Project 3: Helmholtz Resonators (Bottle Lab)

In this project, we will look at the phenomenon of Helmholtz resonance: the tendency of a bottle (or any cavity with a small opening) to vibrate when air passes across the opening, producing sound. Specifically, you will be measuring the frequency of the sound produced by a bottle when you blow on it, and then record how this frequency changes as you increase or decrease the amount of water inside the bottle. Finally, you will compare your results to the theoretical model to determine its predictive power.

There are a few reasons why we chose this lab for QEA3:

- This is a practical example of how to use the **FFT** (which we will study soon) to analyze the frequency content of time series data (in this case a sound signal).
- The Helmholtz resonator model is an application of **linearization** and **second-order systems** that can generate surprisingly accurate predictions.
- The additional materials (besides your laptop and MATLAB) needed for this experiment are pretty cheap: a bottle, a food scale, a ruler, some water, and something to pour the water.

We begin with a mathematical model for the system followed by a description of the experimental procedure and a walk-through of how to analyze the data you collected. The final section of this document lists the required deliverables (plots, calculations and discussion points) you must include in your lab report.

3.1 System Model

Consider the bottle illustrated on the next page. This bottle consists of a neck (a narrow opening) which leads into a larger cavity/chamber. In a Helmholtz resonator, the air inside the neck acts as an oscillating mass. As this air moves back and forth within the neck, it pushes and pulls on the air inside the cavity, causing the air inside the cavity to contract and expand. This change in volume results in a change in the cavity pressure that resists the motion of air in the neck, effectively acting as a spring.

By treating the air in the neck and cavity as single lumped elements we can derive a nonlinear second-order differential equation that describes the motion of the system. Linearizing this model will allow us to compute an effective "mass" and "stiffness" for the system, which can then be used to calculate the resonant frequency. In deriving this model, we will need to account for the following system parameters and variables:

- The length, L, and cross-sectional area, A of the bottle neck.
- The the atmospheric pressure, P_0, and density, ρ, of air (both assumed to be constant).
- The heat capacity ratio, γ of air at STP, which is approximately $\gamma \approx 7/5$.
- The pressure, $P(t)$, and volume, $V(t)$, of the air inside the cavity. This air expands and contracts as it's being displaced by the air inside the neck of the bottle, which causes both its pressure and volume to change as a function of time. When the system is at rest, $P(t) = P_0$ (the ambient pressure), and $V(t) = V_0$ (the static cavity volume). V_0 will vary with how much water is in the bottle.
- The translational displacement of the air inside the neck, $\Delta x(t)$, and the resulting volume displacement, $\Delta V(t)$ of the air inside the cavity.

