CS251 Assignment-1
Problem 1

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August 22, 2022

Problem 1

For each of the following distributions, do:

1. Plot the probability density function (PDF) based on the analytical expression. The PDF must appear smooth enough and without apparent signs of discretization.

2. Plot the cumulative distribution function (CDF) using Riemann-sum approximation. The CDF must appear smooth enough and without apparent signs of discretization.

3. Use Riemann-sum approximations to compute the approximate variance (if finite) within a tolerance of 0.01 of its true value known analytically.

Approach

(i) The respective analytical function was applied to the entire range at once to obtain the PDF. We took a step size of `int_size = 0.1` for the input values.

(ii) With Riemann-sum approximation,

$$P(X \leq x) = \int_{-\infty}^{x} p(t)dt \sim \sum_{i=-a}^{x} p(i) \cdot h \tag{1}$$

Where h is the rectangle width, and a an appropriately chosen lower bound.

For getting the cumulative sum for each x together we used the `cumsum()` function. Then multiplied the entire array with `int_size` ($= h$) to get the CDF function.

(iii) For the mean and variance, the Riemann-sum approximations were

$$\text{Mean} = E[X] = \int_{-\infty}^{\infty} x \cdot p(x)dx \sim \sum_{i=-a}^{b} i \cdot p(i)h \tag{2}$$

$$E[X^2] = \int_{-\infty}^{\infty} x^2 \cdot p(x)dx \sim \sum_{i=-a}^{b} i^2 \cdot p(i)h \tag{3}$$

$$\text{Var}(X) = E[X^2] - E[X]^2 \tag{4}$$

With h again taken as `int_size` , we computed `x.*pdf` and `x.*x.*pdf`, took their sums, and multiplied with `int_size` to get the respective expectations.

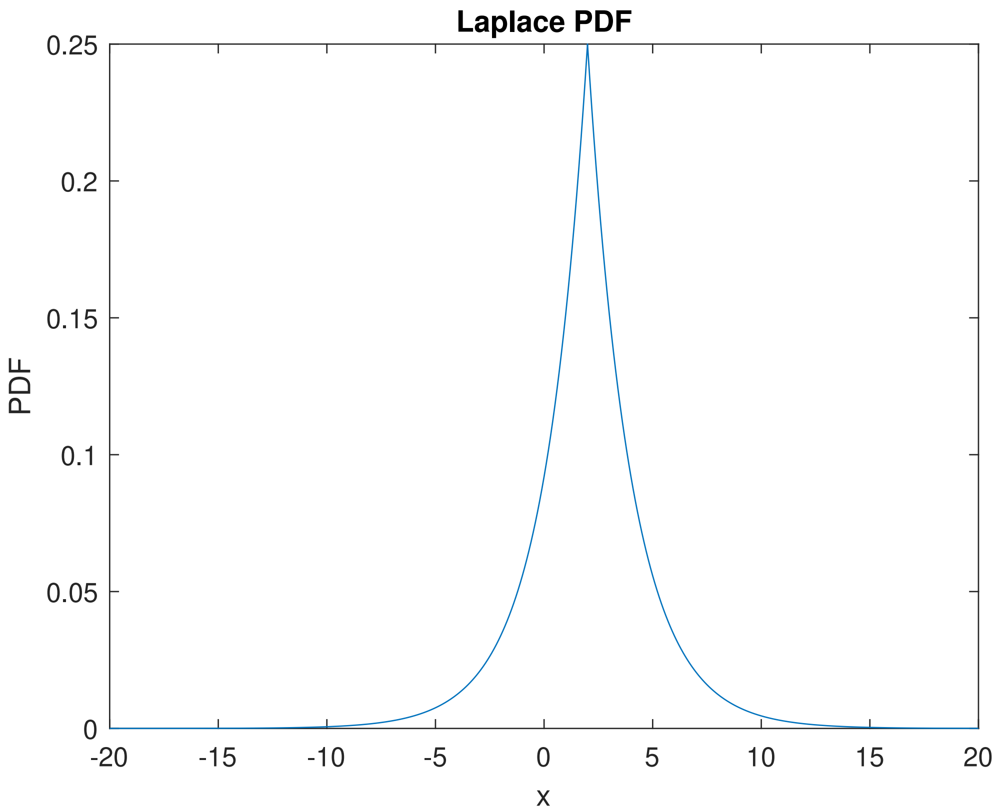
Refer `problem1.mlx` for the code and data.

