Day 3 : Conditional Expectations and Stopping Times

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Today's Reading

[B] Chapter 1.2. Conditional Expectations,
Chapter 1.4. Stopping Times

Conditional Probability and Independency

For an event B with $\mathbf{P}(B) > 0$, the *conditional probability* of A given B is

$$\mathbf{P}(A|B) = \frac{\mathbf{P}(A \cap B)}{\mathbf{P}(B)}.$$

Since $\mathbf{P}(\Omega|B) = 1$, the conditional probability $\mathbf{P}(\cdot|B)$ is also a probability measure on the σ-algebra $\mathcal{F}$.

The event A is *independent* of the event B with $\mathbf{P}(B) > 0$ if

$$\mathbf{P}(A|B) = \mathbf{P}(A).$$

Equivalently,

$$\mathbf{P}(A \cap B) = \mathbf{P}(A)\mathbf{P}(B).$$

Conditional Expectation

Let X be a random variable. The *conditional expectation* of X given an event B is

$$\mathbf{E}\{X|B\} = \int_\Omega X(\omega)\mathbf{P}(d\omega|B) = \frac{\mathbf{E}\{X\mathbb{1}_B\}}{\mathbf{P}(B)}.$$

Exercise. $\mathbf{E}\{\mathbb{1}_A|B\} = \mathbf{P}(A|B)$.

The *conditional expectation* $\mathbf{E}\{X|\mathcal{Q}\}$ of X given a σ-algebra $\mathcal{Q}$ generated by disjoint sets B_k, $k = 1, \ldots, m$, is for $\omega \in B_k$,

$$\mathbf{E}\{X|\mathcal{Q}\} = \mathbf{E}\{X|B_k\} = \frac{\mathbf{E}\{X\mathbb{1}_{B_k}\}}{\mathbf{P}(B_k)}.$$

Exercise

Exercise. ([B] Exercise 2.1)
Let $\Omega = \{\omega : \omega \in [-1/2, 1/2]$, $\mathcal{F} = \mathcal{B}([-1/2, 1/2])$, $\mathbf{P}(d\omega) = d\omega$. Let $X(\omega) = \omega^2$. Prove that

$$\mathbf{P}(A|\sigma(X)) = \frac{1}{2}\mathbb{1}_A(\omega) + \frac{1}{2}\mathbb{1}_A(-\omega),$$
$$\mathbf{E}(Y|\sigma(X)) = \frac{1}{2}Y(\omega) + \frac{1}{2}Y(-\omega).$$

Properties of Conditional Expectations

1) Linearity.

$$\mathbf{E}\{aX + bY|\mathcal{Q}\} = \alpha\mathbf{E}\{X|\mathcal{Q}\} + \beta\mathbf{E}\{Y|\mathcal{Q}\} \quad \text{a.s.}$$

2) If X does not depend on $\mathcal{Q}$, then

$$\mathbf{E}\{X|\mathcal{Q}\} = \mathbf{E}X \quad \text{a.s.}$$

3) If Y is $\mathcal{Q}$-measurable, then

$$\mathbf{E}\{XY|\mathcal{Q}\} = Y\mathbf{E}\{X|\mathcal{Q}\} \quad \text{a.s.}$$

4) For $\mathcal{Q} \subset \mathcal{M}$,

$$\mathbf{E}\{X|\mathcal{Q}\} = \mathbf{E}\{\mathbf{E}\{X|\mathcal{M}\}|\mathcal{Q}\} \quad \text{a.s.}$$

4') For $A \in \mathcal{M}$,

$$\mathbf{E}\{X|A\} = \mathbf{E}\{\mathbf{E}\{X|\mathcal{M}\}|A\}.$$

Properties of Conditional Expectations

5) If $X \leq Y$ a.s., then

$$\mathbf{E}\{X|\mathcal{Q}\} \leq \mathbf{E}\{|X||\mathcal{Q}\} \quad \text{a.s.}$$

6) $|\mathbf{E}\{X|\mathcal{Q}\}| \leq \mathbf{E}\{|X||\mathcal{Q}\}$ a.s.