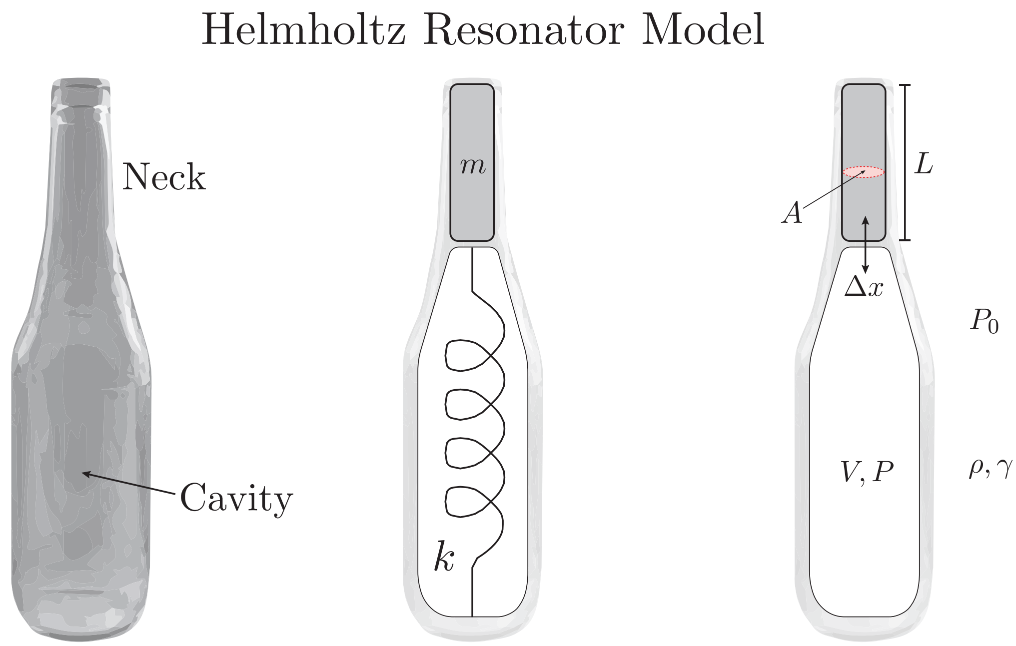

Figure 3.1: **Left:** The bottle consists of a neck and a cavity. **Center:** We will model the bottle as a lumped second-order system: the air inside the neck is the mass and the air inside the cavity acts as a spring. **Right:** the parameters and variables we will need to consider when modeling the system.

The mass of air in the neck is equal to the product of its density and volume:

$$m = \text{Density} \cdot \text{Volume} = \rho V_{neck} \tag{3.1}$$

Assuming the neck is prismatic, its volume is the product of its length and cross-sectional area:

$$V_{neck} = \text{Length} \cdot \text{Area} = LA \tag{3.2}$$

Plugging in, we see that the mass of air inside the neck is given by:

$$m = \rho LA \tag{3.3}$$

When the air inside the neck translates by an amount, $\Delta x(t)$, it displaces a volume inside the cavity that is proportional to the cross-sectional area of the neck:

$$\Delta V(t) = A\Delta x(t) \tag{3.4}$$

Note the sign convention: a positive value of Δx correspond to the neck air moving outside of the bottle, which results in an expansion of the cavity air (thus the positive value of ΔV). Let V_0 be the volume of the air inside the cavity when the system is at rest (i.e. $\Delta x = 0$, meaning that no cavity air is being displaced), in other words the static cavity volume. From this, we see that the volume of air inside the cavity is given by:

$$V(t) = V_0 + \Delta V(t) = V_0 + A\Delta x(t) \tag{3.5}$$

If we assume that the vibrations of the neck mass occur too quickly for a significant amount of heat to transfer between the cavity and the outside through the walls of the bottle (i.e. the expansion/compression of the cavity is an adiabatic process), then the pressure and volume inside the cavity satisfy:

$$P(t)V(t)^\gamma = \text{constant} = c \tag{3.6}$$

When the system is at rest, no air is being displaced, meaning that $V = V_0$, and the cavity pressure is equal to the atmospheric pressure: $P(t) = P_0$. Plugging these into the previous equation, we get:

$$P_0 V_0^\gamma = c, \qquad \rightarrow \qquad P(t)V(t)^\gamma = P_0 V_0^\gamma \tag{3.7}$$

We can rearrange this equation to find an expression of pressure of the cavity air as a function of its volume:

$$P(t) = P_0 \left(\frac{V_0}{V(t)}\right)^\gamma \tag{3.8}$$

Substituting in $V_0 + A\Delta x(t)$ for $V(t)$ gives us:

$$P(t) = P_0 \left(\frac{V_0}{V_0 + A\Delta x(t)}\right)^\gamma = P_0 \left(\frac{V_0 + A\Delta x(t)}{V_0}\right)^{-\gamma} = P_0 \left(1 + \frac{A}{V_0}\Delta x(t)\right)^{-\gamma} \tag{3.9}$$

This equation relates the pressure of the air inside the cavity as a function of the displacement of air inside the neck. To convert this into an equation of motion, we need to construct the free-body diagram for the air inside the neck:

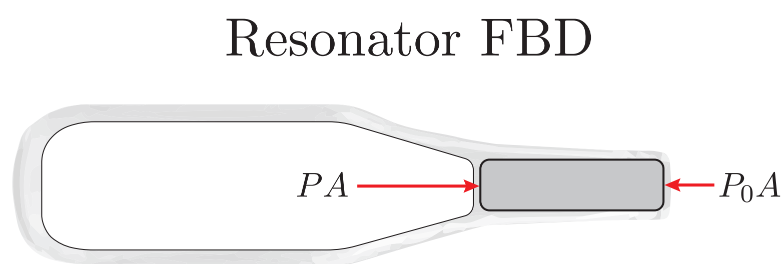

Figure 3.2: Free-body diagram of a Helmholtz Resonator. We need to consider two forces: the force of the air inside the cavity, $P(t)A$, and the force of the atmosphere, P_0A.

