Day 11 : Review

Dain Kim

January 12th

Gaussian Random Variables

Gaussian variable $X \sim N(\mu, \sigma)$ has density

$$p_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-\mu)^2}{2}\right).$$

For $X \sim N(0,1)$, we have

$$\mathbf{E}[X^{2n}] = \frac{(2n)!}{2^n n!}, \quad \mathbf{E}[X^{2n+1}] = 0.$$

In particular,

$$\mathbf{E}[X] = 0, \quad \mathbf{E}[X^2] = 1, \quad \mathbf{E}[X^3] = 0, \quad \mathbf{E}[X^4] = 3.$$

Gaussian Processes and Gaussian Spaces

A *(centered) Gaussian space* $L^2(\Omega, \mathcal{F}, \mathbf{P})$ is a closed linear subspace consisting of centered Gaussian variables.

A real-valued stochastic process (X_t), $t \in \Sigma$, is a *Gaussian process* if any finite linear combination of X_t is a centered Gaussian.

Gaussian White Noise

Let $(E, \mathcal{E})$ be a measurable space and μ a σ-finite measure on $(E, \mathcal{E})$. A *Gaussian white noise* with intensity μ is an isometry

$$L^2(E, \mathcal{E}, \mu) \to \text{a centered Gaussian space.}$$

That is, for $f \in L^2(E, \mathcal{E}, \mu)$,

$$\mathbf{E}[G(f)^2] = ||G(f)||^2_{L^2(\Omega, \mathcal{F}, \mathbf{P})} = ||f||^2_{L^2(E, \mathcal{E}, \mu)} = \int f^2 d\mu,$$

and for $f, g \in L^2(E, \mathcal{E}, \mu)$,

$$\mathbf{E}[G(f)G(g)] = \langle f, g \rangle_{L^2(E, \mathcal{E}, \mu)} = \int fg d\mu.$$

Pre-Brownian Motion

Let G be a Gaussian White Noise on $\mathbb{R}^+$ whose intensity is Lebesgue measure. The stochastic process (B_t), $t \in \mathbb{R}^+$ defined as

$$B_t = G(\mathbb{1}_{[0,t]})$$

is *pre-Brownian motion*.

Its covariance is

$$\mathbf{E}[B_s B_t] = \min(s, t) = s \wedge t.$$

Characterizations of pre-Brownian motions

Proposition

Let $X(t)$, $t \geq 0$, be a real-valued stochastic process. The followings are equivalent:

1) $X(t)$, $t \geq 0$, is a pre-Brownian motion
2) $X(t)$, $t \geq 0$, is a centered Gaussian process with covariance $K(s,t) = \min(s,t)$
3) $X(0) = 0$ a.s., and for every $0 \leq s < t$, the random variable $X(t) - X(s)$ is independent of $\sigma(X(r), r \leq s)$ and distributed according to $N(0, t-s)$.
4) $X(0) = 0$ a.s., and for every $0 = t_0 < t_1 < \ldots < t_p$, the variables $X_{t_i} - X_{t_{i-1}}$, $1 \leq i \leq p$, are independent, and are distributed according to $N(0, t_i - t_{i-1})$.

Pre-Brownian Motion

1) $-B_t$ is also a Brownian motion
2) For every $\lambda > 0$, $B_t^\lambda = \dfrac{1}{\lambda} B_{\lambda^2 t}$ is a Brownian motion
3) For every $s \geq 0$, $B_t^{(s)} = B_{s+t} - B_s$ is a Brownian motion and is independent of $\sigma(B_r, r \leq s)$.

Let X_t, $t \in \Sigma$, be a stochastic process with values in E. The *sample paths* are the mappings $\Sigma \ni t \mapsto X_t(\omega)$ obtained when $\omega \in \Omega$ is fixed.

The sample paths of pre-Brownian Motions are not necessarily continuous.

