CSE/Math 340 Algorithms — Spring 2013

Homework Assignment No. 11

This assignment is optional.
Attempt as many, or as few, of these problems as you wish.
All points you earn (max: 125) are extra credit added to your HW+Presentation total,
up to a maximum of 1035 points (= sum of perfect scores on all HWs #1-10).
(HW+Presentation points contribute 20% to the final course grade.)

Posted on CourseSite: **Friday, April 12.**
Due: **Friday, April 19:** *at 11:10 AM, in class, as hardcopy*

Assigned Reading: CLRS Ch.35 "Approximation Algorithms" (§§35.1 & 35.2 only).

1. *[NP-Complete Problems]* [**25 points maximum**]

   The *Clique Cover* decision problem is:

   Given any graph $G = (V, E)$ and any fixed known integer constant k, *decide* does there exist a partition of V into k disjoint sets $V_1, V_2, \ldots, V_k$ such that for each $i = 1, 2, \ldots, k$, for every pair of vertices $u, v \in V_i$, there exists an edge $(u, v) \in E$.

   Prove that the *Clique Cover* decision problem is NP-complete.

   [Hint: consider a reduction from the *Coloring* decision problem.]

2. *[Restrictions of NP-Complete Problems]* [**25 points maximum**]

   Recall that an *Independent Set* of a graph $G = (V, E)$ is a subset of vertices $I \subseteq V$ such that for all pairs of vertices $u, v \in I$, $(u, v) \notin E$. The *size* of an independent set I is equal to $|I|$. The problem of finding the largest independent set in general graphs is NP-Complete.

   Answer the following two questions.... (In both, you may describe the algorithm informally using pseudocode mixed with English; but please give careful rigorous arguments for correctness and asymptotic runtime.)

   (a) Suppose that the degree of each node in G is at most a known constant d. Design a polynomial-time algorithm to find an independent set in G which is no smaller than $1/(d+1)$ of the optimal (largest) size. (That is, it is a $(d+1)$-*approximation* algorithm for this problem.)

[Hint: consider this greedy heuristic:

initialize $I = \emptyset$ (the empty set)

while $|V| > 0$

pick any $v \in V$ which has the smallest degree

add v to I

delete v and all its edges from $G = (V, E)$

]

(b) Now suppose that G is a **tree** but each node's degree is arbitrary (unconstrained). Design a polynomial-time algorithm to find the maximum-size independent set. (That is, find an *exact* algorithm for this restricted problem.)

3. *[Restrictions of NP-complete Problems]* **[25 points maximum]**

Recall this definition:

> A *Vertex Cover* of a graph $G = (V, E)$ is a subset of vertices $C \subseteq V$ such that for each edge $(u, v) \in E$, either vertex $u \in C$ or vertex $v \in C$ or both. The *size* of a vertex cover C is equal to $|C|$.

The *Vertex Cover* decision problem is:

> **Given** any graph $G = (V, E)$ and any budget B (any real number),
> **find** a vertex cover of G of size $\leq B$.

For general graphs (*i.e.*, when an input instance can be *any graph, without restriction*), the *Vertex Cover* decision problem is NP-complete.

Give a polynomial-time algorithm for finding a minimum-size vertex cover in *trees* (which are, of course, a *restricted class* of graphs). [Note: there exists a greedy linear-time algorithm: if you can find one, you'll earn an **extra 30** extra-credit points (on top of the 25 points for finding any polytime algorithm).]

4. *[Approximation Algorithms]* **[25 points maximum]**

Vertex Cover has a "ρ-approximation algorithm": a polynomial-time algorithm which always finds a vertex cover C of size no more than some constant $\rho \geq 1$ times the optimal size $|O|$ (where O is an optimal vertex cover of the graph), *i.e.* it guarantees that:

$$\frac{|C|}{|O|} \leq \rho$$

(One example of this is given in CLRS Theorem 35.1 (pp. 1109–1111), which proves that the heuristic APPROX-VERTEX-COVER is a 2-approximation algorithm.)

Of course, the *Clique* problem is NP-complete too.

However, from the proof of Theorem 34.12 (CLRS pp. 1090–1091), we know that *Vertex Cover* and *Clique* are "complementary" in the sense that a minimum-size vertex cover is simply the complement of a maximum-size clique in the complementary graph (*i.e.* the graph made up of the edges missing in the original graph).

Do these facts imply that there is a ρ-approximation algorithm for *Clique* too? (It's OK if its ρ is different from the one for *Vertex Cover*.) *Yes* or *No*? Justify your answer carefully and rigorously.

5. *[Heuristics for NP-complete Problems]* [**25 points maximum**]

Suppose that the vertices for an instance of the *Traveling Salesperson* problem are points in the plane and that the cost $c(u, v)$ is the Euclidean (straight-line) distance between u and v.

Answer the following two questions:

(a) Show that an optimal tour never crosses itself: that is, no two edges on the tour intersect except at endpoints, and then only when they are adjacent along the tour. [Hint: apply the triangle inequality to replace pairs of crossing edges with non-crossing edges, for no greater cost.]

(b) Apply the above fact in the design of a heuristic to generate tours that *may be* better than the ones found by the APPROX-TSP-TOUR algorithm (CLRS pp. 1112–1115). (By "heuristic" here, I mean a procedure that is *not guaranteed* to find a better solution: more weakly, it only *might*.)