Acceleration of convolution in FBP algorithm for CT

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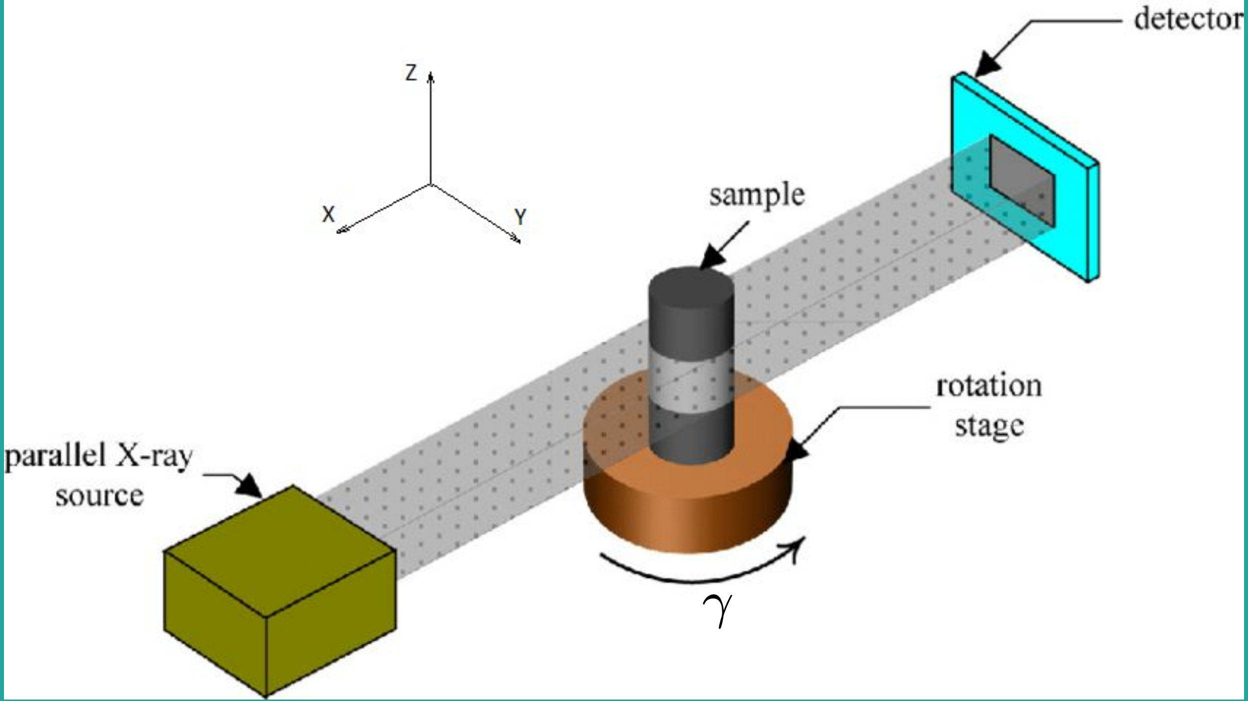
X-ray tomography with parallel projection scheme



https://www.researchgate.net/figure/Schematic-of-a-typical-parallel-beam-X-ray-mCT-facility_fig1_339852360

Intensity decrease is described by Beer–Lambert law:

$$I(l) = I_0 e^{-K \cdot l}$$

So the measured value of attenuation coefficient is:

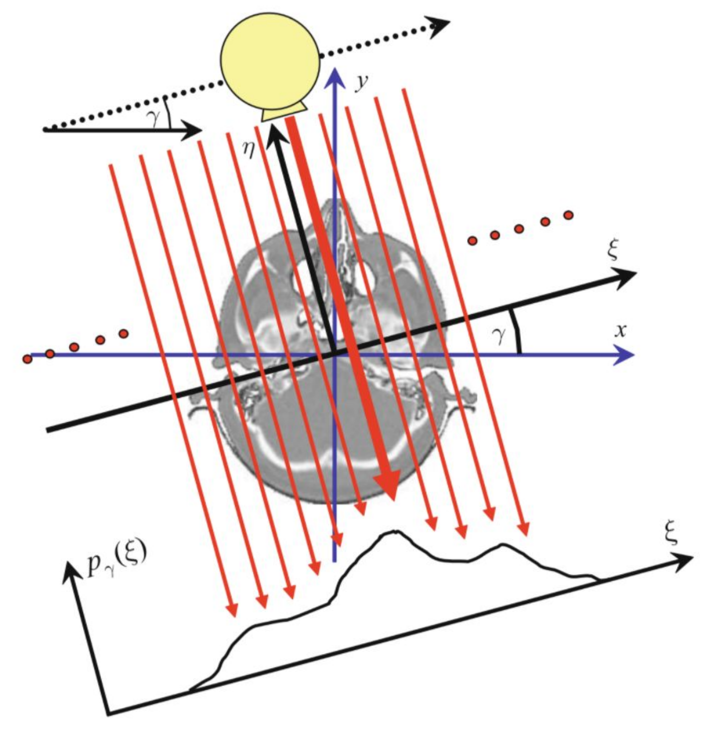
$$K = \frac{1}{l} \ln \left(\frac{I_0}{I(l)} \right)$$

