CS3063 Theory of Computing

Semester 4 (21 Intake), Jan – May 2024

Lecture 4

Regular Languages & Finite Automata – Session 3

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Announcements

- Assignment 1
  – Will be out this week (check on Moodle)
  – Due after 2 weeks

Today's Outline

Lecture 4

- NFA-Λ (NFA with Λ–transitions)
- Equivalence among FA, NFA, NFA-Λ
- RE→FA, via Thompson's Algorithm

PART 1

Today's Outline

Lecture 4

- NFA-Λ (NFA with Λ–transitions)
- Equivalence among FA, NFA, NFA-Λ
- RE→FA, via Thompson's Algorithm

Review

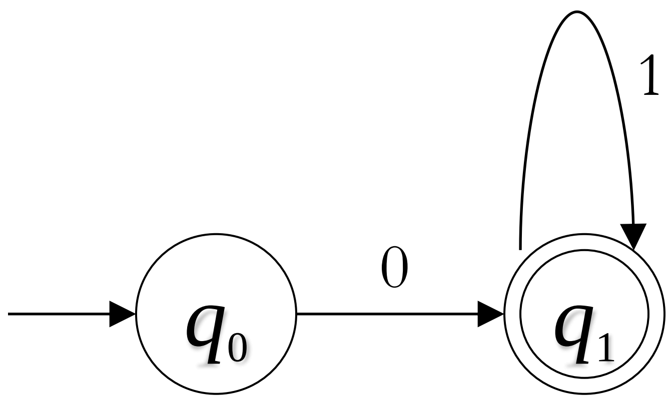
- Regular expressions/languages
- Finite automata (FA): deterministic version, DFA
- A language is regular iff it is the set of strings accepted by some FA
- FA $\leftrightarrow$ Regular Expressions
- Set operations on languages
- NFA
- Equivalence between NFA and DFA

NFA with Λ–transitions (NFA-Λ)

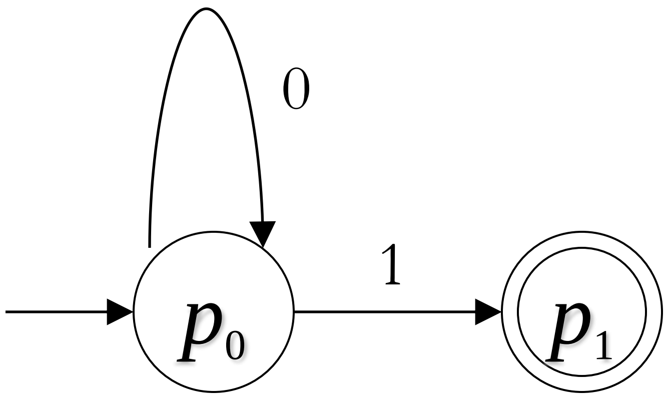
- NFA with even more freedom
  - Transitions without any input (i.e., with Λ)
  - More general than NFA
  - Denoted by NFA-Λ

- But no more powerful than NFA (or DFA) with respect to languages accepted!

