Day 7 : Brownian Motions

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Today's Reading

[L] Chapter 2.2-2.3

Review: Sample Paths

Let E be a metric space equipped with its Borel σ-field.

Let (X_t), $t \in T$, be a stochastic process with values in E. The *sample paths* of X are the mappings $t \mapsto X_t(\omega)$ for a fixed $\omega \in \Omega$.

Two stochastic processes X_t and Y_t are *modifications* to each other if

$$\mathbf{P}(X_t = Y_t) = 1.$$

Review: Kolmogorov Continuity Criterion

Theorem (Kolmogorov Continuity Criterion)

For a process $X(t)$, $t \in [a, b]$, assume that there exist positive constants q, ϵ, and C such that

$$\mathbf{E}|X(t) - X(s)|^q \leq C|t-s|^{1+\epsilon} \quad \text{for all } s, t \in [a, b].$$

Then the process X has a continuous modification $\widetilde{X}$.

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Then the process X has a continuous modification $\widetilde{X}$. Moreover, there is a modification whose sample paths are α-Hőlder continuous for $\alpha \in (0, \epsilon/q)$, that is, for each sample path ω, there exists a constant $C_\alpha(\omega)$ such that

$$|\tilde{X}_t(\omega), \tilde{X}_s(\omega)| \leq C_\alpha(\omega)|t - s|^\alpha.$$

Review: Pre-Brownian Motions

If G is a Gaussian white noise on $\mathbb{R}^+$ with intensity its Lebesgue measure, the random process $(B_t)_{t\in(0,\infty)}$ defined by

$$B_t = G(\mathbb{1}_{[0,t]})$$

is a *pre-Brownian motion*.

The covariance is

$$\mathbf{E}[B_s B_t] = \mathbf{E}[G([0,s])G([0,t])] = \int_0^\infty \mathbb{1}_{[0,s]}\mathbb{1}_{[0,t]}(r)dr = \min(s,t).$$

Review: Characterizations of pre-Brownian motions

Proposition

Let $X(t)$, $t \geq 0$, be a real-valued stochastic process. The followings are equivalent:

1) $X(t)$, $t \geq 0$, is a pre-Brownian motion
2) $X(t)$, $t \geq 0$, is a centered Gaussian process with covariance $K(s,t) = \min(s,t)$
3) $X(0) = 0$ a.s., and for every $0 \leq s < t$, the random variable $X(t) - X(s)$ is independent of $\sigma(X(r), r \leq s)$ and distributed according to $N(0, t-s)$.
4) $X(0) = 0$ a.s., and for every $0 = t_0 < t_1 < \ldots < t_p$, the variables $X_{t_i} - X_{t_{i-1}}$, $1 \leq i \leq p$, are independent, and are distributed according to $N(0, t_i - t_{i-1})$.

Brownian Motions

A stochastic process (B_t), $t \geq 0$, is a *Brownian motion* if

1) (B_t), $t \geq 0$, is a pre-Brownian motion.
2) All sample paths of B are continuous.

Brownian Motions

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So, does Brownian motion exist?

Modifications of pre-Brownian motions

If (B_t), $t \geq 0$, is a pre-Brownian motion, then (B_t) satisfies the Kolmogorov Continuity Criterion for $q > 2$ and $\epsilon = \frac{q}{2} - 1$. Let $X \sim N(0,1)$, then $B_t - B_s = \sqrt{t-s}X$ for any $s < t$. Therefore,

$$\mathbf{E}[|B_t - B_s|^q] = (t-s)^{q/2}\mathbf{E}[|X|^q] < C_q(t-s)^{q/2}.$$

Hence, each pre-Brownian motion has a modification whose sample paths are continuous.

Basic Propoerties

Proposition

1) $-B_t$ is also a Brownian motion
2) For every $\lambda > 0$, $B_t^\lambda = \dfrac{1}{\lambda} B_{\lambda^2 t}$ is a Brownian motion
3) For every $s \geq 0$, $B_t^{(s)} = B_{s+t} - B_s$ is a Brownian motion and is independent of $\sigma(B_r, r \leq s)$.

It turns out that Brownian motion is not unique!

The Wiener measure

$C(\mathbb{R}^+, \mathbb{R})$: the space of all continuous functions $f : \mathbb{R}^+ \to \mathbb{R}$ equipped with the σ-algebra $\mathcal{C}$ generated by coordinate mappings $\omega \mapsto \omega(t)$.

Given a Brownian motion B, consider the mapping

$$\begin{aligned} \Omega &\to C(\mathbb{R}^+, \mathbb{R}) \\ \omega &\mapsto (t \mapsto B_t(\omega)). \end{aligned}$$

The *Wiener measure* $W(d\omega)$ is defined as the image of the probability measure $\mathbf{P}(d\omega)$.

