An analytic height fog model for real-time rendering

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The Earth's atmosphere is often expressed by Rayleigh scattering and Mie scattering, and the participating media with exponentially decreasing air density with altitude[NSTN93, BN08, EK10]. In this document we present that the volume rendering equation[NGHJ18] is solved analytically when imposing further restrictions: the attenuation direction is parallel to an axis and the scale height is constant.

1 The volume rendering equation

Let us start with the radiative transfer equation [Cha60] with the context of [NGHJ18]:

$$(\vec{\omega} \cdot \nabla)L(\vec{x}, \vec{\omega}) = -\mu_t(\vec{x})L(\vec{x}, \vec{\omega}) + \mu_a(\vec{x})L_e(\vec{x}, \vec{\omega}) + \mu_s(\vec{x})L_s(\vec{x}, \vec{\omega}). \tag{1}$$

Here $L(\vec{x}, \vec{\omega})$ expresses the radiance of a light ray at position $\vec{x} = (x, y, z)$ and direction $\vec{\omega} = (\cos\phi \sin\theta, \sin\phi \sin\theta, \cos\theta)$. The absorption coefficient μ_a, scattering coeffient μ_s and total extinction coefficient $\mu_t = \mu_a + \mu_s$ describes the optical property of the participating media. L_e is the emitted radiance and L_s is the in-scattered radiance. The in-scattered radiance L_s collects the incident radiance L_i from all directions:

$$L_s(\vec{x}, \vec{\omega}) = \oint p(\vec{\omega}, \vec{\omega}')L_i(\vec{x}, \vec{\omega}')d\Omega(\vec{\omega}') \tag{2}$$

where $d\Omega(\vec{\omega})$ is the differential solid angle, and the integral $\oint$ is defined over a unit sphere: $\oint d\Omega(\vec{\omega}) = \int_0^{2\pi} \int_0^{\pi} \sin\theta d\theta d\phi$.

Next, we assume that the light ray travels in the media along the direction $\vec{\omega}$. Then we can parametrize $\vec{x}$ as $\vec{x}(s) = \vec{x}(0) + \vec{\omega}s$, and the radiative transfer equation (1) can be written as

$$\frac{dL\left(\vec{x}(s), \vec{\omega}\right)}{ds} = -\mu_t\left(\vec{x}(s)\right) L\left(\vec{x}(s), \vec{\omega}\right) + \mu_a\left(\vec{x}(s)\right) L_e\left(\vec{x}(s), \vec{\omega}\right) + \mu_s\left(\vec{x}(s)\right) L_s\left(\vec{x}(s), \vec{\omega}\right), \tag{3}$$

where we used the chain rule to rewrite the directional derivative [1].

Since this is a first-order linear differential equation, we can solve it immediately [2]:

$$\begin{aligned} L(\vec{x}(s), \vec{\omega}) = {} & T(\vec{x}(0) \to \vec{x}(s))L\left(\vec{x}(0), \vec{\omega}\right) \\ & + \int_0^s T(\vec{x}(s') \to \vec{x}(s)) \left[\mu_a\left(\vec{x}(s)\right) L_e\left(\vec{x}(s), \vec{\omega}\right) + \mu_s\left(\vec{x}(s')\right) L_s\left(\vec{x}(s'), \vec{\omega}\right)\right] ds', \end{aligned} \tag{4}$$

[1] $\left(\vec{\omega} \cdot \nabla\right) L\left(\vec{x}, \vec{\omega}\right) = \left(\frac{d\vec{x}(s)}{ds} \cdot \nabla\right) L\left(\vec{x}(s), \vec{\omega}\right) = \frac{dL\left(\vec{x}(s), \vec{\omega}\right)}{ds}$

[2] The general solution to the first order linear differential equation

$$\frac{dy(x)}{dx} + p(x)y(x) = q(x)$$

is given by

$$y(x) = ce^{-\int^x p(\xi)d\xi} + \int^x q(\zeta)e^{-\int_\zeta^x p(\xi)d\xi}d\zeta$$

where

$$T(\vec{x}(a) \to \vec{x}(b)) = \exp\left(-\int_a^b \mu_t\left(\vec{x}(t)\right) dt\right). \tag{5}$$

(4) is called volume rendering equation, and (5) is called transmittance.

2 The height fog model

2.1 Assumptions

Atmospheric scattering

We consider the conditions for the volume rendering equation (4) to simply express the Earth's atmosphere. First, we assume that there is no self-emission

$$L_e\left(\vec{x}\right) = 0. \tag{6}$$

Next, the participating media decays exponentially with altitude[NSTN93, BN08, EK10]

$$\mu_s\left(\vec{x}\right) = \beta_{sR} e^{-\frac{h(\vec{x})}{H_R}} + \beta_{sM} e^{-\frac{h(\vec{x})}{H_M}} = \sum_j^{R,M} \beta_{sj} e^{-\frac{h(\vec{x})}{H_j}}, \tag{7}$$

where $h(\vec{x})$ is the height in the vertical direction, H is called scale height, R and M are Rayleigh and Mie scattering component. This assumption applies to μ_a and μ_t as well:

$$\mu_a\left(\vec{x}\right) = \sum_j^{R,M} \beta_{aj} e^{-\frac{h(\vec{x})}{H_j}}, \qquad \mu_t\left(\vec{x}\right) = \sum_j^{R,M} \beta_{tj} e^{-\frac{h(\vec{x})}{H_j}}. \tag{8}$$

Height fog

In order to obtain an analytical solution of the volume rendering equation (4), we impose further restrictions on the above assumptions.

