Cooling an atom in a weakly driven high-Q cavity

Gerald Hechenblaikner, Markus Gangl, Peter Horak,* and Helmut Ritsch
Institut für Theoretische Physik, Universität Innsbruck, Technikerstrasse 25, A-6020 Innsbruck, Austria
(Received 1 April 1998)

We investigate the external and internal dynamics of a two-level atom in a standing wave cavity. In the strong coupling regime, where the atom field coupling g dominates the atomic and cavity decay rates (Γ,κ), a cooling mechanism entirely different from free-space Doppler cooling appears. Under suitable operating conditions, the cavity dynamics induces a Sisyphus type cooling, which is the dominant contribution to the total friction force acting on a moving atom. Simple equations describing the key properties of this effect are derived from a completely classical picture and confirmed by a semiclassical approach. The model is investigated in the bad and good cavity limits, and analytic expressions for the friction coefficient and the momentum diffusion for slow atoms are derived. Using a continued fractions expansion the cooling force for arbitrary velocities is evaluated numerically. The result is used to calculate the equilibrium temperatures of the atom of the order of $kT \approx \hbar\kappa$, which can be much lower than the free-space Doppler limit and agree well to those obtained by quantum wave-function simulations. [S1050-2947(98)02310-5]

PACS number(s): 32.80.Pj, 42.50.Vk, 42.50.Lc

I. INTRODUCTION

It is well known that radiative properties of atoms are changed inside a cavity [1], since the electric field eigenmodes in a cavity are different from free space. Already in 1946 Pourcel predicted an enhancement in the spontaneous decay rate for an atom placed in a resonant cavity [2]. Later Kleppner pointed out the opposite possibility of inhibiting spontaneous emission [3]. Since the early days of quantum optics the simplest theoretical representation of such a system, namely the Jaynes-Cummings model [4] of a two-level atom coupled to a single radiation mode of an optical resonator, has become one of the most thoroughly investigated models both theoretically (cavity-QED) and experimentally, using microwave cavities [5] and recently also optical high finesse resonators [6]. With the advent of laser cooling of free atoms also the question of atomic motion and cavity-induced light forces was considered in various papers. For example, a possible scheme to cool free atoms in so-called colored vacua characteristic for cavities was proposed in [7,8]. Laser cooling in a squeezed vacuum [9–11] and adiabatic cooling in cavities [12] has also been investigated. Similarly Cirac *et al.* dealt with the interaction between the cavity mode and an atom, which was additionally strongly driven by a laser, in the bad cavity limit [13]. Many dynamical aspects of atomic motion in cavities have been carefully investigated also in a series of recent papers by Parkins and co-workers [14,15]. They show that spontaneous transitions together with the dipole force leads to strong heating and finally expulsion of the atom from the cavity mode. In the tight-binding (Lamb-Dicke) regime, where the atom is confined to less than an optical wavelength, they also consider the influence of quantized atomic motion on the dynamics [16].

In the meantime advances in cavity technology have allowed one to experimentally reach the so-called strong-coupling limit in the optical domain [17,18], so that already a single atom strongly modifies the cavity dynamics [7,19–26]. As a striking example the influence of a single atom on transmission and intracavity intensity of a driven cavity can be measured [6]. The cavity output may be continuously monitored to determine the atomic position at high resolutions [23,27]. Spontaneous emission of an atom moving in a cavity standing wave has also been examined [28,29]. In some recent theoretical work we have suggested that light forces which appear in the coupled atom-cavity dynamics in this strong-coupling regime can be used to cool and trap a single atom exactly at the antinodes of the cavity field, where one has the strongest atom-field coupling [30]. In this work we extended this treatment to a much larger parameter regime, including blue detuned light fields and faster atoms. In the extended semiclassical model explicit analytical expressions for forces, diffusion, and kinetic temperature are derived.

This work is organized as follows: in Sec. II the model of an atom interacting with a quantized standing wave in a weakly driven cavity will be introduced and a simple classical analog of this system will be discussed. In Sec. III a semiclassical version of this model, where the light field and the internal atomic degrees of freedom are quantized, is presented. The friction force is analytically calculated and the physical origin of its various contributions is discussed. Finally the two limiting cases of a strongly and a weakly damped cavity are elaborated. In Sec. IV the so-called dressed state approach is used to help in the physical interpretation of the results in analogy to the well-known Sisyphus cooling [31,32] in optical molasses. The cooling force for arbitrary velocities using the method of continued fractions is presented in Sec. V, while Sec. VI is devoted to a derivation of the momentum diffusion using the quantum regression theorem, which contains additional terms arising from the cavity dynamics. This allows one to give estimates of the equilibrium temperature and allows us to make comparisons to results obtained through a numerical fully quan-

*Present address: Laboratoire Kastler-Brossel, Ecole Normale Superieure, 24 rue Lhomond, 75231 Paris Cedex 05, France.

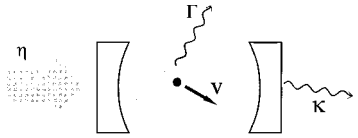

FIG. 1. Particle moving in a weakly driven cavity with losses from spontaneous emission Γ and cavity decay κ.

tum treatment of the problem including quantization of the external motion of the atom.

II. CLASSICAL MODEL

Let us in the following look at a two-level atom strongly coupled to a single isolated mode of frequency ω_0 of a weakly driven cavity. Losses from spontaneous emission (Γ) and cavity decay (κ) are included in our model via coupling to external reservoirs. The atom is allowed to move along the standing wave to which it is coupled with a strength determined by a local coupling function $g(x)=gu(x)$, where $u(x)$ is the normalized mode function. The cavity is assumed to be externally driven by a monochromatic field of frequency ω_p and effective amplitude η. A graphic scheme of our model is given in Fig. 1.

Before going to a quantum description let us start from a completely classical point of view of a massive pointlike dipole in an optical resonator. As we will see later, the equations derived here strongly resemble those found in a semiclassical treatment in the good cavity limit.

Let us consider a cavity field mode coupled to an atom, driven by an external pump field and damped due to the conductivity σ of the cavity mirrors. The cavity field induces a dipole moment of the atom, which in turn contributes to the electric field as a driving term in the wave equation [33],

$$\left(-\Delta+\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\vec{E}(\vec{x},t)+\mu_0\sigma\frac{\partial}{\partial t}\vec{E}(\vec{x},t) =-\mu_0\frac{\partial^2}{\partial t^2}\vec{P}(\vec{x},t)+\mu_0\sigma\frac{\partial}{\partial t}\vec{E}^{\rm ext}(\vec{x},t), \tag{1}$$

where the damping term was modeled as $\mu_0(\partial/\partial t)(\sigma\vec{E})$ and the external pump was taken into account by $\mu_0\sigma(\partial/\partial t)\vec{E}^{\rm ext}$. For simplicity we assume a quasi-1D situation, where the field is a plane wave propagating in the x direction, which leads to

$$\frac{\partial^2}{\partial t^2}E(x,t)+\frac{\sigma}{\epsilon_0}\frac{\partial}{\partial t}E(x,t)-c^2\frac{\partial^2}{\partial x^2}E(x,t) =-\frac{1}{\epsilon_0}\frac{\partial^2}{\partial t^2}P(x,t)+\frac{\sigma}{\epsilon_0}\frac{\partial}{\partial t}E^{\rm ext}(x,t). \tag{2}$$

We can expand this into normal modes

$$E(x,t)=\sum_{n=0}^{\infty}\tilde{E}_n(t)\cos(K_n x), \tag{3}$$

with eigenfrequencies $\omega_n=cK_n=n\pi c/d$, where d denotes the cavity length. After some straightforward manipulations we arrive at the equation for the dynamics of the cavity mode with eigenfrequency ω_c and wave vector K, which is assumed to be quasiresonant with the pump field of frequency ω_p,

$$\ddot{\tilde{E}}+\frac{\sigma}{\epsilon_0}\dot{\tilde{E}}+\omega_c^2\tilde{E}=-\frac{1}{\epsilon_0}\ddot{\tilde{P}}+\frac{\sigma}{\epsilon_0}\frac{\partial}{\partial t}\tilde{E}^{\rm ext}, \tag{4}$$

with $\tilde{P}(t)=(2/d)\int_{-d/2}^{d/2}dx\,P(x,t)\cos(Kx)$ and analogously for $\tilde{E}^{\rm ext}$.

