Itô's Formula

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Today's Reading

[L] Section 5.1.4. Convergence of Stochastic Integrals
Section 5.2. Itô's Formula

Review : Continuous Semimartingales

A process $X = (X_t)$, $t \geq 0$, is a *continuous semimartingale* if there is a decomposition

$$X_t = M_t + A_t$$

so that M is a continuous local martingale and A is a finite variation process.

Such decomposition is unique and called the *canonical decomposition.*

For two $X = M + A$, $Y = M' + A'$ continuous semimartingales with canonical decompositions, the *bracket* is the finite variation process defined by

$$\langle X, Y \rangle_t = \langle M, M' \rangle_t.$$

Review : Stochastic Integrals for Semimartingales

A progressive process H is *locally bounded* if

$$\forall t \geq 0, \quad \sup_{s \leq t} |H_s| < \infty, \quad \text{a.s.}$$

Note that if H is progressive and locally bounded, then for any finite variation process V, we have

$$\forall t \geq 0, \quad \int_0^t |H_s||dV_s| < \infty, \quad \text{a.s.}$$

and similarly $H \in L^2_{\text{loc}}(M)$ for every continuous local martingale M.

Hence, we may define $H \cdot X$ for a continuous semimartingale X with canonical decomposition $X = M + V$ by

$$H \cdot X = H \cdot M + H \cdot V.$$

We write $(H \cdot X)_t = \int_0^t H_s dX_s$.

Recall : Lebesgue Dominated Convergence Theorem

Theorem

Given a measure μ on $\mathbb{R}$, for a sequence of measurable function $f_n \colon \mathbb{R} \to [-\infty, \infty]$ with pointwise limit $f(x) = \lim_{n\to\infty} f_n(x)$, if there exists an integrable function g, i.e., $\int g d\mu < +\infty$, such that $|f_n(x)| \leq g(x)$ for all n and x, then f_n and f are integrable and

$$\lim_{n\to\infty} \int_{\mathbb{R}} f_n d\mu = \int_{\mathbb{R}} \lim_{n\to\infty} f_n d\mu = \int_{\mathbb{R}} f d\mu.$$

Dominated Convergence Theorem for Stochastic Integrals

Proposition

Given a semimartingale X, let $X = M + V$ be the canonical decomposition of X. Let $t > 0$. Let $(H^n)_{n\geq 1}$ and H be locally bounded progressive processes and K a nonnegative progressive process. Assume the following properties hold a.s.:

(i) $H_n^2 \to H_s$ as $n \to \infty$, for every $s \in [0, t]$,

(ii) $|H_s^n| \leq K_s$, for every $n \geq 1$ and $s \in [0, t]$,

(iii) $\int_0^t (K_s)^2 d\langle M, M\rangle_s < \infty$ and $\int_0^t K_s|dV_s| < \infty$.

Then

$$\int_0^t H_s^n dX_s \xrightarrow[n\to\infty]{} \int_0^t H_s dX_s$$

in probability.

Remark. The assumption (iii) automatically holds if K is locally bounded.

An Approximation of Continuous Integrands

Proposition

Let X be a continuous semimartingale and H an adapted process with continuous sample paths. Then for every $t > 0$ and subdivisions $0 = t_0^n < \cdots < t_{p_n}^n = t$,

$$\lim_{\sup_i |t_{i+1} - t_i| \to 0} \sum_{i=0}^{p_n - 1} H_{t_{i+1}^n} (X_{t_{i+1}^n} - X_{t_i^n}) = \int_0^t H_s dX_s.$$

Proof Sketch. Define a process H^n by

$$H_s^n = \begin{cases} H_{t_i^n} & \text{if } s \in (t_i^n, t_{i+1}^n] \\ H_0 & \text{if } s = 0 \\ 0 & \text{if } s > t \end{cases}$$

and $K_s = \max_{0 \le r \le s} |H_s|$. Then apply the Dominated Convergence Theorem. □

