CS251 Assignment-1
Problem 2

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August 22, 2022

Problem 2

Consider two independent Poisson random variables X and Y , with parameters $\lambda_X := 3$ and $\lambda_Y := 4$. The PMF for the poisson distribution is given as

$$p_\lambda(X = k) = e^{-\lambda}\frac{\lambda^k}{k!} \tag{1}$$

- Define a random variable $Z := X + Y$, having a probability mass function (PMF) $P(Z)$.

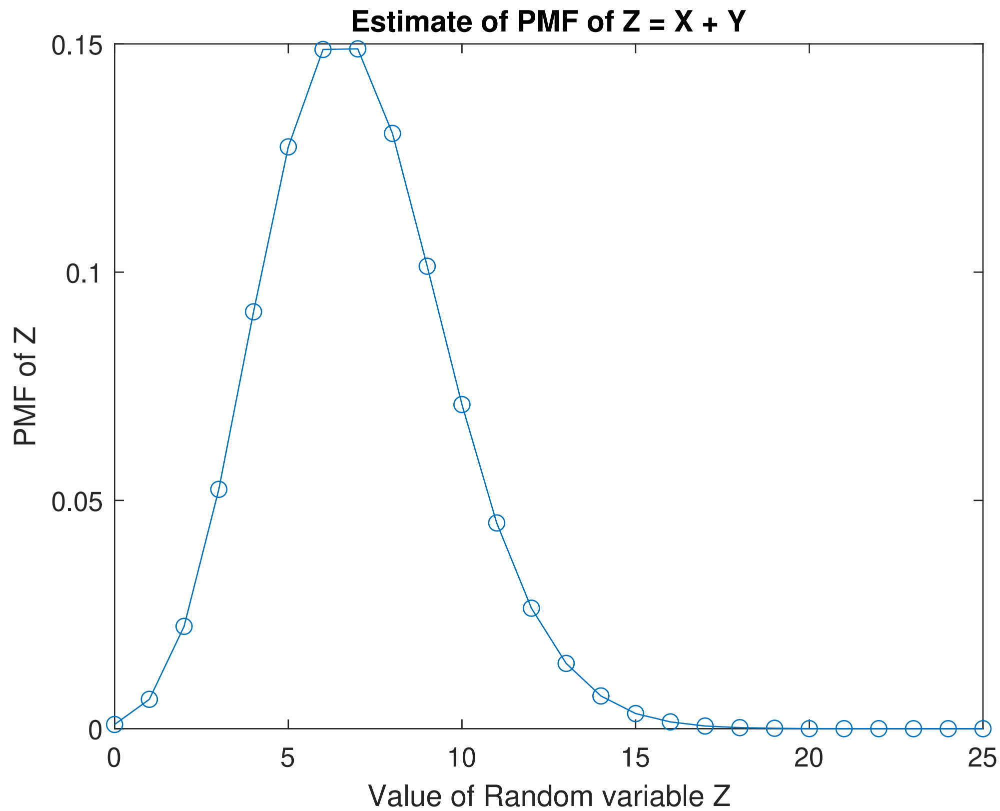
  1. Empirically obtain an estimate $\hat{P}(Z)$ of the PMF $P(Z)$, by drawing $N = 10^6$ instances (sample points) of X and Y both. You may use the poissrnd$(\cdot)$ function in Matlab. Report the values of $\hat{P}(Z = k)$ for $k = 0, 1, 2, \cdots, 25$.

     We first generated the N instances as a size N array of `poissrnd(`λ_x`)+poissrnd(`λ_y`)`. To count the frequencies of these values we simply used the `hist()` function. Divide by N to get the empirical probability.

     Values generated are also reported in `problem2.mlx` too.

```
P_emp_1 = 1x26
  0 to  4
    0.0009    0.0064    0.0224    0.0524    0.0913
  5 to  9
    0.1274    0.1488    0.1489    0.1304    0.1013
 10 to 14
    0.0710    0.0451    0.0264    0.0143    0.0072
 15 to 19
    0.0033    0.0015    0.0006    0.0002    0.0001
 20 to 24
    0.0000    0.0000    0.0000    0         0
 25
    0
```

