

Historical and Mathematical Overview of Einstein's General Relativity

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Description of Work

Newton's physics was the widely accepted way to interpret movement of bodies, and the action of gravitational force. However, observations showed that his model was flawed. His model claimed that gravity was the result of attractive forces between objects, but this was not the case, and was especially problematic for large-scale computations. Einstein conceptualized the theory of special relativity, which described the relationship between space and time, and described motion in terms of inertial frames of reference. He was able to make corrections to Newton's theory, and a more accurate model of motion when it approaches light speed. This was still a simplified model of reality. In general relativity, Einstein takes gravity into account and shows how gravitational action is actually the warping of space and time caused by massive objects. Einstein used Riemannian and differential geometry to develop his field equations, which describe the theory of general relativity. His theory helps to describe the motion of planetary bodies, black holes, time dilation, and is very important for modern astrophysics.

This work will present the motivation for general relativity, applications of the theory, and the Schwarzschild solution for the path of a photon in the presence of a single massive object.

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2 Einstein's Theory

2.1 Special Relativity

It was shown that neither electromagnetic nor mechanical phenomena could determine a state of absolute rest, and so Einstein reasoned that such a state did not exist. In short, his theory of relativity states that all observers moving at constant speeds, in inertial frames of reference, will witness the same laws of physics. Someone standing on earth and someone in a rocket accelerating at 9.8 m/s will witness the same laws of physics because the forces acting on their frame of reference are the same. In addition, Einstein postulated that the speed of light is identical for all observers. Experiments showed that light always travelled at the same speed, no matter the speed of the observer. [5, p. 136]

Einstein's theory of relativity states that motion is relative to the observer (fig 2) and that an observer in an inertial reference frame cannot, through experimentation, determine their speed (fig 3). Since the speed of light is always constant, speeds cannot add as in Newton's mechanics. Since there is a "universal speed limit," the speed of light, if something is moving at or near the speed of light, we cannot perform the following calculation: $u_{\text{ground}} = c + v$, where u is the speed measured by an observer, and a ball is thrown at the speed of light, c , on a train traveling at speed v . Instead, the correct measurement is: $u_{\text{ground}} = \frac{u_{\text{train}} + v}{1 + vu_{\text{train}}/c^2}$, and the "speed limit" is taken into account. [5, p. 140]

2.2 General Relativity

In the general theory, Einstein allows uniform acceleration of reference frames, and takes gravity into account. [6, p. 114] The curvature of space is, in fact, due to energy and gravitation. Figure 4 illustrates an extension of the special theory, as shown in figure 3. Figure 3 illustrated that one could not test their speed using physics; Figure 4 shows that at constant acceleration, in a constant gravitational field, one cannot use physics to tell the difference between the fields, if they are at the same acceleration.

Einstein's field equations describe the geometry of space-time, which depends on the amount of gravitating matter in the region under investigation. [8, p. 112] His field equation is [8, p. 113].

$$G^{\mu\nu} = R^{\mu\nu} - \frac{1}{2}Rg^{\mu\nu} = -8\pi G/c^4 T^{\mu\nu}$$

Where,

$G^{\mu\nu}$ is the Einstein tensor

$R^{\mu\nu}$ is the Ricci curvature tensor

R is the scalar curvature

$g^{\mu\nu}$ is the metric tensor

G is the gravitational constant

c is the speed of light

$T^{\mu\nu}$ is the stress-energy tensor

The Einstein field equations relates a set of 4×4 tensors, each with 10 individual components. They are a system of 10 coupled, nonlinear, partial differential equations, and thus very difficult to solve.

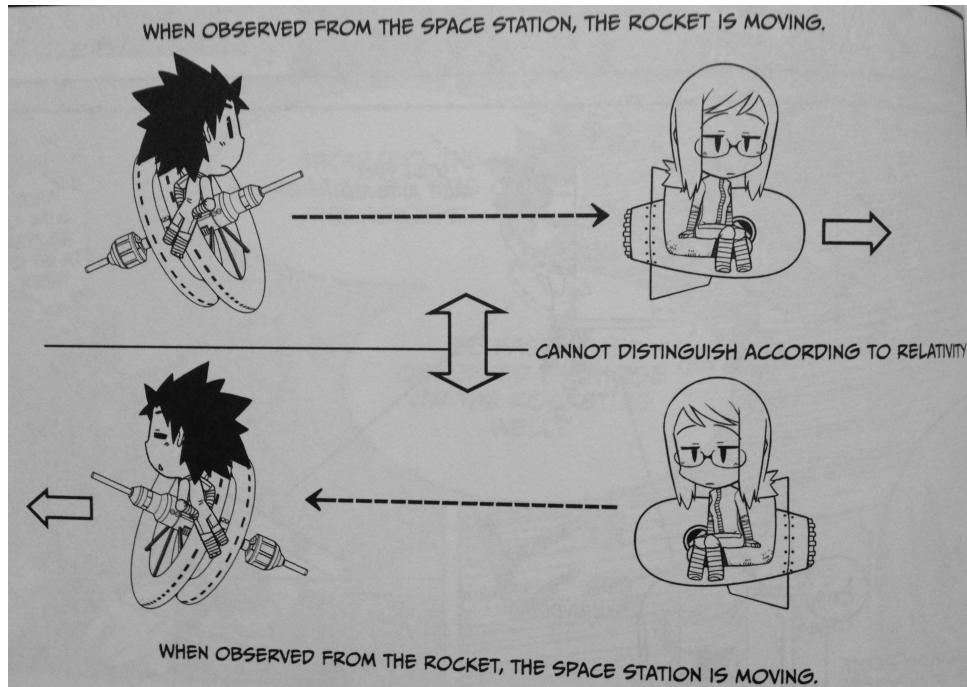


Figure 2: Motion is relative to the observer [4, p. 66]

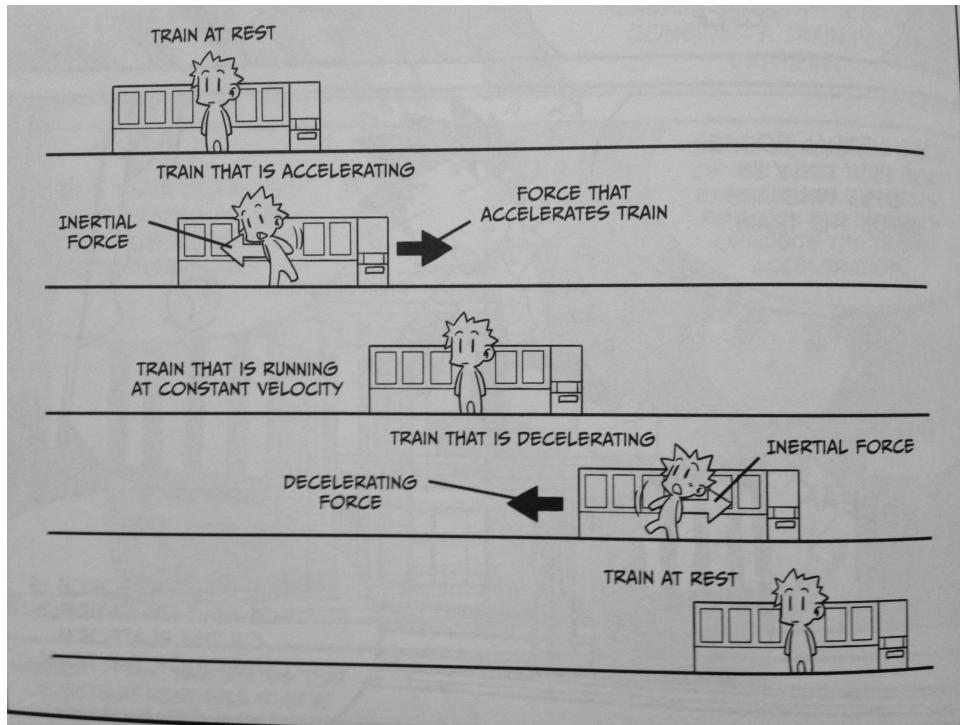


Figure 3: Cannot tell difference between train at rest and at constant velocity [4, p. 124]

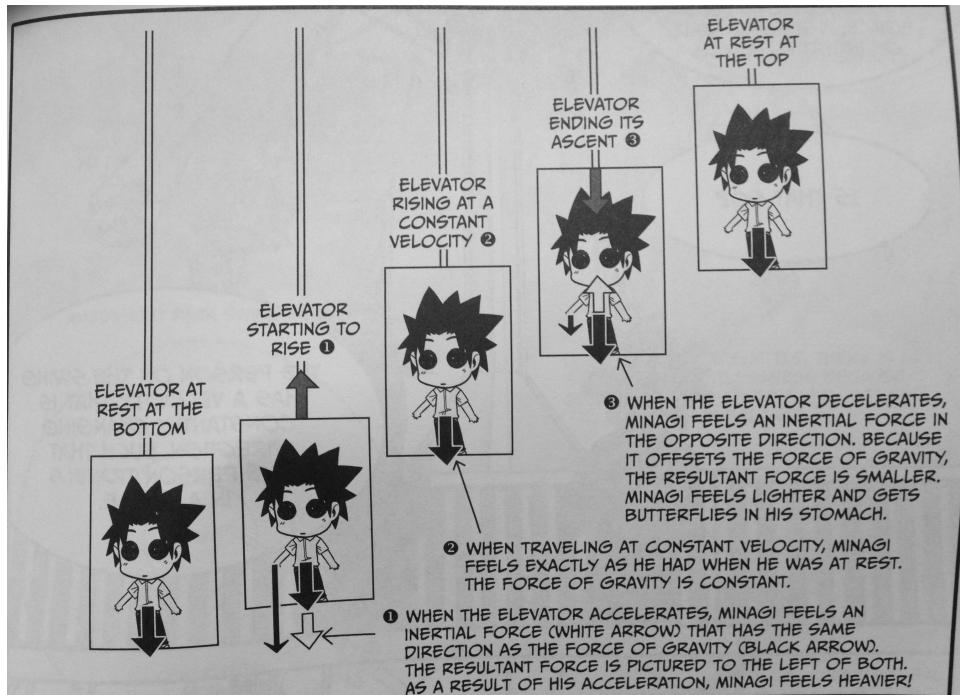


Figure 4: Cannot tell difference between elevator at rest and at constant velocity [4, p. 127]

2.3 Results of general relativity

Mercury's path around the sun differed very greatly and notably from that predicted by Newton's theory. Previously, it was believed that Mercury's shift in perihelion may have been due to the gravitational pull of an extra planet between the sun and Mercury. However, Einstein's theory showed that the orbit of the planets would shift slightly in the sun's gravitational field. Mercury's perihelion shift was most noticeable due to its proximity to the sun.

