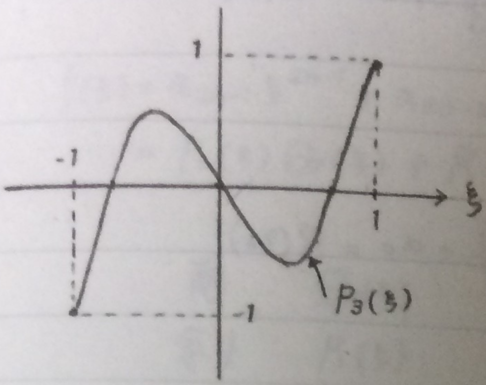
3点積分公式の導出



3次のLegendre関数

$$P_3(\xi) = \frac{1}{2}(5\xi^3 - 3\xi)$$

$$f(\xi) = a_5\xi^5 + a_4\xi^4 + a_3\xi^3 + a_2\xi^2 + a_1\xi + a_0$$

$$= \left(\frac{5}{2}\xi^3 - \frac{3}{2}\xi\right)\left(\frac{2}{5}a_5\xi^2 + \frac{2}{5}a_4\xi + \frac{2}{5}a_3 + \frac{6}{25}a_5\right) + \left(a_2 + \frac{3}{5}a_4\right)\xi^2$$

$$+ \left(a_1 + \frac{3}{5}a_3 + \frac{9}{25}a_5\right)\xi + a_0$$

$$
\begin{array}{r|l}
 & \frac{2}{5}a_5\xi^2 + \frac{2}{5}a_4\xi + \frac{2}{5}\left(a_3 + \frac{3}{5}a_5\right) \\
\hline
\frac{5}{2}\xi^3 - \frac{3}{2}\xi & a_5\xi^5 + a_4\xi^4 + a_3\xi^3 + a_2\xi^2 + a_1\xi + a_0 \\
 & a_5\xi^5 \qquad - \frac{3}{5}a_5\xi^3 \\
\hline
 & a_4\xi^4 + \left(a_3 + \frac{3}{5}a_5\right)\xi^3 + a_2\xi^2 + a_1\xi + a_0 \\
 & a_4\xi^4 \qquad - \frac{3}{5}a_4\xi^2 \\
\hline
 & \left(a_3 + \frac{3}{5}a_5\right)\xi^3 + \left(a_2 + \frac{3}{5}a_4\right)\xi^2 + a_1\xi + a_0 \\
 & \left(a_3 + \frac{3}{5}a_5\right)\xi^3 \qquad - \frac{3}{5}\left(a_3 + \frac{3}{5}a_5\right)\xi \\
\hline
 & \left(a_2 + \frac{3}{5}a_4\right)\xi^2 + \left(a_1 + \frac{3}{5}a_3 + \frac{9}{25}a_5\right)\xi + a_0
\end{array}
$$

$$\int_{-1}^{1} f(\xi)\,d\xi = \underbrace{\int_{-1}^{1}\left(\frac{5}{2}\xi^3 - \frac{3}{2}\xi\right)\left(\frac{2}{5}a_5\xi^2 + \frac{2}{5}a_4\xi + \frac{2}{5}a_3 + \frac{6}{25}a_5\right)d\xi}_{=0}$$

$$+ \underbrace{\int_{-1}^{1}\left[\left(a_2 + \frac{3}{5}a_4\right)\xi^2 + \left(a_1 + \frac{3}{5}a_3 + \frac{9}{25}a_5\right)\xi + a_0\right]d\xi}_{=R(\xi)}$$