Example

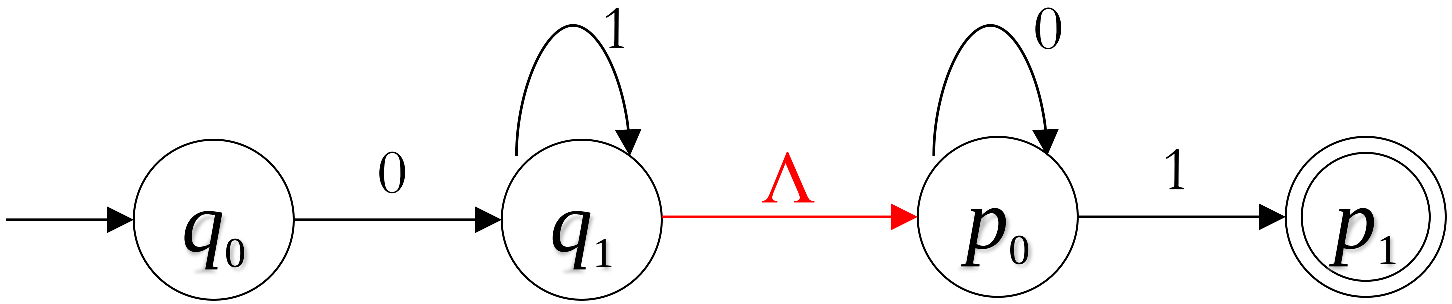


(a) NFA accepting 01*

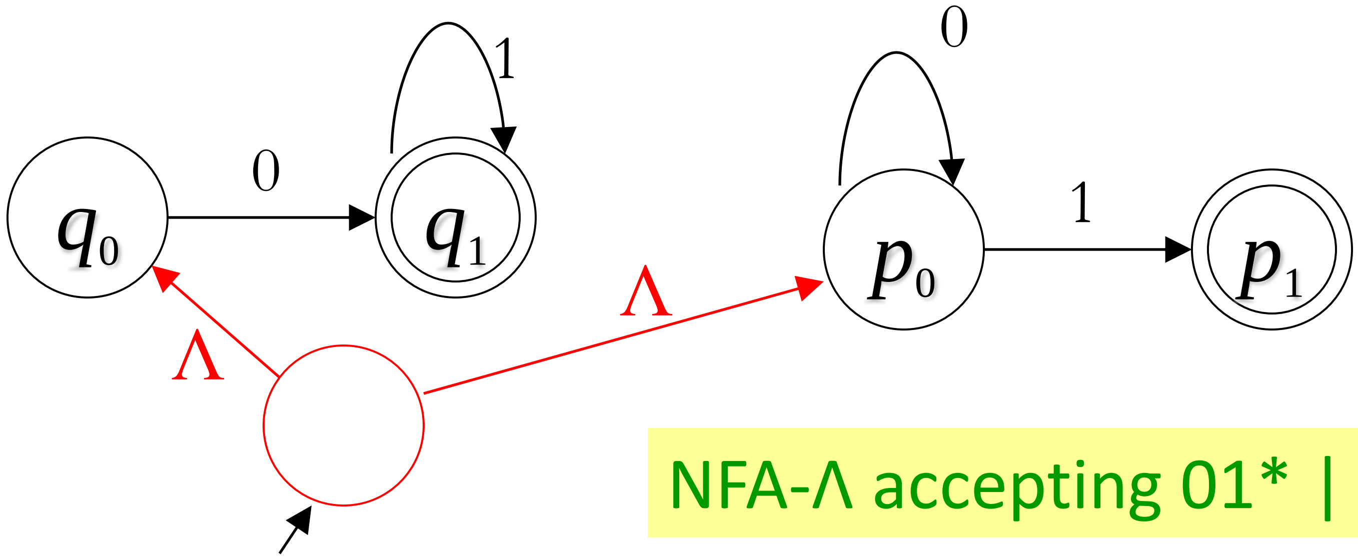


(b) NFA accepting 0*1

Example ...contd



NFA-Λ accepting 01*0*1



NFA-Λ accepting 01* | 0*1

NFA-Λ: Definition

- NFA-Λ is a 5-tuple $(Q, \Sigma, q_0, A, \delta)$, where:
  - Q, Σ, q_0 and A are the same as for an NFA
  - But …
  - δ, the transition function, maps $Q \times (\Sigma \cup \{\Lambda\})$ to 2^Q

Extended Transition Function δ^*

- Can we extend δ, as with NFA, to obtain δ^* that describes the status of an NFA-Λ on an input string x ?

- Not as easy to define recursively, because Λ-transitions are involved

- Let us define the notion of **Λ-closure** first

Λ-Closure

- **Definition**: Let M=$(Q, \Sigma, q_0, A, \delta)$ be an NFA-Λ and S be a subset of Q. The **Λ-closure** of S is the set Λ(S) defined as follows:
  - Every element of S is an element of Λ(S)
  - For any q in Λ(S), every element of $\delta(q, \Lambda)$ is in Λ(S)
  - No other element of Q is in Λ(S)

- For a state q, $\Lambda(q)$ is the set of all states that can be reached from q using Λ-transitions
  - There is a path labeled Λ from q to all such states

Λ-Closure ...contd

- If $\delta^*(q, y)$ is the set of all the states that can be reached from q using the symbols of y and Λ–transitions, then

$$R = \bigcup_{r \in \delta^*(q,y)} \delta(r, a)$$

is the set of states R we can reach in one more step by using the symbol a

- The Λ-closure of this R includes any additional states that we can reach with Λ–transitions subsequently

Defining δ* for an NFA-Λ

- Definition: Let M=(Q, Σ, q_0, A, δ) be an NFA-Λ; the function δ^* : $Q \times \Sigma^* \rightarrow 2^Q$ is such that:
  - For any $q \in Q$, $\delta^*(q, \Lambda) = \Lambda(\{q\})$
  - For any $q \in Q$, $y \in \Sigma^*$and, $a \in \Sigma$

$$\delta^*(q, ya) = \Lambda\left(\bigcup_{r \in \delta^*(q,y)} \delta(r, a)\right)$$

A string x is **accepted** by M if

$$\delta^*(q_0, x) \cap A \neq \emptyset$$

PART 2

Today's Outline

Lecture 4

- NFA-Λ (NFA with Λ–transitions)
- **Equivalence among FA, NFA, NFA-Λ**
- RE→FA, via Thompson's Algorithm

Equivalence of NFA-Λ & NFA

- Theorem: For an NFA-Λ, $M=(Q, \Sigma, q_0, A, \delta)$ accepting a language $L \subseteq \Sigma^*$, there is an NFA, $M_1=(Q_1, \Sigma, q_1, A_1, \delta_1)$ that accepts L
- Proof involves showing M_1 will be the NFA $(Q, \Sigma, q_0, A_1, \delta_1)$ where, for $q \in Q$ and $a \in \Sigma$, $\delta_1(q,a) = \delta^*(q,a)$ and

$$A_1 = \begin{cases} A \cup \{q_0\} & \text{if } \Lambda(\{q_0\}) \cap A \neq \emptyset \text{ in } M \\ A & \text{otherwise} \end{cases}$$

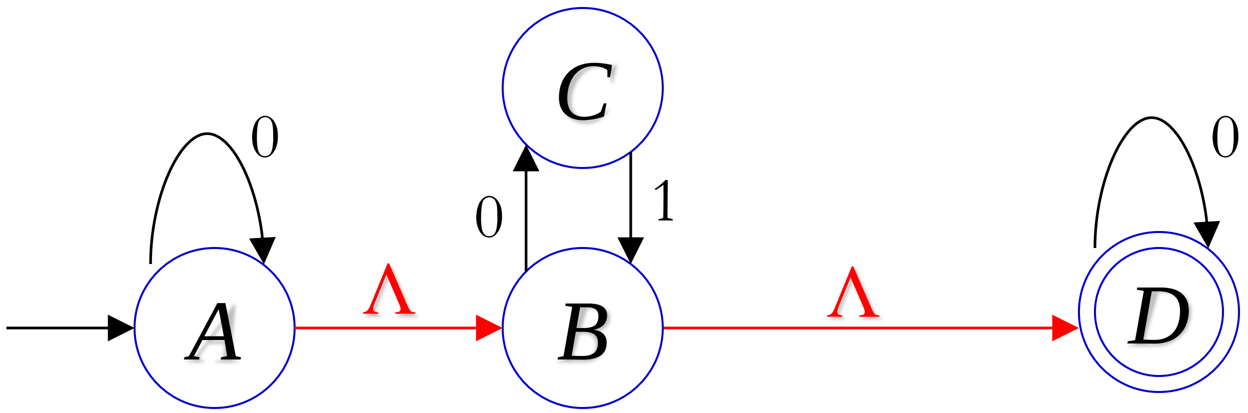
NFA Equivalent to NFA-Λ: How?