From other side, K can be expressed as Radon transform:

$$K_\gamma(y, z) = \int_0^l k_{(\gamma,y,z)}(s) ds$$

The result of CT is the values of k(γ, y, z)(s) in the volume of the object under examination.

Algorithm of tomographic reconstruction: Filtered Back-Projection (FBP)

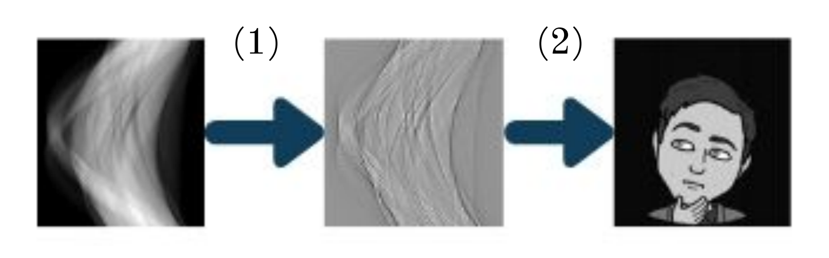


https://www.springer.com/gp/book/9783642072574

FBP implements the inverse Radon transform:

$$f(x,y) = \int_0^{\pi} \left\{ \int_{-\infty}^{+\infty} p_\gamma(z) h(\xi - z) dz \right\} d\gamma$$

FBP algorithm has 2 steps:



FBP: (1) - O(N^3), (2) - O(N^3)

HFBP: (1) - M*N^2, (2) - O(N^2 log N)

FIR filtering

$$x_\gamma(i) = \begin{cases} p_\gamma(i), & i \in \overline{0, N-1} \\ 0, & i \notin \overline{0, N-1} \end{cases}$$

$$\forall i \in \overline{0, N-1} \quad [x_\gamma * h](i) = \sum_{k=-N}^{N} h(k) x_\gamma(i-k)$$

Ramp filter:

$$h(k) = \begin{cases} 1/4, & k = 0 \\ 0, & \text{for even k} \\ -1/(\pi k)^2, & \text{for odd k} \end{cases} \qquad k \in \overline{-N, N}$$

IIR filtering

$$y_i^+(x) = \sum_{k=0}^{M-1} a_k x_{i-k} + \sum_{k=1}^{M} b_k y_{i-k}^+(x)$$

$$y_i^-(x) = \sum_{k=0}^{M-1} a_k x_{i+k} + \sum_{k=1}^{M} b_k y_{i+k}^-(x)$$

$$\hat{y}_i(x) = y_i^-(x) + y_i^+(x), \quad i \in \overline{0, N-1}$$

Project aim

- is to decrease the number of multiplication and addition operations for convolution step in HFBP algorithm,

  namely:

- to decrease the number of operations for convolution of a signal of length N from N^2 to NM operations where M << N

  under condition that:

- reconstruction error is fixed (while measured as RMSE on phantom images).