$P_3(\xi)$の零点

$P_3(\xi_i) = 0$ より

$$\underline{\xi_1 = -\sqrt{\frac{3}{5}}}\,,\quad \underline{\xi_2 = 0}\,,\quad \underline{\xi_3 = \sqrt{\frac{3}{5}}}$$

$$
\begin{cases}
f(\xi_1) = \left(a_2 + \dfrac{3}{5}a_4\right)\dfrac{3}{5} - \left(a_1 + \dfrac{3}{5}a_3 + \dfrac{9}{25}a_5\right)\sqrt{\dfrac{3}{5}} + a_0 = R(\xi_1) \\[2ex]
f(\xi_2) = a_0 = R(\xi_2) \\[2ex]
f(\xi_3) = \left(a_2 + \dfrac{3}{5}a_4\right)\dfrac{3}{5} + \left(a_1 + \dfrac{3}{5}a_3 + \dfrac{9}{25}a_5\right)\sqrt{\dfrac{3}{5}} + a_0 = R(\xi_3)
\end{cases}
$$

Lagrange 補間

$$
\tilde{R}(\xi) = \frac{(\xi-\xi_2)(\xi-\xi_3)}{(\xi_1-\xi_2)(\xi_1-\xi_3)}R(\xi_1) + \frac{(\xi-\xi_1)(\xi-\xi_3)}{(\xi_2-\xi_1)(\xi_2-\xi_3)}R(\xi_2) + \frac{(\xi-\xi_1)(\xi-\xi_2)}{(\xi_3-\xi_1)(\xi_3-\xi_2)}R(\xi_3)
$$

$$
\int_{-1}^{1} f(\xi)\,d\xi = \int_{-1}^{1} R(\xi)\,d\xi = \int_{-1}^{1} \tilde{R}(\xi)\,d\xi = \sum_{i=1}^{3} w_i R(\xi_i) = \underline{\underline{\sum_{i=1}^{3} w_i f(\xi_i)}}
$$

$$
\begin{cases}
w_1 = \displaystyle\int_{-1}^{1} \frac{\xi\left(\xi - \sqrt{\frac{3}{5}}\right)}{\left(\sqrt{\frac{3}{5}}\right)\left(2\sqrt{\frac{3}{5}}\right)}\,d\xi = \int_{-1}^{1} \frac{5}{6}\xi\left(\xi - \sqrt{\frac{3}{5}}\right)d\xi = \frac{5}{6}\left[\frac{\xi^3}{3} - \sqrt{\frac{3}{5}}\,\frac{\xi^2}{2}\right]_{-1}^{1} = \underline{\frac{5}{9}} \\[3ex]
w_2 = \displaystyle\int_{-1}^{1} \frac{\left(\xi + \sqrt{\frac{3}{5}}\right)\left(\xi - \sqrt{\frac{3}{5}}\right)}{\left(\sqrt{\frac{3}{5}}\right)\cdot\left(-\sqrt{\frac{3}{5}}\right)}\,d\xi = -\int_{-1}^{1} \frac{5}{3}\left(\xi^2 - \frac{3}{5}\right)d\xi = -\frac{5}{3}\left[\frac{\xi^3}{3} - \frac{3}{5}\xi\right]_{-1}^{1} = \underline{\frac{8}{9}} \\[3ex]
w_3 = \displaystyle\int_{-1}^{1} \frac{\left(\xi + \sqrt{\frac{3}{5}}\right)\xi}{\left(2\sqrt{\frac{3}{5}}\right)\left(\sqrt{\frac{3}{5}}\right)}\,d\xi = \int_{-1}^{1} \frac{5}{6}\xi\left(\xi + \sqrt{\frac{3}{5}}\right)d\xi = \frac{6}{5}\left[\frac{\xi^3}{3} + \sqrt{\frac{3}{5}}\,\frac{\xi^2}{2}\right]_{-1}^{1} = \underline{\frac{5}{9}}
\end{cases}
$$

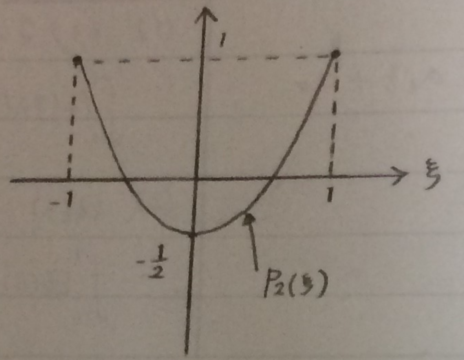
$$
\sum_{i=1}^{3} w_i f(\xi_i) = \frac{2}{3}\left(a_2 + \frac{3}{5}a_4\right) + 2a_0 = \underline{\frac{2}{5}a_4 + \frac{2}{3}a_2 + 2a_0}
$$