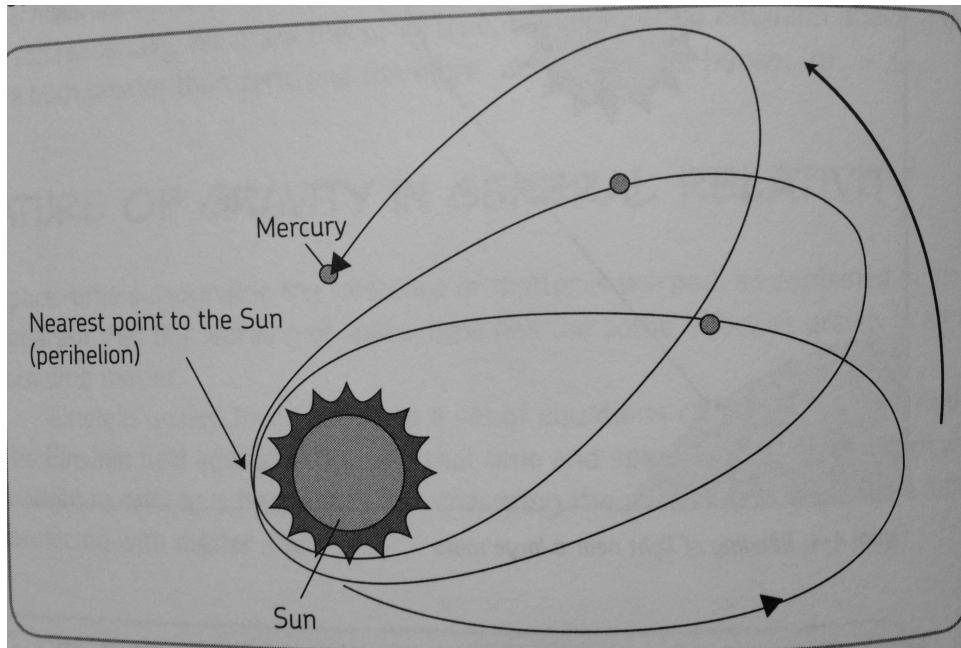


Figure 5: Anomalous shift in perihelion of Mercury [4, p. 164]

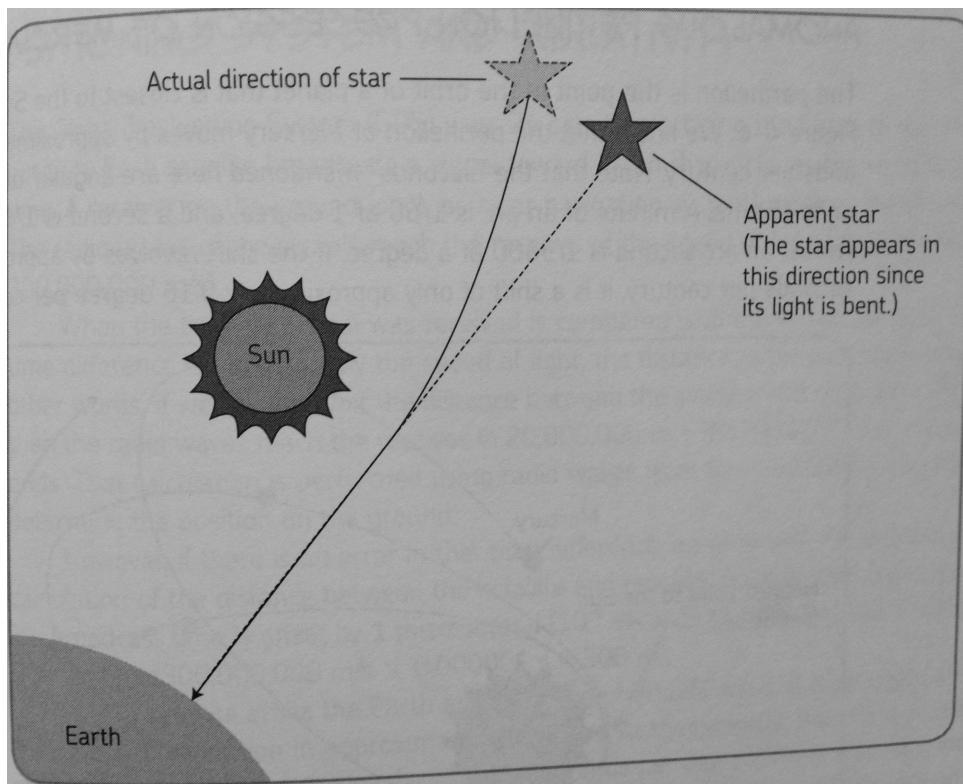


Figure 6: Bending of light near a large mass [4, p. 163]

Though we may observe stars or planets at a given location, due to the effect of the bending of light near massive objects, stars are actually located in other directions.

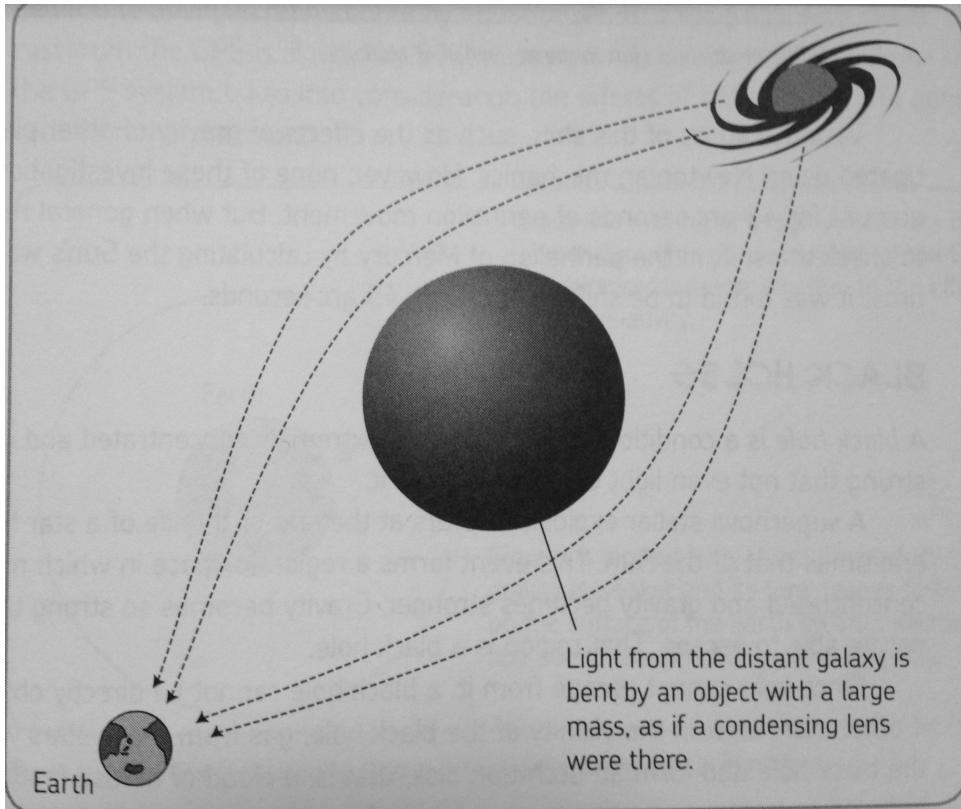


Figure 7: Gravitational lensing [4, p. 163]

Our use of GPS today depends on calculations that derive from general relativity. The GPS satellites are far from the surface of the earth so their experience of time is slightly different. Thus, they have to be calibrated to agree with clocks on the earth.

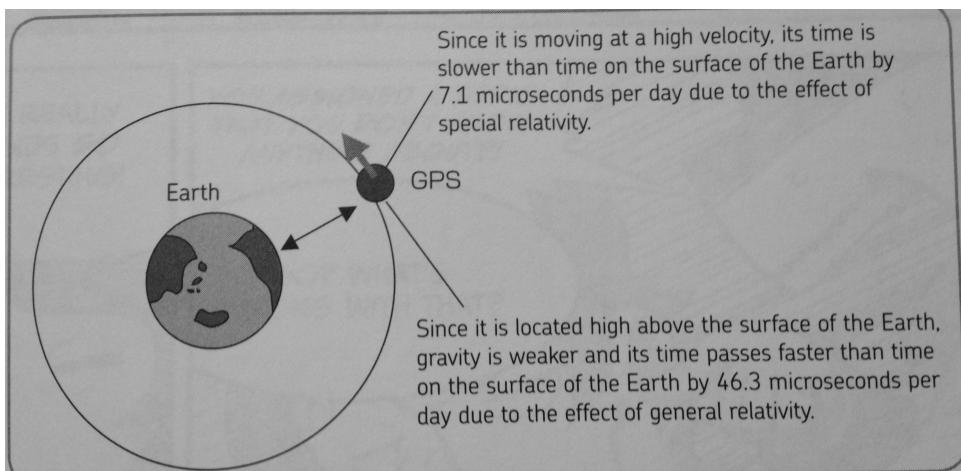


Figure 8: GPS compensates for the effects of general relativity [4, p. 165]

3 Solving for the Schwarzschild Solution

There is no general solution to Einstein's equations, so we must investigate specific cases, in which there is assumed symmetry, and other simplifying assumptions, such as about distribution of matter. [7, p. 137] The first exact solution was obtained by Karl Schwarzschild in 1916. He assumed that the metric tensor field was that for a static, spherically symmetric gravitational field in the empty space-time surrounding a massive spherical object. [8, p. 117] Schwarzschild assumed that: the field was static; that field was spherically symmetric; the space-time was empty; the space-time was asymptotically flat. [8, p. 117] He coordinated by (r, t, θ, φ) , and proposed the metric, which is described below. [8, p. 117]

Applications of the Schwarzschild solution include perihelion advance, the bending of light, time delay in radar sounding, and the geodesic effect. [8, p. 123] We can also apply the Schwarzschild solution to a black hole, in which we have a Schwarzschild radius of a massive body. [8, p. 152] If a star, or test particle, is near enough to a dense star that has a Schwarzschild radius, the star will lose mass, which will flow into the dense star. [7, p. 196] Since the system is rotating, the material will approach the black hole in a spiralic motion. The matter will lose energy as it is pulled closer and closer to the Schwarzschild radius, until it is absorbed into the black hole. [7, p. 196]

3.1 The Schwarzschild metric

$$c^2 d\tau^2 = \left(1 - \frac{r_s}{r}\right) c^2 dt^2 - \left(1 - \frac{r_s}{r}\right)^{-1} dr^2 - r^2(d\theta^2 + \sin^2 \theta d\varphi^2)$$

c^2 : speed of light

τ : proper time, time measured by a clock moving along the world line of the test particle

r_s : Schwarzschild radius of the massive body

r : the radial coordinate

t : time coordinate, measured by a stationary clock infinitely far from the massive body

θ : the colatitude

φ : the longitude

3.2 The metric tensor matrix

$$\begin{pmatrix} g_{11} & g_{12} & g_{13} & g_{14} \\ g_{21} & g_{22} & g_{23} & g_{24} \\ g_{31} & g_{32} & g_{33} & g_{34} \\ g_{41} & g_{42} & g_{43} & g_{44} \end{pmatrix}$$

The metric tensor matrix is used to describe the metric of the system, and is used to describe the geometry of the space.

$$g_{ij}[\mathbf{f}] = g \left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \right)$$

$$\mathbf{f} = \left(\frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}, \frac{\partial}{\partial x^3}, \frac{\partial}{\partial x^4} \right) = \left(\frac{\partial}{\partial t}, \frac{\partial}{\partial r}, \frac{\partial}{\partial \theta}, \frac{\partial}{\partial \varphi} \right)$$

Using the Schwarzschild metric, we arrive at:

$$g \left(\frac{\partial}{\partial t}, \frac{\partial}{\partial t} \right) = \left(1 - \frac{r_s}{r} \right) c^2$$

$$g \left(\frac{\partial}{\partial r}, \frac{\partial}{\partial r} \right) = - \left(1 - \frac{r_s}{r} \right)^{-1}$$

$$g \left(\frac{\partial}{\partial \theta}, \frac{\partial}{\partial \theta} \right) = -r^2$$

$$g \left(\frac{\partial}{\partial \varphi}, \frac{\partial}{\partial \varphi} \right) = -r^2 \sin^2 \theta$$

And all other values for $g_{ij} = \text{zero}$.

