On the solvability of Burton–Miller-type boundary integral equations

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1 Exterior problems

Throughout this note, we assume that Ω is a bounded subset of $\mathbb{R}^d$ ($d = 2$ or 3) which is the open complement of an unbounded domain of class C^2. Namely, the open and bounded subset Ω is not necessarily connected.

Given a constant $k \in \mathbb{C}$, we shall consider

- Exterior Dirichlet problem: find $u \in C^2(\mathbb{R}^d \setminus \overline{\Omega}) \cap C^1(\mathbb{R}^d \setminus \Omega)$ such that

$$\begin{cases} \mathit{\Delta} u + k^2 u = 0 & \text{in } \Omega, \\ u = f & \text{on } \partial\Omega, \\ r^{(d-1)/2}\left(\dfrac{\partial u}{\partial r} - \mathrm{i}ku\right) \to 0 & \text{uniformly as } |x| =: r \to \infty \end{cases} \tag{1}$$

- Exterior Neumann problem: find $u \in C^2(\mathbb{R}^d \setminus \overline{\Omega}) \cap C^1(\mathbb{R}^d \setminus \Omega)$ such that

$$\begin{cases} \mathit{\Delta} u + k^2 u = 0 & \text{in } \Omega, \\ \left.\dfrac{\partial u}{\partial \nu}\right|_+ = g & \text{on } \partial\Omega, \\ r^{(d-1)/2}\left(\dfrac{\partial u}{\partial r} - \mathrm{i}ku\right) \to 0 & \text{uniformly as } |x| =: r \to \infty \end{cases}, \tag{2}$$

where f and g are given functions on $\partial\Omega$. The normal derivative should be understood in the sense of uniform convergence with parallel surfaces. For a function $\varphi \in C^1(\Omega)$, we define

$$\left.\frac{\partial \varphi}{\partial \nu}\right|_- (x) := \lim_{h \to +0} \nu(x) \cdot \nabla\varphi(x - h\nu(x)) \quad \text{on } \partial\Omega$$

when it converges uniformly on $\partial\Omega$, where ν is the unit normal vector on $\partial\Omega$ outward to Ω. Similarly, for a function $\varphi \in C^1(\mathbb{R}^d \setminus \overline{\Omega})$, we define

$$\left.\frac{\partial \varphi}{\partial \nu}\right|_+ (x) := \lim_{h \to +0} \nu(x) \cdot \nabla\varphi(x + h\nu(x)) \quad \text{on } \partial\Omega$$

when it converges uniformly on $\partial\Omega$.

The well-posedness of the exterior Dirichlet and Neumann problems can be established via layer-potential techniques.

Theorem 1.1 (Colton and Kress [1], Theorem 3.11)**.** *Let $k > 0$ and $f \in C(\partial\Omega)$. Then the exterior Dirichlet problem* (1) *is uniquely solvable.*

Theorem 1.2 (Colton and Kress [1], Theorem 3.12)**.** *Let $k > 0$ and $g \in C(\partial\Omega)$. Then the exterior Neumann problem* (2) *is uniquely solvable.*

As a consequence, we immediately obtain the following results.

Corollary 1.3. *Suppose that a function $\varphi \in C^2(\mathbb{R}^d \setminus \overline{\Omega})$ solves the Helmholtz equation*

$$\Delta\varphi + k^2\varphi = 0 \quad \text{in } \mathbb{R}^d \setminus \overline{\Omega}$$

with radiation condition

$$r^{(d-1)/2}\left(\frac{\partial\varphi}{\partial r} - \mathrm{i}k\varphi\right) \to 0 \quad \text{uniformly as } |x| =: r \to \infty$$

for some $k > 0$ and

$$\lim_{h\to+0} \varphi(x + h\nu(x)) = 0 \tag{3}$$

uniformly on $\partial\Omega$. Then $\varphi(x) = 0$ for all $x \in \mathbb{R}^d \setminus \overline{\Omega}$.

