Tree Structures and Tree Traversals

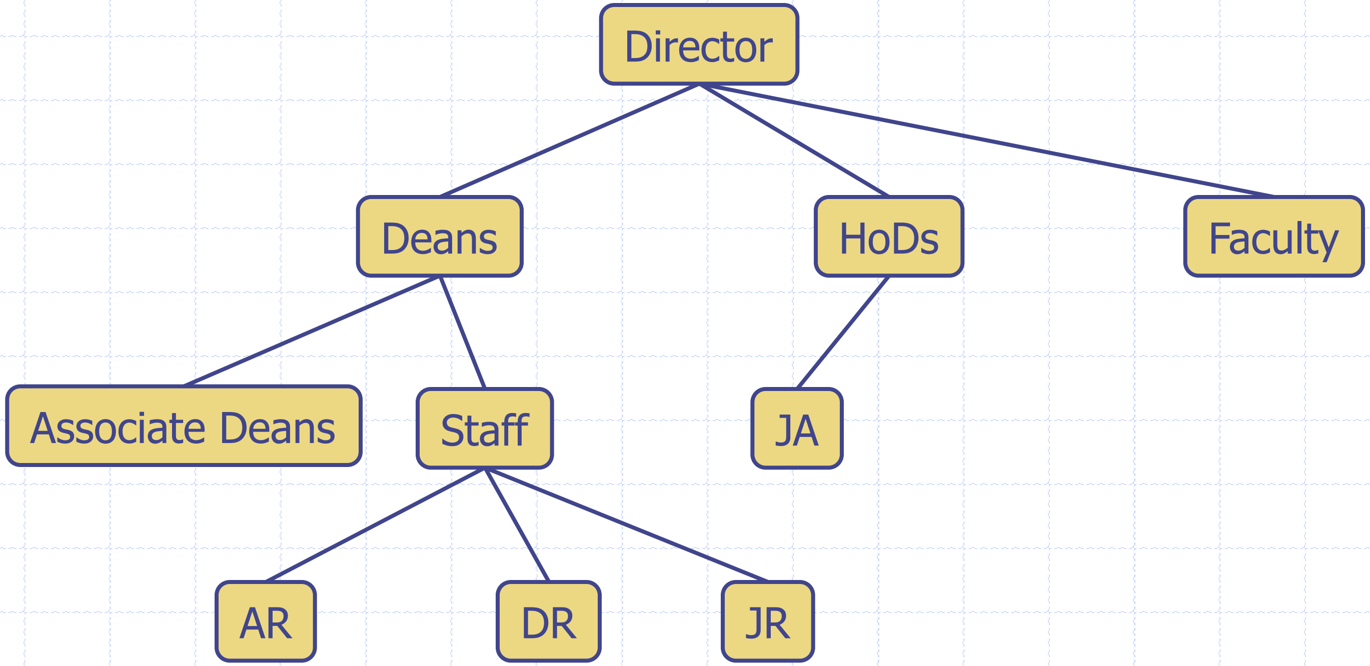
Readings - Chapter 8

Trees

- Abstract model of a hierarchical structure
- A tree consists of nodes with a parent-child relation

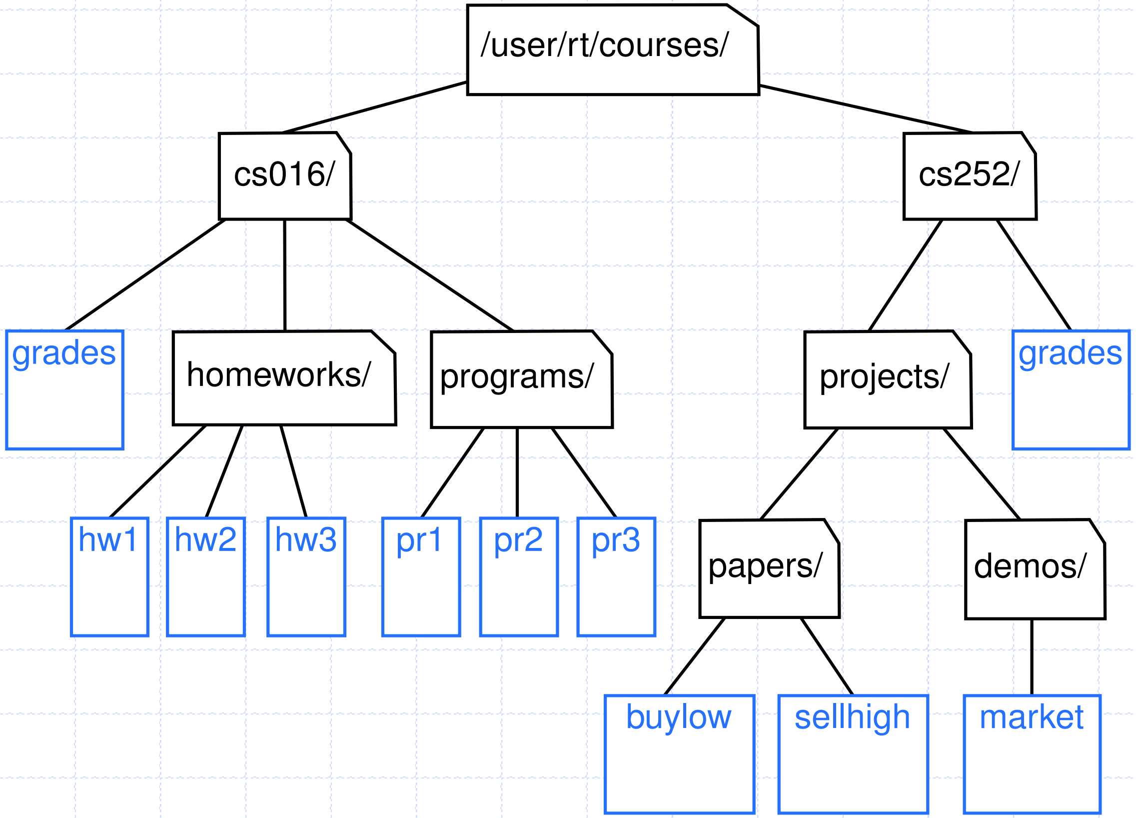
google images

Trees - Examples



organization structure of a corporation

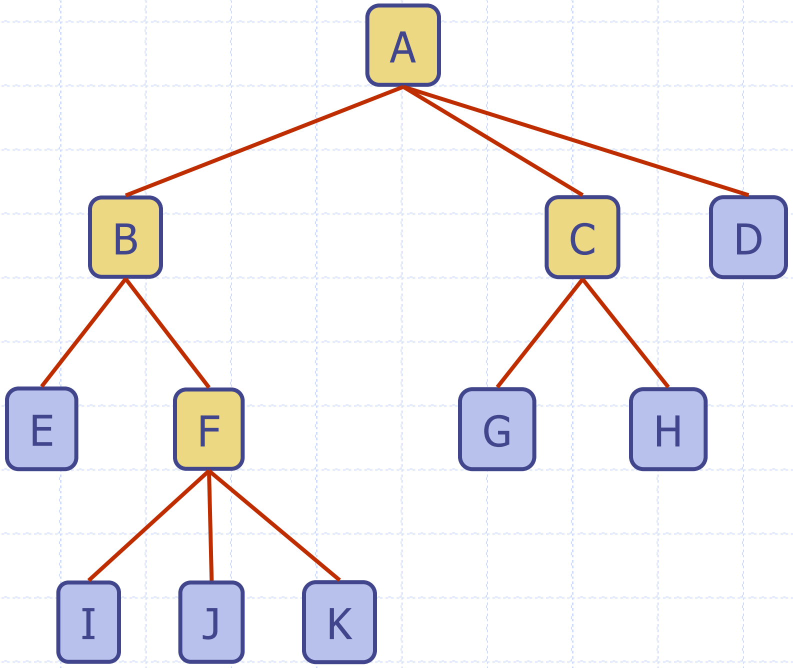
Trees - Examples (2)



Portion of a file system

Trees - Terminology

- *A* is the *root* node
- *B* is *parent* of *E* and *F*
- *A* is *ancestor* of *E* and *F*
- *E* and *F* are *descendants* of *A*
- *C* is the *sibling* of *B*
- *E* and *F* are *children* of *B*
- *E, I, J, K, G, H,* and *D* are *leaves*
- *A, B, C,* and *F* are *internal nodes*



Trees - Terminology (2)

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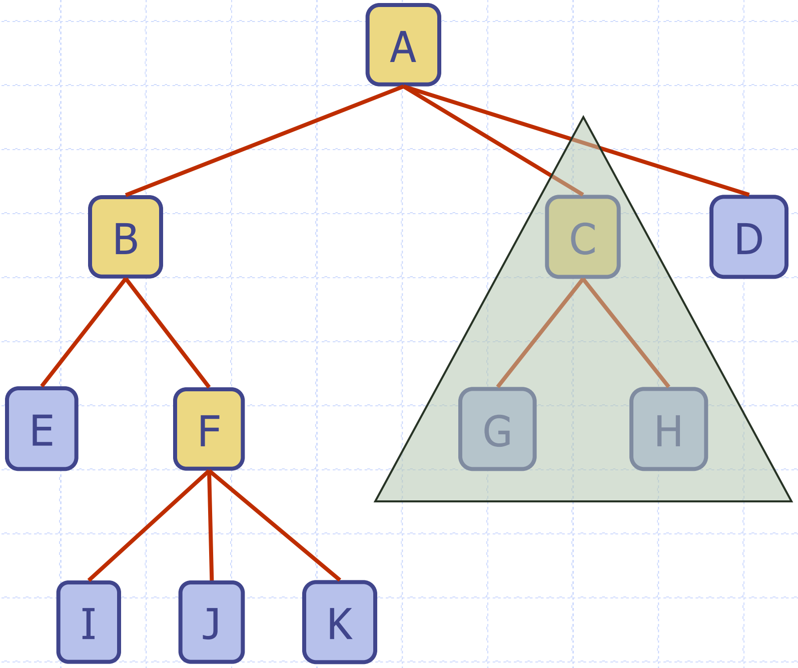
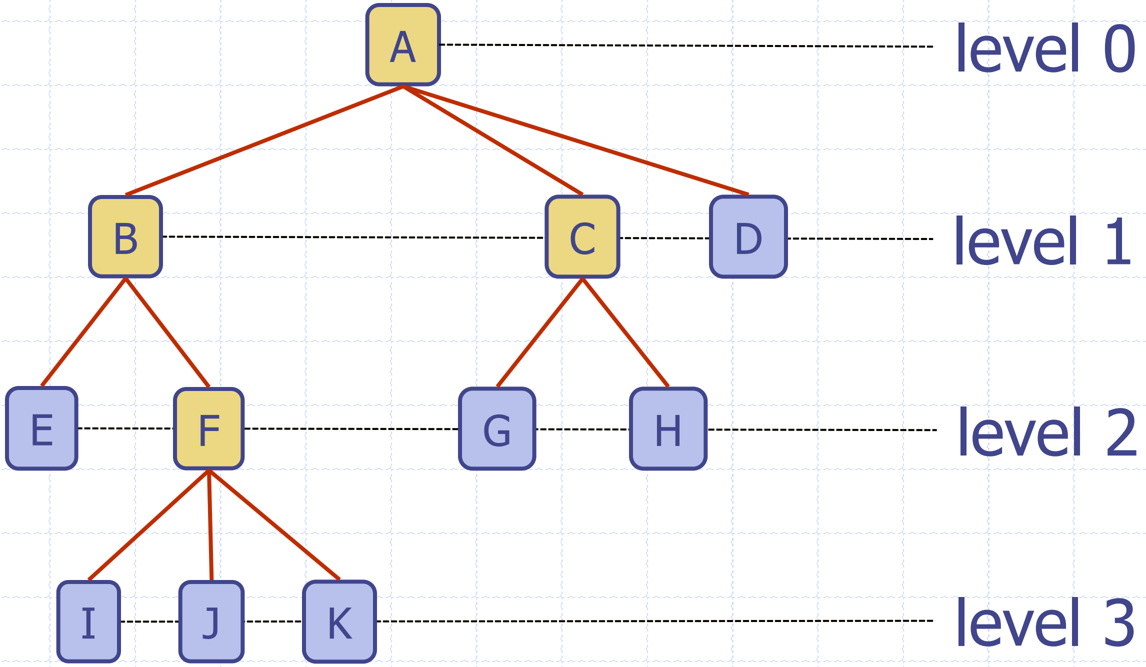
- *Subtree:* tree consisting of node and its descendants



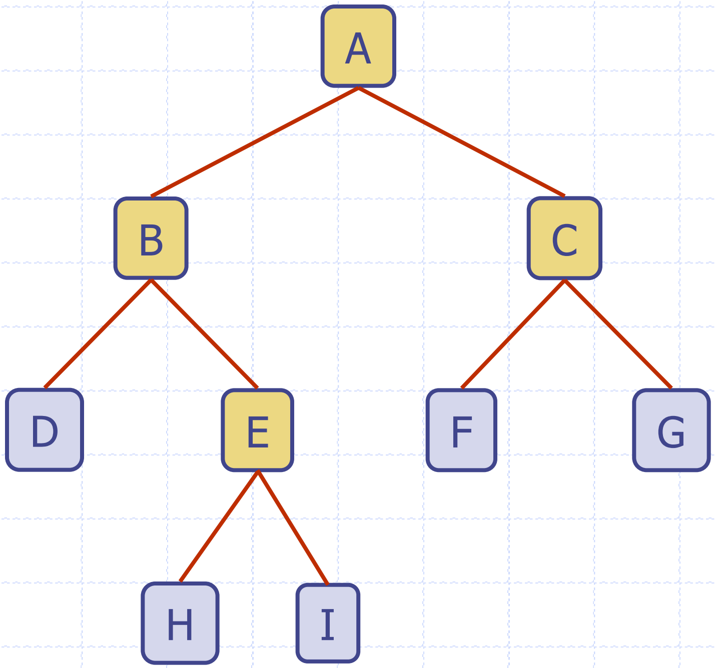
Trees - Terminology (3)

- The *depth* (*level*) of *E* is 2
- The *height* of the tree is 3
- The *degree* of node F is 3



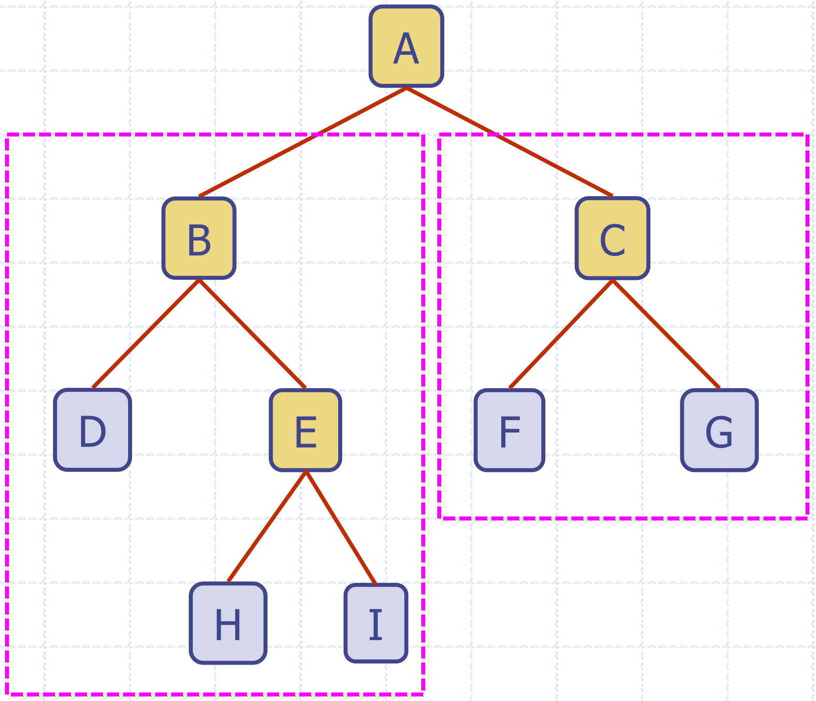
Binary Trees

- An **ordered tree** is one in which the children of each node are ordered
- **Binary tree**: ordered tree with all nodes having at most 2 children
  - left child and right child



