Day 8 : The Strong Markov Property of Brownian Motions

Dain Kim

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Today's Reading

[L] Chapter 2.4

Review: Stopping Time

A *stopping time* with respect to a filtration $\{\mathcal{F}_t, t \in [0, \infty)\}$ is a mapping $T : \Omega \to [0, \infty]$ such that $\{T \le t\} \in \mathcal{F}_t$ for every $t \in [0, \infty)$.

Review: Basic Propoerties of Brownian Motions

Proposition

1) $-B_t$ is also a Brownian motion
2) For every $\lambda > 0$, $B_t^\lambda = \dfrac{1}{\lambda} B_{\lambda^2 t}$ is a Brownian motion
3) For every $s \geq 0$, $B_t^{(s)} = B_{s+t} - B_s$ is a Brownian motion and is independent of $\sigma(B_r, r \leq s)$.

The σ-algebras of stopping times

Let $(B_t)_{t\geq 0}$ be a Brownian motion. We use the notation

$$\mathcal{F}_t = \sigma\{B_s : s \leq t\}, \quad \mathcal{F}_\infty = \sigma\{B_s : s \geq 0\}.$$

Let T be a stopping time. The *σ-algebra of the past before T* is

$$\mathcal{F}_T = \{A \in \mathcal{F}_\infty : \forall t \geq 0, A \cap \{T \leq t\} \in \mathcal{F}_t\}.$$

Define the real random variable $\mathbb{1}_{\{T<\infty\}} B_T(\omega)$ by

$$\mathbb{1}_{\{T<\infty\}} B_T(\omega) = \begin{cases} B_{T(\omega)}(\omega), & T(\omega) < \infty \\ 0, & T(\omega) = \infty \end{cases}.$$

The σ-algebras of stopping times

The random variable $\mathbb{1}_{\{T<\infty\}}B_T(\omega)$ is $\mathcal{F}_T$-measurable:

$$\begin{aligned}\mathbb{1}_{\{T<\infty\}}B_T &= \lim_{n\to\infty}\sum_{i=0}^{\infty}\mathbb{1}_{\{i2^{-n}\le T\le (i+1)2^{-n}\}}B_{i2^{-n}} \\ &= \lim_{n\to\infty}\sum_{i=0}^{\infty}\mathbb{1}_{\{T\le (i+1)2^{-n}\}}\mathbb{1}_{\{i2^{-n}\le T\}}B_{i2^{-n}}\end{aligned}$$

and for any interval A not containing 0,

$$\{B_s\mathbb{1}_{\{s\le T\}}\in A\}\cap\{T\le t\} = \begin{cases}\emptyset, & t<s \\ \{B_s\in A\}\cap\{s\le T\le t\}, & t\ge s.\end{cases}$$

Strong Markov Property

Theorem (Strong Markov Property)

Let T be a stopping time such that $\mathbf{P}(T < \infty) > 0$. Set, for $t \geq 0$,

$$B_t^{(T)} = \mathbb{1}_{\{T<\infty\}}(B_{T+t} - B_T).$$

Then under the probability measure $\mathbf{P}(\cdot | T < \infty)$, the process $(B_t^{(T)})$, $t \geq 0$, is a Brownian motion independent of $\mathcal{F}_T$.

Proof. We prove for the case $T < \infty$, a.s. Fix $A \in \mathcal{F}_T$ and $0 \leq t_1 \leq \cdots \leq t_p$. Let $F \colon \mathbb{R}^p \to \mathbb{R}^+$ be a bounded continuous function. We show that

$$\mathbf{E}[\mathbb{1}_A F(B_{t_1}^{(T)}, \ldots, B_{t_p}^{(T)})] = P(A)\mathbf{E}[F(B_{t_1}, \ldots, B_{t_p})].$$

Strong Markov Property

Denote by $[t]_n$ the smallest real of the form $k2^{-n}$ at least t. By convention, we let $[\infty]_n = \infty$. By the continuity of F,

$$F(B_{t_1}^{(T)}, \ldots B_{t_p}^{(T)}) = \lim_{n\to\infty} F(B_{t_1}^{([T]_n)}, \ldots, B_{t_p}^{([T]_n)}),$$

