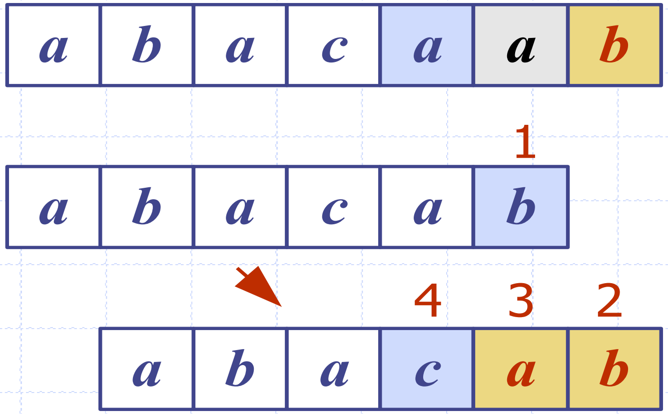
Pattern Matching



Strings

- A string is a sequence of characters
- Examples of strings:
  - Python program
  - HTML document
  - DNA sequence
  - Digitized image
- An alphabet Σ is the set of possible characters for a family of strings
- Example of alphabets:
  - ASCII
  - Unicode
  - {0, 1}
  - {A, C, G, T}

- Let P be a string of size m
  - A substring $P[i \,..\, j]$ of P is the subsequence of P consisting of the characters with ranks between i and j
  - A prefix of P is a substring of the type $P[0 \,..\, i]$
  - A suffix of P is a substring of the type $P[i \,..\, m - 1]$
- Given strings T (text) and P (pattern), the pattern matching problem consists of finding a substring of T equal to P
- Applications:
  - Text editors
  - Search engines
  - Biological research

Brute-Force Pattern Matching

- The brute-force pattern matching algorithm compares the pattern P with the text T for each possible shift of P relative to T, until either
  - a match is found, or
  - all placements of the pattern have been tried
- Brute-force pattern matching runs in time $O(nm)$
- Example of worst case:
  - T □ $aaa \ldots ah$
  - P □ $aaah$
  - may occur in images and DNA sequences
  - unlikely in English text

```
Algorithm BruteForceMatch(T, P)
    Input text T of size n and pattern
        P of size m
    Output starting index of a
        substring of T equal to P or −1
        if no such substring exists
    for i ← 0 to n − m
        { test shift i of the pattern }
        j ← 0
        while j □ m ∧ T[i □ j] □ P[j]
            j ← j □ 1
        if j □ m
            return i {match at i}
        else
            break while loop {mismatch}
    return -1 {no match anywhere}
```

Right to Left Matching

- Matching the pattern from right to left
- For a pattern abc:

T: `bbacdcbaabcddcdaddaaabcbcb`

P: `abc`

- Worst case is still O(n m)

Bad Character Rule (BCR) (1)

- On a mismatch between the pattern and the text, we can shift the pattern by more than one place.

```
ddbbacdcbaabcddcdaddaaabcbcb
acabc
```

Bad Character Rule (BCR) (2)

- Preprocessing –
  - A table, for each position in the pattern and a character, last occurrence of the mismatched character in P preceding the mismatch . O(n |Σ|) space. O(1) access time.

a c a b c

1 2 3 4 5

	1	2	3	4	5
a	1	1	3	3	3
b				4	4
c		2	2	2	5

Bad Character Rule (BCR) (3)

- On a mismatch, shift the pattern to the right until the first occurrence of the mismatched character in P.

- Still O(n m) worst case running time:

```
T: aaaaaaaaaaaaaaaaaaaaaaaaa
P: abaaaa
```

Good Suffix Rule (GSR) (1)

- We want to use the knowledge of the matched characters in the pattern’s suffix.
- If we matched S characters in T, what is (if exists) the smallest shift in P that will align a sub-string of P of the same S characters ?

Good Suffix Rule (GSR) (2)

- **Example 1 – how much to move:**

```
T:  bbacdcbaabcddcdaddaaabcbcb
P:  cabbabdbab
          cabbabdbab
```

Good Suffix Rule (GSR) (3)

- **Example 2 – what if there is no alignment:**

↓

```
T:  bbacdcbaabcbbabdbabcaabcbcb
P:  bcbbabdbabc
                 bcbbabdbabc
```

Good Suffix Rule (GSR) (4)

T: a b c d e b a

P: d a c

t

- We mark the matched sub-string in T with *t* and the mismatched char with *x*
  1. In case of a mismatch: shift right until the first occurrence of *t* in P such that the next char *y* in P holds $y \neq x$
  2. Otherwise, shift right to the largest prefix of P that aligns with a suffix of *t*.

Boyer Moore Algorithm

- Preprocess(P)
- k := n

while (k ≤ m) do

- Match P and T from right to left starting at k
- If a mismatch occurs: shift P right (advance k) by max(good suffix rule, bad char rule).
- else, print the occurrence and shift P right (advance k) by the good suffix rule.

Boyer-Moore Algorithm Demo

GTTATAGCTGATCGCGGCGTAGCGGCGAA

GTAGCGGCG

Bad Character Rule – shift by 7
Good Suffix Rule – shift by 0

Boyer-Moore Algorithm Demo

GTTATAGCTGATCGCGGCGTAGCGGCGAA

GTAGCGGCG *t*

Bad Character Rule – shift by 1

Good Suffix Rule – shift by 3 (9-6)

Pattern P – GTAGCGGCG ***t*** **– GCG**

Prefixes of P

G

GT

GTA

GTAG

GTAGC

GTAGCG

Find the longest prefix of P which has *t* as the suffix.

Boyer-Moore Algorithm Demo

GTTATAGCTGATCGCGGCGTAGCGGCGAA

GTAGCGGCG *t*

Pattern P – GTAGCGGCG

t – GCGGCG

Prefixes of P

G
GT
GTA
GTAG
GTAGC
GTAGCG
GTAGCGG
GTAGCGGC
GTAGCGGCG

Find the longest prefix of P which has t as the suffix.

Boyer-Moore Algorithm Demo

GTTATAGCTGATCGCGGCGTAGCGGCGAA

GTAGCGGCG *t*

Bad Character Rule – shift by 3
Good Suffix Rule – shift by 8 (9-1)

Pattern P – GTAGCGGCG *t* – GCGGCG

suffixes of *t*	Prefixes of P
G	G
CG	GT
GCG	GTA
GCGG	GTAG
GCGGC	GTAGC
GCGGCG	GTAGCG

Find the longest prefix of P which is a suffix of *t*.

Boyer-Moore Algorithm Demo

```
GTTATAGCTGATCGCGGCGTAGCGGCGAA
                  GTAGCGGCG
```

Good Suffix Rule (GSR) (5)

- L(i) – The biggest index j, such that j < n and prefix P[1..j] contains suffix P[i..n] as a suffix but not suffix P[i-1..n]

i	1	2	3	4	5	6	7	8	9
P	G	T	A	G	C	G	G	C	G
L(i)	**0**	**0**	**0**	**0**	**0**	**0**	**6**	**0**	**7**

Good Suffix Rule (GSR) (6)

- l(i) – The length of the longest suffix of P[i..n] that is also a prefix of P

i	1	2	3	4	5	6	7	8	9
P	G	T	A	G	C	G	G	C	G
l(i)	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**

Good Suffix Rule (GSR) (7)

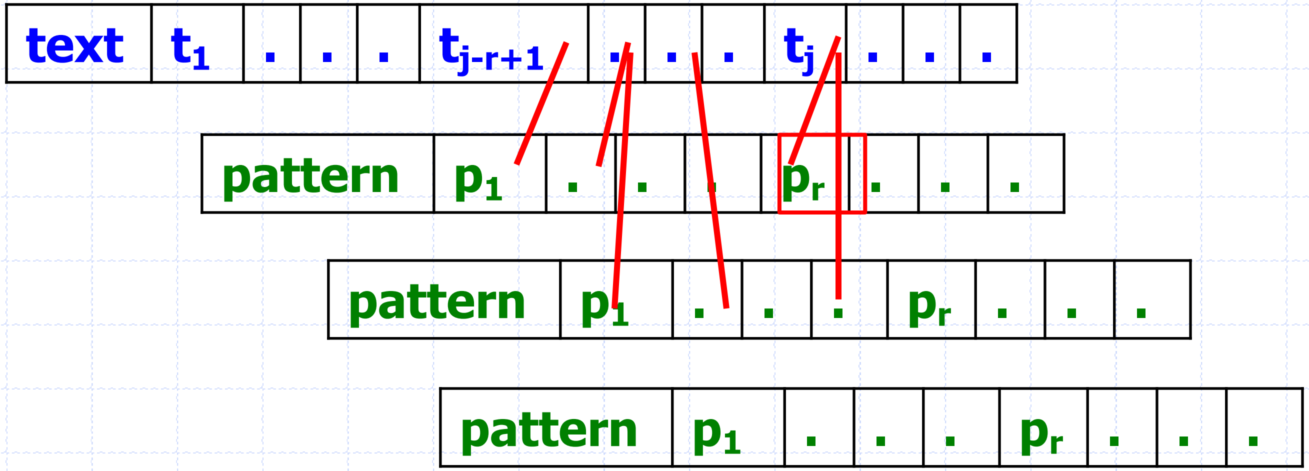
- Putting it together
  - If mismatch occurs at position n, shift P by 1
  - If a mismatch occurs at position i-1 in P:
    - If L(i) > 0, shift P by n – L(i)
    - else shift P by n – l(i)
  - If P was found, shift P by n – l(2)

Boyer-Moore Algorithm Analysis

- Worst case
  - O(n+m) if the pattern does not occur in the text
  - O(nm) if the pattern does occur in the text
  - For patterns of small size, the algorithm might not be efficient.