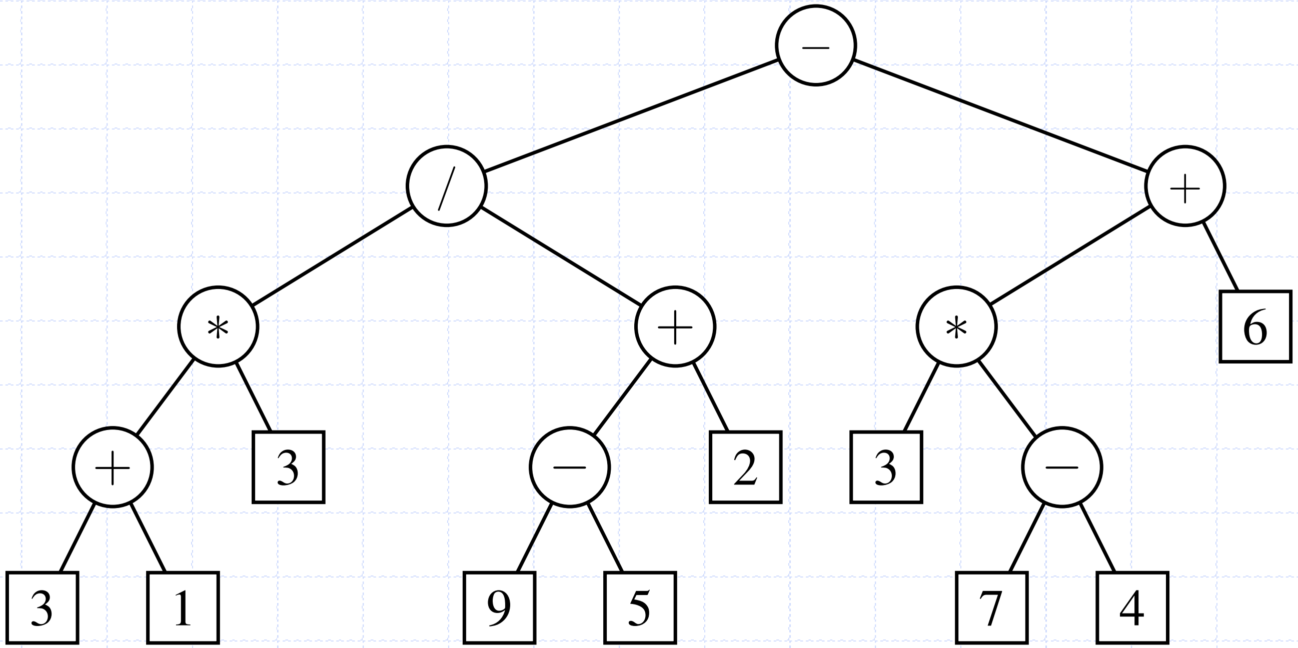
Binary Trees

- Recursive definition of binary tree
  - either a leaf or
  - an internal node (the root) and one/two binary trees (left subtree and/or right subtree)



Example of Binary Trees - Arithmetic Expression Tree

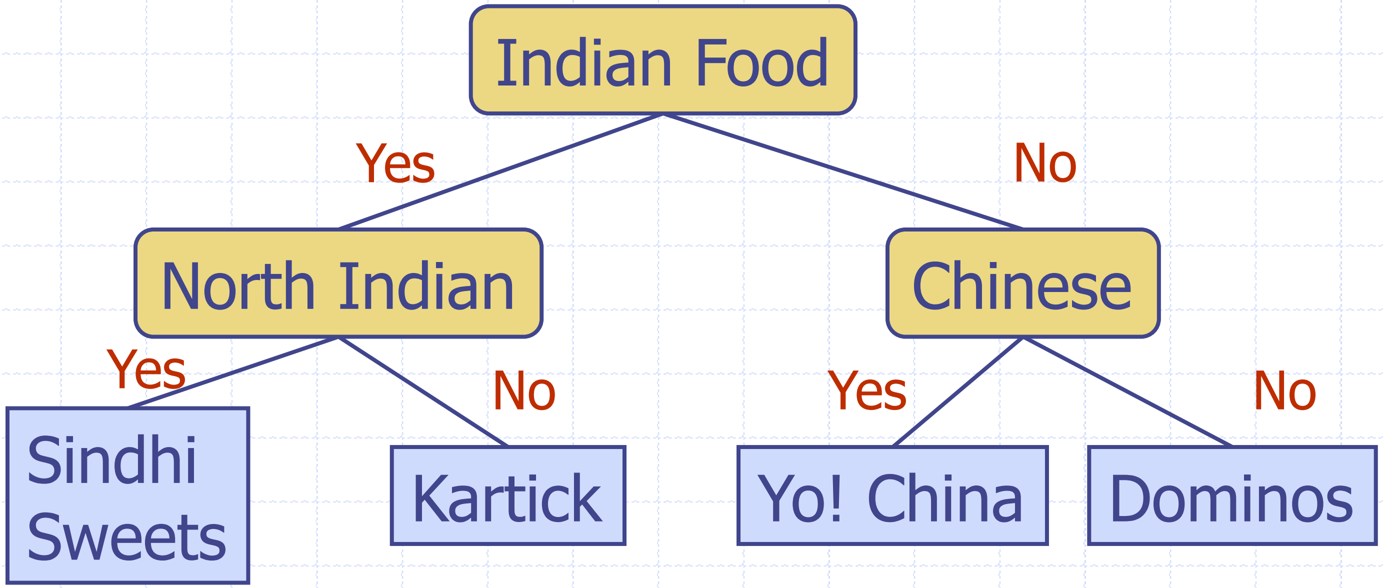
- Binary tree associated with an arithmetic expression
  - internal nodes: operators
  - external nodes: operands



((((3 + 1) *3) / (9 - 5)+2))-((3*(7 - 4)) +6))

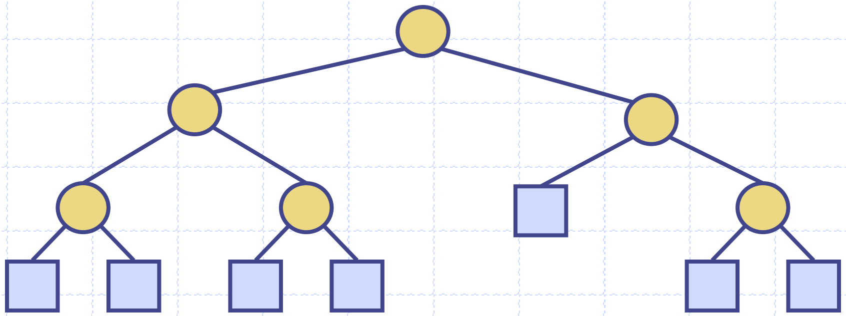
Example of Binary Trees - Decision Tree

- Binary tree associated with a decision process
  - internal nodes: questions with yes/no answer
  - external nodes: decisions
- Example: dining decision

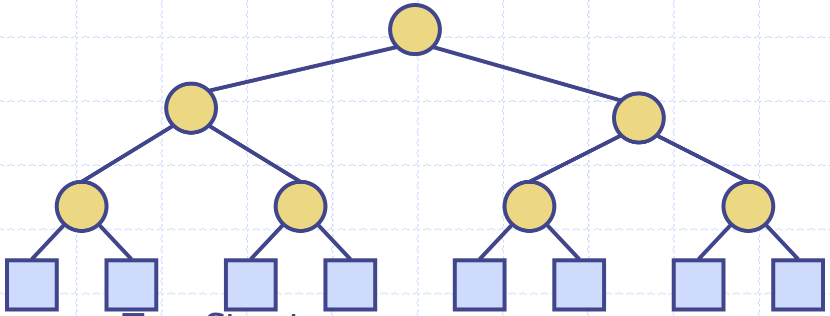


Proper, Full, Complete Binary Trees

- Proper/Full - Every node has either zero or two children

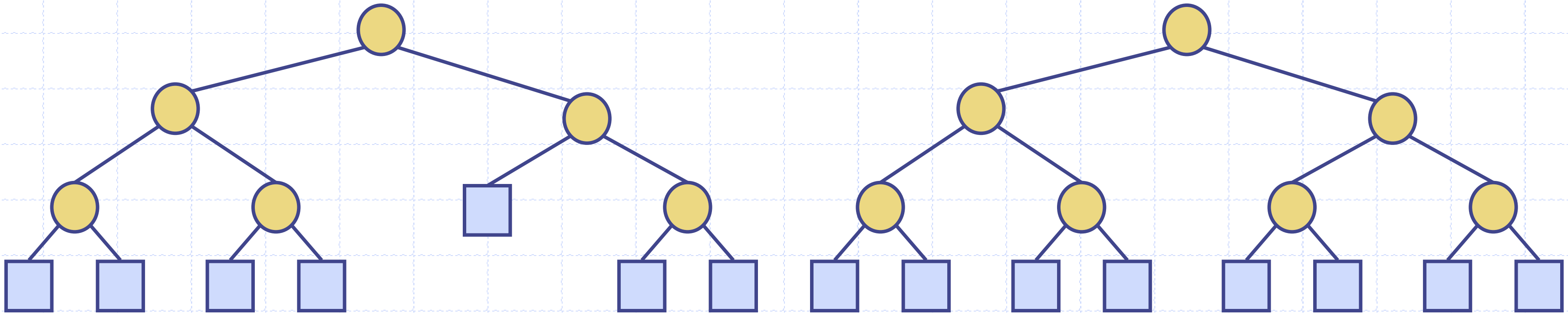


- Complete - every level except possibly the last is completely filled and all leaf nodes are as left as possible.



Binary tree from a complete binary tree

- A binary tree can be obtained from appropriate complete binary tree by pruning.



Properties of a Binary Tree

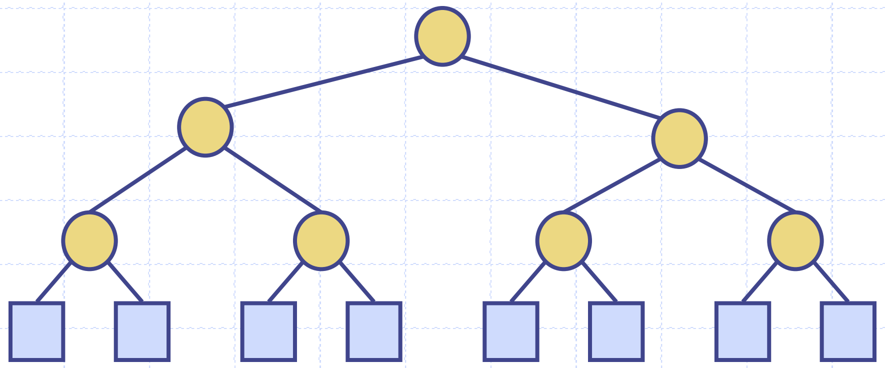
- Notations
  - n - number of nodes
  - n_E - number of leaves (external nodes)
  - n_I - number of internal nodes
  - h - height of the tree
- $h+1 \le n \le 2^{h+1} - 1$
- $1 \le n_E \le 2^h$
- $h \le n_I \le 2^h - 1$
- $\log(n+1) - 1 \le h \le n-1$

Properties of Binary Trees (2)

- $n_E \leq n_I + 1$
- proof by induction on n_I
  - Tree with 1 node has a leaf but no internal node
  - Assume $n_E \leq n_I + 1$ for tree with $k\text{-}1$ internal nodes
  - A tree with k internal nodes has k_1 internal nodes in the left subtree and $k\text{-}k_1\text{-}1$ internal nodes in the right subtree
  - By induction $n_E \leq (k_1+1) + (k-k_1-1+1) = k+1$

Complete Binary Tree

- level i has 2^i nodes
- In a tree of height h
  - leaves are at level h
  - $n_E = 2^h$
  - $n_I = 1 + 2 + 2^2 + \ldots + 2^{h-1} = 2^h - 1$
  - $n_I = n_E - 1$
  - $n = 2^{h+1} - 1$
- In a tree of n nodes
  - n_E is $(n+1)/2$
  - $h = \log_2 (n_E)$



Tree ADT

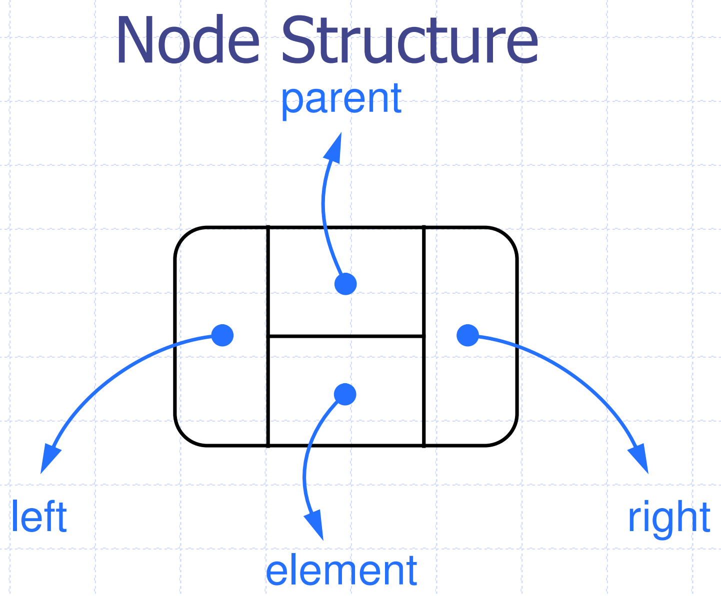
- We use positions to abstract nodes
- Generic methods:
  - Integer len()
  - Boolean is_empty()
  - Iterator positions()
  - Iterator iter()
- Accessor methods:
  - position root()
  - position parent(p)
  - Iterator children(p)
  - Integer num_children(p)

- Query methods:
  - Boolean is_leaf(p)
  - Boolean is_root(p)
- Update method:
  - element replace (p, o)
- Additional update methods may be defined by data structures implementing the Tree ADT

Abstract Tree Class in Python

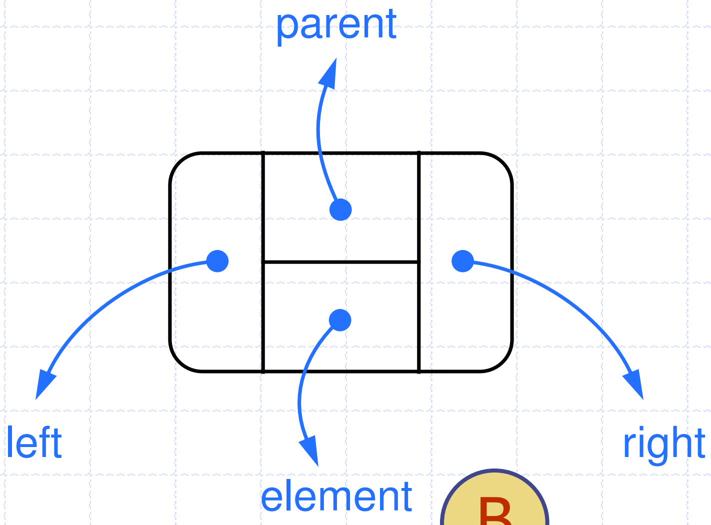
```python
class Tree:
  """Abstract base class representing a tree structure."""

  #------------------------------- nested Position class -------------------------------
  class Position:
    """An abstraction representing the location of a single element."""

    def element(self):
      """Return the element stored at this Position."""
      raise NotImplementedError('must be implemented by subclass')

    def __eq__(self, other):
      """Return True if other Position represents the same location."""
      raise NotImplementedError('must be implemented by subclass')

    def __ne__(self, other):
      """Return True if other does not represent the same location."""
      return not (self == other)            # opposite of __eq__

  # ---------- abstract methods that concrete subclass must support ----------
  def root(self):
    """Return Position representing the tree's root (or None if empty)."""
    raise NotImplementedError('must be implemented by subclass')

  def parent(self, p):
    """Return Position representing p's parent (or None if p is root)."""
    raise NotImplementedError('must be implemented by subclass')

  def num_children(self, p):
    """Return the number of children that Position p has."""
    raise NotImplementedError('must be implemented by subclass')

  def children(self, p):
    """Generate an iteration of Positions representing p's children."""
    raise NotImplementedError('must be implemented by subclass')

  def __len__(self):
    """Return the total number of elements in the tree."""
    raise NotImplementedError('must be implemented by subclass')

  # ---------- concrete methods implemented in this class ----------
  def is_root(self, p):
    """Return True if Position p represents the root of the tree."""
    return self.root( ) == p

  def is_leaf(self, p):
    """Return True if Position p does not have any children."""
    return self.num_children(p) == 0

  def is_empty(self):
    """Return True if the tree is empty."""
    return len(self) == 0
```