Project is relevant

- in nano-tomography when monitoring reconstruction is used
- in medicine when monitoring reconstruction is used
- for real-time support of surgeries

Problem statement

For the rows of sinogram S, which was obtained in practice, the following problem should be solved:

$$\frac{1}{|S|}\sum_{s\in S}\|\hat{y}(s) - h * s\|_2 \to \min_{a,b}$$

The following function is minimized in order to solve this problem:

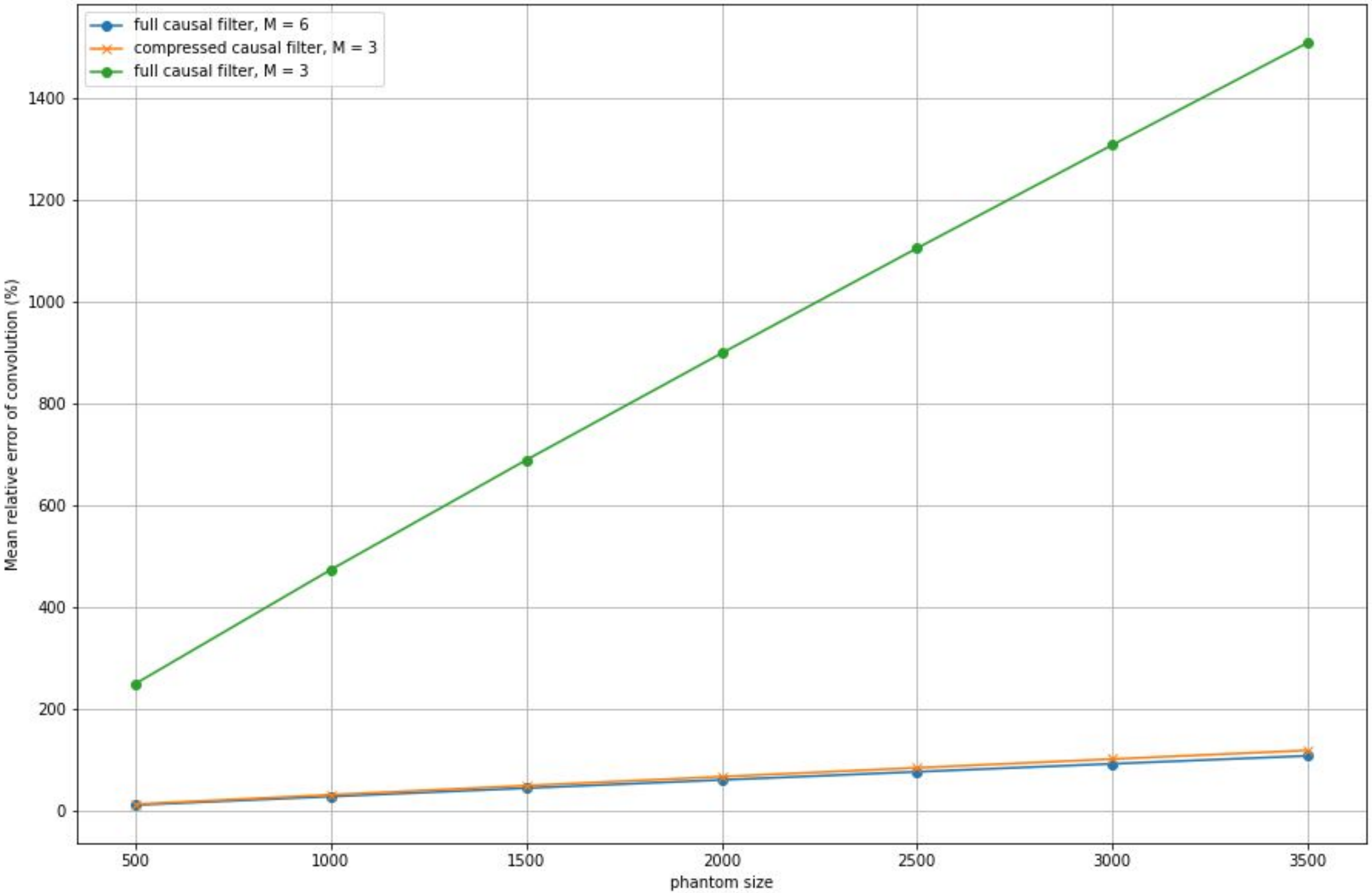
$$Q[\beta](a,b) = \|\hat{y}(\beta) - \beta * h\|_2^2 = \sum_{i=0}^{N-1}[\hat{y}_i(\beta) - (\beta * h)_i]^2$$

Baseline algorithm:

$$Z(a,b) = \|y^{+}(\alpha) - \alpha * h^{+}\|_2^2 \to \min_{a,b}$$

$$\alpha_k = \delta_k + 1 = \begin{cases} 2, & k = N \div 2 \\ 1, & k \neq N \div 2 \end{cases}, \quad k \in \overline{0, N-1}$$

Result 1: the number of arithmetical operations performed at filtering step was decreased **twice** for the same accuracy.



