CS251 Assignment-1
Problem 4

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Problem 4

Consider a continuous random variable X that has an M-shaped probability density function (PDF) $P_X(\cdot)$ as follows:

$$P_X(x) := 0 \text{ for } |x| > 1, \text{ and } P_X(x) := |x| \text{ for } x \in [-1, 1] \tag{1}$$

Consider independent continuous random variables $\{X_i : i = 1, 2, \cdots, \infty\}$ with PDFs identical to that of X. Define random variables

$$Y_N := \frac{1}{N} \sum_{i=1}^{N} X_i, \tag{2}$$

for $N = 1, 2, \cdots, \infty$, which have associated distributions $P_{Y_N}(\cdot)$.

- Write code to generate independent draws from $P_X(\cdot)$. Your code can use only the uniform random number generator rand() (no other generator). Submit this code.

  Let x be the variable representing the values of rand() (X, uniform distribution on $[0, 1]$).

  Now, we have to find a random variable Y, $y = f(x)$ such that $P(y) = |y|, -1 \leq y \leq 1$.

  Every interval Δx should map to a corresponding Δy such that the probabilities of X being in $x \to x + \Delta x$ and Y being in $y \to y + \Delta y$ are the same.

  Which implies

  $$P_Y(y)dy = P_X(x)dx \tag{3}$$

  As

  $$P(x) = 1, 0 \leq x \leq 1 \text{ and } P(y) = |y|, -1 \leq y \leq 1, \tag{4}$$

  $$P_Y(y)dy = P_X(x)dx \tag{5}$$
  $$\implies |y|dy = dx \tag{6}$$
  $$\implies y = \sqrt{2(x + c_1)}, -\sqrt{2(c_2 - x)} \tag{7}$$

  We assume that the solution y is continuous. We know that the range of y is $[-1, 1]$ and the two solution functions are continuous. Let us call the first solution y_1 and the second solution y_2. $y_1 \geq 0$ and $y_2 \leq 0$. Hence, $\exists\ \alpha \in [0, 1]$ such that $y_1(\alpha) = y_2(\alpha) = 0$. We also know that the mass to the left of $y = 0$ is the same as the mass to the right of $y = 0$. Hence, $\alpha = 0.5$ by symmetry.

  $$\implies c_1 = -0.5 \text{ and } c_2 = 0.5 \tag{8}$$

  The transformation function $y = y(x)$ can be written in a simplified form as:

  $$y = \text{sgn}(2x - 1) \cdot \sqrt{|2x - 1|} \tag{9}$$

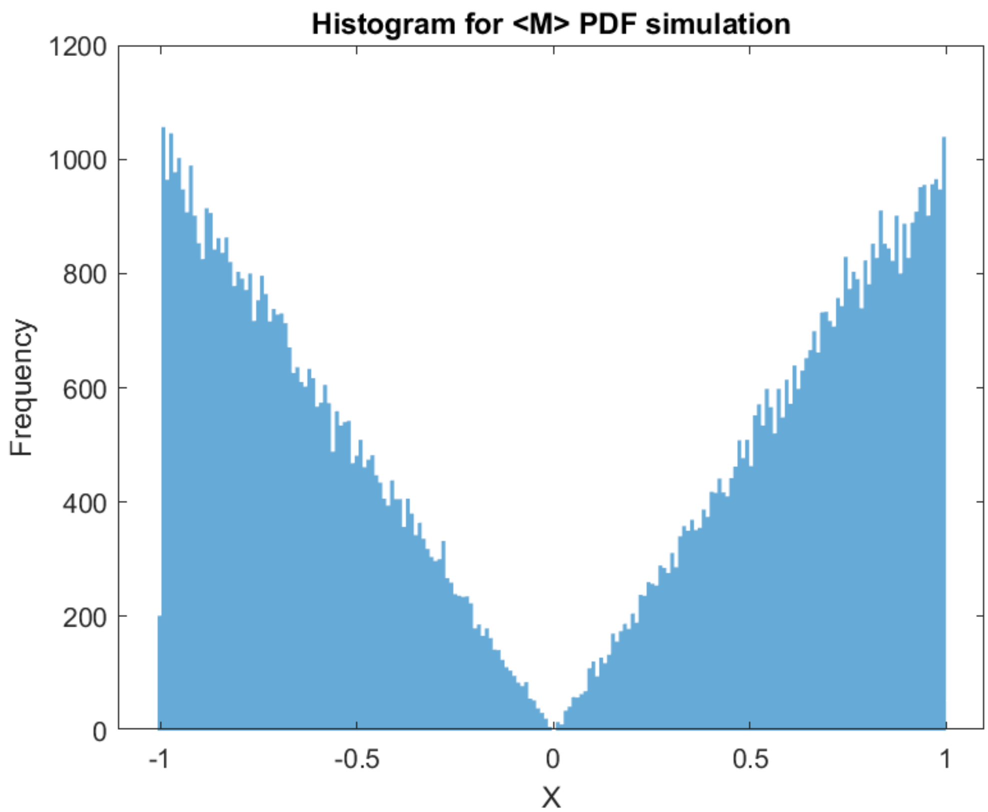
To generate on instance of this random variable –