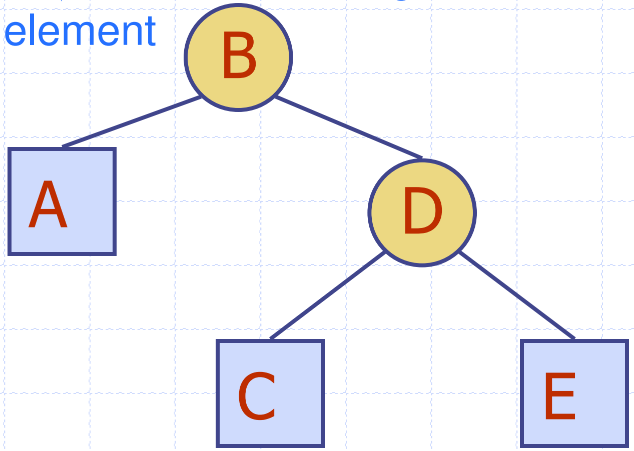
Linked Structure for Binary Trees

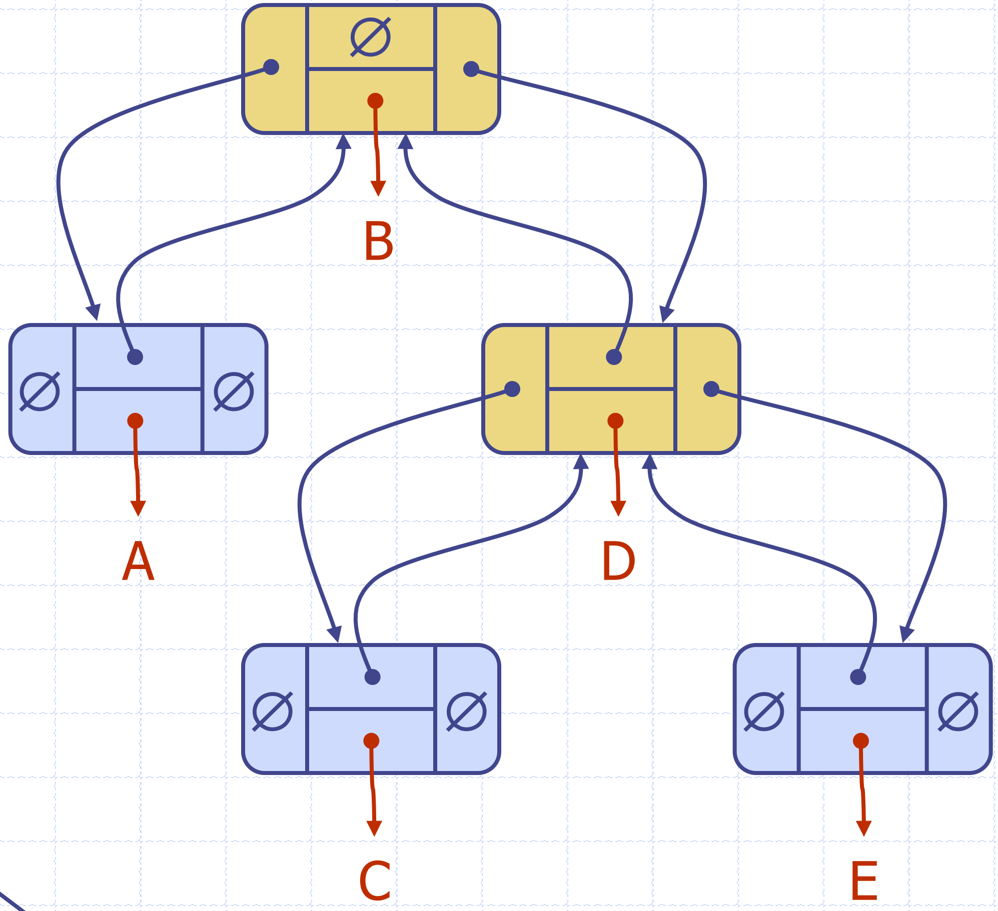


Linked Structure for Binary Trees

Node Structure



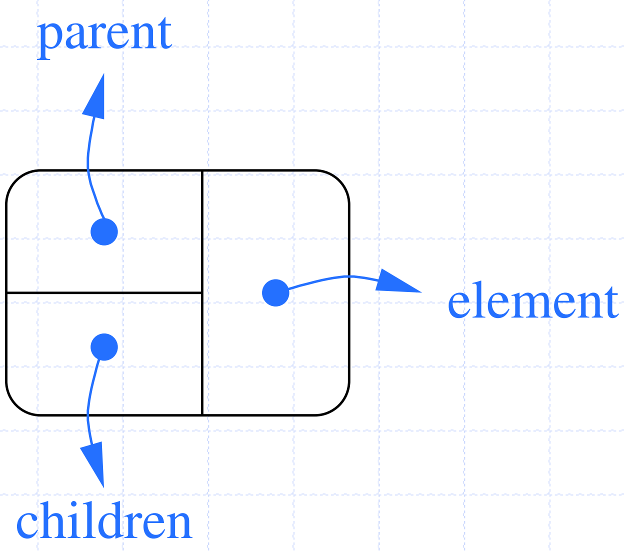




Code Fragments
8.8, 8.9, 8.10, 8.11

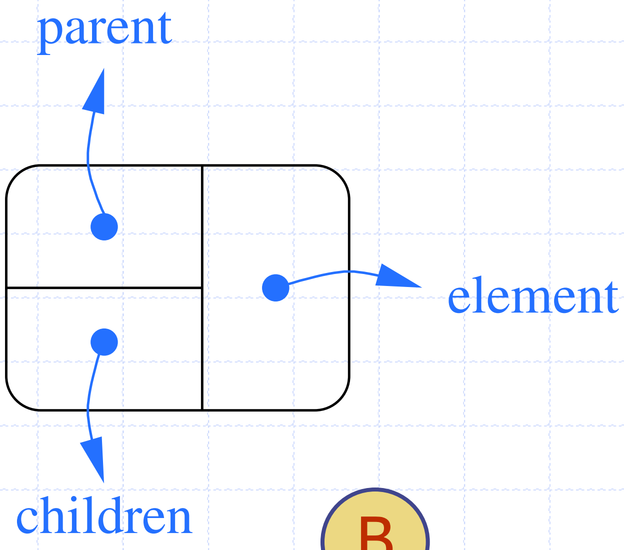
Linked Structure for General Trees

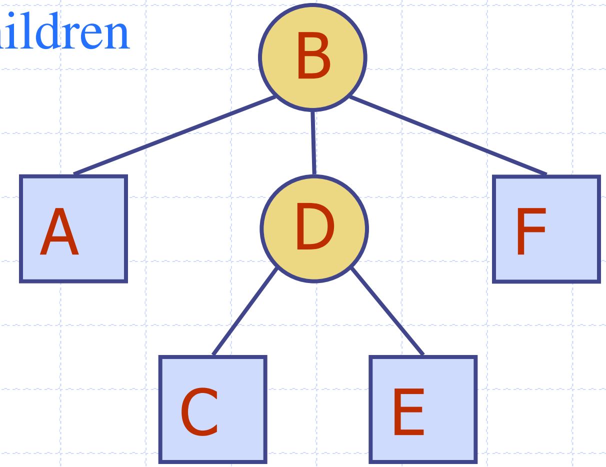
Node Structure

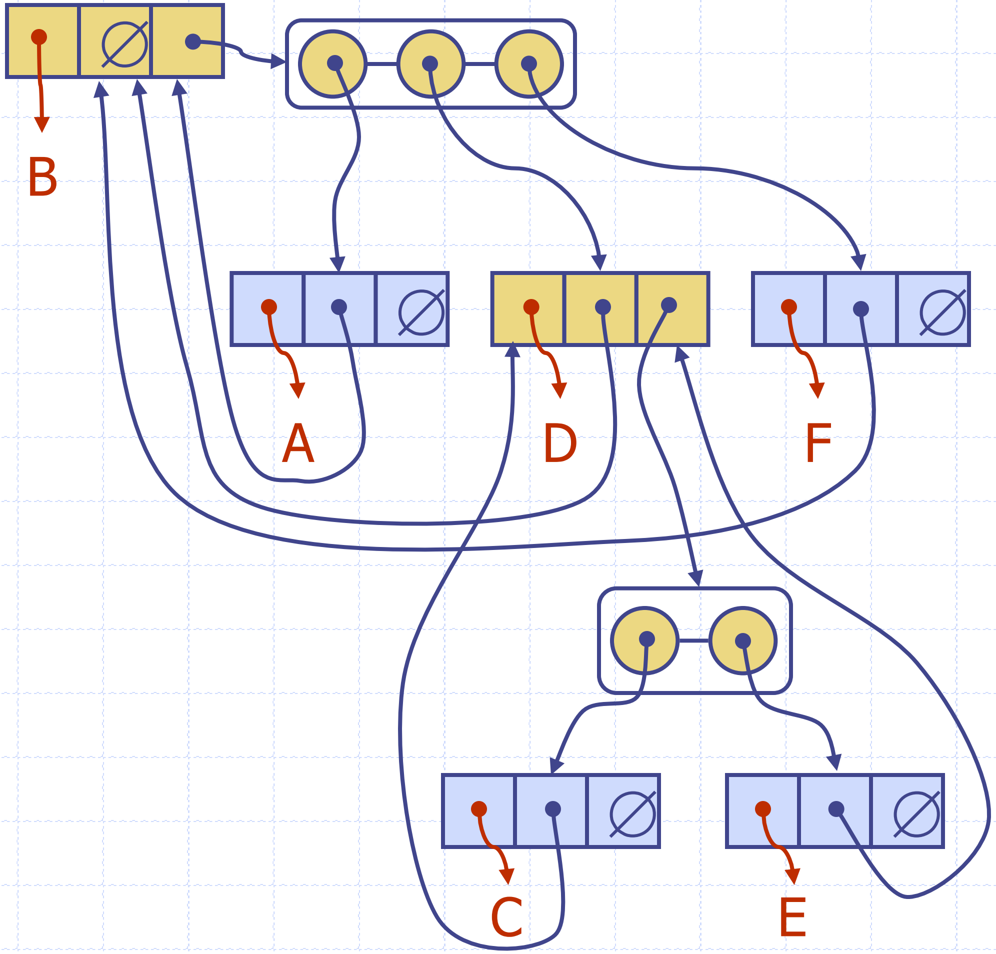


Linked Structure for General Trees

- Node Structure







Computing Depth

- p be a position within the tree T
- calculate depth(p)

```
def depth(self, p):
"""Return the number of levels separating Position p
from the root."""
     if self.is_root(p):
          return 0
     else:
          return 1 + self.depth(self.parent(p))
```

Computing Height

```
def _height1(self): # works, but O(n^2) worst-case time
"""Return the height of the tree."""
return max(self.depth(p) for p in self.positions() if
self.is_leaf(p))
```

Analysis:

$O(n + \Sigma_{p \varepsilon L}(d_p+1))$ is $O(n^2)$

Computing Height

```
def _height2(self, p): # time is linear in size of subtree
"""Return the height of the subtree rooted at Position
p."""
if self.is_leaf(p):
return 0
else:
return 1 + max(self._height2(c) for c in self.children(p))
```

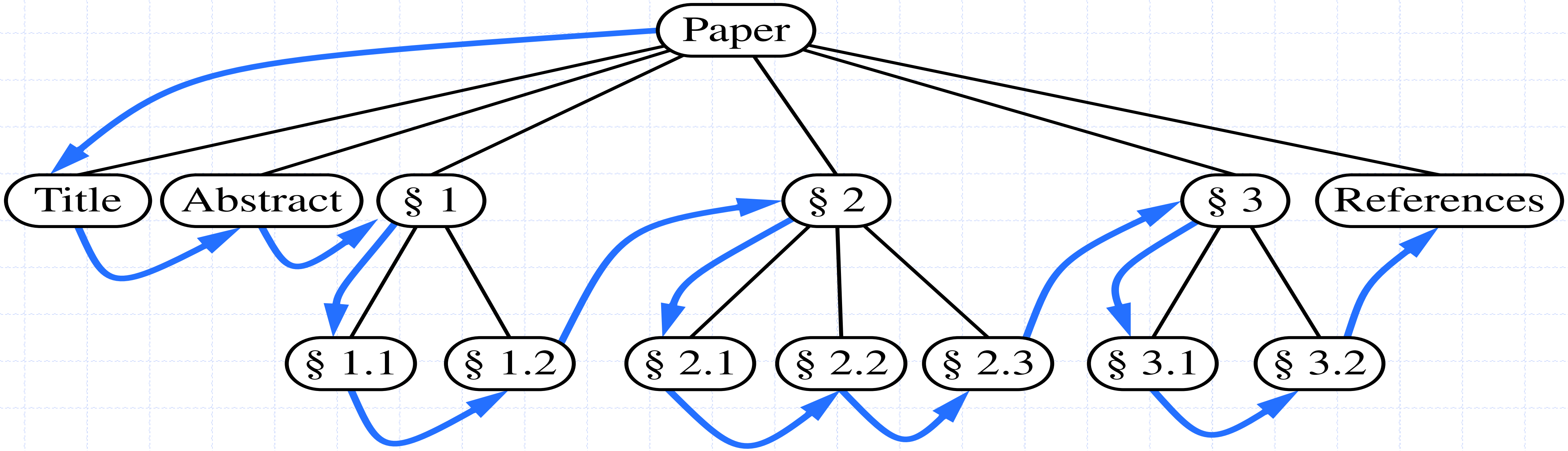
Analysis

$O(\Sigma_p(c_p+1))$ is $O(n)$

Tree Traversals

- Systematic way of visiting all nodes in a tree in a specified order
  - preorder - processes each node before processing its children
  - postorder - processes each node after processing its children

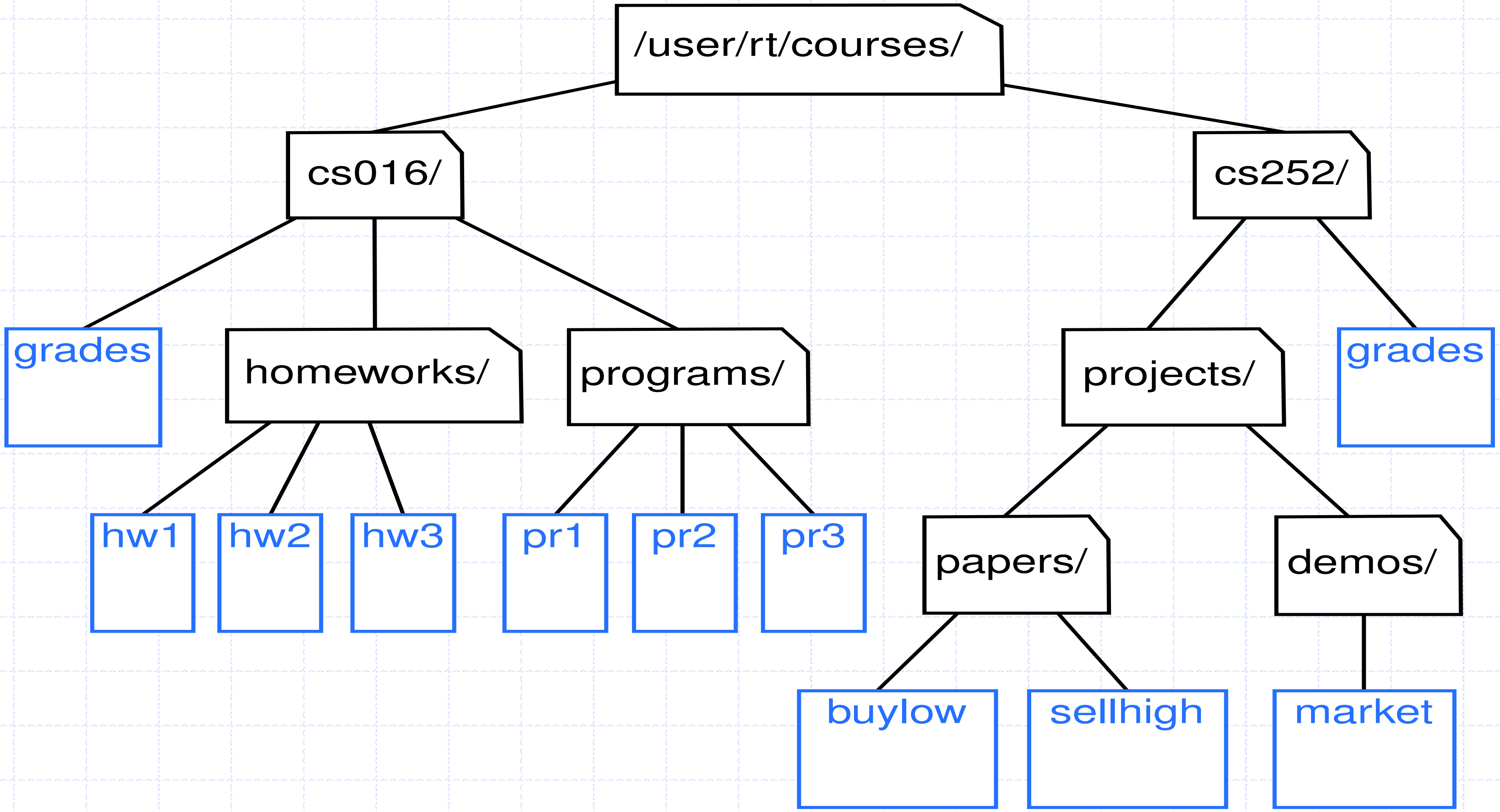
Preorder Traversal



Preorder Traversal - Algorithm

- Algorithm preorder(p)
  - perform the “visit” action for position p
  - for each child c in children(p) do
    - preorder(c)
- Example:
  - reading a document from beginning to end