- From the theorem on equivalency
  - Proof gives a method to obtain equivalent NFA
  - Proof by induction (on length of input string)
- Method based on eliminating Λ–transitions without changing states
  - E.g., if we have $p \rightarrow q$ for 0 and $q \rightarrow r$ for Λ, eliminate the Λ-transition and add $p \rightarrow r$ for 0
  - The transition function δ_1 shows how this is done

Constructing an NFA Equivalent to an NFA-Λ ...contd

- For a state q and symbol a, $\delta^*(q, a)$ is the set of states that can be reached from q, using a and Λ-transitions before and after
- Can say similarly for a string x
- Specifying the accepting states A_1 is to be done with care
  - Check whether it is possible to go to one of A from q_0 using only Λ-transitions
  - If yes, make q_0 an element of A_1

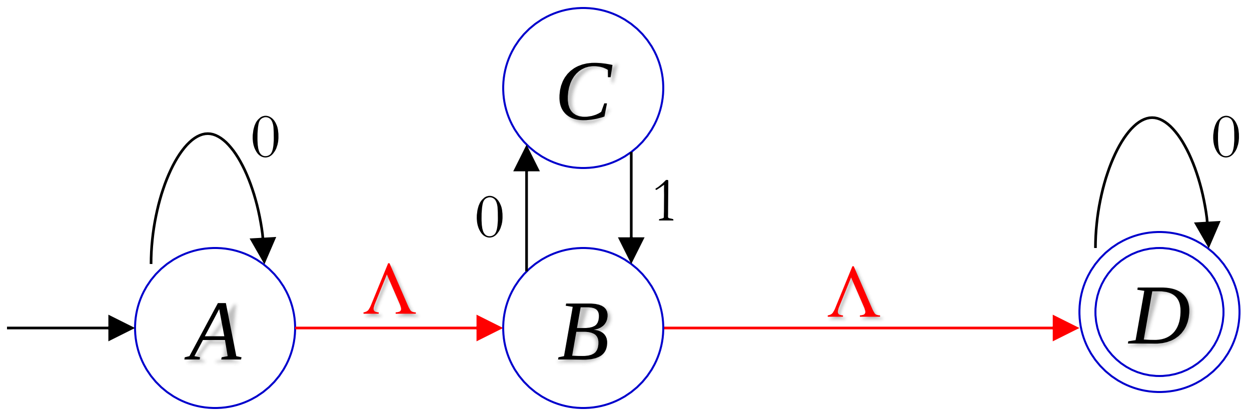
Example: NFA-Λ accepting 0*(01)*0*



Find an equivalent NFA

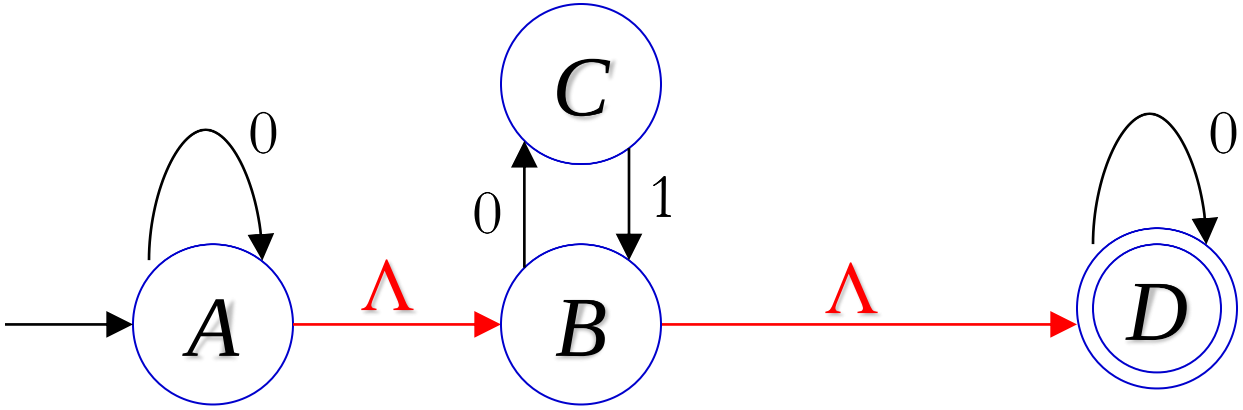
q	$\delta(q, \Lambda)$	$\delta(q, 0)$	$\delta(q, 1)$
A	$\{B\}$	$\{A\}$	$\emptyset$
B	$\{D\}$	$\{C\}$	$\emptyset$
C	$\emptyset$	$\emptyset$	$\{B\}$
D	$\emptyset$	$\{D\}$	$\emptyset$