Kolmogorov Continuity Criterion

Theorem (Kolmogorov Continuity Criterion)

For a process $X(t)$, $t \in [a, b]$, assume that there exist positive constants q, ϵ, and C such that

$$\mathbf{E}|X(t) - X(s)|^q \leq C|t - s|^{1+\epsilon} \quad \text{for all } s, t \in [a, b].$$

Then the process X has a continuous modification $\widetilde{X}$. Moreover, there is a modification whose sample paths are α-Hőlder continuous for $\alpha \in (0, \epsilon/q)$, that is, for each sample path ω, there exists a constant $C_\alpha(\omega)$ such that

$$|\tilde{X}_t(\omega), \tilde{X}_s(\omega)| \leq C_\alpha(\omega)|t - s|^\alpha.$$

Modifications of pre-Brownian Motions

If (B_t), $t \geq 0$, is a pre-Brownian motion, then (B_t) satisfies the Kolmogorov Continuity Criterion for $q > 2$ and $\epsilon = \frac{q}{2} - 1$.

Let $X \sim N(0,1)$, then $B_t - B_s = \sqrt{t-s}X$ for any $s < t$. Therefore,

$$\mathbf{E}[|B_t - B_s|^q] = (t-s)^{q/2}\mathbf{E}[|X|^q] < C_q(t-s)^{q/2}.$$

Hence, each pre-Brownian motion has a modification whose sample paths are continuous.

A stochastic process (B_t), $t \geq 0$, is a *Brownian motion* if

1) (B_t), $t \geq 0$, is a pre-Brownian motion.
2) All sample paths of B are continuous.

The Wiener measure

Let $C(\mathbb{R}^+, \mathbb{R})$ be the space of all continuous functions $f : \mathbb{R}^+ \to \mathbb{R}$. For a given Brownian motion (B_t), $t \geq 0$, we may consider the following map:

$$\begin{aligned} \Omega &\to C(\mathbb{R}^+, \mathbb{R}) \\ \omega &\mapsto (t \mapsto B_t(\omega)). \end{aligned}$$

Let $\mathcal{C}$ be the smallest σ-field on $C(\mathbb{R}^+, \mathbb{R})$ for which the coordinate mappings $w \mapsto w(t)$ are measurable for every $t \geq 0$. The *Wiener measure* $W(d\omega)$ is defined as the image of the probability measure $\mathbf{P}(d\omega)$.

Now we make a special choice

$$\Omega = C(\mathbb{R}^+, \mathbb{R}), \quad \mathcal{F} = \mathcal{C}, \quad \mathbf{P}(dw) = W(dw),$$

then the *canonical process* $X_t(\omega) = \omega(t)$ is a Brownian motion. This is a *canonical construction* of Brownian motion.

Strong Markov Property of Brownian Motion and Reflection Property

Theorem (Strong Markov Property)

Let T be a stopping time such that $\mathbf{P}(T < \infty) > 0$. Set, for $t \geq 0$,

$$B_t^{(T)} = \mathbb{1}_{\{T<\infty\}}(B_{T+t} - B_T).$$

Then under the probability measure $\mathbf{P}(\cdot | T < \infty)$, the process $(B_t^{(T)})$, $t \geq 0$, is a Brownian motion independent of $\mathcal{F}_T$.

Theorem

For every $t > 0$, let $S_t = \sup_{s \leq t} B_s$. Then if $a \geq 0$ and $b \in (-\infty, a]$, we have

$$\mathbf{P}(S_t \geq a, B_t \leq b) = \mathbf{P}(B_t \geq 2a - b).$$

In particular, S_t has the same distribution as $|B_t|$.

The Wiener Integral

Note that pre-Brownian motion is defined as $B_t = G(\mathbb{1}_{[0,t]})$.
Conversely, for any given pre-Brownian motion (B_t), the associated Gaussian white noise is determined fully: for any step function $f = \sum_{i=1}^{n} \lambda_i \mathbb{1}_{(t_{i-1},t_i]}$, where $0 = t_0 < t_1 < \cdots < t_n$,

$$G(f) = \sum_{i=1}^{n} \lambda_i (B_{t_i} - B_{t_{i-1}}).$$

We write for $f \in L^2(\mathbb{R}^+, \mathcal{B}(\mathbb{R}^+), dt)$,

$$G(f) = \int_0^\infty f(s) dB_s.$$

Similarly,

$$G(f\mathbb{1}_{[0,t]}) = \int_0^t f(s) dB_s, \quad G(f\mathbb{1}_{(s,t]}) = \int_s^t f(r) dB_r.$$

This integration is called *the Wiener integral*.