Postorder Traversal



Postorder Traversal - Algorithm

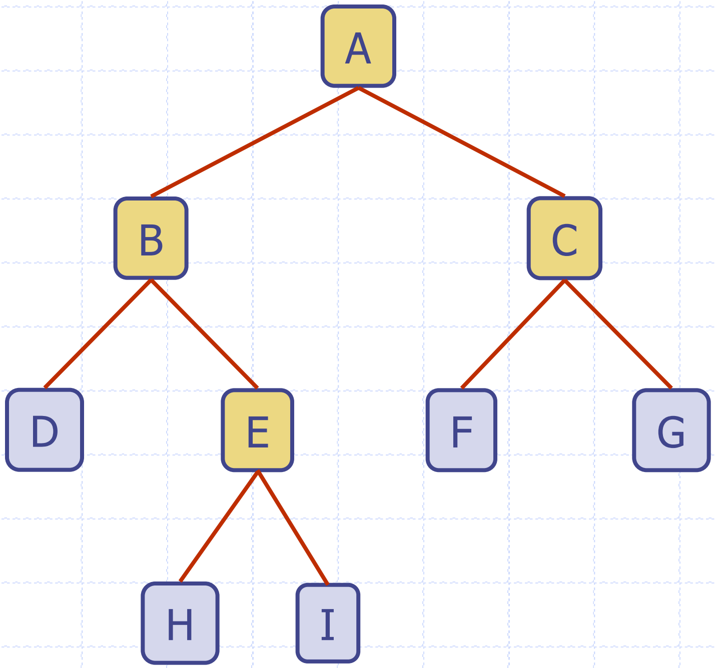
- Algorithm postorder(p)
  - for each child c in children(p) do
    - postorder(c)
  - perform the “visit” action for position p
- Example
  - du - disk usage command in Unix

Traversals of Binary Trees

- preorder(v)
  - visit(v)
  - preorder(v.leftchild())
  - preorder(v.rightchild())
- postorder(v)
  - postorder(v.leftchild())
  - postorder(v.rightchild())
  - visit(v)

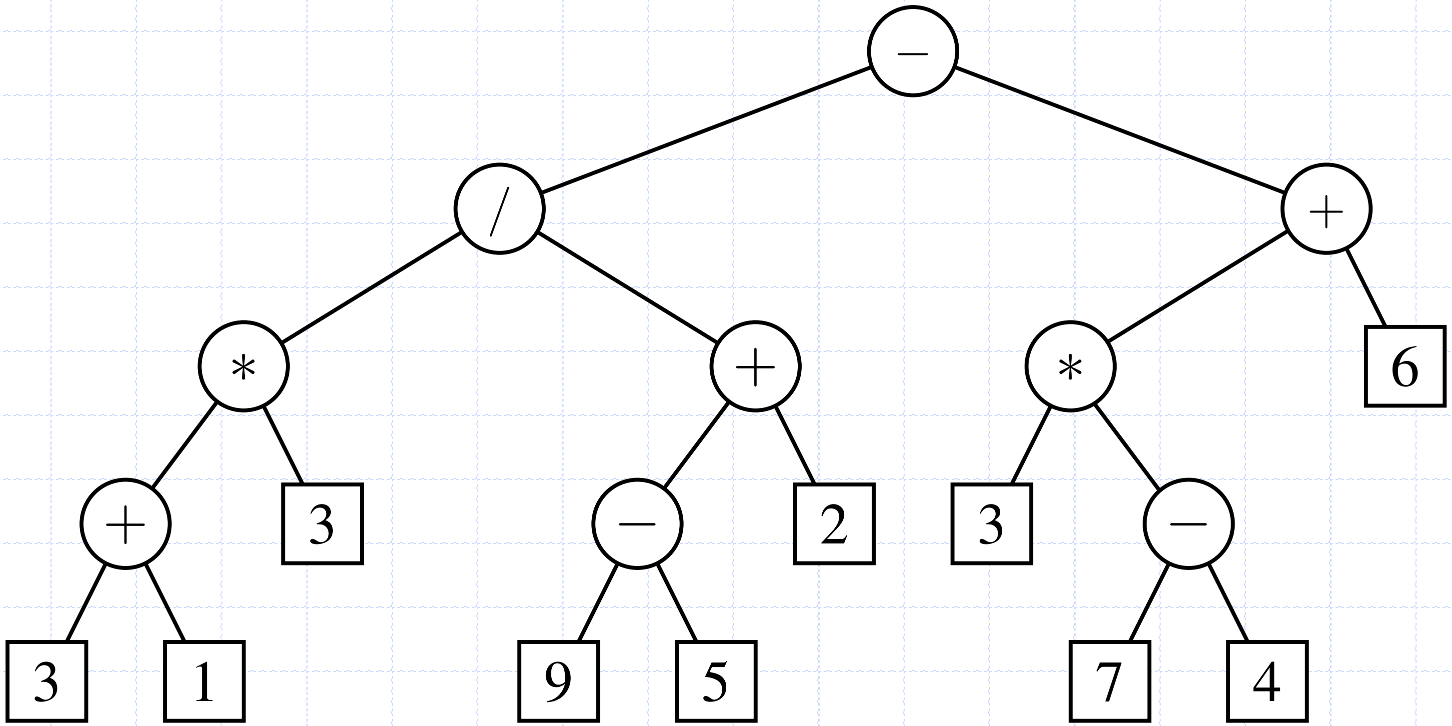
More Example of Traversals

- Visit - printing the data in the node
- Preorder traversal
  - a b d e h i c f g
- Postorder traversal
  - d h i e b f g c a



Application of Postorder Traversal

- Evaluating Arithmetic Expressions



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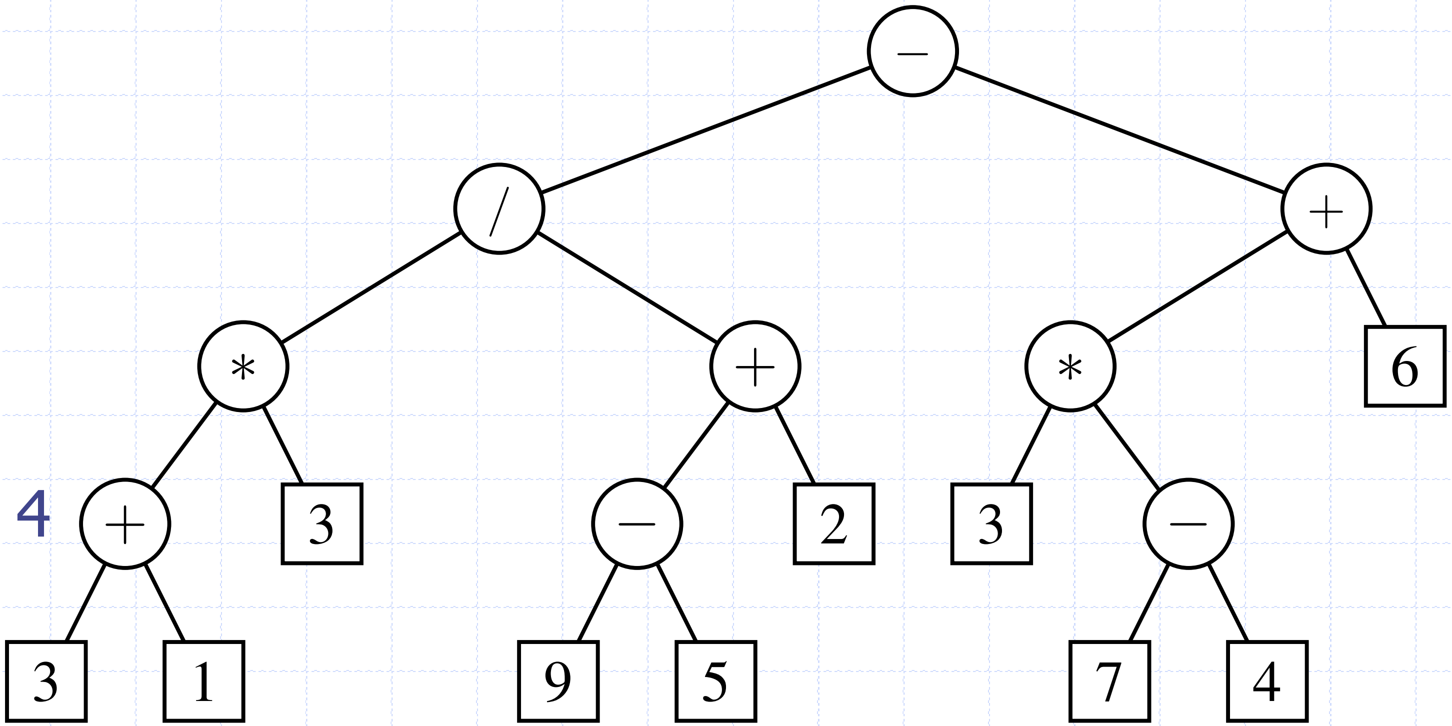

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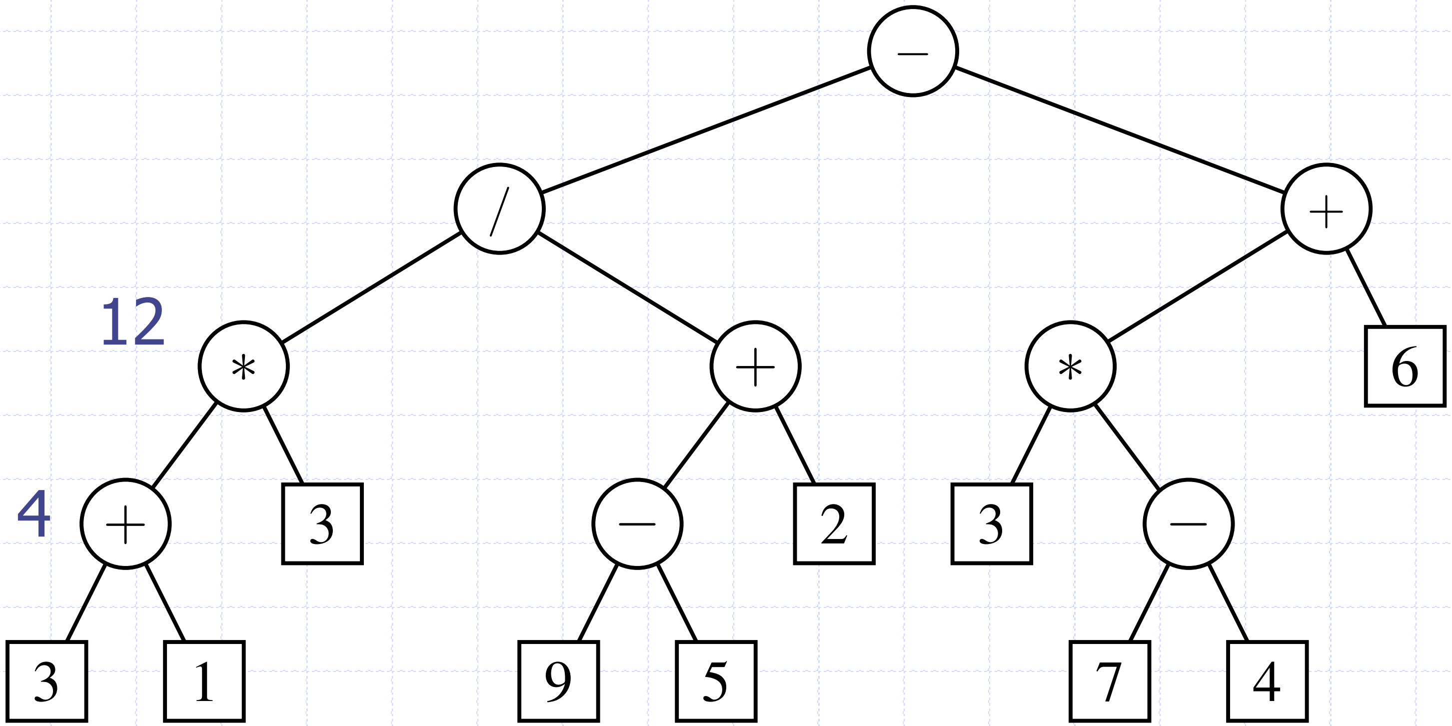

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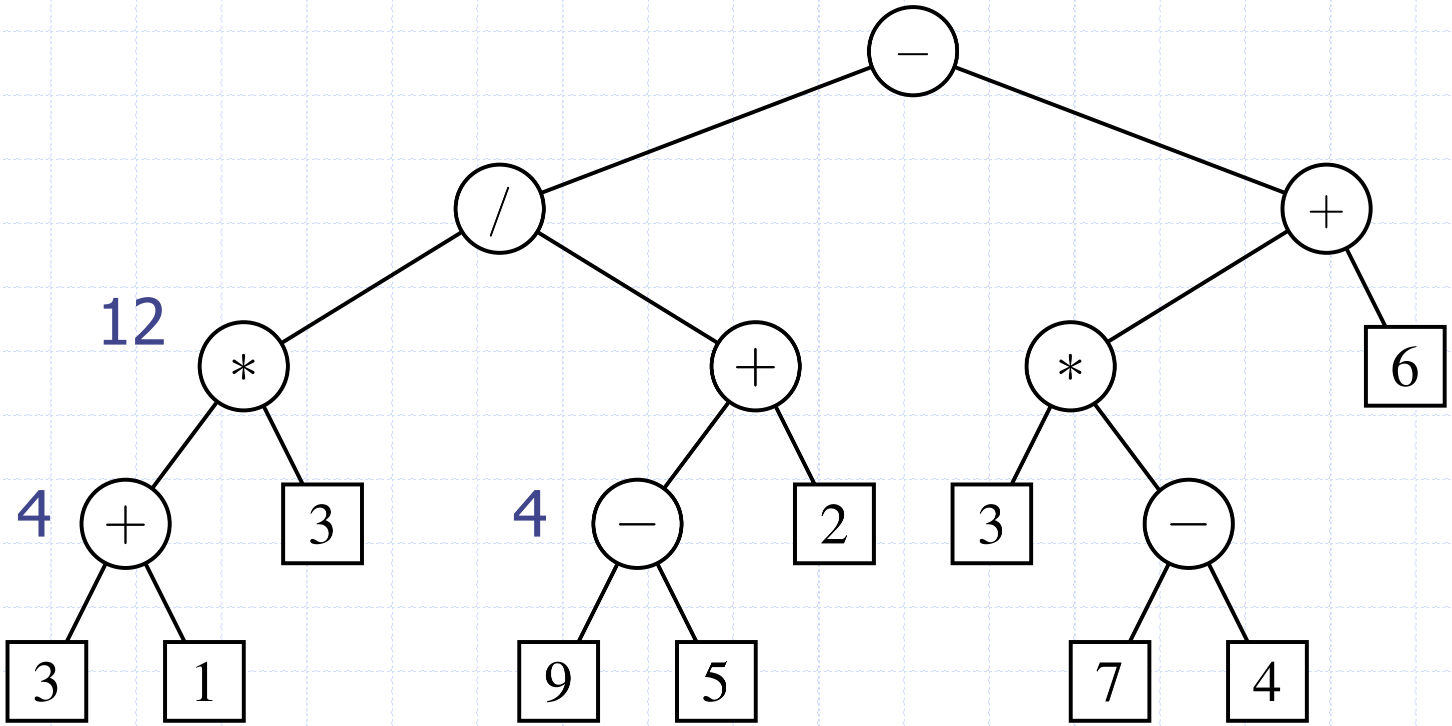

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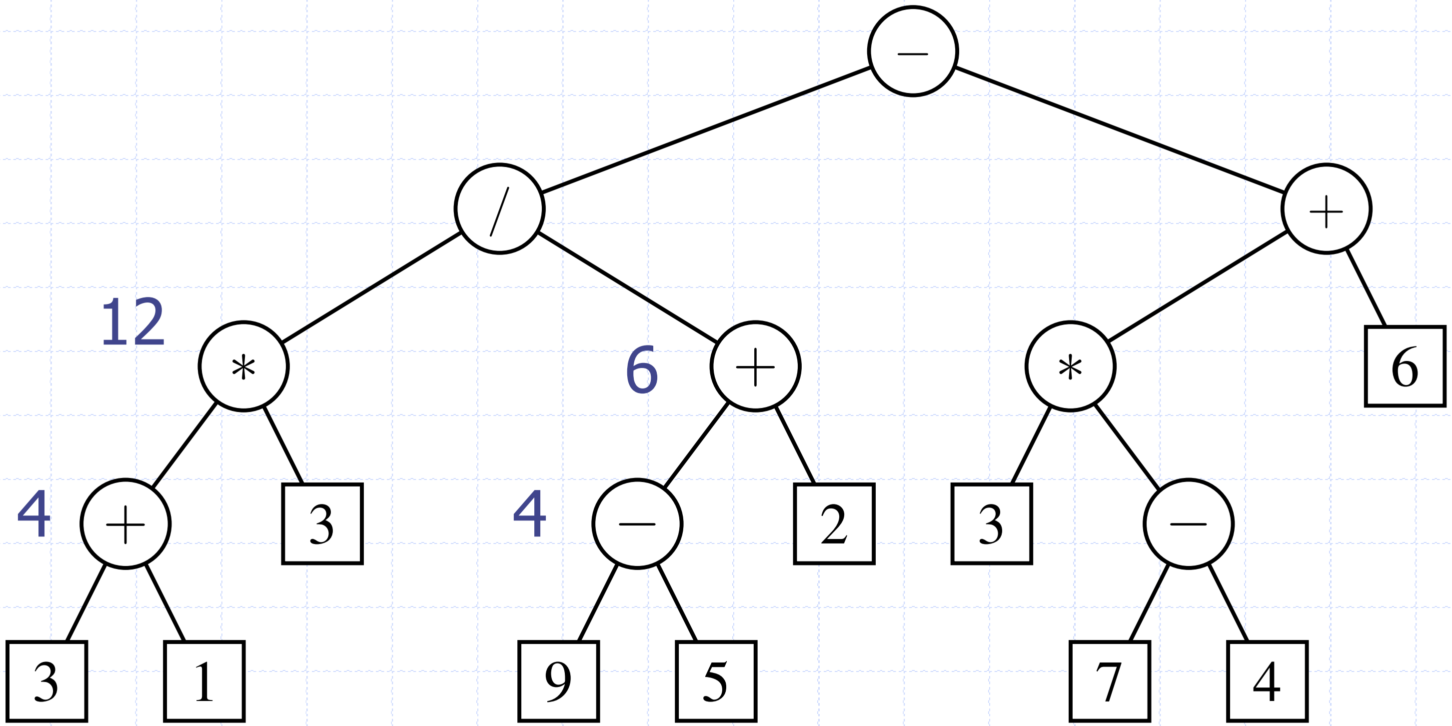