1 Laplace PDF

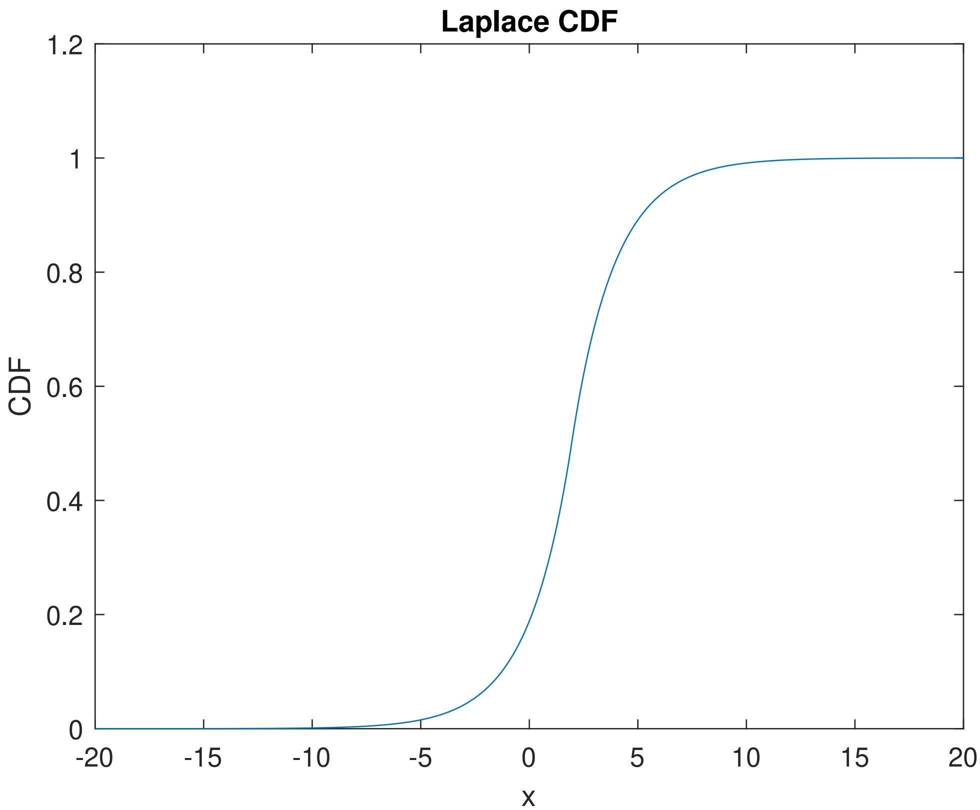
with location parameter $\mu := 2$ and scale parameter $b := 2$.
The analytical expression for the Laplace PDF is

$$P_X(x) = \frac{1}{2b} e^{-|x-\mu|} \tag{5}$$

For the estimation we take discrete points at intervals of `int_size = 0.1` We got the following plot:



For the CDF we used the same `int_size` as the width of the rectangles for Riemann summing. The following plot is the calculated CDF



Analytically, the variance would be (location parameter is inconsequential, so consider $\mu = 0$)

$$\begin{aligned}
\mathrm{Var}(X) &= E[X^2] - E[X]^2 && (6)\\
&= \int_{-\infty}^{\infty} x^2 \cdot \frac{1}{2b} e^{-|x|/b} dx - \left(\int_{-\infty}^{\infty} x \cdot \frac{1}{2b} e^{-|x|/b} dx \right)^2 && (7)\\
&= 2b^2 && (8)
\end{aligned}$$

For our given parameters, we get: 8
The value we obtained empirically was: 7.9992
Relative Error: 0.0001

