verified, and it may fairly be concluded that the turbulence at 6 ft. 10 in. behind a 3×3 in. grid in a wind tunnel is isotropic.

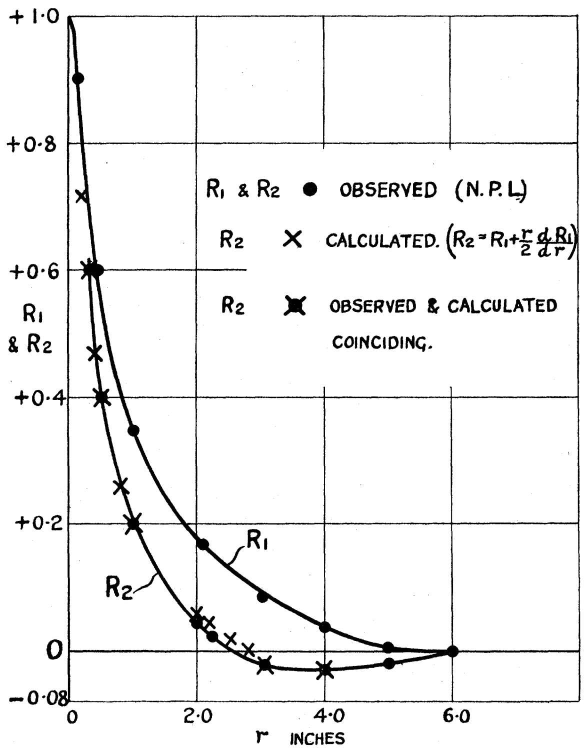

Fig. 4

Calculation of λ from measured rate of dissipation of energy

In the case of isotropic turbulence λ can be found by measuring the rate of dissipation of energy. This can be found by measuring the rate of decay of the mean kinetic energy of turbulent motion.

Fig. 5 shows the measured values of U/u', $(u' = \sqrt{\overline{u^2}})$, in the air stream in which $F(n)$ and R_x were measured. It will be seen that in this case U/u' increases linearly* with x, the distance down stream from the grid. I have shown (Taylor 1935) that U/u' increases linearly with x when λ satisfies the following relationship:

$$\frac{\lambda}{M} = A\sqrt{\frac{\nu}{u'M}}, \tag{21}$$

* This is not a general law. Cases where U/u' does not increase linearly have been observed.

where M is the mesh size of the grid producing the turbulence. Conversely, when U/u' increases linearly with x, λ must be related to u' by the equation (21). By measuring the slope of the line which passes through the observed points in fig. 5, I find that A in (21) is 2·12. At the point where the spectrum measurements were made, 6 ft. 10 in. from the grid, $U/u' = 33{\cdot}5$. Since $M = 3$ in. $= 7{\cdot}62$ cm. and $\nu = 0{\cdot}148$, (21) becomes

$$\lambda = 2{\cdot}12\sqrt{\frac{33{\cdot}5 \times 0{\cdot}148 \times 7{\cdot}62}{U}} \text{ cm.}$$

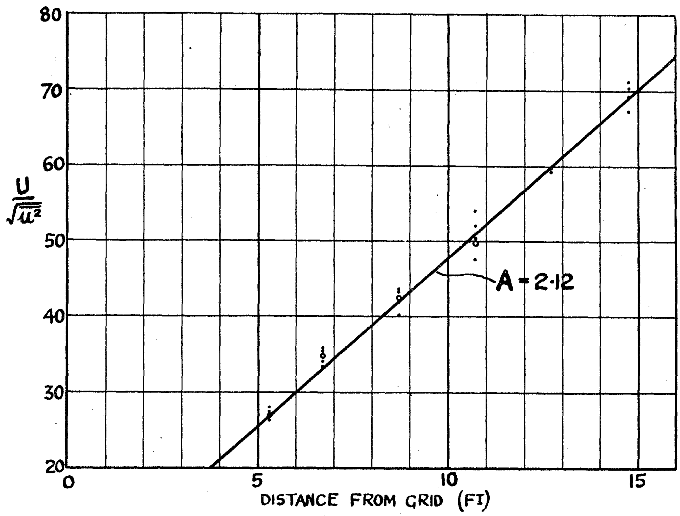

Fig. 5

The values given in column 3, Table II, are calculated from this formula. Comparing the values of λ calculated from the measured dissipation (column 3) with those calculated from the measured spectrum curves using equation (18) (column 2), it will be seen that the agreement is fairly good.