Now, the g_{ij} matrix =

$$\begin{pmatrix} \left(1 - \frac{r_s}{r} \right) c^2 & 0 & 0 & 0 \\ 0 & - \left(1 - \frac{r_s}{r} \right)^{-1} & 0 & 0 \\ 0 & 0 & -r^2 & 0 \\ 0 & 0 & 0 & -r^2 \sin^2 \theta \end{pmatrix}$$

And the inverse matrix, $g^{im} =$

$$\begin{pmatrix} \left(\left(1 - \frac{r_s}{r} \right) c^2 \right)^{-1} & 0 & 0 & 0 \\ 0 & -\left(1 - \frac{r_s}{r} \right) & 0 & 0 \\ 0 & 0 & -r^{-2} & 0 \\ 0 & 0 & 0 & \left(-r^2 \sin^2 \theta \right)^{-1} \end{pmatrix}$$

3.3 Christoffel symbols

$$0 = \Gamma_{kl}^i = \frac{1}{2} g^{im} \left(\frac{\partial g_{mk}}{\partial x^l} + \frac{\partial g_{ml}}{\partial x^k} - \frac{\partial g_{kl}}{\partial x^m} \right)$$

example calculation:

Choose $i = 1$, and run through $k,l,m = 1,2,3,4$

$$\Gamma_{11}^1 = \frac{1}{2} g^{1m} \left(\frac{\partial g_{m1}}{\partial t} + \frac{\partial g_{m1}}{\partial t} - \frac{\partial g_{11}}{\partial x^m} \right) = 0$$

$$m = 1 : \frac{1}{2} g^{11} \left(\frac{\partial g_{11}}{\partial t} + \frac{\partial g_{11}}{\partial t} - \frac{\partial g_{11}}{\partial t} \right) = \frac{1}{2} \left(1 - \frac{r_s}{r} \right)^{-1} \left(\frac{\partial \left(1 - \frac{r_s}{r} \right) c^2}{\partial t} + \frac{\partial \left(1 - \frac{r_s}{r} \right) c^2}{\partial t} - \frac{\partial \left(1 - \frac{r_s}{r} \right) c^2}{\partial t} \right)$$

$$m = 2 : \frac{1}{2} g^{12} \left(\frac{\partial g_{21}}{\partial t} + \frac{\partial g_{21}}{\partial t} - \frac{\partial g_{11}}{\partial r} \right) = \frac{1}{2}(0) \left(0 + 0 - \frac{\partial \left(1 - \frac{r_s}{r} \right) c^2}{\partial r} \right) = 0$$

$$m = 3 : \frac{1}{2} g^{13} \left(\frac{\partial g_{31}}{\partial t} + \frac{\partial g_{13}}{\partial t} - \frac{\partial g_{11}}{\partial \theta} \right) = \frac{1}{2}(0) \left(0 + 0 - \frac{\partial \left(1 - \frac{r_s}{r} \right) c^2}{\partial \theta} \right) = 0$$

$$m = 4 : \frac{1}{2} g^{14} \left(\frac{\partial g_{41}}{\partial t} + \frac{\partial g_{14}}{\partial t} - \frac{\partial g_{11}}{\partial \varphi} \right) = \frac{1}{2}(0) \left(0 + 0 - \frac{\partial \left(1 - \frac{r_s}{r} \right) c^2}{\partial \varphi} \right) = 0$$

$$\Gamma_{21}^1 = \Gamma_{12}^1 = \frac{1}{2} g^{1m} \left(\frac{\partial g_{m1}}{\partial t} + \frac{\partial g_{m1}}{\partial r} - \frac{\partial g_{21}}{\partial x^m} \right) = \frac{\frac{r_s}{r}}{2 \left(1 - \frac{r_s}{r} \right)}$$

$$m = 1 : \frac{1}{2} g^{11} \left(\frac{\partial g_{12}}{\partial t} + \frac{\partial g_{11}}{\partial r} - \frac{\partial g_{21}}{\partial t} \right) = \frac{1}{2} \left(1 - \frac{r_s}{r} \right)^{-1} \left(0 + \frac{r_s}{r^2} c^2 - 0 \right) = \frac{\frac{r_s}{r} c^2}{2 \left(1 - \frac{r_s}{r} \right) c^2}$$

$$m = 2 : \frac{1}{2} g^{12} \left(\frac{\partial g_{22}}{\partial t} + \frac{\partial g_{21}}{\partial r} - \frac{\partial g_{21}}{\partial r} \right) = 0$$

$$m = 3 : \frac{1}{2} g^{13} \left(\frac{\partial g_{32}}{\partial t} + \frac{\partial g_{23}}{\partial r} - \frac{\partial g_{21}}{\partial \theta} \right) = 0$$

$$m = 4 : \frac{1}{2} g^{14} \left(\frac{\partial g_{42}}{\partial t} + \frac{\partial g_{24}}{\partial r} - \frac{\partial g_{21}}{\partial \varphi} \right) = 0$$

$$\Gamma_{13}^1 = \Gamma_{31}^1 = \frac{1}{2} g^{1m} \left(\frac{\partial g_{m1}}{\partial \theta} + \frac{\partial g_{m1}}{\partial t} - \frac{\partial g_{13}}{\partial x^m} \right) = 0$$

$$m = 1 : \frac{1}{2} g^{11} \left(\frac{\partial g_{11}}{\partial \theta} + \frac{\partial g_{11}}{\partial t} - \frac{\partial g_{13}}{\partial t} \right) = 0$$

$$m = 2 : \frac{1}{2} g^{12} \left(\frac{\partial g_{21}}{\partial \theta} + \frac{\partial g_{21}}{\partial t} - \frac{\partial g_{13}}{\partial r} \right) = 0$$

$$m = 3 : \frac{1}{2} g^{11} \left(\frac{\partial g_{31}}{\partial \theta} + \frac{\partial g_{13}}{\partial t} - \frac{\partial g_{13}}{\partial \theta} \right) = 0$$

$$m = 4 : \frac{1}{2} g^{14} \left(\frac{\partial g_{41}}{\partial \theta} + \frac{\partial g_{14}}{\partial t} - \frac{\partial g_{13}}{\partial \varphi} \right) = 0$$

$$\begin{aligned}
\Gamma_{14}^1 &= \Gamma_{41}^1 = \frac{1}{2}g^{1m} \left(\frac{\partial g_{m1}}{\partial \varphi} + \frac{\partial g_{m4}}{\partial t} - \frac{\partial g_{41}}{\partial x^m} \right) = 0 \\
m=1 &: \frac{1}{2}g^{11} \left(\frac{\partial g_{11}}{\partial \varphi} + \frac{\partial g_{14}}{\partial t} - \frac{\partial g_{41}}{\partial t} \right) = 0 \\
m=2 &: \frac{1}{2}g^{12} \left(\frac{\partial g_{21}}{\partial \varphi} + \frac{\partial g_{24}}{\partial t} - \frac{\partial g_{41}}{\partial r} \right) = 0 \\
m=3 &: \frac{1}{2}g^{13} \left(\frac{\partial g_{31}}{\partial \varphi} + \frac{\partial g_{34}}{\partial t} - \frac{\partial g_{41}}{\partial \theta} \right) = 0 \\
m=4 &: \frac{1}{2}g^{14} \left(\frac{\partial g_{41}}{\partial \varphi} + \frac{\partial g_{44}}{\partial t} - \frac{\partial g_{41}}{\partial \varphi} \right) = 0
\end{aligned}$$

$$\begin{aligned}
\Gamma_{22}^1 &= \frac{1}{2}g^{1m} \left(\frac{\partial g_{m2}}{\partial r} + \frac{\partial g_{m2}}{\partial r} - \frac{\partial g_{22}}{\partial x^m} \right) = 0 \\
m=1 &: \frac{1}{2}g^{11} \left(\frac{\partial g_{12}}{\partial r} + \frac{\partial g_{12}}{\partial r} - \frac{\partial g_{22}}{\partial t} \right) = 0 \\
m=2 &: \frac{1}{2}g^{12} \left(\frac{\partial g_{22}}{\partial r} + \frac{\partial g_{22}}{\partial r} - \frac{\partial g_{22}}{\partial r} \right) = 0 \\
m=3 &: \frac{1}{2}g^{13} \left(\frac{\partial g_{32}}{\partial r} + \frac{\partial g_{32}}{\partial r} - \frac{\partial g_{22}}{\partial \theta} \right) = 0 \\
m=4 &: \frac{1}{2}g^{14} \left(\frac{\partial g_{42}}{\partial r} + \frac{\partial g_{42}}{\partial r} - \frac{\partial g_{22}}{\partial \varphi} \right) = 0
\end{aligned}$$

$$\begin{aligned}
\Gamma_{23}^1 &= \Gamma_{32}^1 = \frac{1}{2}g^{1m} \left(\frac{\partial g_{m2}}{\partial \theta} + \frac{\partial g_{m3}}{\partial r} - \frac{\partial g_{23}}{\partial x^m} \right) = 0 \\
m = 1 &: \frac{1}{2}g^{11} \left(\frac{\partial g_{12}}{\partial \theta} + \frac{\partial g_{13}}{\partial r} - \frac{\partial g_{23}}{\partial t} \right) = 0 \\
m = 2 &: \frac{1}{2}g^{12} \left(\frac{\partial g_{22}}{\partial \theta} + \frac{\partial g_{23}}{\partial r} - \frac{\partial g_{23}}{\partial r} \right) = 0 \\
m = 3 &: \frac{1}{2}g^{13} \left(\frac{\partial g_{32}}{\partial \theta} + \frac{\partial g_{33}}{\partial r} - \frac{\partial g_{23}}{\partial \theta} \right) = 0 \\
m = 4 &: \frac{1}{2}g^{14} \left(\frac{\partial g_{42}}{\partial \theta} + \frac{\partial g_{43}}{\partial r} - \frac{\partial g_{23}}{\partial \varphi} \right) = 0
\end{aligned}$$