     The following plot was generated from the data–



2. What will the PMF $P(Z)$ be theoretically/analytically ?
The poisson distribution is

$$P_\lambda(a) = \frac{\lambda^a}{a!} e^{-\lambda} \tag{2}$$

$P(Z = z)$ is

$$P(Z = z) = \sum_{x=0}^{z} P(X = x) \cdot P(Y = z - x) \tag{3}$$

$$= \sum_{x=0}^{z} \frac{\lambda_x^x}{x!} \frac{\lambda_Y^{z-x}}{(z-x)!} e^{-\lambda_X - \lambda_Y} \tag{4}$$

$$= \sum_{x=0}^{z} \frac{(\lambda_X/\lambda_Y)^x \cdot {}^zC_x}{z!} \lambda_y^z e^{-\lambda_X - \lambda_Y} \tag{5}$$

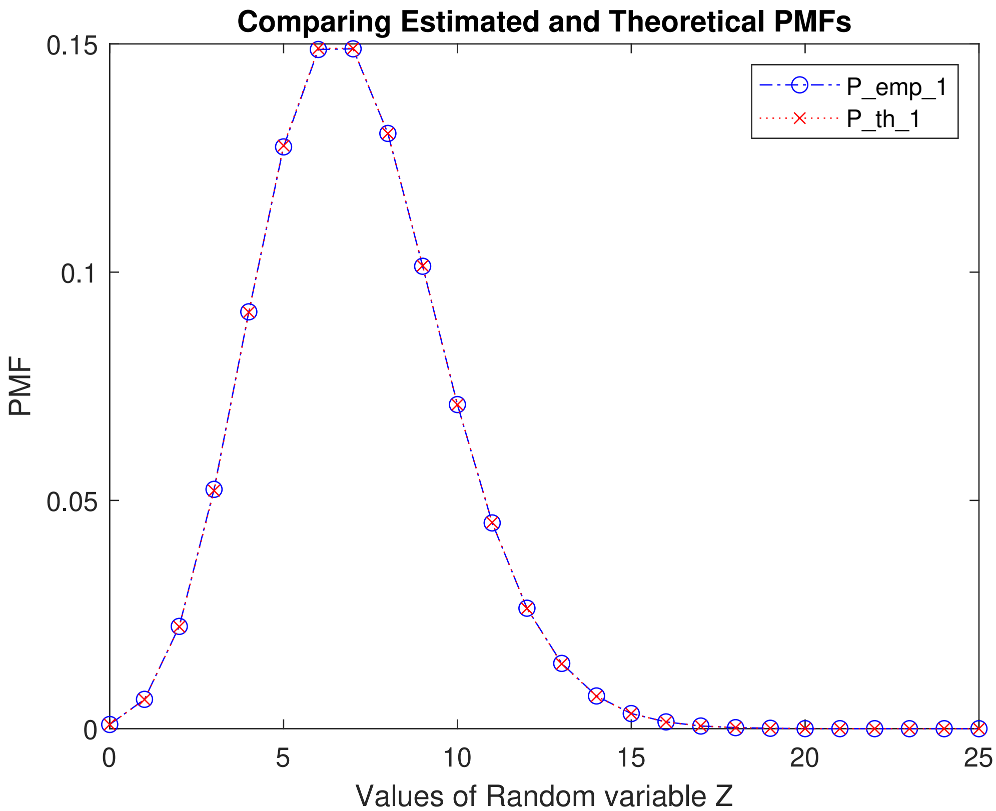
$$= \frac{(\lambda_X + \lambda_Y)^z}{z!} e^{-(\lambda_X + \lambda_Y)} \tag{6}$$

This is again a poisson distribution, which also agrees with the general intuition.

3. Show and compare the values for $\hat{P}(Z = k)$ and for $k = 0, 1, 2, \cdots, 25$.

	k	Theoretical	Empirical
1	0	0.0009	0.0009
2	1	0.0064	0.0064
3	2	0.0223	0.0224
4	3	0.0521	0.0524
5	4	0.0912	0.0913
6	5	0.1277	0.1274
7	6	0.1490	0.1488
8	7	0.1490	0.1489
9	8	0.1304	0.1304
10	9	0.1014	0.1013
11	10	0.0710	0.0710
12	11	0.0452	0.0451
13	12	0.0263	0.0264
14	13	0.0142	0.0143
15	14	0.0071	0.0072
16	15	0.0033	0.0033
17	16	0.0014	0.0015
18	17	0.0006	0.0006
19	18	0.0002	0.0002
20	19	0.0001	0.0001
21	20	0	0
22	21	0	0
23	22	0	0
24	23	0	0
25	24	0	0
26	25	0	0

Values generated are also reported in `problem2.mlx`. For comparing the true and estimated PMFs, we plotted the two on a graph–.


