Day 6 : Gaussian Processes and pre-Brownian Motions

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Today's Reading

[L] Le Gall, Jean-François. Brownian motion, martingales, and stochastic calculus. Vol. 274. New York: Springer, 2016.
[L] Chapter 1,2

Gaussian variables

A real random variable X is a *standard Gaussian variable* if its density is

$$p_X(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right).$$

A real random variable Y is *Gaussian* with $N(m, \sigma^2)$-distribution if

$$Y = \sigma X + m,$$

or equivalently,

$$p_Y(y) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(y-m)^2}{2\sigma^2}\right).$$

The moments of Gaussian variables

For any $\lambda \in \mathbb{R}$,

$$\mathbf{E}[e^{\lambda X}] = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{\lambda x} e^{-x^2/2} dx = e^{\lambda^2/2} \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-(x-\lambda)^2/2} dx = e^{\lambda^2/2}.$$

Since $\mathbf{E}[e^{zX}]$ is well defined for $z \in \mathbb{C}$ and holomorphic,

$$\mathbf{E}[e^{zX}] = e^{z^2/2}$$

for all $z \in \mathbb{C}$. Let $z = i\xi$, $\xi \in \mathbb{R}$, then by Taylor expansion,

$$\mathbf{E}[e^{i\xi X}] = 1 + i\xi \mathbf{E}[X] + \cdots + \frac{(i\xi)^n}{n!} \mathbf{E}[X^n] + O(|]\xi|^{n+1})$$

as $\xi \to 0$. On the other hand,

$$\mathbf{E}[e^{i\xi X}] = e^{-\xi^2/2} = \sum_{n=0}^{\infty} (-1)^n \frac{\xi^{2n}}{2^n n!}.$$

Threfore,

$$\mathbf{E}[X^{2n}] = \frac{(2n)!}{2^n n!}, \quad \mathbf{E}[X^{2n+1}] = 0.$$

Gaussian Vectors

Let $E = \mathbb{R}^d$. A random variable X with values in E is a *Gaussian vector* if $\langle u, X \rangle$ is a Gaussian variable for every $u \in E$.

Example. A random vector $X = (X_1, \ldots, X_d)$ consisting of d independent Gaussian variables is a Gaussian vector.

Gaussian Processes

A *(centered) Gaussian space* is a closed linear subspace of $L^2(\Omega, \mathcal{F}, \mathbf{P})$ whose elements are all centered Gaussian variables.

Example. If $X = (X_1, \ldots, X_d)$ is a centered Gaussian vector, then

$$\mathrm{Span}\{X_1, \ldots, X_d\}$$

is a Gaussian space.

A real-valued stochastic process $X(t)$, $t \in T$, is called a *(centered) Gaussian process* if any finite linear combination of the variables is centered Gaussian.

Gaussian White Noise

Let $(E, \mathcal{E})$ be a measurable space, μ a σ-finite measure. A *Gaussian white noise* with intensity μ is an isometry

$$G\colon L^2(E, \mathcal{E}, \mu) \to \text{(a (centered) Gaussian space)}.$$

That is, for $f \in L^2(E, \mathcal{E}, \mu)$, $G(f)$ is centered Gaussian with variance

$$\mathbf{E}[G(f)^2] = ||G(f)||^2_{L^2(\Omega, \mathcal{F}, \mathbf{P})} = ||f||^2_{L^2(E, \mathcal{E}, \mu)} = \int f^2 d\mu.$$

If $f, g \in L^2(E, \mathcal{E}, \mu)$, the covariance is

$$\mathbf{E}[G(f), G(g)] = \langle f, g \rangle_{L^2(E, \mathcal{E}, \mu)} = \int fgd\mu.$$

Pre-Brownian Motions

If G is a Gaussian white noise on $\mathbb{R}^+$ with intensity its Lebesgue measure, the random process $(B_t)_{t\in(0,\infty)}$ defined by

$$B_t = G(\mathbb{1}_{[0,t]})$$

is a *pre-Brownian motion*.

