On the solvability of Burton–Miller-type boundary integral equations

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1 Exterior problems

Throughout this note, we assume that Ω is a bounded subset of $\mathbb{R}^d$ ($d = 2$ or 3) which is the open complement of an unbounded domain of class C^2. Namely, the open and bounded subset Ω is not necessarily connected.

Given a constant $k \in \mathbb{C}$, we shall consider

- Exterior Dirichlet problem: find $u \in C^2(\mathbb{R}^d \setminus \overline{\Omega}) \cap C^1(\mathbb{R}^d \setminus \Omega)$ such that

$$\begin{cases} \mathit{\Delta} u + k^2 u = 0 & \text{in } \Omega, \\ u = f & \text{on } \partial\Omega, \\ r^{(d-1)/2}\left(\dfrac{\partial u}{\partial r} - \mathrm{i}ku\right) \to 0 & \text{uniformly as } |x| =: r \to \infty \end{cases} \tag{1}$$

- Exterior Neumann problem: find $u \in C^2(\mathbb{R}^d \setminus \overline{\Omega}) \cap C^1(\mathbb{R}^d \setminus \Omega)$ such that

$$\begin{cases} \mathit{\Delta} u + k^2 u = 0 & \text{in } \Omega, \\ \left.\dfrac{\partial u}{\partial \nu}\right|_+ = g & \text{on } \partial\Omega, \\ r^{(d-1)/2}\left(\dfrac{\partial u}{\partial r} - \mathrm{i}ku\right) \to 0 & \text{uniformly as } |x| =: r \to \infty \end{cases}, \tag{2}$$

where f and g are given functions on $\partial\Omega$. The normal derivative should be understood in the sense of uniform convergence with parallel surfaces. For a function $\varphi \in C^1(\Omega)$, we define

$$\left.\frac{\partial \varphi}{\partial \nu}\right|_-(x) := \lim_{h \to +0} \nu(x) \cdot \nabla\varphi(x - h\nu(x)) \quad \text{on } \partial\Omega \tag{3}$$

when it converges uniformly on $\partial\Omega$, where ν is the unit normal vector on $\partial\Omega$ outward to Ω. Similarly, for a function $\varphi \in C^1(\mathbb{R}^d \setminus \overline{\Omega})$, we define

$$\left.\frac{\partial \varphi}{\partial \nu}\right|_+(x) := \lim_{h \to +0} \nu(x) \cdot \nabla\varphi(x + h\nu(x)) \quad \text{on } \partial\Omega \tag{4}$$

when it converges uniformly on $\partial\Omega$.

The well-posedness of the exterior Dirichlet and Neumann problems can be established via layer-potential techniques.

Theorem 1.1 (Colton and Kress [1], Theorem 3.11)**.** *Let $k > 0$ and $f \in C(\partial\Omega)$. Then the exterior Dirichlet problem* (1) *is uniquely solvable.*

Theorem 1.2 (Colton and Kress [1], Theorem 3.12)**.** *Let $k > 0$ and $g \in C(\partial\Omega)$. Then the exterior Neumann problem* (2) *is uniquely solvable.*

As a consequence, we immediately obtain the following results.

Corollary 1.3. *Suppose that a function $\varphi \in C^2(\mathbb{R}^d \setminus \overline{\Omega})$ solves the Helmholtz equation*

$$\Delta\varphi + k^2\varphi = 0 \qquad \text{in } \mathbb{R}^d \setminus \overline{\Omega}$$

for some $k > 0$ and

$$\lim_{h \to +0} \varphi(x + h\nu(x)) = 0 \tag{5}$$

uniformly on $\partial\Omega$. Then $\varphi(x) = 0$ for all $x \in \mathbb{R}^d \setminus \overline{\Omega}$.

Proof. Define a function u by

$$u(x) := \begin{cases} \varphi(x) & x \in \mathbb{R}^d \setminus \overline{\Omega} \\ 0 & x \in \partial\Omega \end{cases}.$$

It is easy to see that u belongs to $C^2(\mathbb{R}^d \setminus \overline{\Omega}) \cap C^1(\mathbb{R}^d \setminus \Omega)$ from the uniform convergence (6). Since $u \in C^2(\mathbb{R}^d \setminus \overline{\Omega}) \cap C^1(\mathbb{R}^d \setminus \Omega)$ solves the exterior Dirichlet problem (1), the unique solvability (Theorem 1.1) ensures that $u = 0$ in $\mathbb{R}^d \setminus \overline{\Omega}$. □

Corollary 1.4. *Suppose that a function $\varphi \in C^2(\mathbb{R}^d \setminus \overline{\Omega})$ solves the Helmholtz equation*

$$\Delta\varphi + k^2\varphi = 0 \qquad \text{in } \mathbb{R}^d \setminus \overline{\Omega}$$

for some $k > 0$ and

$$\left.\frac{\partial\varphi}{\partial\nu}\right|_+ := \lim_{h \to +0} \nu(x) \cdot \nabla\varphi(x + h\nu(x)) = 0 \tag{6}$$

uniformly on $\partial\Omega$. Then $\varphi(x) = 0$ for all $x \in \mathbb{R}^d \setminus \overline{\Omega}$.

Proof. It is easy to see that there exists a unique extension $u \in C^2(\mathbb{R}^d \setminus \overline{\Omega}) \cap C^1(\mathbb{R}^d \setminus \Omega)$ of $\varphi \in C^2(\mathbb{R}^d \setminus \overline{\Omega})$ such that $\varphi = u$ in $\mathbb{R}^d \setminus \overline{\Omega}$ with the uniform convergence

$$\left.\frac{\partial u}{\partial\nu}\right|_+ = \lim_{h \to +0} \nu(x) \cdot \nabla u(x + h\nu(x)) = 0.$$

The statement immediately follows from Theorem 1.2, i.e., the unique solvability of the exterior Neumann problem (2). □

2 Layer potentials and boundary integral operators

References

[1] David L Colton and Rainer Kress. *Inverse acoustic and electromagnetic scattering theory*, volume 93. Springer, fourth edition, 2019.