so by Lebesgue Dominated Convergence Theorem and simple Markov property,

$$\begin{aligned}
\mathbf{E}[\mathbb{1}_A F(B_{t_1}^{(T)}, \ldots, B_{t_p}^{(T)})] &= \lim_{n\to\infty} \mathbf{E}[\mathbb{1}_A F(B_{t_1}^{([T]_n)}, \ldots, B_{t_p}^{([T]_n)})] \\
&= \lim_{n\to\infty} \sum_{k=0}^{\infty} \mathbf{E}[\mathbb{1}_A \mathbb{1}_{\{(k-1)2^{-n} \le T \le k2^{-n}\}} \\
&\qquad \cdot F(B_{k2^{-n}+t_1} - B_{k2^{-n}}, \ldots, B_{k2^{-n}+t_p} - B_{k2^{-n}})] \\
&= \lim_{n\to\infty} \sum_{k=0}^{\infty} \mathbf{P}(A \cap \{(k-1)2^{-n} < T \le k2^{-n}\}) \mathbf{E}[F(B_{t_1}, \ldots, B_{t_p})] \\
&= \mathbf{P}(A)\mathbf{E}[F(B_{t_1}, \ldots, B_{t_p})].
\end{aligned}$$

Strong Markov Property

If $\mathbf{P}(T = \infty) > 0$, we may substitute A by $A \cap \{T < \infty\}$ and the desired result follows.

Once we have

$$\mathbf{E}[\mathbb{1}_A F(B_{t_1}^{(T)}, \dots, B_{t_p}^{(T)})] = P(A)\mathbf{E}[F(B_{t_1}, \dots, B_{t_p})],$$

for $A = \Omega$, since $B^{(T)}$ has the same finite-dimensional marginal distributions as B, it is also a Brownian motion. Then the above equation shows that $B^{(T)}$ is independent of $\mathcal{F}_T$. $\square$

Reflection Principle

Theorem

For every $t > 0$, let $S_t = \sup_{s \leq t} B_s$. Then if $a \geq 0$ and $b \in (-\infty, a]$, we have

$$\mathbf{P}(S_t \geq a, B_t \leq b) = \mathbf{P}(B_t \geq 2a - b).$$

In particular, S_t has the same distribution as $|B_t|$.

Proof. Define a stopping time

$$T_a = \inf\{t \geq 0 : B_t = a\}.$$

Note that $T_a < \infty$. We have

$$\mathbf{P}(S_t \geq a, B_t \leq b) = \mathbf{P}(T_a \leq t, B_t \leq b) = \mathbf{P}(T_a \leq t, B^{(T_a)}_{t-T_a} \leq b - a).$$

Reflection Principle

Write $B' = B^{(T_a)}$ and note that B' is a Brownian motion independent of $\mathcal{F}_{T_a}$. Note also that $-B'$ is a Brownian motion as well, and that (T_a, B') and $(T_a, -B')$ have the same distribution. Define

$$H = \{(s, w) \in \mathbb{R}^+ \times C(\mathbb{R}^+, \mathbb{R}) : s \leq t, w(t-s) \leq b - a\}.$$

Then

$$\begin{aligned} \mathbf{P}((T_a, B') \in H) &= \mathbf{P}((T_a, -B') \in H) \\ &= \mathbf{P}(T_a \leq t, -B^{(T_a)}_{t-T_a} \leq b - a) \\ &= \mathbf{P}(T_a \leq t, B_t \geq 2a - b) \\ &= \mathbf{P}(B_t \geq 2a - b). \end{aligned}$$

Finally,

$$\begin{aligned} \mathbf{P}(S_t \geq a) &= \mathbf{P}(S_t \geq a, B_t \geq a) + \mathbf{P}(S_t \geq a, B_t \leq a) \\ &= 2\mathbf{P}(B_t \geq a) = \mathbf{P}(|B_t| \geq a). \end{aligned}$$

So S_t has the same distribution as $|B_t|$. $\square$

Distribution of T_a

Corollary

For every $a > 0$, T_a has the same distribution as a^2/B_1^2 and has density

$$f(t) = \frac{a}{\sqrt{2\pi t^3}} \exp\left(-\frac{a^2}{2t}\right) \mathbb{1}_{\{t>0\}}.$$

Proof. For every $t \geq 0$,

$$\begin{aligned}
\mathbf{P}(T_a \leq t) &= \mathbf{P}(S_t \geq a) \\
&= \mathbf{P}(|B_t| \geq a) \\
&= \mathbf{P}(B_t^2 \geq a^2) \\
&= \mathbf{P}(tB_1^2 \geq a^2) \\
&= \mathbf{P}\left(\frac{a^2}{B_1^2} \leq t\right).
\end{aligned}$$

Real Brownian Motions

If Z is a real random variable, a stochastic process (X_t) is a *real Brownian motion* if

$$X_t = Z + B_t$$

where B is a real Brownian motion started from 0 and independent of Z.

A stochastic process $B_t = (B_t^1, \dots, B_t^d)$ with values in $\mathbb{R}^d$ is a *d-dimensional Brownian motion* started from 0 if its components $B^1, \dots, B^d$ are independent real Brownian motions started from 0.