Day 9 : Continuous Local Martingales

Dain Kim

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Today's Reading

[L] Chapter 3.3. Continuous Time Martingales and Supermartingales
Chapter 4.1. Finite Variation Processes
Chapter 4.2. Continuous Local Martingales

Review: Gaussian White Noise

Let $(E, \mathcal{E})$ be a measurable space, μ a σ-finite measure. A *Gaussian white noise* with intensity μ is an isometry

$$G\colon L^2(E, \mathcal{E}, \mu) \to \text{(a (centered) Gaussian space)}.$$

That is, for $f \in L^2(E, \mathcal{E}, \mu)$, $G(f)$ is centered Gaussian with variance

$$\mathbf{E}[G(f)^2] = ||G(f)||^2_{L^2(\Omega, \mathcal{F}, \mathbf{P})} = ||f||^2_{L^2(E, \mathcal{E}, \mu)} = \int f^2 d\mu.$$

If $f, g \in L^2(E, \mathcal{E}, \mu)$, the covariance is

$$\mathbf{E}[G(f), G(g)] = \langle f, g \rangle_{L^2(E, \mathcal{E}, \mu)} = \int fg d\mu.$$

The Wiener Integral

Note that pre-Brownian motion is defined as $B_t = G(\mathbb{1}_{[0,t]})$.
Conversely, for any given pre-Brownian motion (B_t), the associated Gaussian white noise is determined fully: for any step function $f = \sum_{i=1}^n \lambda_i \mathbb{1}_{(t_{i-1},t_i]}$, where $0 = t_0 < t_1 < \cdots < t_n$,

$$G(f) = \sum_{i=1}^{n} \lambda_i (X_{t_i} - X_{t_{i-1}}).$$

We write for $f \in L^2(\mathbb{R}^+, \mathcal{B}(\mathbb{R}^+), dt)$,

$$G(f) = \int_0^\infty f(s)dB_s.$$

Similarly,

$$G(f\mathbb{1}_{[0,t]}) = \int_0^t f(s)dB_s, \quad G(f\mathbb{1}_{(s,t]}) = \int_s^t f(r)dB_r.$$

This integration is called *the Wiener integral*.

Review: Martingales

$(\Omega, \mathcal{F}, \mathbf{P})$: a probability space
$\{\mathcal{F}_t\}_{t\in\Sigma}$: a filtration

A stochastic process $X(t)$, $t \in \Sigma$, is called a *martingale* w.r.t. the filtration $\{\mathcal{F}_t\}_{t\in\Sigma}$ if

1) $\mathbf{E}|X(t)| < \infty$ for every $t \in \Sigma$,
2) For every $t \in \Sigma$, the r.v. $X(t)$ is $\mathcal{F}_t$-measurable.
3) $\mathbf{E}\{X(t)|X(s)\} = X(s)$.

If 1) holds, we say $X(t) \in L^1$.
If 2) holds, we say $X(t)$ is adapted

Martingales

An adapted real-valued process (X_t), $t \geq 0$, such that $X_t \in L^1$ for every $t \geq 0$ is called

- a *martingale* if $\mathbf{E}[X_t|\mathcal{F}_s] = X_s$ for every $0 \leq s < t$.
- a *supermartingale* if $\mathbf{E}[X_t|\mathcal{F}_s] \leq X_s$ for every $0 \leq s < t$.
- a *submartingale* if $\mathbf{E}[X_t|\mathcal{F}_s] \geq X_s$ for every $0 \leq s < t$.

Examples

A process (Z_t), $t \geq 0$, with values in $\mathbb{R}$ or $\mathbb{R}^d$ has *independent increments* w.r.t. a filtration $(\mathcal{F}_t)$ if for every $0 \leq s < t$,

$$Z_t - Z_s \text{ is independent of } \mathcal{F}_s.$$

If Z is a real valued process having independent increments w.r.t. $(\mathcal{F}_t)$,

1) If $Z_t \in L^1$ for every $t \geq 0$, $\tilde{Z}_t = Z_t - \mathbf{E}[Z_t]$ is a martingale.
2) If $Z_t \in L^2$ for every $t \geq 0$. $Y_t = \tilde{Z}_t^2 - \mathbf{E}[\tilde{Z}_t^2]$ is a martingale.
3) If, for some $\theta \in \mathbb{R}$, we have $\mathbf{E}[e^{\theta Z_t}] < \infty$ for every $t \geq 0$, then $X_t = e^{\theta Z_t}/\mathbf{E}[e^{\theta Z_t}]$ is a martingale.