2 Gumbel PDF

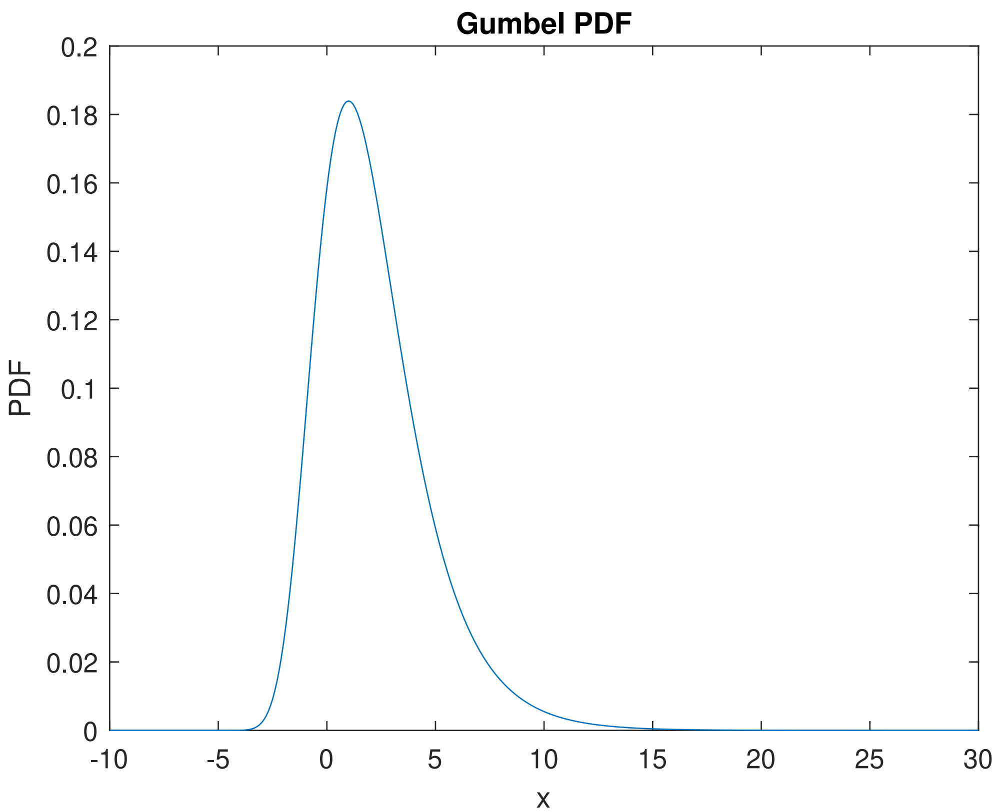
with location parameter $\mu := 1$ and scale parameter $\beta := 2$.
The analytical expression for the Gumbel PDF is