Approximation of compressed causal component of ramp-filter

$$\mathbf{d}_k = \frac{-1}{[\pi * (2k+1)]^2}, \quad k \in \{0, 1, 2, \dots\}.$$

$$\|y^+(\delta) - \mathbf{D}\|_2^2 = \sum_{i=0}^{N-1} [y_i^+(\delta) - \mathbf{D}_i]^2 \to \min_{\tilde{a},\tilde{b}}$$

$$A_{2k+1} = \tilde{a}_k, A_{2k} = 0, k \in \overline{0, M-1}$$

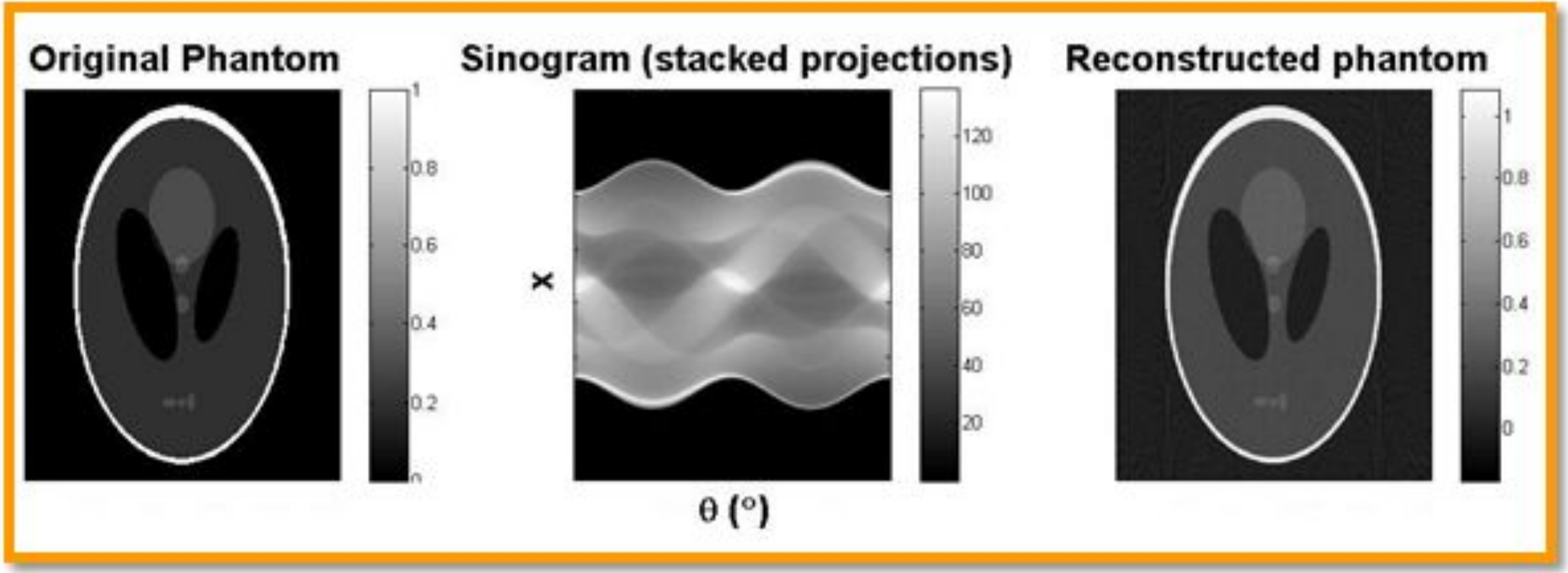
$$B_{2k} = \tilde{b}_k, B_{2k+1} = 0, k \in \overline{0, M-1}$$