Martingales

An adapted real-valued process (X_t), $t \geq 0$, such that $X_t \in L^1$ for every $t \geq 0$ is called

- a *martingale* if $\mathbf{E}[X_t|\mathcal{F}_s] = X_s$ for every $0 \leq s < t$.
- a *supermartingale* if $\mathbf{E}[X_t|\mathcal{F}_s] \leq X_s$ for every $0 \leq s < t$.
- a *submartingale* if $\mathbf{E}[X_t|\mathcal{F}_s] \geq X_s$ for every $0 \leq s < t$.

Continuous Local Martingales

An adapted process $M = (M_t)$, $t \geq 0$, with continuous sample paths and $M(0) = 0$ a.s. is called a *continuous local martingale* if there exists a nondecreasing sequence $(T_n)_{n \geq 0}$ of stopping times such that

1) $T_n \nearrow \infty$, i.e., $T_n(\omega) \nearrow \infty$ for every $\omega \in \Omega$
2) for every n, the stopped process M^{T_n} is a uniformly integrable martingale.

We call T_n *reduces* M if $T_n \nearrow \infty$ and M^{T_n} is a uniformly integrable martingale for every n.

Properties of Continuous Local Martingales

Proposition

1) If M is a nonnegative continuous local martingale such that $M_0 \in L^1$, then M is a supermartingale.
2) If M is a continuous local martingale and there exists a random variable $Z \in L^1$ such that $|M_t| \leq Z$ for every $t \geq 0$, then M is a uniformly integrable martingale.
3) If M is a continuous local martingale and $M_0 \in L^1$, the sequence of stopping times

$$T_n = \inf\{t \geq 0 : |M_t| \geq n\}$$

reduces M.

The Quadratic Variation of Continuous Local Martingales

We assume that $(\mathcal{F}_t)$ is a complete filtration. That is, for every $A \subset \Omega$ such that there exists $A \subset B \subset \Omega$ with $\mathbf{P}(B) = 0$, $A \in \mathcal{F}_t$ for all t.

Theorem

Let $M = (M_t)$, $t \geq 0$, be a continuous local martingale. There exists an increasing process $(\langle M, M \rangle_t$, $t \geq 0$, which is unique up to indistinguishability, such that $M_t^2 - \langle M, M \rangle_t$ is a continuous local martingale. Moreover,

$$\langle M, M \rangle_t = \lim_{\sup_i |t_{i+1} - t_i| \to 0} \sum_{i=1}^{n-1} (M_{t_i^n} - M_{t_{i-1}^n})^2$$

in probability. The process $\langle M, M \rangle$ is called the *quadratic variation* of M.

Examples. $\langle B, B \rangle_t = t$.

The Bracket of Two Continuous Local Martingales

Let M and N be two continuous local martingales. The *bracket* $\langle M, N \rangle$ is the finite variation process defined by

$$\langle M, N \rangle_t = \frac{1}{2} \left(\langle M + N, M + N \rangle_t - \langle M, M \rangle_t - \langle N, N \rangle_t \right)$$

for every $t \geq 0$.

The bracket $\langle M, N \rangle$ is the unique finite variation process up to indistinguishability such that $M_t N_t - \langle M, N \rangle_t$ is a continuous local martingale.

Examples. $\langle B, B' \rangle_t = 0$.

Exercises

Problem 1. [Exercise 2.25]
1. Show that the process (W_t), $t \geq 0$, defined by $W_0 = 0$ and $W_t = tB_{1/t}$ for $t > 0$ is (indistinguishable of) a real Brownian motion started from 0.
2. Infer that $\lim_{t\to\infty} \dfrac{B_t}{t} = 0$.

Problem 2. [Exercise 2.29]
Show that

$$\limsup_{t \searrow 0} \frac{B_t}{\sqrt{t}} = +\infty, \quad \limsup_{t \searrow 0} \frac{B_t}{\sqrt{t}} = -\infty.$$

Deduce that, for every $s \geq 0$, the function $t \mapsto B_t$ has a.s. no right derivative at s.