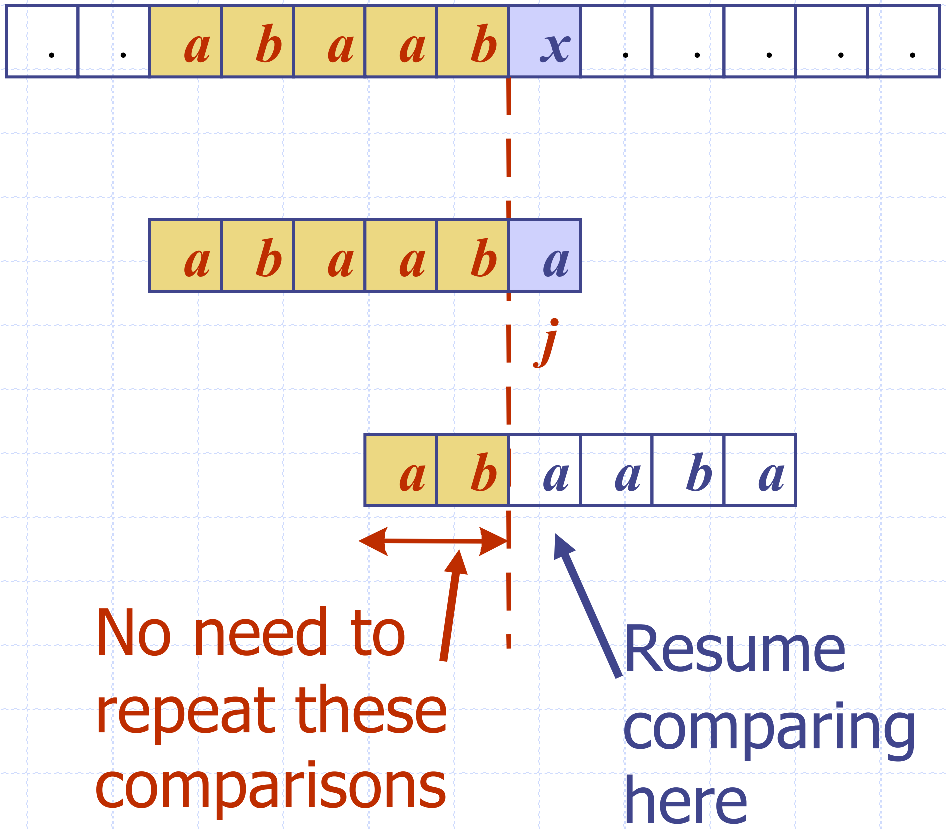
Knuth-Morris-Pratt - Algorithm

- KMP Algorithm
  - compares the pattern to the text in left-to-right, but shifts the pattern more intelligently than the brute-force algorithm.



The KMP Algorithm

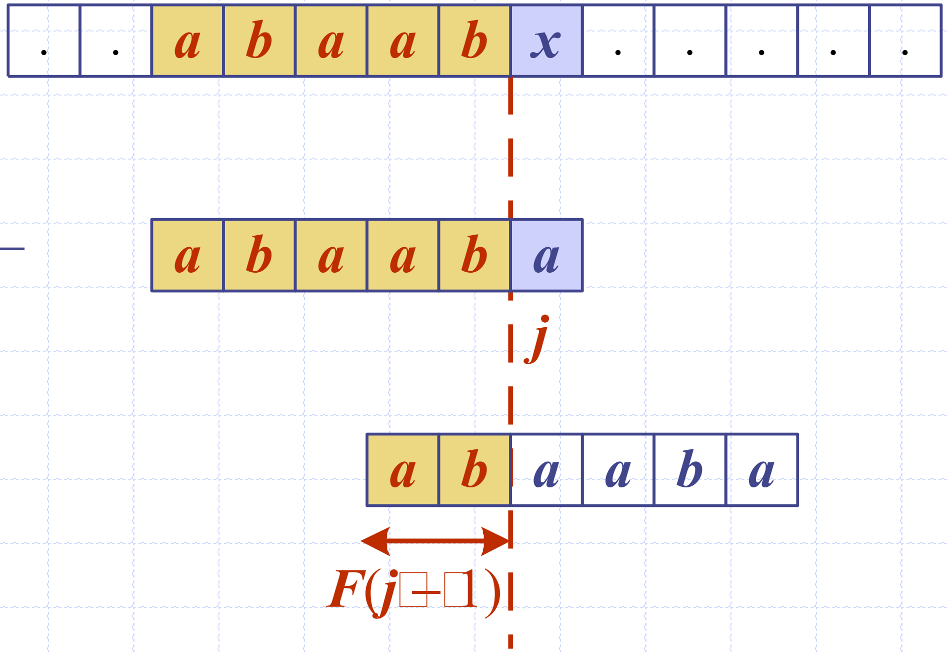
- Knuth-Morris-Pratt's algorithm compares the pattern to the text in left-to-right, but shifts the pattern more intelligently than the brute-force algorithm.
- When a mismatch occurs, what is the **most** we can shift the pattern so as to avoid redundant comparisons?
- Answer: the largest prefix of $P[0..j]$ that is a suffix of $P[1..j]$



KMP Failure Function

- Knuth-Morris-Pratt's algorithm preprocesses the pattern to find matches of prefixes of the pattern with the pattern itself
- The failure function $F(j)$ is defined as the size of the largest prefix of $P[0..j]$ that is also a suffix of $P[1..j]$
- Knuth-Morris-Pratt's algorithm modifies the brute-force algorithm so that if a mismatch occurs at $P[j] \neq T[i]$ we set $j \leftarrow F(j-1)$

j	0	1	2	3	4	□
$P[j]$	a	b	a	a	b	a
$F(j)$	0	0	1	1	2	□



Computing the Failure Function

- The failure function can be represented by an array and can be computed in $O(m)$ time
- At each iteration of the while-loop, either
  - i increases by one, or
  - the shift amount $i - j$ increases by at least one (observe that $F(j - 1) < j$)
- Hence, there are no more than $2m$ iterations of the while-loop

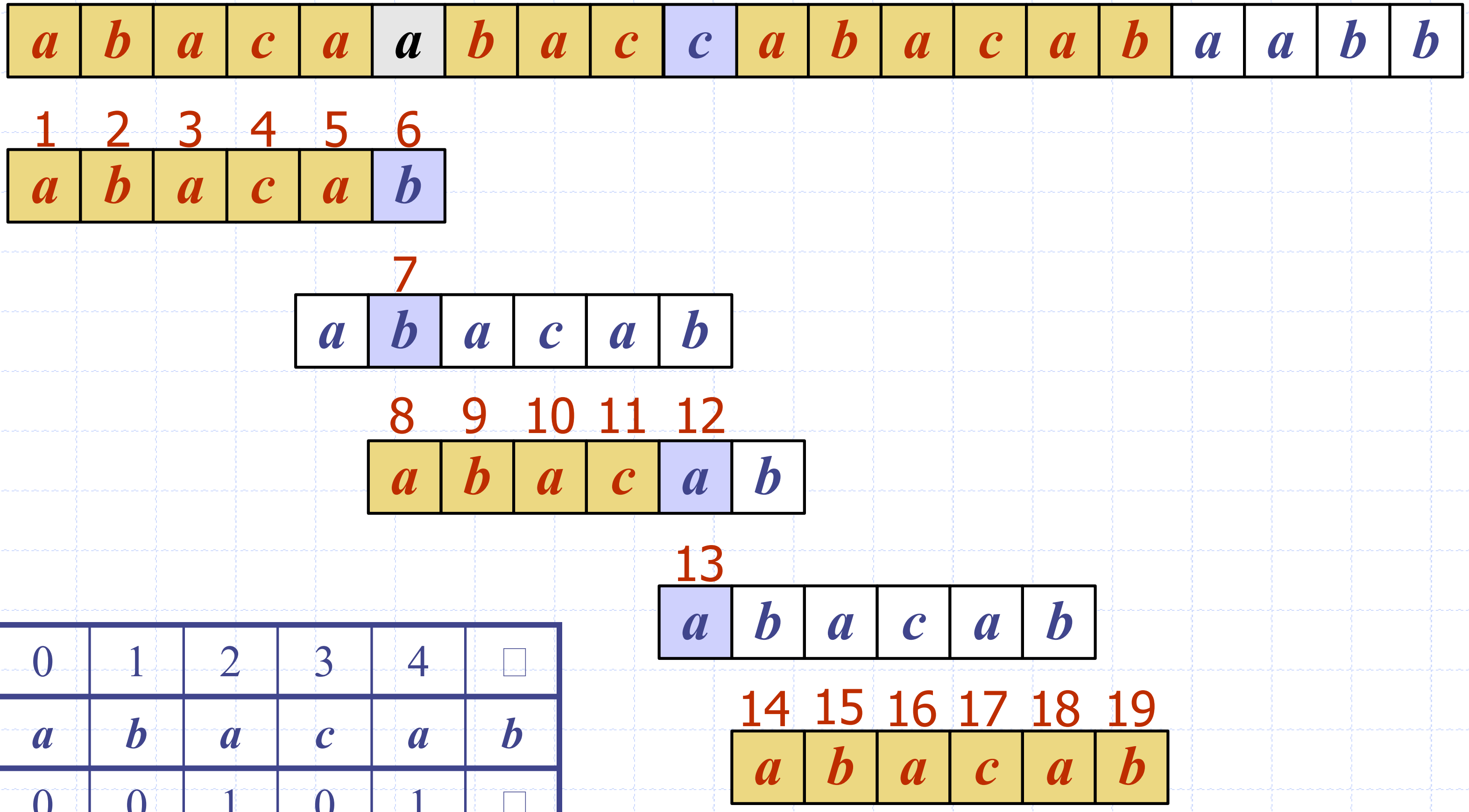
```
Algorithm failureFunction(P)
    F[0] ← 0
    i ← 1
    j ← 0
    while i □ m
        if P[i] □ P[j]
            {we have matched j + 1 chars}
            F[i] ← j + 1
            i ← i □ 1
            j ← j □ 1
        else if j □ 0 then
            {use failure function to shift P}
            j ← F[j − 1]
        else
            F[i] ← 0 { no match }
            i ← i □ 1
```