$$\begin{aligned}
\Gamma_{24}^1 &= \Gamma_{42}^1 = \frac{1}{2}g^{1m} \left(\frac{\partial g_{m2}}{\partial \varphi} + \frac{\partial g_{m4}}{\partial r} - \frac{\partial g_{24}}{\partial x^m} \right) = 0 \\
m = 1 &: \frac{1}{2}g^{11} \left(\frac{\partial g_{12}}{\partial \varphi} + \frac{\partial g_{14}}{\partial r} - \frac{\partial g_{24}}{\partial t} \right) = 0 \\
m = 2 &: \frac{1}{2}g^{12} \left(\frac{\partial g_{22}}{\partial \varphi} + \frac{\partial g_{24}}{\partial r} - \frac{\partial g_{24}}{\partial r} \right) = 0 \\
m = 3 &: \frac{1}{2}g^{13} \left(\frac{\partial g_{32}}{\partial \varphi} + \frac{\partial g_{34}}{\partial r} - \frac{\partial g_{24}}{\partial \theta} \right) = 0 \\
m = 4 &: \frac{1}{2}g^{14} \left(\frac{\partial g_{42}}{\partial \varphi} + \frac{\partial g_{44}}{\partial r} - \frac{\partial g_{24}}{\partial \varphi} \right) = 0
\end{aligned}$$

$$\begin{aligned}\Gamma_{33}^1 &= \frac{1}{2}g^{1m} \left(\frac{\partial g_{m3}}{\partial \theta} + \frac{\partial g_{m3}}{\partial \theta} - \frac{\partial g_{33}}{\partial x^m} \right) = 0 \\ m=1 &: \frac{1}{2}g^{11} \left(\frac{\partial g_{13}}{\partial \theta} + \frac{\partial g_{13}}{\partial \theta} - \frac{\partial g_{33}}{\partial t} \right) = 0 \\ m=2 &: \frac{1}{2}g^{12} \left(\frac{\partial g_{23}}{\partial \theta} + \frac{\partial g_{23}}{\partial \theta} - \frac{\partial g_{33}}{\partial r} \right) = 0 \\ m=3 &: \frac{1}{2}g^{13} \left(\frac{\partial g_{33}}{\partial \theta} + \frac{\partial g_{33}}{\partial \theta} - \frac{\partial g_{33}}{\partial \theta} \right) = 0 \\ m=4 &: \frac{1}{2}g^{14} \left(\frac{\partial g_{43}}{\partial \theta} + \frac{\partial g_{43}}{\partial \theta} - \frac{\partial g_{33}}{\partial \varphi} \right) = 0\end{aligned}$$

$$\begin{aligned}\Gamma_{34}^1 = \Gamma_{43}^1 &= \frac{1}{2}g^{1m} \left(\frac{\partial g_{m3}}{\partial \varphi} + \frac{\partial g_{m4}}{\partial \theta} - \frac{\partial g_{34}}{\partial x^m} \right) = 0 \\ m=1 &: \frac{1}{2}g^{11} \left(\frac{\partial g_{13}}{\partial \varphi} + \frac{\partial g_{14}}{\partial \theta} - \frac{\partial g_{34}}{\partial t} \right) = 0 \\ m=2 &: \frac{1}{2}g^{12} \left(\frac{\partial g_{23}}{\partial \varphi} + \frac{\partial g_{24}}{\partial \theta} - \frac{\partial g_{34}}{\partial r} \right) = 0 \\ m=3 &: \frac{1}{2}g^{13} \left(\frac{\partial g_{33}}{\partial \varphi} + \frac{\partial g_{34}}{\partial \theta} - \frac{\partial g_{34}}{\partial \theta} \right) = 0 \\ m=4 &: \frac{1}{2}g^{14} \left(\frac{\partial g_{43}}{\partial \varphi} + \frac{\partial g_{44}}{\partial \theta} - \frac{\partial g_{34}}{\partial \varphi} \right) = 0\end{aligned}$$

$$\begin{aligned}
\Gamma_{44}^1 &= \frac{1}{2}g^{1m} \left(\frac{\partial g_{m4}}{\partial \varphi} + \frac{\partial g_{m4}}{\partial \varphi} - \frac{\partial g_{44}}{\partial x^m} \right) = 0 \\
m=1 &: \frac{1}{2}g^{11} \left(\frac{\partial g_{14}}{\partial \varphi} + \frac{\partial g_{14}}{\partial \varphi} - \frac{\partial g_{44}}{\partial t} \right) = 0 \\
m=2 &: \frac{1}{2}g^{12} \left(\frac{\partial g_{24}}{\partial \varphi} + \frac{\partial g_{24}}{\partial \varphi} - \frac{\partial g_{44}}{\partial r} \right) = 0 \\
m=3 &: \frac{1}{2}g^{13} \left(\frac{\partial g_{34}}{\partial \varphi} + \frac{\partial g_{34}}{\partial \varphi} - \frac{\partial g_{44}}{\partial \theta} \right) = 0 \\
m=4 &: \frac{1}{2}g^{14} \left(\frac{\partial g_{44}}{\partial \varphi} + \frac{\partial g_{44}}{\partial \varphi} - \frac{\partial g_{44}}{\partial \varphi} \right) = 0
\end{aligned}$$

Using the metric tensor above, our Christoffel symbols are:

set i=1, $\Gamma_{kl}^1 =$

$$\begin{bmatrix} & & r_s & \\ & & \frac{r_s}{r^2} & 0 & 0 \\ 0 & & 2(1 - \frac{r_s}{r}) & & \\ & & & & \\ \frac{r_s}{r^2} & & 0 & 0 & 0 \\ & & 2(1 - \frac{r_s}{r}) & & \\ & & 0 & 0 & 0 \\ & & 0 & 0 & 0 \end{bmatrix}$$

set i=2, $\Gamma_{kl}^2 =$

$$\begin{bmatrix} \frac{r_s c^2}{r^2} & 0 & 0 & 0 \\ 0 & -\left(\frac{r_s}{r^2}\right) & 0 & 0 \\ 0 & 0 & -r\left(1-\frac{r_s}{r}\right) & 0 \\ 0 & 0 & 0 & -\left(1-\frac{r_s}{r}\right)(2r \sin^2 \theta) \end{bmatrix}$$

set i=3, $\Gamma_{kl}^3 =$

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{r} & 0 \\ 0 & \frac{1}{r} & 0 & 0 \\ 0 & 0 & 0 & -\sin \theta \cos \theta \end{bmatrix}$$

set i=4, $\Gamma_{kl}^4 =$

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{r} \\ 0 & 0 & 0 & \cot \theta \\ 0 & \frac{1}{r} & \cot \theta & 0 \end{bmatrix}$$

3.4 Geodesic equation

The geodesic equation describes lines of motion in our metric. These are like straight lines as we are used to them on Earth. Geodesic lines determine the lines of shortest distance in curved space. [9, p. 74] The following calculation is used to find the geodesics for the Schwarzschild solution.

$$0 = \frac{\partial^2 u^i}{\partial \tau^2} + \Gamma_{kl}^i \frac{\partial u^l}{\partial \tau} \frac{\partial u^k}{\partial \tau}$$

sample calculation:

pick $i = 1$

$$\begin{aligned} 0 &= \frac{\partial^2 u^1}{\partial \tau^2} + 2\Gamma_{12}^1 \frac{\partial u^1}{\partial \tau} \frac{\partial u^2}{\partial \tau} \\ 0 &= \frac{\partial^2 t}{\partial \tau^2} + 2 \frac{\frac{r_s}{r^2}}{2(1 - \frac{r_s}{r})} \frac{\partial t}{\partial \tau} \frac{\partial r}{\partial \tau} \\ 0 &= \frac{\partial^2 t}{\partial \tau^2} + \frac{\frac{r_s}{r^2}}{(1 - \frac{r_s}{r})} \frac{\partial t}{\partial \tau} \frac{\partial r}{\partial \tau} \end{aligned}$$

This is the geodesic equation for when $i = 1$. Our sum is just $2\Gamma_{12}^1$ since $\Gamma_{12}^1 = \Gamma_{21}^1$ by symmetry of Christoffel symbols of the second kind, and all other values for $\Gamma_{kl}^1 = 0$.

The geodesic equations for the Schwarzschild metric:

$$\begin{aligned} 0 &= \frac{\partial^2 t}{\partial \tau^2} + \frac{w}{v} \frac{\partial t}{\partial \tau} \frac{\partial r}{\partial \tau} \\ 0 &= \frac{\partial^2 r}{\partial \tau^2} + \frac{w c^2}{2v} \left(\frac{\partial t}{\partial \tau} \right)^2 + \frac{-w}{2v} \left(\frac{\partial r}{\partial \tau} \right)^2 + -v(r \sin^2 \theta) \left(\frac{\partial \varphi}{\partial \tau} \right)^2 \\ 0 &= -(\sin \theta \cos \theta) \left(\frac{\partial \varphi}{\partial \tau} \right)^2 \\ 0 &= \frac{\partial^2 \varphi}{\partial \tau^2} + 2 \left(\frac{1}{r} \right) \frac{\partial \varphi}{\partial \tau} \frac{\partial r}{\partial \tau} \end{aligned}$$

$$\text{where } w = \frac{r_s}{r^2} \text{ and } v = \left(1 - \frac{r_s}{r^2} \right)$$

These equations can be solved as a system of ordinary differential equations. θ is set to $\frac{\pi}{2}$

in order to simplify the equations and make them solvable. The metric is used to determine initial conditions.

3.5 Initial conditions

In order to solve a system of ODEs, we need initial conditions for our variables. The variables are r , t , φ . Since the equations to be solved include second derivatives, initial conditions are needed for the initial values and first derivatives at $\tau = 0$.

We choose a value for $r(0) = r_0$, in terms of initial distance of the photon from the Schwarzschild radius, and set

$$t(0) = 0 \text{ and}$$

$$\varphi(0) = 0.$$

To solve for $\frac{\partial r}{\partial \tau}$, the equation of a circle, $r^2 = x^2 + y^2$ is used. Differentiating, we have:

$2r \frac{\partial r}{\partial \tau} = 2x \frac{\partial x}{\partial \tau} + 2y \frac{\partial y}{\partial \tau}$. But we know that the initial distance, x , is the same as our initial radius, r , and initial $y = 0$, since the initial position is along the x -axis. Thus, $\frac{\partial r}{\partial \tau} = \frac{\partial x}{\partial \tau}$.