$$P_X(x) = e^{-\left(\tilde{x}+e^{-\tilde{x}}\right)} \tag{9}$$

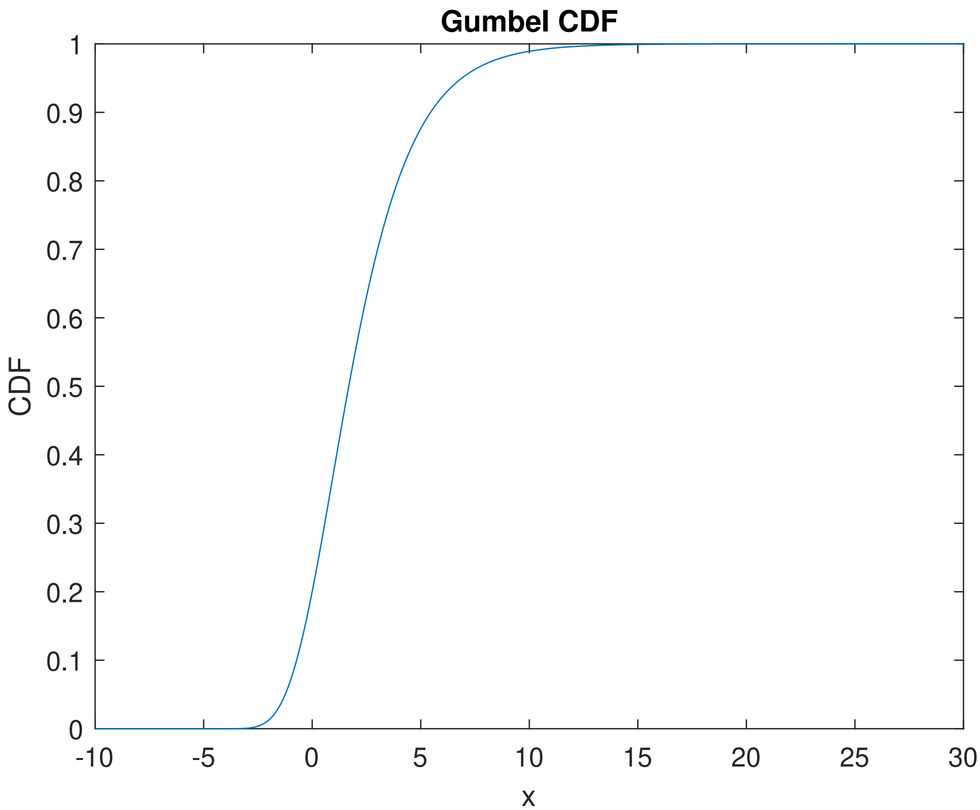
where

$$\tilde{x} = \frac{x-\mu}{\beta} \tag{10}$$

For the estimation we take discrete points at intervals of `int_size = 0.1` again. We got the following plot:



For the CDF we use the same `int_size` as the width of the rectangles for Riemann summing. The following plot



Analytically variance would be

$$\begin{aligned}
\mathrm{Var}(X) &= E[X^2] - E[X]^2 && (11)\\
&= \int_{-\infty}^{\infty} x^2 \cdot e^{-\left(\frac{x-\mu}{\beta}+e^{\frac{x-\mu}{\beta}}\right)} dx - \left(\int_{-\infty}^{\infty} x \cdot e^{-\left(\frac{x-\mu}{\beta}+e^{-\frac{x-\mu}{\beta}}\right)} dx\right)^2 && (12)\\
&= \frac{\pi^2}{6}\beta^2 && (13)
\end{aligned}$$

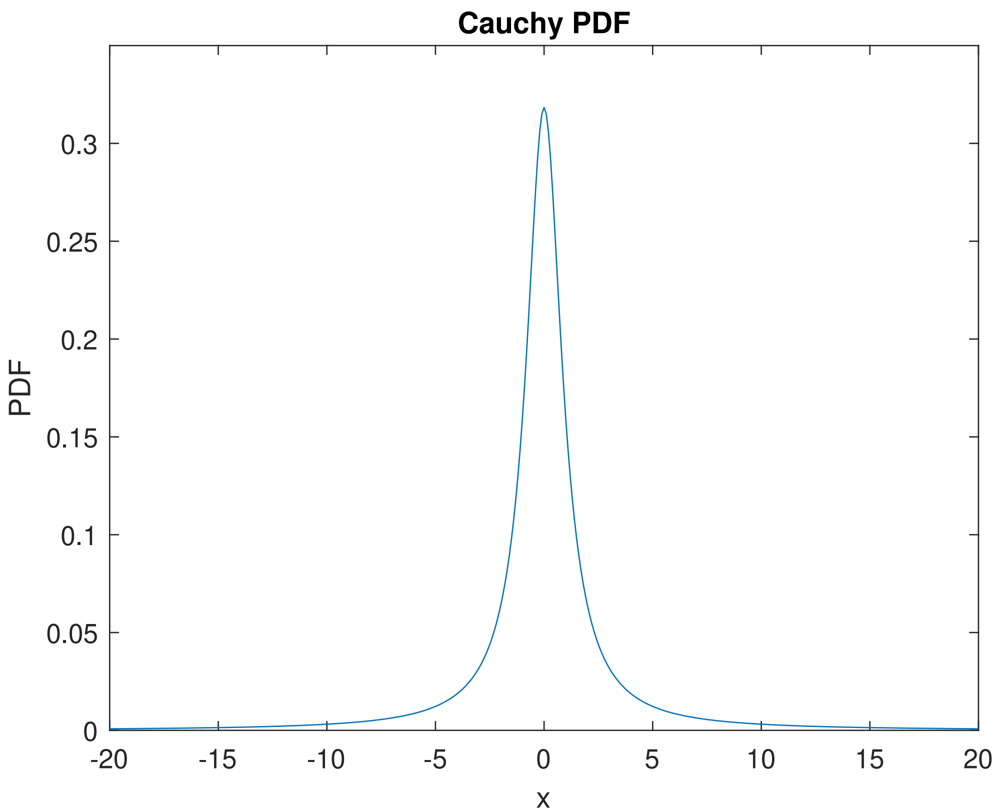
For our given parameters, we get: 6.5797
The value we obtained empirically was: 6.5797