Solution

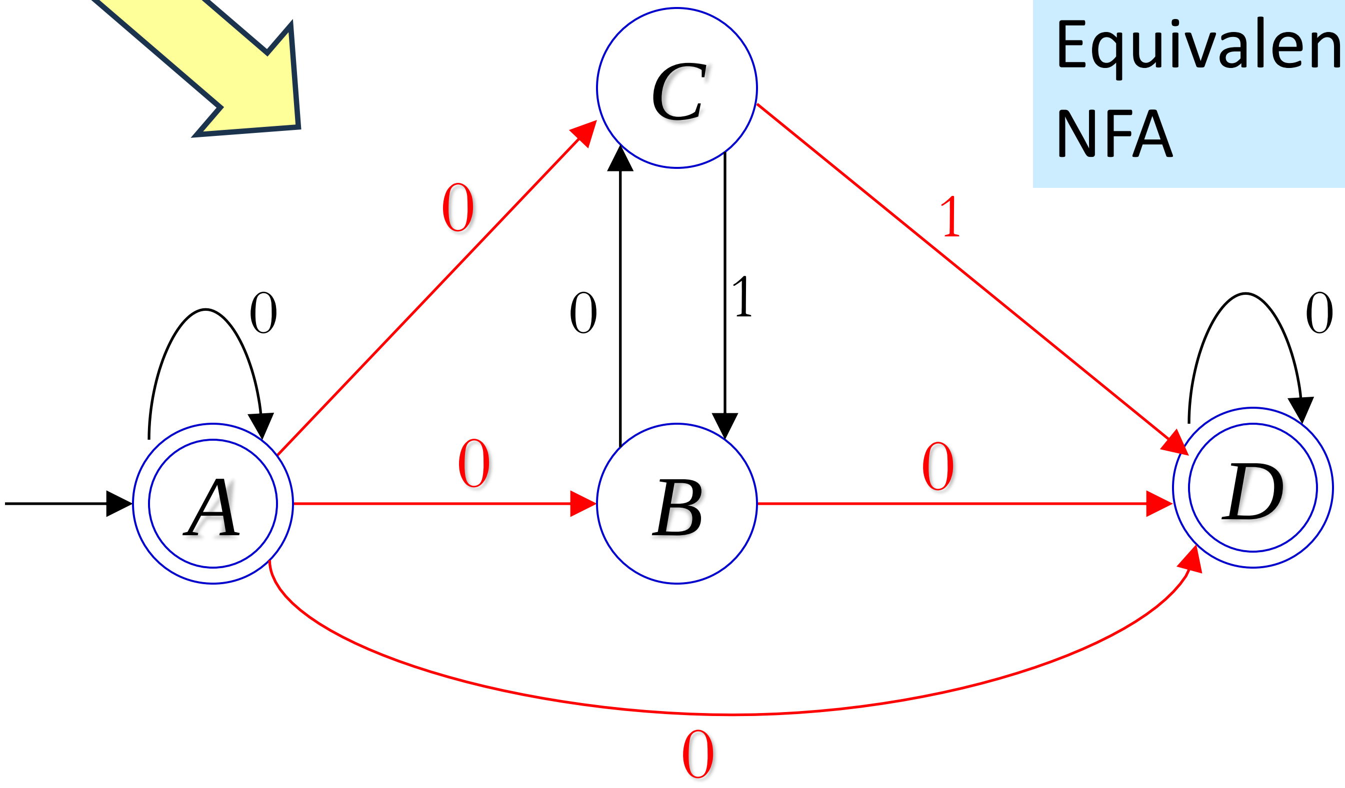


q	$\delta(q, \Lambda)$	$\delta(q, 0)$	$\delta(q, 1)$	$\delta^*(q, 0)$	$\delta^*(q, 1)$
A	$\{B\}$	$\{A\}$	Ø	$\{A,B,C,D\}$	Ø
B	$\{D\}$	$\{C\}$	Ø	$\{C,D\}$	Ø
C	Ø	Ø	$\{B\}$	Ø	$\{B,D\}$
D	Ø	$\{D\}$	Ø	$\{D\}$	Ø

Solution ...contd



Given NFA-Λ



Equivalent NFA

Summary on FA

- Theorem
  - For any alphabet Σ, and a language $L \subset \Sigma^*$, the following statements are equivalent:

1. L can be recognized by a DFA
2. L can be recognized by an NFA
3. L can be recognized by an NFA-Λ

Kleene’s Theorem (again)

- ***A language L is regular if and only if there is an FA that accepts L***

- Part 1
  – Any regular language can be recognized by an FA
- Part 2
  – The language accepted by any FA is regular

PART 3

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- NFA-Λ (NFA with Λ–transitions)
- Equivalence among FA, NFA, NFA-Λ
- RE→FA, via Thompson's Algorithm

RE→DFA : How? (Again)

- Steps
  1. Thompson's construction: RE → NFA-Λ
  2. Convert NFA-Λ → NFA → DFA

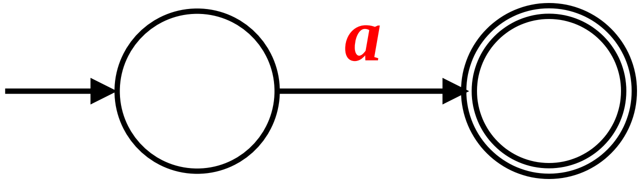
- Thompson's construction
  – Also known as the McNaughton-Yamada-Thompson algorithm

Thompson's Construction

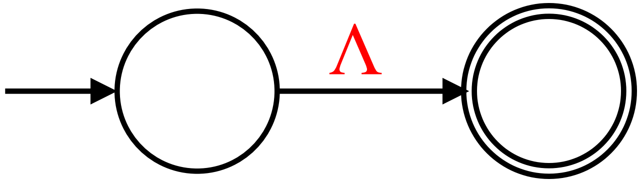
- Input: regular expression ***r*** over alphabet Σ
- Output: NFA-Λ for ***r***
- Method
  - Break ***r*** into subexpressions, recursively until no operators are present in them
    - (get basic regular languages as elements)
  - Apply inductive rules to construct larger NFA-Λ for an expression from NFA-Λ of its subexpressions

Thompson's Construction: Basis

- Basic languages: {***a***}, {Λ}, ∅   (given ***a*** in Σ)
- Their NFA-Λ



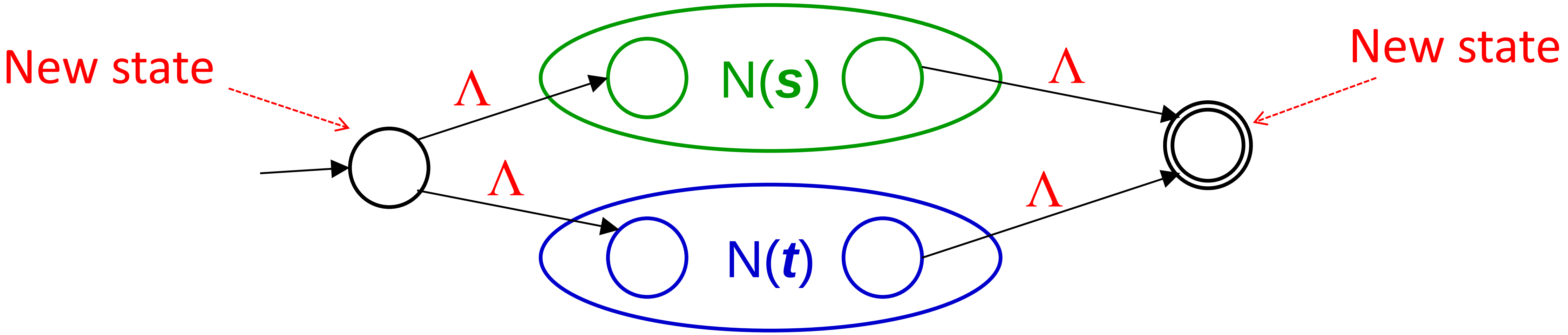
{***a***}



{Λ}

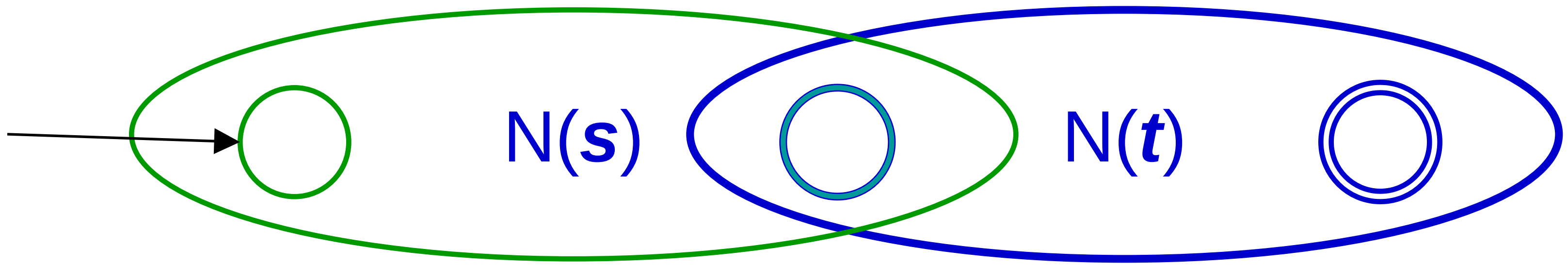
Thompson’s Construction: Induction

- Suppose N(***s***) and N(***t***) are NFA-Λ for regular expressions ***s*** and ***t***, respectively
- **Case (a)**: ***r*** = ***s*** | ***t*** (Union)
  - N(***r***), the NFA-Λ for ***r*** is constructed as follows