Your first task is to finish this derivation in order to compute the resonant frequency of the system. To accomplish this, apply Newton's second law to generate the nonlinear second-order differential equation for the system. Then, linearize this system w/respect to the equilibrium state $\Delta x = 0$. The linearized differential equation should be an undamped second-order system of the form:

$$m\Delta\ddot{x} + k\Delta x = 0 \tag{3.10}$$

where the effective mass, m and stiffness, k, are given by:

$$m = \rho A L, \qquad k = \frac{\gamma A^2 P_0}{V_0} \tag{3.11}$$

From here, the natural/resonant frequency, ω_n is given by:

$$\boxed{\omega_n = \sqrt{\frac{k}{m}}, \qquad m = \rho A L, \qquad k = \frac{\gamma A^2 P_0}{V_0}} \tag{3.12}$$

When air flows across the opening of the bottle, it excites this resonant frequency, producing a sound. Note that ω_n will be inversely proportional to the square root of the static cavity volume, V_0. We can reduce this volume by adding water to the bottle, which should increase the resonant frequency. Given the volume of water inside the bottle, we can predict the frequency of sound that the bottle makes when we blow on it.

3.2 Experimental Procedure

We have provided a script that records a sound using your laptop microphone, determines its primary frequency component using the FFT, and then prints out the primary frequency and plays it back at you through your laptop speakers. The frequency analysis function has some settings that you can tweak in order to filter out any frequencies that are outside of a desired range (to help keep ambient noise from interfering with your experiments). Please read through the script, and test it by humming a sound into the microphone and making sure the script plays the primary frequency component of that sound through the speakers. Once you understand how to use the frequency analysis script, we can begin our data collection process:

1. Empty the bottle completely (either by drinking whatever was in it or pouring it out).
2. Using a ruler and the food scale, measure and record the mass of the bottle, the **inner** diameter of its opening, and the length of its neck.
3. At some point (either at the start or end of the experiment), make sure to measure the mass of the bottle when it is filled to the brim (all the way up to the opening) with water.
4. To collect a single data point, pour some water into the bottle. Measure the combined mass of the bottle+water. Use the frequency analysis script to compute the primary frequency component of the sound that is produced when you blow on the bottle. Record the combined bottle+water mass and the sound frequency in a table.
5. Repeat the previous step, collecting multiple data points (make sure to collect at least $15-20$ data points, but closer to $30-40$ would be even better!). When collecting data, make sure to capture a range of volumes. Try to include a few data points where the water height is past the neck of the bottle (nearly completely full). Additionally, please include a frequency measurement for when the bottle is completely empty (no water).

3.3 Data Analysis

In this section, we will walk you through the steps of how to write a MATLAB script to interpret the data you collected. To begin with, you need to load your mass/frequency data into MATLAB. Assuming that you recorded your measurements in an excel spreadsheet, you can use MATLAB's readmatrix function.

```
%THE VALUES HERE ARE JUST PLACEHOLDERS
%You will need to change them!!!
fpath = 'C:\Users\taylorott\'; %path string (must end with \)
fname = 'bottle_data01.xls'; %spreadsheet file name
sheet_name = 'Sheet1';
cell_range = 'A2:F7';

%concatenate path and filename
file_string = [fpath,fname]

data_mat = readmatrix(file_string,'Sheet',sheet_name,'Range',cell_range)

%pull the lists of masses and frequencies out of
%data_mat, so each can manipulated independently
total_mass_list =
freq_list =
```

Don't accidentally load the column titles of your spreadsheet (if you had any) into your matrix! This can be avoided by making sure you select the correct cell range. Please read the linked documentation for further guidance on how to use the readmatrix function.

