Day 4 : Martingales

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December 8th

Today's Reading

[B] Chapter 1.5. Martingales

Martingales

$(\Omega, \mathcal{F}, \mathbf{P})$: a probability space
$\{\mathcal{F}_t\}_{t \in \Sigma}$: a filtration

A stochastic process $X(t)$, $t \in \Sigma$, is called a *martingale* w.r.t. the filtration $\{\mathcal{F}_t\}_{t \in \Sigma}$ if

1) $\mathbf{E}|X(t)| < \infty$ for every $t \in \Sigma$,
2) For every $t \in \Sigma$, the r.v. $X(t)$ is $\mathcal{F}_t$-measurable.
3) $\mathbf{E}\{X(t)|X(s)\} = X(s)$.

Examples of Martingales

Let η_l, $l = 1, 2, \ldots$ be i.i.d. r.v.s and $\mathcal{F}_k = \sigma(\eta_l : 1 \leq l \leq k)$.

1) If $\mathbf{E}\eta_1 = 0$, the process

$$X(k) = \sum_{l=1}^{k} \eta_l,$$

$l = 1, 2, \ldots$, is a martingale.

2) If $\mathbf{E}\eta_1 = 0$, $\mathbf{E}\eta_1^2 = \sigma^2 < \infty$, the process

$$Y(k) = \left(\sum_{l=1}^{k} \eta_l \right)^2 - k\sigma^2,$$

$k = 1, 2, \ldots$, is a martingale.

Examples of Martingales

3) Let $\phi(\alpha) = \mathbf{E}e^{i\alpha\eta_1}$ be the characteristic function of the r.v. η_1. Then the process

$$Z(k) = \frac{1}{\phi^k(\alpha)} \exp\left(i\alpha \sum_{l=1}^{k} \eta_l\right),$$

$l = 1, 2, \ldots$ is a martingale.

4) Let η_l be Bernoulli's random variables s.t. $\mathbf{P}(\eta_1 = 1) = p$ and $\mathbf{P}(\eta_1 = -1) = 1 - p$. Then the process

$$U(k) = \left(\frac{1-p}{p}\right)^{\sum_{l=1}^{k} \eta_l},$$

$k = 1, 2, \ldots,$ is a martingale.

Supermartingales and Submartingales

$(\Omega, \mathcal{F}, \mathbf{P})$: a probability space
$\{\mathcal{F}_t\}_{t\in\Sigma}$: a filtration

A stochastic process $X(t)$, $t \in \Sigma$, is called a *supermartingale* (resp. *submartingales*) w.r.t. the filtration $\{\mathcal{F}_t\}_{t\in\Sigma}$ if

1) $\mathbf{E}|X(t)| < \infty$ for every $t \in \Sigma$,
2) For every $t \in \Sigma$, the r.v. $X(t)$ is $\mathcal{F}_t$-measurable.
3) $\mathbf{E}\{X(t)|X(s)\} \leq X(s)$ (resp. $\mathbf{E}\{X(t)|X(s)\} \geq X(s)$).

A random time change

Theorem

Let $(X(k), \mathcal{F}_k)$, $k = 1, 2, \ldots$ be a supermartingale. If σ and τ are integer-valued bounded stopping times w.r.t. $\{\mathcal{F}_k\}_{k=1}^{\infty}$ such that

$$1 \leq \sigma(\omega) \leq \tau(\omega) \leq n$$

for almost all $\omega \in \Omega$ and some integer n. Then $X(\sigma)$ is $\mathcal{F}_\sigma$-measurable and

$$\mathbf{E}\{X(\tau)|\mathcal{F}_\sigma\} \leq X(\sigma) \quad \text{a.s.}$$

Corollary

If $(X(k), \mathcal{F}_k)$, $k = 1, 2, \ldots$ is a supermartingale and $1 \leq \rho \leq n$ be an integer-valued stopping time w.r.t. $\{\mathcal{F}_k\}_{k=1}^{n}$, then

$$\mathbf{E}X(1) \geq \mathbf{E}X(\rho) \geq \mathbf{E}X(n).$$

Decomposition of submartingales

Theorem (Doob's decomposition)

Any submartingale $(X(k), \mathcal{F}_k)$, $k = 0, 1, 2, \ldots$ has a unique decomposition as

$$X(k) = M(k) + A(k),$$

where $(M(k), \mathcal{F}_k)$ is a martingale and $A(k)$ is an $\mathcal{F}_{k-1}$-measurable nondecreasing process, $A(0) = 0$.

Construction. $M(0) = X(0)$, $A(0) = 0$, and

$$\begin{aligned} M(k) &= M(k-1) + (X(k) - \mathbf{E}\{X(k)|\mathcal{F}_{k-1}\}) \\ A(k) &= A(k-1) + (\mathbf{E}\{X(k)|\mathcal{F}_{k-1}\} - X(k-1)). \end{aligned}$$

Convergence of martingales

Theorem

Let $X(k), \mathcal{F}_k)$, $k = 1, 2, \ldots$ be a submartingale such that

$$\sup_k \mathbf{E} X^+(k) < \infty.$$

Then $X(k)$ converges a.s. as $k \to \infty$ to a limit X_∞ and $\mathbf{E}|X_\infty| < \infty$.

Strong Law of Large Numbers

Theorem

Let X_k, $k = 1, 2, \ldots$, be i.i.d. r.v.s with $\mathrm{Var}\, X_1 < \infty$. Then

$$\frac{1}{n} \sum_{k=1}^{n} X_k \to \mathbf{E} X_1.$$

Lemma. Let x_k, $k = 1, 2, \ldots$ be a sequence of real numbers such that $\sum_{k=1}^{\infty} x_k$ converges. Let b_k, $k = 1, 2, \ldots$ be a monotone sequence of positive numbers tending to infinity. Then

$$\frac{1}{b_n} \sum_{k=1}^{n} b_k x_k \to 0.$$