Comparing Estimated and Theoretical PMFs
P_emp_1
P_th_1
PMF
0.15
0.1
0.05
0
0
5
10
15
20
25
Values of Random variable Z

- Implement a Poisson thinning process (as discussed in class) on the random variable Y , where the thinning process uses probability parameter 0.8. Let the thinned random variable be Z.

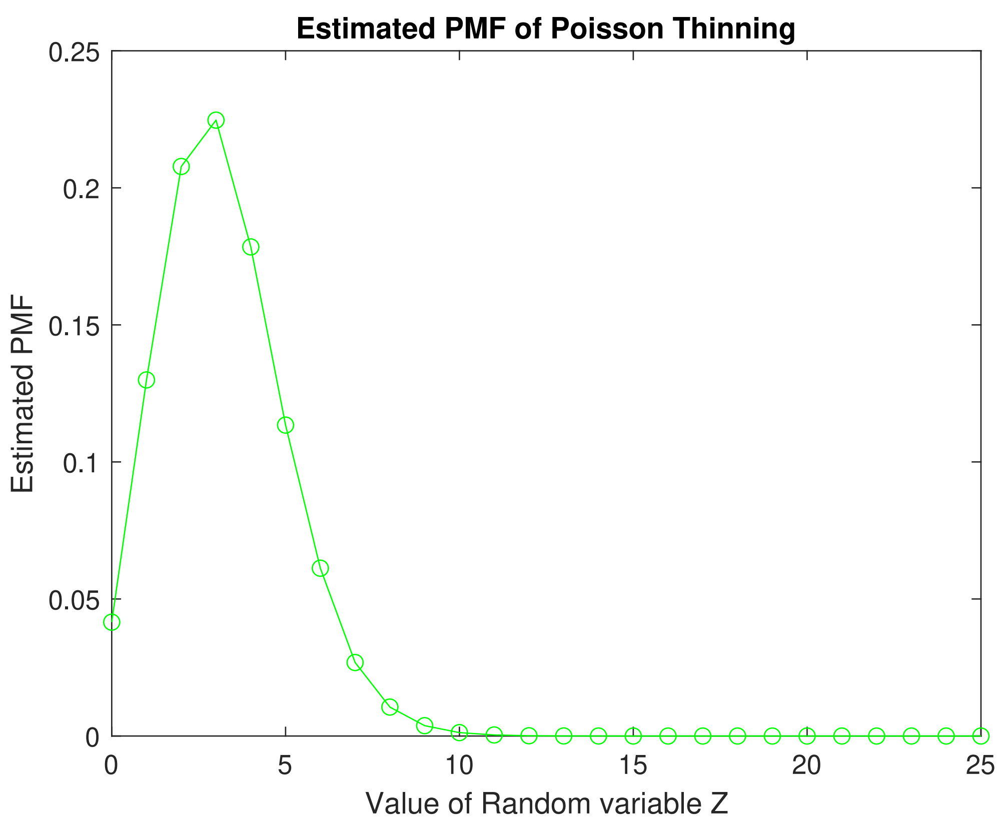
  1. Empirically obtain an estimate $\hat{P}(Z)$ of the PMF $P(Z)$, by drawing $N := 10^5$ instances (sample points) from Y . You may use the poissrnd($\cdot$) and binornd($\cdot$) functions in Matlab. Report the values of $\hat{P}(Z = k)$ for $k = 0, 1, 2, \cdots , 25$.

     We first generated the pre-thinned variables as an array of size N, then applying binornd to this array to get our thinned variables. To count the frequencies of these values we again used the `hist()` function and then divided by N to get the empirical probability.

```
P_emp_2 = 1x26
  0 to  5
    0.0415    0.1299    0.2078    0.2247    0.1784    0.1134
  6 to 11
    0.0612    0.0268    0.0106    0.0038    0.0012    0.0004
 12 to 17
    0.0001    0.0000    0.0000    0         0         0
 18 to 23
    0         0         0         0         0         0
 24 to 25
    0         0
```

     Values generated are also reported in `problem2.mlx`.

     The following plot was generated from the data–



2. What will the PMF $P(Z)$ be theoretically/analytically ?

   Poisson thinning is a composition of a Poisson and Binomial experiments.