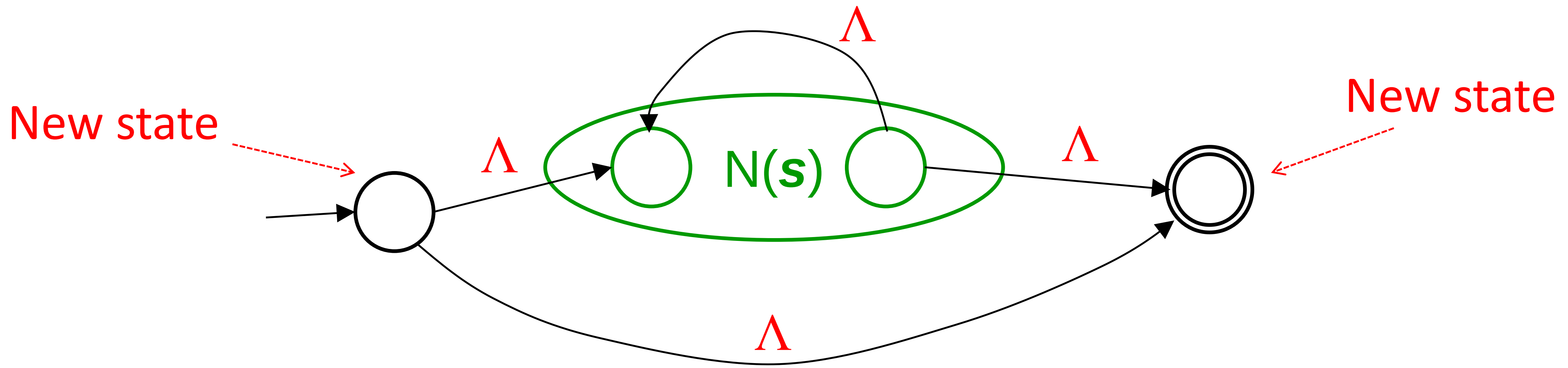
Thompson's Construction: Induction

- Suppose N(***s***) and N(***t***) are NFA-Λ for regular expressions ***s*** and ***t***, respectively
- **Case (b)**: ***r*** = ***st*** (Concatenation)
  - – N(***r***), the NFA-Λ for ***r*** is constructed as follows



Thompson's Construction: Induction

- Suppose N(***s***) is the NFA-Λ for regular expression ***s***
- **<u>Case (c)</u>**: ***r*** = ***s**** (Closure)
  - N(***r***), the NFA-Λ for ***r*** is constructed as follows



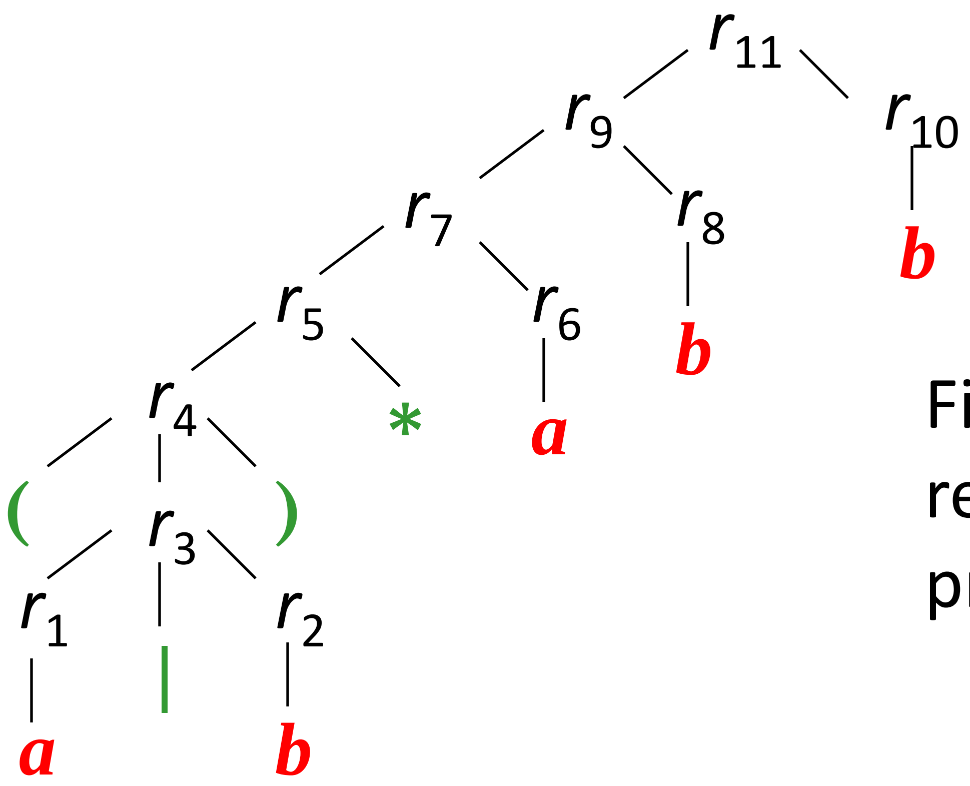
Thompson's Construction: Induction

- Suppose N(***s***) is the NFA-Λ for regular expression ***s***
- **<u>Case (d)</u>**: ***r* = (*s*)**
  - N(***r***), the NFA-Λ for ***r*** is the same as N(***s***)

- Note:
  - # states in N(***r***) ≤ 2x # operators, operands
  - N(***r***) has 1 start state (no incoming edge), 1 accepting state (no outgoing edge)