In terms of coordinates, $\frac{\partial x}{\partial \tau} = v \cos \alpha$, where v = velocity of the photon, α is the angle of initial velocity vector, and $\cos \alpha = \frac{\partial x}{\partial \tau}$. We have

$$\frac{\partial r}{\partial \tau} = v \cos \alpha.$$

To calculate $\frac{\partial \varphi}{\partial \tau}$, we differentiate the relations $\cos \varphi = \frac{x}{r}$ and $\sin \varphi = \frac{y}{r}$. We have

$-\frac{\partial \sin \varphi}{\partial \tau} = \frac{r \frac{\partial x}{\partial \tau} - x \frac{\partial r}{\partial \tau}}{r^2}$ and $\frac{\partial \cos \varphi}{\partial \tau} = \frac{r \frac{\partial y}{\partial \tau} - y \frac{\partial r}{\partial \tau}}{r^2}$ respectively. Using initial values $\varphi = 0$, $x = r$, $y = 0$ tells us that the first equation gives no information and the second equation yields:

$$\frac{\partial \varphi}{\partial \tau} = \frac{\partial y}{\partial r}.$$

The final initial value is solved using the Schwarzschild metric. Rearranging and substituting, we get:

$$\frac{\partial t}{\partial \tau} = \sqrt{\frac{1}{1 - \frac{r_s}{r}} \left(1 + \frac{(\cos \alpha)^2}{r_s} r + \sin^2 \alpha \right)}.$$

Our initial conditions are:

$$r(0) = r_0$$

$$t(0) = 0$$

$$\varphi(0) = 0$$

$$\frac{\partial r}{\partial \tau} = v \cos \alpha$$

$$\frac{\partial \varphi}{\partial \tau} = \frac{\partial y}{\partial r}$$

$$\frac{\partial t}{\partial \tau} = \sqrt{\frac{1}{1 - \frac{r_s}{r}} \left(1 + \frac{(\cos \alpha)^2}{r_s} r + \sin^2 \alpha \right)}$$

3.6 MATLAB code to solve equations

In order to solve the Schwarzschild solution using the MATLAB ODE solver, all equations must be solved in terms of y . We define the variables:

$$r = y_1$$

$$t = y_2$$

$$\varphi = y_3$$

$$\frac{\partial r}{\partial \tau} = dy_1 = y_4$$

$$\frac{\partial t}{\partial \tau} = dy_2 = y_5$$

$$\frac{\partial \varphi}{\partial \tau} = dy_3 = y_6$$

$$\frac{\partial^2 r}{\partial \tau^2} = dy_4 = y_7$$

$$\frac{\partial^2 t}{\partial \tau^2} = dy_5 = y_8$$

$$\frac{\partial^2 \varphi}{\partial \tau^2} = dy_6 = y_9$$

Substituting, the Schwarzschild solution equations are:

$$\begin{aligned} 0 &= y_8 + \frac{w}{v} y_5 y_6 \\ 0 &= y_7 + \frac{w}{2v} c^2 y_5^2 - v(y_1 \sin^2 \theta) y_6^2 \\ 0 &= -(\sin \theta \cos \theta) y_6^2 \\ 0 &= y_9 + \frac{2}{y_1} y_6 y_4 \end{aligned}$$

The third equation gives us no information, and we must solve in terms of y's. So the ordinary differential equations to be solved are:

$$\begin{aligned} y_8 &= -\frac{w}{v} y_5 y_6 \\ y_7 &= -\frac{w}{2v} c^2 y_5^2 + v(y_1 \sin^2 \theta) y_6^2 \\ y_9 &= -\frac{2}{y_1} y_6 y_4 \end{aligned}$$

The ODE was solved using MATLAB, plotting the path of a photon, $(\frac{\partial r}{\partial \tau}, \frac{\partial \varphi}{\partial \tau})$ and the Schwarzschild radius. The code and some generated figures follow:

```

1 function geo
2 %%geodesic equation
3 %venusaur
4 %****set options****
5 options = odeset('RelTol', ...
    1e-3, 'AbsTol', 1e-6, 'NormControl', 'on', 'MaxOrder', 5);
6 %2GM/c^2 = r_s
7 %***make sure to change RS in function below!!***
8 %1.27e10; %Schwarzschild radius (m) radius of Sagittarius A* (SMBH)
9 %2.95e+3; %Shwarzschild radius (m) of sun
10 RS=8.87e-3; %Schwarzschild radius (m) of earth (radius of earth = ...
    6371 m)
```

```

11 v = 299792458; %speed of photon (light) (m/s);
12
13 % ***things to change***
14 r0 = 700000000000000000000000000000000000000000000000000*RS; %initial radius (m)
15 a = 120; %initial angle of photon (deg)
16 time = 10000000000000000000000000000000; %time of plot
17
18 % vector of initial conditions:
19 y0=[r0 0 0 v*cosd(a) sqrt( (1/(1-(RS/r0))) * (1 + ...
    (cosd(a)^2)/(1-(RS/r0)) ...
20     + sind(a)^2)) v*sind(a)/(r0)];
21 %r0, t0, p0, dr/dT (<dx,dy>), dt/dT, dp/dT
22
23 %solve system using 15s
24 [T,y]= ode15s(@defineODE,[0 time],y0,options);
25     r=[y(:,1)];
26     t=[y(:,2)];
27     p=[y(:,3)];
28     dr=[y(:,4)];
29     dt=[y(:,5)];
30     dp=[y(:,6)];
31
32 %plot the solution for r and p as a function of proper time T.
33 polar(p,r,'b'); %graph of photon path
34 hold on
35
36     xc = 0;
37     yc = 0;
38     theta = linspace(0,2*pi);
39     x = RS*cos(theta) + xc;
40     y = RS*sin(theta) + yc;
41     plot(x,y,'r') %graph of schwarzschild radius
42
43 set(0,'DefaultAxesFontSize',15)
44 xlabel('angle')
45 ylabel('radius');
46 title('path of photon in presence of massive body');
47 legend('path of photon','Schwarzschild horizon');
48 grid on;
49
50 hold off
51
52     function dy = defineODE(T,y)

```

```

53 dy = zeros(length(y0),1);
54 %y1=r %y2=t %y3=phi
55 c = 299792458; %speed of light (m/s)
56 %1.27e+10; %Schwarzschild radius (m) radius of Sagittarius A* ...
57 % (SMBH)
58 RS= 8.87e-3; %Schwarzschild radius (m) of sun
59 RS= 8.87e-3; %Schwarzschild radius (m) of earth
60
61 % theta = pi/2
62 w = (RS/(y(1).^2));
63 v = (1 - (RS./y(1)));
64 dy(1) = y(4);
65 dy(2) = y(5);
66 dy(3) = y(6);
67 dy(4) = (-1.* (w./(2.*v))).*( (c).^2).* (y(5)).^2+ ...
68 (w./(2.*v)).*(y(4)).^2-(v.* (y(1).*sin(pi/2).^2).*y(6).^2); % = y(7)
69 dy(5) = (-1.* (w./v)).*y(5).*y(4); % = y(8)
70 dy(6) = (-2/y(1)).*y(6).*y(4); % = y(9)
71 end
72 end

```

3.7 Figures of solution

The following figures were generated by the above MATLAB code, using the Schwarzschild radius of the earth. The caption gives information about initial radius of photon, r_0 , initial angle, α , and observation time, T .

3.7.1 Non-escape examples

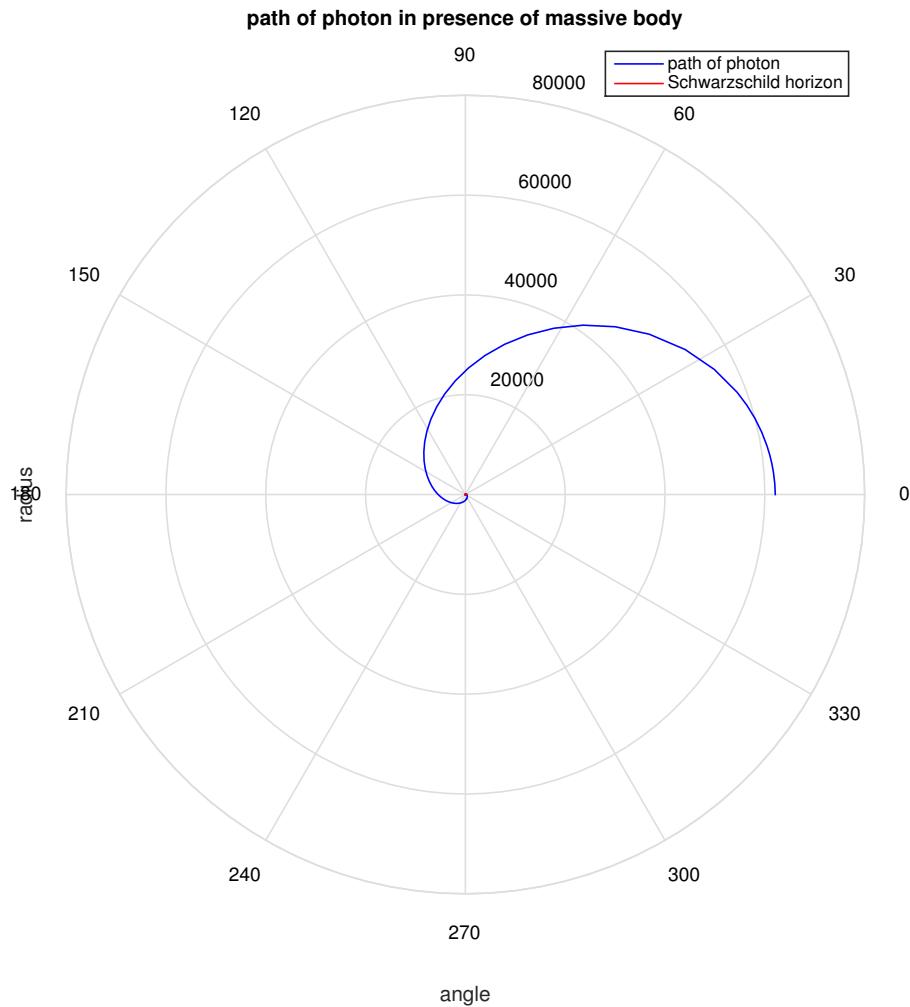


Figure 9: initial radius = $7 \times 10^6 r_s$, initial angle = 90 deg, running time = 1×10^3

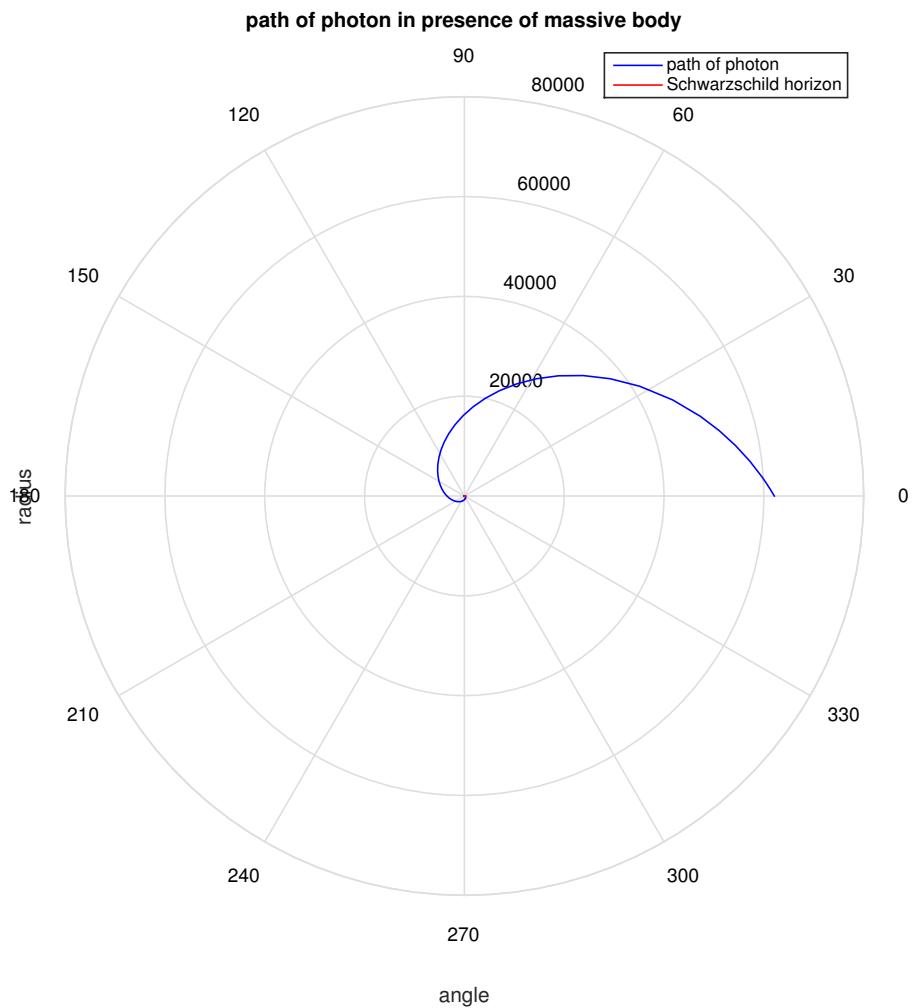


Figure 10: initial radius = $7 \times 10^6 r_s$, initial angle = 120 deg, running time = 1×10^3

