The Math Behind Fields to Flow

QEA 2 Final Teach-In Project

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DIVERGENCE : $\begin{bmatrix}\partial/\partial x\\ \partial/\partial y\end{bmatrix} \cdot \begin{bmatrix}F_x\\ F_y\end{bmatrix} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y}$

dot product

$$\text{div } F = \nabla \cdot F$$

"how fluid flows out or into small regions of a point."

- Divergence is scalar with only magnitude that represents the intensity of a fluid's flow from a source.

POSITIVE DIVERGENCE

NEGATIVE DIVERGENCE

CURL : $\begin{bmatrix}\partial/\partial x\\ \partial/\partial y\end{bmatrix} \times \begin{bmatrix}F_x\\ F_y\end{bmatrix} = \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}$

cross product

$$\text{curl } F = \nabla \times F$$

"how much a fluid rotates around a point."

- Curl is a vector with both a magnitude and direction.
- The magnitude determines how strong the flow field is.
- The direction determines whether the flow field is clockwise or anti-clockwise.

POSITIVE CURL

NEGATIVE CURL

MAXWELL'S EQUATIONS:

E = electrical field B = magnetic field

① Gauss's Law:

$$\text{div } E = \frac{\rho}{\varepsilon_0}$$

ρ: charge density; ε_0: constant: permittivity of free space $= 8.85 \times 10^{-12}$

② $\text{div } B = 0$ incompressible

③ Faraday's Law:

$$\text{curl } E = -\frac{\partial B}{\partial t}$$

④ Ampere-Maxwell's Law:

$$\text{curl } B = \mu_0 \left[J + \varepsilon_0 \left(\frac{\partial E}{\partial t} \right) \right]$$

μ_0: permittivity; J: current density

change of E or B affects curl of other field

NAVIER-STOKES EQUATIONS:

- Assumptions made for the fluid:
  - → newtonian : applying shear force does not affect fluid's viscosity (thickness)
  - → incompressible : pressure on fluid does affect volume
  - → isothermal : as fluid flows, there is no loss/gain of heat

① $\nabla \cdot u = 0$
  - → divergence of a velocity vector field of a fluid.
  - → mass is conserved within the fluid.
  - → impossible for an incompressible fluid to disappear, hence 0.

② newton's 2nd law $ma = \Sigma F$

$$\rho \frac{du}{dt} = -\nabla p + \mu \nabla^2 u + F$$

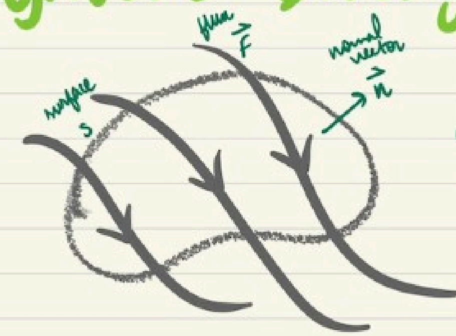
① replace mass with density
② acc = $(u)'$ (velocity's deriv)

aka

$$\rho \cdot \left(\frac{\partial u}{\partial t} + u \cdot \nabla u \right)$$

$-\nabla p$: change of pressure; $\mu \nabla^2 u$: friction; F: external forces aka ρg (gravity)

GAUSS'S DIVERGENCE THEOREM:



$$\iiint_V \vec{\nabla} \cdot \vec{F} \, dV = \iint_S \vec{F} \cdot \vec{n} \, dS$$

divergence of flux over enclosed volume; flux thru closed surface area

As a partial derivative equation (PDE):
"mass conservation of volume"

$$\text{Total mass in } V = \iiint_V \rho \, dV$$

ρ: density

rate of change of mass in volume; how much fluid is leaving thru SA

$$\frac{d}{dt} \iiint_V \rho \, dV = -\iint_S \rho \vec{F} \cdot \vec{n} \, dS$$

div of flux over V

$$= -\iiint_V \vec{\nabla} \cdot (\rho \vec{F}) \, dV$$

CONTINUITY EQUATION:

- Adding a source term (Q) as a supplier of energy.