厳密解

$$
\int_{-1}^{1} f(\xi)\,d\xi = \underline{\frac{2}{5}a_4 + \frac{2}{3}a_2 + 2a_0}
$$

5次式の $f(\xi)$ まで厳密解は等しい

2点積分公式の導出



2次のLegendre関数

$$P_2(\xi) = \frac{1}{2}(3\xi^2 - 1)$$

$$f(\xi) = a_3\xi^3 + a_2\xi^2 + a_1\xi + a_0$$

$$= \left(\frac{3}{2}\xi^2 - \frac{1}{2}\right)\left(\frac{2}{3}a_3\xi + \frac{2}{3}a_2\right) + \left(a_1 + \frac{1}{3}a_3\right)\xi + \left(a_0 + \frac{1}{3}a_2\right)$$

$$\begin{array}{r l}
 & \frac{2}{3}a_3\xi + \frac{2}{3}a_2 \\
\frac{3}{2}\xi^2 - \frac{1}{2} \Big) & a_3\xi^3 + a_2\xi^2 + a_1\xi + a_0 \\
 & a_3\xi^3 \qquad\qquad - \frac{1}{3}a_3\xi \\
\hline
 & a_2\xi^2 + \left(a_1 + \frac{1}{3}a_3\right)\xi + a_0 \\
 & a_2\xi^2 \qquad\qquad\qquad - \frac{1}{3}a_2 \\
\hline
 & \left(a_1 + \frac{1}{3}a_3\right)\xi + \left(a_0 + \frac{1}{3}a_2\right)
\end{array}$$

$$\int_{-1}^{1} f(\xi)\,d\xi = \underbrace{\int_{-1}^{1}\left(\frac{3}{2}\xi^2 - \frac{1}{2}\right)\left(\frac{2}{3}a_3\xi + \frac{2}{3}a_2\right)d\xi}_{=0}$$

$$+ \underbrace{\int_{-1}^{1}\left[\left(a_1 + \frac{1}{3}a_3\right)\xi + \left(a_0 + \frac{1}{3}a_2\right)\right]d\xi}_{=R(\xi)}$$

$P_2(\xi)$の零点

$$P_2(\xi_i) = 0$$

$$\underline{\xi_1 = -\sqrt{\frac{1}{3}}}\ ,\quad \xi_2 = \underline{\sqrt{\frac{1}{3}}}$$

$$\begin{cases} f(\xi_1) = -\left(a_1 + \frac{1}{3}a_3\right)\sqrt{\frac{1}{3}} + \left(a_0 + \frac{1}{3}a_2\right) \\ f(\xi_2) = \left(a_1 + \frac{1}{3}a_3\right)\sqrt{\frac{1}{3}} + \left(a_0 + \frac{1}{3}a_2\right) \end{cases}$$

Lagrange補間

$$\tilde{R}(\xi) = \frac{(\xi - \xi_2)}{(\xi_1 - \xi_2)} R(\xi_1) + \frac{(\xi - \xi_1)}{(\xi_2 - \xi_1)} R(\xi_2)$$

$$\int_{-1}^{1} f(\xi)\,d\xi = \int_{-1}^{1} R(\xi)\,d\xi = \int_{-1}^{1} \tilde{R}(\xi)\,d\xi = \sum_{i=1}^{2} w_i R(\xi_i) = \sum_{i=1}^{2} w_i f(\xi_i)$$

$$\begin{cases} w_1 = \int_{-1}^{1} \frac{\left(\xi - \sqrt{\frac{1}{3}}\right)}{\left(-2\sqrt{\frac{1}{3}}\right)}\,d\xi = -\frac{\sqrt{3}}{2}\left[\frac{\xi^2}{2} - \sqrt{\frac{1}{3}}\,\xi\right]_{-1}^{1} = 1 \\ w_2 = \int_{-1}^{1} \frac{\left(\xi + \sqrt{\frac{1}{3}}\right)}{\left(2\sqrt{\frac{1}{3}}\right)}\,d\xi = \frac{\sqrt{3}}{2}\left[\frac{\xi^2}{2} + \sqrt{\frac{1}{3}}\,\xi\right]_{-1}^{1} = 1 \end{cases}$$

$$\sum_{i=1}^{2} w_i f(\xi_i) = 2\left(a_0 + \frac{1}{3}a_2\right) = \frac{2}{3}a_2 + 2a_0$$