$$Y_i^+(x) = \sum_{k=1}^{2M} A_k x_{i-k} + \sum_{k=1}^{2M} B_k y_{i-k}^+(x)$$

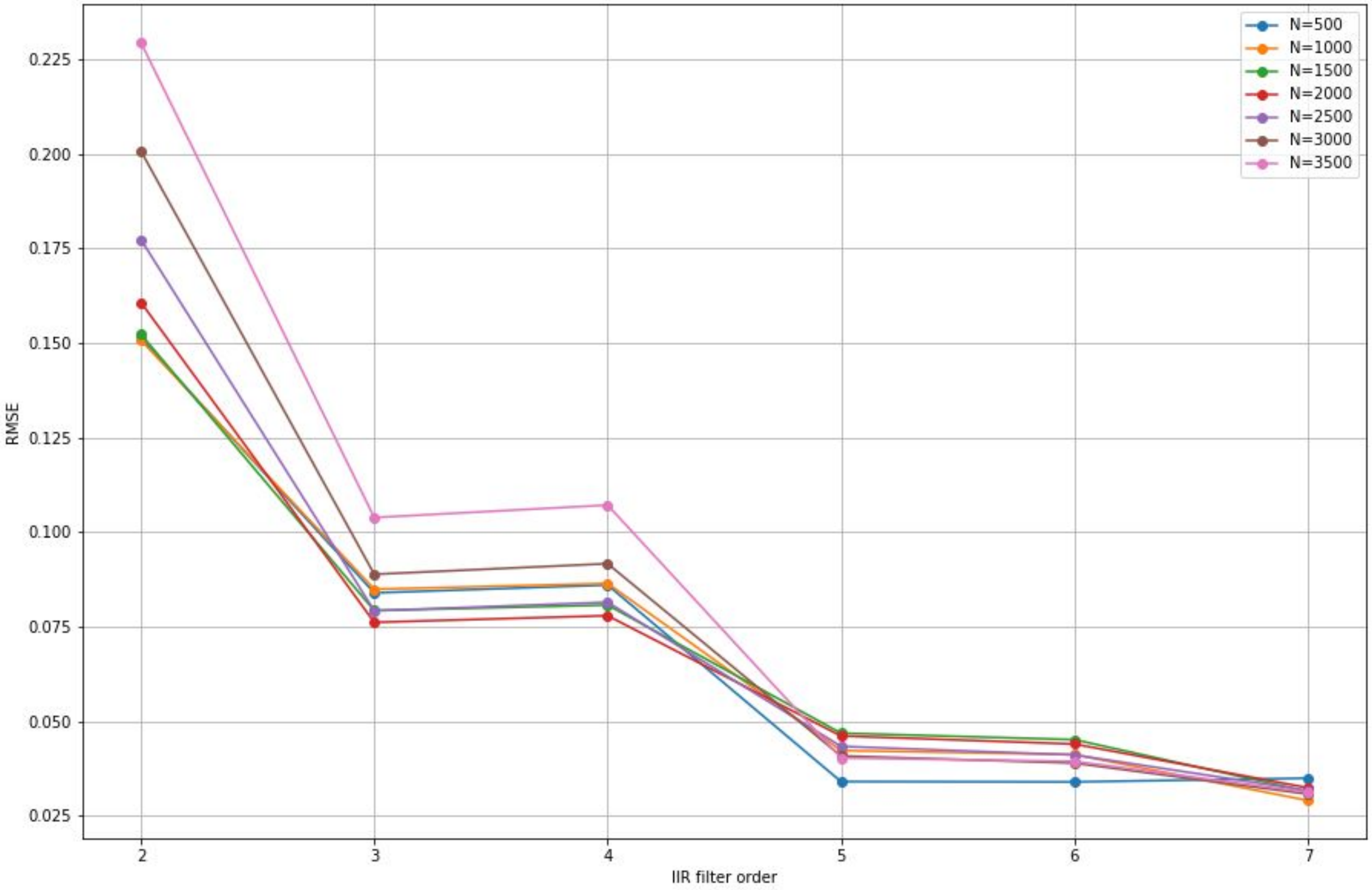
$$Y_i^-(x) = \sum_{k=1}^{2M} A_k x_{i+k} + \sum_{k=1}^{2M} B_k y_{i+k}^-(x)$$