Table II. Values of λ

U ft./sec.	λ calculated from spectrum curves	λ calculated from observed dissipation
15	0·86 cm.	0·61 cm.
20	0·63	0·53
35	0·50	0·40

Correlation measured with band filter circuits

Recently Dryden (1937) has made measurements of R_x using various band filter circuits in his amplifier. The action of the band filter is to cut out all disturbances except those whose frequencies lie between certain limits. By supplying truly sinusoidal disturbances of known frequency and amplitude to the filter circuit the characteristic curve showing the response of the circuit to unit input were obtained. If $\phi(n)$ is the ratio of $\overline{u^2}$ measured with the filter to $\overline{u^2}$ measured without it, the characteristic curve $\{\phi(n), n\}$ for one of Dryden's circuits which passes frequencies between 250 and 500 cycles is shown in fig. 6. The measured values of R_x using this filter circuit and with wind speed $U = 20$ ft./sec. are shown in fig. 7.

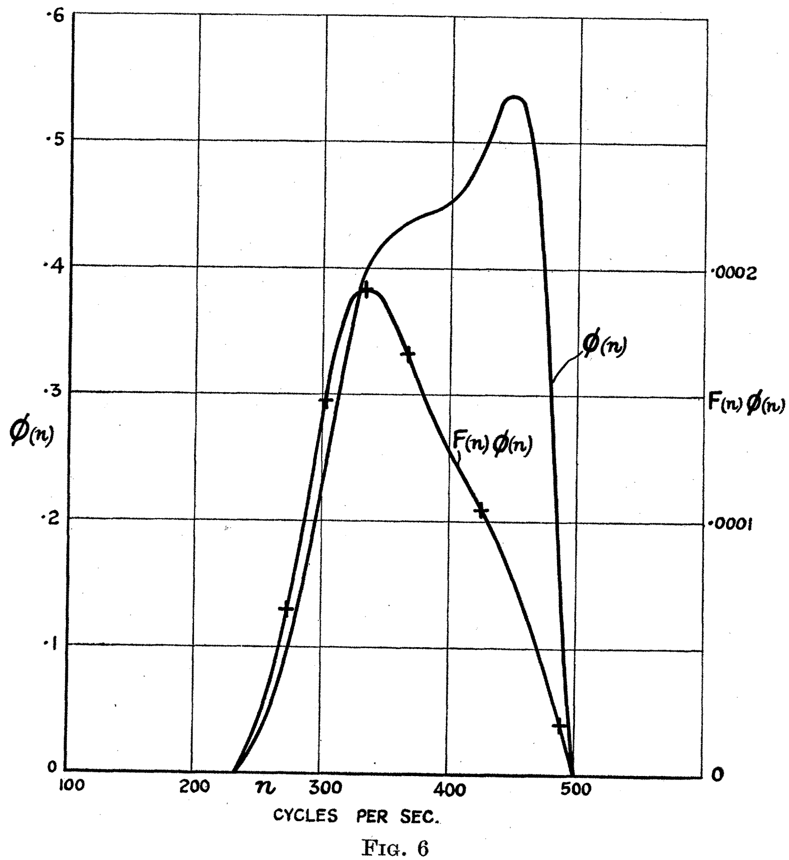

FIG. 6

We have already seen how R_x is related to $F(n)$. It is clear that the same relationship will still hold when the filter circuit is inserted, but $F(n)$ must now be replaced by

$$\frac{F(n)\,\phi(n)}{\int_0^\infty F(n)\,\phi(n)\,dn}.$$

If $R_{x\phi}$ is the value of R_x measured with the band filter circuit ϕ, the formula analogous to (11) is

$$R_{x\phi} = \frac{\int_0^\infty F(n)\,\phi(n)\cos\frac{2\pi nx}{U}\,dn}{\int_0^\infty F(n)\,\phi(n)\,dn}. \tag{22}$$