7) If $\mathbf{E}(\sup_{n\in\mathbb{N}} |X_n|) < \infty$ and $X_n \to X$ a.s., then

$$\mathbf{E}\{X_n|\mathcal{Q}\} \to \mathbf{E}\{X|\mathcal{Q}\} \quad \text{a.s.}$$

7') If $\{X_n\}_{n\in\mathbb{N}}$ is a uniformly integrable family of random variables and $X_n \to X$ in probability, then $\mathbf{E}\{X_n|\mathcal{Q}\} \to \mathbf{E}\{X|\mathcal{Q}\}$ in mean.

Filtrations

A family of σ-algebras $\{\mathcal{F}\}_{t\in\Sigma}$ on $(\Omega, \mathcal{F})$ is called a *filtration* if

$$\mathcal{F}_s \subseteq \mathcal{F}_t \subseteq \mathcal{F}$$

for every $s, t \in \Sigma$ with $s < t$.

For $\Sigma = [0, T]$, a filtration is *right continuous* if for every $t \in [0, T)$,

$$\mathcal{F}_t = \mathcal{F}_{t+} := \bigcap_{\epsilon > 0} \mathcal{F}_{t+\epsilon}.$$

The collection $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbf{P})$ is called a *filtered probability space*.

Filtered Probability Space and usual conditions

A filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbf{P})$ is said to satisfy the *usual conditions* if

1) $\mathcal{F}$ is $\mathbf{P}$-complete, [1]
2) $\mathcal{F}_0$ contains all $\mathbf{P}$-null sets of $\mathcal{F}$,
3) $\{\mathcal{F}_t\}$ is right continuous.

A process $X(t)$, $t \in \Sigma$, defined on a filtered probability space is *adapted* to the filtration $\{\mathcal{F}_t\}$ if for every $t \in \Sigma$ the r.v. $X(t)$ is $\mathcal{F}_t$-measurable.

A process $X(t)$, $t \in [0, T]$, defined on a filtered probability space is *progressively measurable* if for every $t \in [0, T]$ the mapping $(s, \omega) \mapsto X(s, \omega)$ from $[0, t] \times \Omega$ to $\mathbb{R}$ is $\mathcal{B}([0, t]) \times \mathcal{F}_t$-measurable.

Proposition

An adapted process with right or left continuous paths is progressively measurable.

[1] If there is a set A with $A_1 \subseteq A \subseteq A_2$ and $\mathbf{P}(A_1) = \mathbf{P}(A_2)$, then $A \in \mathcal{F}$.

Stopping Times

A *stopping time* with respect to a filtration $\{\mathcal{F}_t, t \in \Sigma \subseteq [0, \infty)\}$ is a mapping $\tau : \Omega \to \Sigma \cup \{\infty\}$ such that $\{\tau \leq t\} \in \mathcal{F}_t$ for every $t \in \Sigma$.

Examples.

1) The first hitting time of a level z : $H_z = \min\{s : X(s) = z\}$.

1') $H_{a,b} = \min\{s : X(s) \notin (a, b)\}$.

2) The moment inverse of integral functional

$$\nu(t) = \min\{s : \int_0^s g(X(v))dv = t\},$$

where g is a nonnegative measurable function.

3) The inverse range time

$$\theta_v = \min\{t : \sup_{0 \leq s \leq t} X(s) - \inf_{0 \leq s \leq t} X(s) \geq v\}.$$

Properties of stopping times

1) If τ is a stopping time, then $\{\tau < t\} \in \mathcal{F}_t$ and $\{\tau = t\} \in \mathcal{F}_t$.
2) If t_0 is a nonnegative constant, then $\tau = t_0$ is a stopping time.
3) If τ is a stopping time, then $\tau + t_0$ is a stopping time for a nonnegative constant t_0.
4) If σ and τ are stopping times, then $\sigma \vee \tau = \max\{\sigma, \tau\}$ and $\sigma \wedge \tau = \min\{\sigma, \tau\}$ are stopping times.
5) If $\tau_n, n \in \mathbb{N}$, are stopping times, then $\inf \tau_n$, $\sup \tau_n$, $\liminf \tau_n$, and $\limsup \tau_n$ are stopping times.