Day 14 : Stochastic Integrals for Martingales 2

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Today's Reading

[L] Section 5.1.

Review : Stochastic Integrals for Martingales

For any $M \in \mathbb{H}^2$ and $H \in \mathcal{E}$ of the form

$$H_s(\omega) = \sum_{i=0}^{p-1} H_{(i)}(\omega) \mathbb{1}_{(t_i, t_{i+1}]}(s),$$

the process $H \cdot M$ defined as

$$(H \cdot M)_t = \sum_{i=0}^{p-1} H_{(i)}(M_{t_{i+1} \wedge t} - M_{t_i \wedge t})$$

belongs to $\mathbb{H}^2$.

The mapping $H \mapsto H \cdot M$ extends to an isometry from $L^2(M)$ to $\mathbb{H}^2$. We write

$$\int_0^t H_s dM_s = (H \cdot M)_t.$$

A Characterization of Stochastic Integrals

Proposition

The process $H \cdot M$ is the unique martingale of $\mathbb{H}^2$ such that for any $N \in \mathbb{H}^2$,

$$\langle H \cdot M, N \rangle = H \cdot \langle M, N \rangle.$$

Proof sketch. It is known that

$$\mathbf{E}\left[\int_0^\infty |H_s||d\langle M, N\rangle_s\right] \leq ||H||_{L^2(M)}||N||_{\mathbb{H}^2},$$

which gives that $\int_0^\infty H_s d\langle M, N\rangle_s = (H \cdot \langle M, N\rangle)_\infty$ is well-defined. Assume that H is an elementary process of the form

$$H = \sum_{i=0}^{p-1} H_{(i)} \mathbb{1}_{(t_i, t_{i+1}]}.$$

A Characterization of Stochastic Integrals

Define $M^i_t = H_{(i)}(M_{t_{i+1}\wedge t} - M_{t_i \wedge t})$. Then

$$\begin{aligned}
\langle H \cdot M, N \rangle_t &= \sum_{i=0}^{p-1} \langle M^i_t, N \rangle \\
&= \sum_{i=0}^{p-1} H_{(i)} \left(\langle M, N \rangle_{t_{i+1}\wedge t} - \langle M, N \rangle_{t_i \wedge t} \right) \\
&= \int_0^t H_s d\langle M, N \rangle_s \\
&= (H, \langle M, N \rangle).
\end{aligned}$$

Then we may show the equality for general $H \in L^2(M)$ using that $\mathcal{E}$ is dense in $L^2(M)$.

A Characterization of Stochastic Integrals

By the proposition, in particular we have

$$\langle H \cdot M, K \cdot N \rangle = H \cdot (K \cdot \langle M, N \rangle) = HK \cdot \langle M, N \rangle.$$

This gives that

$$\langle \int_0^\bullet H_s dM_s, \int_0^\bullet K_s dN_s \rangle_t = \int_0^t H_s K_s d\langle M, N \rangle_s.$$

Associativity of Stochastic Integrals

Proposition

Let $H \in L^2(M)$ and K a progressive process. Then $KH \in L^2(M)$ if and only if $K \in L^2(H \cdot M)$. If the latter holds, then

$$(KH) \cdot M = K \cdot (H \cdot M).$$

Proof. The first assertion follows from

$$\mathbf{E}\left[\int_0^\infty K_s^2 H_s^2 d\langle M, M\rangle_s\right] = \mathbf{E}\left[K_s^2 d\langle H \cdot M, H \cdot M\rangle_s\right].$$

For any $N \in \mathbb{H}^2$,

$$\langle (KH) \cdot M, N\rangle = KH \cdot \langle M, N\rangle = K \cdot (H \cdot \langle M, N\rangle) = K \cdot \langle H \cdot M, N\rangle.$$

By the uniqueness, we conclude that $(KH) \cdot M = K \cdot (H \cdot M)$.

Moments of Stochastic Integrals

Suppose that $M, N \in \mathbb{H}^2$ and $H \in L^2(M)$, $K \in L^2(N)$. Since $H \cdot M$ and $K \cdot N$ are martingales, by the proposition, we have

$$\mathbf{E}\left[\int_0^t H_s dM_s\right] = 0,$$

$$\mathbf{E}\left[\left(\int_0^t H_s dM_s\right)\left(\int_0^t K_s dN_s\right)\right] = \mathbf{E}\left[\int_0^t H_s K_s d\langle M, N\rangle_s\right].$$

Next : Stochastic Integrals for Local Martingales

Recall that if M is a martingale with continuous paths, we defined

$$L^2(M) = \left\{ \text{progressive process } H \text{ s.t. } \mathbf{E}\left[\int_0^\infty H_s^2 d\langle M, M\rangle_s\right] < \infty \right\} / \sim$$

where $H \sim H'$ if $H = H' = 0$, $d\langle M, M\rangle_s$ a.e., a.s.

If M is a continuous local martingale, we define

$$L^2_{\text{loc}}(M) = \left\{ \text{progressive process } H \text{ s.t. } \mathbf{E}\left[\int_0^t H_s^2 d\langle M, M\rangle_s\right] < \infty \text{ a.s.} \right\} / \sim .$$

Then we may define the stochastic integrals for continuous local martingales similarly.