Day 10 : Continuous Local Martingales 2

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January 6th

Today's Reading

[L] Chapter 4.2. Continuous Local Martingales
Chapter 4.3. The Quadratic Variation of a Continuous Local Martingale
Chapter 4.4. The Bracket of Two Continuous Local Martingales
Chapter 4.5. Continuous Semimartingales

Review: Continuous Local Martingales

$(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbf{P})$: a filtered probability space

Let T be a stopping time and (X_t), $t \geq 0$, an adapted process with continuous sample paths.

X^T : the process X stopped at T.
$X_t^T := X_{t \wedge T}$.

If S is another stopping time,

$$(X^T)^S = (X^S)^T = X^{S \wedge T}.$$

Review: Continuous Local Martingales

An adapted process $M = (M_t)$, $t \geq 0$, with continuous sample paths and $M(0) = 0$ a.s. is called a *continuous local martingale* if there exists a nondecreasing sequence $(T_n)_{n\geq 0}$ of stopping times such that

1) $T_n \nearrow \infty$, i.e., $T_n(\omega) \nearrow \infty$ for every $\omega \in \Omega$
2) for every n, the stopped process M^{T_n} is a uniformly integrable martingale.

A class $\mathcal{C}$ of random variables is *uniformly integrable* if

1) there exists a constant $C > 0$ such that $\mathbf{E}[|X|] < C$ for all $X \in \mathcal{C}$,
2) for every $\epsilon > 0$, there exists $\delta > 0$ such that for every measurable set A with $\mathbf{P}(A) < \delta$,

$$\mathbf{E}[|X| I_A] < \epsilon$$

for all $X \in \mathcal{C}$.

This is equivalent to

$$\lim_{K\to\infty} \sup_{X\in\mathcal{C}} \mathbf{E}[|X| \mathbb{1}_{\{|X|\geq K\}}] = 0.$$

Continuous Local Martingales

An adapted process $M = (M_t)$, $t \geq 0$, with continuous sample paths and $M(0) = 0$ a.s. is called a *continuous local martingale* if there exists a nondecreasing sequence $(T_n)_{n \geq 0}$ of stopping times such that

1) $T_n \nearrow \infty$, i.e., $T_n(\omega) \nearrow \infty$ for every $\omega \in \Omega$
2) for every n, the stopped process M^{T_n} is a uniformly integrable martingale.

If we don't assume $M_0 = 0$ a.s., M_t is a *continuous local martingale* if $N_t = M_t - M_0$ is a continuous local martingale.

We call T_n *reduces* M if $T_n \nearrow \infty$ and M^{T_n} is a uniformly integrable martingale for every n.

Continuous Local Martingales Remarks

- A continuous local martingale M_t need not satisfy $M_t \in L^1$.
- Martingales with continuous sample paths are continuous local martingales, but not vice versa.
  - For martingales, $T_n = n$ reduces M.
  - If B is a $(\mathcal{F}_t)$-Browniam motion and Z is a $\mathcal{F}_0$-measurable random variable, then $Z_t = Z + B_t$ is a continuous local martingale. But Z_t is not a martingale if $\mathbf{E}[|Z|] = \infty$.
- In the definition, if $M_0 = 0$ a.s., the "uniformly integrable martingale" can be replaced by "martingale."
  - $M^{T_n \wedge n}$ is a uniformly integrable martingale.
- The space of all continuous local martingales is a vector space.

Properties

Proposition

1) If M is a nonnegative continuous local martingale such that $M_0 \in L^1$, then M is a supermartingale.
2) If M is a continuous local martingale and there exists a random variable $Z \in L^1$ such that $|M_t| \leq Z$ for every $t \geq 0$, then M is a uniformly integrable martingale.
3) If M is a continuous local martingale and $M_0 \in L^1$, the sequence of stopping times
$$T_n = \inf\{t \geq 0 : |M_t| \geq n\}$$
reduces M.

