Day 15 : Quadratic Variation revisited

Dain Kim

January 27th

Today's Reading

[L] Section 5.1.

Processes of our interests

- Finite Variation Processes
- Brownian Motions
- Cádlág Processes

Uniformly Integrability

A martingale (X_t), $t \geq 0$, is *closed* if there is a random variable $Z \in L^1$ such that

$$X_t = \mathbf{E}[Z|\mathcal{F}_t].$$

Proposition

If X is a martingale with right continuous paths, then the followings are equivalent:

- X is closed
- the collection (X_t) is uniformly integrable
- X_t converges in L^1 a.s. as $t \to \infty$.

Review : Continuous Local Martingales

An adapted process $M = (M_t)$, $t \geq 0$, with continuous sample paths and $M(0) = 0$ a.s. is called a *continuous local martingale* if there exists a nondecreasing sequence $(T_n)_{n\geq 0}$ of stopping times such that

1) $T_n \nearrow \infty$, i.e., $T_n(\omega) \nearrow \infty$ for every $\omega \in \Omega$
2) for every n, the stopped process M^{T_n} is a uniformly integrable martingale.

Review : The Quadratic Variation of Continuous Local Martingales

We assume that $(\mathcal{F}_t)$ is a complete filtration. That is, for every $A \subset \Omega$ such that there exists $A \subset B \subset \Omega$ with $\mathbf{P}(B) = 0$, $A \in \mathcal{F}_t$ for all t.

Theorem

Let $M = (M_t)$, $t \geq 0$, be a continuous local martingale. There exists an increasing process $(\langle M, M \rangle_t$, $t \geq 0$, which is unique up to indistinguishability, such that $M_t^2 - \langle M, M \rangle_t$ is a continuous local martingale. Moreover,

$$\langle M, M \rangle_t = \lim_{\sup_i |t_{i+1} - t_i| \to 0} \sum_{i=1}^{n-1} (M_{t_i^n} - M_{t_{i-1}^n})^2$$

in probability. The process $\langle M, M \rangle$ is called the *quadratic variation* of M.

Finite Variation Differentiable Real Functions

Let $f : [a, b] \to \mathbb{R}$ be a differentiable function with bounded derivatives. Then f has finite variation: for $a = t_0 < t_1 < \cdots < t_p = b$,

$$\begin{aligned}\lim_{\sup|t_{i+1}-t_i|\to 0} \sum_{i=0}^{p-1} |f(t_{i+1}) - f(t_i)|^2 &= \lim_{\sup|t_{i+1}-t_i|\to 0} \sum_{i=0}^{p-1} |f'(t_i^*)|^2 |t_{i+1} - t_i|^2 \\ &= \int_a^b |f'(x)| dx \leq C(b-a).\end{aligned}$$

The quadratic Variation is zero:

$$\begin{aligned}\lim_{\sup|t_{i+1}-t_i|\to 0} \sum_{i=0}^{p-1} |f(t_{i+1}) - f(t_i)|^2 &= \lim_{\sup|t_{i+1}-t_i|\to 0} \sum_{i=0}^{p-1} |f'(t_i^*)|^2 |t_{i+1} - t_i|^2 \\ &\leq \sup_{|t_{i+1}-t_i|} \int_a^b |f'(x)|^2 dx \to 0.\end{aligned}$$

Review : Finite Variation Process

$(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbf{P})$: a filtered probability space

An adapted process $A = (A_t)$, $t \geq 0$, is called a *finite variation process* if all its sample paths are finite variation functions on $\mathbb{R}^+$. If in addition the sample paths are nondecreasing functions, the process A is called an *increasing process*.

Theorem

If M is a continuous local martingale and also a finite variation process (in particular, $M_0 = 0$), then $M_t = 0$ for every $t \geq 0$, a.s.

Review : The Bracket of Two Continuous Local Martingales

The *bracket* of two continuous local martingales M, N is the finite variation process defined by

$$\langle M, N \rangle_t = \frac{1}{2}(\langle M+N, M+N \rangle_t - \langle M, M \rangle_t - \langle N, N \rangle_t).$$

- $\langle M, N \rangle$ is unique up to indistinguishability finite variation process s.t. $M_t N_t - \langle M, N \rangle_t$ is a continuous local martingale.
- The mapping $(M, N) \mapsto \langle M, N \rangle$ is bilinear and symmetric.
- We may compute it as

$$\langle M, N \rangle_t = \lim_{\sup |t_{i+1} - t_i| \to 0} \sum_{i=0}^{p-1} (M_{t_{i+1}} - M_{t_i})(N_{t_{i+1}} - N_{t_i})$$

in probability. We also call the bracket as "covariation."

Review : Quadratic Variation of Brownian Motions

Example. Let B be a $(\mathcal{F}_t)$-Browniam motion. Then B is a martingale with continuous sample paths, so it is a continuous local martingale.

Proposition

For a fixed t,

$$\langle B, B\rangle_t = \lim_{\sup_i |t_{i+1}-t_i|\to 0} \sum_{i=1}^{n-1} (B_{t_i^n} - B_{t_{i-1}^n})^2 = t$$

where $0 = t_0^n < t_1^n < \cdots < t_n^n = t$.

Proof. Define $f_j = \mathbb{1}_{(t_{j-1},t_j]}$ for $1 \le j \le n$. The functions are independent. Since G is a Gaussian White Noise, note that $\mathbf{E}[G(f_j)^2] = |t_j - t_{j-1}|$. So

$$\mathbf{E}\left[\left(\sum_{j=1}^{n} G(f_j)^2 - |t_j - t_{j-1}|\right)^2\right] = \sum_{j=1}^{n} \mathrm{Var}\left(G(f_j)^2\right) = 2\sum_{j=1}^{n} |t_j - t_{j-1}|^2 \to 0.$$

Cádlág Processes

A function $f : \mathbb{R}_{\geq 0} \to \mathbb{R}$ is called *cádlág* if it is right continuous and has left limit at every $t > 0$.

A stochastic process (X_t), $t \geq 0$, is called a *cádlág process* if the sample paths are cádlág with probability 1.

What is the quadratic variation of cádlág processes?