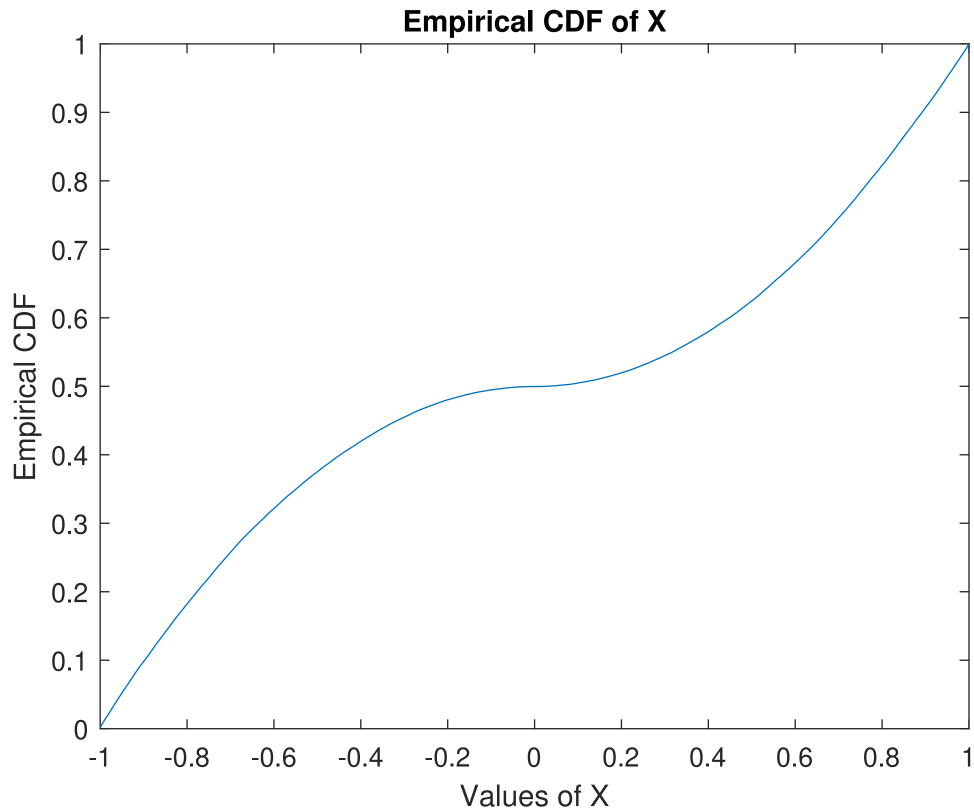
```
mprob = sign(2*rand()-1).*sqrt(abs(2*rand()-1));
```

Refer `problem4.mlx`.

- Show plots of

  1. The histogram (with 200 bins)





2. Cumulative distribution function (CDF), both using $M := 10^5$ draws from the PDF $P_X(\cdot)$.

- Use the code written in the previous sub-question to write code to generate independent draws from $P_{Y_N}(\cdot)$. Submit this code.

  Refer `problem4.mlx`. We used this function to generate values of Y_N (as a function of N)–

```
function pyn = avg_M(N)
    pyn = mean(sign(2*rand(N,1)-1).*sqrt(abs(2*rand(N,1)-1)));
end
```

- Show plots (separately) of histograms using draws from each of the PDFs $P_{Y_N}(\cdot)$ for $N = 2, 4, 8, 16, 32, 64$.

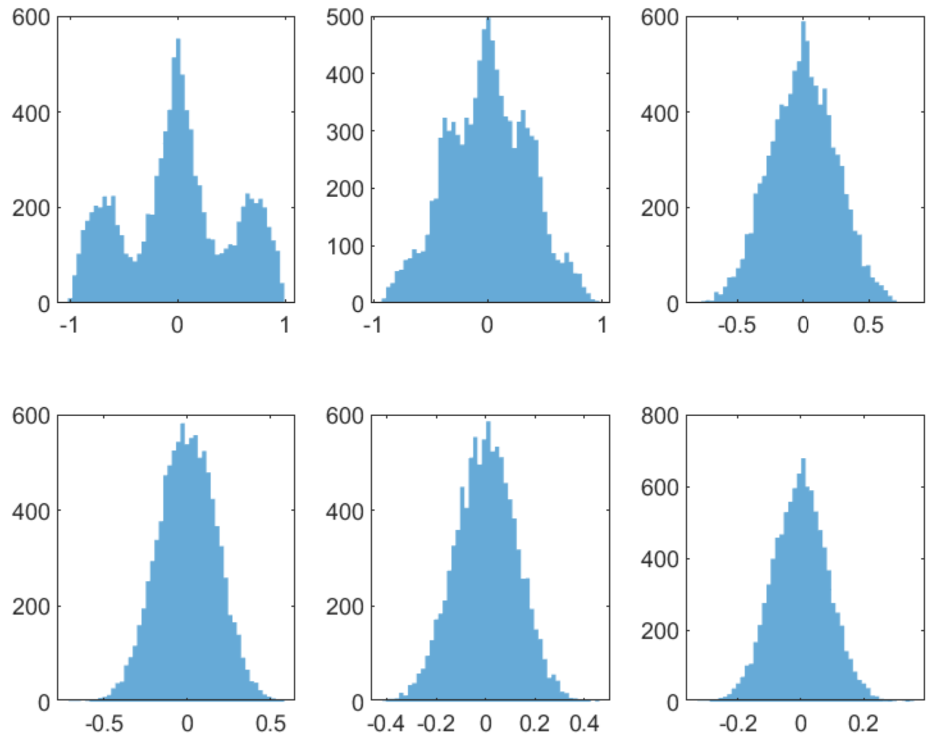


Figure 1: Histograms corresponding to $N = 2, 4, 8, 16, 32, 64$ respectively (L-R T-B)

Show plots, on the same graph, of all the CDFs associated with Y_N for $N = 1, 2, 4, 8, 16, 32, 64$, computed using 10^4 draws from each $P_{Y_N}(\cdot)$. Plot each CDF curve using a different color. You may use the cdfplot($\cdot$) function in Matlab.