Day 13 : Stochastic Integrals for Martingales 1

Dain Kim

January 20th

Today's Reading

[L] Section 5.1.

Overview

In this lecture, we want to define "the stochastic integral of H w.r.t. M," denoted by $H \cdot M$, where M is a martingale with continuous sample paths bounded in L^2 and H is a progressive process in L^2 space related to M.

The integral will be defined as

$$(H \cdot M)_t = \int_0^t H_s dM_s.$$

Why do we care $d\langle M, M\rangle_s$?

For any analytic function $f : \mathbb{R} \to \mathbb{R}$, by Taylor series, we may approximate the fucntion as follows:

$$
\begin{aligned}
f(x_1) - f(x_0) &\approx f'(x_0)(x_1 - x_0) + \frac{f''(x_0)}{2}(x_1 - x_0)^2 + \cdots \\
f(x_2) - f(x_1) &\approx f'(x_1)(x_2 - x_1) + \frac{f''(x_1)}{2}(x_2 - x_1)^2 + \cdots \\
&\vdots \\
f(x_n) - f(x_{n-1}) &\approx f'(x_{n-1})(x_n - x_{n-1}) + \frac{f''(x_{n-1})}{2}(x_n - x_{n-1})^2 + \cdots
\end{aligned}
$$

The fundamental theorem of calculus gives that

$$f(x_n) - f(x_0) \approx f'(x_0)(x_1 - x_0) + f'(x_1)(x_2 - x_1) + \cdots + f'(x_{n-1})(x_n - x_{n-1}),$$

which holds because the higher term vanishes as we partition the interval $[x_0, x_n]$ into short enough intervals. However, the second term does not vanish for stochastic integrals.

The space of martingales bounded in L^2

We define a space

$$\mathbb{H}^2 = \{\text{continuous martingales } M \text{ bounded in } L^2 \text{ s.t. } M_0 = 0\}/\sim$$

where $M \sim N$ if M and N are indistinguishable.

Theorem

A continuous local martingale M with $M_0 \in L^2$ is a martingale if and only if $\mathbf{E}[\langle M, M\rangle_\infty] < \infty$.

Example. The Brownian motion B is not bounded in L^2 as $B_t \sim N(0, t)$ and so $\int_\Omega |B_t|^2 d\mu = t$. Moreover, $\mathbf{E}[\langle B, B\rangle_\infty] = \infty$ as $\langle B, B\rangle_t = t$.

The space of martingales bounded in L^2

Proposition

If M and N are martingales with continuous paths bounded in L^2, then $\langle M, N \rangle_\infty$ is well defined and is the limit of $\langle M, N \rangle_t$ as $t \to \infty$ a.s. Moreover,

$$\mathbf{E}[M_\infty N_\infty] = \mathbf{E}[M_0 N_0] + \mathbf{E}[\langle M, N \rangle_\infty].$$

Therefore, we may give an inner product on $\mathbb{H}^2$ defined by

$$(M, N)_{\mathbb{H}^2} = \mathbf{E}[\langle M, N \rangle_\infty] = \mathbf{E}[M_\infty N_\infty].$$

Exercise. We need to check:

1) Symmetry. $(M, N)_{\mathbb{H}^2} = (N, M)_{\mathbb{H}^2}$.
2) Linearity. $(aM_1 + bM_2, N)_{\mathbb{H}^2} = a(M_1, N)_{\mathbb{H}^2} + b(M_2, N)_{\mathbb{H}^2}$.
3) Positive definite. $(M, M)_{\mathbb{H}^2} = 0$ if and only if $M = 0$.

Hilbert space

A vector space H equipped with an inner product $\langle \, , \, \rangle$ is a *Hilbert space* if the metric induced by the inner product is complete.

Theorem

The space $\mathbb{H}^2$ is a Hilbert space.

The progressive σ-algebra

Recall that an adapted stochastic process (X_t), $t \geq 0$ to a filtration $(\mathcal{F}_t)$ is *progressive* if for every $t \geq 0$, the mapping $(\omega, s) \mapsto X_s(\omega)$ is measurable for $\mathcal{F}_t \otimes \mathcal{B}([0, t])$.

The progressive σ-algebra is the collection $\mathcal{P}$ of sets $A \in \mathcal{F} \otimes \mathcal{B}(\mathbb{R}^+)$ such that $X_t(\omega) = \mathbb{1}_A(\omega, t)$ is progressive.

Exercise. Check that $\mathcal{P}$ is a σ-algebra.

The space of progressive processes

Let $\mathcal{P}$ be the progressive σ-algebra on $\Omega \times \mathbb{R}^+$. For $M \in \mathbb{H}^2$, we define

$$L^2(M) = \left\{ \text{progressive process } H \text{ s.t. } \mathbf{E}\left[\int_0^\infty H_s^2 d\langle M, M\rangle_s\right] < \infty \right\} / \sim$$

where $H \sim H'$ if $H = H' = 0$, $d\langle M, M\rangle_s$ a.e., a.s.

The space $L^2(M)$ is also a Hilbert space equipped with an inner product

$$(H, K)_{L^2(M)} = \mathbf{E}\left[\int_0^\infty H_s K_s d\langle M, M\rangle_s\right].$$

Elementary processes

A progressive process H is an *elementary process* if H can be written as

$$H_s(\omega) = \sum_{i=0}^{p-1} H_{(i)}(\omega) \mathbb{1}_{(t_i, t_{i+1}]}(s)$$

for $0 = t_0 < t_1 < \cdots < t_p$ and each $H_{(i)}$ is a bounded $\mathcal{F}_{t_i}$-measurable random variable.

Denote by $\mathcal{E}$ the space of all elementary processes. Note that $\mathcal{E}$ is a linear subspace of $L^2(M)$ for any $M \in \mathbb{H}^2$. Moreover,

Proposition

For any $M \in \mathbb{H}^2$, $\mathcal{E}$ is dense in $L^2(M)$.

Stochastic Integral

Exercise. If $M \in \mathbb{H}^2$ and T a stopping time, then $M^T \in \mathbb{H}^2$.
Exercise. If $H \in L^2(M)$, the process $\mathbb{1}_{[0,T]}H$ defined by

$$(\mathbb{1}_{[0,T]}H)_s(\omega) = \mathbb{1}_{\{0 \le s \le T(\omega)\}} H_s(\omega)$$

belongs to $L^2(M)$.

For any $M \in \mathbb{H}^2$ and $H \in \mathcal{E}$ of the form

$$H_s(\omega) = \sum_{i=0}^{p-1} H_{(i)}(\omega) \mathbb{1}_{(t_i, t_{i+1}]}(s),$$

the process $H \cdot M$ defined as

$$(H \cdot M)_t = \sum_{i=0}^{p-1} H_{(i)}(M_{t_{i+1} \wedge t} - M_{t_i \wedge t})$$

belongs to $\mathbb{H}^2$.

Stochastic Integral

The mapping $H \mapsto H \cdot M$ extends to an isometry from $L^2(M)$ to $\mathbb{H}^2$.

The process $H \cdot M$ is the unique martingale of $\mathbb{H}^2$ such that for any $N \in \mathbb{H}^2$,

$$\langle H \cdot M, N \rangle = H \cdot \langle M, N \rangle.$$

If T is a stopping time, we have

$$(\mathbb{1}_{[0,T]} H) \cdot M = (H \cdot M)^T = H \cdot M^T.$$

We finally use the notation

$$\int_0^t H_s dM_s = (H \cdot M)_t,$$

and call $H \cdot M$ the *stochastic integral* of H w.r.t. M.