- The participating media decays along the y direction

$$h\left(\vec{x}(s)\right) = y(s). \tag{9}$$

- The scale height is constant regardless of the participating medium

$$H_R = H_M = H. \tag{10}$$

- There are two types of light source, directional light L_{sun} and ambient light L_{amb}

$$\begin{aligned} L_s\left(\vec{x}(s), \vec{\omega}\right) &= \oint p\left(\vec{\omega}, \vec{\omega}'\right) L_i\left(\vec{x}(s), \vec{\omega}'\right) d\Omega\left(\vec{\omega}'\right) \\ &\to \oint \left(p(\vec{\omega}, \vec{\omega}') L_{\text{sun}}\, \delta(\vec{\omega}' - \vec{\omega}_{\text{sun}}) + \frac{1}{4\pi} L_{\text{amb}}\right) d\Omega\left(\vec{\omega}'\right) \\ &= p(\vec{\omega}, \vec{\omega}_{\text{sun}}) L_{\text{sun}} + L_{\text{amb}}, \end{aligned} \tag{11}$$

where we have assumed that the phase function for L_{amb} is $p = \frac{1}{4\pi}$, and the delta function $\delta(\vec{n} - \vec{m})$ with the normal vector represents $\delta(\vec{n} - \vec{m}) = \frac{1}{\sin\theta_n}\delta(\theta_n - \theta_m)\delta(\phi_n - \phi_m)$.

2.2 Height fog model

By imposing conditions (6, 7, 8, 9, 10, 11) on the volume rendering equation (4), we obtain

$$\begin{aligned} L\left(\vec{x}(s), \vec{\omega}\right) &= T(\vec{x}(0) \to \vec{x}(s)) L\left(\vec{x}(0), \vec{\omega}\right) \\ &\quad + \int_0^s T(\vec{x}(s') \to \vec{x}(s)) \sum_j^{R,M} \beta_{sj} e^{-\frac{y(s')}{H}} \left(p_j(\vec{\omega}, \vec{\omega}_{\mathrm{sun}}) L_{\mathrm{sun}} + L_{\mathrm{amb}}\right) ds', \end{aligned} \tag{12}$$

$$T(\vec{x}(a) \to \vec{x}(b)) = \exp\left(-\int_a^b \sum_j^{R,M} \beta_{tj} e^{-\frac{y(t)}{H}} dt\right). \tag{13}$$

This model (12) is the equation we are going to solve. Hereafter we call it as height fog model.

2.3 Analytic solution

Transmittance

Let us first solve the transmittance (13) analytically:

$$\begin{aligned} T(\vec{x}(a) \to \vec{x}(b)) &= \exp\left(-\int_a^b \sum_j^{R,M} \beta_{tj} e^{-\frac{y(t)}{H}} dt\right) \\ &= \exp\left(-e^{-\frac{1}{H} y(0)} \sum_j^{R,M} \beta_{tj} \int_a^b e^{-\frac{\omega_y t}{H}} dt\right) \\ &= \exp\left[-\frac{H}{\omega_y} \left(\sum_j^{R,M} \beta_{tj}\right) e^{-\frac{1}{H} y(0)} \left(e^{-\frac{1}{H}\omega_y a} - e^{-\frac{1}{H}\omega_y b}\right)\right], \end{aligned} \tag{14}$$

where we have used $y(s) = y(0) + \omega_y s$.

In-scatter term

Next, we solve the second term of the height fog model (12). We write the term as $L_{\mathrm{inscatter}}$ for convenience sake:

$$L_{\mathrm{inscatter}}\left(\vec{x}(s), \vec{\omega}\right) = \int_0^s T(\vec{x}(s') \to \vec{x}(s)) \sum_j^{R,M} \beta_{sj} e^{-\frac{y(s')}{H}} \left(p_j(\vec{\omega}, \vec{\omega}_{\mathrm{sun}}) L_{\mathrm{sun}} + L_{\mathrm{amb}}\right) ds'. \tag{15}$$