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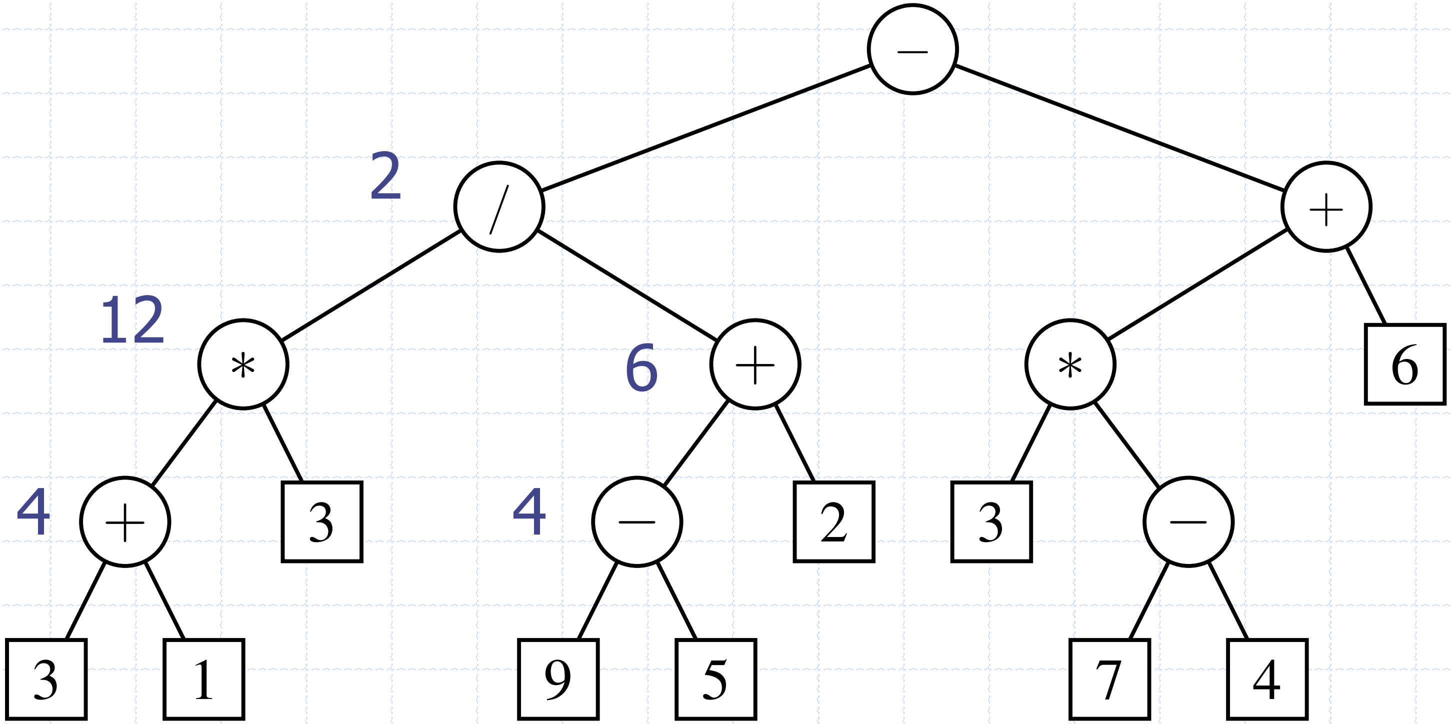

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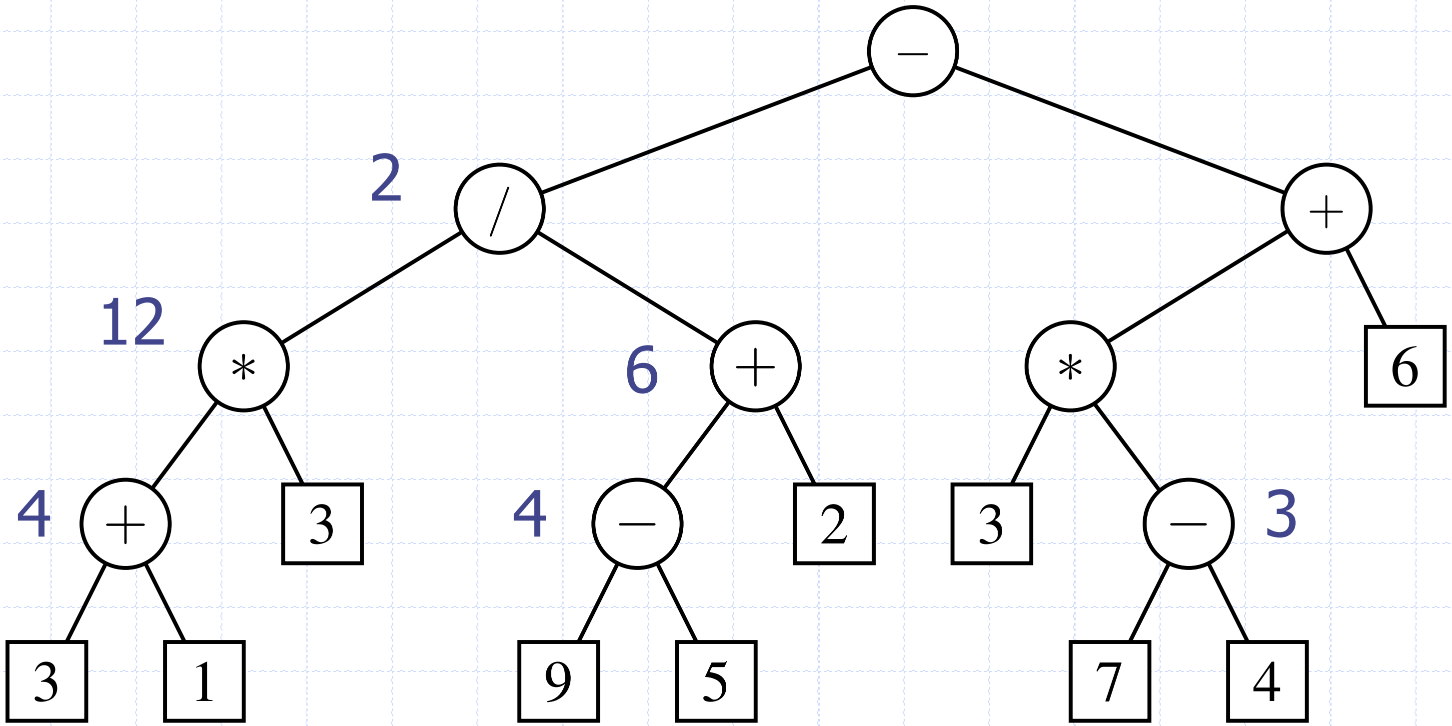

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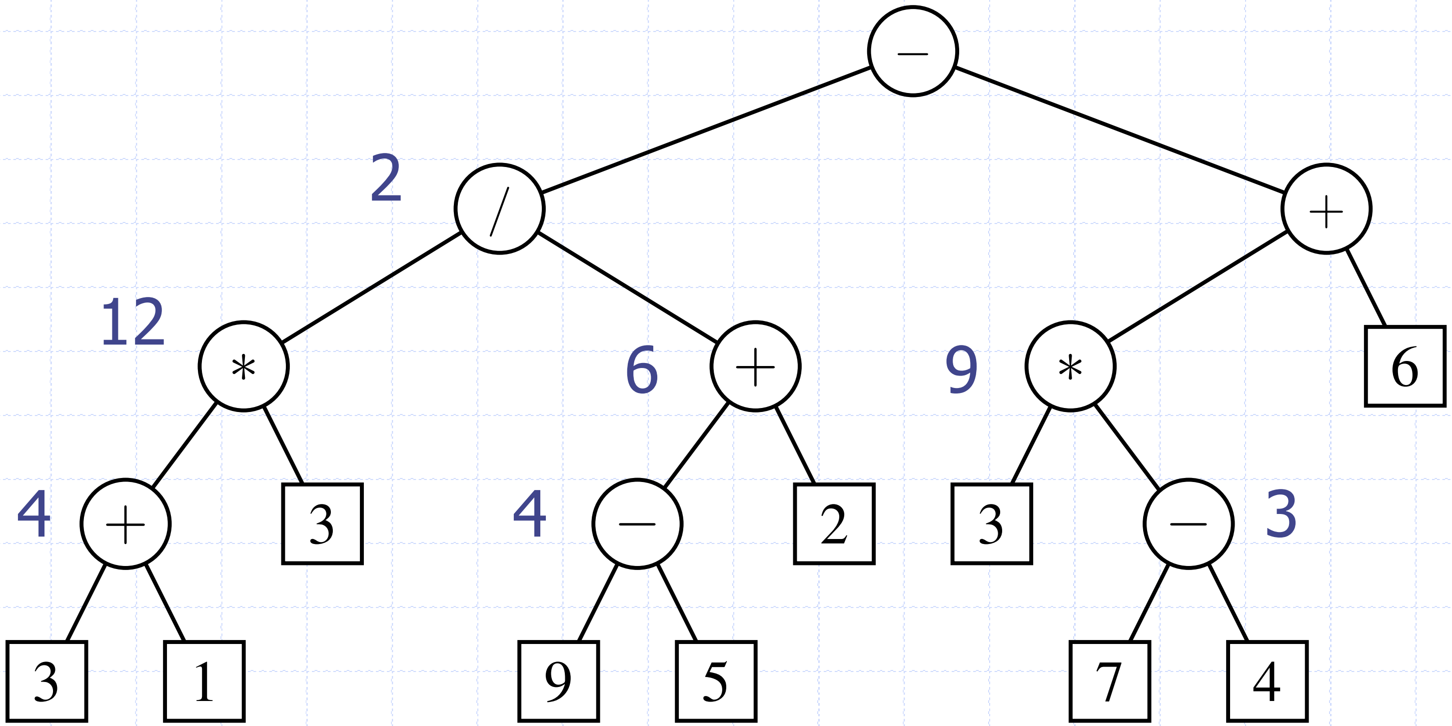

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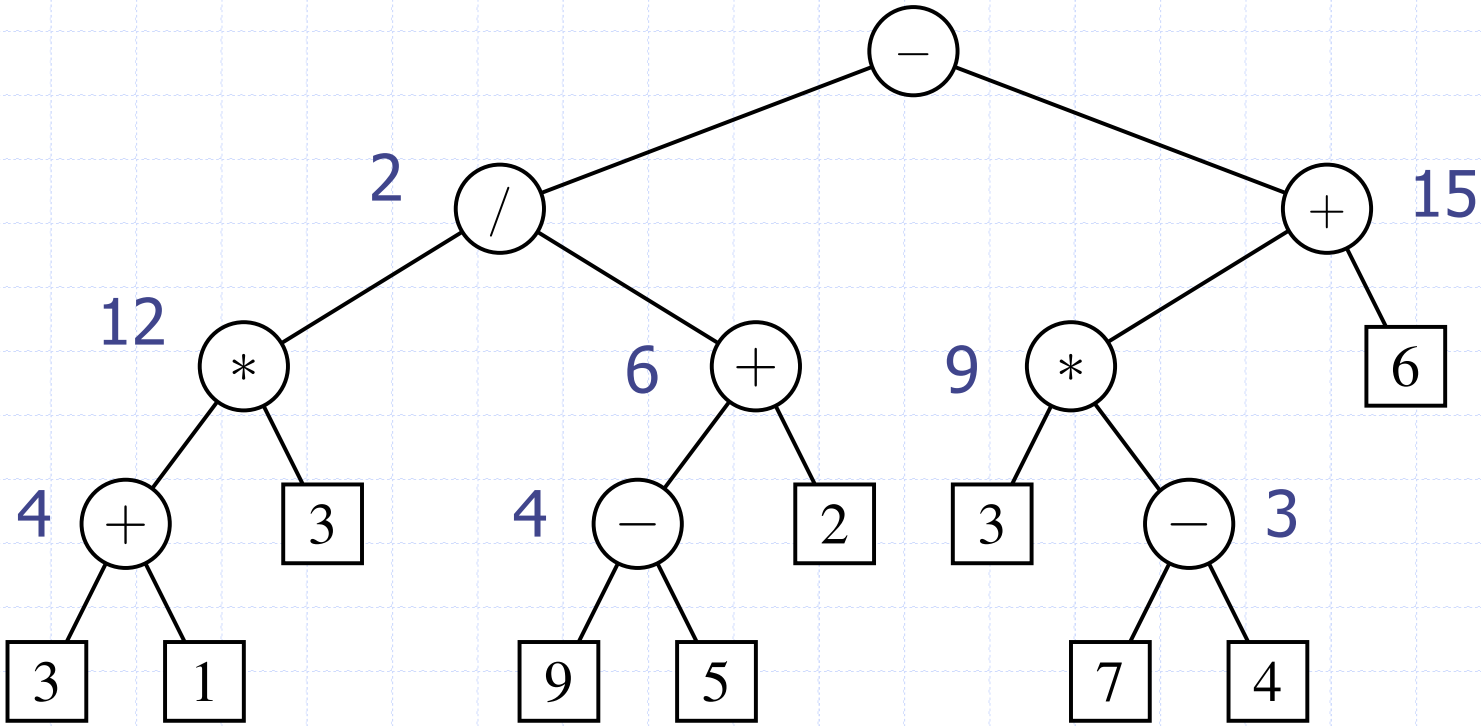