Proof. Define a function u by

$$u(x) := \begin{cases} \varphi(x) & x \in \mathbb{R}^d \setminus \overline{\Omega} \\ 0 & x \in \partial\Omega \end{cases}.$$

It is easy to see that u belongs to $C^2(\mathbb{R}^d \setminus \overline{\Omega}) \cap C^1(\mathbb{R}^d \setminus \Omega)$ from the uniform convergence (4). Since $u \in C^2(\mathbb{R}^d \setminus \overline{\Omega}) \cap C^1(\mathbb{R}^d \setminus \Omega)$ solves the exterior Dirichlet problem (1), the unique solvability (Theorem 1.1) ensures that $u = 0$ in $\mathbb{R}^d \setminus \overline{\Omega}$. □

Corollary 1.4. *Suppose that a function $\varphi \in C^2(\mathbb{R}^d \setminus \overline{\Omega})$ solves the Helmholtz equation*

$$\Delta\varphi + k^2\varphi = 0 \quad \text{in } \mathbb{R}^d \setminus \overline{\Omega}$$

with radiation condition

$$r^{(d-1)/2}\left(\frac{\partial\varphi}{\partial r} - \mathrm{i}k\varphi\right) \to 0 \quad \text{uniformly as } |x| =: r \to \infty$$

for some $k > 0$ and

$$\left.\frac{\partial\varphi}{\partial\nu}\right|_{+} := \lim_{h\to+0} \nu(x)\cdot\nabla\varphi(x + h\nu(x)) = 0 \tag{4}$$

uniformly on $\partial\Omega$. Then $\varphi(x) = 0$ for all $x \in \mathbb{R}^d \setminus \overline{\Omega}$.

Proof. It is easy to see that there exists a unique extension $u \in C^2(\mathbb{R}^d \setminus \overline{\Omega}) \cap C^1(\mathbb{R}^d \setminus \Omega)$ of $\varphi \in C^2(\mathbb{R}^d \setminus \overline{\Omega})$ such that $\varphi = u$ in $\mathbb{R}^d \setminus \overline{\Omega}$ with the uniform convergence

$$\left.\frac{\partial u}{\partial \nu}\right|_{+} = \lim_{h \to +0} \nu(x) \cdot \nabla u(x + h\nu(x)) = 0.$$

The statement immediately follows from Theorem 1.2, i.e., the unique solvability of the exterior Neumann problem (2). □

2 Layer potentials and boundary integral operators

Let G be the fundamental solution of the Helmholtz equation, given by

$$G(x,y) = \frac{\mathrm{i}}{4} H_0^{(1)}(k|x-y|), \quad x, y \in \mathbb{R}^2, x \neq y$$

for $d = 2$ or

$$G(x,y) = \frac{1}{4\pi} \frac{\mathrm{e}^{\mathrm{i}k|x-y|}}{|x-y|}, \quad x, y \in \mathbb{R}^3, x \neq y$$

for $d = 3$, where $H_n^{(1)}$ is the Hankel function of the first kind and order n. Given $\varphi \in C(\partial\Omega)$, the function

$$\mathbb{R}^d \setminus \partial\Omega \ni x \mapsto \int_{\partial\Omega} G(x,y)\varphi(y)\mathrm{d}s(y)$$

is called a *single-layer potential* with density $\varphi \in C(\partial\Omega)$. Analogously, for $\varphi \in C(\partial\Omega)$,

$$\mathbb{R}^d \setminus \partial\Omega \ni x \mapsto \int_{\partial\Omega} \frac{\partial G}{\partial \nu(y)}(x,y)\varphi(y)\mathrm{d}s(y)$$

is called a *double-layer potential* with density $\varphi \in C(\partial\Omega)$.

We are interested in the limiting case where the point x lies on the boundary $\partial\Omega$. In view of this, we formally write

$$\begin{aligned}
(S\varphi)(x) &:= \int_{\partial\Omega} G(x,y)\varphi(y)\mathrm{d}s(y) \quad x \in \partial\Omega, \\
(D\varphi)(x) &:= \int_{\partial\Omega} \frac{\partial G}{\partial \nu(y)}(x,y)\varphi(y)\mathrm{d}s(y) \quad x \in \partial\Omega, \\
(D'\varphi)(x) &:= \int_{\partial\Omega} \frac{\partial G}{\partial \nu(x)}(x,y)\varphi(y)\mathrm{d}s(y) \quad x \in \partial\Omega, \\
(N\varphi)(x) &:= \frac{\partial}{\partial \nu(x)} \int_{\partial\Omega} \frac{\partial G}{\partial \nu(y)}(x,y)\varphi(y)\mathrm{d}s(y) \quad x \in \partial\Omega.
\end{aligned}$$

In order that the operators S, D, D', and N are well-defined, the function φ requires sufficient regularity on $\partial\Omega$. We have the following *mapping properties* of boundary integral operators:

Theorem 2.1 (Colton and Kress [1], Theorem 3.4)**.** *Let $k > 0$, $\alpha \in (0, 1]$ and let $\partial\Omega$ be of class C^2. Then the following operators are well-defined and linear with some additional properties:*