$$Y(x) = Y_i^+(x) + Y_i^-(x) + 0.25 \times x$$

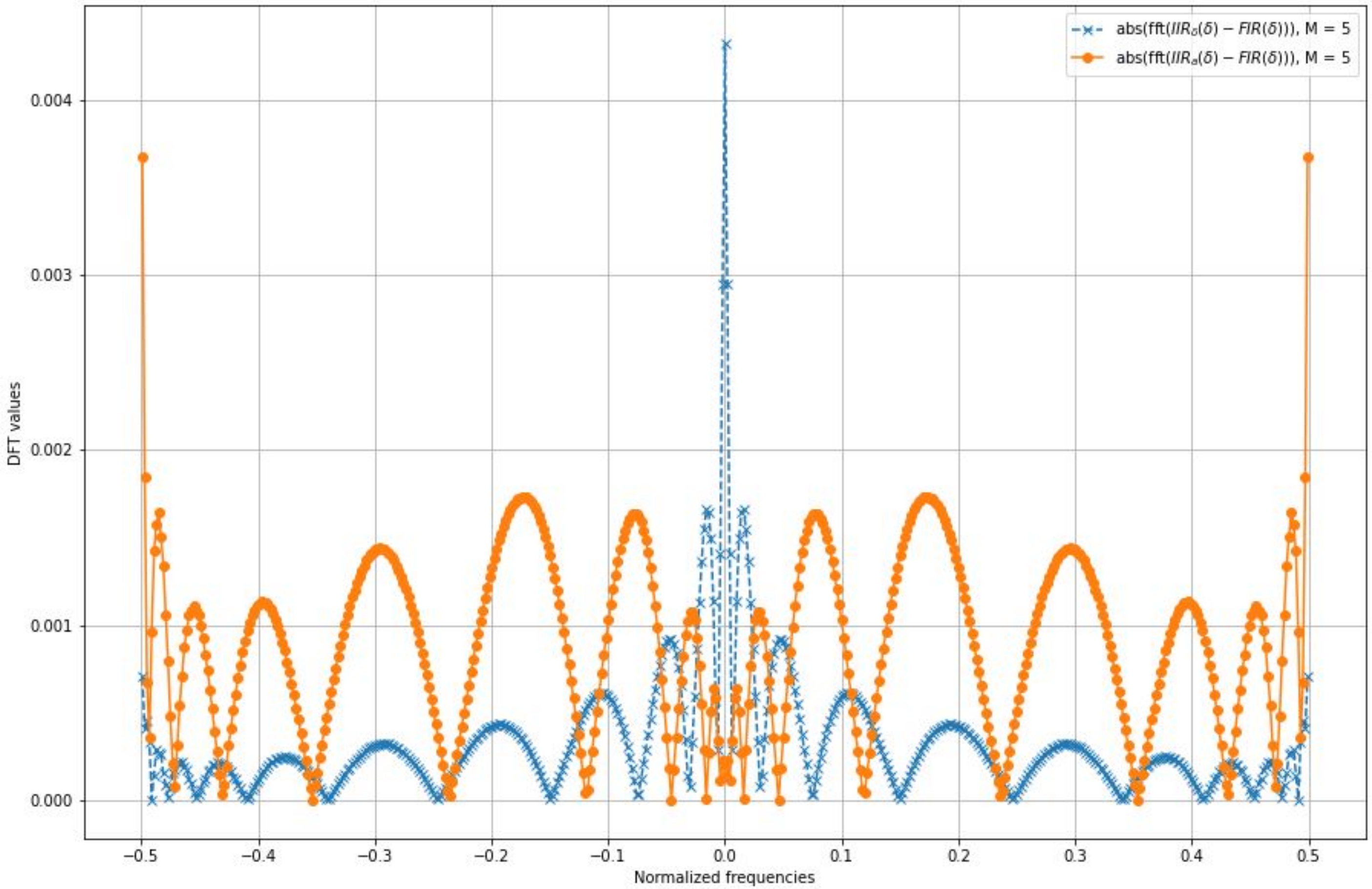
Phantom of Shepp-Logan.



Reconstruction error for phantom of several sizes N as function of recurrent filter order



Result 2: Filter approximation, that minimized the 2nd norm of impulse response, was shown to be non optimal for reconstruction.



$x \setminus \beta$	δ	α
δ	0.0017	0.0036
α	0.3923	0.0105
s	1.0199	0.0306

Table 1. Relative error RE(x) of IIR filtering for inputs x with different spectrum.

$$RE(x) = \frac{\|\hat{y}(x) - \beta * x\|_2}{\|\beta * x\|_2}$$

Thanks for attention!