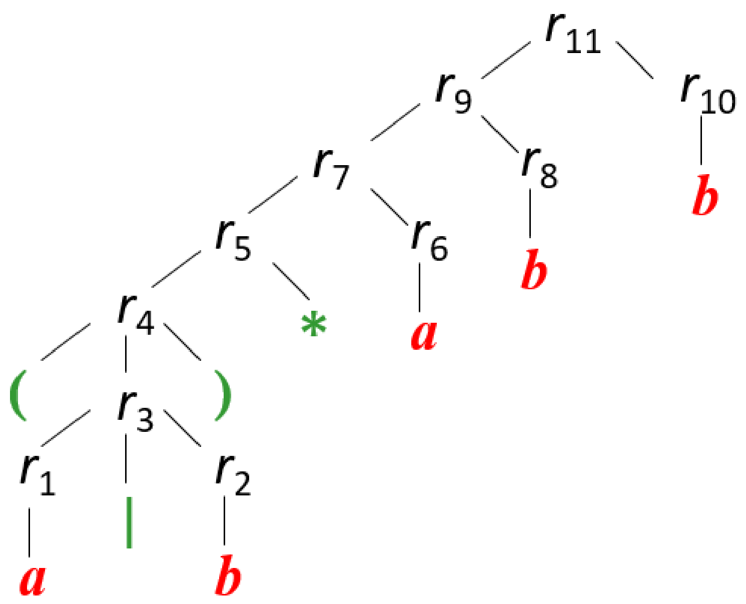
Example

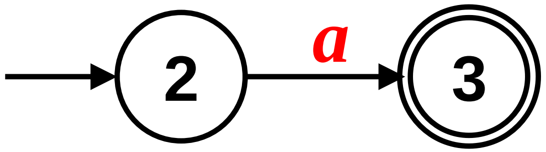
- Use Thompson's construction algorithm to construct an NFA-Λ for r = $(a|b)^*abb$



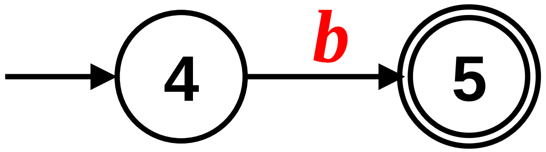
First, break r into subexpressions, recursively until no operators are present in them