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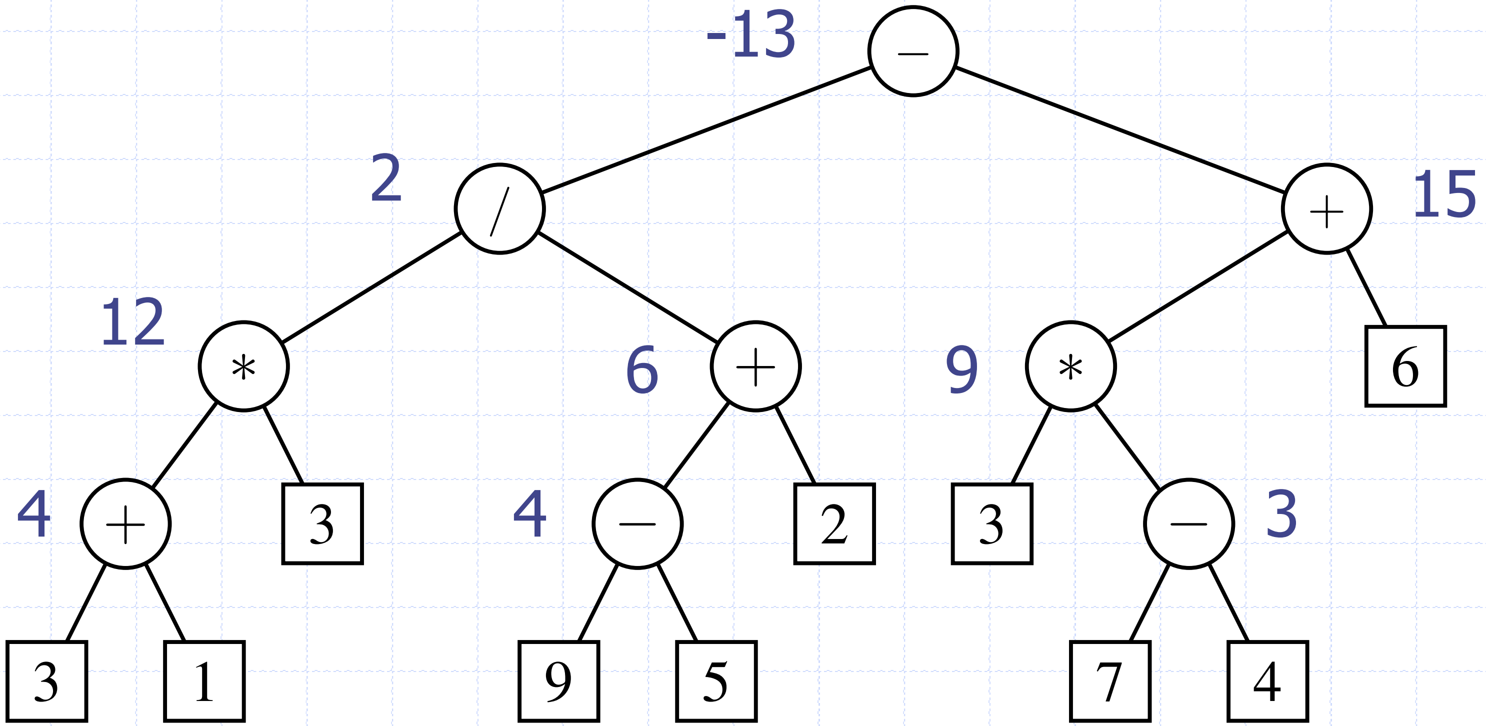
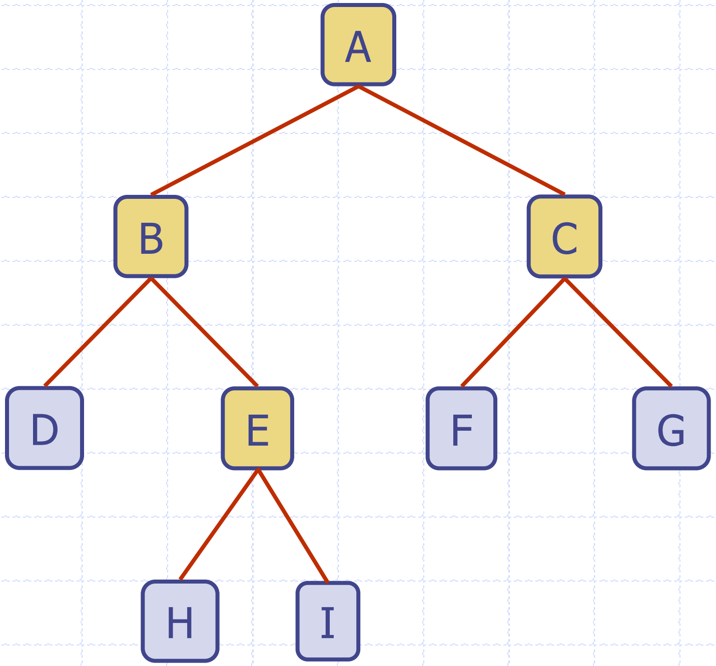


Inorder traversals

- Visit the node between the visit to the left and right subtree
- Algorithm inorder(p)
  - If p has a left child lc then
    - inorder(lc)
  - perform “visit” action for position p
  - If p has a right child rc then
    - inorder(rc)

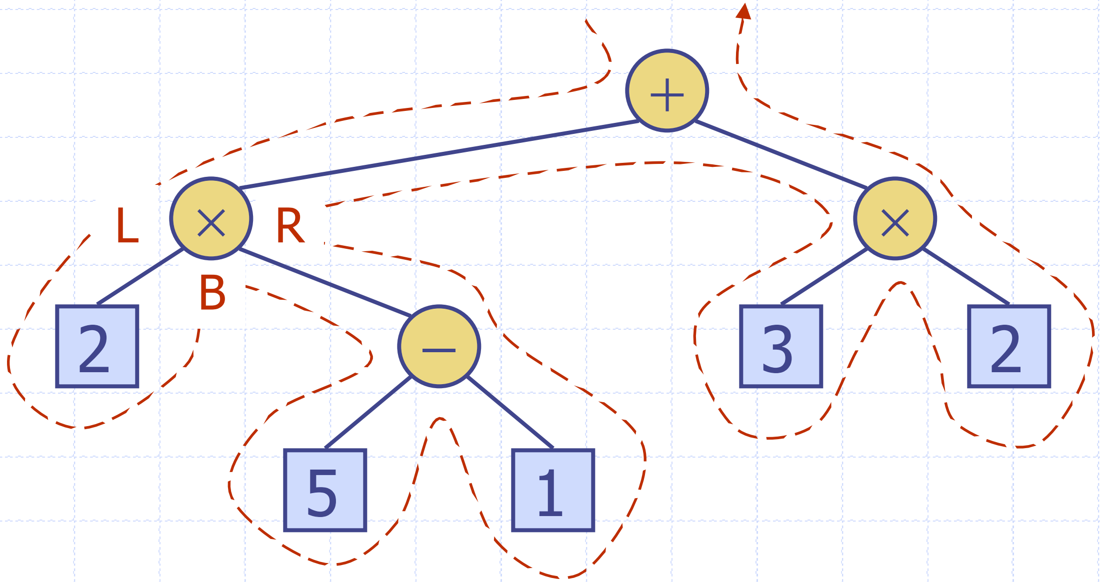
Example - Inorder Traversal

- Inorder
  - d b h e i a f c g



Euler Tour Traversal

- Generic traversal of a binary tree
- Includes as special cases the preorder, postorder and inorder traversals
- Walk around the tree and visit each node three times:
  - on the left (preorder)
  - from below (inorder)
  - on the right (postorder)



Building Tree from Preorder Traversal

- Given the preorder traversal, can we uniquely determine the binary tree?

Preorder

a b d e h i c f g

Building Tree from Postorder Traversal

- Given the postorder traversal, can we uniquely determine the binary tree?

Postorder

d h i e b f g c a

Building Tree from Pre- and In- Order Traversals

- Given the preorder and inorder traversals of a binary tree we can uniquely determine the tree

Preorder	Inorder
a b d e h i c f g	d b h e i a f c g

Building Tree from Pre- and In- Order Traversals

- Given the preorder and inorder traversals of a binary tree we can uniquely determine the tree

Preorder	Inorder
a b d e h i c f g	d b h e i a f c g

A

Building Tree from Pre- and In- Order Traversals

- Given the preorder and inorder traversals of a binary tree we can uniquely determine the tree

Preorder

a	b	d	e	h	i	c	f	g

Inorder

d	b	h	e	i	a	f	c	g

A

Building Tree from Pre- and In- Order Traversals

- Given the preorder and inorder traversals of a binary tree we can uniquely determine the tree

Preorder	Inorder
a b d e h i c f g	d b h e i a f c g
c f g	f c g

A

Building Tree from Pre- and In- Order Traversals

- Given the preorder and inorder traversals of a binary tree we can uniquely determine the tree

Preorder

a b d e h i c f g

c f g

Inorder

d b h e i a f c g

f c g

A

Building Tree from Pre- and In- Order Traversals

- Given the preorder and inorder traversals of a binary tree we can uniquely determine the tree

Preorder

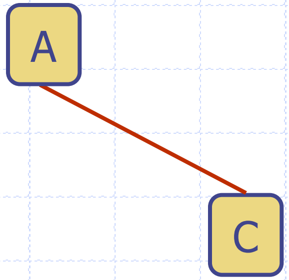
a b d e h i c f g

c f g

Inorder

d b h e i a f c g

f c g



Building Tree from Pre- and In- Order Traversals

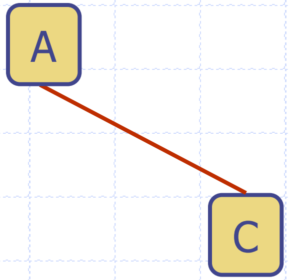
- Given the preorder and inorder traversals of a binary tree we can uniquely determine the tree

Preorder

a	b	d	e	h	i	c	f	g
						c	f	g

Inorder

d	b	h	e	i	a	f	c	g
						f	c	g



Building Tree from Pre- and In- Order Traversals

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Preorder

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						c	f	g

Inorder

d	b	h	e	i	a	f	c	g
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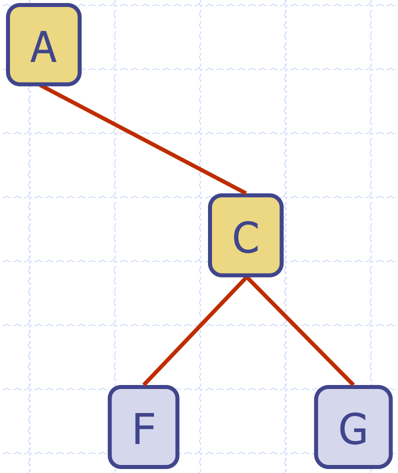


Building Tree from Pre- and In- Order Traversals

- Given the preorder and inorder traversals of a binary tree we can uniquely determine the tree

Preorder

a b d e h i c f g

c f g

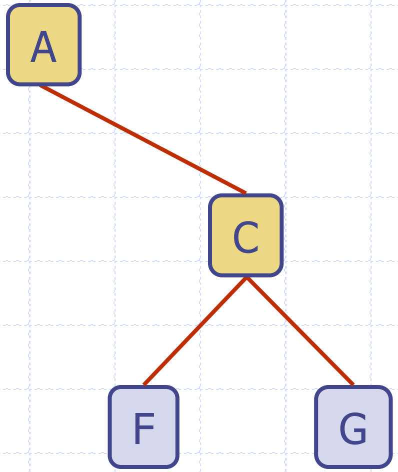
b d e h i

Inorder

d b h e i a f c g

f c g

d b h e i



Building Tree from Pre- and In- Order Traversals

- Given the preorder and inorder traversals of a binary tree we can uniquely determine the tree

Preorder

a b d e h i c f g

c f g

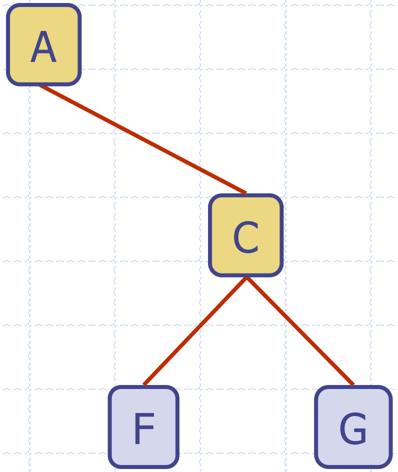
b d e h i

Inorder

d b h e i a f c g

f c g

d b h e i



Building Tree from Pre- and In- Order Traversals

- Given the preorder and inorder traversals of a binary tree we can uniquely determine the tree

Preorder

a b d e h i c f g

c f g

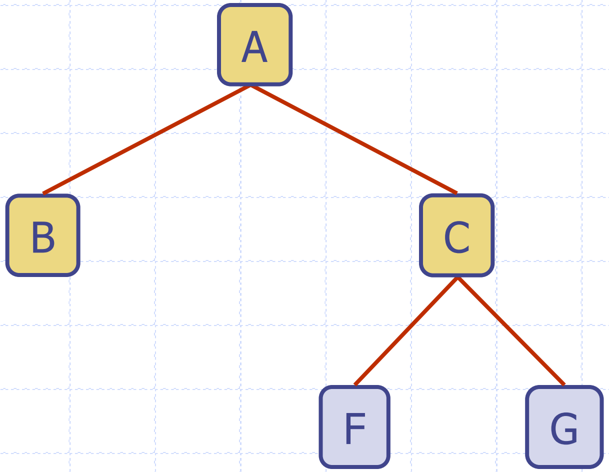
b d e h i

Inorder

d b h e i a f c g

f c g

d b h e i



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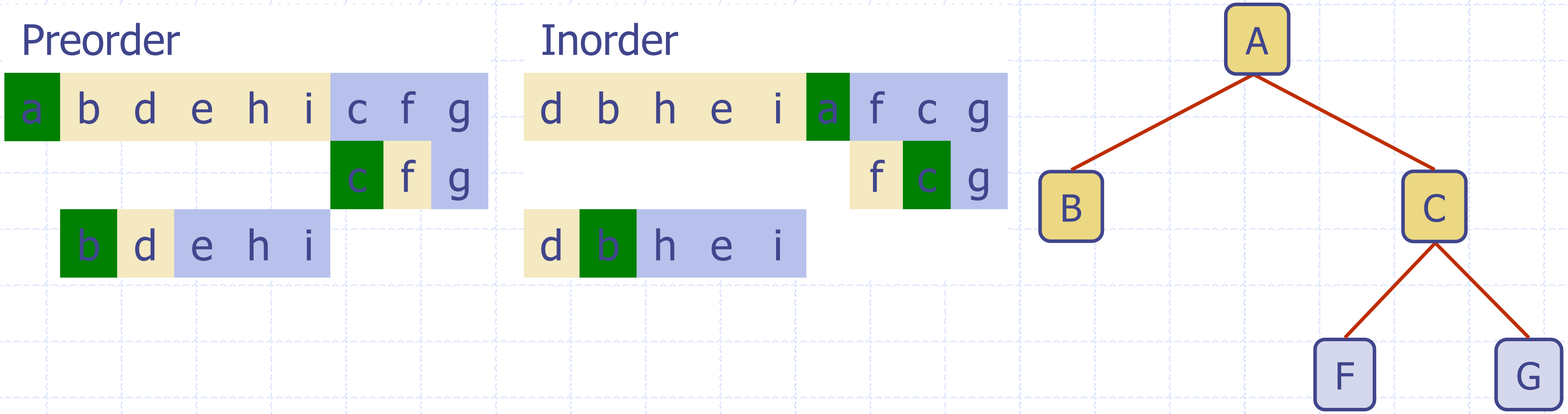

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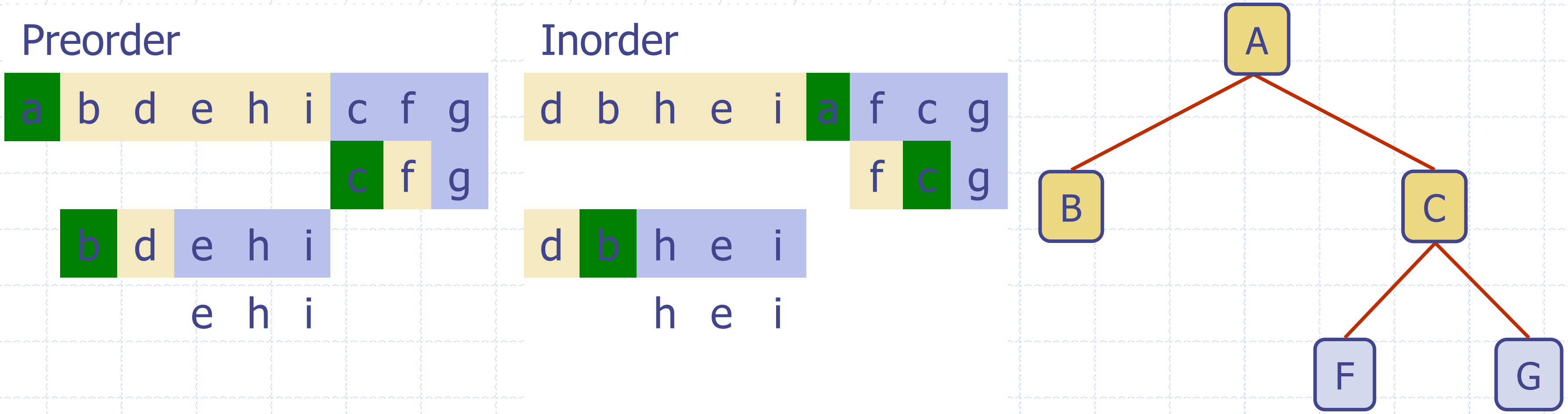

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Preorder

a b d e h i c f g

c f g

b d e h i

e h i

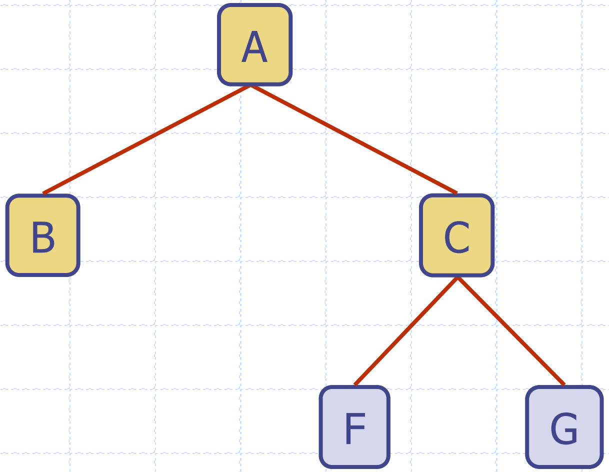
Inorder

d b h e i a f c g

f c g

d b h e i

h e i



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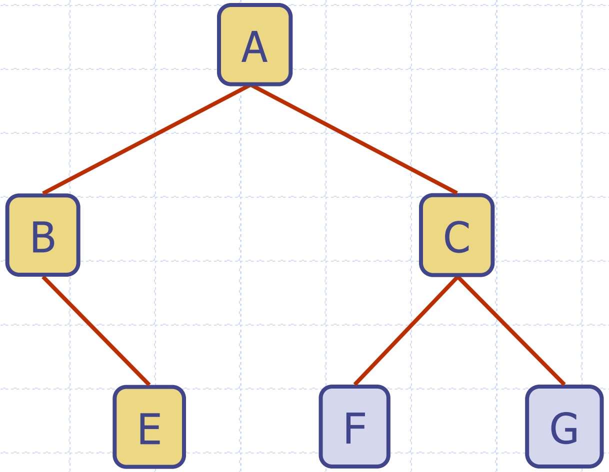