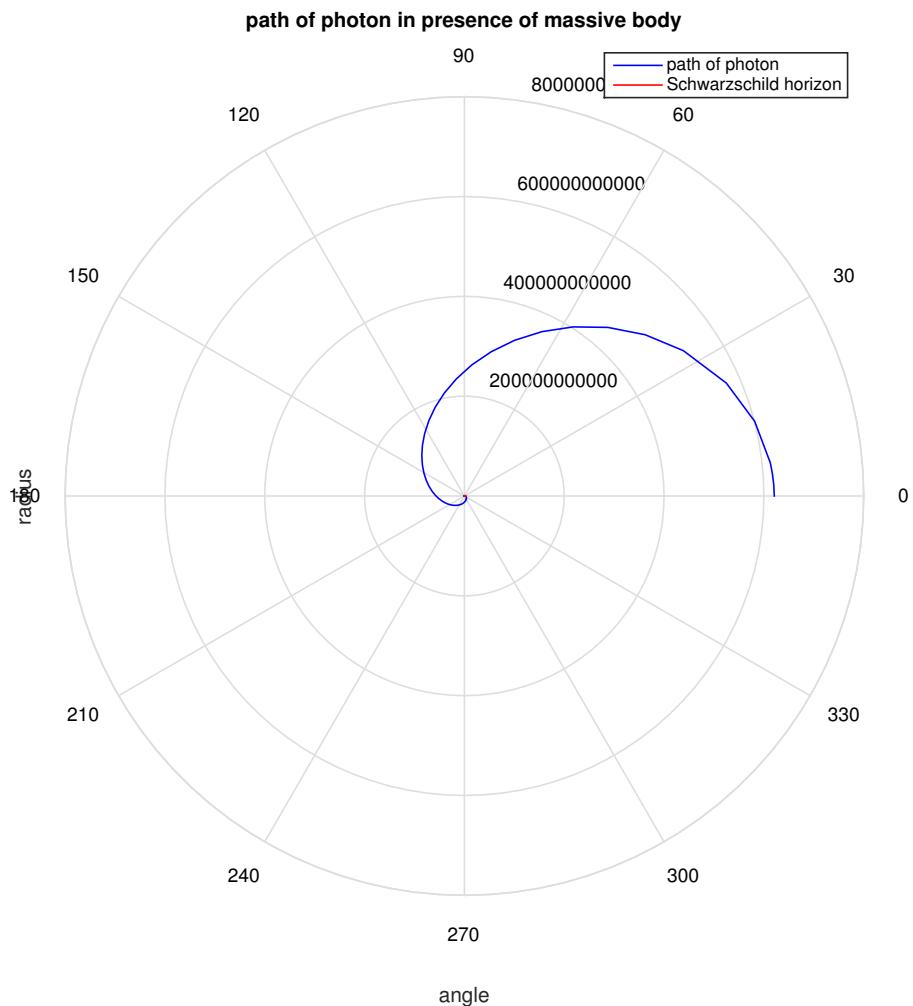


Figure 11: initial radius = $7 \times 10^{13} r_s$, initial angle = 90 deg, running time = 1×10^4

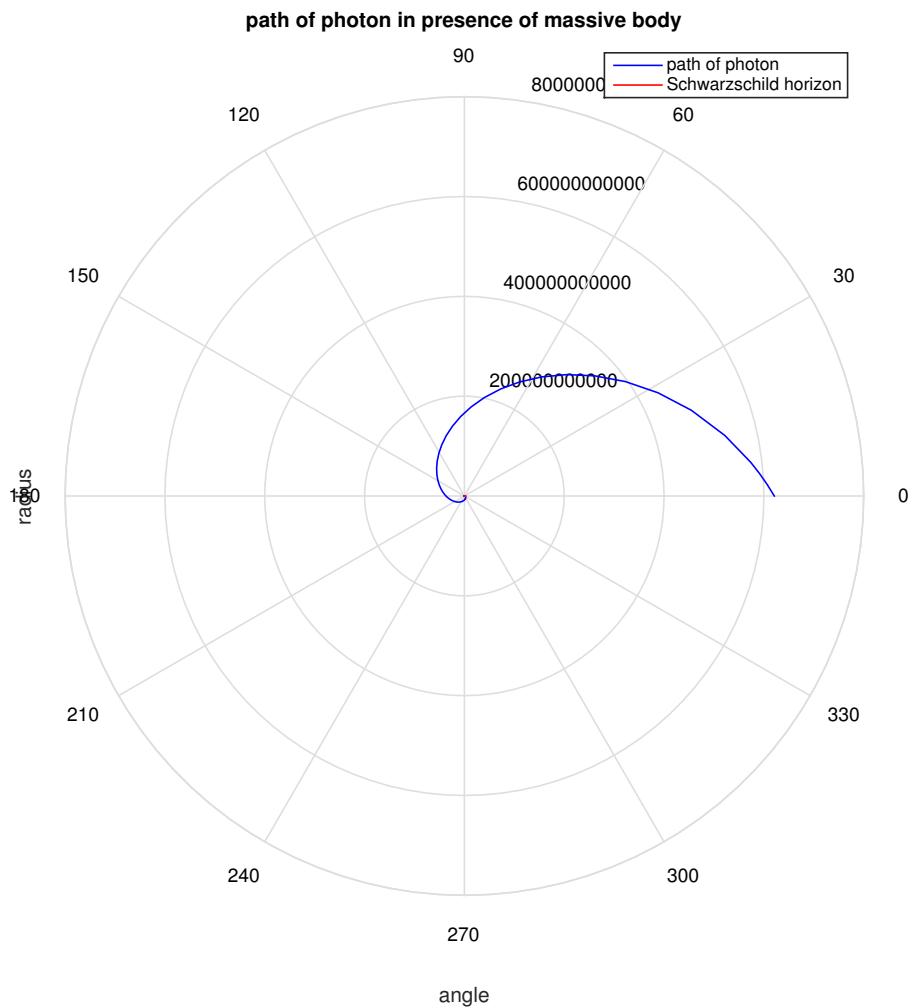


Figure 12: initial radius = $7 \times 10^{13} r_s$, initial angle = 120 deg, running time = 1×10^4

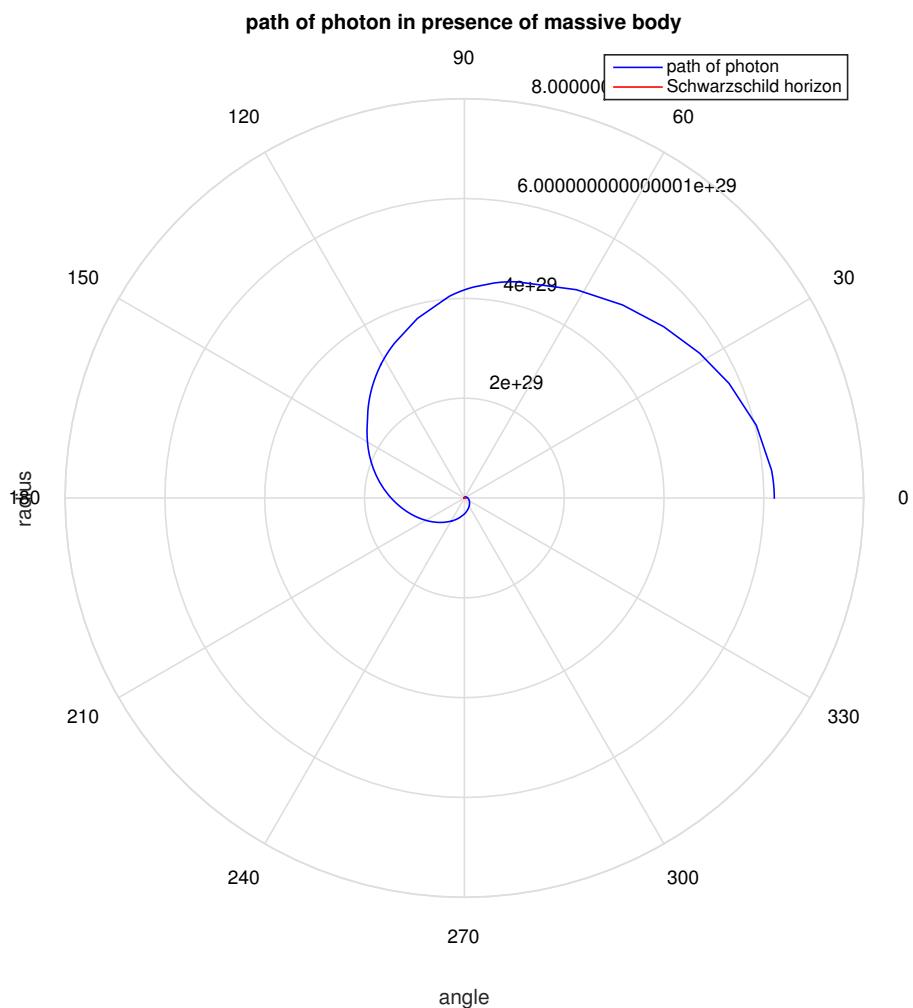


Figure 13: initial radius = $7 \times 10^{31} r_s$, initial angle = 90 deg, running time = 1×10^{25}

