CS3063 Theory of Computing

Semester 4 (21 Intake), Jan – May 2024

Lecture 7

Context-Free Languages: Session 2

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Announcement

- Week **8**: Quiz **6** (based on **L7**, this lecture)
  - Thursday **14th March**
  - 8.15am: Group 2
  - 9.15am: Group 1

Previous Lecture

- Context-free Languages
  - Context-free Grammars (CFGs)
  - Derivations
  - Properties of CFLs
  - CFG for a Regular Language
  - CFG from an FA
  - Regular Grammars
  - Derivation Trees
  - Ambiguous CFGs

Today’s Outline:

Lecture 7

Context-free Languages (CFLs) - 2

- Ambiguous CFGs (continued)
- Simplified Forms and Normal Forms
- Pushdown Automata
  - Definition
  - Acceptance
- Examples

PART 1

Today's Outline:

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- Ambiguous CFGs (continued)
- Simplified Forms and Normal Forms
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Ambiguous CFGs: Review

- Ambiguous CFG means the grammar can produce a string that has more than 1 derivation tree

- For some ambiguous CFGs, it is possible to find an equivalent unambiguous CFG

Example (from last time)

- Suppose we have the ambiguous CFG (for algebraic-expressions), G:

  $$S \rightarrow S + S \mid S * S \mid (S) \mid a$$

- Is there an equivalent unambiguous CFG?
  – Avoid $S \rightarrow S + S$ and $S \rightarrow S * S$ because these produce ambiguity
  – Can impose
    - rules of order; e.g., $a+a+a$ means $(a+a)+a$
    - operator precedence; e.g., * has higher precedence than +

Solution

- An ***expression*** is a ***sum of terms***
  - Replace $S \rightarrow S + S$ by $S \rightarrow S + T \mid T$ where T stands for ***term***
- A ***term*** is a ***product of factors***
  - $T \rightarrow T * F \mid F$ where F stands for ***factor***

- Precedence of * over + incorporated
- Association from left-to-right incorporated

Solution ...contd

- How about parenthesized expressions?
  - Most appropriate to consider as a factor
  - That is, it should get high precedence
- So we have the CFG, G' :

$$\left.\begin{aligned} S &\rightarrow S + T \mid T \\ T &\rightarrow T * F \mid F \\ F &\rightarrow (S) \mid a \end{aligned}\right\}$$

Can prove G' is equivalent to the original G and that G' is unambiguous

Associativity of Operators

- Associativity
  - Tells us how to group operators with same precedence when parentheses are not used
- **Left associativity**
  - Examples

$5 + 3 + 2 = (5 + 3) + 2 = 5 + (3 + 2) = 10$

$9 – 3 – 2 = (9 – 3) – 2 = 4$ but $9 - (3 - 2) = 8$

$5 – 3 + 2 = (5 – 3) + 2 = 4$ but $5 - (3 + 2) = 0$

$16 ÷ 4 * 2 = (16 ÷ 4) * 2 = 8$ but $16 ÷ (4 * 2) = 2$

Associativity …

- Subtraction and division: we normally consider as inherently *left associative*
- Addition, multiplication
  - No inherent associativity, but usually considered as left associative
- **Right associativity**
  - Examples: exponentiation, assignment

    4^3^2 = 4^(3^2)=4^9=2^18 but (4^3)^2=2^12

    a = b = c means a = (b = c)

PART 2

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Simplified & Normal Forms

- Possible improvements to grammars (without changing the language)
  - Eliminate Λ-productions
    - Productions of the form $A \rightarrow \Lambda$
  - Eliminate unit productions
    - Productions in which one non-terminal is replaced simply by another
  - Standardize productions for a "normal form"

Λ-Productions

- Example (E.g., 6.14, p. 233 in text book)
- Let G be a CFG such that:

$S \rightarrow ABCBCDA$

$A \rightarrow CD$

$B \rightarrow C\mathit{b}$

$C \rightarrow \mathit{a} \mid \Lambda$

$D \rightarrow \mathit{b}D \mid \Lambda$

- Cannot simply remove the Λ-productions
- Rewrite the1st one as $S \rightarrow A_1BC_1BC_2DA_2$
  - A_1, A_2, C_1, C_2 and D have Λ-productions
  - These can also derive non-null strings

Λ-Productions ...contd

- Without Λ-productions, we need to allow all possibilities by adding productions of the form $S \rightarrow \alpha$
  - α is a string obtained from ABCBCDA by deleting some subset of $\{A_1, A_2, C_1, C_2, D\}$
  - There are $2^5=32$ subsets
  - From the original $S \rightarrow$ ABCBCDA , obtain 31 others and add to G