- $S : C(\partial\Omega) \to C(\partial\Omega)$, *compact*
- $S : C(\partial\Omega) \to C^{0,\alpha}(\partial\Omega)$, *bounded*
- $S : C^{0,\alpha}(\partial\Omega) \to C^{1,\alpha}(\partial\Omega)$, *bounded*
- $S : C^{1,\alpha}(\partial\Omega) \to C^{1,\alpha}(\partial\Omega)$, *compact*
- $D : C(\partial\Omega) \to C(\partial\Omega)$, *compact*
- $D : C(\partial\Omega) \to C^{0,\alpha}(\partial\Omega)$, *bounded*
- $D : C^{0,\alpha}(\partial\Omega) \to C^{1,\alpha}(\partial\Omega)$, *bounded*
- $D : C^{1,\alpha}(\partial\Omega) \to C^{1,\alpha}(\partial\Omega)$, *compact*
- $D' : C(\partial\Omega) \to C(\partial\Omega)$, *compact*
- $D' : C(\partial\Omega) \to C^{0,\alpha}(\partial\Omega)$, *bounded*
- $D' : C^{1,\alpha}(\partial\Omega) \to C^{1,\alpha}(\partial\Omega)$, *compact*
- $N : C^{1,\alpha}(\partial\Omega) \to C^{0,\alpha}(\partial\Omega)$, *bounded*
- $SN : C^{1,\alpha}(\partial\Omega) \to C^{1,\alpha}(\partial\Omega)$, $SN = D^2 - I$, *Fredholm of index zero*
- $NS : C^{0,\alpha}(\partial\Omega) \to C^{0,\alpha}(\partial\Omega)$, $NS = (D')^2 - I$, *Fredholm of index zero*

The layer potentials are associated with the boundary integral operators via the following *jump relations*:

Theorem 2.2 (Colton and Kress [1], Theorem 3.1)**.** *Let $k > 0$ and $\varphi \in C(\partial\Omega)$. Then the single-layer potential*

$$w(x) := \int_{\partial\Omega} G(x, y)\varphi(y)\mathrm{d}s(y) \quad x \in \mathbb{R}^d \setminus \partial\Omega$$

is twice-continuously differentiable and solves the Helmholtz equation

$$\Delta w(x) + k^2 w(x) = 0 \quad \text{for all } x \in \mathbb{R}^d \setminus \partial\Omega$$

with radiation condition

$$r^{(d-1)/2}\left(\frac{\partial w}{\partial r} - \mathrm{i}kw\right) \to 0 \quad \text{uniformly as } |x| =: r \to \infty.$$

Moreover, the limit

$$\lim_{h\to+0} w(x \pm h\nu(x)) = (S\varphi)(x)$$

converges uniformly to the continuous function on $\partial\Omega$. Analogously, the limit

$$\left.\frac{\partial w}{\partial \nu}\right|_{\pm}(x) := \lim_{h\to+0} \nu(x)\cdot\nabla w(x \pm h\nu(x)) = \mp\frac{1}{2}\varphi(x) + (D'\varphi)(x)$$

converges uniformly to the continuous function on $\partial\Omega$.

Theorem 2.3 (Colton and Kress [1], Theorem 3.1)**.** *Let $k > 0$ and $\varphi \in C(\partial\Omega)$. Then the double-layer potential*

$$v(x) := \int_{\partial\Omega} \frac{\partial G}{\partial\nu(y)}(x,y)\varphi(y)\mathrm{d}s(y) \quad x \in \mathbb{R}^d \setminus \partial\Omega$$

is twice-continuously differentiable and solves the Helmholtz equation

$$\Delta v(x) + k^2 v(x) = 0 \quad \text{for all } x \in \mathbb{R}^d \setminus \partial\Omega$$

with radiation condition

$$r^{(d-1)/2}\left(\frac{\partial v}{\partial r} - \mathrm{i}kv\right) \to 0 \quad \text{uniformly as } |x| =: r \to \infty.$$

Moreover, the limit

$$\lim_{h\to+0} v(x \pm h\nu(x)) = \pm\frac{1}{2}\varphi(x) + (D\varphi)(x)$$

converges uniformly to the continuous function on $\partial\Omega$, and

$$\lim_{h\to+0}\left[\nu(x)\cdot\nabla v(x + h\nu(x)) - \nu(x)\cdot\nabla v(x - h\nu(x))\right] = 0$$

uniformly on $\partial\Omega$.

References

[1] David L Colton and Rainer Kress. *Inverse acoustic and electromagnetic scattering theory*, volume 93. Springer, fourth edition, 2019.