Due 23:59 Nov 30 (Sunday). There are 100 points in this assignment.
Submit your answers **(must be typed)** in pdf file to CourSys
`https://coursys.sfu.ca/2025fa-cmpt-705-x1/`.
Submissions received after 23:59 will get penalty of reducing points: 20 and 50 points deductions for submissions received at $[00:00, 00:10]$ and $(00:10, 00:30]$ of Dec 1, respectively; no points will be given to submissions after $00:30$ of Dec 1.

1. (Chapter 10 Problem 1 of the text book) 15 points

   The Hitting Set problem is defined as follows: Let $A = \{a_1, .., a_n\}$ and $B_1, .., B_m$ be a collection of subsets of A, a subset H of A is a hitting set for $B_1, .., B_m$ is $H \cap B_i \neq \emptyset$ for every $1 \leq i \leq m$. The Hitting Set problem is that given A and $B_1, .., B_m$, and integer $k > 0$, whether there is a hitting set of size k or not. The problem is NP-complete. Assume that $|B_i| \leq c$ for some constant $c > 0$. Give an algorithm that solves the problem with a running time of $O(f(c, k) \cdot p(n, m))$, where $p(n, m)$ is a polynomial in n and m, and $f(c, k)$ is an arbitrary function depending only on c and k, not on n or m, and analyze the algorithm.

2. Chapter 10 Problem 8 of the text book) 15 points

   Consider the class of 3-SAT instances in which each clause has exactly three literals and each of the n variables appears (counting x and $\bar{x}$) in exactly three clauses. Prove that every such 3-SAT instance is satisfiable and a satisfying assignment can be found in polynomial time.

3. (Chapter 13 Problem 4 of text book) 10 points

   A peer-to-peer network (digraph) G is constructed as follows:
   $V(G) = \{v_1\}$;
   for $i = 2$ to n select randomly with same probability a node v_j in G, $V(G) = V(G) \cup \{v_i\}$ and $E(G) = E(G) \cup \{(v_i, v_j)\}$;

   Give a formula on the expected indegree of each node v_j in terms n and j for $1 \leq j \leq n$.

4. (Chapter 13 Problem 7 of the text book) 10 points

   Given a set of clauses $C_1, .., C_k$ of Boolean variables $X = \{x_1, .., x_n\}$ with each clause has at least one literal and the literals in single literal clauses are distinct, the MAX SAT problem is to find a truth assignment satisfying as many clause as possible. A randomized algorithm for the MAX SAT is to assign each x_i independently to 0 or 1 with probability 1/2. Show that the expected number of clauses satisfied by this algorithm is at least $k/2$. Give an example to show that there are MAX SAT instances such that no assignment satisfies more than $k/2$ clauses.

5. (Chapter 13 Problem 10 of text book) 10 points

   There are n bidding agents; agent i has a distinct integer bid $b_i > 0$. The bidding agents appear in an order chosen uniformly at random, each proposes its bid b_i in turn, and at all times the system maintains a variable b^* equal to the highest bid seen so far. (Initially b^* is set to 0.) What is the expected number of times that b^* is updated when this process is executed, as a function of n?

6. (Chapter 13 Problem 11 of the text book) 15 points

   There are k machines and k jobs. Each job is assigned to one of the k machines independently at random (with each machine equally likely).

   (a) Let $N(k)$ be the expected number of machines that do not receive any jobs, so that $N(k)/k$ is the expected fraction of machines with no job. What is the limit $\lim_{k\to\infty} N(k)/k$? Give a proof of your answer.

   (b) Suppose that machines are not able to queue up excess jobs, so if the random assignment of jobs to machines sends more than one job to a machine M, then M will do the first of the jobs it receives and reject the rest. Let $R(k)$ be the expected number of rejected jobs; so $R(k)/k$ is the expected fraction of rejected jobs. What is $\lim_{k\to\infty} R(k)/k$? Give a proof of your answer.

   (c) Now assume that machines have slightly larger buffers; each machine M will do the first two jobs it receives, and reject any additional jobs. Let $R_2(k)$ denote the expected number of rejected jobs under this rule. What is $\lim_{k\to\infty} R_2(k)/k$? Give a proof of your answer.

7. 15 points

   Given a graph G, we consider the following problem: color the nodes of G using $k > 1$ colors; an edge $e = \{u, v\}$ is called satisfied if u and v are colored by different colors; and we want to color the nodes to satisfy as many edges as possible. A randomized algorithm RA for the maximization problem is as follows: for every node v of G, select one of the k colors independently with probability $1/k$ and assign the color to v.

   (a) Prove that the expected number of edges satisfied by RA is $(1-1/k)m$, where m is the number of edges in G. Assume $(1-1/k)m$ is an integer, prove that the probability that RA satisfies at least $(1-1/k)m$ edges is at least $1/m$.

   (b) Give a Monte Carlo algorithm which satisfies at least $(1-1/k)m$ edges with probability at least $1-1/m$. Give a Las Vegas algorithm which satisfies at least $(1-1/k)m$ edges.

8. 10 points

   Derandomize the randomized algorithm RA in the previous question to get an $O(km)$ time deterministic $(1-1/k)$-approximation algorithm for the maximization problem for k colors and a graph G of n nodes and m edges, and analyze the algorithm.