Before solving $L_{\mathrm{inscatter}}$, we rewrite the transmittance (14) to a simpler form:

$$\begin{aligned} T(\vec{x}(s') \to \vec{x}(s)) &= \exp\left[-\frac{H}{\omega_y} \left(\sum_j^{R,M} \beta_{tj}\right) e^{-\frac{1}{H} y(0)} \left(e^{-\frac{1}{H}\omega_y s'} - e^{-\frac{1}{H}\omega_y s}\right)\right] \\ &= e^{-\frac{B}{A}\left(e^{-As'} - e^{-As}\right)}, \end{aligned} \tag{16}$$

where

$$A = \frac{\omega_y}{H}, \quad B = \left(\sum_j^{R,M} \beta_{tj}\right) e^{-\frac{1}{H} y(0)}. \tag{17}$$

Also, we keep the light sources in l_j:

$$l_j = p_j(\vec{\omega}, \vec{\omega}_{\text{sun}})L_{\text{sun}} + L_{\text{amb}}. \tag{18}$$

Substituting (16) (18) into $L_{\text{inscatter}}$ (15), we obtain

$$\begin{aligned} L_{\text{inscatter}}\left(\vec{x}(s), \vec{\omega}\right) &= \left(\sum_j^{R, M} \beta_{sj} l_j\right) \int_0^s e^{-\frac{B}{A}\left(e^{-As'} - e^{-As}\right)} e^{-\frac{y(s')}{H}} ds' \\ &= \left(\sum_j^{R, M} \beta_{sj} l_j\right) e^{\frac{B}{A}e^{-As}} e^{-\frac{y(0)}{H}} \int_0^s e^{-\frac{B}{A}e^{-As'}} e^{-As'} ds', \end{aligned} \tag{19}$$

where we have used $y(s') = y(0) + \omega_y s'$. Here, since the integral can be solved as

$$\int e^{-\frac{B}{A}e^{-Ax}} e^{-Ax} dx = \frac{1}{B} e^{-\frac{B}{A}e^{-Ax}},$$

thus we obtain the analytic solution by applying it to the above equation:

$$\begin{aligned} L_{\text{inscatter}}\left(\vec{x}(s), \vec{\omega}\right) &= \left(\sum_j^{R, M} \beta_{sj} l_j\right) e^{\frac{B}{A}e^{-As}} e^{-\frac{y(0)}{H}} \left(\frac{1}{B} e^{-\frac{B}{A}e^{-As}} - \frac{1}{B} e^{-\frac{B}{A}}\right) \\ &= \left(\sum_j^{R, M} \beta_{sj} l_j\right) e^{-\frac{y(0)}{H}} \frac{1}{B} \left(1 - e^{-\frac{B}{A}\left(1 - e^{-As}\right)}\right). \end{aligned} \tag{20}$$

Writing back (16) (17) (18) to the above equation, we obtain the analytic solution of $L_{\text{inscatter}}$:

$$L_{\text{inscatter}}\left(\vec{x}(s), \vec{\omega}\right) = \frac{\sum_j^{R, M} \beta_{sj} \left(p_j(\vec{\omega}, \vec{\omega}_{\text{sun}})L_{\text{sun}} + L_{\text{amb}}\right)}{\sum_j^{R, M} \beta_{tj}} \left(1 - T(\vec{x}(0) \to \vec{x}(s))\right). \tag{21}$$

Analytic solution

Finally, we substitute the analytic solution of the transmittance (14) and the in-scatter term (21) into the height fog model (12). Doing so we obtain

$$\begin{aligned} L\left(\vec{x}(s), \vec{\omega}\right) = {} & L\left(\vec{x}(0), \vec{\omega}\right) T(\vec{x}(0) \to \vec{x}(s)) \\ & + \frac{\sum_j^{R, M} \beta_{sj} \left(p_j(\vec{\omega}, \vec{\omega}_{\text{sun}})L_{\text{sun}} + L_{\text{amb}}\right)}{\sum_j^{R, M} \beta_{tj}} \left(1 - T(\vec{x}(0) \to \vec{x}(s))\right), \end{aligned} \tag{22}$$

where

$$T(\vec{x}(0) \to \vec{x}(s)) = \exp\left[-\frac{H}{\omega_y}\left(\sum_j^{R, M} \beta_{tj}\right) e^{-\frac{1}{H}y(0)} \left(1 - e^{-\frac{\omega_y}{H}s}\right)\right]. \tag{23}$$

(22) is the analytic solution of the height fog model [3].

[3] We can get an expression of the same form as [dCI17] by rewriting as follows:

$$L_{\text{background}} = L\left(\vec{x}(0), \vec{\omega}\right), \qquad \theta_{sv} = \cos^{-1}\left(\vec{\omega} \cdot \vec{\omega}_{\text{sun}}\right),$$

$$\alpha = T(\vec{x}(0) \to \vec{x}(s)) = \exp\left[-\frac{H}{y(s) - y(0)}\left(\sum_j^{R, M} \beta_{tj}\right) e^{-\frac{1}{H}y(0)} \left(1 - e^{-\frac{1}{H}(y(s) - y(0))}\right) s\right],$$

where we have used $\omega_y = \frac{y(s) - y(0)}{|\vec{x}(s) - \vec{x}(0)|}$, $s = |\vec{x}(s) - \vec{x}(0)|$.

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