The KMP Algorithm

- The failure function can be represented by an array and can be computed in $O(m)$ time
- At each iteration of the while-loop, either
  - i increases by one, or
  - the shift amount $i - j$ increases by at least one (observe that $F(j-1) < j$)
- Hence, there are no more than $2n$ iterations of the while-loop
- Thus, KMP's algorithm runs in optimal time $O(m + n)$

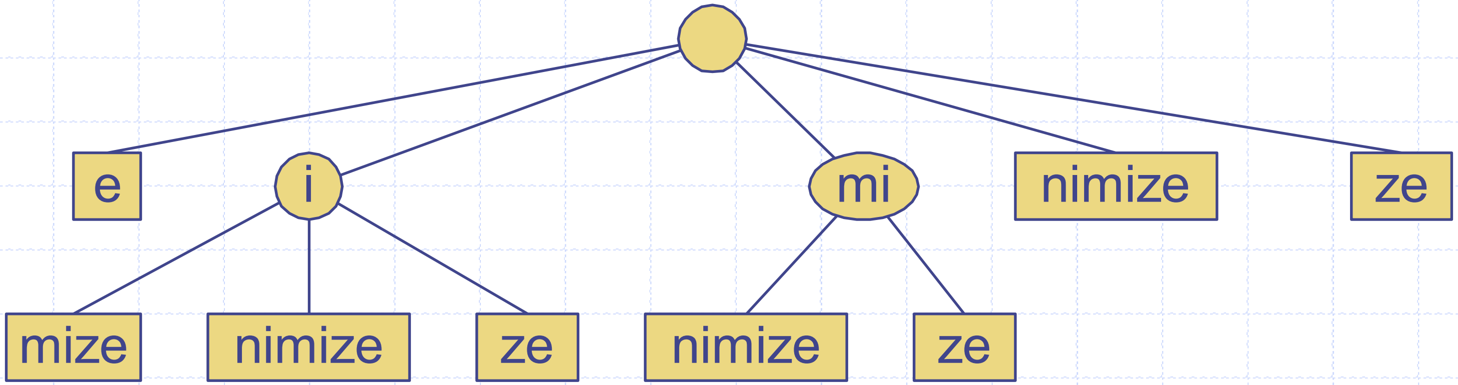
```
Algorithm KMPMatch(T, P)
    F ← failureFunction(P)
    i ← 0
    j ← 0
    while i < n
        if T[i] = P[j]
            if j = m − 1
                return i − j { match }
            else
                i ← i + 1
                j ← j + 1
        else
            if j > 0
                j ← F[j − 1]
            else
                i ← i + 1
    return −1 { no match }
```

Example



j	0	1	2	3	4	□
$P[j]$	a	b	a	c	a	b
$F(j)$	0	0	1	0	1	□

Tries

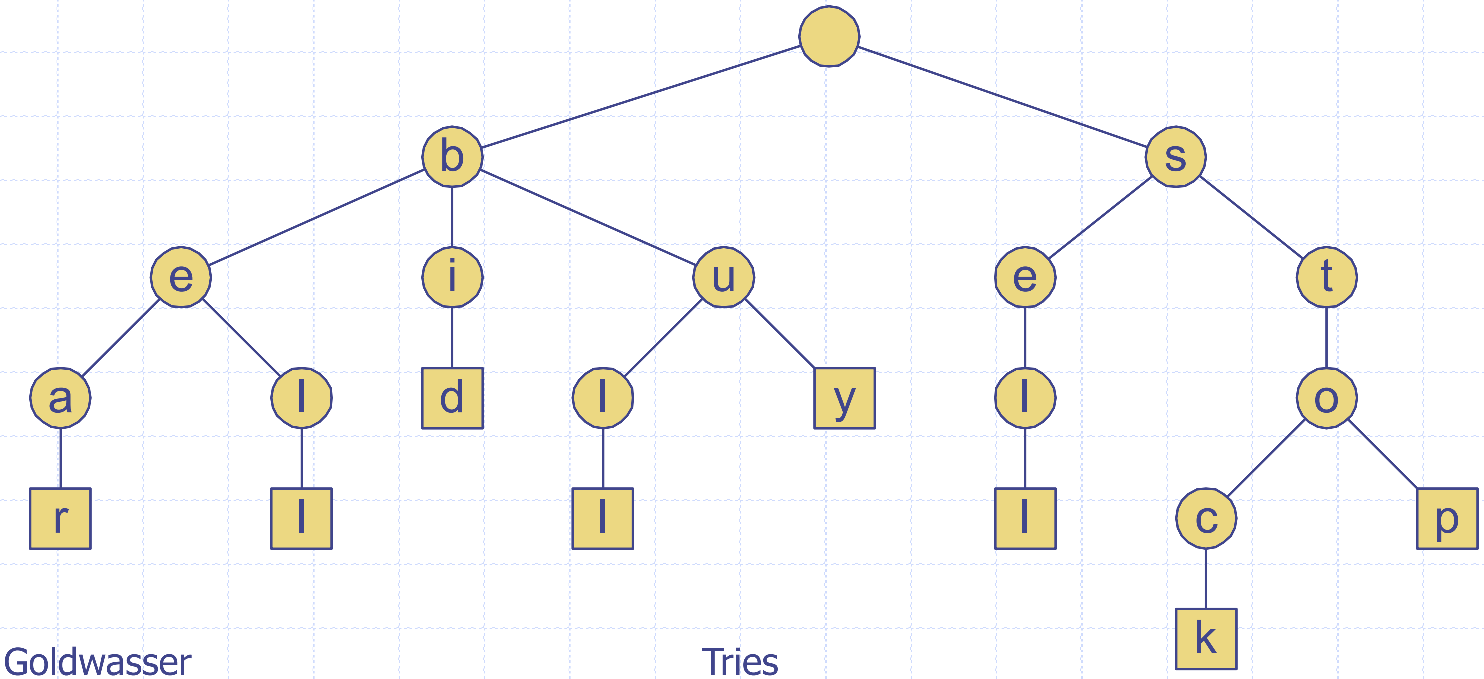


Preprocessing Strings

- Preprocessing the pattern speeds up pattern matching queries
  - After preprocessing the pattern, KMP's algorithm performs pattern matching in time proportional to the text size
- If the text is large, immutable and searched for often (e.g., works by Shakespeare), we may want to preprocess the text instead of the pattern
- A trie is a compact data structure for representing a set of strings, such as all the words in a text
  - A trie supports pattern matching queries in time proportional to the pattern size