Λ-Productions ...contd

- Do similarly with other original productions
- The final CFG has 40 productions, including the above 32 and:

$$\text{A} \rightarrow \text{CD} \mid \text{C} \mid \text{D}$$
$$\text{B} \rightarrow \text{C}b \mid b$$
$$\text{C} \rightarrow a$$
$$\text{D} \rightarrow b\text{D} \mid b$$

Λ-Productions ...contd

- Definition: A *nullable* non-terminal in a CFG, $G=(V, \Sigma, S, P)$, is defined as:
  - Any non-terminal A for which P contains the production $A \rightarrow \Lambda$ is *nullable*
  - If P contains the production $A \rightarrow B_1B_2...B_n$ and $B_1, B_2,\ldots, B_n$ are *nullable*, then A is *nullable*
  - No other non-terminal in V is *nullable*

- Nullable non-terminals are those A for which $A \Rightarrow^* \Lambda$

Algorithm *FindNull*

- Finding the nullable non-terminals in a given CFG, G=(V, Σ, S, P) [p. 234 of text book]

FindNull

N_0 = { A ∈V | production A → Λ exists in P}

$i = 0$

do {

$i = i + 1$

$N_i = N_{i-1} \bigcup$ { A | P has A → α for some α ∈ N_{i-1}*}

} **while** $N_i \neq N_{i-1}$

N_i is the set of nullable non-terminals

Λ-Productions ...contd

Given a CFG, $G=(V, \Sigma, S, P)$, algorithm to get an equivalent CFG, $G1=(V, \Sigma, S, P1)$ where $L(G1)=L(G) - \{\Lambda\}$, G1 has no Λ-productions [pp. 234-235]

1. Initialize P1 to P
2. Find all nullable non-terminals in V using *FindNull*
3. For every $A \rightarrow \alpha$ in P, add to P1 every production that can be obtained from this one by deleting from α one or more occurrences of nullable non-terminals in α
4. Delete all Λ-productions from P1, any duplicates and productions of the form $A \rightarrow A$

Λ-Productions ...contd

- Eliminating Λ-productions will likely increase the number of productions substantially
  - Example on slides 14-16 (E.g., 6.14, p.233)
- If G is an ambiguous CFG, then the grammar produced by eliminating Λ-productions in G is also ambiguous

Eliminating Unit Productions

- Similar to eliminating Λ-productions
  - Need to consider all pairs of non-terminals A, B for which $A \Rightarrow^* B$ (and also if $A \rightarrow B$ exists)

- Must ensure we do not eliminate legitimate strings in the language
  - Whenever $B \rightarrow \alpha$ is a non-unit production and $A \Rightarrow^* B$, add the production $A \rightarrow \alpha$

Unit Productions ...contd

Given a CFG, $G=(V, \Sigma, S, P)$, algorithm to get an equivalent CFG, $G1=(V, \Sigma, S, P1)$ having no unit productions [pp. 236-237]

1. Initialize P1 to P
2. For each A in V, find the set of A-*derivable* non-terminals (those that can be derived from A)
3. For every pair (A,B) such that B is A-*derivable*, and every non-unit production $B \rightarrow \alpha$ in P, add $A \rightarrow \alpha$ to P1 if it is not already in P1
4. Delete all unit productions from P1

Example

- Consider a previous example we studied

$S \rightarrow S + T \mid T$

$T \rightarrow T * F \mid F$

$F \rightarrow (S) \mid a$

Unit productions

- After deleting unit productions we have:

$S \rightarrow S + T \mid T * F \mid (S) \mid a$

$T \rightarrow T * F \mid (S) \mid a$

$F \rightarrow (S) \mid a$

Normal Forms

- Normal forms impose further restrictions upon the forms of productions in a CFG

- Examples
  - Chomsky Normal Form (CNF)
  - Greibach Normal Form (GNF)

Chomsky Normal Form (CNF)

- A CFG is in CNF if every production is one of the two types:

  $A \rightarrow BC$

  $A \rightarrow a$

  where A, B and C are non-terminals and a is a terminal

Converting a CFG to CNF

- Follow 4 steps
  1. Eliminate Λ-productions
  2. Eliminate unit productions
  3. Restrict the RHS of productions to single terminals or strings of ≥ 2 non-terminals
  4. Replace each production having > 2 non-terminals on RHS by an equivalent set of productions each having exactly 2 non-terminals on the RHS