Solution (for $r = (a|b)^*abb$) …

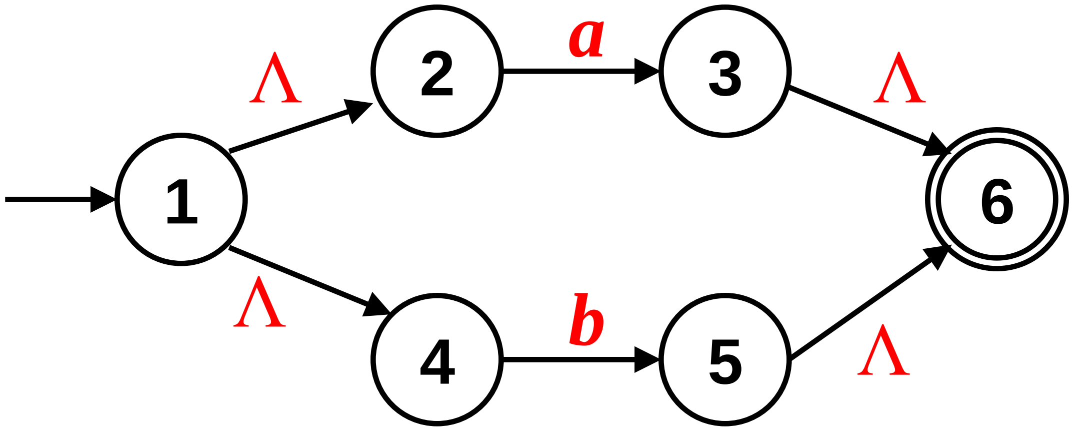




NFA-Λ for r_1

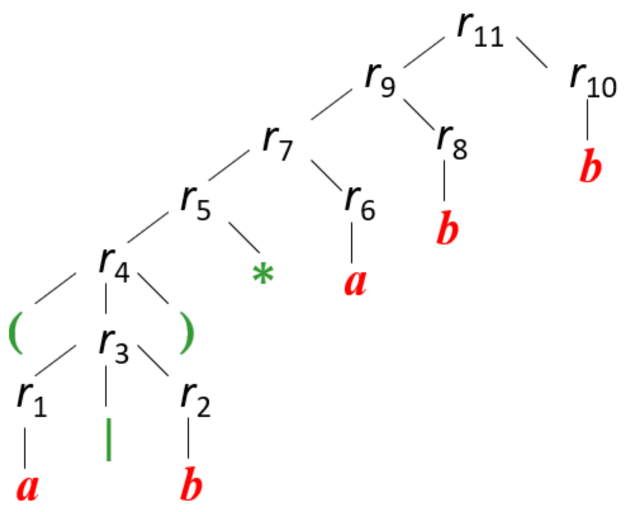


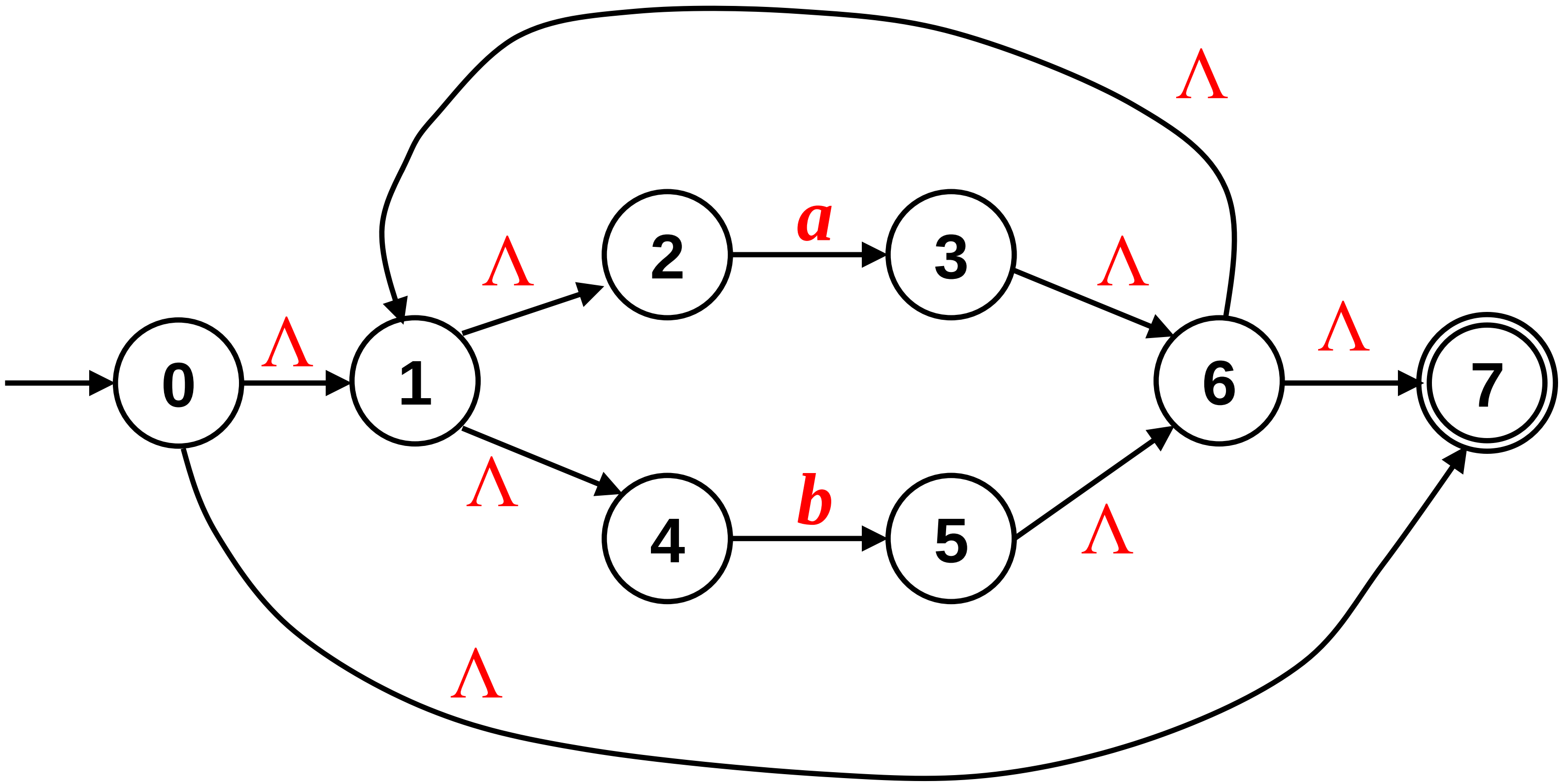
NFA-Λ for r_2



NFA-Λ for $r_3 = a|b$

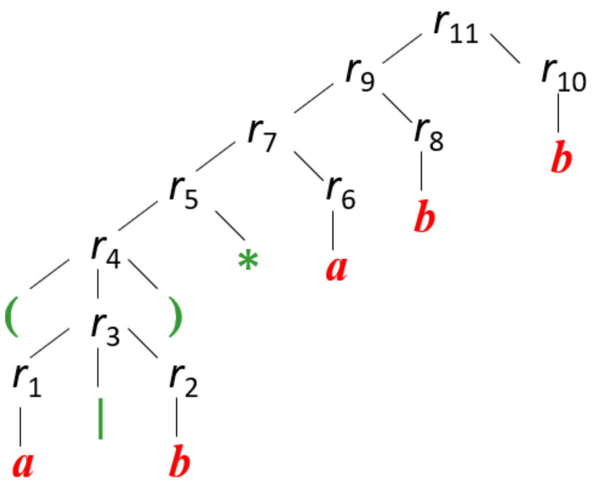
Solution (for $r = (a|b)^*abb$) …

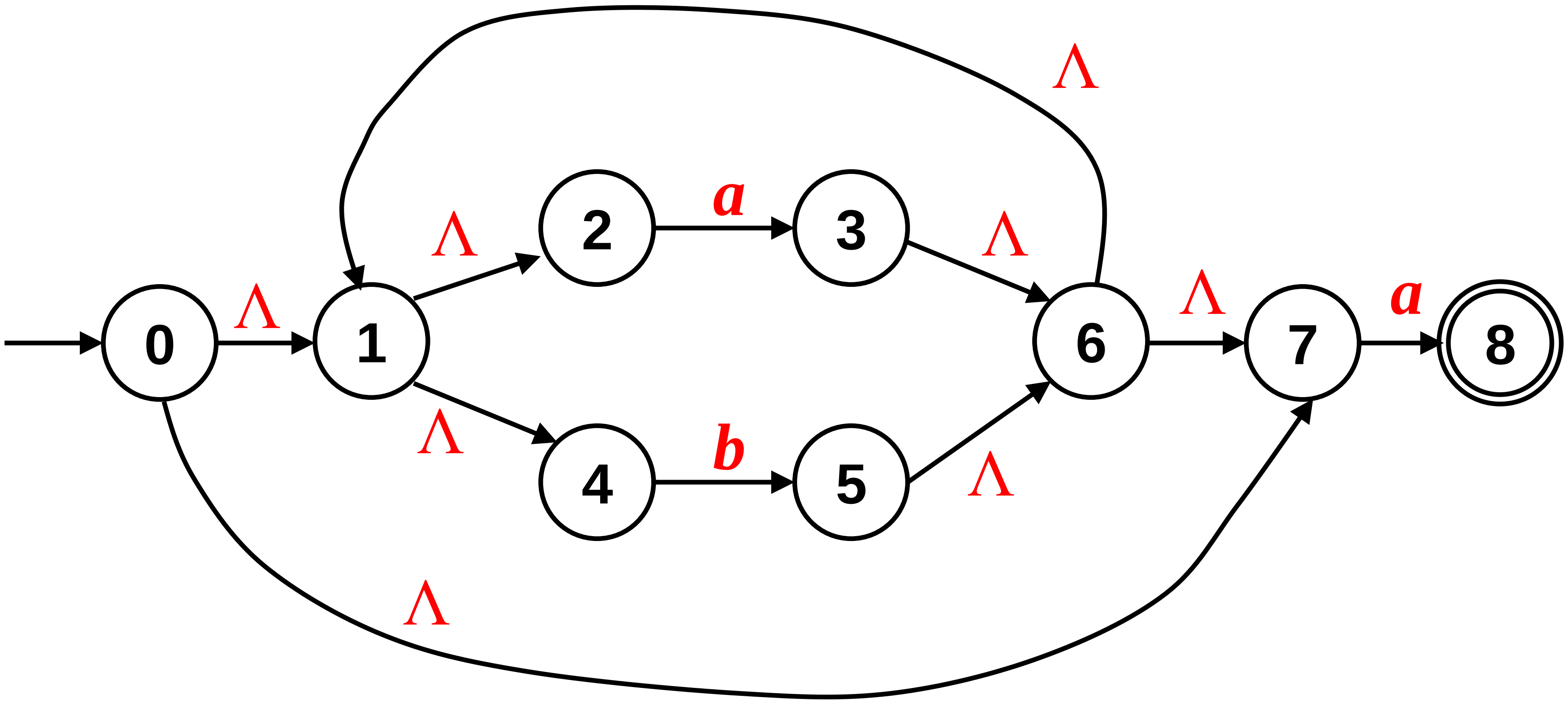




NFA-Λ for $r_5 = (a|b)^*$