Now we can make the following ansatz for the electric fields and the polarization:

$$\tilde{E}^{\rm ext}(z,t)=\mathcal{E}^{\rm ext}(t)e^{-i\omega_p t}+\text{c.c.},\quad \tilde{E}(t)=\mathcal{E}(t)e^{-i\omega_p t}+\text{c.c.},$$
$$\tilde{P}(t)=\mathcal{P}(t)e^{-i\omega_p t}+\text{c.c.}, \tag{5}$$

where the amplitudes are assumed to vary slowly compared to the rapidly oscillating exponential functions, i.e., $|\dot{\mathcal{E}}|\ll\omega_p|\mathcal{E}|$ and analogously for $\mathcal{P}$ and $\mathcal{E}^{\rm ext}$. Using the above approximations we find the following equation for the mode amplitude:

$$\dot{\mathcal{E}}+(\kappa-i\Delta_c)\mathcal{E}\approx\frac{i\omega_p\mathcal{P}}{2\epsilon_0}+\kappa\mathcal{E}^{\rm ext}, \tag{6}$$

where $\Delta_c=\omega_p-\omega_c$ and $\kappa=\sigma/(2\epsilon_0)$. As a next step we have to introduce the dynamics of the atomic dipole, which is simply modeled as an elastically bound electron. Thus its elongation is governed by the equation of a damped harmonic oscillator driven by an electric field at the atomic position x_a:

$$\ddot{y}(t)+2\Gamma\dot{y}(t)+\omega_0^2 y(t)=\frac{e}{m}E(x_a,t). \tag{7}$$

Let us now introduce the slowly varying complex amplitude $Y(t)$ via the ansatz

$$y(t)=Y(t)e^{-i\omega_p t}+Y^*(t)e^{i\omega_p t}. \tag{8}$$

Upon inserting this into Eq. (7) and solving for the steady state, we obtain

$$Y(t)=\frac{e\mathcal{E}(t)/m}{2\omega_p(-i\Gamma+(\omega_0^2-\omega_p^2)/2\omega_p)}\cos(Kx_a). \tag{9}$$

For a pointlike particle the polarization density can be defined as $P(x,t)=ey(t)\delta(x-x_a)/A$ where A is the cavity cross section and V is the cavity volume. This yields the following expression for the slowly varying amplitude:

$$\mathcal{P}(t)=i\frac{e^2}{m\omega_p V}\frac{\cos^2(Kx_a)}{(\Gamma-i\Delta_a)}\mathcal{E}(t), \tag{10}$$

where $\Delta_a=\omega_p-\omega_0$ and the approximation $(\omega_0^2-\omega_p^2)/(2\omega_p)=(\omega_0-\omega_p)(\omega_0+\omega_p)/(2\omega_p)\approx-\Delta_a$ was made. Putting everything together and splitting the polarization into a real and an imaginary part, Eq. (6) leads to the following equation for the slowly varying field amplitude:

$$\dot{\mathcal{E}}(t)=[-\kappa-\gamma(x_a)+i\Delta_c-iU(x_a)]\mathcal{E}(t)+\kappa\mathcal{E}^{\rm ext}, \tag{11}$$

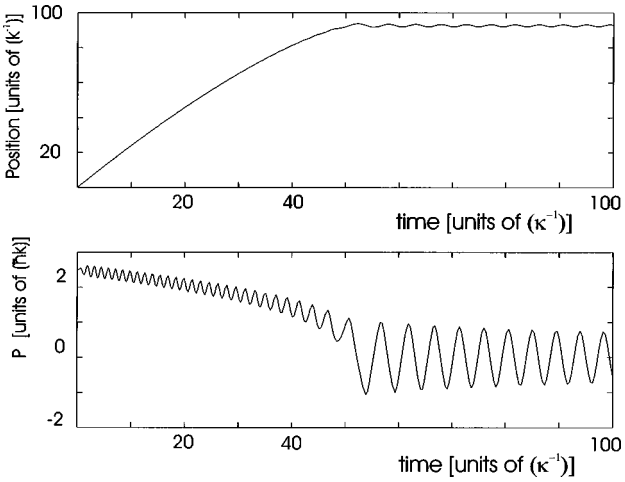

FIG. 2. The particle is cooled while moving along the cavity standing wave until its kinetic energy gets so low that it is trapped in a potential well where it oscillates back and forth.

where $\Delta_c=\omega_p-\omega_c$ and

$$U(x_a)=\frac{\Delta_a}{\Gamma^2+\Delta_a^2}g_0^2\cos^2(Kx_a),$$

$$\gamma(x_a)=\frac{\Gamma}{\Gamma^2+\Delta_a^2}g_0^2\cos^2(Kx_a). \tag{12}$$

Here $g_0^2=e^2/(2V\epsilon_0 m)$ is a measure for the cavity-atom coupling strength. The force on the atom is then given by

$$f(x_a)=\nabla[ey(x_a,t)E(x_a,t)]$$
$$=-\nabla\left[\frac{e^2}{2\omega_p m}\frac{\Delta_a}{\Gamma^2+\Delta_a^2}\mathcal{E}^2\cos^2(Kx_a)\right]. \tag{13}$$

We rewrite Eqs. (11) and (13) in terms of a dimensionless parameter α whose squared absolute value is associated with the average cavity photon number,

$$|\alpha^2|=\frac{\epsilon_0|\mathcal{E}^2|V}{\hbar\omega_p}, \tag{14}$$

and obtain the following set of equations for the atom-cavity dynamics:

$$\dot{\alpha}=[-\kappa-\gamma(x)+i\Delta_c-iU(x)]\alpha+\eta,$$

$$\dot{p}=-|\alpha|^2\frac{d}{dx}U(x), \tag{15}$$

$$\dot{x}=\frac{p}{m},$$

where η characterizes the driving laser strength and the index a for the atomic position has been omitted for convenience. These equations can be easily generalized to three dimensions. Integration of Eq. (15) gives a useful physical picture of the particle being cooled down until it is trapped in a periodic potential. This is shown in Fig. 2, where a particle moves along a laser standing wave with detunings and decay rates chosen such that this ''cavity cooling'' force is optimized. Note that it does not take into account diffusion of any kind nor does it include Doppler cooling, which is, however, small for the chosen parameters. The cooling force can be understood from the purely classical viewpoint as arising from the strong dependence of the intracavity intensity on the position of the atom within the cavity standing wave. For certain parameters the intensity reaches a maximum for an atom at a node whereas the potential $U(x)$ is a minimum there. However, if the particle is slowly moving along the potential $U(x)$ the maximum field intensity will be reached after it has passed the potential minimum due to the finite cavity response time. Accordingly the particle sees a higher intensity and thus a stronger damping force when going up $U(x)$ than it sees an antidamping one when going down $U(x)$. On average this leads to a damping force.

III. SEMICLASSICAL ANALYSIS OF POTENTIALS AND FORCES

For a better understanding of this cooling force found above we will in the following develop a more appropriate semiclassical model of the atom-cavity system, where the internal atomic degrees of freedom as well as the cavity mode are treated quantum mechanically, but the atomic motion is treated classically. The master equation describing our model, cf. Fig. 1, is given in the rotating-wave approximation and in a frame rotating with the pump frequency ω_p as

$$\dot{\rho}=-\frac{i}{\hbar}[H_{\mathrm{JC}},\rho]-\frac{i}{\hbar}[H_P,\rho]-\Gamma(\{\sigma^+\sigma^-,\rho\}_+-2\sigma^-\rho\sigma^+)$$
$$-\kappa(\{a^+a,\rho\}_+-2a\rho a^+), \tag{16}$$

where

$$H_{\mathrm{JC}}=-\hbar\Delta_a\sigma^+\sigma^--\hbar\Delta_c a^\dagger a+\hbar g(x)(a\sigma^++\sigma^-a^\dagger),$$

$$H_P=-i\hbar\eta(a-a^\dagger) \tag{17}$$

are the Jaynes-Cummings and the pump Hamiltonian, respectively, a is the annihilation operator of the cavity mode, and σ^- that of the atomic transition. $g(x)=g_0\cos(kx)$ denotes the atom-cavity coupling strength, which varies sinusoidally according to the cavity standing wave. Instead of solving the master equation one can equivalently deal with the Heisenberg equations for atomic and mode operators. The coupling to the vacuum field resulting in spontaneous emission and cavity decay then accounts for the appearance of noise terms in the Heisenberg equations [34]. One finds

$$\dot{a}=i\Delta_c a-ig(x)\sigma^--\kappa a+\eta+F_1, \tag{18a}$$

$$\dot{\sigma}^-=i\Delta_a\sigma^-+ig(x)\sigma_z a-\Gamma\sigma^-+\sigma_z F_2, \tag{18b}$$

where $\sigma_z=[\sigma^+,\sigma^-]$. F_1 and F_2 are noise operators that essentially contain only free input field operators. Here, it is only important to know that they yield zero when applied to the vacuum,

$$F_1|\mathrm{vac}\rangle=F_2|\mathrm{vac}\rangle=0. \tag{19}$$

A. Steady-state force on a motionless atom

In order to linearize Eqs. (18) we assume that the cavity is very weakly driven. In this case there is at most one photon in the cavity and only the combined atom-cavity states $|1,g\rangle$ (one photon in the cavity mode, atom in the ground state), $|0,e\rangle$ (no photon, atom in the excited state), and $|0,g\rangle$ contribute to the system dynamics. Thus,

$$\langle \sigma_z a\rangle = -\langle a\rangle. \tag{20}$$

One has to keep in mind that this is valid only for low saturation when the excited state is hardly occupied. Then the time evolution of the expectation values may be written as

$$\langle \dot{\vec{Y}}\rangle = \mathbf{A}\langle \vec{Y}\rangle + \vec{Z}_\eta, \tag{21}$$

where

$$\vec{Y}=\begin{pmatrix} a \\ \sigma^- \end{pmatrix}, \quad \mathbf{A}=\begin{pmatrix} i\Delta_c-\kappa & -ig(x) \\ -ig(x) & i\Delta_a-\Gamma \end{pmatrix}, \quad \vec{Z}_\eta=\begin{pmatrix} \eta \\ 0 \end{pmatrix}. \tag{22}$$