Example

- If $\text{A} \rightarrow a\text{A}b$ and $\text{B} \rightarrow ab$ then the CNF will be:

$$\text{A} \rightarrow \text{XZ}$$
$$\text{Z} \rightarrow \text{AY}$$
$$\text{B} \rightarrow \text{XY}$$
$$\text{X} \rightarrow a$$
$$\text{Y} \rightarrow b$$

PART 3

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Pushdown Automata (PDA)

- **Example**: Consider the following CFG

  $$S \rightarrow aSa \mid bSb \mid c$$

  – Generates odd-length palindromes of $\{a, b\}$ with c being the middle symbol

  – How to recognize this language?

    - Scan a string L-to-R, pushing symbols in the 1^{st} half onto a stack as we read each one
    - After reaching the middle "c", begin matching next ones with those on stack
    - Can do this with an automaton of 3 states: start state, state after seeing "c" and accepting state

Pushdown Automata (PDA)

- A PDA is a 7-tuple M=$(Q, \Sigma, \Gamma, q_0, Z_0, A, \delta)$, where:
  - Q is a finite set of states
  - Σ and Γ are finite sets (the input and stack alphabets, respectively)
  - q_0, the initial state, is an element of Q
  - Z_0, the initial stack symbol, is an element of Γ
  - A, the set of accepting states, is a subset of Q
  - δ is the transition function

Chapter 7 in textbook

Pushdown Automata (PDA)

- Transition δ maps $Q \times (\Sigma \cup \{\Lambda\}) \times \Gamma$ to the set of finite subsets of $Q \times \Gamma^*$

- Current state is not enough to specify the status of the machine
- A **move** depends on
  - The current state
  - The next input
  - The symbol currently on top of the stack

Pushdown Automata (PDA)

- A move consists of
  - Changing states (can stay in current state)
  - Replacing the symbol on top of the stack by a string of zero or more symbols
- Popping the top symbol off the stack
  - Means replacing it by Λ
- Pushing Y onto the stack
  - Means replacing the top symbol X by YX (assume left end of string is on top of stack)

Pushdown Automata (PDA)

- We allow the possibility that stack alphabet is different from input alphabet
- For a state q, an input a and a stack symbol X,

  $$\delta(q, a, X) = (p, \beta)$$

  means in state q, with X on top of the stack, we read the symbol a, move to state p, and replace X on the stack by string β

Pushdown Automata (PDA)

- Initially special start symbol Z_0 on the stack
  - Z_0 never removed from stack
  - No additional copies of Z_0 pushed onto stack
  - Z_0 on top means the stack is empty
  - No move when stack is empty

- We allow moves when Λ is input

Terminology / Notations

- A ***configuration*** of the PDA M=$(Q, \Sigma, \Gamma, q_0, Z_0, A, \delta)$, is a triple

  $$(q, x, \alpha)$$

  where q is in Q, x is in Σ^* and α is in Γ^*
- "(q, x, α) is the **current configuration**" →
  - q is the current state
  - x is the string of remaining unread input
  - α is the current stack contents (leftmost is top)

Terminology / Notations ...contd

$$(p, x, \alpha) \vdash_{\mathrm{M}} (q, y, \beta)$$

means, one of the possible moves in the first configuration takes M to the second

- This can happen in 2 ways
  - Either with an input symbol
  - Or, a Λ-transition

Notations ...contd

- In both cases, $x=ay$ for some a in Σ U {Λ}
  - β is obtained from α by replacing the 1st symbol X by some string μ
  - That is (q, μ) is in $\delta(p, a, X)$
- Then,

  $$(p, x, \alpha) \vdash^{*}_{M} (q, y, \beta)$$

  means, there is a sequence of moves that takes M from the first to the second configuration

Acceptance by a PDA

- **Definition**: If M=(Q, Σ, Γ, q_0, Z_0, A, δ) is a PDA and x is in Σ^*, x is accepted by M if

$$(q_0, x, Z_0) \vdash^*_{\mathrm{M}} (q, \Lambda, \alpha)$$

  for some α in Γ^* and some q in A
  - There *exists* a sequence of moves
  - The stack may or may not be empty because α=Λ or α≠Λ ; this is *acceptance by final state*
  - Possible to have *acceptance by empty stack*
  - Are the two forms equivalent?