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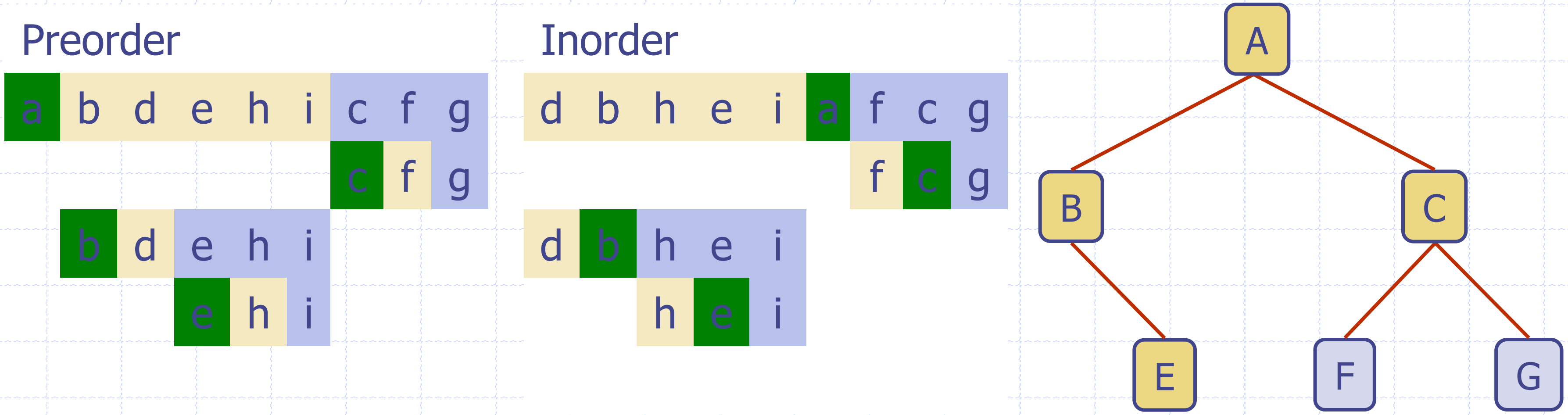

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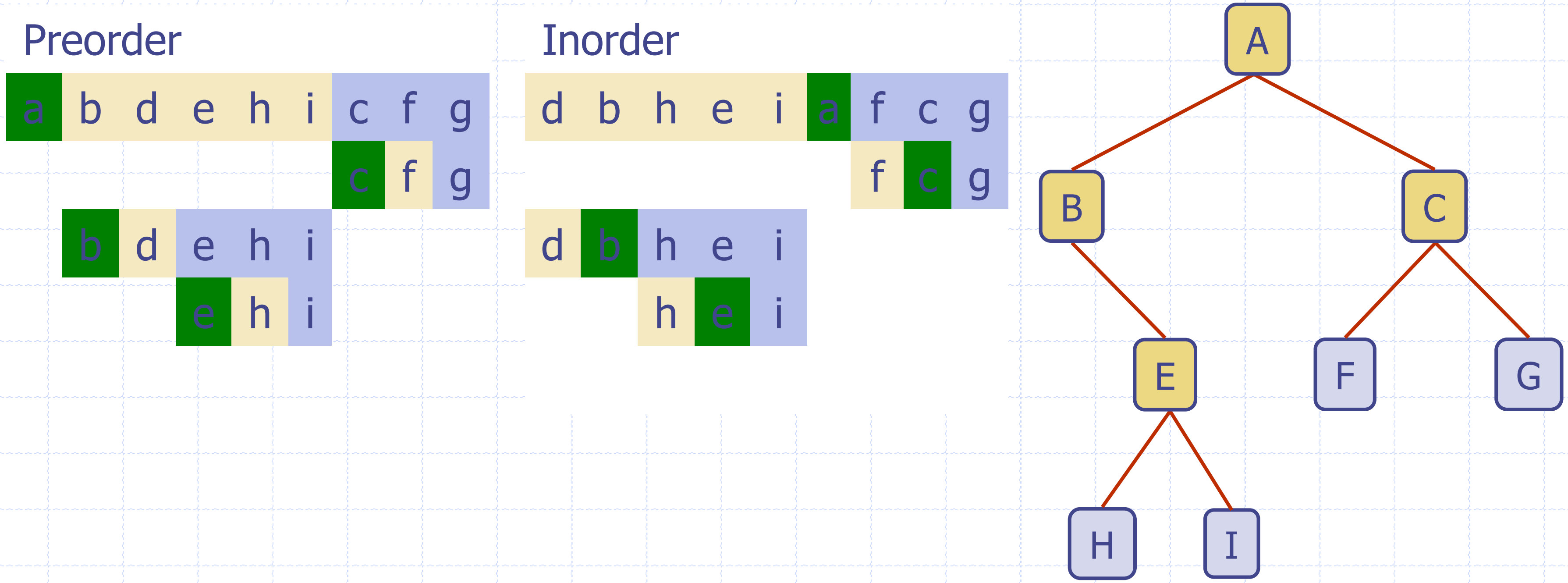


Building Tree from Pre- and In- Order Traversals

- Given the preorder and inorder traversals of a binary tree we can uniquely determine the tree

Preorder

a b d e h i c f g

c f g

b d e h i

e h i

d

Inorder

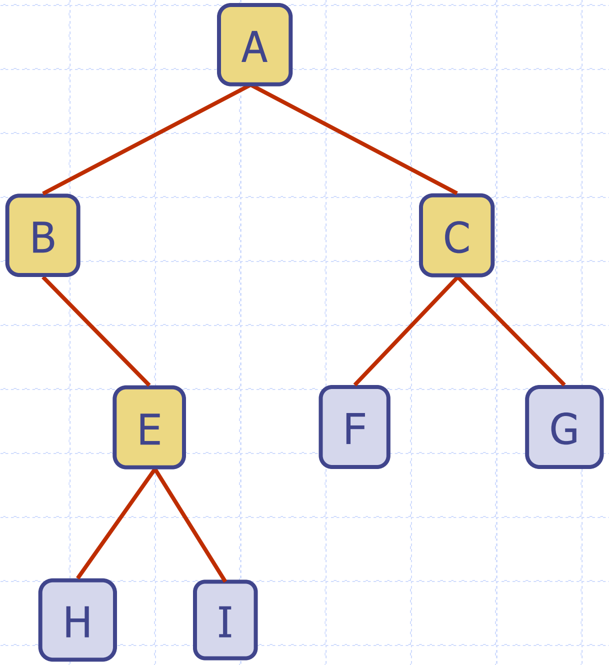
d b h e i a f c g

f c g

d b h e i

h e i

d



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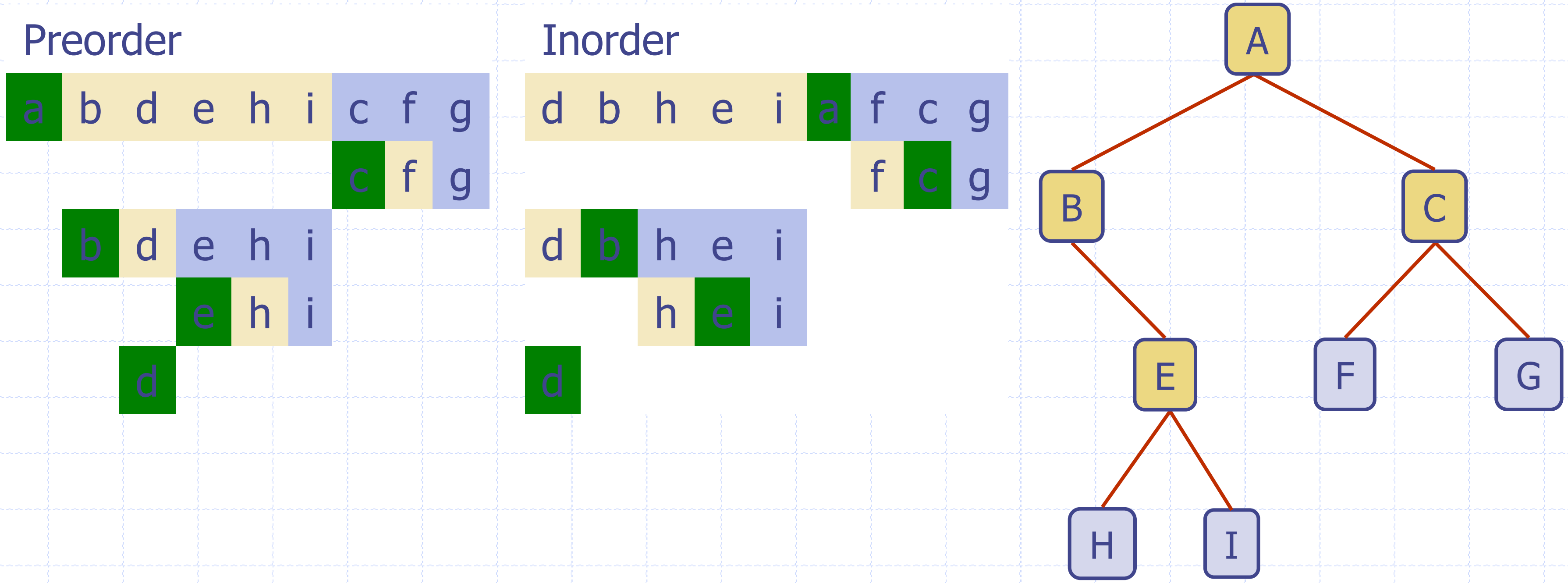

Building Tree from Pre- and In- Order Traversals

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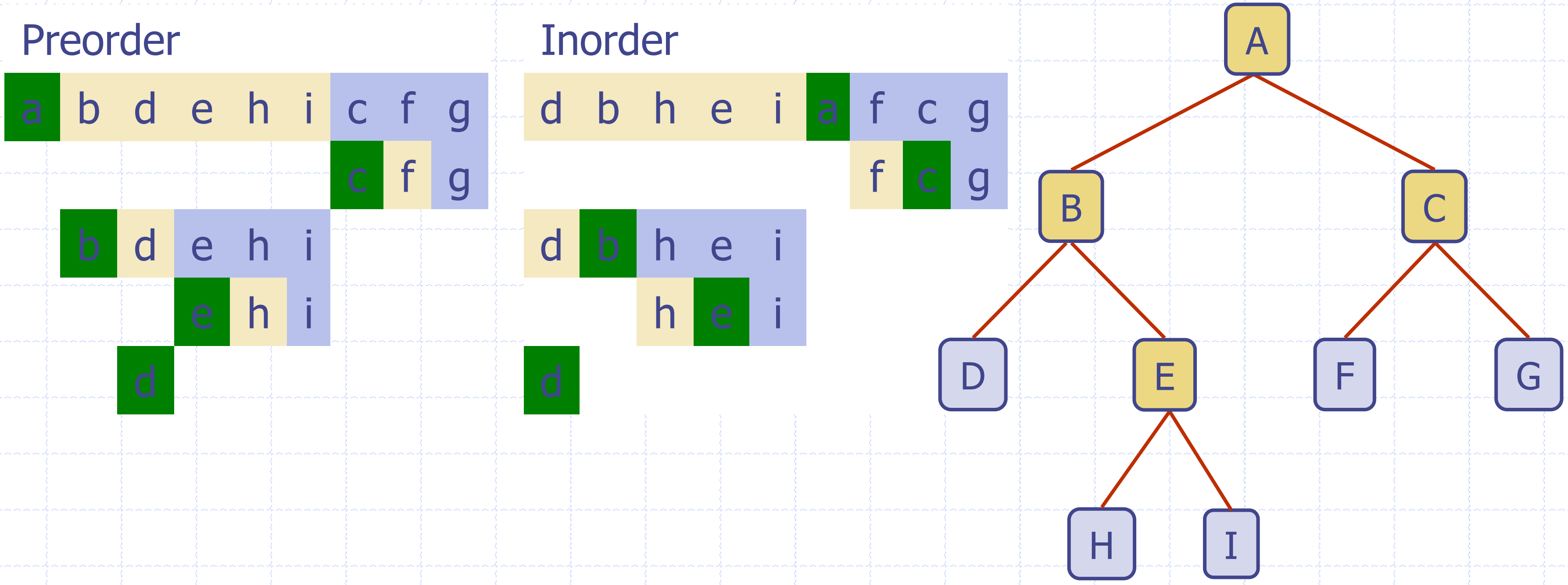

Building Tree from Post- and In- Order Traversals

- Given the postorder and inorder traversals of a binary tree we can uniquely determine the tree
- The last node visited in the postorder traversal is the root of the binary tree

Postorder	Inorder
d h i e b f g c a	d b h e i a f c g

Building Tree from Post- and In- Order Traversals

- Given the postorder and inorder traversals of a binary tree we can uniquely determine the tree
- The last node visited in the postorder traversal is the root of the binary tree

Postorder	Inorder
d h i e b f g c a	d b h e i a f c g

A

Building Tree from Post- and In- Order Traversals

- Given the postorder and inorder traversals of a binary tree we can uniquely determine the tree
- The last node visited in the postorder traversal is the root of the binary tree

Postorder

d	h	i	e	b	f	g	c	a

Inorder

d	b	h	e	i	a	f	c	g

Building Tree from Post- and In- Order Traversals

- Given the postorder and inorder traversals of a binary tree we can uniquely determine the tree
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Postorder

d	h	i	e	b	f	g	c	a
					f	g	c	

Inorder

d	b	h	e	i	a	f	c	g
						f	c	g

A

Building Tree from Post- and In- Order Traversals

- Given the postorder and inorder traversals of a binary tree we can uniquely determine the tree
- The last node visited in the postorder traversal is the root of the binary tree

Postorder

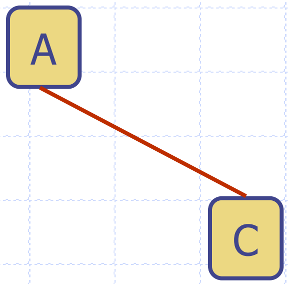
d h i e b f g c a

f g c

Inorder

d b h e i a f c g

f c g



Building Tree from Post- and In- Order Traversals

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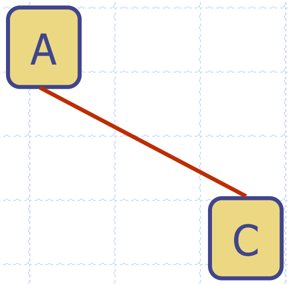
Postorder

d	h	i	e	b	f	g	c	a
					f	g	c	

Inorder

d	b	h	e	i	a	f	c	g
						f	c	g



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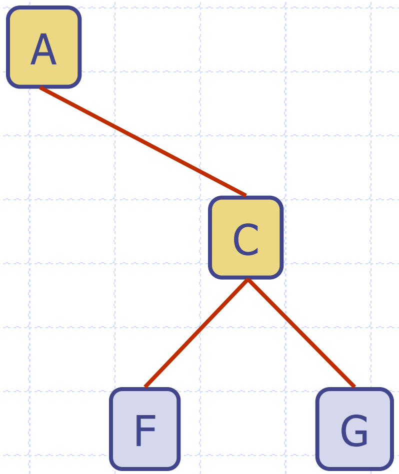
Postorder

d	h	i	e	b	f	g	c	a
					f	g	c	

Inorder

d	b	h	e	i	a	f	c	g
						f	c	g



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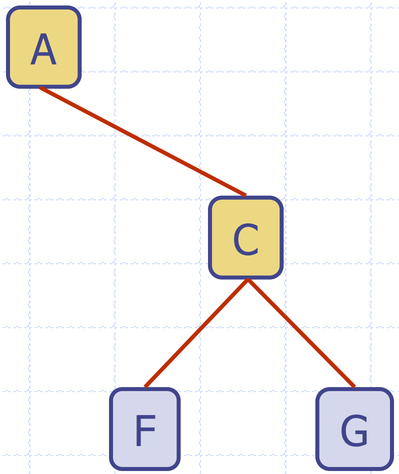
Postorder	Inorder
d h i e b f g c a	d b h e i a f c g
f g c	f c g
d h i e b	d b h e i