Similarly one can derive equations for the expectation values of the normally ordered operator products

$$\vec{X}=\begin{pmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{pmatrix} = \begin{pmatrix} a^\dagger\sigma^- + \sigma^+ a \\ \frac{1}{i}(a^\dagger\sigma^- - \sigma^+ a) \\ a^\dagger a \\ \sigma^+\sigma^- \end{pmatrix}, \tag{23}$$

which read

$$\langle \dot{\vec{X}}\rangle = \mathbf{B}\langle \vec{X}\rangle + \eta\langle \vec{I}\rangle, \tag{24}$$

where

$$\mathbf{B}=\begin{pmatrix} -\gamma & -\Delta & 0 & 0 \\ \Delta & -\gamma & -2g & 2g \\ 0 & g & -2\kappa & 0 \\ 0 & -g & 0 & -2\Gamma \end{pmatrix}, \quad \vec{I}=\begin{pmatrix} \sigma^- + \sigma^+ \\ \frac{1}{i}(\sigma^- - \sigma^+) \\ a + a^\dagger \\ 0 \end{pmatrix}, \tag{25}$$

with $\Delta=\Delta_a-\Delta_c$ and $\gamma=\Gamma+\kappa$. Note that the normal order is important to obtain vanishing averages of products involving noise operators. Solving those equations for $\langle \dot{\vec{X}}\rangle = 0$ one obtains for the expectation values of the motionless atom at steady state

$$\langle \vec{X}\rangle_0 = \frac{\eta^2}{|\det(A)|^2}[2\Delta_a g(x), -2g(x)\Gamma, \Delta_a^2+\Gamma^2, g(x)^2], \tag{26}$$

where $\det(A)$ is the determinant of $\mathbf{A}$, given by

$$\det(A)=\Gamma\kappa + g(x)^2 - \Delta_a\Delta_c - i(\Delta_c\Gamma + \Delta_a\kappa). \tag{27}$$

Comparing Eq. (26) with the solution of Eq. (21) one finds the equalities

$$\langle a^\dagger\sigma^-\rangle = \langle a^\dagger\rangle\langle\sigma^-\rangle, \tag{28}$$

$$\langle a^\dagger a\rangle = \langle a^\dagger\rangle\langle a\rangle.$$

This shows that the operator products factorize for the steady state of a motionless atom in the limit of a weakly driven cavity. One finds for the average photon number from Eq. (26)

$$\langle a^\dagger a\rangle = \eta^2 \frac{\Delta_a^2+\Gamma^2}{[\Gamma\kappa + g(x)^2 - \Delta_a\Delta_c]^2 + (\Delta_a\kappa + \Delta_c\Gamma)^2}. \tag{29}$$

This result differs from what one obtains from ordinary input-output formalism with no atom inside the cavity, where maximum transmission occurs when the cavity is driven resonantly and most light is reflected back at the input mirror when there is a large detuning between cavity and pump laser. If the atom is present inside the cavity, it may shift the cavity resonance frequency due to its interaction with the cavity mode. If the pump is detuned from cavity resonance, the strong atom-cavity coupling for an atom at an antinode may shift the pump into resonance so that maximum transmission occurs. For the force acting on the atom we find [31]

$$F(x)=\dot{P}=\frac{i}{\hbar}[H,P]=-\hbar\nabla g(x)(a^\dagger\sigma^- + \sigma^+ a), \tag{30}$$

and inserting Eq. (26) yields

$$f(x)=\langle F(x)\rangle = -\hbar\eta^2 \frac{\Delta_a \nabla g(x)^2}{[\Gamma\kappa + g(x)^2 - \Delta_a\Delta_c]^2 + (\Delta_a\kappa + \Delta_c\Gamma)^2}. \tag{31}$$

This expression for the force may be integrated to find the potential. For negative detuning $\Delta_a<0$ the atom is attracted to the antinodes (*high field seeker*), whereas the opposite holds for positive detuning.

B. Friction force on a slowly moving atom

It is straightforward to find an expression for the friction coefficient, i.e., the linear velocity dependence of the force for small velocities. We will follow a procedure completely analogous to the one outlined in [31,35], which is valid if the atom moves only a fraction of a wavelength before either a cavity or an atomic decay occurs, i.e., if

$$kv \ll \Gamma, \kappa. \tag{32}$$

In general it is not necessary for both ratios to be much smaller than one, depending on which decay channel predominates. We define the friction coefficient as the force term linear in v and expand the density matrix in powers of v with ρ_0 denoting the zeroth and ρ_1 the first order in v, respectively. We may then write for the master equation to first order in v:

$$v\,\frac{\partial}{\partial x}\rho_0 = \mathcal{L}\rho_1, \tag{33}$$

where

$$\mathcal{L}\rho = -\frac{i}{\hbar}[H,\rho] - \Gamma(\sigma^+\sigma^-\rho + \rho\sigma^+\sigma^- - 2\sigma^-\rho\sigma^+), \tag{34}$$

and the system was assumed to be in steady state with a fixed atomic velocity. We then find the following equations:

$$\begin{aligned} v\,\frac{\partial}{\partial x}\langle\vec{Y}\rangle_0 &= \mathbf{A}\langle\vec{Y}\rangle_1, \\ v\,\frac{\partial}{\partial x}\langle\vec{X}\rangle_0 &= \mathbf{B}\langle\vec{X}\rangle_1 + \eta\langle\vec{I}\rangle_1, \end{aligned} \tag{35}$$

where $\langle\cdots\rangle_0$, $\langle\cdots\rangle_1$ denote the zeroth-order expectation values, which have been calculated in the previous subsection, and first-order expectation values, respectively. Thus with the notation of Eqs. (23) the friction force is given by

$$f_1(x) = -\hbar\nabla g\langle X_1\rangle_1. \tag{36}$$

The full expression for Eq. (36) is listed in Appendix A. Here we only want to note that f_1 consists of two parts, where each of these can be obtained as the full force in a certain limit, which also give physical interpretations for these terms. These limits will be discussed in the following two subsections.

C. Strongly damped cavity

The first limit in which we want to study the behavior of our system is the limit of a strongly damped cavity; i.e., the cavity relaxation is assumed to be much faster than the atomic relaxation so that the cavity adiabatically follows the atomic evolution. As a consequence one can get rid of the cavity operators and find a master equation for the atomic degrees of freedom alone. This can be derived without a weak driving field approximation from the Heisenberg equation for the mode operator (18a), which is already linear. Setting $\dot{a}=0$ in Eq. (18a) and solving for a, one expresses a through atomic and noise operators, which may subsequently be substituted into the Heisenberg equation for the atomic operator (18b). From this one is able to deduce the master equation for the atomic density operator

$$\frac{d\rho}{dt} = -\frac{i}{\hbar}[H_{\mathrm{at}},\rho] + \mathcal{L}\rho, \tag{37}$$

where

$$\begin{aligned} H_{\mathrm{at}} &= -\hbar\tilde{\Delta}(x)\sigma^+\sigma^- + \hbar\tilde{g}(x)(\sigma^+ + \sigma^-), \\ \mathcal{L}\rho &= -\tilde{\Gamma}(x)(\sigma^+\sigma^-\rho + \rho\sigma^+\sigma^- - 2\sigma^-\rho\sigma^+), \end{aligned} \tag{38}$$

and

$$\begin{aligned} \tilde{\Delta}(x) &= \Delta_a - \Delta_c\frac{g(x)^2}{\Delta_c^2+\kappa^2}, \\ \tilde{g}(x) &= \frac{g(x)\,\eta}{\sqrt{\Delta_c^2+\kappa^2}}, \\ \tilde{\Gamma}(x) &= \Gamma + \kappa\frac{g(x)^2}{\Delta_c^2+\kappa^2}. \end{aligned} \tag{39}$$

The general expression for the force with the mode adiabatically eliminated reads

$$\begin{aligned} F(x) &= -\frac{\partial H_{at}}{\partial x} \\ &= -\hbar\Delta_c\frac{\nabla g(x)^2}{\Delta_c^2+\kappa^2}\sigma^+\sigma^- \\ &\quad -\hbar\nabla g(x)\frac{\eta}{\sqrt{\Delta_c^2+\kappa^2}}(\sigma^+ + \sigma^-). \end{aligned} \tag{40}$$

Noting the analogy of Eqs. (37) and (38) to the optical Bloch equations we can immediately take their well-known solutions [31] and apply them to Eq. (40) to find for the force

$$\begin{aligned} f(x) = \langle F(x)\rangle &= -\hbar\Delta_c\frac{\nabla g(x)^2}{\Delta_c^2+\kappa^2}\frac{1}{2}\frac{s}{s+1} \\ &\quad -\hbar\nabla g(x)\frac{\eta}{\sqrt{\Delta_c^2+\kappa^2}}\frac{\tilde{\Delta}}{2\tilde{g}(x)}\frac{s}{s+1}, \end{aligned} \tag{41}$$

where s is defined as $s = 2\tilde{g}(x)^2/(\tilde{\Delta}^2+\tilde{\Gamma}^2)$.

