# Day 2 : Stochastic Processes

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# Today's Reading

[B] Chapter 1.3. Stochastic Processes, Continuity criterion

# Stochastic Processes

Let $\Sigma$ be an arbitrary set. A *stochastic process* is a family

$$X = \{X(t, \omega), t \in \Sigma\}$$

of random variables depending on some parameter $t$.

- $\Sigma \subset [0, \infty)$ a subinterval :
  $t$ is the *time* and $X$ is a *continuous time process*
- $\Sigma \subset \mathbb{R}^k$ :
  $X$ is a *multiparameter process*
- $\Sigma = \mathbb{N}$ :
  $X$ is a *stochastic sequence*.

# Examples of Stochastic Processes

- Bernoulli process : a sequence of i.i.d. random variables

$$X_i = \begin{cases} 1 & \text{with probability } p \\ 0 & \text{otherwise.} \end{cases}$$

- Simple Random Walk : For i.i.d. binary random variables $X_1, X_2, \ldots$, a sequence of random variables

$$S_n = X_1 + X_2 + \cdots + X_n.$$

- Brownian motion
- Poisson process

# Sample Paths

For a given sample point $\omega \in \Omega$, the mapping $t \mapsto X(t, \omega)$ is called a *sample path* of the process $X(t)$, $t \in \Sigma$.

# Equivalence of Stochastic Processes

Two stochastic processes $X(t)$ and $Y(t)$, $t \in \Sigma$, defined on the same probability space, are said to be *stochastically equivalent* (or *modifications* of each other) if

$$\mathbf{P}(X(t) = Y(t)) = 1 \quad \text{for all } t \in \Sigma.$$

Two stochastic processes $X(t)$ and $Y(t)$ with the same condition are said to be *indistinguishable* (or *equivalent*) if there exists a set $\Lambda \in \mathcal{F}$ such that $\mathbf{P}(\Lambda) = 0$ and

$$X(t, \omega) = Y(t, \omega) \quad \text{for all } t \in \Sigma \text{ and } \omega \in \Omega \setminus \Lambda.$$

# Equivalence of Stochastic Processes

*Example.*
$\Omega = [0, 1]$
$\mathcal{F} = \mathcal{B}([0, 1])$
$\mathbf{P}$ : the Lebesgue measure
$\Sigma = [0, \infty)$.
Define

$$X_t(\omega) = 0 \quad \text{for all } \omega$$
$$Y_t(\omega) = \begin{cases} 1 & \text{if } t - \lfloor t \rfloor \neq \omega \\ 0 & \text{otherwise.} \end{cases}$$

Then $X_t$ and $Y_t$ are stochastically equivalent (modifications of each other) but not equivalent.

# Stochastically Continuity

A stochastic process $X(t)$, $t \in [a, b]$, is called *measurable* if the mapping $(t, \omega) \to X(t, \omega)$ is $\mathcal{B}([a, b]) \times \mathcal{F}$-measurable.

## Definition

A process $X(t)$, $t \in [a, b]$, is said to be *stochastically continuous* (or *continuous in probability*) if for any $t \in [a, b]$ and $\epsilon > 0$,

$$\lim_{s \to t} \mathbf{P}(|X(s) - X(t)| > \epsilon) = 0,$$

and *continuous in mean square* if for every $t \in [a, b]$,

$$\lim_{s \to t} \mathbf{E}|X(s) - X(t)|^2 = 0.$$

# Finite-dimensional Distributions

Given a stochastic process $X$, there is associated the family of finite-dimensional distributions

$$\mathcal{P}_{t_1,t_2,\ldots,t_n}(\Delta_1 \times \Delta_2 \times \cdots \times \Delta_n) \\ = \mathbf{P}(X(t_1) \in \Delta_1, X(t_2) \in \Delta_2, \ldots, X(t_n) \in \Delta_n), \quad \Delta_k \in \mathcal{B}(\mathbb{R})$$

for all $t_k \in \Sigma$, $k = 1, \ldots, n$.

The distribution satisfies the following conditions:

1) Symmetry :
   $\mathcal{P}_{t_1,t_2,\ldots,t_n}(\Delta_1 \times \Delta_2 \times \cdots \times \Delta_n) = \mathcal{P}_{t_{l_1},t_{l_2},\ldots,t_{l_n}}(\Delta_{l_1} \times \Delta_{l_2} \times \cdots \times \Delta_{l_n})$
   for every permutation $\{l_1, \cdots, l_n\}$ of $\{1, \ldots, n\}$.
2) Consistency :

$$\mathcal{P}_{t_1,\ldots,t_{k-1},t_k,t_{k+1},\ldots,t_n}(\Delta_1 \times \cdots \times \Delta_{k-1} \times \mathbb{R} \times \Delta_{k+1} \times \cdots \times \Delta_n) \\ = \mathcal{P}_{t_1,\ldots,t_{k-1},t_{k+1},\ldots,t_n}(\Delta_1 \times \cdots \times \Delta_{k-1} \times \Delta_{k+1} \times \cdots \times \Delta_n)$$

   for every $1 \leq k \leq n$.

# Finite-dimensional Distributions

Let $\mathcal{P}_{t_1,\dots,t_n}(\Delta_1 \times \cdots \times \Delta_n)$ be a family of finite-dimensional distributions satisfying the symmetry and consistency.

**Theorem**

There exists a probability space $(\Omega, \mathcal{F}, \mathbf{P})$ and a process $X(t), t \in \Sigma$, defined on this space such that its family of finite-dimensional distributions coincide with the given one.

# Kolmogorov Continuity Criterion

**Theorem**

For a process $X(t)$, $t \in [a, b]$, assume that there exist positive constants $\alpha, \beta$, and $M$ such that

$$\mathbf{E}|X(t) - X(s)|^{\alpha} \leq M|t - s|^{1+\beta} \quad \text{for all } s, t \in [a, b].$$

Then the process $X$ has a continuous modification $\widetilde{X}$.

*Proof.* See [B] Theorem 3.2.