Day 5 : Martingales 2

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Today's Reading

[B] Chapter 1.5. Martingales

Martingales

$(\Omega, \mathcal{F}, \mathbf{P})$: a probability space
$\{\mathcal{F}_t\}_{t\in\Sigma}$: a filtration

A stochastic process $X(t)$, $t \in \Sigma$, is called a *martingale* w.r.t. the filtration $\{\mathcal{F}_t\}_{t\in\Sigma}$ if

1) $\mathbf{E}|X(t)| < \infty$ for every $t \in \Sigma$,
2) For every $t \in \Sigma$, the r.v. $X(t)$ is $\mathcal{F}_t$-measurable.
3) $\mathbf{E}\{X(t)|X(s)\} = X(s)$.

Examples of Martingales

Let η_l, $l = 1, 2, \ldots$ be i.i.d. r.v.s and $\mathcal{F}_k = \sigma(\eta_l : 1 \leq l \leq k)$.

1) If $\mathbf{E}\eta_1 = 0$, the process

$$X(k) = \sum_{l=1}^{k} \eta_l,$$

$l = 1, 2, \ldots$, is a martingale.

2) If $\mathbf{E}\eta_1 = 0$, $\mathbf{E}\eta_1^2 = \sigma^2 < \infty$, the process

$$Y(k) = \left(\sum_{l=1}^{k} \eta_l\right)^2 - k\sigma^2,$$

$k = 1, 2, \ldots$, is a martingale.

Examples of Martingales

3) Let $\phi(\alpha) = \mathbf{E}e^{i\alpha\eta_1}$ be the characteristic function of the r.v. η_1. Then the process

$$Z(k) = \frac{1}{\phi^k(\alpha)} \exp\left(i\alpha \sum_{l=1}^{k} \eta_l\right),$$

$l = 1, 2, \ldots$ is a martingale.

4) Let η_l be Bernoulli's random variables s.t. $\mathbf{P}(\eta_1 = 1) = p$ and $\mathbf{P}(\eta_1 = -1) = 1 - p$. Then the process

$$U(k) = \left(\frac{1-p}{p}\right)^{\sum_{l=1}^{k} \eta_l},$$

$k = 1, 2, \ldots,$ is a martingale.

Supermartingales and Submartingales

$(\Omega, \mathcal{F}, \mathbf{P})$: a probability space
$\{\mathcal{F}_t\}_{t \in \Sigma}$: a filtration

A stochastic process $X(t)$, $t \in \Sigma$, is called a *supermartingale* (resp. *submartingales*) w.r.t. the filtration $\{\mathcal{F}_t\}_{t \in \Sigma}$ if

1) $\mathbf{E}|X(t)| < \infty$ for every $t \in \Sigma$,
2) For every $t \in \Sigma$, the r.v. $X(t)$ is $\mathcal{F}_t$-measurable.
3) $\mathbf{E}\{X(t)|X(s)\} \leq X(s)$ (resp. $\mathbf{E}\{X(t)|X(s)\} \geq X(s)$).

Optional Stopping Theorem

Theorem

Let $(X(k), \mathcal{F}_k)$, $k = 1, 2, \ldots$ be a martingale. If τ is an integer-valued bounded stopping times w.r.t. $\{\mathcal{F}_k\}_{k=1}^{\infty}$ such that one of the following holds:

- $\tau \leq c$ a.s. for a constant c
- $\mathbf{E}\tau < \infty$ and $\mathbf{E}[|X_{t+1} - X_t| \; \mathcal{F}_t] < c$ a.s. for a constant c
- There exists a constant c such that $|X_{\min(t,\tau)}| \leq c$ for all $t \in \mathbb{Z}^+$.

Then $\mathbf{E}[X_T] = \mathbf{E}[X_0]$.

Gambler's Ruin

Problem

Suppose that a gambler plays a fair game. In each round, the gambler either earns or loses \$1. He wins the game if he earns \$a and is ruined if he loses \$b. What is the expected number of rounds the gambler plays before he either wins or is ruined?

Let X_t be the earning of the gambler at time t. Define a stopping time T as the first moment the gambler either wins \$a or loses \$b. Note that (X_t) is a Martingale. Therefore,

$$\mathbf{E}[X_T] = p(-a) + (1-p)b = \mathbf{E}[X_0] = 0.$$

Hence, $p = \frac{b}{a+b}$.

Gambler's Ruin

Claim. $X_t^2 - t$ is a Martingale.

Proof. exercise.

Therefore,

$$\mathbf{E}[T] = \mathbf{E}[X_T^2] = a^2 \frac{b}{a+b} + b^2 \frac{a}{a+b} = ab.$$

Wald's Identity

Theorem

Let X_t be i.i.d. random variables and T a stopping time. If $\mathbf{E}[T], \mathbf{E}[X_1] < \infty$, then

$$\mathbf{E}[\sum_{i=1}^{T} X_i] = \mathbf{E}[T]\mathbf{E}[X_1].$$