It should be emphasized again that Eq. (41) is valid for any driving amplitude η. It can be shown by expanding Eq. (41) in powers of η^2 that the expression for the force we found before in Eq. (31) is just the contribution proportional to η^2 of Eq. (41). Alternatively it is possible to linearize Eq. (37) for weak pumping and to obtain the corresponding Heisenberg equation for the atomic operator σ^-:

$$\langle\dot{\sigma}^-\rangle = (i\tilde{\Delta}-\tilde{\Gamma})\langle\sigma^-\rangle - i\tilde{g}(x). \tag{42}$$

As the expectation values factorize for weak pumping we can use

$$\langle\sigma^+\sigma^-\rangle = \langle\sigma^+\rangle\langle\sigma^-\rangle \tag{43}$$

and Eq. (40) to calculate the force on the motionless atom. To do so we can set $\langle\dot{\sigma}^-\rangle_0 = 0$ and solve for $\langle\sigma^-\rangle_0$ and we find again Eq. (31) for the force on the motionless atom. To obtain the friction coefficient we expand Eq. (42) in powers of v to obtain the operator expectation values to first order, yielding

$$\langle\sigma^-\rangle_1 = \frac{v\nabla\langle\sigma^-\rangle_0}{(i\tilde{\Delta}-\tilde{\Gamma})}, \tag{44}$$

and insert this into the expression for the force (40), which gives the friction force f_{at} in the bad cavity limit

$$\begin{aligned} f_{\mathrm{at}} &= -\hbar\Delta_c\frac{\nabla g(x)^2}{\Delta_c^2+\kappa^2}(\langle\sigma^+\rangle_1\langle\sigma^-\rangle_0 + \langle\sigma^+\rangle_0\langle\sigma^-\rangle_1 + \text{c.c.}) \\ &\quad -\hbar\nabla g(x)\frac{\eta}{\sqrt{\Delta_c^2+\kappa^2}}(\langle\sigma^+\rangle_1 + \text{c.c.}). \end{aligned} \tag{45}$$

The somewhat unhandy expression is given in Appendix A.

In a previous paper [36] dealing with the same system the cavity dynamics was assumed to be unaffected by the atomic motion and the cavity mode was assumed to be in a quasicoherent state. Hence the cavity mode operator a can be replaced by a c-number α, which is calculated for the steady state of the motionless atom and then reinserted into the master equation. The resulting equations can be considered as optical Bloch equation analogs where the effective coupling strength is given by $g_0\alpha$. The cooling or heating of the atom is then solely governed by the Doppler effect. This is the ultimate case of the limit discussed in this subsection when the cavity mode completely decouples from the system dynamics. Following these assumptions one finds for the steady-state force on the motionless atom again Eq. (31), and the friction force coincides with the friction force in the bad cavity limit $f_{\rm at}$ for $\kappa \gg \Gamma, g$. Hence we will from now on also refer to $f_{\rm at}$ as the ''Doppler cooling force.'' As a final remark let us note that in general the bad cavity limit is a good approximation if $\kappa \gg g$ and the *atom* is driven near resonance.

D. Weakly damped cavity

In the opposite limit of a very good cavity one can assume the internal atomic dynamics to be much faster than the cavity dynamics. Proceeding analogously to the preceding section one sets $\dot{\sigma}^- = 0$ in the linearized Heisenberg equation of the atomic operator σ^- (18b) and uses this to eliminate σ^- in Eq. (18a). Finally one obtains the master equation of the cavity mode density operator

$$\frac{d\rho}{dt} = -\frac{i}{\hbar}[H_{ca}, \rho] + \mathcal{L}\rho, \tag{46}$$

with

$$H_{ca} = -\hbar a^\dagger a \left(\Delta_c - \Delta_a \frac{g(x)^2}{\Delta_a^2 + \Gamma^2} \right) - i\eta(a - a^\dagger),$$
$$\mathcal{L}\rho = -\left(\kappa + \Gamma \frac{g(x)^2}{\Delta_a^2 + \Gamma^2} \right)(a^\dagger a \rho + \rho a^\dagger a - 2a\rho a^\dagger). \tag{47}$$

Equivalently the equation for $\alpha = \langle a \rangle$ can be derived,

$$\dot{\alpha} = [-\kappa - \gamma(x) + i\Delta_c - iU(x)]\alpha + \eta, \tag{48}$$

where $\gamma(x)$ and $U(x)$ are given by Eq. (12). For the force one finds

$$F(x) = -\frac{\partial}{\partial x} H_{ca} = -a^\dagger a \frac{d}{dx} U(x). \tag{49}$$

Upon substitution of the complex number α for the mode operator a one obtains exactly the classical equations of motion (15). One can see now that in the limit in which the cavity dynamics is predominant and the driving field is sufficiently weak the semiclassical model can be seen in complete analogy to the classical treatment of an atomic dipole oscillating in a cavity standing wave.

Now one can calculate the steady-state force for a motionless or slowly moving atom noting again that operator products factorize for a weakly driven cavity, i.e., $\langle a^\dagger a \rangle = \langle a^\dagger \rangle \langle a \rangle$. After solving Eq. (48) for $\langle a \rangle_0$, where $\langle \dot{a} \rangle_0 = 0$, one finds again Eq. (31) for the force on the motionless atom. For very small velocities one proceeds analogously to the previous section. From Eq. (48) one calculates the operator expectation values to first order in v,

$$\langle a \rangle_1 = \frac{v \nabla \langle a \rangle_0}{-\kappa - \gamma(x) + i\Delta_c - iU(x)}, \tag{50}$$

and inserts this into an expansion of the force (49). Finally one obtains the friction force in the good cavity limit

$$f_{ca} = -(\langle a^\dagger \rangle_0 \langle a \rangle_1 + \langle a^\dagger \rangle_1 \langle a \rangle_0 + \text{c.c.}) \frac{d}{dt} U(x), \tag{51}$$

which is fully listed in Appendix A.

The friction force in the bad cavity limit $f_{\rm at}$ and the friction force in the good cavity limit $f_{\rm ca}$ add up to the total friction force f_1 obtained in Eq. (36), i.e.,

$$f_1 = f_{\rm at} + f_{\rm ca}. \tag{52}$$

In general the good cavity limit is a good approximation, if $g, \Gamma \gg \kappa$ and the *cavity* is driven near resonance, i.e., spontaneous decay rate and detuning from the atomic transition are larger than the cavity decay.

A numerical example of the total friction force f_1 can be found in Fig. 3, where we give a contour plot for this quantity versus the cavity and atomic detunings. We find that only for small detunings the Doppler force $f_{\rm at}$ predominates, see the features marked ''Doppler peak'' in Fig. 3, whereas for most parameter regimes the friction force $f_{\rm ca}$, which is induced by the strong atom-cavity coupling, is the major contribution to the total force. In the following section we will investigate this force in more detail and give a more intuitive interpretation of its properties in terms of a Sisyphus cooling mechanism invoking the dressed state formalism.

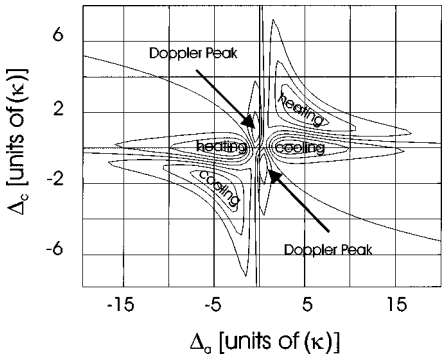

FIG. 3. Contour plot of the friction force for $\Gamma = \kappa$, $g_0 = 3\frac{1}{3}\kappa$. The long stretched features are due to the cavity-induced force $f_{\rm ca}$ and the two sharp peaks are due the Doppler force $f_{\rm at}$.

IV. THE DRESSED-STATE APPROACH: SISYPHUS COOLING

In this section we will interpret the new cavity cooling force found in the previous section using dressed states [31], which are defined as the energy eigenstates of the compound

atom-cavity system. The basic idea is to describe the cooling mechanism as due to transitions between a few of these dressed states, where these transitions are either induced by the pump field or occur due to radiative decays of the atom or the cavity.

A. Adiabatic potentials

The interaction between the cavity mode and the atom shifts the energies of the unperturbed system. The unperturbed eigenvectors can be grouped pairwise into manifolds such that the interaction term in the Jaynes-Cummings Hamiltonian (17) only acts between states of the same manifold.

On diagonalizing the Jaynes-Cummings Hamiltonian expressed in the basis of a given manifold we find the eigenvalues corresponding to the upper and the lower eigenenergy, respectively,

$$\lambda_+ = -(n+1)\Delta_c - \frac{\Delta}{2} + \frac{\Omega_n}{2},$$
$$\lambda_- = -(n+1)\Delta_c - \frac{\Delta}{2} - \frac{\Omega_n}{2}, \tag{53}$$

where the effective Rabi frequency

$$\Omega_n = \sqrt{\Delta^2 + 4g(x)^2(n+1)} \tag{54}$$

was introduced. The corresponding eigenvectors are

$$|+,n\rangle = \cos\,\theta_n|e,n\rangle + \sin\,\theta_n|g,n+1\rangle,$$
$$|-,n\rangle = -\sin\,\theta_n|e,n\rangle + \cos\,\theta_n|g,n+1\rangle, \tag{55}$$

where the angle θ is defined by

$$\sin 2\theta_n = \frac{2g(x)\sqrt{n+1}}{\Omega_n}, \quad \cos 2\theta_n = -\frac{\Delta}{\Omega_n}. \tag{56}$$