PART 4

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Example 1

- Consider the previous CFG again

$$S \rightarrow aSa \mid bSb \mid c$$

  – Generates odd-length palindromes of $\{a, b\}$ with c being the middle symbol (let's call this language *SimplePal*)

- What is the PDA to accept *SimplePal*?
  – Discussion from E.g. 7.1 (p. 251)

Example 1 ...contd

- Simple palindrome (*SimplePal*) recognizer
  - 3 states $Q=\{q_0, q_1, q_2\}$
  - q_0 : initial state
    - Processes first ½ of string, each input symbol pushed to stack
    - Go to q_1 upon receiving *c*, the middle symbol
  - q_1 : state for processing second ½ of string
    - Each input symbol compared to top stack symbol; if they match pop and discard, else crash (reject)
    - Go to q_2 when stack is empty
  - q_2 : accepting state

Example 1 ...contd

- Simple palindrome (*SimplePal*) recognizer
  - Input alphabet Σ={a, b, c}
  - Stack alphabet Γ={a, b, Z_0}
  - Transition function δ: Table 7.1 (p. 254), next slide
  - Machine will crash when no move specified
  - Trace the moves for sample inputs *abcba*, *ab*

Transition Function/Table

Move #	State	Input	Stack Symbol	Move(s)
1	q_0	a	Z_0	(q_0 , aZ_0)
2	q_0	b	Z_0	(q_0 , bZ_0)
3	q_0	a	a	(q_0 , aa)
4	q_0	b	a	(q_0 , ba)
5	q_0	a	b	(q_0 , ab)
6	q_0	b	b	(q_0 , bb)
7	q_0	c	Z_0	(q_1 , Z_0)
8	q_0	c	a	(q_1 , a)
9	q_0	c	b	(q_1 , b)
10	q_1	a	a	(q_1 , Λ)
11	q_1	b	b	(q_1 , Λ)
12	q_1	Λ	Z_0	(q_2 , Z_0)
	All other combinations			none

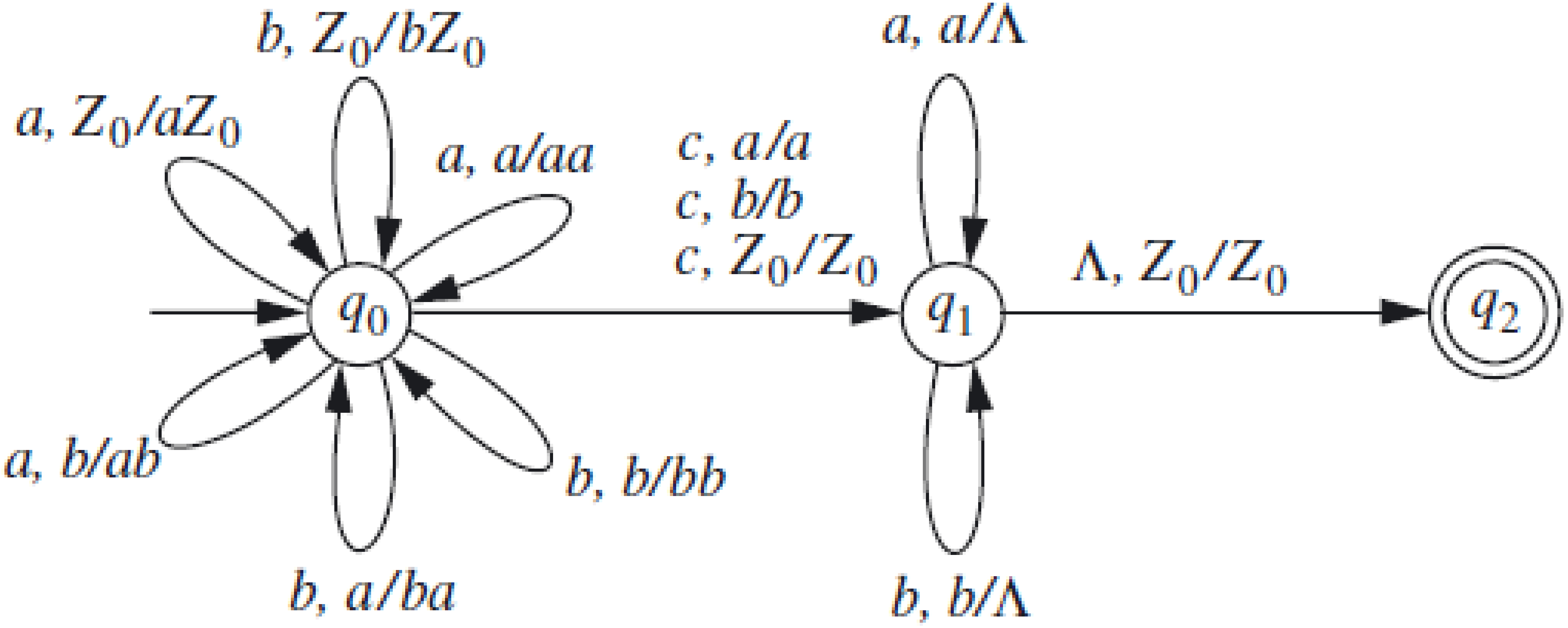
Tracing Moves; “abcba”, “ab”