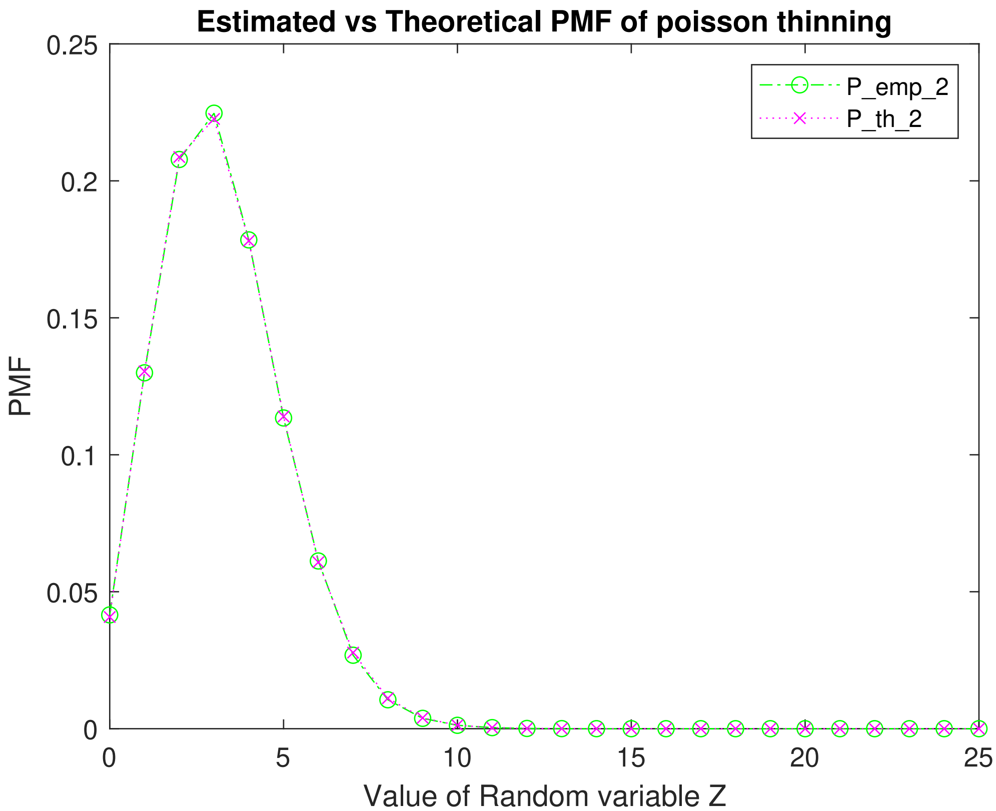
$$
\begin{aligned}
P(Z=k) &= \sum_{j=k}^{\infty} P(Y=j, Z=k) && (7)\\
&= \sum_{j=k}^{\infty} P(Z=k|Y=j)P(Y=j) && (8)\\
&= \sum_{j=k}^{\infty} \binom{j}{k} p^k (1-p)^{j-k} \cdot \frac{\lambda_Y^j}{j!} e^{-\lambda} && (9)\\
&= e^{-\lambda} \frac{p^k \lambda^k}{k!} \sum_{(j-k)=0}^{\infty} \frac{(1-p)^{j-k}}{(j-k)!} && (10)\\
&= \frac{e^{-\lambda p}(\lambda p)^k}{k!} && (11)
\end{aligned}
$$

   This is again a poisson distribution which agrees with the general intuition.

3. Show and compare the values for $\hat{P}(Z=k)$ and $P(Z=k)$ for $k = 0, 1, 2, \cdots, 25$.

	k	Theoretical	Empirical
1	0	0.0408	0.0415
2	1	0.1304	0.1299
3	2	0.2087	0.2078
4	3	0.2226	0.2247
5	4	0.1781	0.1784
6	5	0.1140	0.1134
7	6	0.0608	0.0612
8	7	0.0278	0.0268
9	8	0.0111	0.0106
10	9	0.0040	0.0038
11	10	0.0013	0.0012
12	11	0.0004	0.0004
13	12	0.0001	0.0001
14	13	0	0
15	14	0	0
16	15	0	0
17	16	0	0
18	17	0	0
19	18	0	0
20	19	0	0
21	20	0	0
22	21	0	0
23	22	0	0
24	23	0	0
25	24	0	0
26	25	0	0

   Values generated are also reported in `problem2.mlx`. For comparing the true and estimated PMFs, we plotted the two on a graph–.


Estimated vs Theoretical PMF of poisson thinning
0.25
0.2
0.15
0.1
0.05
0
PMF
0
5
10
15
20
25
Value of Random variable Z
P_emp_2
P_th_2