$$\frac{d}{dt} \iiint_V \rho \, dV = -\iiint_V \vec{\nabla} \cdot (\rho \vec{F}) \, dV + \iiint_V Q \, dV$$

- Mass continuity equation becomes:

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot (\rho \vec{F}) = Q$$

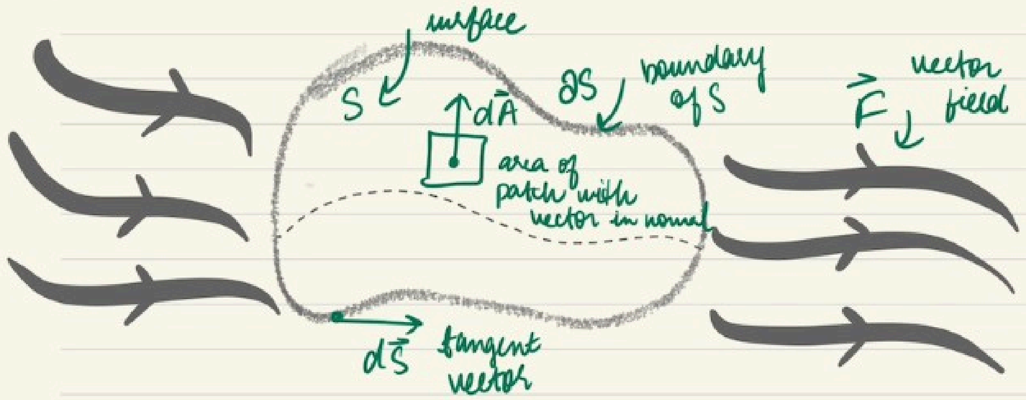
- Mass continuity for incompressible fluids:

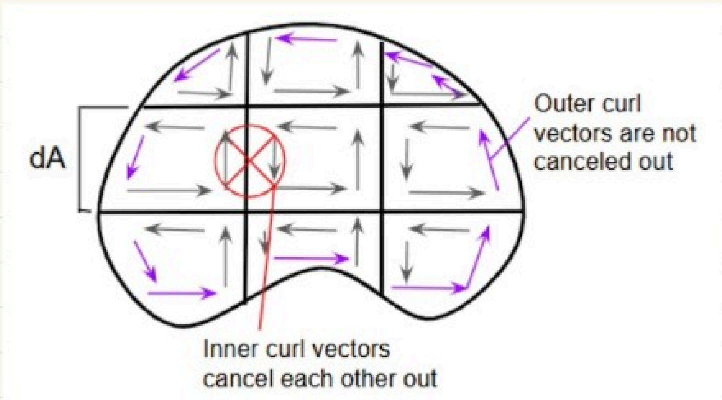
$$\rho \left(\vec{\nabla} \cdot \vec{F} \right) = 0$$

no divergence

3D ⟷ heavy correlation ⟷ 2D

STOKES' THEOREM:





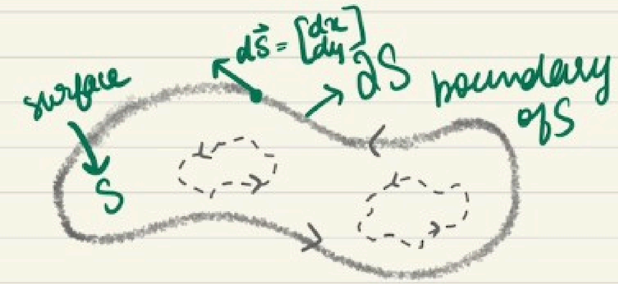
$$\boxed{\iint_S}\ \boxed{(\vec{\nabla} \times \vec{F})}\cdot \boxed{d\vec{A}} = \boxed{\int_{\partial s} \vec{F} \cdot d\vec{s}}$$

Surface integral | Curl | In direction Normal to surface (dot product) | Force vectors tangent to contour

surface integral of the curl of vector field at the patches = tangent vectors around the perimeter of surface defining overall circulation

GREEN'S THEOREM:

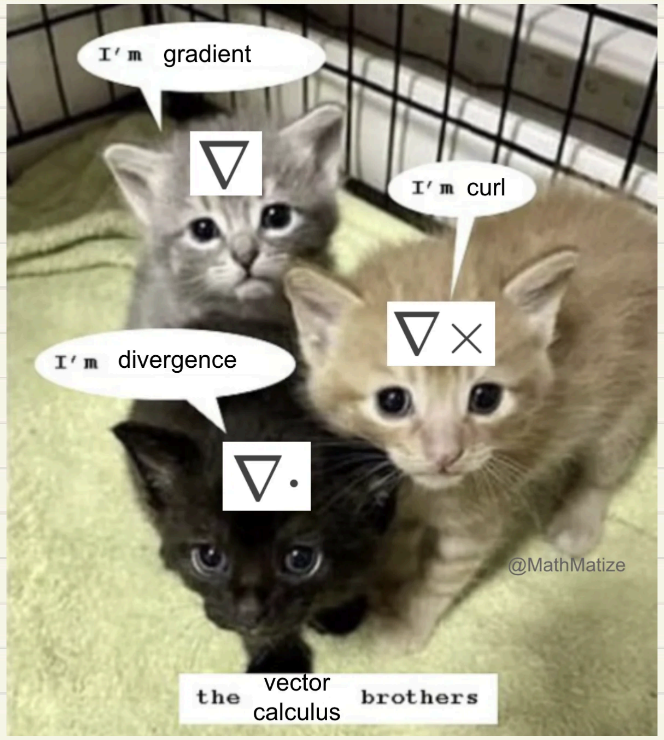
"Stokes theorem, but on a flat surface"



$$\iint_S \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dx\,dy = \int_{\partial S} F_1 dx + F_2 dy$$

all of the curl in the existing vector field

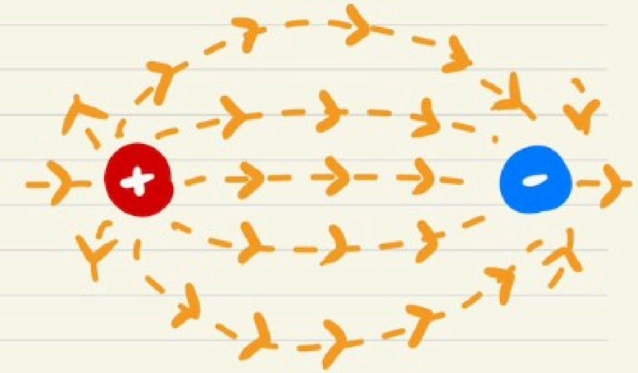
add up all the x & y vectors around the perimeter



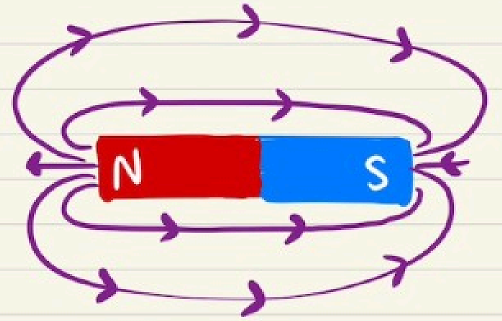
electro-magnetic field

LORENTZ FORCE LAW:

ELECTRIC FIELD (E)

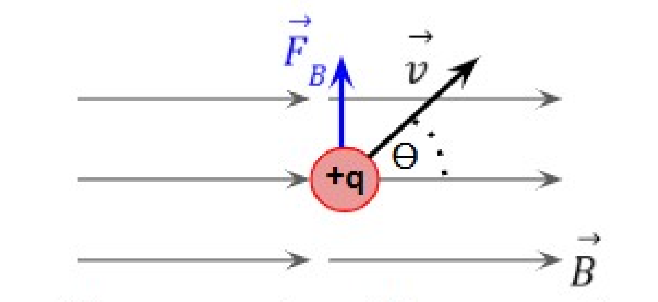


MAGNETIC FIELD (B)



$$\vec{F} = q(\vec{E} + \vec{V} \times \vec{B})$$

$\vec{F}$: force exerted on a charged particle
q: charge
$\vec{E}$: electric field
$\vec{V}$: velocity of charged particle
$\vec{B}$: magnetic field



Charges moving within a magnetic field experience a force due to the magnetic field.