厳密解

$$\int_{-1}^{1} f(\xi)\,d\xi = \frac{2}{3}a_2 + 2a_0$$

3次式の$f(\xi)$まで厳密解は等しい

No.

Date

2次元への拡張

$$I = \int_{-1}^{1}\int_{-1}^{1} f(\xi,\eta)\,d\xi\,d\eta$$

$$\fallingdotseq \int_{-1}^{1}\left(\sum_{i=1}^{n} w_i f(\xi_i,\eta)\right)d\eta$$

$$\fallingdotseq \sum_{j=1}^{n} w_j \left(\sum_{i=1}^{n} w_i f(\xi_i,\eta_j)\right) = \underline{\underline{\sum_{i=1}^{n}\sum_{j=1}^{n} w_i w_j f(\xi_i,\eta_j)}}$$

3次元への拡張

$$I = \int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} f(\xi,\eta,\zeta)\,d\xi\,d\eta\,d\zeta$$

$$\fallingdotseq \int_{-1}^{1}\int_{-1}^{1}\left(\sum_{i=1}^{n} w_i f(\xi_i,\eta,\zeta)\right)d\eta\,d\zeta$$

$$\fallingdotseq \int_{-1}^{1}\left\{\sum_{j=1}^{n} w_j \left(\sum_{i=1}^{n} w_i f(\xi_i,\eta_j,\zeta)\right)\right\}d\zeta$$

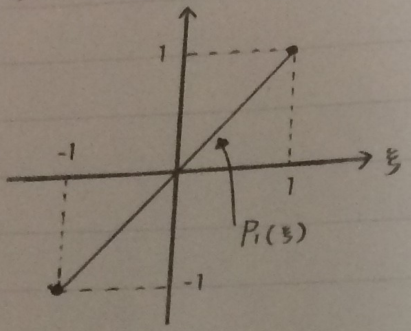
$$\fallingdotseq \sum_{k=1}^{n} w_k \left\{\sum_{j=1}^{n} w_j \left(\sum_{i=1}^{n} w_i f(\xi_i,\eta_j,\zeta_k)\right)\right\} = \underline{\underline{\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n} w_i w_j w_k f(\xi_i,\eta_j,\zeta_k)}}$$

$\boxed{\text{まとめ}}$

n	i	ξ_i	w_i
2	1	$-\sqrt{1/3}$	1
	2	$\sqrt{1/3}$	1
3	1	$-\sqrt{3/5}$	$5/9$
	2	0	$8/9$
	3	$\sqrt{3/5}$	$5/9$

Date

1点積分公式の導出



1次のLegendre関数

$P_1(\xi)=\xi$

$$f(\xi)=a_1\xi+a_0$$
$$=\xi a_1+a_0$$

$$\int_{-1}^{1}f(\xi)\,d\xi=\underbrace{\int_{-1}^{1}\xi a_1\,d\xi}_{=0}+\underbrace{\int_{-1}^{1}a_0\,d\xi}_{=R(\xi)}$$

$P_1(\xi)$の零点

$$P_1(\xi_i)=0$$

$$\xi_1=0$$

$$f(\xi_1)=a_0=R(\xi_1)$$

$$\int_{-1}^{1}f(\xi)\,d\xi=\int_{-1}^{1}a_0\,d\xi=2a_0=2R(\xi_1)=2f(\xi_1)$$

厳密解

$$\int_{-1}^{1}f(\xi)\,d\xi=2a_0$$

1次式の$f(\xi)$のみ厳密解は等しい