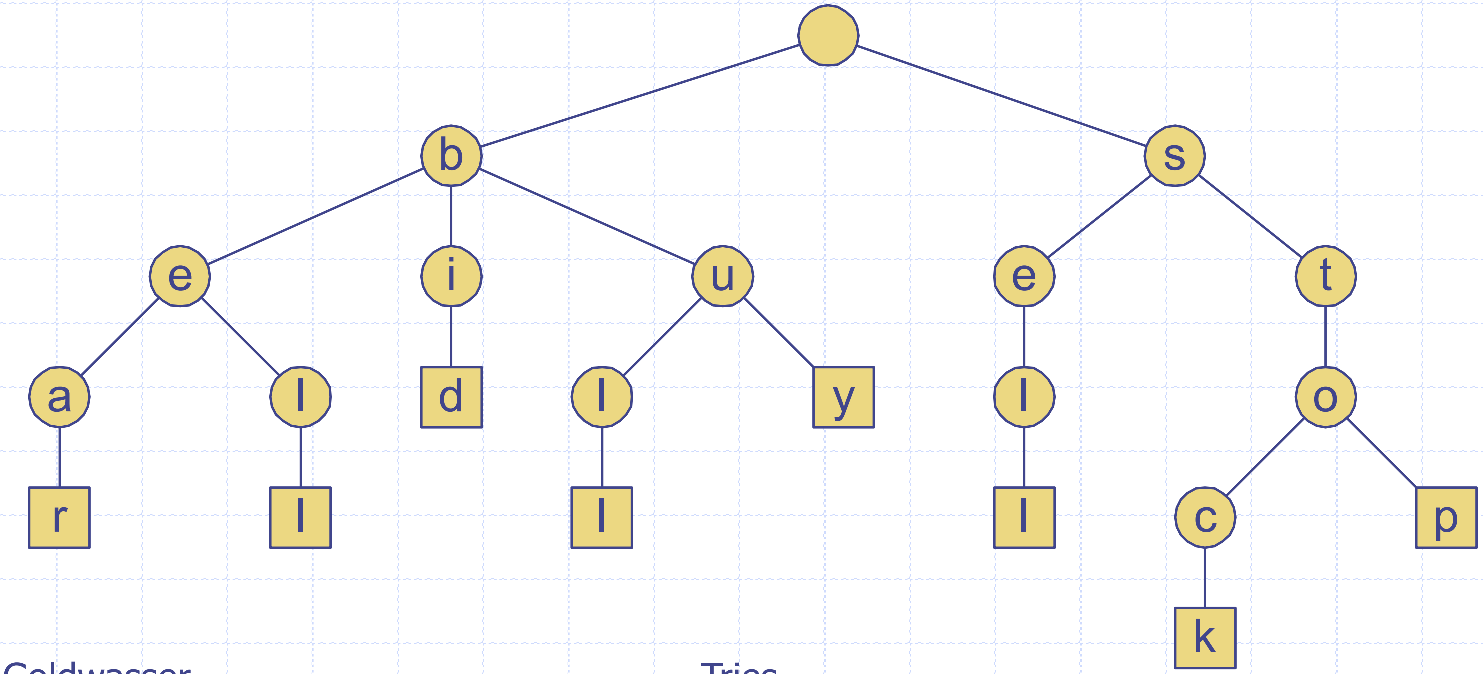
Standard Tries

- The standard trie for a set of strings S is an ordered tree such that:
  - Each node but the root is labeled with a character
  - The children of a node are alphabetically ordered
  - The paths from the external nodes to the root yield the strings of S
- Example: standard trie for the set of strings
  S = { bear, bell, bid, bull, buy, sell, stock, stop }



Analysis of Standard Tries

- A standard trie uses $O(n)$ space and supports searches, insertions and deletions in time $O(dm)$, where:
  - n total size of the strings in S
  - m size of the string parameter of the operation
  - d size of the alphabet



Application of Tries

- A standard trie supports the following operations on a processed text in O(m) time, where m is the length of the string.
  - word matching: find the first occurrence of word X in the text.
  - prefix matching: find the first occurrence of the longest prefix of word X in the text.

Word Matching with a Trie

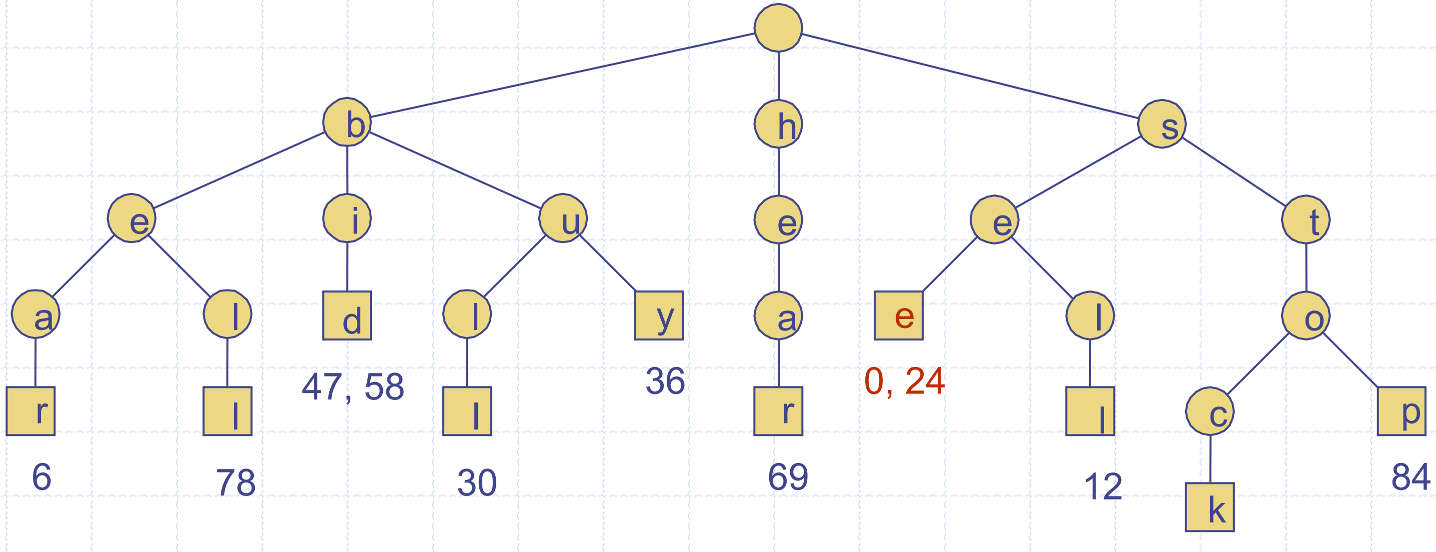
- insert the words of the text into trie
- Each leaf is associated w/ one particular word
- leaf stores indices where associated word begins ("see" starts at index 0 & 24, leaf for "see" stores those indices)

s	e	e		a		b	e	a	r	?		s	e	l	l		s	t	o	c	k	!	
0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23

s	e	e		a		b	u	l	l	?		b	u	y		s	t	o	c	k	!	
24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40	41	42	43	44	45	46

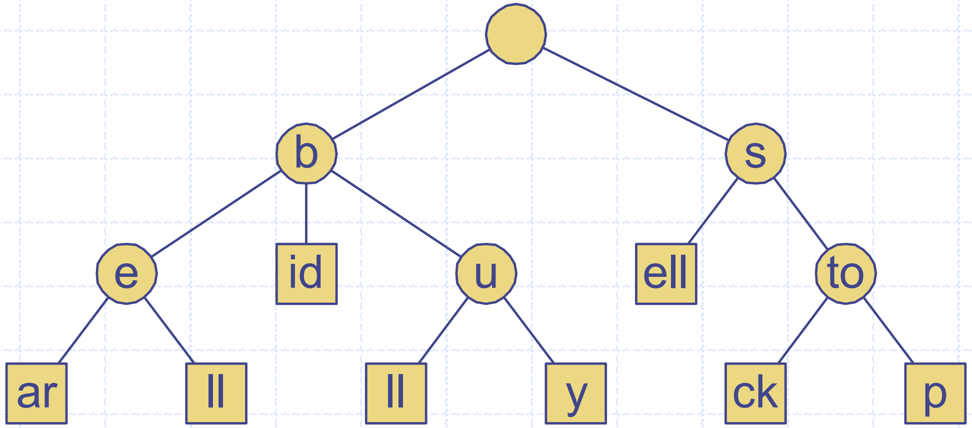
b	i	d		s	t	o	c	k	!		b	i	d		s	t	o	c	k	!	
47	48	49	50	51	52	53	54	55	56	57	58	59	60	61	62	63	64	65	66	67	68

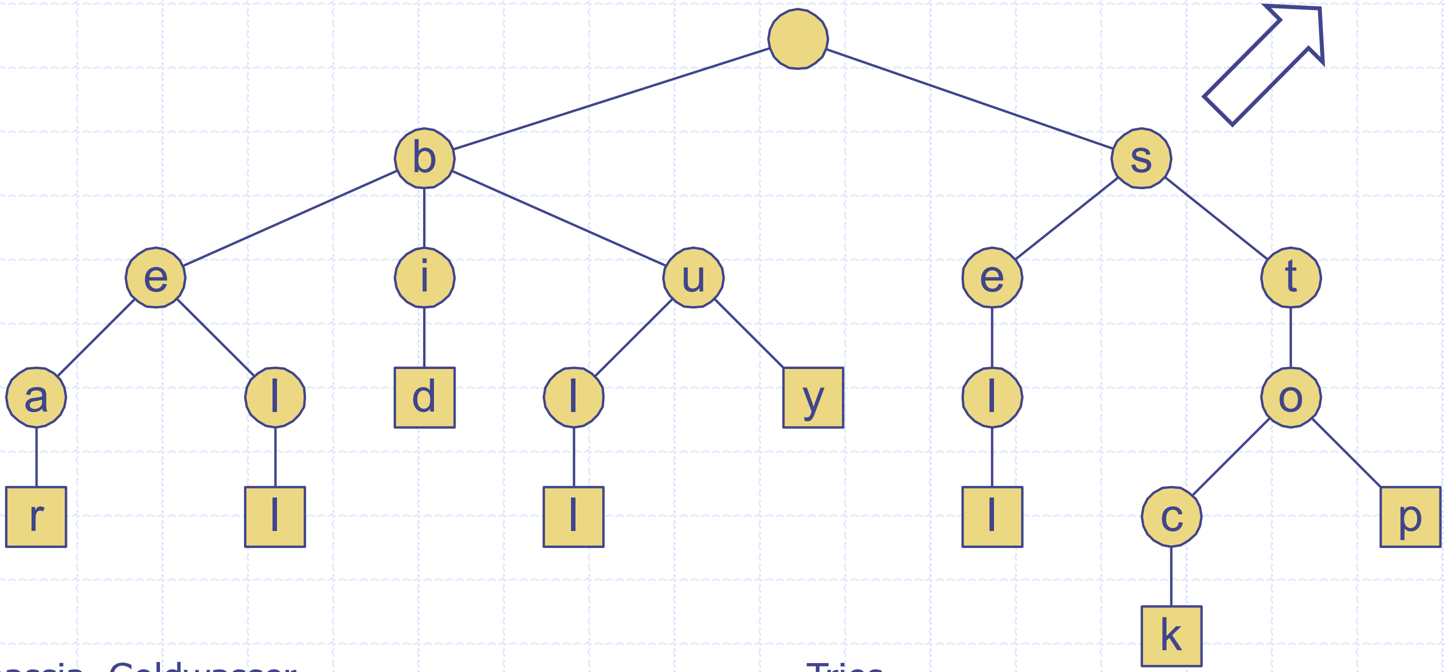
h	e	a	r		t	h	e		b	e	l	l	?		s	t	o	p	!
69	70	71	72	73	74	75	76	77	78	79	80	81	82	83	84	85	86	87	88