The Wiener measure

For $0 = t_0 < t_1 < \cdots < t_n$ and $A_0, A_1, \ldots, A_n \in \mathcal{B}(\mathbb{R})$, define a cylinder set of the form

$$A = \{\omega \in C(\mathbb{R}^+, \mathbb{R}) : \omega(t_0) \in A_0, \omega(t_1) \in A_1, \ldots, \omega(t_n) \in A_n\}.$$

Then A is measurable and

$$\begin{aligned} &W(\{\omega : \omega(t_0) \in A_0, \omega(t_1) \in A_1, \ldots, \omega(t_n) \in A_n\}) \\ &\quad = \mathbf{P}(B_{t_0} \in A_9, B_{t_1} \in A_1, \ldots, B_{t_n} \in A_n) \\ &\quad = \mathbb{1}_{A_0}(0) \int_{A_1 \times \cdots \times A_n} \frac{dx_1 \cdots dx_n}{(2\pi)^{n/2} \sqrt{t_1(t_2 - t_1) \cdots (t_n - t_{n-1})}} \\ &\qquad\qquad \cdot \exp\left(- \sum_{i=1}^{n} \frac{(x_i - x_{i-1})^2}{2(t_i - t_{i-1})} \right) \end{aligned}$$

The probability is independent of the choice of Brownian motions.

Canonical process

Consider a special probability space

$$\Omega = C(\mathbb{R}^+, \mathbb{R}), \quad \mathcal{F} = \mathcal{C}, \quad \mathbf{P}(d\omega) = W(d\omega).$$

Then the *canonical process*

$$X_t(\omega) = \omega(t)$$

is a Brownian motion. This is a *canonical construction* of Brownian motion.

Blumenthal's zero-one law

Let (B_t), $t \geq 0$, be a Brownian motion. Define σ-algebras

$$\mathcal{F}_t = \sigma(B_s, s \leq t)$$

and let

$$\mathcal{F}_{0+} = \bigcap_{s>0} \mathcal{F}_s.$$

Then $\mathcal{F}_{0+}$ is trivial. That is, for any $A \in \mathcal{F}_{0+}$,

$$P(A) = 0 \text{ or } 1.$$

Blumenthal's zero-one law

Theorem

The σ-algebra $\mathcal{F}_{0+}$ is trivial. That is, for any $A \in \mathcal{F}_{0+}$, $P(A) = 0$ or 1.

Proof. Fix time steps $0 < t_1 < t_2 < \cdots < t_k$. Let $g \colon \mathbb{R}^k \to \mathbb{R}$ be a bounded continuous function. Fix $A \in \mathcal{F}_{0+}$. Then for $\epsilon < t_1$,

$$\begin{aligned}
\mathbf{E}[\mathbb{1}_A g(B_{t_1}, \ldots, B_{t_k})] &= \lim_{\epsilon \to 0+} \mathbf{E}[\mathbb{1}_A g(B_{t_1} - B_\epsilon, \ldots, B_{t_k} - B_\epsilon)] \\
&= \lim_{\epsilon \to 0+} \mathbf{P}(A)\mathbf{E}[g(B_{t_1} - B_\epsilon, \ldots, B_{t_k} - B_\epsilon)] \\
&= \mathbf{P}(A)E[g(B_{t_1}, \ldots, B_{t_k}].
\end{aligned}$$

Therefore, $\mathcal{F}_{0+}$ is independent of $\sigma(B_{t_1}, \ldots, B_{t_k})$. Since this is true for arbitrary $t_1 < t_2 < \cdots < t_k$, we conclude that $\mathcal{F}_{0+}$ is independent of $\sigma(B_t, t > 0) = \sigma(B_t, t \geq 0)$. Hence, $\mathcal{F}_{0+} \subset \sigma(B_t, t \geq 0)$ is independent of itself, which gives the theorem.

Stopping time and Brownian motion

Proposition

1) For every $\epsilon > 0$,

$$\sup_{0 \le s \le \epsilon} B_s > 0, \quad \inf_{0 \le s \le \epsilon} B_s < 0, \quad \text{a.s.}$$

2) For every $a \in \mathbb{R}$, let the stopping time $T_a = \inf\{t \ge 0 : B_t = a\}$. Then

$$T_a < \infty, \quad \text{a.s.}$$

Therefore,

$$\limsup_{t \to \infty} B_t = +\infty, \quad \liminf_{t \to \infty} B_t = -\infty, \quad \text{a.s.}$$

Corollary

For any nontrivial interval $I \subset \mathbb{R}$, $t \mapsto B_t$ is not monotone on I, a.s.

Infinite variation

The function $t \mapsto B_t$ has infinite variation, a.s.