Make sure to include any constant system parameters as variables in your script. **Be careful and consistent with your units!** If you find that something is off, there is a good chance the problem is an inconsistency in how you were are your units (ex. you were using grams when it should have been kilograms or meters instead of centimeters). Probably the easiest way to be consistent about your units is to convert everything to SI immediately (keep everything in kilograms, meters, and seconds), but how you manage your units is up to you. Wolfram Alpha is a great online tool for both unit conversion and looking up various parameters like the density of air at STP.

```
water_density = %density of water
air_density = %density of air
gamma_air = %heat capacity ratio of air
P0 = %the ambient (atmospheric) pressure in Needham
neck_diameter = %inner diameter of bottle opening
neck_length = %length of the bottle neck
empty_bottle_mass = %mass of completely empty bottle
filled_bottle_mass = %mass of completely filled bottle
```

Once you have loaded in the measured parameters, you can compute some of relevant geometric parameters that would be hard to measure directly:

```
area_cx = %cross-sectional area of bottle neck
neck_volume = %volume of bottle neck
neck_air_mass = %mass of air in bottle neck

bottle_volume = %total volume that the bottle can hold
empty_cavity_volume = %volume of the cavity (without any water)
```

We now need to use the list of measured total mass (bottle+water) to infer the corresponding total volume of air (in both the cavity **and** neck) for each amount of water that was poured in the bottle:

```
water_mass_list = %mass of water inside the bottle (list)
water_volume_list = %volume of water inside the bottle (list)

%total volume of air inside the bottle (cavity AND neck)
%for different amounts of water (list)
air_volume_list =
```

At this point, you should plot the frequency as a function of the total volume of air inside the bottle:

```
max_volume = max(air_volume_list);
max_freq = max(freq_list);

figure;
hold on;
axis([0,1.1*max_volume,0,1.1*max_freq]);
plot(air_volume_list,freq_list,'ro','markerfacecolor','r');

%remember to include axis labels and a title!!!
xlabel('');
ylabel('');
title('');
```

Let's compare our measurements to the predictions made by the model. Consider a range of cavity volumes:

```
n_points = %number of points

%generate a range of cavity volumes from empty to full
V0_list = linspace(empty_cavity_volume/100, empty_cavity_volume, n_points);
```

Compute the predicted frequency, $\omega_n = \sqrt{\frac{k}{m}}$ for each of these cavity volumes:

```
%list of predicted stiffnesses for each cavity volume
%remember to use the ./ (elementwise division) when
%doing division, instead of just using / !
predicted_k_list =

%list of predicted frequencies for each cavity volume
predicted_freq_list =

%total volume (cavity + neck) of air inside the bottle
%for each cavity volume
predicted_total_volume =
```

Finally, let's plot the predicted volume vs. frequency curve and compare the results:

```
%plot the predicted values on the same axes as the measured values
plot(predicted_total_volume,predicted_freq_list,'k','linewidth',2);

%please include a legend!
mylegend = legend('','');

%set the legend location to the top-right
legend_loc = 'northeast';
set(mylegend,'location',legend_loc);
```

Qualitatively, how well do the predicted and measured values align with one another?

3.4 Lab Report Deliverables

1. Please include a picture of your bottle in your lab report.
2. Please fill in the following table of values, which should be included in your lab report:

Quantity name	**Value** **(units)**
density of water	
density of air	
heat capacity ratio of air	
ambient pressure	
neck diameter	
neck length	
empty bottle mass	
filled bottle mass	
cross-sectional area of bottle neck	
neck volume	
neck air mass	
bottle volume	
empty cavity volume	

3. Separate from your lab report, make sure to upload a spreadsheet file of the raw data that you collected (mass and frequency measurements), as well as the MATLAB script that you wrote to analyze the data.

4. Please include a summary of your derivation in which you applied Newton's second law to generate the nonlinear differential equation, linearized the system to find the equivalent mass-spring system (for small displacements), and the used the effective mass and stiffness of the linearized system to compute the resonant frequency.

5. Please include a plot comparing the predicted and measured values of the sound frequency as a function of the total volume of air inside the bottle. The plot should have labeled axes (with units), a **descriptive** title, and a legend.

6. Please include a written description of your experimental procedure.

7. Please include a discussion of your analysis of the data. This description should include any relevant equations that you used to compute the system parameters from your measurements.

8. Is the Helmoholtz resonator model accurate? In which regime of air volumes is it good at predicting the resonant frequency? Where does it start to break down?

9. When deriving our model, are there any assumptions that we made about the system that aren't actually true? Come up with at least two potential sources of error that might explain mismatches between the measured and predicted frequencies.