The covariance is

$$\mathbf{E}[B_s B_t] = \mathbf{E}[G([0,s])G([0,t])] = \int_0^\infty \mathbb{1}_{[0,s]}\mathbb{1}_{[0,t]}(r)dr = \min(s,t).$$

Characterizations of pre-Brownian motions

Proposition

Let $X(t)$, $t \geq 0$, be a real-valued stochastic process. The followings are equivalent:

1) $X(t)$, $t \geq 0$, is a pre-Brownian motion
2) $X(t)$, $t \geq 0$, is a centered Gaussian process with covariance $K(s,t) = \min(s,t)$
3) $X(0) = 0$ a.s., and for every $0 \leq s < t$, the random variable $X(t) - X(s)$ is independent of $\sigma(X(r), r \leq s)$ and distributed according to $N(0, t-s)$.
4) $X(0) = 0$ a.s., and for every $0 = t_0 < t_1 < \ldots < t_p$, the variables $X_{t_i} - X_{t_{i-1}}$, $1 \leq i \leq p$, are independent, and are distributed according to $N(0, t_i - t_{i-1})$.

Proof

1) $\Rightarrow$ 2) and 3) $\Rightarrow$ 4) are clear.

Theorem

Let H be a centered Gaussian space and $(H_i)_{i \in I}$ a collection of linear subspaces. The subspaces are pairwise orthogonal in L^2 if and only if $\sigma(H_i)$ are independent.

2) $\Rightarrow$ 3) : Let $H_s = \text{Span}\{X_r : r \leq s\}$ and $\tilde{H}_s = \text{Span}\{X_{s+u} - X_s : u \geq 0\}$. These two subspaces are orthogonal as

$$\mathbf{E}[X_r(X_{s+u} - X_s)] = \mathbf{E}[X_r X_{s+u}] - \mathbf{E}[X_r X_s] = r - r = 0$$

for all $r \leq s$. Therefore, $X_t - X_s$ is independent of $\sigma(X(r), r \leq s)$. Note that

$$\mathbf{E}[(X_t - X_s)^2] = \mathbf{E}[X_t^2] - \mathbf{E}[X_t X_s] + \mathbf{E}[X_s^2] = t - \min(t, s) + s = t - s.$$

Therefore, $X_t - X_s \sim N(0, t - s)$.

Proof

4) $\Rightarrow$ 1) : Let f be a step function $f = \sum_{i=1}^{n} \lambda_i \mathbb{1}_{(t_{i-1}, t_i]}$, where $0 = t_0 < t_1 < \cdots < t_n$, and set

$$G(f) = \sum_{i=1}^{n} \lambda_i (X_{t_i} - X_{t_{i-1}}).$$

For any other step function g, we can see that

$$\mathbf{E}[G(f)G(g)] = \int_{\mathbb{R}^+} f(t)g(t)dt,$$

and the set of step functions is dense in L^2, so G is a Gaussian White Noise. Moreover, by the construction,

$$G(\mathbb{1}_{[0,t]}) = X_t.$$

Basic Propoerties

Let B_t, $t \geq 0$, be a pre-Brownian motion. For a choice $0 = t_0 < t_1 < \ldots < t_n$, the vector $(B_{t_1}, \ldots, B_{t_n})$ has density

$$p(x_1, \ldots, x_n) = \frac{1}{(2\pi)^{n/2}\sqrt{(t_1 - t_0)\cdots(t_n - t_{n-1})}} \exp\left(-\sum_{i=1}^{n} \frac{(x_i - x_{i-1})^2}{2(t_i - t_{i-1})}\right)$$

Proposition

1) $-B_t$ is also a pre-Brownian motion
2) For every $\lambda > 0$, $B_t^{\lambda} = \frac{1}{\lambda} B_{\lambda^2 t}$ is a pre-Brownian motion
3) For every $s \geq 0$, $B_t^{(s)} = B_{s+t} - B_s$ is a pre-Brownian motion and is independent of $\sigma(B_r, r \leq s)$.