Theorem

If M is a continuous local martingale and also a finite variation process (in particular, $M_0 = 0$), then $M_t = 0$ for every $t \geq 0$, a.s.

The Quadratic Variation of Continuous Local Martingales

We assume that $(\mathcal{F}_t)$ is a complete filtration. That is, for every $A \subset \Omega$ such that there exists $A \subset B \subset \Omega$ with $\mathbf{P}(B) = 0$, $A \in \mathcal{F}_t$ for all t.

Theorem

Let $M = (M_t)$, $t \geq 0$, be a continuous local martingale. There exists an increasing process $(\langle M, M \rangle_t$, $t \geq 0$, which is unique up to indistinguishability, such that $M_t^2 - \langle M, M \rangle_t$ is a continuous local martingale. Moreover,

$$\langle M, M \rangle_t = \lim_{\sup_i |t_{i+1} - t_i| \to 0} \sum_{i=1}^{n-1} (M_{t_i^n} - M_{t_{i-1}^n})^2$$

in probability. The process $\langle M, M \rangle$ is called the *quadratic variation* of M.

The Quadratic Variation of Continuous Local Martingales

Example. Let B be a $(\mathcal{F}_t)$-Browniam motion. Then B is a martingale with continuous sample paths, so it is a continuous local martingale.

Proposition

For a fixed t,

$$\langle B, B\rangle_t = \lim_{\sup_i |t_{i+1}-t_i|\to 0} \sum_{i=1}^{n-1} (B_{t_i^n} - B_{t_{i-1}^n})^2 = t$$

where $0 = t_0^n < t_1^n < \cdots < t_n^n = t$.

Proof. Define $f_j = \mathbb{1}_{(t_{j-1},t_j]}$ for $1 \leq j \leq n$. The functions are independent. Since G is a Gaussian White Noise, note that $\mathbf{E}[G(f_j)^2] = |t_j - t_{j-1}|$. So

$$\mathbf{E}\left[\left(\sum_{j=1}^{n} G(f_j)^2 - |t_j - t_{j-1}|\right)^2\right] = \sum_{j=1}^{n} \mathrm{Var}\left(G(f_j)^2\right) = 2\sum_{j=1}^{n} |t_j - t_{j-1}|^2 \to 0.$$

The Bracket of Two Continuous Local Martingales

Let M and N be two continuous local martingales. The *bracket* $\langle M, N \rangle$ is the finite variation process defined by

$$\langle M, N \rangle_t = \frac{1}{2} \left(\langle M + N, M + N \rangle_t - \langle M, M \rangle_t - \langle N, N \rangle_t \right)$$

for every $t \geq 0$.

The bracket $\langle M, N \rangle$ is the unique finite variation process up to indistinguishability such that $M_t N_t - \langle M, N \rangle_t$ is a continuous local martingale.

The Bracket of Two Continuous Local Martingales

Proposition

Let B and B' be two independent $(\mathcal{F}_t)$-Brownian motions. Then

$$\langle B, B' \rangle_t = 0$$

for every $t \geq 0$.

Proof. We may assume $B_0 = B'_0 = 0$. Note that

$$X_t = \frac{1}{\sqrt{2}}(B_t + B'_t)$$

is a martingale. Moreover, X is a Brownian motion. Then $\langle X, X \rangle_t = t$, so using the bilinearity, $\langle B, B' \rangle_t = 0$. $\square$

Two continuous local martingales M and N are *orthogonal* if $\langle M, N \rangle = 0$.

Continuous Semimartingales

A process $X = (X_t)$, $t \geq 0$, is a *continuous semimartingale* if there is a decomposition

$$X_t = M_t + A_t$$

so that M is a continuous local martingale and A is a finite variation process.

Such decomposition is unique.

For two $X = M + A$, $Y = M' + A'$ continuous semimartingales with canonical decompositions, the *bracket* is the finite variation process defined by

$$\langle X, Y \rangle_t = \langle M, M' \rangle_t.$$