Move #	Resulting state	Unread input	Stack
(initially)	q_0	abcba	Z_0
1	q_0	bcba	aZ_0
4	q_0	cba	baZ_0
9	q_1	ba	baZ_0
11	q_1	a	aZ_0
10	q_1	-	Z_0
12	q_2	-	Z_0
(accept)			
(initially)	q_0	ab	Z_0
1	q_0	b	aZ_0
4	q_0	-	baZ_0
(crash)			

Example 1 ...contd

- Simple palindrome (*SimplePal*) recognizer
  - Consider an input string like *acaa*
    - Portion of the string "*aca*" can be considered as accepted, but not the whole string
  - "Transition diagram" in Fig. 7.1 (p. 255)
    - Cannot be seen in the same way as for an FA
    - Special notations used for transitions
      - *Input symbol, stack symbol* / *string for top of stack*
    - Stack contents not seen; full status not shown
    - Form of this discussed again later

Transition Diagram



Example 2

- Example 7.2 in text book p. 257
- A PDA accepting ***the language of all palindromes*** over {a, b} even or odd-length (let's call this ***Pal***)
  - When we reach the mid-point, if odd-length, discard mid symbol
  - Push all symbols in first half to stack
  - Match second half symbols to symbols on stack

Example 2 ...contd

- How does PDA know mid-point reached?
  - PDA has to **guess**
  - Guessing OK if non-palindromes not accepted
  - First a sequence of "***not yet***" guesses
    - Push each symbol to stack (first half)
  - Then a "***yes***" guess stops the "not yet" sequence
    - Odd-length (xsx^r): discard next, match after that
    - Even-length (xx^r): : match from next
  - After that no more guesses

Example 2 ...contd

- Consequences of above guessing
  - Non-palindromes will not be accepted
  - Can accept all palindromes
    - There is a sequence of choices involving making the correct “yes” guess at the right time
  - PDA may also guess at the wrong time for a given palindrome
    - May not accept or accept a different palindrome

Example 2 ...contd

- PDA for ***Pal*** (similar to that of Example 1)
  - 3 states $Q=\{q_0, q_1, q_2\}$
  - q_0 = initial (guessing) state, q_1 = comparison-making state, q_2 = accepting state
  - Input alphabet $\Sigma=\{a, b\}$
  - Stack alphabet $\Gamma=\{a, b, Z_0\}$
  - Transition function δ: Table 7.2 (p. 258), next slide