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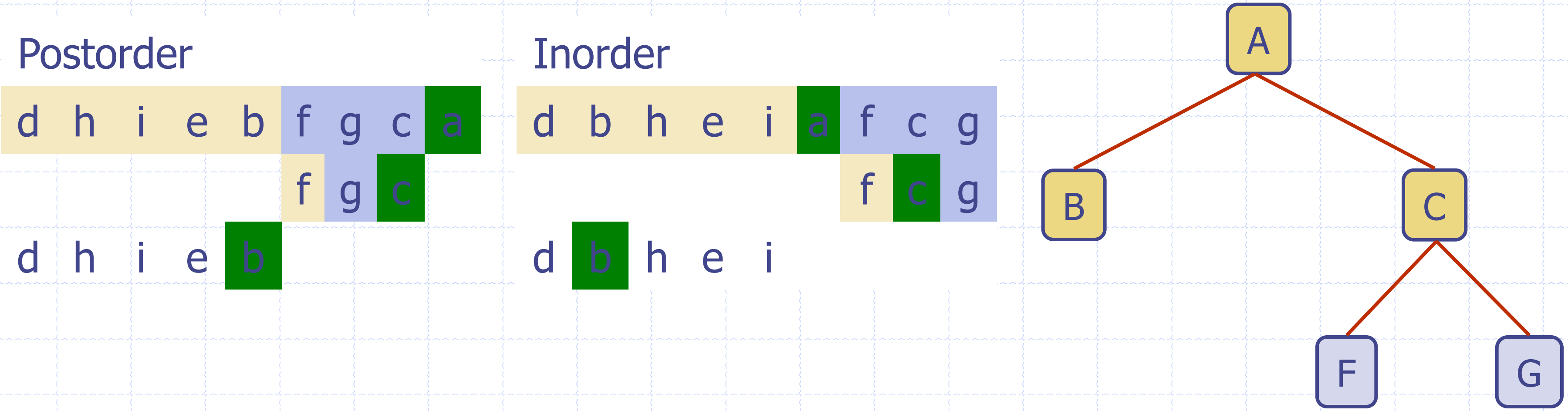

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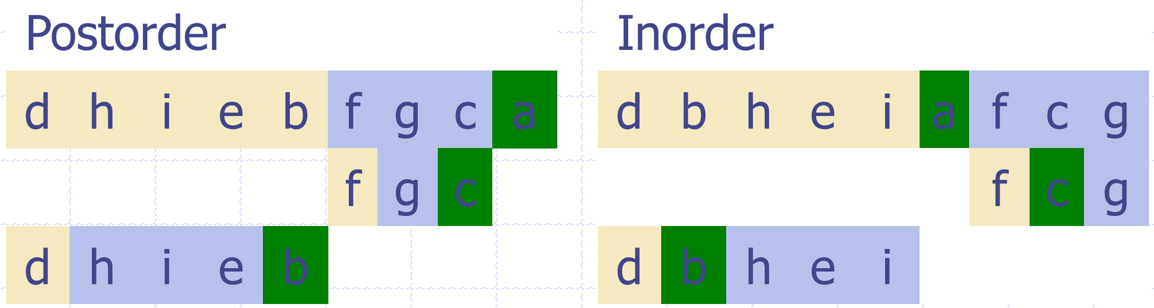

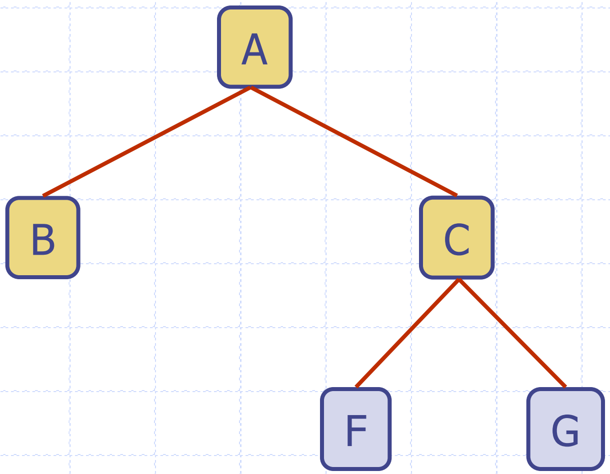

- and so on..

Building Tree from Pre- and Post- Order Traversals

- Given the pre and postorder traversal of a binary tree, can we uniquely reconstruct the tree?

Preorder	Postorder
a b d e h i c f g	d h i e b f g c a

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- Given the pre and postorder traversal of a binary tree, can we uniquely reconstruct the tree?

Preorder	Postorder
a b d e h i c f g	d h i e b f g c a

A

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Preorder	Postorder
a b d e h i c f g	d h i e b f g c a

A

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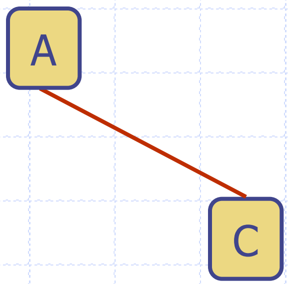
Preorder	Postorder
a b d e h i c f g	d h i e b f g c a
c f g	f g c



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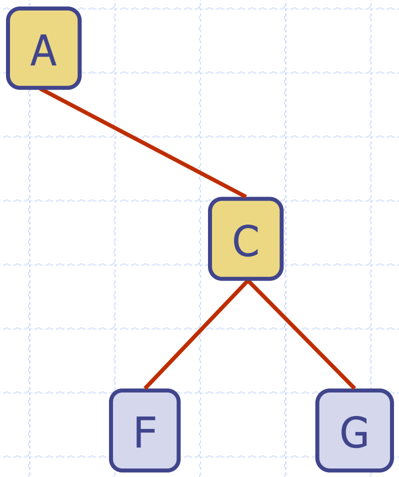
- Given the pre and postorder traversal of a binary tree, can we uniquely reconstruct the tree?

Preorder									Postorder								
a	b	d	e	h	i	c	f	g	d	h	i	e	b	f	g	c	a
						c	f	g						f	g	c	



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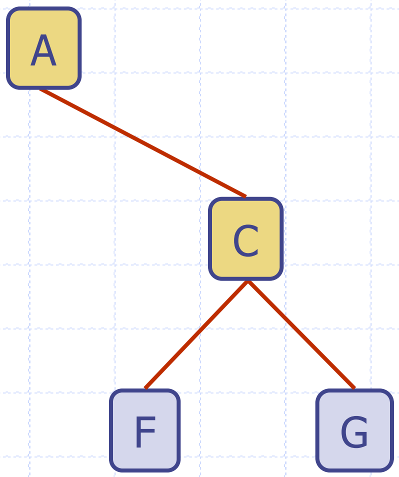
c f g	f g c
b d e h i	d h i e b



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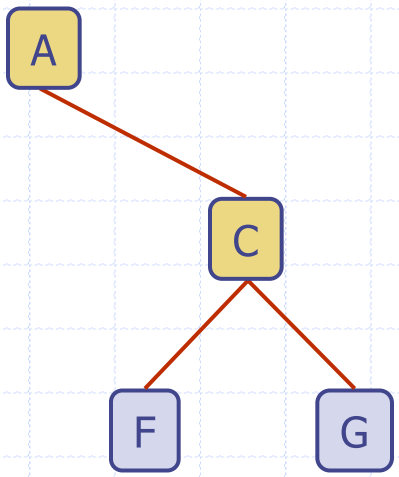
b d e h i	d h i e b



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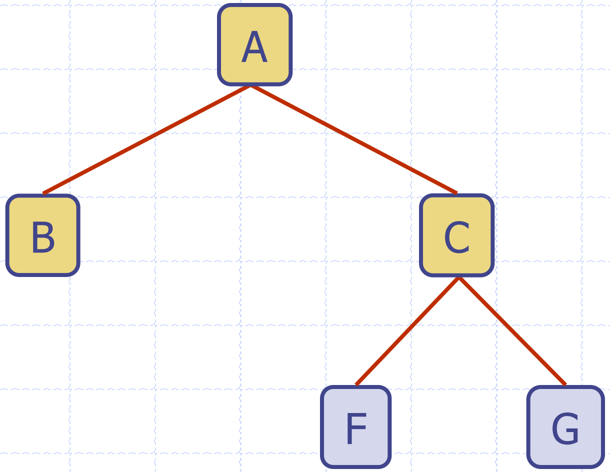
a b d e h i c f g	d h i e b f g c a
c f g	f g c
b d e h i	d h i e b



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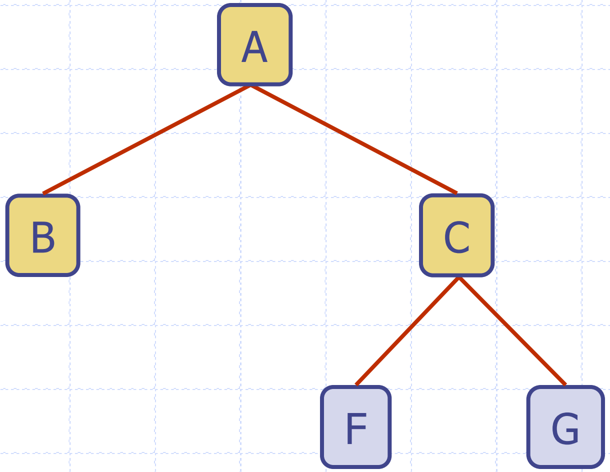
a b d e h i c f g	d h i e b f g c a
c f g	f g c
b d e h i	d h i e b



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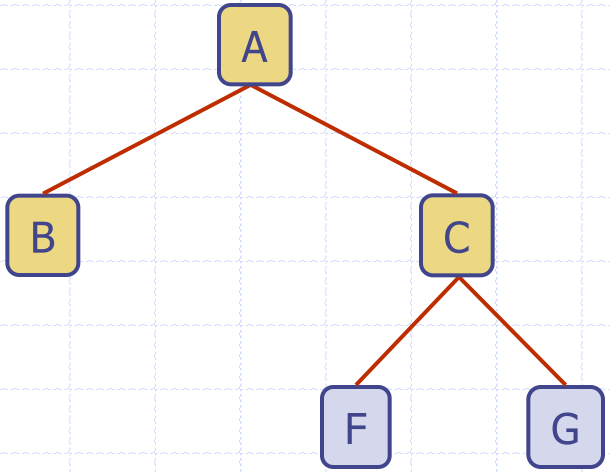
a b d e h i c f g	d h i e b f g c a
c f g	f g c
b d e h i	d h i e b



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Preorder

a b d e h i c f g

c f g

b d e h i

e h i

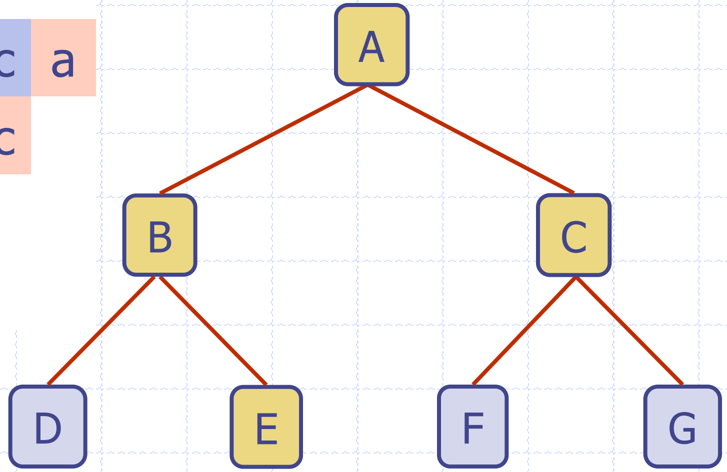
Postorder

d h i e b f g c a

f g c

d h i e b

h i e



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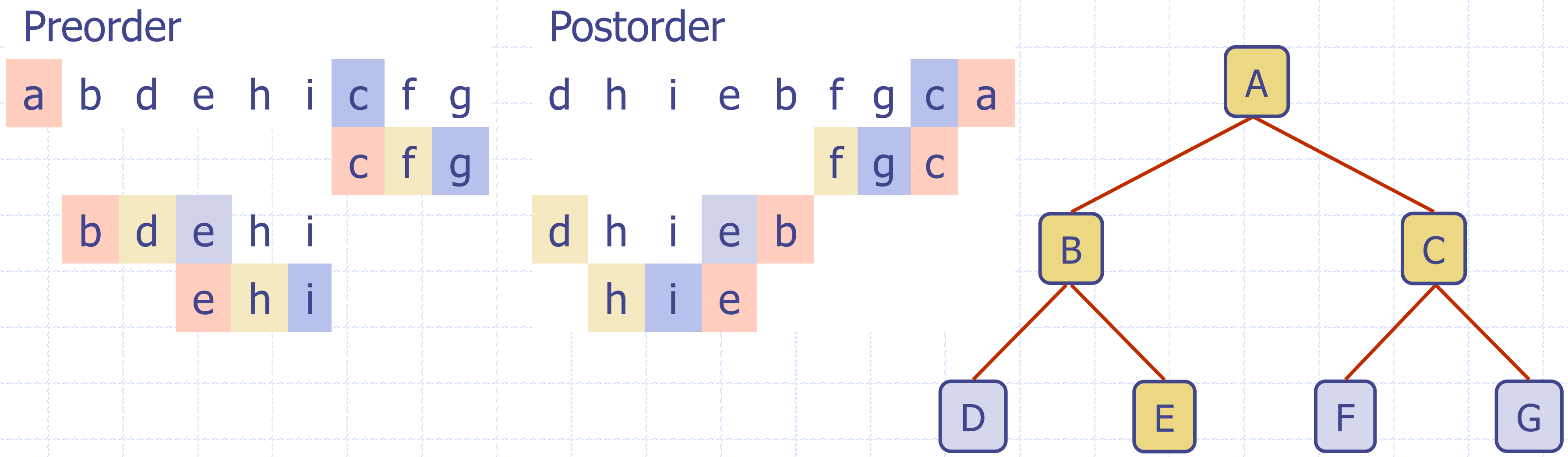

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c f g

b d e h i

e h i

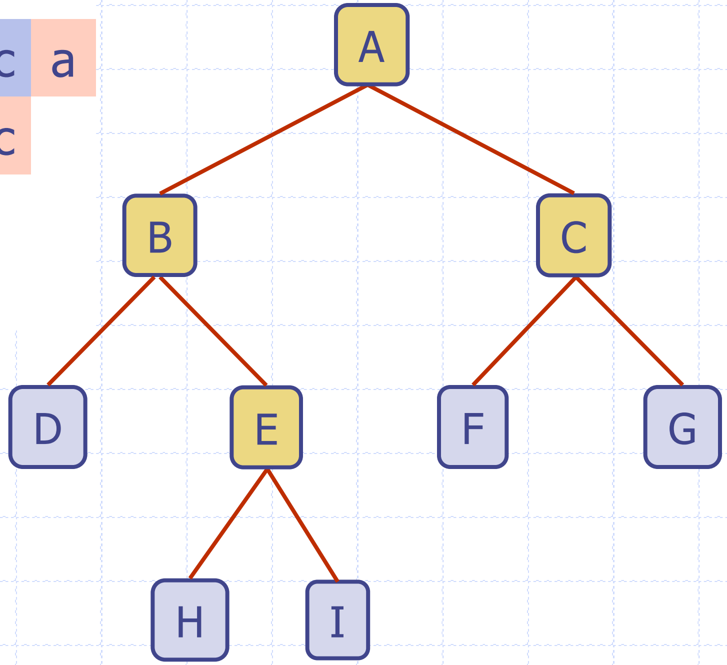
Postorder

d h i e b f g c a

f g c

d h i e b

h i e



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Preorder	Postorder
a b e i c	i e b c a

Building Tree from Pre- and Post- Order Traversals

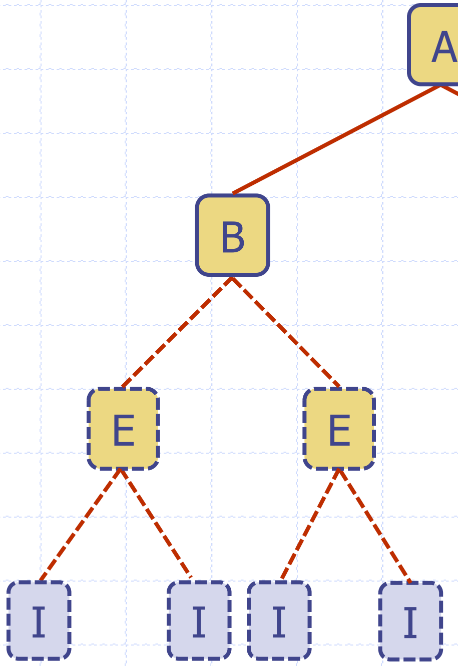
- Given the pre and postorder traversal of a binary tree, can we uniquely reconstruct the tree?

Preorder

a b e i c

Postorder

i e b c a



Building Tree from Pre- and Post- Order Traversals

- Given the pre and postorder traversal of a binary tree, can we uniquely reconstruct the tree?

Preorder

a b e i c

Postorder

i e b c a

Only if the internal nodes in a binary tree have exactly two children