3.7.2 Escape examples

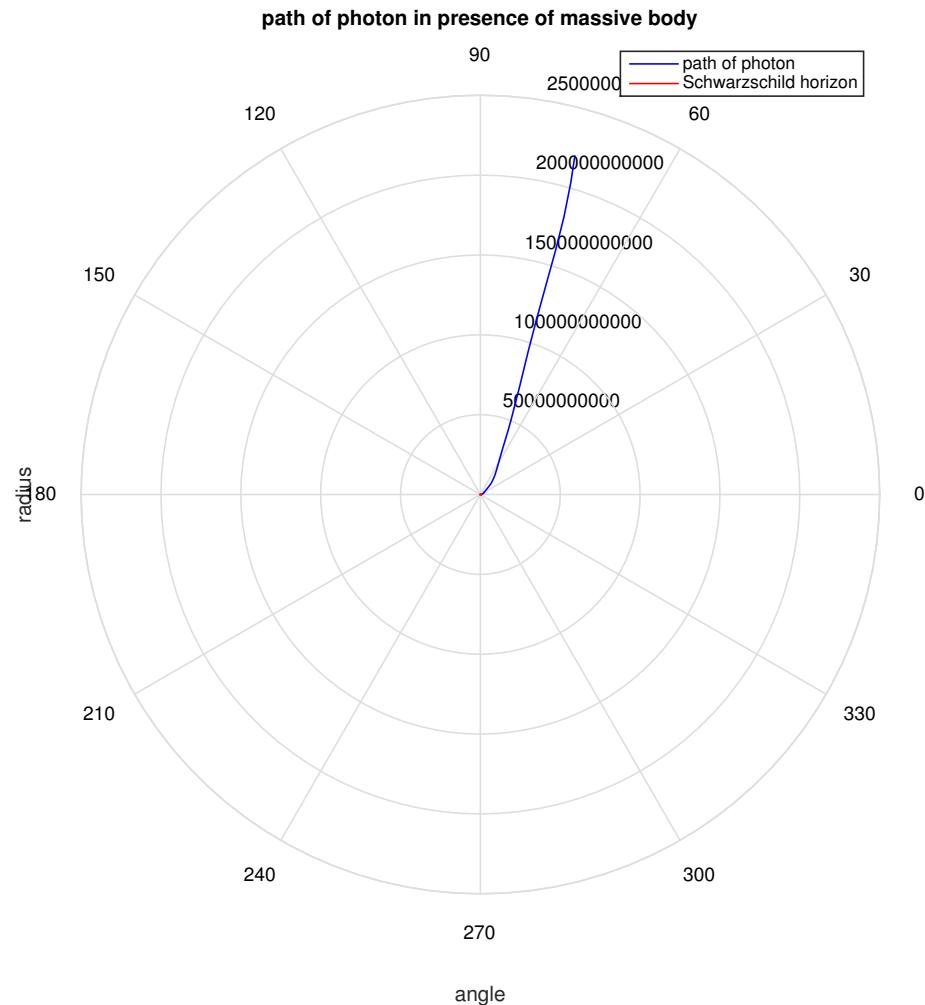


Figure 14: initial radius = $7 \times 10^6 r_s$, initial angle = 10 deg, running time = 1×10^3

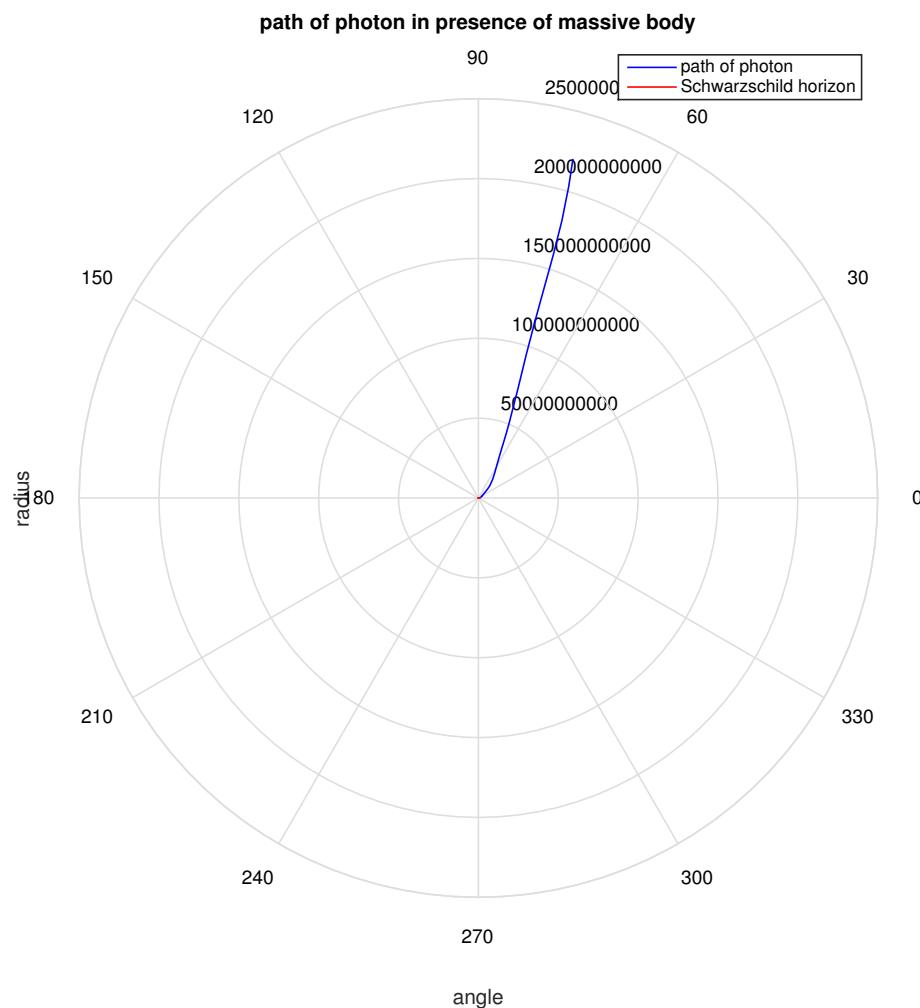


Figure 15: initial radius = $7 \times 10^6 r_s$, initial angle = 40 deg, running time = 1×10^3

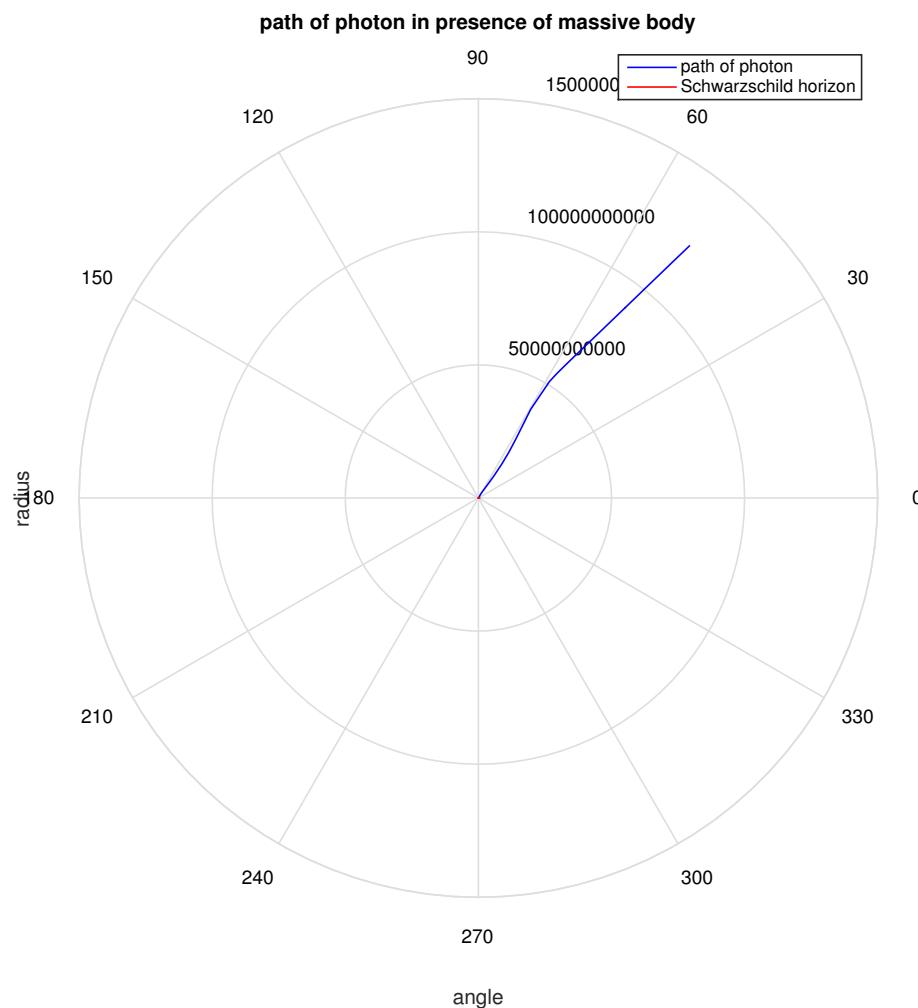


Figure 16: initial radius = $7 \times 10^9 r_s$, initial angle = 40 deg, running time = 1×10^3

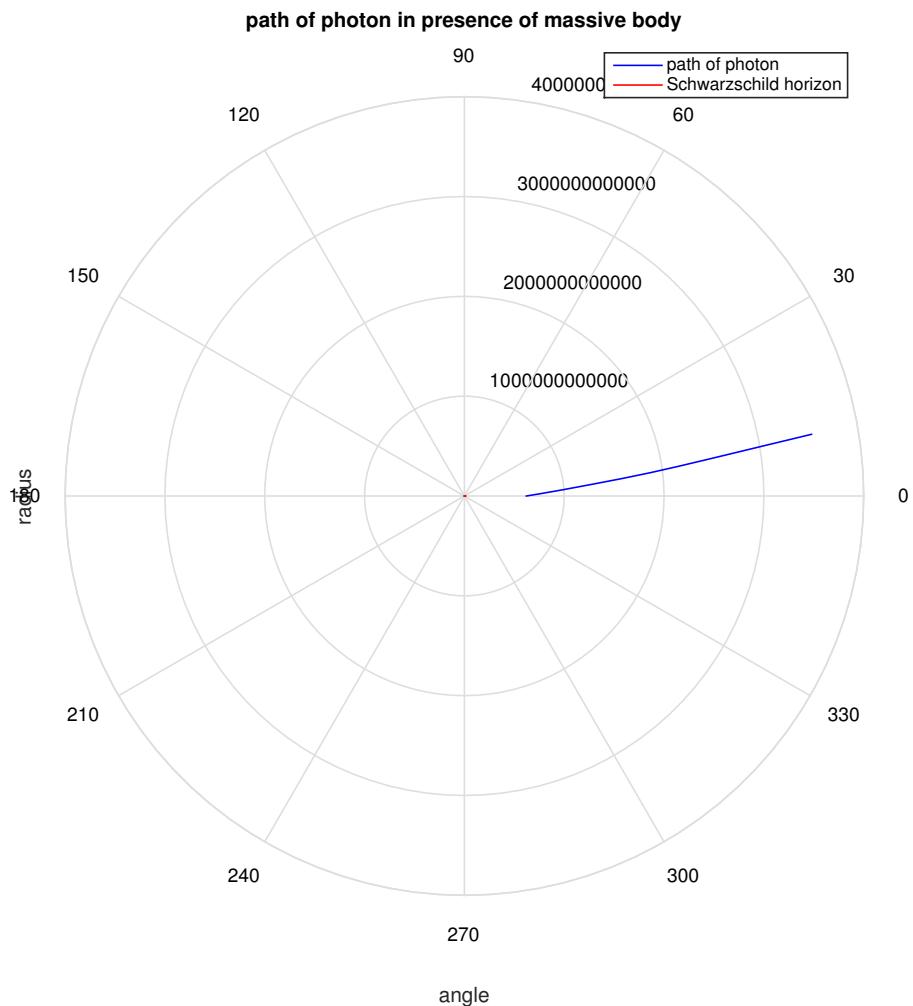


Figure 17: initial radius = $7 \times 10^{13} r_s$, initial angle = 10 deg, running time = 1×10^4