Compressed Tries

- A compressed trie has internal nodes of degree at least two
- It is obtained from standard trie by compressing chains of “redundant” nodes
- ex. the “i” and “d” in “bid” are “redundant” because they signify the same word



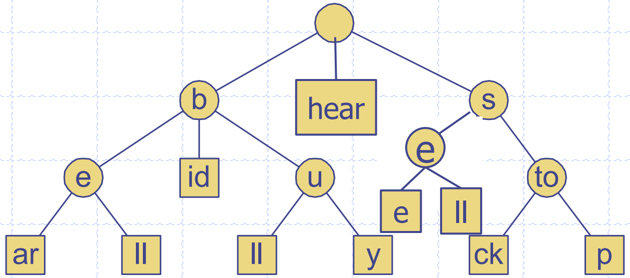


Why Compressed Tries?

- A compressed trie storing a collection S of s strings from an alphabet of size d has the following properties
  - Every internal node of T has at least two children and at most d children
  - T has s leaf nodes
  - The number of internal nodes of T is O(s)

Compact Representation

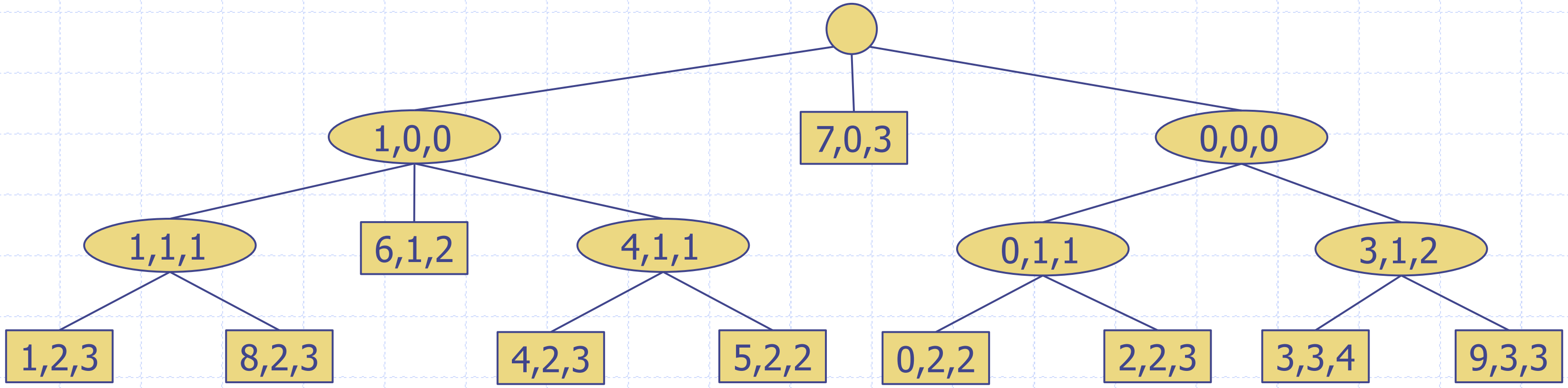
- Compact representation of a compressed trie for an array of strings:
  - Stores at the nodes ranges of indices instead of substrings
  - Uses $O(s)$ space, where s is the number of strings in the array
  - Serves as an auxiliary index structure



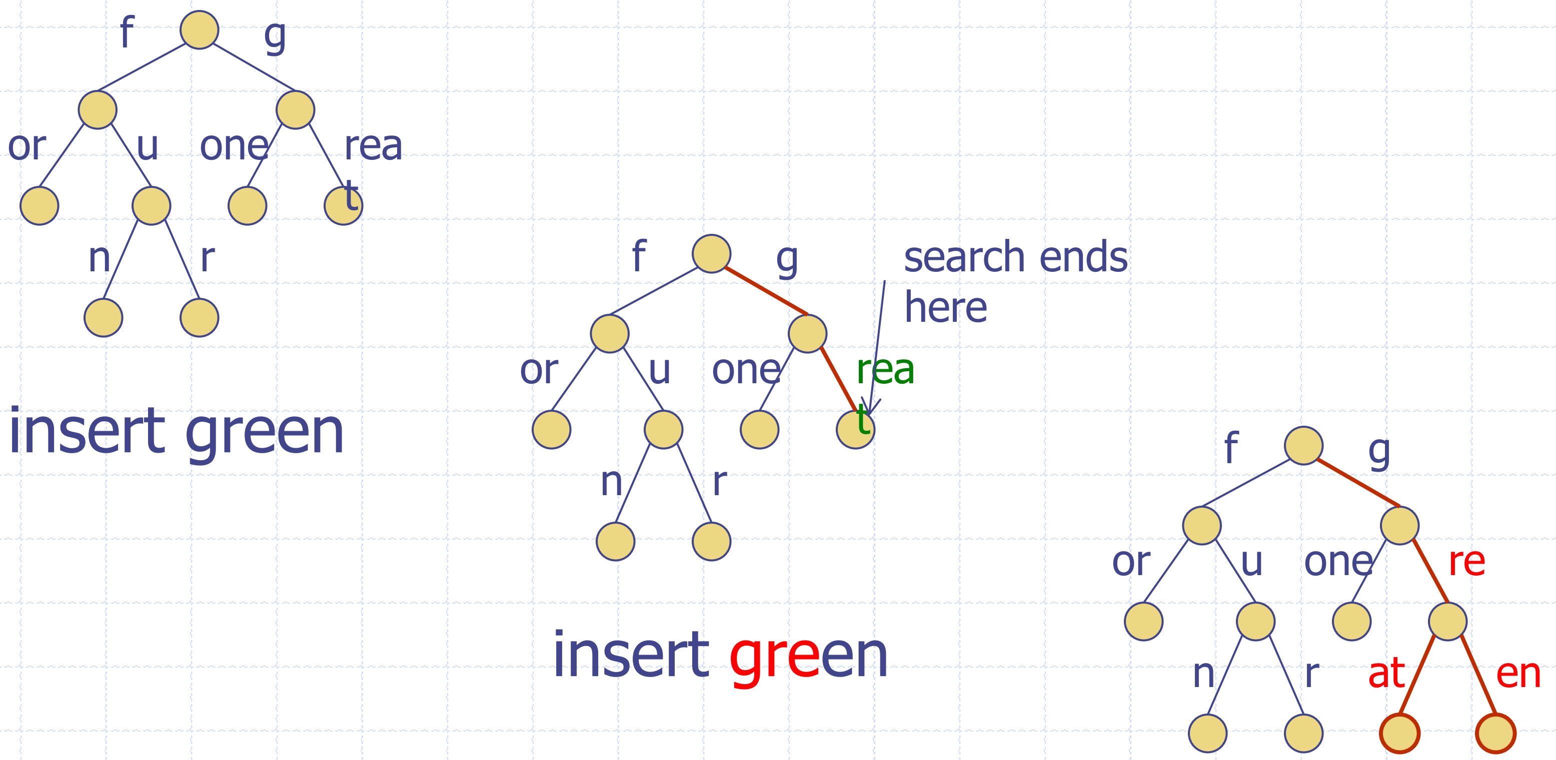
	0	1	2	3	4
S[0] =	s	e	e		
S[1] =	b	e	a	r	
S[2] =	s	e	l	l	
S[3] =	s	t	o	c	k