3 Cauchy PDF

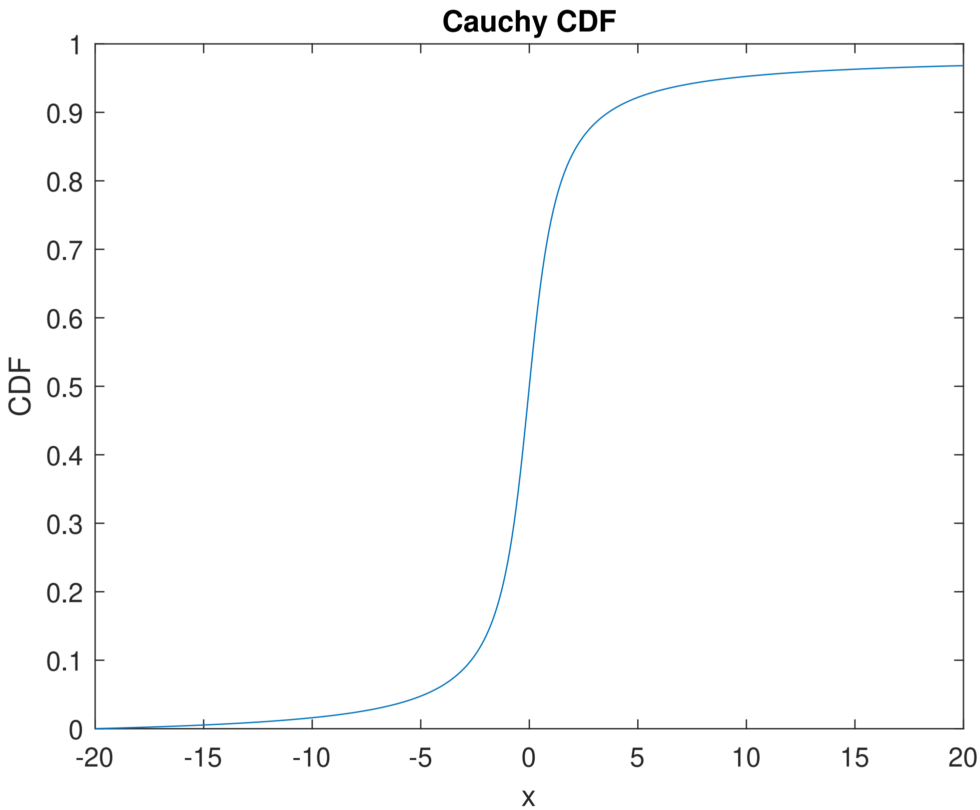
with location parameter $x_0 := 0$ and scale parameter $\gamma := 1$.
The analytical expression for the Laplace PDF is

$$P_X(x) = \frac{1}{\pi\gamma\left(1 + \frac{x-x_0}{\gamma}\right)^2} \tag{14}$$

For the estimation we take discrete points at intervals of the same `int_size = 0.1` We got the following plot:



For the CDF we use the same `int_size` as the width of the rectangles for Riemann summing. The following plot



Analytically variance would be

$$\begin{aligned} \mathrm{Var}(X) &= E[X^2] - E[X]^2 && (15) \\ &= \int_{-\infty}^{\infty} x^2 \cdot \frac{1}{\pi\gamma\left(1+\frac{x-x_0}{\gamma}\right)^2} dx - \left(\int_{-\infty}^{\infty} x \cdot \frac{1}{\pi\gamma\left(1+\frac{x-x_0}{\gamma}\right)^2} dx\right)^2 && (16) \\ &\longrightarrow \infty. && (17) \end{aligned}$$

The above integral expression is a diverging quantity, hence the variance is not finite. This is also confirmed by our code.