Example 2: Transition Table

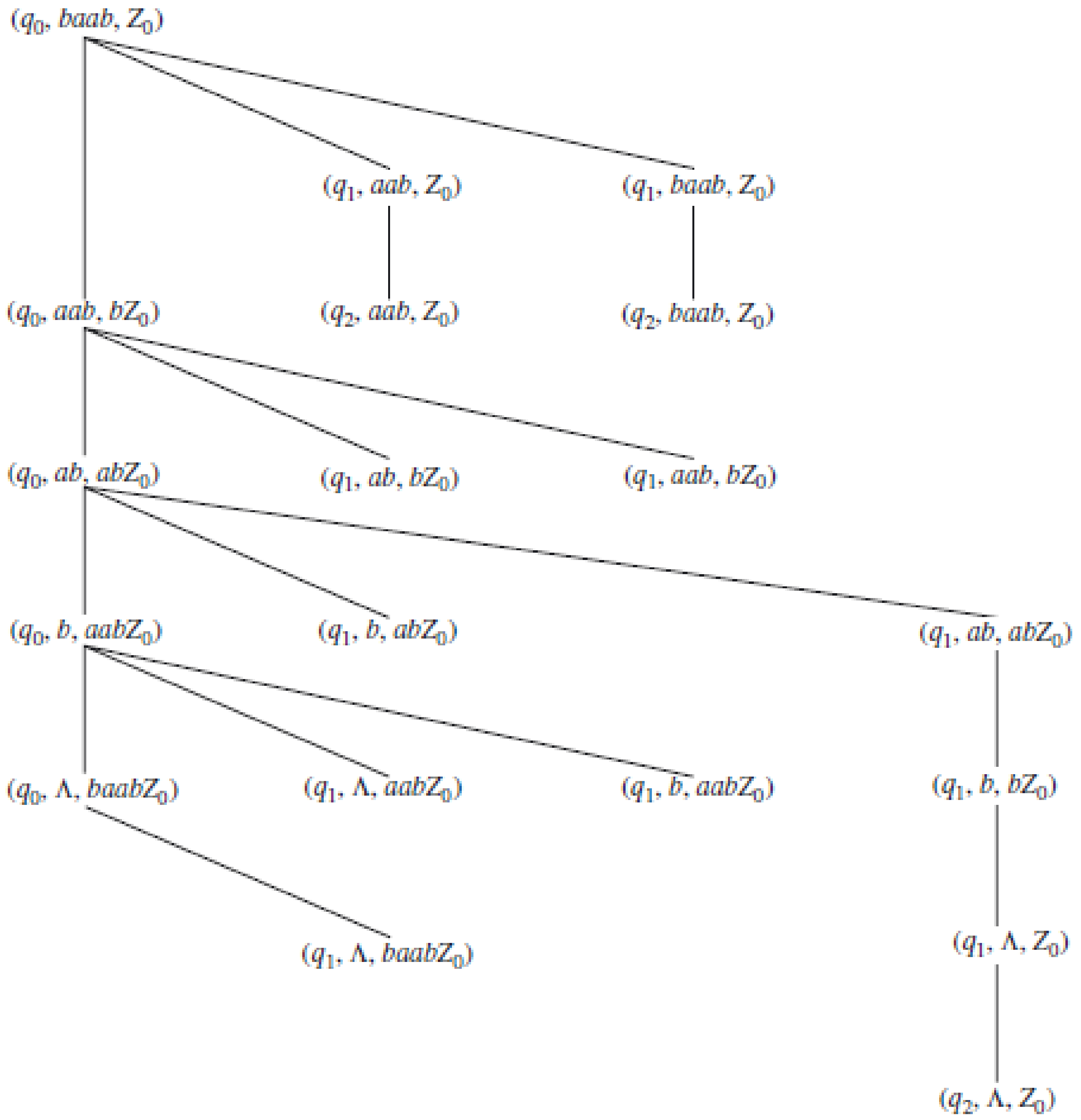
- Table 7.2 Note the non-determinisms

Move Number	State	Input	Stack Symbol	Move(s)
1	q_0	a	Z_0	$(q_0, aZ_0), (q_1, Z_0)$
2	q_0	a	a	$(q_0, aa), (q_1, a)$
3	q_0	a	b	$(q_0, ab), (q_1, b)$
4	q_0	b	Z_0	$(q_0, bZ_0), (q_1, Z_0)$
5	q_0	b	a	$(q_0, ba), (q_1, a)$
6	q_0	b	b	$(q_0, bb), (q_1, b)$
7	q_0	Λ	Z_0	(q_1, Z_0)
8	q_0	Λ	a	(q_1, a)
9	q_0	Λ	b	(q_1, b)
10	q_1	a	a	(q_1, Λ)
11	q_1	b	b	(q_1, Λ)
12	q_1	Λ	Z_0	(q_2, Z_0)
	(all other combinations)			none

Example 2: Computation Tree

- Computation tree shows the configuration at each step and possible choices of moves at each step
- Fig. 7.2 (p.259, next slide) shows the *computation tree* that traces the moves for input *baab*

Computation Tree



Example 2 ...contd

- Sequence of moves leading to acceptance of input string *baab*

$$
\begin{aligned}
(q_0, baab, Z_0) \quad & \vdash (q_0, aab, bZ_0) \\
& \vdash (q_0, ab, abZ_0) \\
& \vdash (q_1, ab, abZ_0) \\
& \vdash (q_1, b, bZ_0) \\
& \vdash (q_1, \Lambda, Z_0) \\
& \vdash (q_2, \Lambda, Z_0) \quad \text{(accept)}
\end{aligned}
$$

Conclusion

- Our discussion today
  - Review on Ambiguity
  - Simplified Forms
  - Normal Forms
  - Pushdown Automata
    - Definition, Acceptance
  - Examples