The index n denotes the corresponding manifold in the dressed states ladder. It is now possible to express the whole master equation in the basis formed by the eigenstates (55). Without the pump beam the system decays to the ground state $|g,0\rangle$ and there is no occupation of higher levels in the dressed ladder. Due to the low intensity of the pump beam driving the cavity we assume that the system is excited only to the first dressed-atom manifold consisting of $|+,0\rangle$ and $|-,0\rangle$, which are linear combinations of $|e,0\rangle$ and $|g,1\rangle$. Thus we only consider the three states [37]

$$|0\rangle \equiv |g,0\rangle, \quad |+\rangle \equiv |+,0\rangle, \quad |-\rangle \equiv |-,0\rangle \tag{57}$$

and their respective couplings. Couplings to or populations of higher manifolds may be omitted in the ''weak driving'' approximation. Using only this three-dimensional state space it is possible to express the master equation in this basis and to obtain the corresponding equations for populations and coherences, e.g.,

$$\begin{aligned}\dot{\rho}_{0+} &= i\lambda_+\rho_{0+} - \Gamma(\rho_{0+}\cos^2\theta - \rho_{0-}\sin\theta\cos\theta)\\ &\quad - \kappa(\rho_{0+}\sin^2\theta + \rho_{0-}\cos\theta\sin\theta) + \eta,\\ \dot{\rho}_{++} &= -\Gamma(\rho_{++}2\cos^2\theta - (\rho_{-+}+\rho_{+-})\sin\theta\cos\theta)\\ &\quad - \kappa(\rho_{++}2\sin^2\theta + (\rho_{-+}+\rho_{+-})\sin\theta\cos\theta)\\ &\quad + \eta(\rho_{+0}+\rho_{0+}).\end{aligned} \tag{58}$$

These equations can be solved easily for the steady state in dependence on the various system parameters, such as detunings and coupling strengths. As expected one finds that, if the dressed levels are well separated in energy, i.e., $|\Delta| \gg g,\Gamma$, and the pump laser is tuned to resonance with one of the dressed energies (53), preferentially this dressed state gets populated whereas the population of the other state remains negligible. However, it turns out that this coupling of the ground state to the lowest dressed states is much more efficient, if the pump is tuned to that state which turns into $|g,1\rangle$ at positions of zero coupling ($g=0$) than if it is tuned to the other. This is due to the fact that the pump laser drives the cavity and thus is only coupled via the cavity mode to the atom. Hence, in the following we will concentrate on the more efficient scheme.

The force acting on an atom in one of the dressed states can be calculated by expressing Eq. (31) in terms of our dressed basis, which yields

$$f(x) = -\frac{\hbar\nabla\Omega}{2}(\rho_{++}-\rho_{--}) - \hbar\Omega\nabla\theta(\rho_{-+}+\rho_{+-}). \tag{59}$$

Expanding Eq. (58) up to first order in g/Δ, and neglecting coherences and populations of any dressed level other than the one the laser is assumed to drive resonantly one can again derive Eq. (11), which we obtained from a completely classical viewpoint and in the ''weakly damped cavity'' limit (48) before. The same can be done with the force (59) so that the dynamics is described by the set of equations (15). Therefore, assuming large detuning Δ and the pump in resonance with the dressed state consisting essentially of the pure state $|g,1\rangle$, one obtains the same expression for the force and the average photon number as in the good-cavity limit discussed in the previous section.

B. Sisyphus interpretation of the cooling mechanism

Analogously to the interpretation given by Dalibard and Cohen-Tannoudji for the movement of an atom in a free standing wave [32], one can create a similar picture for atomic motion along a weakly pumped cavity standing wave. As an example let us assume that initially the atom-cavity system is in the state $|g,0\rangle$ and the atom moves along the cavity axis with a small velocity v. Furthermore we assume $\Delta_c = 0$ and $\Delta > 0$. In this case the dressed state $|+\rangle$, Eq. (57), is resonant with the driving field, if the atom is located at a node of the cavity field. Hence the system will be preferentially pumped into $|+\rangle$ at these positions. After this pumping process the system will remain in this state while the atom moves further and thus the system energy will vary according to λ_+, Eq. (53). If the atom moves so slowly (kv/κ

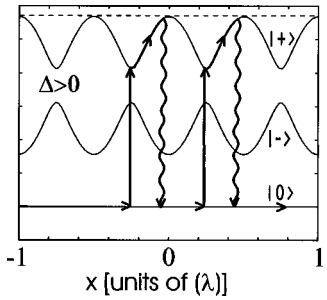

FIG. 4. $\omega_c > \omega_{10}$: The atom is pumped at the minima of the upper dressed level from where it moves upwards and decays back to the flat ground level.

$\ll 1$) that a spontaneous decay of the cavity or of the atom occurs before it reaches the top of the potential λ_+, it loses some of its kinetic energy. This scheme is called Sisyphus cooling and has been treated in detail in connection with atomic movement in a strong standing wave [32] or polarization gradient cooling [38] before.

The sign of Δ and the cavity-pump detuning Δ_c determine whether the atom is mainly moving upwards or downwards the adiabatic potentials $\lambda_\pm$ and thus whether on average it is gaining or losing kinetic energy. As has been seen in Fig. 3 there are two main parameter regions of cooling for a high-Q cavity $(g > \Gamma, \kappa)$.

The first one arises for

$$\Delta_c = 0, \quad \Delta > 0 \tag{60}$$

and corresponds to driving the cavity resonantly, i.e., pumping the atom at the minima of the *upper* dressed level λ_+, from where it loses kinetic energy on the way up. This is shown schematically in Fig. 4.

The second parameter region, which leads to cooling, is around

$$\Delta_c = -\frac{\Delta}{2} - \frac{1}{2}\sqrt{\Delta^2 + 4g_0^2}, \quad \Delta < 0, \tag{61}$$

which corresponds to pumping the system into minima of the *lower* dressed level λ_-, as is shown in Fig. 5. Note that in both cases the level that is pumped is the one that turns into $|g,1\rangle$ for zero atom-mode coupling $[g(x) = 0]$ and contains only small admixtures of $|e,0\rangle$ at the antinodes of the cavity standing wave.

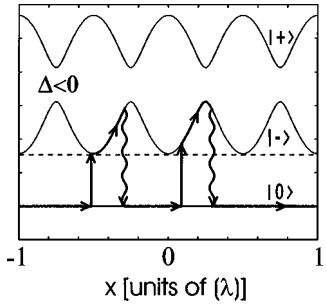

FIG. 5. Sisyphus cooling scheme for $\omega_c < \omega_{10}$: The atom is pumped to the minima of the lower dressed level from where it moves upwards and decays back to the flat ground level.

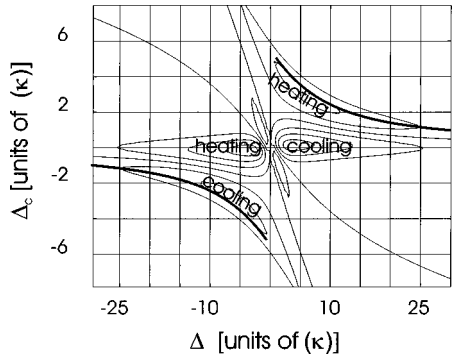

FIG. 6. Contour plot of the friction force for $\Gamma = \kappa$, $g_0 = 5.5\kappa$. The bold lines indicate the pumping of the minima of λ_- (left-hand side) and of the maxima of λ_+ (right-hand side), respectively.

The parameter regimes where the friction force is positive, i.e., accelerating the atom, can be explained analogously. We find for these heating regions

$$\Delta_c = 0, \quad \Delta < 0 \quad \text{and} \quad \Delta_c = -\frac{\Delta}{2} + \frac{1}{2}\sqrt{\Delta^2 + 4g_0^2}, \quad \Delta > 0. \tag{62}$$

The very good agreement of the friction force maxima with the predictions of the dressed state scheme is again found in Fig. 6 for similar parameters, where the bold lines are drawn according to Eqs. (61) and (62) for $\Delta < 0$ (cooling) and $\Delta > 0$ (heating), respectively.

Figure 6 shows that the features of the total friction force can be explained very well by the predictions of the Sisyphus cooling picture. The two peaks around the origin arise for small detunings $\Delta_a \approx \Gamma$ and result from the Doppler cooling force. They can be plotted independently by using the Doppler cooling force f_{at} calculated above. However, for *all* parameters a small contribution of the Doppler force exists. This contribution is a damping one, increasing the overall cooling force if $\Delta_a < 0$ and the minima of the lower dressed level are pumped as shown in Fig. 5, because the laser is red detuned in this case. On the other hand, for $\Delta_a > 0$ and pumping of the minima of the upper level as shown in Fig. 4, the opposite holds and the additional Doppler contribution is an antidamping one because the pump laser is blue detuned.

V. FRICTION FORCE FOR ARBITRARY VELOCITIES

So far we have only calculated the forces on a motionless or a very slowly moving particle. If one is, however, interested in a particle moving with a speed $kv/\kappa > 1$ it is necessary to use the method of continued fractions as introduced in [39] to calculate the force. We will follow a procedure analogous to the one outlined in [40] and only briefly sketch it.

First, mode and atomic operators are combined with their adjoints to form the Hermitian operator $\vec{Z}$ whose eigenvalues are real, which is a prerequisite for the above-mentioned procedure [40]. Starting from Eq. (21) we find

$$\dot{\langle \vec{Z} \rangle} = \mathbf{C}\langle \vec{Z} \rangle + 2\,\eta \vec{e}_1, \tag{63}$$

where

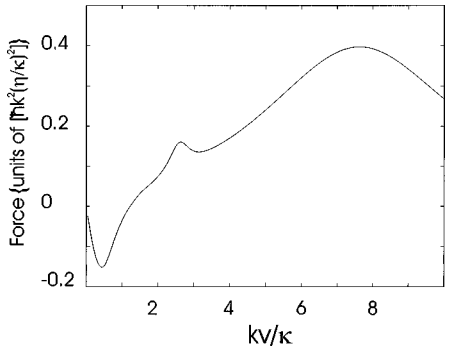

FIG. 7. The averaged dipole force is plotted against the velocity for $\Gamma=\kappa$, $g_0=3+\frac{1}{3}\kappa$, $\Delta=7\kappa$, $\Delta_c=0$.