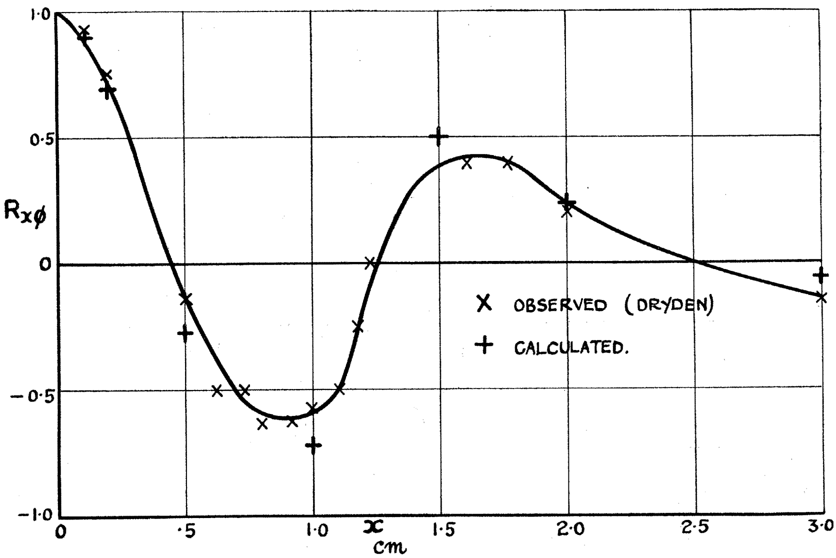

FIG. 7

Before this formula can be used in comparison with Dryden's observed values of $R_{x\phi}$ it is necessary to find $F(n)$. Dryden has not measured $F(n)$ except very roughly, but he has measured R_x in the same air stream and without the filter circuit. His measurements are shown in fig. 8. To calculate $F(n)$ we may use the expression (16) with the observed R_x. The values of $\frac{U}{4}F(n)$ so found is also shown in fig. 8. Taking the values of $F(n)$ from a smooth curve through the calculated points, the values of $F(n)\,\phi(n)$ together can be found for any given values of U and m by multiplying the ordinates of the $F(n)$ and $\phi(n)$ curves. The values of $F(n)\,\phi(n)$ at $U = 20$ ft./sec. ($=610$ cm./sec.) found in this way are shown in fig. 6.

Using a series of values of x, the values of $R_{x\phi}$ have been calculated by numerical integration of (22). The values so calculated are shown in Table III, and are marked in fig. 7. It will be seen that the agreement with Dryden's observations is very good. This agreement provides additional evidence in favour of the main thesis of this paper that R_x and $\frac{U}{2\sqrt{2\pi}}F(n)$ are Fourier transforms of one another, so that each can be predicted when the other has been measured.

TABLE III—CALCULATED VALUES OF $R_{x\phi}$

x (cm.)	0·1	0·2	0·5	1·0	1·5	2·0	3·0
$R_{x\phi}$	+0·91	+0·79	−0·27	−0·72	+0·51	+0·23	−0·06

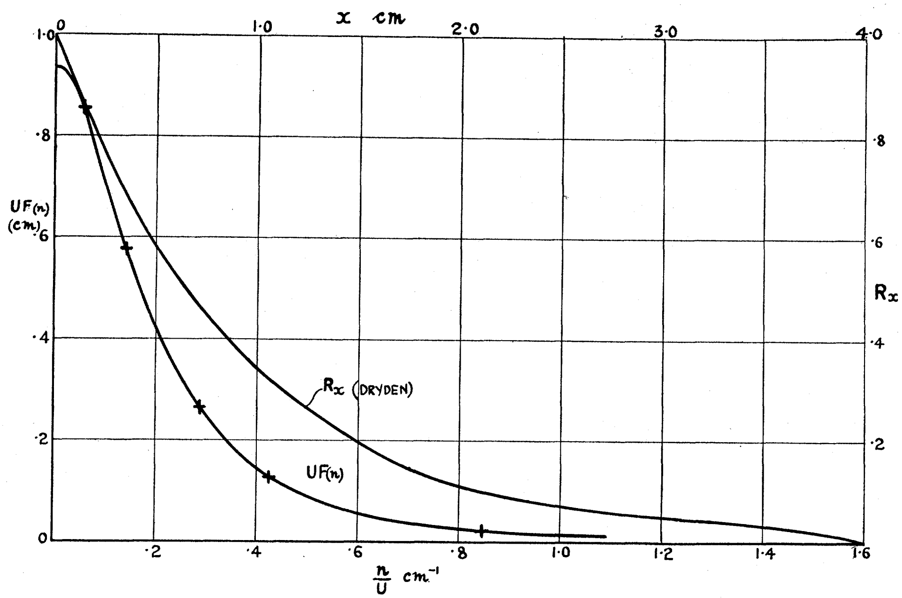

FIG. 8

BEARING OF SPECTRUM MEASUREMENTS ON THEORY OF DISSIPATION

The fact that the $\{UF(n), n/U\}$ curve is independent of U over nearly the whole range indicates that the turbulent flow at a fixed point behind a regular grid is similar, so far as the main features of the flow are concerned, at all speeds. On the other hand the fact that small quantities of very high frequency disturbances appear, and increase as the speed increases, seems to confirm the view frequently put forward by the author that the dissipation of energy is due chiefly to the formation of very small regions where the vorticity is very high. Apart from these very small regions the turbulence behind a grid is similar at all speeds.

Summary

It is shown that a definite connexion exists between the spectrum of the time variation in wind at a fixed point in a wind stream and the curve of correlation between the wind variations at two fixed points. The spectrum curve and the correlation curve are, in fact, Fourier transforms of one another.

As an example of the use of this relationship the spectrum of turbulence in an American wind tunnel was calculated from measurements of correlation by Dryden. In some further experiments Dryden modified this spectrum by inserting a filter circuit and then measured the correlation with this filter in circuit. The modified spectrum is here calculated from the filter characteristics and the Fourier transform theorem is used to calculate the modified correlation curve. The agreement with Dryden's measurements is very good indeed.

The paper ends with some remarks on the bearing of the spectrum measurements on the theory of dissipation of energy in turbulent flow.

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