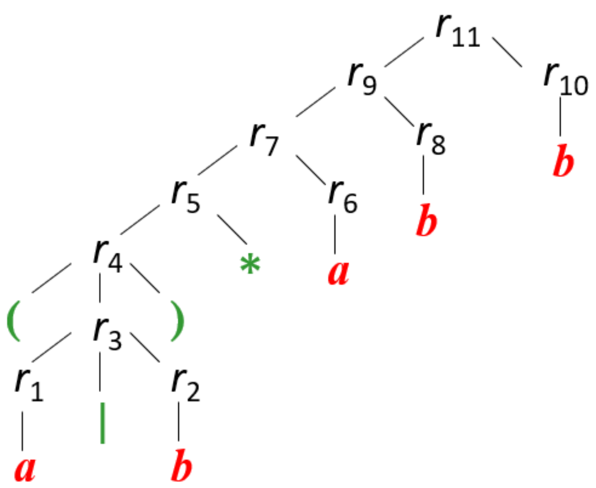
Solution (for r = (a|b)*abb) …

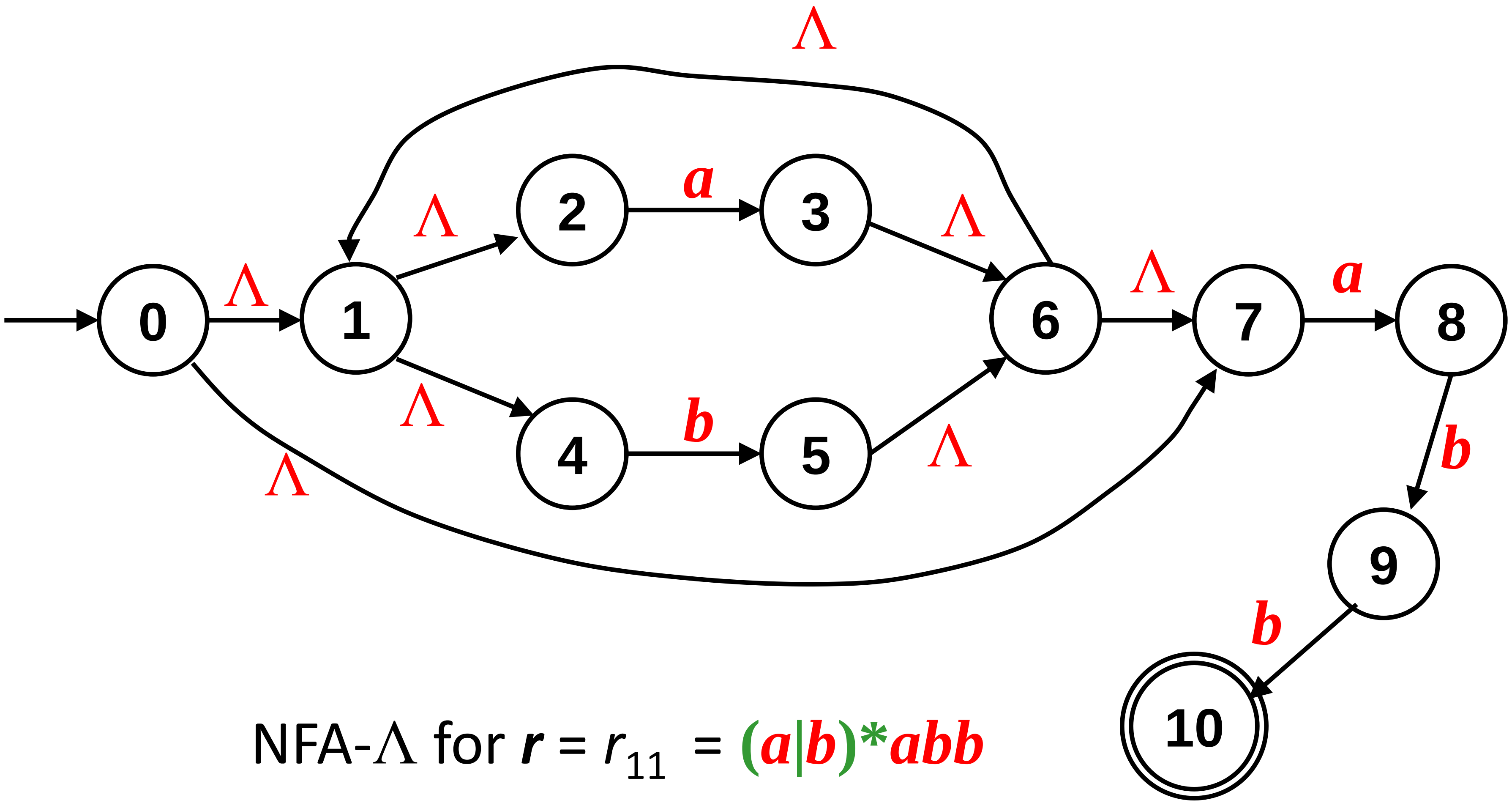




NFA-Λ for r_7 = (a|b)*a

Final Solution (for $r = (a|b)^*abb$)





NFA-Λ for $r = r_{11} = (a|b)^*abb$

Conclusion

- Today we discussed
  - NFA-Λ (NFA with Λ–transitions)
  - Converting NFA-Λ to equivalent NFA
  - Equivalence among DFA, NFA, NFA-Λ
  - Thompson's construction