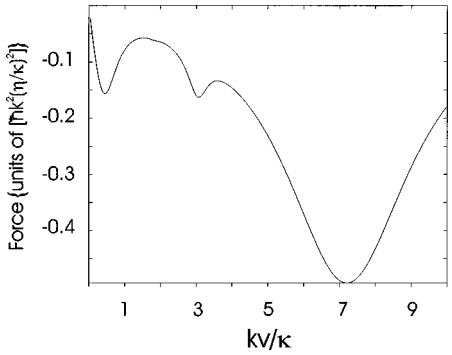

FIG. 8. The averaged dipole force is plotted against the velocity for $\Gamma=\kappa$, $g_0=3\frac{1}{3}\kappa$, $\Delta=-7\kappa$, $\Delta_c=-1\frac{1}{3}$.

$$\langle\vec{Z}\rangle=\begin{pmatrix}\langle a+a^+\rangle\\ \frac{1}{i}\langle a-a^+\rangle\\ \langle\sigma^-+\sigma^+\rangle\\ \frac{1}{i}\langle\sigma^--\sigma^+\rangle\end{pmatrix},$$

$$\mathbf{C}=\begin{pmatrix}-\kappa & -\Delta_c & 0 & g\\ \Delta_c & -\kappa & -g & 0\\ 0 & g & -\Gamma & -\Delta_a\\ -g & 0 & \Delta_a & -\Gamma\end{pmatrix} \tag{64}$$

and $\vec{e}_1=(1,0,0,0)$. Then the atom and mode operator expectation values are expanded into Fourier series according to

$$\langle\vec{Z}\rangle=\sum_n e^{inkvt}\vec{z}_n,\quad \langle\vec{X}\rangle=\sum_n e^{inkvt}\vec{x}_n, \tag{65}$$

where $\vec{X}$ was defined in Eq. (23).

These Fourier expansions are substituted into Eqs. (24) and (63) to find recurrence relations for the Fourier coefficients. The recurrence relations for $\vec{z}_n$ may be solved for a given number of recursions and the result inserted into the recurrence relations for $\vec{x}_n$, which may in turn be solved recursively. As a result we obtain the position averaged force on the atom through the first Fourier component of the first vector element x_1^1. This procedure may be repeated for different values of the atomic velocity for given detunings and system parameters to yield the velocity dependence of the cooling force. In Fig. 7 the force is plotted against the atomic velocity where the driving laser is tuned to the minima of the upper dressed level, cf. the discussion in Sec. IV B. In this case Sisyphus *cooling* appears for low atomic velocities $kv/\kappa<1$. However, for increasing velocity the Sisyphus effect becomes small and Doppler *heating* becomes predominant. Hence the force changes from a damping to an antidamping one and reaches its heating maximum for a speed $kv\approx\Delta_a$ before the Doppler effect becomes smaller again for larger velocities.

Following the discussions of Sec. IV B cooling can alternatively be achieved by pumping the minima of the *lower* level. An example for this is given in Fig. 8. For low velocities Sisyphus cooling prevails again, but for higher velocities it becomes less efficient and Doppler cooling becomes dominant, thereby increasing the overall cooling effect.

VI. STEADY-STATE MOMENTUM DISTRIBUTION AND TEMPERATURE

In the previous sections we have only dealt with the friction force which for properly chosen parameters decelerates the moving atom. However, momentum diffusion counteracts this cooling and leads to a finite steady-state temperature. There are two major contributions to the momentum diffusion. One of these is the random momentum transfer of absorbed and emitted photons and the second process, referred to as dipole heating, is associated with the random jumps between different optical potentials for the atom. These processes prohibit that the atom completely stops at a field antinode, and a finite equilibrium momentum distribution is reached when the contributions of friction and heating cancel.

A. Diffusion

Similarly to Brownian motion an approximate equilibrium temperature can be inferred from the knowledge of the momentum diffusion D and the friction coefficient β. The derivation of the total diffusion D (for an atom with $v=0$) is rather long and is therefore given in Appendix B. As a result we obtain

$$\begin{aligned}D_{\mathrm{tot}}&=D_{\mathrm{SE}}+D_{dp}\\ &=\hbar^2k^2g^2\frac{\eta^2\Gamma}{|\det(A)|^2}\\ &\quad+\hbar^2(\nabla g)^2\frac{\eta^2\Gamma}{|\det(A)|^2}\left(1+\frac{4\Delta_a g^2}{\Gamma}\frac{\Delta_c\Gamma+\Delta_a\kappa}{|\det(A)|^2}\right).\end{aligned} \tag{66}$$

Apart from the last term appearing in the brackets in the expression for D_{dp} this expression for the total diffusion is equivalent to the expression that is found when the cavity mode is treated classically and assumed to decouple from the atomic dynamics, as discussed at the end of Sec. III C. It is the same diffusion in the weak intensity limit as was calculated in [31,35] for an atom moving in a free standing wave.

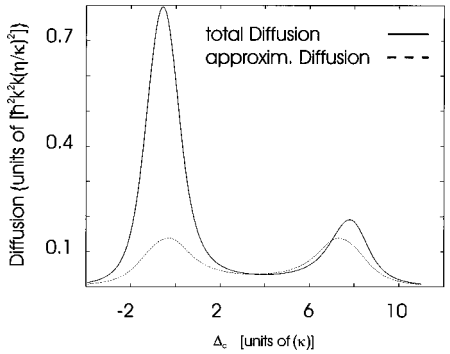

FIG. 9. The total diffusion and the diffusion without the correction term due to the quantum nature of the cavity mode are plotted against the cavity detuning Δ_c for $\Delta=-7\kappa$, $\Gamma=\kappa$, $g_0=3\frac{1}{3}\kappa$.

The expression derived above now shows that the quantum nature of the mode introduces a correction term.

In fact, this correction term can be much larger than the remaining terms, if the cavity is driven near resonance and the detuning Δ between cavity and atomic transition is large. This corresponds exactly to the parameter regime where the cavity-induced cooling mechanism predominates, i.e., where f_{ca} [Eq. (51)] is the dominant contribution of the total friction force. On the other hand, if the atom is driven resonantly and the Doppler cooling force f_{at} [Eq. (45)] is dominant, $\Delta_a \approx 0$ and the correction term disappears. This is shown in Fig. 9 where the wavelength averaged diffusion with and without the correction term is plotted against the pump-laser–cavity detuning Δ_c for fixed Δ. When Δ_c is large and the atom is driven resonantly the correction term is small whereas it is significant for $\Delta_c \approx 0$ when Sisyphus cooling dominates over Doppler cooling. Figure 10 shows a plot of the total friction force and the Doppler and cavity cooling components of which it is composed. One can see the strong Sisyphus effect for $\Delta_c \approx 0$. The cooling is a maximum when the lower dressed level is excited at its minima and turns into a heating force when the cavity is pumped resonantly, $\Delta_c = 0$. For larger detunings the cavity mediated force becomes diminishingly small and the total force is dominated by the Doppler component, which has its peaks for atom detunings $\Delta_a \approx \pm\Gamma$.

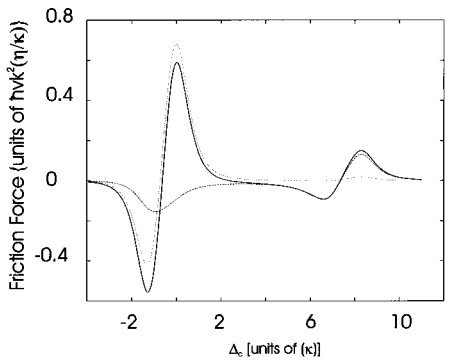

FIG. 10. The total friction force f_1 (solid curve), the Doppler force f_{at} (dashed), and the cavity-mediated force f_{ca} (dotted) are plotted against the laser-cavity detuning Δ_c for the same parameters as in Fig. 9

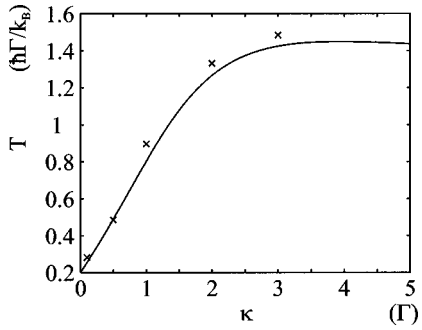

FIG. 11. Going beyond the Doppler limit for $\Gamma \gg \kappa$. $g_0=2\Gamma$, $\Delta=-1.9\Gamma$, $\Delta_c=-1.3\Gamma$. T is given in units of $\hbar\Gamma/k_B$ and κ in units of Γ. The crosses mark the results obtained from quantum Monte Carlo simulations.

Let us note again that, as discussed above and as can be clearly seen from Figs. 9 and 10, the cavity-induced friction force and the quantum correction term of the momentum diffusion coefficient assume their respective maxima for the same parameters.

B. Steady-state temperature

The values for the friction coefficient and the diffusion averaged over a wavelength may be related to the temperature by

$$k_B T = \frac{\overline{D_{\mathrm{tot}}}}{\overline{f_1}}, \tag{67}$$

as has been investigated in more detail in [31].