$(\mathcal{F}_t)$-Brownian motion

A real-valued process $B = (B_t), t \geq 0$ is an *$(\mathcal{F}_t)$-Brownian motion* if

1) B is a Brownian motion
2) B is adapted
3) B has independent increments w.r.t. $(\mathcal{F}_t)$

If B is a Brownian motion and $(\mathcal{F}_t^B)$ is the canonical filtration of B, then B is an $(\mathcal{F}_t^B)$-Brownian motion. Therefore,

$$B_t, \quad B_t^2 - t, \quad e^{\theta B_t - \frac{\theta^2}{2} t}$$

are martingales. Moreover, if $f \in L^2(\mathbb{R}^+, \mathcal{B}(\mathbb{R}^+), dt)$ and $Z_t = \int_0^t f(s) dB_s$, then

$$\int_0^t f(s) dB_s, \quad \left(\int_0^t f(s) dB_s \right)^2 - \int_0^t f(s)^2 ds,$$
$$\exp\left(\theta \int_0^t f(s) dB_s - \frac{\theta^2}{2} \int_0^t f(s)^2 ds \right)$$

are martingales.

Finite Variation

Let $T \geq 0$. A continuous function $a \colon [0, T] \to \mathbb{R}$ with $a(0) = 0$ has *finite variation* if there is a signed measure μ on $[0, T]$ such that $a(t) = \mu([0, t])$ for every $t \in [0, T]$.

Proposition

For every $0 < t \leq T$

$$\int_0^t |da(s)| = \sup \left\{ \sum_{i=1}^{n} |a(t_i) - a(t_{i-1})| \right\}$$

where $0 = t_0 < t_1 < \cdots < t_n = t$.

Counterexample. $f(x) = x \sin\left(\frac{1}{x}\right)$ has infinite variation.

Finite Variation Process

$(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbf{P})$: a filtered probability space

An adapted process $A = (A_t)$, $t \geq 0$, is called a *finite variation process* if all its sample paths are finite variation functions on $\mathbb{R}^+$. If in addition the sample paths are nondecreasing functions, the process A is called an *increasing process*.

If A is a finite variation process, the process

$$V_t = \int_0^t |dA_s|$$

is an increasing process.

Finite Variation Process

A stochastic process is called *progressive* if for every $t \geq 0$, the mapping

$$(w, s) \mapsto X_s(w)$$

on $\Omega \times [0, t]$ is $(\mathcal{F}_t \otimes \mathcal{B}([0, t]))$-measurable.

Proposition

If A is a finite variation process and H is a progressive process such that for all $t \geq 0$ and $\omega \in \Omega$,

$$\int_0^t |H_s(\omega)||dA_s(\omega)| < \infty.$$

Then the process $H \cdot A$ is defined as

$$(H \cdot A)_t = \int_0^t H_s dA_s$$

is also a finite variation process.

Continuous Local Martingales

$(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbf{P})$: a filtered probability space

Let T be a stopping time and (X_t), $t \geq 0$, an adapted process with continuous sample paths.

X^T : the process X stopped at T.
$X_t^T := X_{t \wedge T}$.

If S is another stopping time,

$$(X^T)^S = (X^S)^T = X^{S \wedge T}.$$

Continuous Local Martingales

An adapted process $M = (M_t)$, $t \geq 0$, with continuous sample paths and $M(0) = 0$ a.s. is called a *continuous local martingale* if there exists a nondecreasing sequence $(T_n)_{n \geq 0}$ of stopping times such that

1) $T_n \nearrow \infty$, i.e., $T_n(\omega) \nearrow \infty$ for every $\omega \in \Omega$
2) for every n, the stopped process M^{T_n} is a uniformly integrable martingale.

A class $\mathcal{C}$ of random variables is *uniformly integrable* if

1) there exists $M > 0$ such that $\mathbf{E}[|X|] < M$,
2) for every $\epsilon > 0$, there exists $\delta > 0$ such that for every measurable set A with $\mathbf{P}(A) < \delta$,
$$\mathbf{E}[|X| I_A] < \epsilon$$
for all $X \in \mathcal{C}$.