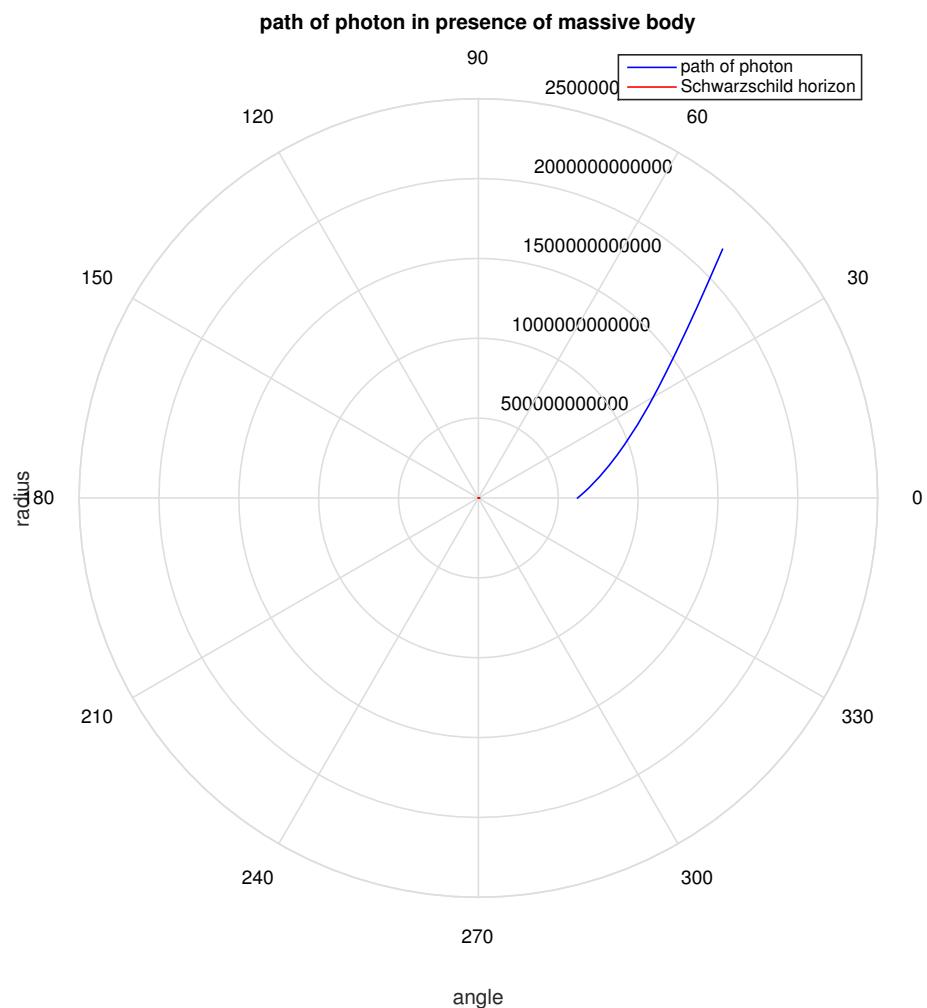


Figure 18: initial radius = $7 \times 10^{13} r_s$, initial angle = 40 deg, running time = 1×10^4

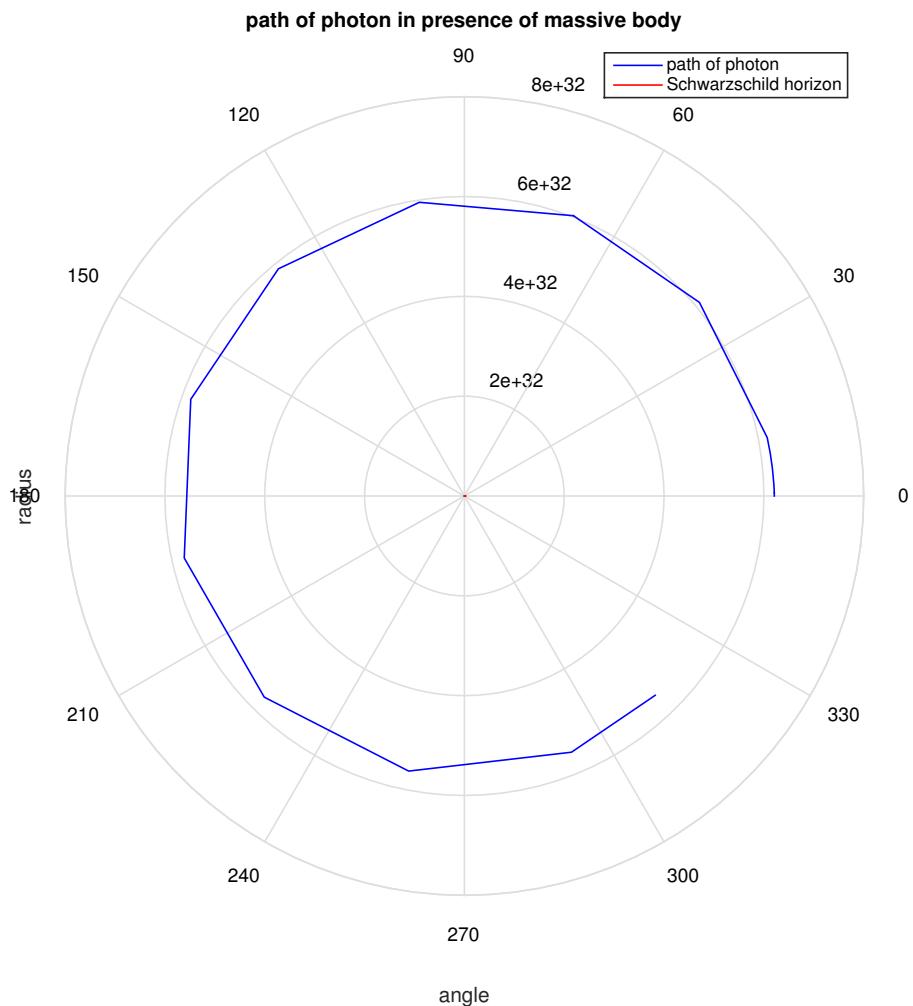


Figure 19: initial radius = $7 \times 10^{34} r_s$, initial angle = 90 deg, running time = 1×10^{24}

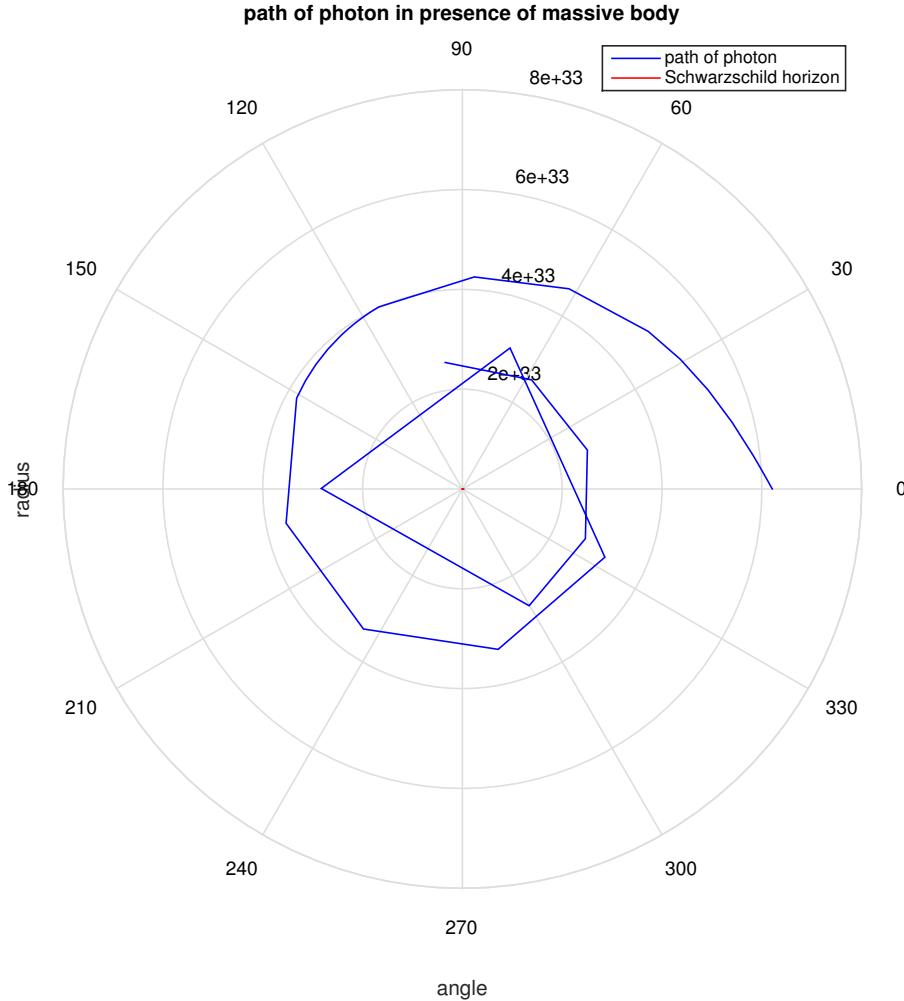


Figure 20: initial radius = $7 \times 10^{35} r_s$, initial angle = 120 deg, running time = 1×10^{26}

3.7.3 Discussion

From the figures above, we observe that a photon directed towards the Earth at 90 and 120 degrees, from a distance 7×10^6 to 7×10^{31} times the Schwarzschild radius, or 7×10^3 m to 7×10^{27} m away from the surface of the earth, will be pulled in to the Earth. It will not actually be swallowed into the Schwarzschild radius, as illustrated in the images, since the *actual surface* of the Earth would inhibit that. However, if we were observing a black hole with the same properties, the photon would be pulled in to the center of the blackhole.

A photon directed towards the Earth at 10 and 40 degrees, from a distance 7×10^6 to

7×10^{13} times the Schwarzschild radius, or 7×10^3 m to 7×10^{10} m away from the surface of the earth, will escape the pull of Earth's gravitational field.

We observe interesting motion in figures 20 and 21. In figure 20, a photon is directed towards the earth at 90 degrees, 7×10^{31} m away from the surface of the earth. It makes a circle around the Earth, which is pulled in to the center slightly. If the figure was observed for longer, such as 1×10^{26} T, the photon's observed motion becomes chaotic, crossing lines, but the MATLAB code does not determine that the photon is pulled in to the Earth.

In figure 21, a photon is directed towards the earth at 120 degrees, 7×10^{32} m away from the surface of the earth. We observe a spiralic, and the MATLAB code does not determine that the photon is pulled in to the Earth. Though at smaller radii, the 90 and 120 degree cases were non-escape examples, if the photons are initially far enough from a massive object, they may not be pulled in to the Schwarzschild radius.

shewn.

4 References

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