For cooling a two-level atom in a free standing wave the so-called Doppler limit, given by $k_B T_{\mathrm{min}} = \hbar\Gamma$, imposes a lower limit on the attainable temperatures. One must consider that for Doppler cooling the energy transfer channel is the atomic spontaneous decay and equilibrium temperatures can be as low as the width of this transfer channel, i.e., the natural linewidth.

In our case of cavity-induced cooling the dominant transfer channel is through the cavity decay and temperatures should therefore be limited by the cavity decay rate. This can be seen clearly in Fig. 11 where the equilibrium temperature is plotted against the cavity loss rate κ and where a linear dependence is observed for small κ giving rise to final temperatures of the order of $\hbar\kappa/k_B$. If κ becomes larger than the spatial modulation of the dressed levels the effect washes out and Doppler cooling becomes dominant with final temperatures around the Doppler limit. The results are nicely confirmed by fully quantum Monte Carlo simulations, denoted by the crosses in Fig. 11, where the external degrees of freedom are also quantized.

VII. CONCLUSION

We have shown that one of the key effects leading to a reduction of the kinetic energy of a particle strongly coupled to a cavity field can be understood from a purely classical model of a point dipole coupled to the field inside an optical

resonator, where the only dissipative process is cavity damping. Via this damping kinetic energy will be extracted from the moving dipole until it comes to a complete stop at a field antinode. Including quantum mechanics in the atomic and field dynamics implies spontaneous emission of the atom and quantizing the cavity field as well as atomic motion. This leads to various heating processes as well as extra cooling forces such as, e.g., the Doppler cooling mechanism. Hence the atom reaches (at least approximately) a certain steady-state temperature determined by all these heating and cooling mechanisms. This temperature of the order of $kT \approx \hbar\kappa$ can be much smaller then the optical potential depth so that the atom can be well trapped. Of course in a realistic setup the transverse motion of the atom has to be included as well. Due to the finiteness of the effective mode volume the particle will of course eventually escape. We have performed some preliminary calculations in 2D and 3D, which show long confinement times ($>$ms) for a transverse Gaussian profile of 10–100 μm width. This should provide for enough time for typical cavity QED experiments. A second interesting question considers the dynamics of several atoms in such a cavity. In this case one finds cooling but also a strong cavity mediated correlation effect on the dynamics of the atoms. Of course one could go to the extreme limit of storing a Bose condensate in such a cavity dipole trap. Here the requirements on the cavity finesse and coupling are reduced because of the large number of atoms, which will lead to strong cavity field phase shifts anyway. On the other hand to avoid spontaneous emission, large detunings between pump field and atomic transition have to be used. Eventually this could provide an excellent method to store, observe, and manipulate a condensate.

ACKNOWLEDGMENTS

We thank K. M. Gheri and K. Ellinger for stimulating discussions, and H. Stecher for computational support. This work was supported by the Österreichischer Fonds zur Förderung der wissenschaftlichen Forschung under Project No. S06506-PHY.

APPENDIX A: FRICTION FORCE FOR THE GOOD AND BAD CAVITY LIMITS

The total friction force f_1 was shown to be the sum of the cavity mediated force f_{ca}, which is obtained after adiabatic elimination of the atomic operators and the Doppler force f_{at}, which is obtained after adiabatic elimination of the mode operators,

$$f_1 = f_{\mathrm{ca}} + f_{\mathrm{at}}. \tag{A1}$$

The respective expressions are given below:

$$\begin{aligned} f_{at} = &-\hbar(\nabla g)^2 \eta^2 4(-\Delta_a^3\Delta_c^4\Gamma - \Delta_a^2\Delta_c^3 g^2\Gamma + \Delta_a\Delta_c^2 g^4\Gamma \\ &+ \Delta_c g^6\Gamma - \Delta_a\Delta_c^4\Gamma^3 + \Delta_c^3 g^2\Gamma^3 - 2\Delta_a^3\Delta_c^2 g^2\kappa + 2\Delta_a g^6\kappa \\ &+ 2\Delta_c g^4\Gamma^2\kappa - 2\Delta_a^3\Delta_c^2\Gamma\kappa^2 - \Delta_a^2\Delta_c g^2\Gamma\kappa^2 + 3\Delta_a g^4\Gamma\kappa^2 \\ &- 2\Delta_a\Delta_c^2\Gamma^3\kappa^2 + \Delta_c g^2\Gamma^3\kappa^2 - 2\Delta_a^3 g^2\kappa^3 \\ &- \Delta_a^3\Gamma\kappa^4 - \Delta_a\Gamma^3\kappa^4)\,\frac{1}{|\det(A)|^6}, \end{aligned} \tag{A2}$$

$$\begin{aligned} f_{ca} = &-\hbar(\nabla g)^2 \eta^2 4(-2\Delta_a^2\Delta_c^2\Gamma + 2g^4\Gamma - 2\Delta_c^2\Gamma^3 - 4\Delta_a^3\Delta_c\kappa \\ &+ 4\Delta_a^2 g^2\kappa - 4\Delta_a\Delta_c\Gamma^2\kappa + 4g^2\Gamma^2\kappa \\ &+ 2\Delta_a^2\Gamma\kappa^2 + 2\Gamma^3\kappa^2)\,\frac{\Delta_a g^2}{|\det(A)|^6}. \end{aligned} \tag{A3}$$

APPENDIX B: CALCULATION OF THE DIFFUSION

We will in the following give a short derivation of the diffusion constant in a cavity standing wave by use of the quantum regression theorem (QRT). The QRT states that operator two-time averages $\langle a_\nu(t)a_\mu(t')\rangle$ obey the same equations of motion as single-time averages $\langle a_\nu(t)\rangle$. If one can write

$$\frac{d}{dt}\langle a_\nu(t)\rangle = \Lambda_{\nu\mu}\langle a_\mu(t)\rangle, \tag{B1}$$

it follows that

$$\frac{d}{dt}\langle a_\nu(t)a_\lambda(t')\rangle = \Lambda_{\nu\mu}\langle a_\mu(t)a_\lambda(t')\rangle. \tag{B2}$$

The diffusion is defined as the momentum spread and related to integrals involving force correlations [31,35],

$$2D = \frac{d}{dt}\Delta P^2(t) = \mathrm{Re}\int_0^\infty dt[\langle F(0)F(t)\rangle - \langle F(0)\rangle\langle F(t)\rangle], \tag{B3}$$

where quasistationary conditions were assumed. One contribution to the total diffusion is due to spontaneous emission. Although the force f_{vac} arising from the atom-vacuum coupling,

$$f_{\mathrm{vac}} = -\mathbf{d}\nabla\mathbf{E}_{\mathrm{free}}, \tag{B4}$$

where $\mathbf{d} = \vec{d}(\sigma^+ + \sigma^-)$ is the electric dipole operator, yields zero when averaged over the vacuum field its fluctuations give rise to diffusion. Thus its contribution to Eq. (B3) reads

$$D_{\mathrm{SE}} = \mathrm{Re}\int_0^\infty dt\langle \mathbf{d}(0)\nabla\mathbf{E}_{\mathrm{free}}(0)\mathbf{d}(t)\nabla\mathbf{E}_{\mathrm{free}}(t)\rangle. \tag{B5}$$

Using the rotating-wave approximation one obtains for the integrand

$$\langle\sigma^+(0)\sigma^-(t)\vec{d}\nabla\mathbf{E}^+_{\mathrm{free}}(0)\vec{d}\nabla\mathbf{E}^-_{\mathrm{free}}(t)\rangle = \langle\sigma^+(0)\sigma^-(t)\rangle\sum_{\Delta k}(\vec{d}\vec{\epsilon}_k)^2|\vec{k}|^2\frac{\hbar\omega_k}{2\epsilon_0 V}e^{+i\omega_k t} = \langle\sigma^+(0)\sigma^-(t)\rangle\hbar k^2 2\Gamma\delta(t), \tag{B6}$$

where the last line has been found by taking the sum over the two polarization directions, converting the sum into an integral over frequency, and including the free space density of modes. Then the Markoff approximation was made by assuming the factors appearing next to the exponential to be quasiconstant over the range of integration, so that they can be taken out of the integral, which then can be approximated by a δ function. Taking the characteristic dipole radiation pattern into account this result would have to be modified by a constant factor ($\frac{2}{5}$ for circularly polarized light, for instance). Substituting this result for the integrand back into Eq. (B5) and using Eq. (26) for $\langle \sigma^+ \sigma^- \rangle$ at steady state, we finally obtain

$$D_{\mathrm{SE}} = \hbar^2 k^2 \Gamma \frac{\eta^2 g^2}{|\det(A)|^2}. \tag{B7}$$

The second contribution to the diffusion is due to the fluctuations of the dipole force. With the notations of Eq. (25) the dipole force operator reads

$$F(x) = -\hbar g(x) X_1. \tag{B8}$$

We introduce the abbreviation

$$\langle a_\nu, a_\mu \rangle = \langle \delta a_\nu \delta a_\mu \rangle = \langle a_\nu a_\mu \rangle - \langle a_\nu \rangle \langle a_\mu \rangle, \tag{B9}$$

where $\delta a_\nu = a_\nu - \langle a_\nu \rangle$ is the fluctuation of the operator a_ν around its mean and $\langle a_\nu, a_\mu \rangle$ is the operator covariance. The part of the diffusion due to the dipole fluctuations can be written as

$$D_{dp} = \hbar^2 (\nabla g)^2 \ \mathrm{Re} \int_0^\infty dt \langle \delta X_1(0) \delta X_1(t) \rangle. \tag{B10}$$

Starting from Eqs. (24) and (21) we can derive the following covariance equations where the QRT is invoked to obtain equations for the two-time averages:

$$\frac{d}{dt} \langle X_1(0), \vec{Y}(t) \rangle = \mathbf{A} \langle X_1(0), \vec{Y}(t) \rangle, \tag{B11}$$

where

$$\langle X_1(0), \vec{Y}(t) \rangle = \begin{pmatrix} \langle X_1(0), a(t) \rangle \\ \langle X_1(0), \sigma^-(t) \rangle \end{pmatrix}. \tag{B12}$$

Similarly one finds

$$\frac{d}{dt} \langle X_1(0), \vec{X}(t) \rangle = \mathbf{B} \langle X_1(0), \vec{X}(t) \rangle + \eta \langle X_1(0), \vec{I} \rangle, \tag{B13}$$

with an analogous meaning of $\langle X_1(0), \vec{X}(t) \rangle$.