An Approximation of Continuous Integrands

What if we take the right end of the interval $(t_i^n, t_{i+1}^n]$? Consider a special case $H_t = X_t$, then by the proposition,

$$\lim_{\sup_i |t_{i+1} - t_i| \to 0} \sum_{i=0}^{p_n - 1} X_{t_i^n}(X_{t_{i+1}^n} - X_{t_i^n}) = \int_0^t X_s dX_s \tag{1}$$

in probability. On the other hand, note that

$$\sum_{i=0}^{p_n - 1} X_{t_{i+1}^n}(X_{t_{i+1}^n} - X_{t_i^n}) = \sum_{i=0}^{p_n - 1} X_{t_i^n}(X_{t_{i+1}^n} - X_{t_i^n}) + \sum_{i=0}^{p_n - 1} (X_{t_{i+1}^n} - X_{t_i^n})^2,$$

which gives

$$\lim_{\sup_i |t_{i+1} - t_i| \to 0} \sum_{i=0}^{p_n - 1} X_{t_{i+1}^n}(X_{t_{i+1}^n} - X_{t_i^n}) = \int_0^t X_s dX_s + \langle X, X \rangle_t \tag{2}$$

in probability.

An Approximation of Continuous Integrands

Adding (1) and (2), we get

$$(X_t)^2 - (X_0)^2 = 2\int_0^t X_s dX_s + \langle X, X \rangle_t.$$

This is a special case of Itô's formula.

Itô's Formula

Theorem (1-dimensional Itô's formula)

Let X be a continuous semimartingale and $F\colon \mathbb{R} \to \mathbb{R}$ a twice continuously differentiable real function. Then for every $t \geq 0$,

$$F(X_t) = F(X_0) + \int_0^t F'(X_s)dX_s + \frac{1}{2}\int_0^t F''(X_s)d\langle X, X\rangle_s.$$

Theorem (p-dimensional Itô's formula)

Let $X^1, \dots, X^p$ be p continuous semimartingales and $F\colon \mathbb{R}^p \to \mathbb{R}$ a twice continuously differentiable real function. Then for every $t \geq 0$,

$$\begin{aligned} F(X_t^1, \dots, X_t^p) = F(X_0^1, \dots, X_0^p) &+ \sum_{i=1}^{p} \int_0^t \frac{\partial F}{\partial x^i}(X_s^1, \dots, X_s^p)dX_s^i \\ &+ \frac{1}{2}\sum_{i,j=1}^{p} \int_0^t \frac{\partial^2 F}{\partial x^i \partial x^j}(X_s^1, \dots, X_s^p)d\langle X^i, X^j\rangle_s. \end{aligned}$$

Special Cases of Itô's formula

- Taking $p = 2$ and $F(x, y) = xy$, we get

$$X_t Y_t = X_0 Y_0 + \int_0^t X_s dY_s + \int_0^t Y_s dX_s + \langle X, Y \rangle_t.$$

This is often called the *formula of integration by parts*.

- If $X = Y$ in the above example, we get

$$X_t^2 = X_0^2 + 2 \int_0^t X_s dX_s + \langle X, X \rangle_t.$$

Special Cases of Itô's formula - Brownian motions

Let B be an $(\mathcal{F}_t)$-real Brownian motion.

- Since $\langle B, B\rangle_t = t$, Itô's formula gives

$$F(B_t) = F(B_0) + \int_0^t F'(B_s)dB_s + \frac{1}{2}F''(B_s)ds.$$

- Let $p = 2$ and $X_t^1 = t, X_t^2 = B_t$. For every twice continuously differentiable function $F(t,x)\colon \mathbb{R}_{\geq 0} \times \mathbb{R} \to \mathbb{R}$, Itô's formula gives

$$F(t, B_t) = F(0, B_0) + \int_0^t \frac{\partial F}{\partial x}(s, B_s)dB_s + \int_0^t \left(\frac{\partial F}{\partial t} + \frac{1}{2}\frac{\partial^2 F}{\partial x^2}\right)(s, B_s)ds.$$

Special Cases of Itô's formula - Brownian motions

Let $B_t = (B_t^1, \ldots, B_t^d)$ be a d-dimensional $(\mathcal{F}_t)$-Brownian motion. Since $\langle B^i, B_j \rangle_t = 0$ if $i \neq j$, Itô's formula gives

$$F(B_t^1, \ldots, B_t^d) = F(B_0^1, \ldots, B_d^d) + \sum_{i=1}^{d} \int_0^t \frac{\partial F}{\partial x_i}(B_s^1, \ldots, B_s^d) dB_s^i$$
$$+ \frac{1}{2} \int_0^t \Delta F(B_s^1, \ldots, B_s^d) ds.$$

The formula is often written as

$$F(B_t) = F(B_0) + \int_0^t \nabla F(B_s) \cdot dB_s + \frac{1}{2} \int_0^t \Delta F(B_s) ds.$$