	0	1	2	3
S[4] =	b	u	l	l
S[5] =	b	u	y	
S[6] =	b	i	d	

	0	1	2	3
S[7] =	h	e	a	r
S[8] =	b	e	l	l
S[9] =	s	t	o	p

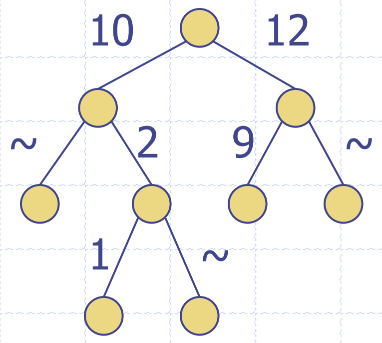


Insertion/Deletion in Compressed Trie



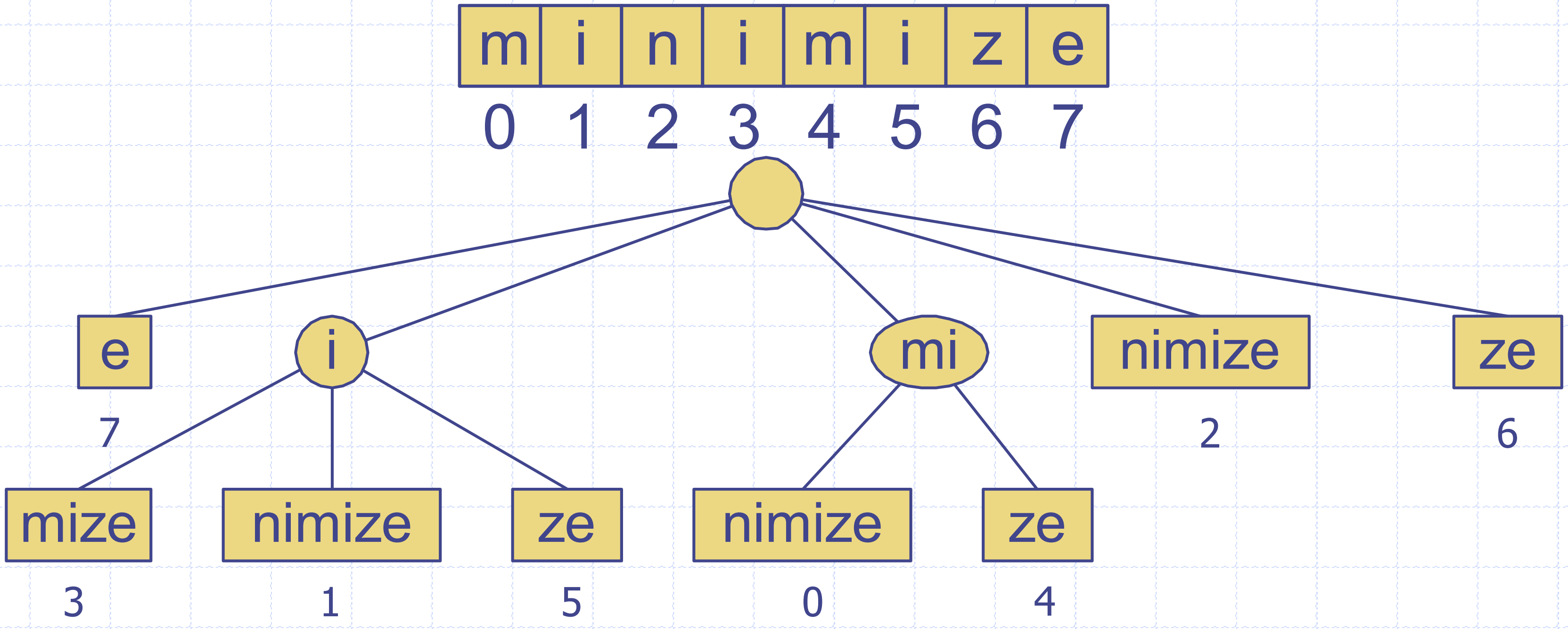
Application - Routing through Tries

- Internet Routers maintain a Trie (table)
- It is not a lookup
  - forwards packets to its neighbors using IP prefix matching rules



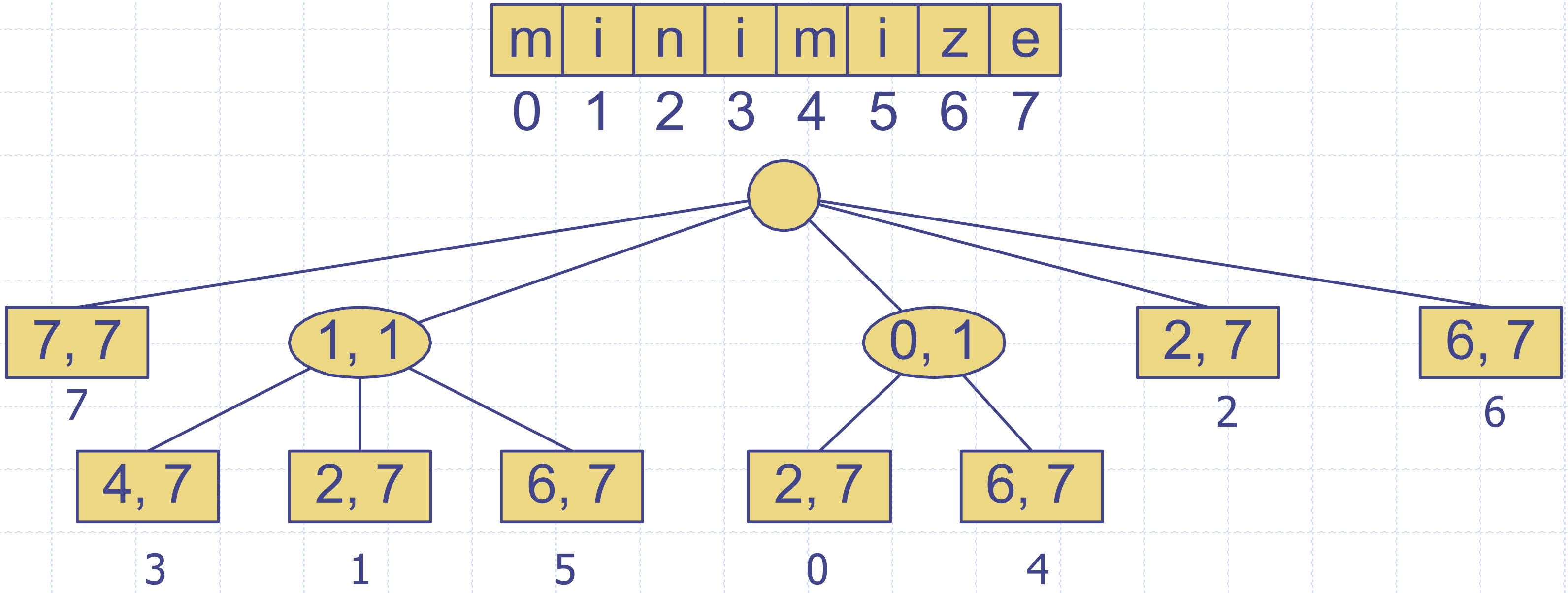
Suffix Trie

- The suffix trie of a string ***X*** is the compressed trie of all the suffixes of ***X***
- Each leaf corresponds to a suffix of ***X***



Analysis of Suffix Tries

- Compact representation of the suffix trie for a string X of size n from an alphabet of size d
  - Uses $O(n)$ space
  - Supports arbitrary pattern matching queries in X in $O(dm)$ time, where m is the size of the pattern
  - Can be constructed in $O(n)$ time



Prefix Matching using Suffix Trie

- If two suffixes have a same prefix, then their corresponding paths are the same at their beginning, and the concatenation of the edge labels of the mutual part is the prefix.

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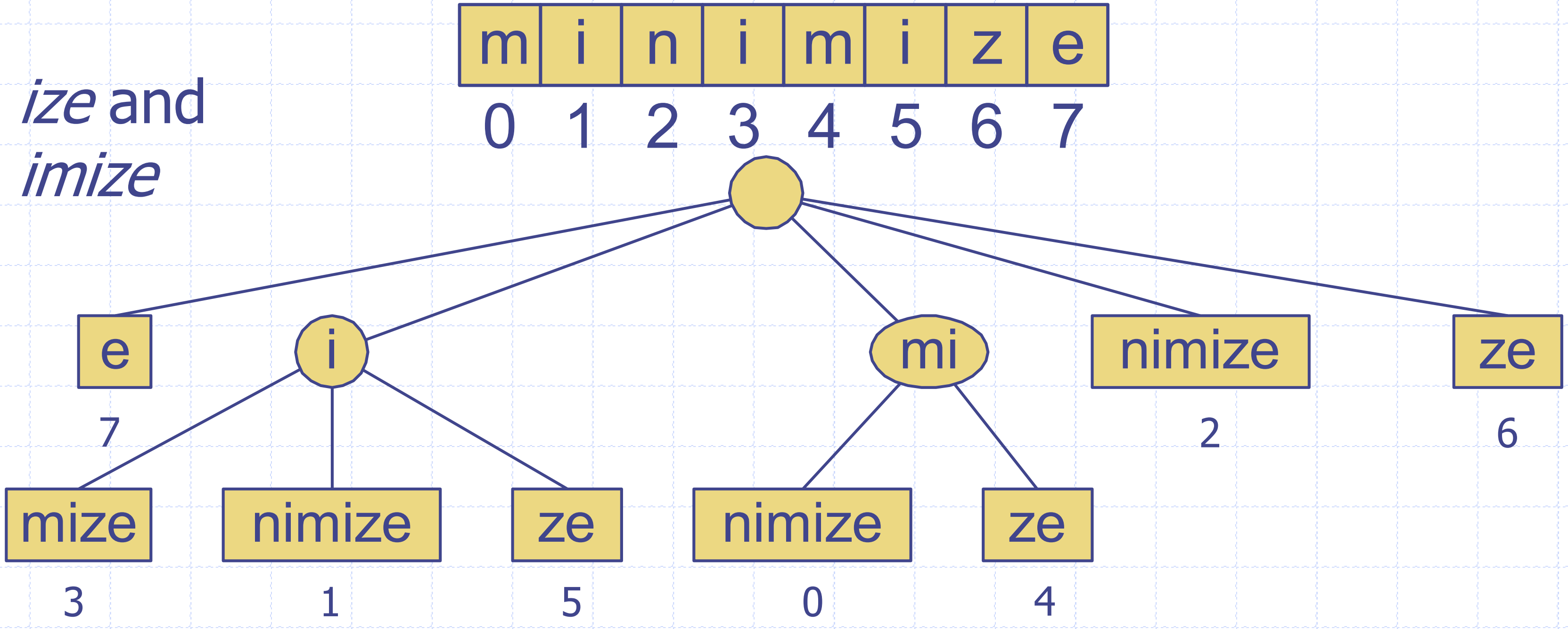
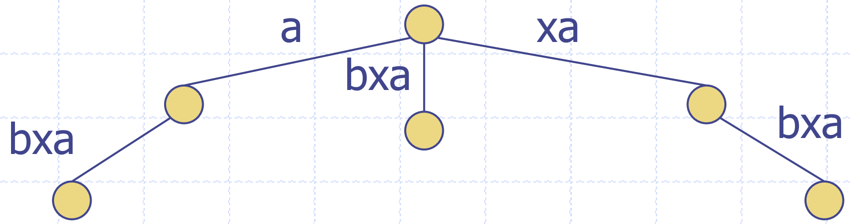
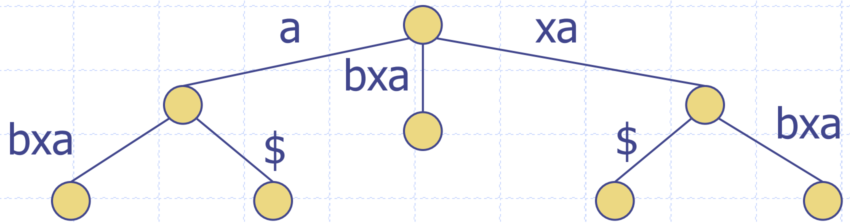