These two linear coupled systems of first-order differential equations can be solved conveniently by performing a Laplace transform that is defined for a function $f(t)$ as

$$L\{g(t)\} = \int_0^\infty dt\, e^{-st} g(t) = \tilde{g}(s), \tag{B14}$$

with the property

$$L\left\{ \frac{d}{dt} g(t) \right\} = -g(0) + s\tilde{g}(s). \tag{B15}$$

We can see from Eq. (B10) that we are only interested in the integral over the force correlation, which amounts to setting $s=0$ in our Laplace transformed sets of equations. (B11) can then be solved by inversion of **A** and the result substituted into Eq. (B13), considering that $\vec{I}$ is a linear combination of $\vec{Y}, \vec{Y}^\dagger$ (25). Equation (B13) in turn can be solved by inversion of **B** and the solution vector contains the desired correlation as its first element. We still have to calculate the initial conditions for the operator covariances that appeared when we Laplace transformed our two sets of equations. They can be calculated as follows.

The mode operators obey the relation

$$[a, a^\dagger] = 1. \tag{B16}$$

As the atom is only very weakly driven we may compare it to a harmonic oscillator of which only the first two levels are occupied at most. Therefore we have

$$[\sigma^-, \sigma^+] = 1. \tag{B17}$$

Now we can solve the linearized equations (18) for steady state, i.e., we demand $\dot{a} = \dot{\sigma}^- = 0$ and obtain expressions of the form

$$a_{st} = c_a + \overline{F_1}, \quad \sigma^-_{st} = c_\sigma + \overline{F_2}, \tag{B18}$$

where c_a, c_σ are c numbers corresponding to the expectation values in steady state and $\overline{F_1}, \overline{F_2}$ are noise operators. Instead of specifying them we require the commutation properties (B16), (B17) to be fulfilled, from which we deduce

$$[\overline{F_1}, \overline{F_1}^+] = 1, \quad [\overline{F_2}, \overline{F_2}^+] = 1. \tag{B19}$$

These properties allow averages over operator products to be calculated easily when applying Eq. (B18) and considering that the noise operators yield zero when applied to the vacuum, as an example:

$$\begin{aligned} \langle a^\dagger \sigma^- \sigma^+ a \rangle &= \langle (c_a^* + \overline{F_1}^+)(c_\sigma + \overline{F_2})(c_\sigma^* + \overline{F_2}^+)(c_a + \overline{F_1}) \rangle \\ &= \langle a^\dagger a \rangle \langle \sigma^+ \sigma^- \rangle + \langle a^+ a \rangle. \end{aligned} \tag{B20}$$

All terms containing noise operators are zero except for the following one, where they appear in antinormal order,

$$\langle c_a^* \overline{F_2 F_2}^+ c_a \rangle = c_a^* c_a = \langle a^\dagger a \rangle. \tag{B21}$$

Consequently the initial conditions may be calculated easily, i.e.,

$$\begin{aligned} \langle X_1(0), X_1(0) \rangle &= \langle X_1(0) X_1(0) \rangle - \langle X_1(0) \rangle \langle X_1(0) \rangle \\ &= \langle \sigma^+ \sigma^- \rangle + \langle a^\dagger a \rangle. \end{aligned} \tag{B22}$$

We find analogous results for all the other initial conditions. Inserting them all into the inverted and Laplace transformed Eq. (B13) and taking the real parts of the first solution vector element as demanded by Eq. (B10) we finally find the total diffusion given in Eq. (66).

[1] S. Haroche, in *Fundamental Systems in Quantum Optics*, Proceedings of the Les Houches Summer School, Session LIII, edited by J. Dalibard, J.-M. Raimond, and J. Zinn-Justin (North-Holland, Amsterdam, 1992).
[2] E. M. Pourcel, Phys. Rev. **69**, 681 (1946).
[3] D. Kleppner, Phys. Rev. Lett. **47**, 233 (1981).
[4] E. T. Jaynes and F. W. Cummings, Proc. IEEE **51**, 89 (1963).
[5] D. Meschede, H. Walther, and G. Mueller, Phys. Rev. Lett. **54**, 551 (1985); M. Brune *et al.*, *ibid.* **59**, 1899 (1987).
[6] Q. A. Turchette, R. J. Thompson, and H. J. Kimble, Appl. Phys. B: Lasers Opt. **60**, S1 (1995).
[7] T. W. Mossberg, M. Lewenstein, and D. J. Gauthier, Phys. Rev. Lett. **67**, 1723 (1991).
[8] M. Lewenstein and L. Roso, Phys. Rev. A **47**, 3385 (1993).
[9] K. Molmer and G. S. Agarwal, Opt. Commun. **120**, 275 (1995).
[10] Y. Shevy, B. Crosignani, and A. Yariv, Phys. Rev. A **46**, 1421 (1992).
[11] R. Graham, D. F. Walls, and W. Zhang, Phys. Rev. A **44**, 7777 (1991).
[12] T. Zaugg, P. Meystre, G. Lenz, and M. Wilkens, Phys. Rev. A **49**, 3011 (1994).
[13] J. I. Cirac, Phys. Rev. A **46**, 4354 (1992).
[14] A. C. Doherty, A. S. Parkins, S. M. Tan, and D. F. Walls, Phys. Rev. A **56**, 833 (1997).
[15] K. S. Wong, M. J. Collett, and D. F. Walls, Opt. Commun. **137**, 269 (1997).
[16] A. C. Doherty, A. S. Parkins, S. M. Tan, and D. F. Walls, Phys. Rev. A **57**, 4804 (1998).
[17] C. M. Savage, Phys. Rev. Lett. **63**, 1376 (1989).
[18] M. Lewenstein, T. W. Mossberg, and R. J. Glauber, Phys. Rev. Lett. **59**, 775 (1987).
[19] K. M. Gheri and H. Ritsch, Phys. Rev. A **56**, 3187 (1997).
[20] J. Jacobson *et al.*, Phys. Rev. Lett. **74**, 4835 (1995).
[21] M. O. Scully, G. M. Meyer, and H. Walther, Phys. Rev. Lett. **76**, 4144 (1996).
[22] J. I. Cirac *et al.*, Opt. Commun. **97**, 353 (1993).
[23] G. Rempe, Appl. Phys. B: Lasers Opt. **60**, 233 (1995).
[24] A. Lambrecht *et al.*, Europhys. Lett. **36**, 93 (1996).
[25] J.-F. Roch *et al.*, Phys. Rev. Lett. **78**, 634 (1997).
[26] K. An *et al.*, Phys. Rev. Lett. **73**, 3375 (1994).
[27] H. Mabuchi, Q. A. Turchette, M. S. Chapman, and H. J. Kimble, Opt. Lett. **21**, 1393 (1996).
[28] W. Ren, J. D. Cresser, and H. J. Carmichael, Phys. Rev. A **46**, 7162 (1992).
[29] W. Ren and H. J. Carmichael, Phys. Rev. A **51**, 752 (1995).
[30] P. Horak *et al.*, Phys. Rev. Lett. **79**, 4974 (1997).
[31] C. Cohen-Tannoudji, in *Fundamental Systems in Quantum Optics*, Proceedings of the Les Houches Summer School, Session LIII, edited by J. Dalibard, J.-M. Raimond, and J. Zinn-Justin (North-Holland, Amsterdam, 1992), pp. 1–164.
[32] J. Dalibard and C. Cohen-Tannoudji, J. Opt. Soc. Am. B **2**, 1707 (1985).
[33] Anthony E. Siegman, *Lasers* (University Science Books, Mill Valley, 1986).
[34] C. W. Gardiner, *Quantum Noise* (Springer, Berlin, 1991).
[35] J. P. Gordon and A. Ashkin, Phys. Rev. A **21**, 1606 (1980).
[36] A. C. Doherty, A. S. Parkins, S. M. Tan, and D. F. Walls, Phys. Rev. A **56**, 833 (1997).
[37] S. Haroche, M. Brune, and J. M. Raimond, Europhys. Lett. **14**, 19 (1991).
[38] J. Dalibard and C. Cohen-Tannoudji, J. Opt. Soc. Am. B **6**, 2023 (1989).
[39] V. G. Minogin and O. T. Serimaa, Opt. Commun. **30**, 373 (1979).
[40] K. Berg-Sørensen, Y. Castin, E. Bonderup, and K. Mølmer, J. Phys. B **25**, 4195 (1992).