Day 12 : Review 2

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Brownian Motion

Problem 1. [Exercise 2.26]
For every real $a \geq 0$, we set $T_a = \inf\{t \geq 0 : B_t = a\}$. Show that the process $(T_a)_{a\geq 0}$ has stationary independent increments, in the sense that, for every $0 \leq a \leq b$, the variable $T_b - T_a$ is independent of the σ-field $\sigma(T_c, 0 \leq c \leq a)$ and has the same distribution as T_{b-a}.

Brownian Motion

Problem 2. [Exercise 2.30]
Let $H := \{t \in [0,1] : B_t = 0\}$. Using a proposition from Lecture 7 (p.15) and the strong Markov property, show that H is a.s. a compact subset of $[0,1]$ with no isolated point and zero Lebesgue measure.

Martingales

Problem 3. [Exercise 3.27]
Let B be an $(\mathcal{F}_t)$-Brownian motion started from 0. Recall the notation $T_x = \inf\{t \geq 0 : B_t = x\}$ for every $x \in \mathbb{R}$. We fix two real numbers a and b with $a < 0 < b$, and we set

$$T = T_a \wedge T_b.$$

1) Show that, for every $\lambda > 0$,

$$\mathbf{E}[\exp(-\lambda T)] = \frac{\cosh(\frac{b+a}{2}\sqrt{2\lambda})}{\cosh(\frac{b-a}{2}\sqrt{2\lambda})}.$$

(Hint: One may consider a martingale of the form

$$M_t = \exp(\sqrt{2\lambda}(B_t - \alpha) - \lambda t) + \exp(-\sqrt{2\lambda}(B_t - \alpha) - \lambda t)$$

with a suitable choice of α.

Martingales

2) Show similarly that, for every $\lambda > 0$,

$$\mathbf{E}[\exp(-\lambda T)\mathbb{1}_{\{T=T_a\}}] = \frac{\sinh(b\sqrt{2\lambda}}{\sinh((b-a)\sqrt{2\lambda}}.$$

3) Recover the formula for $\mathbf{P}(T_a < T_b)$.

Continuous Semimartingales

Problem 4. [Exercise 4.26]
1) Let A be an increasing process (adapted, with continuous sample paths and such that $A_0 = 0$) such that $A_\infty < \infty$ a.s., and let Z be an integrable random variable. We assume that, for every stopping time T,

$$\mathbf{E}[A_\infty - A_T] \leq \mathbf{E}[Z \mathbb{1}_{\{T<\infty\}}].$$

Show, by introducing an appropriate stopping time, that, for every $\lambda > 0$,

$$\mathbf{E}[(A_\infty - \lambda)\mathbb{1}_{\{A_\infty>\lambda\}}] \leq \mathbf{E}[Z \mathbb{1}_{\{A_\infty>\lambda\}}].$$