Day 16 : Stochastic Integrals

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Today's Reading

[L] Section 5.1.

Overview

We extend the definition of stochastic integrals in the order of

$$\text{Martingales bounded in } L^2 \Rightarrow \text{Local Martingales} \Rightarrow \text{Semimartingales}$$

Review: Stochastic Integrals for Martingales bounded in L^2

We defined the following spaces.

$$\mathbb{H}^2 = \{\text{conti martingales } M \text{ bounded in } L^2, M_0 = 0\}/\sim .$$

inner product of $\mathbb{H}^2$: $(M, N)_{\mathbb{H}^2} = \mathbf{E}[\langle M, N\rangle_\infty]$.

$$L^2(M) = \{\text{progressive processes } H \text{ s.t. } \mathbf{E}\left[\int_0^\infty H_s^2 d\langle M, M\rangle_s\right] < \infty.\}$$

inner product of $L^2(M)$: $(H, K)_{L^2(M)} = \mathbf{E}[\int_0^\infty H_s K_s d\langle M, M\rangle_s]$.

Review: Stochastic Integrals for Martingales bounded in L^2

For an elementary process $H \in \mathcal{E}$ of the form

$$H_s(\omega) = \sum_{i=1}^{p-1} H_{(i)}(\omega) \mathbb{1}_{(t_i, t_{i+1}]}(s)$$

define

$$(H \cdot M)_t = \sum_{i=0}^{p-1} H_{(i)}(M_{t_{i+1} \wedge t} - M_{t_i \wedge t}).$$

Then $H \cdot M \in \mathbb{H}^2$. This mapping extends to an isometry $L^2(M) \to \mathbb{H}^2$ using the density of $\mathcal{E}$ in $L^2(M)$.

We use the notation

$$(H \cdot M)_t = \int_0^t H_s dM_s.$$

Review: Stochastic Integrals for Martingales bounded in L^2

Properties:

- $H \cdot M$ is the unique martingale in $\mathbb{H}^2$ such that

$$\langle H \cdot M, N \rangle = H \cdot \langle M, N \rangle \quad \forall N \in \mathbb{H}^2.$$

- If T is a stopping time,

$$(\mathbb{1}_{[0,T]} H) \cdot M = (H \cdot M)^T = H \cdot M^T.$$

- If $M, N \in \mathbb{H}^2$, $H \in L^2(M)$, $K \in L^2(N)$, then

$$\langle \int_0^{\bullet} H_s dM_s, \int_0^{\bullet} K_s dN_s \rangle_t = \int_0^t H_s K_s d\langle M, N \rangle_s.$$

Review: Stochastic Integrals for Martingales bounded in L^2

Properties (cont'd):

- If $H \in L^2(M)$ and K is a progressive process, then

  $$KH \in L^2(M) \Leftrightarrow K \in L^2(H \cdot M).$$

  If the latter properties hold,

  $$(KH) \cdot M = K \cdot (H \cdot M).$$

- If $M, N \in \mathbb{H}^2$, $H \in L^2(M)$, $K \in L^2(N)$, then

  $$\mathbf{E}\left[\int_0^t H_s dM_s\right] = 0$$

  $$\mathbf{E}\left[\left(\int_0^t H_s dM_s\right)\left(\int_0^t K_s dN_s\right)\right] = \mathbf{E}\left[\int_0^t H_s K_s d\langle M, N\rangle_s\right].$$

Stochastic Integrals for Continuous Local Martingales

$$L^2_{\text{loc}}(M) = \{\text{progressive processes } H \text{ s.t.} \int_0^t H_s^2 d\langle M, M\rangle_s < \infty \ \forall t \geq 0, \text{ a.s.}\}$$

Theorem

Let M be a continuous local martingale. For every $H \in L^2_{\text{loc}}(M)$, there exists a unique continuous local martingale with initial value 0, denoted by $H \cdot M$, such that

$$\langle H \cdot M, N\rangle = H \cdot \langle M, N\rangle \quad \forall N : \text{continuous local martingale.}$$

Construction. For each $n \geq 1$, set

$$T_n = \inf\{t \geq 0 : \int_0^t (1 + H_s^2) d\langle M, M\rangle_s \geq n\}.$$

Then $M^{T_n} \in \mathbb{H}^2$ and $H \in L^2(M^{T_n})$.

Stochastic Integrals for Continuous Local Martingales

Therefore, $H \cdot M^{T_n}$ makes sense. Moreover, for any $m > n$,

$$H \cdot M^{T_n} = (H \cdot M^{T_m})^{T_n}.$$

Since $T_n \to +\infty$, it follows that there exists a unique process $H \cdot M$ satisfying

$$(H \cdot M)^{T_n} = H \cdot M^{T_n}.$$

Moreover, $H \cdot M$ has continuous sample paths and is adapted since $(H \cdot M)_t = \lim_{n\to\infty}(H \cdot M^{T_n})_t$. Finally, $H \cdot M^{T_n} \in \mathbb{H}^2$, so $H \cdot M$ is a continuous local martingale.

The Wiener Integral revisited

Note that pre-Brownian motion is defined as $B_t = G(\mathbb{1}_{[0,t]})$. Conversely, for any given pre-Brownian motion (B_t), the associated Gaussian white noise is determined fully: for any step function $f = \sum_{i=1}^{n} \lambda_i \mathbb{1}_{(t_{i-1},t_i]}$, where $0 = t_0 < t_1 < \cdots < t_n$,

$$G(f) = \sum_{i=1}^{n} \lambda_i (B_{t_i} - B_{t_{i-1}}).$$

We write for $f \in L^2(\mathbb{R}^+, \mathcal{B}(\mathbb{R}^+), dt)$,

$$G(f) = \int_0^\infty f(s) dB_s.$$

By viewing f as a deterministic progressive process, the Wiener integral coincides with the stochastic integral we defined.

Stochastic Integrals for Continuous Local Martingales

Properties:

- If T is a stopping time,

$$(\mathbb{1}_{[0,T]}H) \cdot M = (H \cdot M)^T = H \cdot M^T.$$

- If $H \in L^2_{\mathrm{loc}}(M)$ and K is a progressive process, then

$$KH \in L^2_{\mathrm{loc}}(M) \Leftrightarrow K \in L^2_{\mathrm{loc}}(H \cdot M).$$

  If the latter properties hold,

$$(KH) \cdot M = K \cdot (H \cdot M).$$

Review : Continuous Semimartingales

A process $X = (X_t)$, $t \geq 0$, is a *continuous semimartingale* if there is a decomposition

$$X_t = M_t + A_t$$

so that M is a continuous local martingale and A is a finite variation process.

Such decomposition is unique.

For two $X = M + A$, $Y = M' + A'$ continuous semimartingales with canonical decompositions, the *bracket* is the finite variation process defined by

$$\langle X, Y \rangle_t = \langle M, M' \rangle_t.$$

Stochastic Integrals for Semimartingales

A progressive process H is *locally bounded* if

$$\forall t \geq 0, \quad \sup_{s \leq t} |H_s| < \infty, \quad \text{a.s.}$$

Note that if H is progressive and locally bounded, then for any finite variation process V, we have

$$\forall t \geq 0, \quad \int_0^t |H_s||dV_s| < \infty, \quad \text{a.s.}$$

and similarly $H \in L^2_{\text{loc}}(M)$ for every continuous local martingale M.

Hence, we may define $H \cdot X$ for a continuous semimartingale X with canonical decomposition $X = M + V$ by

$$H \cdot X = H \cdot M + H \cdot V.$$

We write $(H \cdot X)_t = \int_0^t H_s dX_s$.