Suffix Trie

- Not all strings are guaranteed to have corresponding suffix trie.
- For example: xabxa

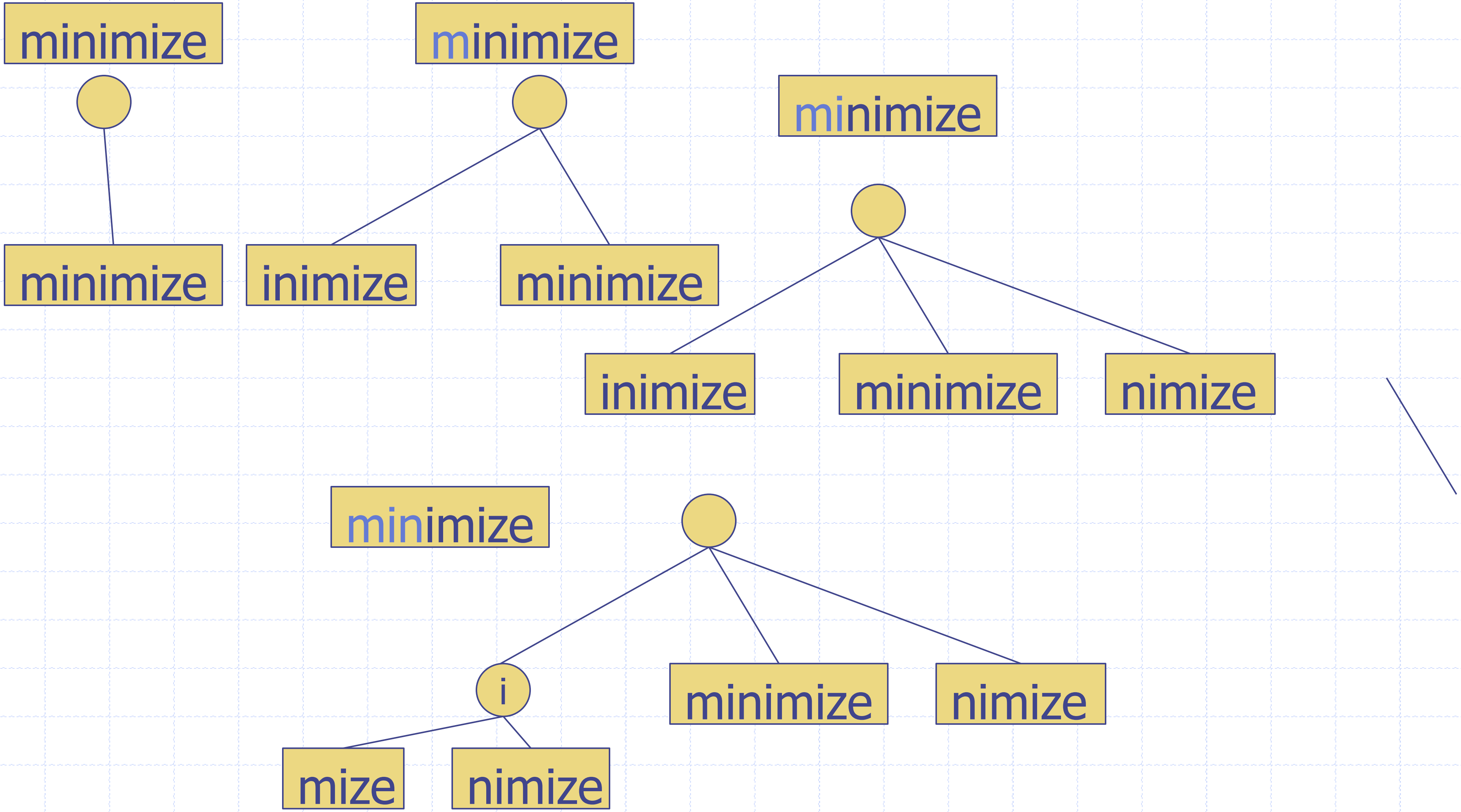




Constructing a Suffix Trie

- S[1..n] is the string
- start with a single edge for S
- enter the edges for the suffix S[i..n] where i goes from 2 to n
  - Starting at the root node find the longest part from the root whose label matches a prefix of S[i..n]. At some point, no further matches are possible
    - If the point is at a node, then denote this node by w
    - If it is in the middle of an edge, then insert a new node called w, at this point
    - create a new edge running from w to a new leaf labeled S[i..n]

Example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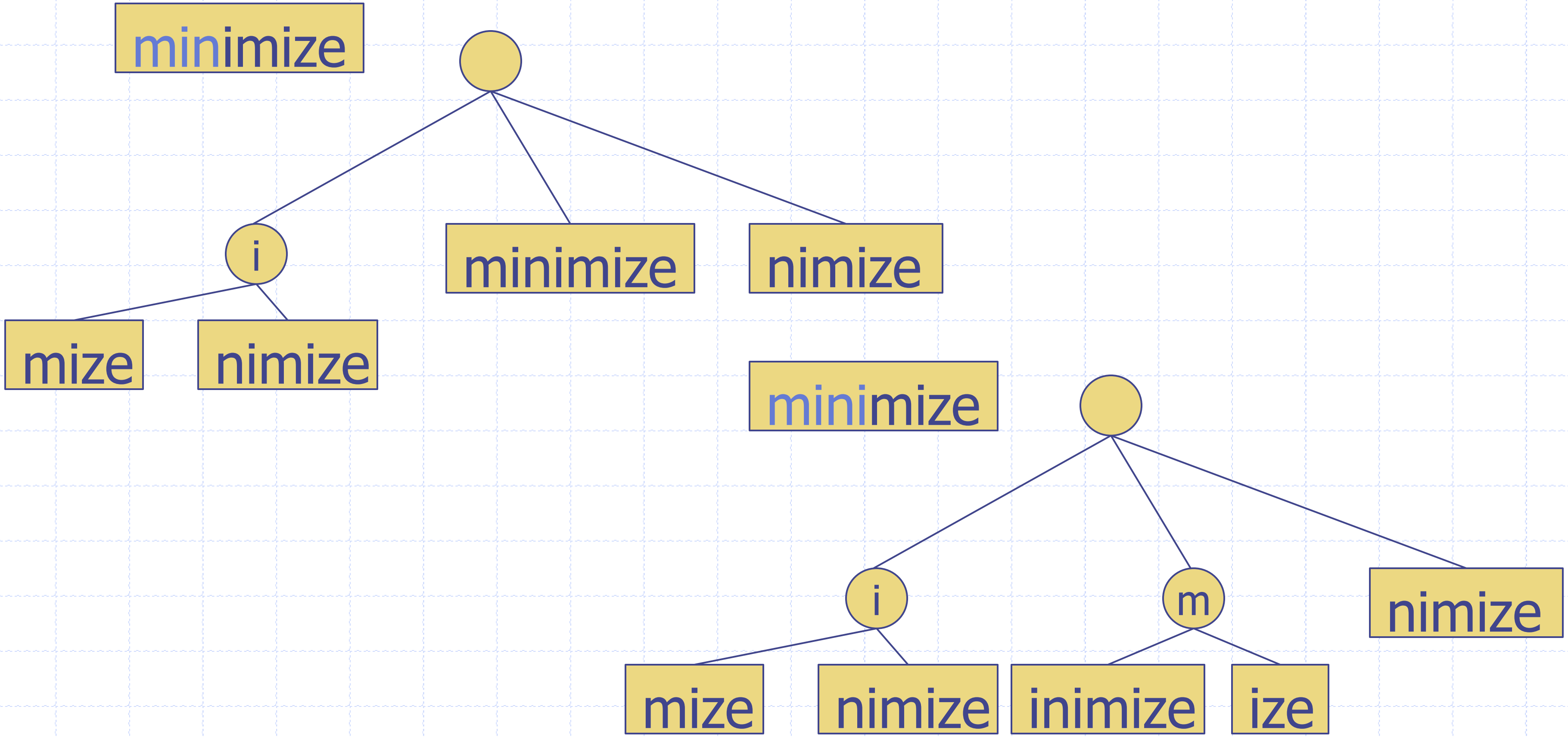
Example (2)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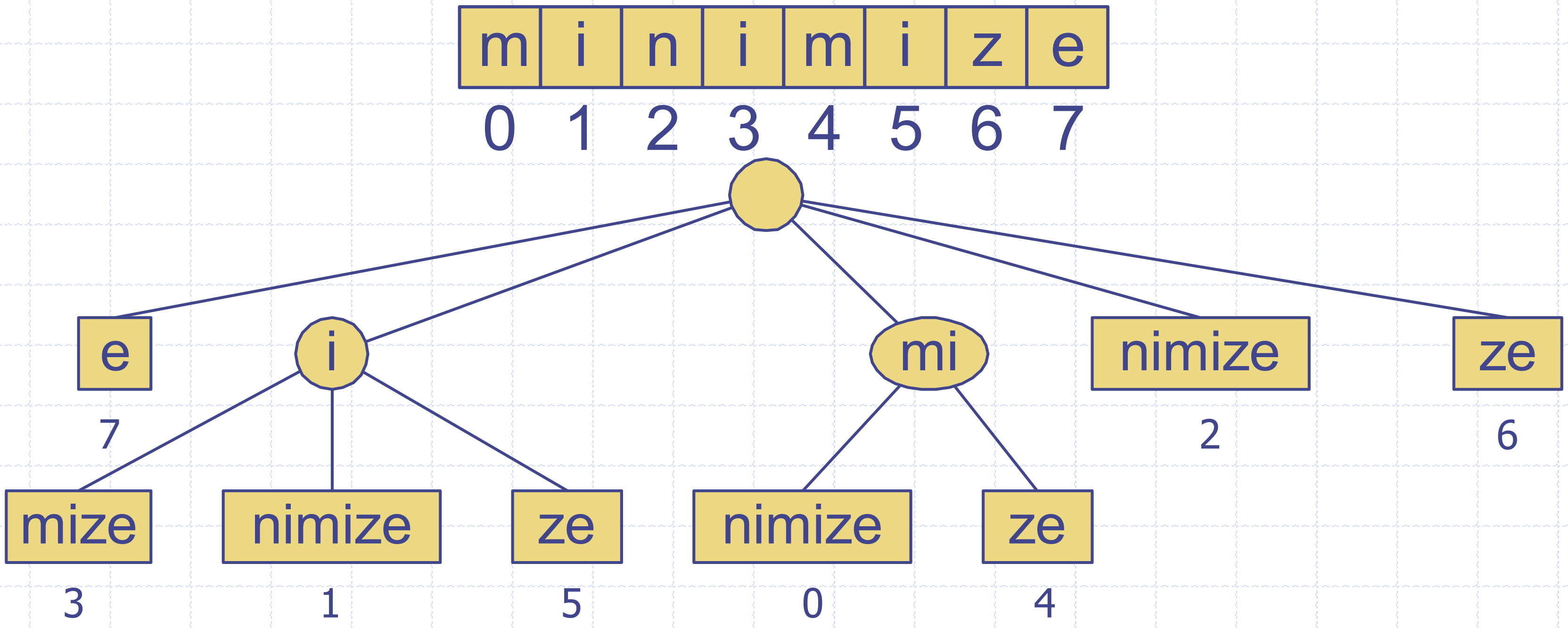
Example (3)



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Constructing a Suffix Trie

Complexity- $O(n^2)$

- S[1..n] is the string
- start with a single edge for S
- enter the edges for the suffix S[i..n] where i goes from 2 to n
  - Starting at the root node find the longest part from the root whose label matches a prefix of S[i..n]. At some point, no further matches are possible
    - If the point is at a node, then denote this node by w
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    - create a new edge running from w to a new leaf labeled S[i..n]

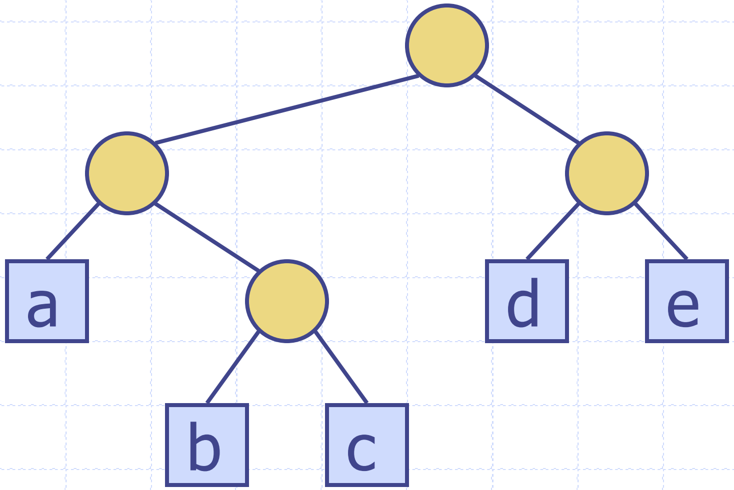
Text Compression

- Given a string X, efficiently encode X into a smaller string Y
  - Saves memory and/or bandwidth
- A good approach: **Huffman encoding**
  - Compute frequency f(c) for each character c.
  - Encode high-frequency characters with short code words
  - No code word is a prefix for another code
  - Use an optimal encoding tree to determine the code words

Encoding Tree Example

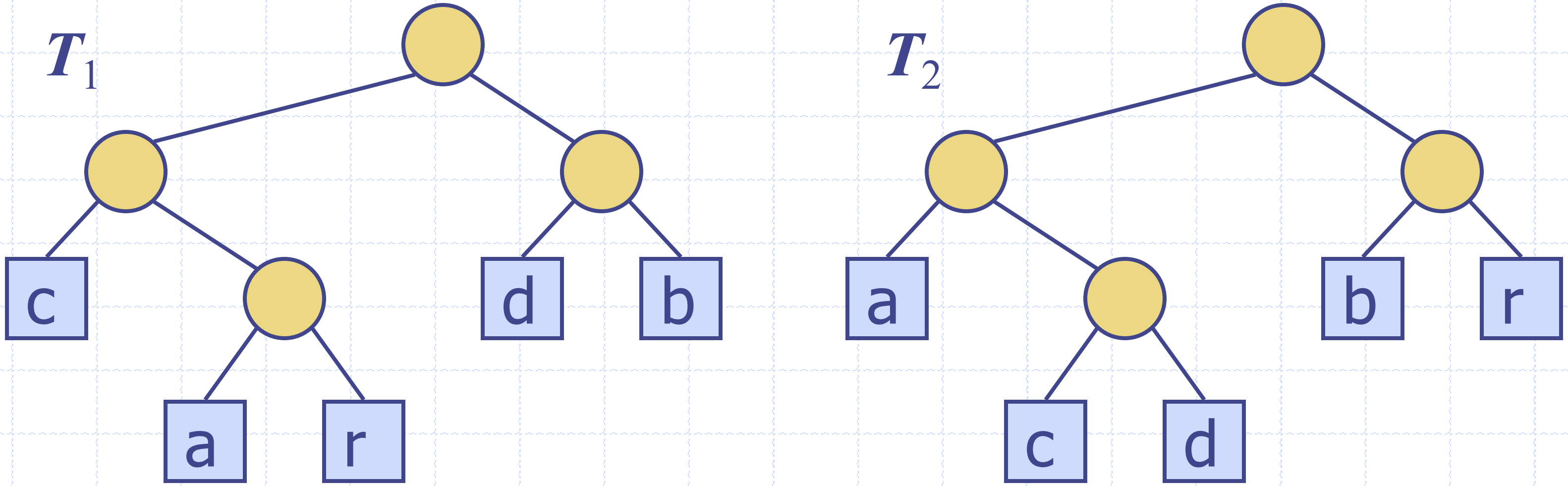
- A **code** is a mapping of each character of an alphabet to a binary code-word
- A **prefix code** is a binary code such that no code-word is the prefix of another code-word
- An **encoding tree** represents a prefix code
  - Each external node stores a character
  - The code word of a character is given by the path from the root to the external node storing the character (0 for a left child and 1 for a right child)

00	010	011	10	11
a	b	c	d	e



Encoding Tree Optimization

- Given a text string X, we want to find a prefix code for the characters of X that yields a small encoding for X
  - Frequent characters should have long code-words
  - Rare characters should have short code-words
- Example
  - $X =$ abracadabra
  - T_1 encodes X into 29 bits
  - T_2 encodes X into 24 bits



Huffman's Algorithm

- Given a string X, Huffman's algorithm construct a prefix code the minimizes the size of the encoding of X
- It runs in time $O(n + d \log d)$, where n is the size of X and d is the number of distinct characters of X
- A heap-based priority queue is used as an auxiliary structure

Huffman's Algorithm

Algorithm Huffman(X):
  Input: String X of length n with d distinct characters
  Output: Coding tree for X

Compute the frequency $f(c)$ of each character c of X.
Initialize a priority queue Q.
for each character c in X **do**
  Create a single-node binary tree T storing c.
  Insert T into Q with key $f(c)$.
while len(Q) > 1 **do**
  $(f_1, T_1) = Q$.remove_min()
  $(f_2, T_2) = Q$.remove_min()
  Create a new binary tree T with left subtree T_1 and right subtree T_2.
  Insert T into Q with key $f_1 + f_2$.
$(f, T) = Q$.remove_min()
return tree T

Example

X = abracadabra

Frequencies

a	b	c	d	r
5	2	1	1	2